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Characterizations of Postman Sets ⋆

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Abstract

Using results by McKee and Woodall on binary matroids, we prove that the set of postman sets has odd cardinality, generalizing a result by Toida on the cardinality of cycles in Eulerian graphs. We study the relationship between T-joins and blocks of the underlying graph, obtaining a decomposition of postman sets in terms of blocks. We conclude by giving several characterizations of T-joins which are postman sets and commenting on practical issues.

Keywords: T-joins, postman sets, cardinality, graph block, decomposition

1 Introduction

We will consider undirected graphs $G = (V, E)$. The set of odd degree vertices of G will be denoted by $O(G)$, or simply by O when it is clear what the underlying graph is. Given a subset T of vertices with $|T|$ even, a set of edges $J \subset E$ is a *T-join* if $O(G_J) = T$ where $G_J = (V, J)$. We will be interested in the family $\mathcal{T}$ of minimal T-joins: an inclusion-wise minimal T-join is just a T-join such that G_J is acyclic. Of course, $\mathcal{T}$ is a *clutter*. When $T = \emptyset$, the empty

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set is the unique minimal $\emptyset$-join, and it is convenient to work instead with the clutter of cycles (regarded as edge-sets) $\mathcal{C}$, so that every non-empty $\emptyset$-join may be written as a union of disjoint cycles. When $T = O(G)$, the minimal T-joins are called *postman sets*, and we will indicate the corresponding clutter by $\mathcal{P}$. We observe that although there are always postman sets, perhaps only the empty set (i.e. $\mathcal{P} = \{\emptyset\}$), we may have $\mathcal{T} = \emptyset$ if some connected component of G contains an odd number of vertices of T. Similarly, $\mathcal{C}$ could be empty.

In 1973, S. Toida [4] proved that in an Eulerian graph there is an odd number of cycles passing through any given edge. This can be shown by deleting the edge, say with endpoints u and v, from the graph and showing that there is an odd number of (simple) u, v-paths in the resulting graph G'. In this case $O(G') = \{u, v\}$, and the u, v-paths in G' are precisely the postman sets in G'. T. McKee [2] showed in 1984 that Toida's result actually characterizes Eulerian graphs: every edge is in an odd number of cycles if and only if $O(G) = \emptyset$. It is worth mentioning that in 1990, D. Woodall [6] gave an alternative proof of McKee's converse, and both McKee and Woodall obtained it as a consequence of more general results in the frame of binary matroids, which we reproduce here as Theorem 2.3.

We use McKee's and Woodall's results directly to show a characterization of the family of postman sets through a condition involving all minimal T-joins and cycles, the precise statement being given in Corollary 2.4. As a consequence of this characterization, in Corollary 2.5 we generalize Toida's result to postman sets in any graph, obtaining that $\mathcal{P}$ has odd cardinality. Although to prove this extension we rely on the McKee's and Woodall's results, it could also be proved inside graph theory (without explicit mention of binary matroids), for example by induction on the number of edges.

In view of McKee's result, it is natural to wonder whether $|\mathcal{T}|$ odd implies $T = O$. However, this is not true. A simple way of looking at McKee's converse of Toida's result is to consider the symmetric difference of all cycles. Similarly, E will be itself a T-join (see Lemma 2.1 below) and therefore $T = O$ if every edge is in an odd number of minimal T-joins. However, even for postman sets not always do we have the latter property.

The blocks of the graph play an important role in the structure of T-joins and postman sets, and we study this interplay in Section 3. We show that if $E_T = \{e \in E : e \in J \text{ for some } J \in \mathcal{T}\}$ and $H_T = \{e \in E : e \notin J \text{ for all } J \in \mathcal{T}\}$, then E_T and H_T are the union of (the edges of) blocks of G, necessarily disjoint. This is strengthened for postman sets, which gives a block decomposition of postman sets. We go on to show in Theorem 4.1 that the set of edges E_O may be written as a symmetric difference of an odd number of postman sets, sharpening Corollary 2.5. Our last result gives further characterizations of postman sets. In the final Section we comment on how our results are reflected on the structure of the matrix associated with the clutter $\mathcal{T}$ of minimal T-joins, and how they could be used in practice to identify those $\mathcal{T}$'s which may be looked at as families of postman sets.

2 Toida and McKee's results for postman sets

Denoting by $A \,\Delta\, Z$ the symmetric difference of the sets A and Z, we will make frequent use of the following well known result (see e.g. [1, p. 168]):

Lemma 2.1 *If J' is a T'-join, then J is a T-join if and only if $J \,\Delta\, J'$ is a $(T \,\Delta\, T')$-join.*

Since non-empty $\emptyset$-joins are disjoint unions of cycles, T-joins and cycles are inter-related:

Corollary 2.2 *If J is a non-minimal T-join, then it is the disjoint union of a minimal T-join and cycles.*

Following Woodall [6], a *binary matroid* is a pair (S, W) where S is a finite set and W is a subspace of 2^S (with scalar operations modulo 2). Also, a *circuit* in a binary matroid (S, W) is a minimal non-empty set in W (identifying subsets and characteristic functions). One of the main results in McKee [2] and Woodall [6] is:

Theorem 2.3 (McKee 1984, Woodall 1990) *Suppose (S, W) is a binary matroid. Then $S \in W$ if and only if each element of S lies in an odd number of circuits. Equivalently, S is the Boolean sum of some set of circuits if and only if S is the Boolean sum of the set of all circuits.*

By considering $S = E$ and W the linear subspace spanned by minimal T-joins and cycles, we have:

Corollary 2.4 *E is the symmetric difference of all postman sets and cycles. Conversely, if $O \neq \emptyset$ and E is the symmetric difference of all minimal T-joins and cycles, then $T = O$.*

Since the symmetric difference of all postman sets and cycles is either an O-join or a $\emptyset$-join depending on whether there is an odd number of postman sets, and since E is an O-join, there must be an odd number of postman sets. This is true even if $O = \emptyset$, where $\mathcal{P} = \{\emptyset\}$. Thus,

Corollary 2.5 *The family of postman sets of G has odd cardinality.*

3 T-joins and Blocks

According to West [5, p. 155], a block of a loopless graph is a maximal connected subgraph with no cut-vertex. For these graphs, the only possible blocks are isolated vertices, cut-edges or maximal 2-connected subgraphs. When loops are present, it is rather tricky to include them within blocks with this definition. As no loop is in any minimal T-join and we are only interested in edges, in this paper we will adopt the following:

Definition 3.1 A *block* of the graph G is a cut-edge, a loop, or the set of edges of a maximal 2-connected loopless subgraph of G.

We will need the following result on the intersection of two clutters:

Lemma 3.2 *Let $\mathcal{Y}$ and $\mathcal{Z}$ be clutters on the same base set X, and suppose $Y \in \mathcal{Y}$ is such that for every $Z \in \mathcal{Z}$ there exist $Y' \in \mathcal{Y}$ and $Z' \in \mathcal{Z}$ with $Y' \cap Z' = \emptyset$ and $Y' \cup Z' \subset Y \,\Delta\, Z$. Then $Y \cap Z = \emptyset$ for all $Z \in \mathcal{Z}$.*

A first consequence of Lemma 3.2 is the following:

Lemma 3.3 *Suppose $\mathcal{T} \neq \emptyset$ and let $e \in E$ be such that $e \notin J$ for all minimal T-join J, i.e. $e \in H_T$. Then $C \subset H_T$, for every cycle C with $e \in C$.*

Lemma 3.3 considers 2-connected blocks of G: either for any edge e in such a block there exists $J \in \mathcal{T}$ with $e \in J$, or else the edges of the block do not intersect any minimal T-join. Since loops are in no minimal T-join, the other interesting blocks to us are the cut-edges (bridges), and these are taken care of by Lemma 3.5. To prove it, we will use the following well known result [1]:

Lemma 3.4 *If $S \subset V$ and J is a T-join, then $|S \cap T| \equiv |\delta(S) \cap J| \pmod 2$, where $\delta(S)$ is the set of edges having exactly one endpoint in S. In particular, if $|S \cap T|$ is odd then $\delta(S) \cap J \neq \emptyset$.*

Lemma 3.5 *Suppose $\mathcal{T} \neq \emptyset$.*

(a) If $e \in E$ is a cut-edge of G, then either $e \in J$ for all T-join J or $e \notin J$ for all $J \in \mathcal{T}$.

(b) If e is not a cut-edge, then there exists a T-join J with $e \notin J$.

Combining Lemmas 3.3 and 3.5 we have:

Theorem 3.6 *With the previous notations, E_T is the union of some of the blocks of G, and H_T is the union of the remaining blocks of G.*

When dealing with postman sets we can say more.

Lemma 3.7 *Let $H_O = \{e \in E : e \notin \mathcal{P}$ for all $P \in \mathcal{P}\}$. Then:*

(a) If $H_O \neq \emptyset$ then H_O is a union of cycles, and if $e \in H_O$ then every cycle containing e is contained in H_O.

(b) H_O is a $\emptyset$-join, i.e. either it is a disjoint union of cycles or $H_O = \emptyset$.

(c) For arbitrary T, either no T-join intersects H_O or else every T-join does.

Lemma 3.8 *$e \in E$ is a cut-edge if and only if $e \in P$ for all $P \in \mathcal{P}$.*

Let $B_1, B_2, \ldots, B_r$ be (the edges of) the blocks of $G = (V, E)$. For $i = 1, \ldots, r$, let $G_i = (V_i, B_i)$, where V_i is the set of endpoints of the edges in B_i, and $\mathcal{P}_i$, the family of postman sets in G_i.

Theorem 3.9 *With the previous notations, there is a one to one correspondence between $\mathcal{P}$ and $\mathcal{P}_1 \times \cdots \times \mathcal{P}_r$, given by*

$$P \to (P \cap B_1, \ldots, P \cap B_r) \quad \textit{and} \quad (P_1, \ldots, P_r) \to P_1 \,\Delta \cdots \Delta\, P_r.$$

4 T-joins and Postman Sets

We now show that we need not consider cycles in Corollary 2.4 if $H_O = \emptyset$:

Theorem 4.1 *Let $E_O = \{e \in E : e \in P \text{ for some } P \in \mathcal{P}\}$. Then there exists $\{P_1, P_2, \ldots, P_s\} \subset \mathcal{P}$ with s odd and $E_O = P_1 \,\Delta\, P_2 \,\Delta \cdots \Delta\, P_s$. Consequently, if $H_O \neq \emptyset$, there also exist $\{C_1, C_2, \ldots, C_t\} \subset \mathcal{C} \cap H_O$ such that $E = P_1 \,\Delta \cdots \Delta\, P_s \,\Delta\, C_1 \,\Delta \cdots C_t$.*

Let us denote by R the symmetric difference $O \,\Delta\, T$ (which may be empty), and by $\mathcal{R}$ the corresponding clutter of minimal R-joins (which may only have the empty set). We have:

Theorem 4.2 *With the previous notations, if $\mathcal{T} \neq \emptyset$ and $G_T = (V, E_T)$, then the following conditions are equivalent:*

(i) $\mathcal{T}$ is the set of postman sets of G_T.

(ii) $E_T = J_1 \,\Delta\, J_2 \,\Delta \cdots \Delta\, J_s$, for some $\{J_1, J_2, \ldots, J_s\} \subset \mathcal{T}$ and odd s.

(iii) $|\mathcal{T}|$ is odd, and $E_T = J_1 \,\Delta\, J_2 \,\Delta \cdots \Delta\, J_s$ for some $\{J_1, J_2, \ldots, J_s\} \subset \mathcal{T}$.

(iv) For every $P \in \mathcal{P}$ there exists $J \in \mathcal{T}$ such that $J \subset P$.

(v) For every $P \in \mathcal{P}$ there exist $J_P \in \mathcal{T}$ and $D_P \in \mathcal{R}$ such that J_P and D_P are disjoint and $P = J_P \cup D_P$.

(vi) E is the disjoint union of E_T, E_R and H_O.

(vii) E_T and E_R are disjoint.

5 The Clutter Matrix associated with $\mathcal{T}$

Many of our results may be visualized via the 0-1 matrix $M(\mathcal{T})$ associated with the clutter $\mathcal{T}$, in which the rows are the characteristic functions of the minimal T-joins and the columns are indexed by the edges of G. If we know that a matrix M is associated with a clutter of T-joins of a graph G, but do not know what G or T is, it is nevertheless very simple to test whether $T = O(G_T)$:

(a) check if $|\mathcal{T}|$ is odd, i.e. if there is an odd number of rows, and,

(b) in case $|\mathcal{T}| > 1$, check if E_T (the set of indices of the non-zero columns of M) can be written as a symmetric difference of minimal T-joins.

Since the symmetric difference of sets corresponds to addition modulo 2 of characteristic vectors, for (b) we may eliminate the zero-columns (and even the all-ones columns) and try to express a row of ones as a sum (mod 2) of some of the rows of the reduced matrix. This can be done quite efficiently (bounded by small powers of $|\mathcal{T}|$ and $|E_T|$) by using Gaussian elimination or matrix triangularization modulo 2.

We should add that, according to Novick and Sebö [3], a clutter may be recognized as a T-join clutter in polynomial time by considering the sixteen nonisomorphic minimally non T-join (binary) clutters. Therefore, the recognition of a matrix as coming from a clutter of postman sets can be done polynomially.

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