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Abstract

We revisit the connection between the complementarity and uncertainty principles of quantum mechanics within the framework of Mach–Zehnder interferometry. We focus our attention on the trade-off relation between complementary path information and fringe visibility. This relation is equivalent to the uncertainty relation of Schrödinger and Robertson for a suitably chosen pair of observables. We show that it is equivalent as well to the uncertainty inequality provided by Landau and Pollak. We also study the relationship of this trade-off relation with a family of entropic uncertainty relations based on Rényi entropies. There is no equivalence in this case, but the different values of the entropic parameter do define regimes that provides us with a tool to discriminate between non-trivial states of minimum uncertainty. The existence of such regimes agrees with previous results of Luis (2011 *Phys. Rev.* A **84** 034101), although their meaning was not sufficiently clear. We discuss the origin of these regimes with the intention of gaining a deeper understanding of entropic measures.

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1. Introduction

The uncertainty principle (UP) [1–3] and the complementarity principle (CP) [4] lie at the heart of quantum mechanics. The former principle establishes that the probability distributions associated with the outcomes of two non-commuting observables cannot be simultaneously sharp; the latter refers to the relationship between pairs of descriptions that are mutually exclusive, but necessary for a complete description of a quantum system. Many years have passed since the original formulation of both principles were established but even today, there is an important debate going on regarding not only their precise formulations, but also their adequate interpretations in several contexts.

The CP has been both theoretically and experimentally studied [5–8], specifically within the framework of Mach–Zehnder (MZ) interferometry. The MZ setting is particularly suitable for discussions about wave–particle duality. In this regard, the wave aspect (related to the fringe-visibility) and the particle aspect (linked to the which-way-has-passed question) are represented by measurable quantities V and P, respectively, which satisfy the duality relation [5, 6]

$$P^2 + V^2 \leqslant 1. \tag{1}$$

This quantitative formulation of CP is expressed in a way that resembles inequalities typical of the UP. However, the derivation of equation (1) *did not* involve any mention of inherent fluctuations in the measured quantities. The connection between these two important principles of quantum mechanics has recently been discussed [7–11]. Specifically, attention is focused on the question: is equation (1) the expression of an uncertainty relation (UR)? Answers in both the affirmative and the negative have been provided. Our goal here is to shed some light on this discussion. We first revisit the link between the duality relation (equation (1)) and the Schrödinger–Robertson (SR) variance-based uncertainty inequalities. Having at our disposal *other* uncertainty relations (URs), such as the Landau–Pollak (LP) one and those based on Rényi entropic measures, it is interesting to address the question of their relationship with (1). We find equivalence in the former case but not in the latter. Indeed, for entropy-based uncertainty inequalities, regimes with distinct qualitative behavior arise according to the different possible values of the entropic parameter. This can be used for detecting special minimum-uncertainty states. The origin of these regimes (already noted in [12]) is not yet clearly understood. Our present discussion delves deeply into the nature of these regimes and may contribute to a better understanding of Rényi entropic measures.

The outline of this paper is as follows. In section 2 we review, for double-slit-like experiments, the derivation of the duality relation (1) and the discussion on which are the relevant operators that account for path information and fringe visibility. In section 3, we address the problem of linkage between CP and UP. After revisiting the equivalence of (1) to the uncertainty inequality prescribed by SR (choosing different pairs of observables), we provide a demonstration of equivalence for the LP case. Additionally, we introduce the analysis of entropic uncertainty inequalities showing that they are not equivalent to the complementarity relation posed by equation (1). After stating clearly the meaning of this non-equivalence, in section 4 we show that, due to this fact, the entropic uncertainty inequalities yield non-trivial information about the system. This is shown by studying states that saturate the UR. We discuss the meaning of the different regimes which appear depending on the value of the entropic parameter and comment on their potential applicability for informational purposes. Some conclusions are drawn in section 5. For self-consistency, we include in the appendix a summary of various quantitative formulations of the UP, by employing variances as well as entropic and other measures.

2. The Mach–Zehnder interferometer scheme and the complementarity relation

The MZ interferometer (see figure 1) is a device that has been used in several branches of physics, in particular, for the study of the CP. In that context, an important quantity is the '*which way*' information, which is quantified by the *predictability* P defined as $P = 2L - 1$, where $L = \max\{w_+, w_-\}$ and w_+ and w_- are the probabilities of the particle taking path '+' or path '−', respectively. On the other hand, the *fringe visibility* is quantified via a natural extension of the usual measure for intensity of light, that is, $V = \frac{p_{\max} - p_{\min}}{p_{\max} + p_{\min}}$ where p stands for the probability that the particle be detected in some position in space, with $p_{\max}$ and $p_{\min}$ denoting, respectively, the maximum and minimum of this probability. The quantitative formulation of CP for the MZ-interferometer scheme is the celebrated duality relation given by equation (1) [5, 6], where the equals sign holds (only) for pure states. This relation was also implicitly alluded to in [13, 14].

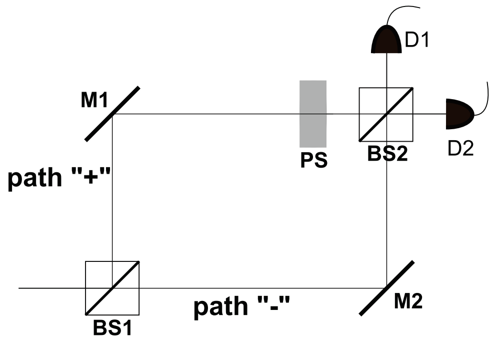

Figure 1. A source emits a photon which, after passing through the beam splitter BS1, splits into paths '+' and '−'. It reflects in mirrors M1 and M2 and is finally observed using detectors D1 and D2. A phase shifter PS and another beam splitter BS2 may be inserted into the setup in order to produce interference.

The MZ interferometer, having two relevant spatial modes, can be represented by a two-dimensional Hilbert space spanned for instance by the set $\{|0\rangle, |1\rangle\}$, which is the so-called computational basis. States $|0\rangle$ and $|1\rangle$ are eigenstates of the Pauli spin operator σ_z, representing the two paths. We use the Bloch representation to describe quantum density operators, namely

$$\rho = \frac{I + \vec{s} \cdot \vec{\sigma}}{2}, \tag{2}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denote the Pauli matrices, I is the 2×2 identity matrix and $\vec{s} = (s_x, s_y, s_z)$ is the Bloch vector (with $\|\vec{s}\| \leqslant 1$) that characterizes the state of the system. The action of a $50:50$ beam splitter can be described by the unitary transformation $U_{\mathrm{BS}} = \mathrm{e}^{-\mathrm{i}\pi\sigma_y/4}$, which implies a rotation of $\pi/2$ of the Bloch vector around the y-axis. A phase shifter introduces a phase difference ϕ between the paths, and it is formally represented by the unitary operator $U_\phi = \mathrm{e}^{-\mathrm{i}\phi\sigma_z/2}$.

Following [9], a single sharp observable $\hat{P}$ can be associated with predictability, while two families of sharp observables $\hat{V}_\phi$ and $\hat{V}_\phi^\perp$ can be associated with visibility. It is possible to express these operators in terms of the Pauli spin ones as

$$\hat{P} = \sigma_z, \tag{3}$$

$$\hat{V}_\phi = (\cos\phi)\,\sigma_x + (\sin\phi)\,\sigma_y, \tag{4}$$

$$\hat{V}_\phi^\perp = -(\sin\phi)\,\sigma_x + (\cos\phi)\,\sigma_y \tag{5}$$

with ϕ ranging, in principle, between 0 and 2π. Note that $\hat{P}$, $\hat{V}_\phi$ and $\hat{V}_\phi^\perp$ are (for each ϕ) a set of mutually complementary observables, that is, if one is certain about the value of one observable, then maximum ignorance reigns concerning the value of any of the other two. We mention that an alternative definition for visibility that could be used is the one given in [11].

For a system in state ρ with Bloch vector $\vec{s}$, equation (2), the predictability P is given by

$$P = |\langle \hat{P} \rangle| = |s_z|.$$

The visibility V can be derived either from observable $\hat{V}_\phi$ or from $\hat{V}_\phi^\perp$ by properly choosing the parameter ϕ. Using an alternative representation for the density ρ as $\begin{pmatrix} \omega_+ & r\,\mathrm{e}^{-\mathrm{i}\theta} \\ r\,\mathrm{e}^{\mathrm{i}\theta} & \omega_- \end{pmatrix}$, the auxiliary state variables $r \equiv \frac{1}{2}\sqrt{s_x^2+s_y^2}$ and $\tan\theta \equiv \frac{s_y}{s_x}$ allow us to write

$$\langle \hat{V}_\phi \rangle = 2r\cos(\theta-\phi) \quad \text{and} \quad \langle \hat{V}_\phi^\perp \rangle = 2r\sin(\theta-\phi).$$

Thus the visibility, which is given by the maximum absolute expectation value of these observables, is equal to $2r$ and can be obtained using $\hat{V}_\phi$ if one sets the phase shifter such that $\phi=\theta$, or using $\hat{V}_\phi^\perp$ and setting $\phi=\theta-\pi/2$. Note that arranging the apparatus PS (see figure 1) with a phase difference of π with respect to these angles gives also the same value of visibility. Finally, due to the positivity of the density matrix, the complementarity relation (1) is directly obtained:

$$P^2+V^2 = s_x^2+s_y^2+s_z^2 \leqslant 1 \tag{6}$$

and it is saturated whenever $\|\vec{s}\|=1$, i.e. for any pure state. Note that the measurements of any two observables (3) and (4), or (3) and (5), can only be carried out in two *incompatible* experimental setups and that joint measurement is not involved. Therefore the trade-off relation (1) expresses *preparation complementarity* [7], that is, the impossibility of preparing the system in a state in which the two observables would simultaneously exhibit sharp values.

3. Connections between complementarity and uncertainty relations

3.1. The P–V duality relation is equivalent to variance-based uncertainty inequalities

The relationship between the predictability–visibility inequality (1) and URs based on variances (A.1) can be readily analyzed using the Bloch representation of the pertinent operators and the density matrix. First of all, the variances of the operators given in equations (3)–(5) are obtained in terms of the predictability P and the visibility V as [9]

$$(\Delta\hat{P})^2 = 1-P^2, \tag{7}$$

$$(\Delta\hat{V}_\phi)^2 = 1-V^2\cos^2(\theta-\phi), \tag{8}$$

$$(\Delta\hat{V}_\phi^\perp)^2 = 1-V^2\sin^2(\theta-\phi), \tag{9}$$

where θ is a state variable (recall section 2). Let us remark that the choice of *adequate observables* (understood as Hermitian operators acting on Hilbert space) is a first step in order to show whether equation (1) is the expression of a UR.

The connection between the CP relation and variance-based URs has been analyzed in [7, 9, 10]. Some critical comments are in order concerning those studies. In [10], the *equivalence* between both principles is highlighted: indeed, the Heisenberg–Robertson (HR) UR is computed there for the pair of observables $\hat{P}$ and $\hat{V}_\theta^\perp$, and also for $\hat{V}_\theta$ and $\hat{V}_\theta^\perp$ (setting the phase shifter to an angle $\phi=\theta$). By doing so, the following uncertainty inequalities are obtained:

$$(\Delta\hat{P})^2(\Delta\hat{V}_\theta^\perp)^2 = 1-P^2 \geqslant V^2, \tag{10}$$

$$(\Delta\hat{V}_\theta)^2(\Delta\hat{V}_\theta^\perp)^2 = 1-V^2 \geqslant P^2 \tag{11}$$

and it comes out that both are equivalent to (1) (for every θ). The main drawback that the authors note in their derivation is the use of $\hat{V}_\theta^\perp$, which has no direct interpretation in terms of either predictability or visibility in connection with the MZ interferometry experiment, because of the results $\langle\hat{V}_\theta^\perp\rangle = 0$ and $\Delta\hat{V}_\theta^\perp = 1$. Moreover, when dealing with $\hat{P}$ and $\hat{V}_\theta$, the corresponding HR UR becomes trivial: $(\Delta\hat{P})^2(\Delta\hat{V}_\theta)^2 \geqslant 0$.

Independently, Björk *et al* [9] also dealt with the problem of connecting CP with UP. The SR UR is deeply connected with the duality relation (1). The analysis consists in obtaining expressions (7) and (8) followed by an appeal to $(\Delta\hat{V}_\phi)^2 \geqslant (\Delta\hat{V}_\theta)^2$ (from basic trigonometry), with the purpose of linking the two fundamental principles of quantum mechanics.

A complete proof of the alluded to equivalence dealing with the appropriate observables $\hat{P}$ and $\hat{V}_\theta$ and the *full* SR UR is given in [7]. We reproduce it here—although in a slightly different way—for the sake of completeness. For arbitrary ϕ, the UR prescribed by Schrödinger and Robertson reads

$$\begin{aligned}(1-P^2)[1-V^2\cos^2(\theta-\phi)] \\ \geqslant P^2V^2\cos^2(\theta-\phi)+V^2\sin^2(\theta-\phi),\end{aligned} \tag{12}$$

where equality holds for any pure state. It is straightforward to show that this family of inequalities reduces to the duality relation (1).

We stress that (12) is valid for *any* phase ϕ introduced by the phase shifter in the MZ interferometer. We then conclude that, in particular, the appropriate choice $\phi=\theta$ implies *equivalence* with the trade-off relation between predictability and visibility. This circumvents the drawback pointed out in [10]. With this simple result, a rather sharp conclusion is drawn from the discussion about complementarity between P and V, including the status of (1) as an UR.

Finally, we mention that in [8] a relationship between wave–particle duality and quantum uncertainty has been investigated, both theoretically and experimentally, by recourse to variances of the operators $\hat{P}$ and $\hat{V}_\theta$. However, this is done without appealing to Heisenberg-like inequalities.

3.2. The P–V duality relation is equivalent to the Landau–Pollak uncertainty inequality

We demonstrate now that inequality (1) becomes equivalent to LP UR [15], a so far uncovered feature, as far as we know. The maximum probabilities associated with observables $\hat{P}$ and $\hat{V}_\theta$, in terms of predictability and visibility, are

$$M_\infty(\hat{P}) = \frac{1+P}{2}, \tag{13}$$

$$M_\infty(\hat{V}_\theta) = \frac{1+V}{2}. \tag{14}$$

Replacing these probabilities in (A.3) and setting $c=1/\sqrt{2}$, we attain the situation of complementary operators. Thus, we

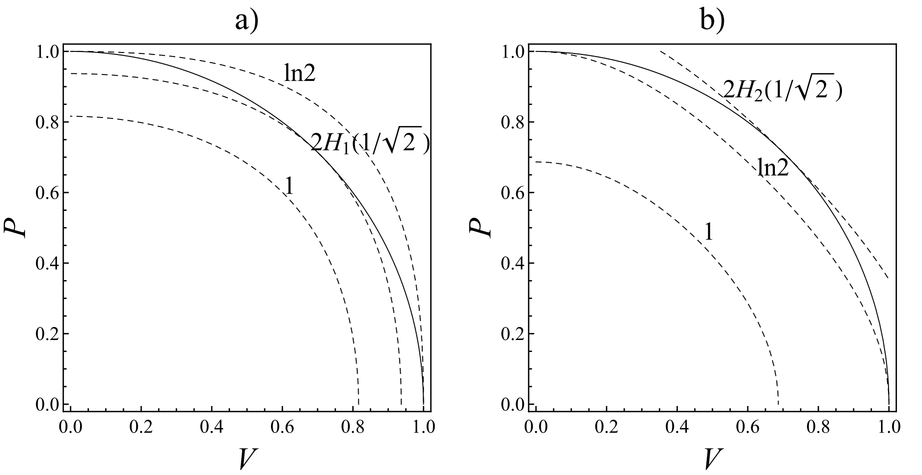

Figure 2. The constraint $P^2 + V^2 = 1$ (solid line) and contour plots (dashed lines) of the sum of Rényi q-entropies for two representative entropic indices: (a) $q = 1$ (Shannon entropy) and (b) $q = 2$ (collision entropy). The values chosen for the entropy sum are indicated next to each contour line: $1, 2H_1(1/\sqrt{2}) \approx 0.833$ (left inset only), $\ln 2 \approx 0.693$ and $2H_2(1/\sqrt{2}) \approx 0.576$ (right inset only).

obtain

$$\sqrt{\left(\frac{1+P}{2}\right)\left(\frac{1+V}{2}\right)} - \sqrt{\left(\frac{1-P}{2}\right)\left(\frac{1-V}{2}\right)} \leqslant \frac{1}{\sqrt{2}}. \quad (15)$$

Squaring both sides of this inequality and regrouping terms conveniently, we immediately arrive at the relation

$$(1 - P^2)(1 - V^2) \geqslant (PV)^2, \quad (16)$$

which coincides with (12) for $\phi = \theta$ and, as mentioned before, can be easily recast in the fashion $P^2 + V^2 \leqslant 1$. This implies that the duality relation (1) can be deduced from the LP inequality, and vice versa. As a consequence, full correspondence between SR and LP URs is obtained, a remarkable fact that is not valid for general pairs of observables.

3.3. Non-equivalence between the P–V duality relation and entropic uncertainty inequalities

Having clarified the above equivalences, we now consider the problem of elucidating the connection between entropic URs (EURs) and the duality relation (1). For an arbitrary state ρ, the expressions for the Rényi q-entropies (A.4) corresponding to $\hat{P}$ and $\hat{V}_\theta$ are

$$H_q(P) = \frac{1}{1-q} \ln \left[\left(\frac{1+P}{2}\right)^q + \left(\frac{1-P}{2}\right)^q \right], \quad (17)$$

$$H_q(V) = \frac{1}{1-q} \ln \left[\left(\frac{1+V}{2}\right)^q + \left(\frac{1-V}{2}\right)^q \right], \quad (18)$$

where, to simplify notation, we have renamed $H_q(\hat{P}; \rho) \equiv H_q(P)$ and $H_q(\hat{V}_\theta; \rho) \equiv H_q(V)$.

For our purposes, we must first find the minimum of the sum of these Rényi entropies over all available states, that is, $\min_\rho \{H_q(\hat{P}; \rho) + H_q(\hat{V}_\theta; \rho)\}$. This constitutes the application of the UR (A.5) for the P–V case of MZ interferometry. Appealing to the concavity property of Rényi entropy when $q \in (0, 2]$, we can restrict our calculations to pure states and then the constrained minimization problem can be recast in the fashion

$$\min_{P^2+V^2=1} \{H_q(P) + H_q(V)\} \quad (19)$$

for every value of q. The final expressions for the EURs take the form

$$H_q(P) + H_q(V) \geqslant \mathcal{B}_q \equiv \begin{cases} \ln 2 & \text{if } 0<q\leqslant q^*, \\ \frac{2}{1-q} \ln \left[\left(\frac{1+1/\sqrt{2}}{2}\right)^q + \left(\frac{1-1/\sqrt{2}}{2}\right)^q \right] & \text{if } q^*<q\leqslant 2, \end{cases} \quad (20)$$

where $q^* \approx 1.4316$ is obtained by solving (numerically) the equation $2H_{q^*}(1/\sqrt{2}) = \ln 2$. Concerning the optimal P–V values, it is seen that three qualitatively different regimes appear:

(i) for $0 < q < q^*$: the minimum sum is attained at $V = 0$ and $P = 1$ or $V = 1$ and $P = 0$;
(ii) at $q = q^*$: the minimum value corresponds to the cases $V = 0$ and $P = 1$, $V = 1$ and $P = 0$ or $V = P = 1/\sqrt{2}$;
(iii) for $q^* < q \leqslant 2$: the minimum sum is attained at $V = P = 1/\sqrt{2}$.

In figure 2 we display, in the V–P plane, the constraint $P^2 + V^2 = 1$ together with several contour lines for the sum of Rényi q-entropies corresponding to two representative values of the entropic parameter, in regimes (i) and (iii) mentioned above. In both cases the contour lines correspond to decreasing values toward the origin. In case (i), $\ln 2$ is the minimum-value contour line that intersects (tangentially) the constraint, at the points $(V, P) = (0, 1)$ or $(V, P) = (1, 0)$. In case (iii), the curve $P^2 + V^2 = 1$ is intersected by the minimum-value contour line $H_q(P) + H_q(V) = 2H_q(1/\sqrt{2})$ precisely at $(V, P) = (1/\sqrt{2}, 1/\sqrt{2})$.

Let us now show that the duality relation cannot be deduced, in a way analogous to the case of SR and LP, from an EUR of the form (20). From the point of view of the values that P and V can take, equations (1) and (12) are equivalent, i.e. they represent the same inequality, but written in different forms. The same can be said about the relationship between equations (1) and (16) for the LP case. Now, in the comparison of equations (1) and (20), one can see that there exist pairs (V, P) in the square $[0, 1] \times [0, 1]$ that fulfill the latter but lie outside the region allowed by CP (then corresponding to non-physical situations). This originates from the fact that the curves where the sum of entropies reaches its minimal value (namely, the curves $H_q(P) + H_q(V) = \mathcal{B}_q$) for each q do not coincide with the curve $P^2 + V^2 = 1$. This fact renders inequalities (1) and (20) *different*. As we shall see below, non-equivalence *in this sense* has interesting consequences, and can be used to explain some features of the entropic measure as an informational quantifier.

4. Minimizing uncertainty states

4.1. Saturation limit of the uncertainty relations

States that saturate an UR are used in several contexts. An important example has to do with coherent states, which saturate the position–momentum Heisenberg UR. This property is one of the reasons why coherent states are usually interpreted as the most 'classical' ones. In this section we analyze, in the framework of the MZ interferometer, the properties of those states that correspond to an equality for the different URs employed, and/or minimize the uncertainty measure chosen in each distinct treatment. We start by clarifying the concepts of *minimizing* states versus saturating (or the so-called *intelligent*) states in the MZ scheme. We use to that effect the operators $\hat{P}$ and $\hat{V}_\theta$, the appropriate observables to be accounted for. An interesting study of the distinction between minimum-uncertainty states and intelligent states for spin observables has been reported in [16]; see also [17] for non-canonical operators.

URs based on variances (such as HR and SR UR) are conceptually different from LP UR and entropy-based ones, in the following sense: in the former case, one has an inequality that is fulfilled *state-by-state* (unless the commutator of the observables is a c-number, in which case the bound is universal), while in the latter cases the uncertainty measure (the sum of arccos functions or of Rényi or other entropies) is lower bounded by a *universal, state-independent* quantity. This crucial difference, which has been essentially the Deutsch criticism that spurred the study of EURs, leads one to make the distinction between minimizing and saturating states in the case of variance-based URs.

Consider variance-based UR for the pair $\hat{P}$– $\hat{V}_\theta$: the SR inequality is given by equation (12) with $\phi = \theta$ and it is seen to saturate *for any pure state*. But the left-hand side of that inequality is a minimum only for eigenstates of the involved operators, and that minimum is zero, as is the bound on the right-hand side. These states correspond physically to having predictability $P = 1$ and visibility $V = 0$, or vice versa, in the interferometry experiment.

Per contra, in the case of LP or entropy-based URs, there is no such distinction as for variances, due to the fact that one has a measure of uncertainty (represented by the sum of arccos functions or entropies) that is always bounded from below by a quantity that depends merely on the observables and not on the state of the system. The states that minimize and at the same time saturate the LP relation are all the pure states; however, this UR does not distinguish any special state among the ones that saturate the relation. With reference to the EURs, since one has the freedom to chose the entropic parameter, one can classify the different sets of minimal-uncertainty states. We discuss this in detail in the following subsection.

4.2. Minimizing states for the case of EURs

In section 3.3, we found three different regimes in the optimization problem (19), depending on the value of the entropic parameter. This fact can be used to shed some light on the meaning of the parameter q, at least within the framework of double-slit-like experiments.

Several generalized entropic uncertainty measures were considered in [12], in particular the sum of Rényi entropies. A study of the states that reach extremal joint uncertainty for pairs of observables A and B is addressed there, finding that they can be classified in the so-called (a) extreme states, which are the eigenstates of A or B (so that $\Delta A = 0$ or $\Delta B = 0$, respectively, with the other variance remaining finite for discrete, bounded operators), and (b) intermediate states, those which deal the same variance for both observables (considered to be dimensionless). The pertinent analysis leads to an apparently contradictory result, as a minimum-uncertainty state for given values of q yields maximum uncertainty for other q-values as well. This behavior seems to be a consequence of a restriction in the concomitant state space, as only a given family of pure states was considered. Recall that the most uncertain states correspond in fact to an entropy sum equal to $2 \ln 2$ for any q, and this is achieved for the completely mixed state, i.e. $\rho = I/2$, a fact not mentioned in [12].

Let us now look in further detail at what happens for different choices of q when EURs of the form (20) are saturated. In section 3.3, by computing the entropy-sum minimum over the whole convex set of quantum states, we obtained the minimizing states for the predictability–visibility case. Employing the classification of [12], we see that the optimal states that we find correspond to: (i) *extreme states*, (ii) *extreme* or *intermediate states* and (iii) *intermediate states*, for q smaller than, equal to and larger than q^*, respectively. Note that the second regime has not been taken into account in [12].

We consider now the characteristic features of intermediate states, which make $|\langle \hat{V}_\theta \rangle| = |\langle \hat{P} \rangle| = \frac{1}{\sqrt{2}}$. They are the pure states of the form (2) with the four different unit Bloch vectors: $\pm\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \pm 1)$. These are precisely the states that saturate the concomitant EURs in the most unbiased way (in the sense of simultaneously having the maximum visibility *and* maximum predictability that is possible). These states correspond to an intermediate situation between the extremal cases of wave behavior ($V = 1$ and $P = 0$) and particle behavior ($V = 0$ and $P = 1$).

As we have seen, the existence of the regimes mentioned above allows one to select special states. These states are 'invisible' for the variance-based or LP formulations of the UP. If the EURs and the duality relation were equivalent, these states could never have been detected. Thus, in a certain sense, the capability of the EUR for detecting these states and the existence of the regimes investigated in [12] are due to the non-equivalence to the duality relation.

Finally, regarding the obtention of the case with joint maximum values for predictability and visibility, note that if one takes into account the restriction given by the CP, $P^2 + V^2 \leqslant 1$, one directly obtains $P = V = 1/\sqrt{2}$ from simple geometric arguments in the V–P plane. However, one might deal with a special situation in which not the whole set of states is available, but only a fraction of them, and one still wants to select the 'best' intermediate states between purely wave and particle aspects. For instance, this situation may appear if the source in figure 1 has limitations for producing certain states. Other interesting situations appear either when the second beam splitter is a Schrödinger cat (as in [18–20]) or in the presence of a noisy environment. In both situations, the kind of states of the system which pass through the interferometer is limited by the state of the environment (even more, in the second case they cannot be controlled). One would like to be endowed with an adequate criterion to deal with this sort of situations. Our proposal is, specifically, to consider the uncertainty measure $H_q(P) + H_q(V)$ with an entropic parameter $q > q^*$ and perform its minimization, constrained to the available set of states. We argue that, in this respect, one obtains a useful selection criterion.

5. Concluding remarks

We have studied here connections between the complementarity and uncertainty principles in the MZ interferometer scheme. Following [9] and related work, we have employed quantum-mechanical operators $\hat{P}$ and $\hat{V}_\phi$ to give an account of the particle and wave aspects of a quantum system, respectively.

The link between the CP inequality (1) and variance-based URs of the form (A.1) has already been considered in [7, 9, 10]. We have thoroughly analyzed some drawbacks of these approaches, and we revised the *equivalence* between the SR UR and the duality relation in the relevant case, i.e. for observables which adequately represent predictability and visibility according to [7].

In the present effort, we have addressed the problem from different viewpoints. Our main findings are the following:

- We have proved the *equivalence* between the duality relation (1) and the LP UR (15) (derived from (A.2)), which is an alternative quantification of the UP.
- We have studied the connection between (1) and a family of EURs (20), which are based on equation (A.5) using Rényi q-entropies (A.4). We have found that these EURs, for the relevant pair of observables $\hat{P}$– $\hat{V}_\theta$, are *not* equivalent to the duality relation.
- We have seen that, when the Rényi uncertainty measures are applied to the MZ scheme, different regimes emerge, depending on the value of the entropic parameter q. We have ascertained that this agrees with previous results [12] showing that the value chosen for q affects the qualitative behavior of the URs.
- Looking at the states which saturate the EURs, we have found non-trivial minimizing states for entropic parameters equal to or above a certain value q^*. We have in this vein established a procedure for solving the problem of finding a state having minimum uncertainty for the observables $\hat{P}$ and $\hat{V}_\theta$ in the most unbiased fashion.
- Finally, we have also discussed the usefulness of such a procedure for information-theoretical purposes, depending on the nature of the source and the beam splitters in the interferometer. The fact that different regimes are found in the canonical example of MZ interferometry seems to provide support to the assertion *there is no preferred value for* q. Indeed, different q-values render the concomitant entropic measures useful for different purposes. We have shown that for the particular example of the MZ interferometer, the non-equivalence between the EURs and the duality relation is, in some sense, the source of these facts.

It is worth stressing then that in the present context the three inequalities (1), (12) and (15) are on an equal footing (which may not be the case for other pairs of observables) in contrast to (20).

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Appendix. Uncertainty relations

Quantitative formulations of the UP are known as URs, and there is now a collection of inequalities that express this principle (see, for instance, the recent review papers [21, 22]). For the sake of self-consistency, in this appendix we summarize the URs employed along the paper.

A.1. Variance-based uncertainty relations

Heisenberg [1] was the first to propose an UR for position and momentum observables in terms of their variances. A generalization of the Heisenberg inequality for an arbitrary pair of Hermitian operators A and B is due to Robertson [2]. A further tighter relation was derived by Schrödinger [3]:

$$(\Delta A)^2(\Delta B)^2 \geqslant \left(\frac{1}{2}\langle\{A,B\}\rangle - \langle A\rangle\langle B\rangle\right)^2 + \left(\frac{1}{2i}\langle[A,B]\rangle\right)^2 \quad \text{(A.1)}$$

with $(\Delta O)^2 = \langle O^2\rangle - \langle O\rangle^2$ being the variance of observable O. If one does not consider the first squared term on the right-hand side of (A.1), one deals with the usual HR UR.

Variance-based UP formulations have been doubly criticized. On the one hand, the lower bound to the product of variances depends, in general, on the state of the system via the expectation values and thus lacks a universal character [23, 24]. Moreover, it can be easily seen [25] that for discrete, bounded operators the lower bound is trivially zero, yielding no valuable information. On the other hand, the use of the variance as a measure of uncertainty (spreading) of a given probability distribution exhibits some limitations (it might be the case, for instance, that the variance is not well defined [22, 26]).

A.2. The Landau–Pollak uncertainty relation

An alternative UP formulation was introduced by Landau and Pollak in the context of time–frequency analysis [15] and adapted to the quantum framework by Maassen and Uffink [24]. Using the notation $M_\infty(A;\rho) = \max_i p_i(A)$ for the maximum probability of the outcomes of observable A, the LP UR reads

$$\arccos\sqrt{M_\infty(A;|\Psi\rangle\langle\Psi|)} + \arccos\sqrt{M_\infty(B;|\Psi\rangle\langle\Psi|)} \geqslant \arccos c, \qquad \text{(A.2)}$$

where $c = \max_{i,j}|\langle a_i|b_j\rangle|$ is the overlap between the N-dimensional eigenbases of A and B, and lies between $\frac{1}{\sqrt{N}}$ and 1. The LP relation captures the essence of the UP for pure states; indeed, the right-hand side is state independent. For two-dimensional systems it can be shown that the LP relation remains valid for mixed states.

For our purposes, we express the LP inequality (A.2) as

$$\sqrt{M_\infty(A)M_\infty(B)} - \sqrt{[1-M_\infty(A)][1-M_\infty(B)]} \leqslant c, \qquad \text{(A.3)}$$

where we have used trigonometric identities.

A.3. Entropy-based uncertainty relations

Information-theory tools have shown their usefulness for the study of URs [27–33]. Consider now, as a measure of uncertainty (ignorance), the one-parameter generalization of Shannon entropy given by Rényi [34] that in the case of an N-dimensional, discrete probability distribution reads

$$H_q(\{p_i\}) = \frac{1}{1-q}\ln\left(\sum_{i=1}^{N} p_i{}^q\right), \qquad \text{(A.4)}$$

where $0 \leqslant p_i \leqslant 1$, $\sum_{i=1}^N p_i = 1$ and the real parameter $q > 0$ with $q \neq 1$. If we let $q \to 1$; then this definition includes by continuity the Shannon case: $H_1(\{p_i\}) = -\sum_{i=1}^N p_i \ln p_i$.

Given a system in a quantum state ρ, the probability distribution associated with an observable A is provided by Born's rule. The corresponding q-entropy is a measure of the degree of uncertainty in the following sense: when one is certain about the observable's value, i.e. $\{p(A)\}$ is a Kronecker delta, the entropy takes its minimum value $H_q = 0$. Contrariwise, for total ignorance concerning the value of A, i.e. $\{p(A)\}$ is the uniform distribution, the entropy is maximal and equal to $H_q = \ln N$ (irrespective of q). For our purposes, let us comment that when $N = 2$, Rényi entropy is a concave function in ρ for $0 < q \leqslant 2$ [35, 36].

An EUR has the form

$$H(A;\rho) + H(B;\rho) \geqslant \mathcal{B}(A,B), \qquad \text{(A.5)}$$

where H is an entropic measure like the ones described above, while $\mathcal{B}$ is a function of the observables. More precisely, the bound depends on the overlap between the eigenbases of both operators and it is state-independent (i.e. it is not a function of the state ρ) and also a positive quantity. Entropic inequalities based on Shannon and Rènyi entropies have been applied for the determination of classicality (or quantumness) of states [37, 38].

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