J. Inverse Ill-Posed Probl. **20** (2012), 765–789
DOI 10.1515/jip-2012-0049

On the existence of global saturation for spectral regularization methods with optimal qualification

Gisela L. Mazzieri, Ruben D. Spies and Karina G. Temperini

Abstract. A family of real functions $\{g_\alpha\}$ defining a spectral regularization method with optimal qualification is considered. Sufficient condition on the family and on the optimal qualification guaranteeing the existence of saturation are established. Appropriate characterizations of both the saturation function and the saturation set are found and some examples are provided.

Keywords. Ill-posed, inverse problem, qualification, saturation.

2010 Mathematics Subject Classification. 47A52, 65J20.

1 Introduction

Let X, Y be infinite dimensional Hilbert spaces and $T : X \to Y$ a bounded linear operator with non-closed range $\mathcal{R}(T)$. It is well known that under these conditions $T^\dagger$, the Moore–Penrose generalized inverse of T, is unbounded ([2]) and therefore the linear operator equation

$$Tx = y \tag{1.1}$$

is ill-posed. The Moore–Penrose generalized inverse can be used to define the least squares solutions of (1.1). In fact equation (1.1) has a least squares solution if and only if $y \in \mathcal{D}(T^\dagger) \doteq \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$ and in that case, $x^\dagger \doteq T^\dagger y$ is the least squares solution of minimum norm and the set of all least-squares solutions of (1.1) is given by $x^\dagger + \mathcal{N}(T)$. Since $T^\dagger$ is unbounded, it follows that $x^\dagger$ does not depend continuously on the data y. Therefore, if instead of the exact data y, a noisy observation y^δ is available, $y^\delta = Tx + \delta\xi$, where the noise ξ is assumed to be bounded, $\|\xi\| \le 1$, then it is possible that $T^\dagger y^\delta$ does not even exist and if it does,

This work was supported in part by Consejo Nacional de Investigaciones Científicas y Técnicas, CONICET, through PIP 2010-2012 Nro. 0219, by Universidad Nacional del Litoral, U.N.L., through project CAI+D 2009-PI-62-315, by Agencia Nacional de Promoción Científica y Tecnológica, ANPCyT, through project PICT-2008-1301 and by the Air Force Office of Scientific Research, AFOSR, through Grant FA9550-10-1-0018.

it will not necessarily be a good approximation of $x^\dagger$ ([13, 14]). This instability becomes evident when trying to approximate $x^\dagger$ by traditional numerical methods and procedures.

Ill-posed problems must be first regularized if one wants to successfully attack the task of numerically approximating their solutions. Regularizing an ill-posed problem such as (1.1) essentially means approximating the operator $T^\dagger$ by a parametric family of bounded operators $\{R_\alpha\}$, where α is the so-called "regularization parameter". If $y \in \mathcal{D}(T^\dagger)$, then the best approximate solution $x^\dagger$ of (1.1) can be written as

$$x^\dagger = \int_0^{\|T\|^2+} \frac{1}{\lambda}\, dE_\lambda T^* y$$

where $\{E_\lambda\}$ is the spectral family associated to the operator T^*T (see [2]). This is mainly why many regularization methods are based on spectral theory and consist of defining

$$R_\alpha \doteq \int_0^{\|T\|^2+} g_\alpha(\lambda)\, dE_\lambda T^*$$

where $\{g_\alpha\}$ is a family of functions appropriately chosen such that

$$\lim_{\alpha\to 0^+} g_\alpha(\lambda) = \frac{1}{\lambda}, \quad \forall \lambda \in (0, \|T\|^2].$$

It is well known, however, that it is not possible to get error estimates nor convergence rates on the whole space X ([1, 15, 16, 18]) for approximate solutions of an inverse ill-posed problem unless some *a-priori* information about the solution is available. This information may come, for instance, in terms of a source representation of the solution ([19]). A special inclusion of available *a-priori* information in Tikhonov regularization method was considered by Leonov and Yagola in 1998 (see [8]). All a regularization method can do is to recover the largest possible amount of information about the solution of the problem, maintaining stability. It is often said that the art of applying regularization methods consist always in maintaining an adequate balance between accuracy and stability. Usually accuracy can be improved with increasing assumptions (or information) on the regularity of the exact solution. In 1994, however, Neubauer ([11]) showed that certain spectral regularization methods "*saturate*", that is, they become unable to continue extracting additional information about the exact solution even upon increasing regularity assumptions. Prior to this, in 1984, Groetsch ([3]) had shown some saturation results for compact operators. The term "saturation" however was introduced for the first time by Vainikko and Veretennikov in 1986 for general spectral methods and illustrated for Tikhonov's and Lavrentiev's methods, as well as for different iterative regularizing algorithms ([17]). Although saturation is strongly related to the best order of convergence that a method can achieve independently of the smoothness assumptions on the exact solution and on the selection of the parameter choice rule, it constitutes a rather subtle and complex issue in the study of regularization methods for inverse ill-posed problems and the concept has, for many years, escaped rigorous formalization in a general context. In 1997, Neubauer ([12]) showed that this saturation phenomenon occurs for instance in the classical Tikhonov–Phillips method. Later on, in 2004, Mathé ([9]) proposed a general definition of the concept of saturation for spectral regularization methods. However, the concept of saturation defined by Mathé is not applicable to general regularization methods and it is not fully compatible with the concept of saturation emerging from the early works of Groetsch ([3]) and Neubauer ([11, 12]). In particular, for instance, the definition of saturation given in [9] does not imply uniqueness and therefore, neither a best global order of convergence. More recently, in 2011, Herdman, Spies and Temperini (see [5]) developed a general theory of global saturation for arbitrary regularization methods. It is also important to mention that in 2008, A. S. Leonov (see [7]) introduced a general approach for eliminating the saturation of accuracy in regularization algorithms by taking into account source conditions of general form.

Related in a dual way to the concept of saturation is the concept of qualification of a spectral regularization method. The term "qualification" was introduced for the first time by Vainikko and Veretennikov in 1986 (see [17]) and later generalized by Mathé and Pereverzev in 2003 ([10]). This concept is strongly related to the optimal order of convergence of the regularization error, under certain "a-priori" assumptions on the exact solution. In 2009 Herdman, Spies and Temperini ([4]) further generalized the concept of qualification and introduced three hierarchical levels of it: weak, strong and optimal qualification. There, it was shown that the weak qualification generalizes the definition introduced in [10].

In this work, some light on the existence of saturation for spectral regularization methods with optimal qualification is shed. In particular, sufficient conditions on the family of real functions $\{g_\alpha\}$ defining the method and on the optimal qualification ρ, which guarantee the existence of saturation, are established. Moreover, in those cases, appropriate characterizations of both the saturation function and the saturation set are provided.

2 Preliminaries

In this section we shall recall some basic concepts on global saturation of regularization methods for inverse ill-posed problems theory (for more details see [5]). In the sequel, $T : X \to Y$ will be a bounded linear operator with non-closed range between two Hilbert spaces X and Y. Without loss of generality we will assume that the operator T is invertible (in the context of inverse problems it is customary to work with the Moore–Penrose generalized inverse of T since one seeks least squares solutions of the problems; therefore the lack of injectivity of T is never a relevant issue). Also, for simplicity of notation and unless otherwise specified, we shall assume that all subsets of the Hilbert space X under consideration are not empty and they do not contain $x = 0$.

Let $M \subset X$. We shall say that a function $\psi : X \times \mathbb{R} \to \mathbb{R}$ belongs to the class $\mathcal{F}_M$ if there exists $a = a(\psi) > 0$ such that ψ is defined in $M \times (0, a)$, with values in $(0, \infty)$ and it satisfies the following conditions:

(i) $\lim_{\delta\to 0^+} \psi(x, \delta) = 0$ for all $x \in M$,

(ii) ψ is continuous and non-decreasing as a function of δ in $(0, a)$ for each fixed $x \in M$.

One may think of $\mathcal{F}_M$ as the collection of all possible δ-"*orders of convergence*" on the set M.

Definition 2.1. Let $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ be a family of regularization operators for the problem $Tx = y$. The "total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ at $x \in X$ for a noise level δ" is defined as

$$\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}(x, \delta) \doteq \inf_{\alpha\in(0,\alpha_0)} \sup_{y^\delta \in \overline{B_\delta(Tx)}} \|R_\alpha y^\delta - x\|,$$

where $\overline{B_\delta(Tx)} \doteq \{y \in Y : \|Tx - y\| \le \delta\}$.

Note that $\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}}$ is the error in the sense of the largest possible discrepancy that can be obtained for an observation of y within noise level δ, with an appropriate choice of the regularization parameter α.

Definition 2.2. Let $M \subset X$ and $\psi, \tilde{\psi} \in \mathcal{F}_M$. We say that "$\psi$ precedes $\tilde{\psi}$ on M", and we denote it with

$$\psi \overset{M}{\preceq} \tilde{\psi},$$

if there exist a constant $r > 0$ and a function $p : M \to (0, \infty)$ such that $\psi(x, \delta) \le p(x)\, \tilde{\psi}(x, \delta)$ for all $x \in M$ and for every $\delta \in (0, r)$.

Definition 2.3. Let $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ be a family of regularization operators for the problem $Tx = y$, $M \subset X$ and $\psi \in \mathcal{F}_M$. We say that ψ is an "upper bound of convergence for the total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on M" if

$$\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}} \overset{M}{\preceq} \psi.$$

With $\mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ we shall denote the set of all functions $\psi \in \mathcal{F}_M$ that are upper bounds of convergence for the total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on M.

The following two definitions formalize certain comparisons of bounds of convergence on different sets of X, which will be needed later to introduce the concept of global saturation.

Definition 2.4. Let $M, \tilde{M} \subset X$, $\psi \in \mathcal{F}_M$ and $\tilde{\psi} \in \mathcal{F}_{\tilde{M}}$.

(i) We say that "ψ on M precedes $\tilde{\psi}$ on $\tilde{M}$", and we denote it with

$$\psi \overset{M,\tilde{M}}{\preceq} \tilde{\psi},$$

if there exist a constant $d > 0$ and a function $k : M \times \tilde{M} \to (0, \infty)$ such that $\psi(x, \delta) \leq k(x, \tilde{x})\, \tilde{\psi}(\tilde{x}, \delta)$ for every $x \in M$, $\tilde{x} \in \tilde{M}$ and $\delta \in (0, d)$.

(ii) We say that "ψ on M is equivalent to $\tilde{\psi}$ on $\tilde{M}$", and we denote it with

$$\psi \overset{M,\tilde{M}}{\approx} \tilde{\psi},$$

if $\psi \overset{M,\tilde{M}}{\preceq} \tilde{\psi}$ and $\tilde{\psi} \overset{\tilde{M},M}{\preceq} \psi$.

Definition 2.5. Let $M \subset X$ and $\psi \in \mathcal{F}_M$. We say that "ψ is invariant over M" if

$$\psi \overset{M,M}{\approx} \psi.$$

Next we recall the concept of global saturation introduced in [5].

Definition 2.6. Let $M_S \subset X$ and $\psi_S \in \mathcal{U}_{M_S}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$. It is said that ψ_S is a "global saturation function of $\{R_\alpha\}$ over M_S" if ψ_S satisfies the following three conditions:

(S1) For every $x^* \in X$, $x^* \neq 0$, $x \in M_S$, $\limsup_{\delta\to 0+} \frac{\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta)}{\psi_S(x,\delta)} > 0$.

(S2) ψ_S is invariant over M_S.

(S3) There is no upper bound of convergence for the total error of $\{R_\alpha\}$ that is a proper extension of ψ_S (in the variable x) and satisfies (S1) and (S2), that is, there exist no $\tilde{M} \supsetneqq M_S$ and $\tilde{\psi} \in \mathcal{U}_{\tilde{M}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ such that $\tilde{\psi}$ satisfies (S1) and (S2) with M_S replaced by $\tilde{M}$ and ψ_S replaced by $\tilde{\psi}$.

The function ψ_S and the set M_S are referred to as the saturation function and the saturation set, respectively.

This conception of global saturation essentially establishes that in no point $x^* \in X$, $x^* \neq 0$, can exist an upper bound of convergence for the total error of the regularization method that is "strictly better" than the saturation function ψ_S at any point of the saturation set M_S.

Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family associated to the linear selfadjoint operator T^*T and $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ a parametric family of functions $g_\alpha : [0, \|T\|^2] \to \mathbb{R}$, $\alpha \in (0, \alpha_0)$, and consider the following standing hypotheses:

(H1) For every $\alpha \in (0, \alpha_0)$ the function g_α is piecewise continuous on $[0, \|T\|^2]$.

(H2) There exists a constant $C > 0$ (independent of α) such that $|\lambda g_\alpha(\lambda)| \le C$ for every $\lambda \in [0, \|T\|^2]$.

(H3) For every $\lambda \in (0, \|T\|^2]$, there exists $\lim_{\alpha\to 0^+} g_\alpha(\lambda) = \frac{1}{\lambda}$.

If $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ satisfies hypotheses (H1)–(H3), then (see [2, Theorem 4.1]) the collection of operators $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$, where

$$R_\alpha \doteq \int_0^{\|T\|^2+} g_\alpha(\lambda)\, dE_\lambda\, T^* = g_\alpha(T^*T)T^*,$$

is a family of regularization operators for $T^\dagger$. In this case we say that $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ is a "family of spectral regularization operators" (FSRO) for $Tx = y$ and that $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ is a "spectral regularization method" (SRM).

The following definitions will be needed both to recall the concept of qualification as introduced in [4], as well as in the rest of the article.

Denote with $\mathcal{O}$ the set of all non-decreasing functions $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{\alpha\to 0^+} \rho(\alpha) = 0$ and with $\mathcal{S}$ the set of all continuous functions $s : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying $s(0) = 0$ and such that $s(\lambda) > 0$ for every $\lambda > 0$. Note that if $s \in \mathcal{S}$ is non-decreasing, then s is an *index function* in the sense of Mathé–Pereverzev ([10]).

Definition 2.7. Let $\rho, \tilde\rho \in \mathcal{O}$. We say that "$\rho$ precedes $\tilde\rho$ at the origin" and we denote it with $\rho \preceq \tilde\rho$ if there exist positive constants c and ε such that $\rho(\alpha) \le c\,\tilde\rho(\alpha)$ for every $\alpha \in (0, \varepsilon)$.

Definition 2.8. Let $\rho, \tilde\rho \in \mathcal{O}$. We say that "$\rho$ and $\tilde\rho$ are equivalent at the origin" and we denote it with $\rho \approx \tilde\rho$ if they precede each other at the origin, that is, if there exist constants $\varepsilon > 0$, c_1, c_2, $0 < c_1 < c_2 < \infty$, such that $c_1\,\rho(\alpha) \le \tilde\rho(\alpha) \le c_2\,\rho(\alpha)$ for every $\alpha \in (0, \varepsilon)$.

Clearly "$\approx$" is an equivalence relation and it introduces in $\mathcal{O}$ a partial ordering. Analogous definitions and notation will be used for $s, \tilde s \in \mathcal{S}$.

Definition 2.9. Let $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ be an SRM, $r_\alpha(\lambda) \doteq 1 - \lambda g_\alpha(\lambda)$, $\rho \in \mathcal{O}$, $s \in \mathcal{S}$.

(i) We say that (s, ρ) is a "weak source-order pair for $\{g_\alpha\}$" if it satisfies

$$\frac{s(\lambda)\,|r_\alpha(\lambda)|}{\rho(\alpha)} = O(1) \quad \text{for } \alpha \to 0^+,\ \forall\, \lambda > 0. \tag{2.1}$$

(ii) We say that (s, ρ) is a "strong source-order pair for $\{g_\alpha\}$" if it is a weak source-order pair and there is no $\lambda > 0$ for which "$O(1)$" in (2.1) can be replaced by "$o(1)$", that is, if (s, ρ) is a weak source-order pair for $\{g_\alpha\}$ and

$$\limsup_{\alpha \to 0^+} \frac{s(\lambda)\,|r_\alpha(\lambda)|}{\rho(\alpha)} > 0 \quad \forall\, \lambda > 0. \tag{2.2}$$

(iii) We say that (ρ, s) is an "order-source pair for $\{g_\alpha\}$" if there exist a constant $\gamma > 0$ and a function $h : (0, \alpha_0) \to \mathbb{R}^+$ with $\lim_{\alpha \to 0^+} h(\alpha) = 0$ such that

$$\frac{s(\lambda)\,|r_\alpha(\lambda)|}{\rho(\alpha)} \geq \gamma \quad \forall\, \lambda \in [h(\alpha), +\infty). \tag{2.3}$$

In the context of the previous definitions we refer to the function ρ as the "order of convergence" and to s as the "source function".

We are now ready to define the concept of qualification in its three different levels as it was introduced in [4].

Definition 2.10. Let $\{g_\alpha\}$ be an SRM.

(i) We say that ρ is "weak qualification of $\{g_\alpha\}$" if there exists a function s such that (s, ρ) is a weak source-order pair for $\{g_\alpha\}$.

(ii) We say that ρ is "strong qualification of $\{g_\alpha\}$" if there exists a function s such that (s, ρ) is a strong source-order pair for $\{g_\alpha\}$.

(iii) We say that ρ is "optimal qualification of $\{g_\alpha\}$" if there exists a function s such that (s, ρ) is a strong source-order pair for $\{g_\alpha\}$ and (ρ, s) is an order-source pair for $\{g_\alpha\}$.

Note that since condition (2.3) implies condition (2.2), in the definition of optimal qualification above the requirement that (s, ρ) be strong source-order pair can be replaced by the one that (s, ρ) be a weak source-order pair.

Now given the SRM $\{g_\alpha\}$ and $\rho \in \mathcal{O}$, we define

$$s_\rho(\lambda) \doteq \liminf_{\alpha \to 0^+} \frac{\rho(\alpha)}{|r_\alpha(\lambda)|} \quad \text{for } \lambda \geq 0. \tag{2.4}$$

Note that $s_\rho(0) = 0$ and if s_ρ is continuous, $s_\rho \in \mathcal{S}$.

The next theorem provides necessary and sufficient condition, in terms of s_ρ, for an order of convergence $\rho \in \mathcal{O}$ to be optimal qualification.

Theorem 2.11 ([4]). *Let $\{g_\alpha\}$ be an SRM and $\rho \in \mathcal{O}$ such that $s_\rho \in \mathcal{S}$. Then ρ is optimal qualification of $\{g_\alpha\}$ if and only if s_ρ verifies* (2.3) *and*

$$0 < s_\rho(\lambda) < +\infty \quad \textit{for every } \lambda > 0. \tag{2.5}$$

The next theorem shows the uniqueness of the source function.

Theorem 2.12 ([4]). *If ρ is optimal qualification of $\{g_\alpha\}$, then there exists only one function s (in the sense of the equivalence classes induced by Definition* 2.8*) such that (s,ρ) is a strong source-order pair and (ρ,s) is an order-source pair for $\{g_\alpha\}$. Moreover if $s_\rho \in \mathcal{S}$, then s_ρ is such a unique function.*

The following converse result, where regularity properties of the exact solution are derived in terms of the rate of convergence of the regularization error, will be needed later. This result states that if the regularization error has order of convergence $\rho(\alpha)$ and (ρ,s) is an order-source pair, then the exact solution belongs to the source set given by the range of the operator $s(T^*T)$.

Theorem 2.13 ([4]). *Let $\{g_\alpha\}$ be an SRM and $R_\alpha = g_\alpha(T^*T)T^*$. If*

$$\|(R_\alpha - T^\dagger)y\| = O(\rho(\alpha)) \quad \text{for } \alpha \to 0^+$$

and (ρ,s) is an order-source pair for $\{g_\alpha\}$, then $T^\dagger y \in \mathcal{R}(s(T^*T))$.

3 Saturation of spectral regularization methods with optimal qualification

The purpose of this section is to shed some light on the saturation of SRM with optimal qualification. More precisely, we will establish sufficient conditions on the family of functions $\{g_\alpha\}$ and on the optimal qualification ρ guaranteeing the existence of saturation. Moreover, for those methods we will provide appropriate characterizations of both the saturation function and the saturation set. Then, let $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ be an SRM and consider the following hypothesis:

(H4) There exists $k>0$ such that $G_\alpha \doteq \|g_\alpha(\cdot)\|_\infty \le \frac{k}{\sqrt{\alpha}}$ $\forall\, \alpha \in (0,\alpha_0)$.

Lemma 3.1. *Let $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ be an SRM satisfying hypothesis* (H4)*. Further let $R_\alpha = g_\alpha(T^*T)T^*$ and let (s,ρ) be a weak source-order pair for $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ where ρ is continuous. Define $X^s \doteq \mathcal{R}(s(T^*T)) \setminus \{0\}$, $\Theta(t) \doteq \sqrt{t}\,\rho(t)$ for $t>0$, $\psi(x,\delta) \doteq \rho\circ\Theta^{-1}(\delta)$ for $x \in X^s$ and $\delta \in (0,\Theta(\alpha_0))$. Then $\psi \in \mathcal{F}_{X^s}$ and, moreover, ψ is an upper bound of convergence for the total error of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on X^s, that is, $\psi \in \mathcal{U}_{X^s}(\mathcal{E}^{\mathrm{tot}}_{\{R_\alpha\}})$.*

Proof. Since ρ is continuous and non-decreasing and $\rho(0^+)=0$, it follows that $\Theta(t)$ is continuous and strictly increasing on $(0,+\infty)$ with $\Theta(0^+)=0$. Therefore Θ^{-1} exists and has the same properties. It then follows that ψ is continuous and non-decreasing as a function of δ in $(0,\Theta(\alpha_0))$ for each fixed $x \in X^s$, and $\psi(x,0^+)=0$ for all $x\in X^s$. Hence $\psi \in \mathcal{F}_{X^s}$.

On the other hand, since (s,ρ) is a weak source-order pair for $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$, there exist positive constants c and $\hat\alpha$ such that $s(\lambda)\,|r_\alpha(\lambda)|\le c\rho(\alpha)$ for every $\alpha\in(0,\hat\alpha)$, $\lambda\in(0,\|T\|^2]$. Moreover, from hypothesis (H2) and the fact that ρ is non-decreasing it follows that the previous inequality holds (perhaps with a different positive constant c) for every $\alpha\in(0,\alpha_0)$, that is,

$$s(\lambda)\,|r_\alpha(\lambda)|\le c\rho(\alpha)\quad \forall\,\alpha\in(0,\alpha_0),\ \forall\,\lambda\in(0,\|T\|^2]. \tag{3.1}$$

Now, for every $p\ge 0$ we define the source sets

$$X^{s,p}\doteq\{x\in X : x=s(T^*T)\zeta,\ \|\zeta\|\le p\}.$$

Then for each $x\in X^s$ there exists $p_x>1$ such that $x\in X^{s,p_x}$. On the other hand, since Θ is continuous and strictly increasing in $(0,\alpha_0)$, there exists a unique $\tilde\alpha_x\in(0,\alpha_0)$ such that $x\in X^{s,p_x}$ and $\Theta(\tilde\alpha_x)=\frac{\delta}{p_x}$. Therefore,

$$\begin{aligned}\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta)&=\inf_{\alpha\in(0,\alpha_0)}\ \sup_{y^\delta\in\overline{B_\delta(Tx)}}\|R_\alpha\, y^\delta-x\| &(3.2)\\ &\le \sup_{y^\delta\in\overline{B_\delta(Tx)}}\|R_{\tilde\alpha_x}\, y^\delta-x\|. &(3.3)\end{aligned}$$

Now, since $y^\delta=Tx+\delta\xi$, $\|\xi\|\le 1$ and $x=s(T^*T)\zeta$ with $\|\zeta\|\le p_x$, it follows immediately that

$$\begin{aligned}\|R_{\tilde\alpha_x}\, y^\delta-x\|&\le\|(g_{\tilde\alpha_x}(T^*T)T^*T-I)s(T^*T)\zeta\|+\delta\|g_{\tilde\alpha_x}(T^*T)T^*\xi\|\\ &\le p_x\sup_{\lambda\in(0,\|T\|^2]}\{s(\lambda)\left|r_{\tilde\alpha_x}(\lambda)\right|\}+\delta\sup_{\lambda\in(0,\|T\|^2]}\{\sqrt{\lambda}\left|g_{\tilde\alpha_x}(\lambda)\right|\},\end{aligned} \tag{3.4}$$

where the last inequality follows from properties of functions of a selfadjoint operator (more precisely, for any piecewise continuous function f there holds $\|f(T^*T)\|\le\sup_\lambda|f(\lambda)|$ and $\|f(T^*T)T^*\|\le\sup_\lambda\{\sqrt{\lambda}\,|f(\lambda)|\}$, see [2, p. 45]). Using (3.1) and hypothesis (H4) in (3.4) it follows that

$$\|R_{\tilde\alpha_x}\, y^\delta-x\|\le p_x\, c\,\rho(\tilde\alpha_x)+\delta\frac{k}{\sqrt{\tilde\alpha_x}}\|T\|. \tag{3.5}$$

Since $\Theta(\tilde\alpha_x)=\sqrt{\tilde\alpha_x}\,\rho(\tilde\alpha_x)=\frac{\delta}{p_x}$, it follows that $\frac{\delta}{\sqrt{\tilde\alpha_x}}=p_x\,\rho(\tilde\alpha_x)$. Hence by virtue of (3.5) one has that

$$\begin{aligned}\|R_{\tilde\alpha_x}\, y^\delta-x\|&\le p_x(c+k\|T\|)\rho(\tilde\alpha_x)\\ &=p_x(c+k\|T\|)\rho\left(\Theta^{-1}\left(\frac{\delta}{p_x}\right)\right)\\ &\le p_x(c+k\|T\|)\rho(\Theta^{-1}(\delta)), &(3.6)\end{aligned}$$

where the last inequality follows from the fact that $p_x > 1$ and both ρ and Θ^{-1} are non-decreasing functions. From inequalities (3.3) and (3.6) it follows that for every $\delta \in (0, \Theta(\alpha_0))$,

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x,\delta) \leq p_x(c + k\|T\|)\rho(\Theta^{-1}(\delta)) = p_x(c + k\|T\|)\,\psi(x,\delta).$$

This proves that $\psi \in \mathcal{U}_{X^s}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$. □

Definition 3.2 ([6]). Let k be a positive constant and $\rho : (0,k] \to (0,+\infty)$ a continuous non-decreasing function such that $\lim_{t\to 0^+} \rho(t) = 0$. We say that ρ is of "local upper type β" ($\beta \geq 0$) if there exists a positive constant d such that $\rho(t) \leq d s^{-\beta}\rho(s\,t)$ for every $s \in (0,1]$, $t \in (0,k]$.

Theorem 3.3 (Saturation for FSRO with optimal qualification). *Let $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ be an SRM satisfying hypothesis* (H4) *and having optimal qualification ρ. Further let $r_\alpha(\lambda) = 1 - \lambda g_\alpha(\lambda)$, $R_\alpha = g_\alpha(T^*T)T^*$ and suppose that $s_\rho \in \mathcal{S}$ (where s_ρ is as defined in* (2.4)*). Assume further that the following hypotheses hold:*

(a) *The function ρ is of local upper type β, for some $\beta \geq 0$.*

(b) *There exist positive constants $\gamma_1, \gamma_2, \lambda^*, c_1$, with $\lambda^* \leq \|T\|^2$ and $c_1 > 1$ such that*

  (i) $0 \leq r_\alpha(\lambda) \leq 1$, *for $\alpha > 0$, $0 \leq \lambda \leq \lambda^*$,*

  (ii) $r_\alpha(\lambda) \geq \gamma_1$, *for $0 \leq \lambda < h(\alpha) \leq \lambda^*$, $\alpha \in (0,\alpha_0)$ where h is as in* (2.3) *(note that by virtue of Theorem* 2.12 *and the fact that $s_\rho \in \mathcal{S}$, there exists only one function $s \in \mathcal{S}$ satisfying* (2.3)*, that is, $s = s_\rho$),*

  (iii) $|r_\alpha(\lambda)|$ *is non-decreasing with respect to α for each $\lambda \in (0, \|T\|^2]$,*

  (iv) $g_\alpha(c_1\alpha) \geq \frac{\gamma_2}{\alpha}$, *for $0 < c_1\alpha \leq \lambda^*$,*

  (v) $g_\alpha(\lambda) \geq g_\alpha(\tilde{\lambda})$, *for $0 < \alpha \leq \lambda \leq \tilde{\lambda} \leq \lambda^*$.*

(c) *There exist $\{\lambda_n\}_{n=1}^\infty \subset \sigma(TT^*)$ and $c \geq 1$ such that $\lambda_n \downarrow 0$ and $\frac{\lambda_n}{\lambda_{n+1}} \leq c$ for every $n \in \mathbb{N}$.*

Let $\Theta(t) \doteq \sqrt{t}\,\rho(t)$ for $t > 0$ and $X^{s_\rho} \doteq \mathcal{R}(s_\rho(T^*T)) \setminus \{0\}$. Then

$$\psi(x,\delta) \doteq \rho \circ \Theta^{-1}(\delta) \quad \textit{for } x \in X^{s_\rho} \textit{ and } \delta \in (0, \Theta(\alpha_0))$$

is saturation function of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on X^{s_ρ}.

To prove this theorem, we will need three previous lemmas. The first one is a somewhat technical result, the second one deals with the existence of an *a priori* parameter choice rule leading to a worst total error having an appropriate order of convergence, while the third one is a converse result.

Lemma 3.4. *Let* $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ *be an SRM,* (ρ, s) *an order-source pair for* $\{g_\alpha\}$ *and suppose hypothesis* (b.ii) *of Theorem* 3.3 *holds. Then for every* $\alpha \in (0, \alpha_0)$ *the operator* $r_\alpha(T^*T)$ *is invertible.*

Proof. Let $\{E_\lambda\}$ be the spectral family of T^*T. It suffices to show that for every $\alpha \in (0, \alpha_0)$, $x \in X$, the function $r_\alpha^{-2}(\lambda)$ is integrable with respect to the measure $d\|E_\lambda x\|^2$. Let $\alpha \in (0, \alpha_0)$ fixed. Since (ρ, s) is an order-source pair for $\{g_\alpha\}$, there exist a constant $\gamma > 0$ and a function $h : (0, \alpha_0) \to \mathbb{R}^+$ with $\lim_{\alpha\to 0^+} h(\alpha) = 0$ such that

$$\frac{s(\lambda)\,|r_\alpha(\lambda)|}{\rho(\alpha)} \ge \gamma \quad \forall\, \lambda \in [h(\alpha), +\infty).$$

Therefore

$$\begin{aligned}\int_{h(\alpha)}^{\|T\|^2+} \frac{1}{r_\alpha^2(\lambda)}\, d\|E_\lambda x\|^2 &\le \frac{1}{\gamma^2\rho^2(\alpha)} \int_{h(\alpha)}^{\|T\|^2+} s^2(\lambda)\, d\|E_\lambda x\|^2 \\ &\le \frac{\|s(T^*T)x\|^2}{\gamma^2\rho^2(\alpha)} < +\infty. \qquad (3.7)\end{aligned}$$

Now, since $\alpha \in (0, \alpha_0)$, it follows from hypothesis (b.ii) of Theorem 3.3 that $r_\alpha(\lambda) \ge \gamma_1 > 0$ for every $\lambda \in [0, h(\alpha))$. Then

$$\int_0^{h(\alpha)} \frac{1}{r_\alpha^2(\lambda)}\, d\|E_\lambda x\|^2 \le \frac{\|x\|^2}{\gamma_1^2} < +\infty. \qquad (3.8)$$

From (3.7) and (3.8) it follows that $\int_0^{\|T\|^2+} r_\alpha^{-2}(\lambda)\, d\|E_\lambda x\|^2 < +\infty$. Hence $r_\alpha(T^*T)$ is invertible. □

Lemma 3.5. *Let* $\{g_\alpha\}$ *be an SRM,* $R_\alpha = g_\alpha(T^*T)T^*$, (ρ, s) *an order-source pair for* $\{g_\alpha\}$ *and assume that hypotheses* (b.ii) *and* (b.iii) *of Theorem* 3.3 *hold. Let* $\varphi : (0, +\infty) \to \mathbb{R}^+$ *be a continuous, non-decreasing function with the property that* $\lim_{\delta\to 0^+} \varphi(\delta) = 0$ *and* $x^* \in X$, $x^* \ne 0$.

(I) *If* $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*, \delta) = o(\varphi(\delta))$ *for* $\delta \to 0^+$, *then there exists an a-priori parameter choice rule* $\tilde\alpha(\delta)$ *such that*

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \|R_{\tilde\alpha(\delta)} y^\delta - x^*\| = o(\varphi(\delta)) \quad \textit{for } \delta \to 0^+.$$

(II) *Part* (I) *remains true with* $o(\varphi(\delta))$ *replaced by* $O(\varphi(\delta))$, *that is, if*

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*, \delta) = O(\varphi(\delta)) \quad \textit{for } \delta \to 0^+,$$

then there exists an a priori parameter choice rule $\tilde\alpha(\delta)$ *such that*

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \|R_{\tilde\alpha(\delta)} y^\delta - x^*\| = O(\varphi(\delta)) \quad \textit{for } \delta \to 0^+.$$

Proof. Let φ and $x^* \in X$ be as in the hypotheses and suppose that

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*, \delta) = o(\varphi(\delta)) \quad \text{for } \delta \to 0^+.$$

Then by the definition of $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}$,

$$\begin{aligned}\lim_{\delta \to 0^+} & \frac{\inf_{\alpha \in (0,\alpha_0)} \sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \|R_\alpha y^\delta - x^*\|}{\varphi(\delta)} \\ &= \lim_{\delta \to 0^+} \inf_{\alpha \in (0,\alpha_0)} \frac{\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \|R_\alpha y^\delta - x^*\|}{\varphi(\delta)} = 0. \end{aligned} \tag{3.9}$$

For the sake of simplicity we define

$$f(\alpha, \delta) \doteq \frac{\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \|R_\alpha y^\delta - x^*\|}{\varphi(\delta)} \quad \text{and} \quad q(\delta) \doteq \inf_{\alpha \in (0,\alpha_0)} f(\alpha, \delta),$$

so that with this notation (3.9) can be written simply as $\lim_{\delta \to 0^+} q(\delta) = 0$ and the objective is to prove the existence of an a priori parameter choice rule $\tilde{\alpha}(\delta)$ such that $\lim_{\delta \to 0^+} f(\tilde{\alpha}(\delta), \delta) = 0$. It can be easily proved that if for certain $\delta_0 > 0$, $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*, \delta_0) = 0$, then $T^\dagger = 0$. Hence $q(\delta) > 0$ for every $\delta \in (0, +\infty)$. Also, $q(\delta)$ is continuous for $\delta \in (0, +\infty)$ since both $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*, \delta)$ and $\varphi(\delta)$ are continuous. Next, for $n \in \mathbb{N}$ we define

$$\delta_n \doteq \sup \left\{ d > 0 : q(\delta) \le \frac{1}{n} \ \forall\, \delta \in (0, d) \right\}.$$

Clearly, $\delta_n \downarrow 0$ and since q is continuous for every $n \in \mathbb{N}$ and every $\delta \in (0, \delta_n]$, $q(\delta) = \inf_{\alpha \in (0,\alpha_0)} f(\alpha, \delta) \le \frac{1}{n}$. Then, there exists an $\alpha_n = \alpha_n(\delta_n) \in (0, \alpha_0)$ such that for all $n \in \mathbb{N}$

$$f(\alpha_n, \delta) \le \frac{2}{n} \quad \forall\, \delta \in (0, \delta_n]. \tag{3.10}$$

Since $\{\alpha_n\} \subset (0, \alpha_0)$ is a bounded sequence, there exist $\alpha^* \in [0, \alpha_0]$ and a subsequence $\{\alpha_{n_k}\} \subset \{\alpha_n\}$ such that $\lim_{k \to +\infty} \alpha_{n_k} = \alpha^*$. We now define $\tilde{\alpha}(\delta) \doteq \alpha_{n_k}$ for $\delta \in (\delta_{n_{k+1}}, \delta_{n_k}]$, $k = 1, 2, \ldots$, and $\tilde{\alpha}(\delta) = \tilde{\alpha}(\delta_{n_1})$ for $\delta > \delta_{n_1}$. Then

$$\lim_{\delta \to 0^+} \tilde{\alpha}(\delta) = \alpha^* \tag{3.11}$$

and

$$0 \le \limsup_{\delta \to 0^+} f(\tilde{\alpha}(\delta), \delta) \le \limsup_{k \to +\infty} \Big[\sup_{\delta \in (0, \delta_{n_k}]} f(\alpha_{n_k}, \delta) \Big] \le \limsup_{k \to +\infty} \frac{2}{n_k} = 0,$$

where the last inequality follows from (3.10).

Hence,

$$\lim_{\delta\to 0^+} f(\tilde{\alpha}(\delta),\delta) = 0. \tag{3.12}$$

It remains to be shown that $\tilde{\alpha}(\delta)$ is an admissible parameter choice rule, for which it suffices to prove that $\lim_{\delta\to 0^+}\tilde{\alpha}(\delta) = 0$, i.e. that $\alpha^* = 0$. If $\alpha^* > 0$, it follows from (3.11) that there exists $\delta_0 > 0$ such that $\tilde{\alpha}(\delta) > \frac{\alpha^*}{2}$ for all $\delta \in (0,\delta_0)$. Hypothesis (b.iii) of Theorem 3.3 then implies that for every $\delta \in (0,\delta_0)$,

$$|r_{\tilde{\alpha}(\delta)}(\lambda)| \geq |r_{\frac{\alpha^*}{2}}(\lambda)| \quad \text{for all } \lambda \in (0, \|T\|^2].$$

Therefore for every $\delta \in (0,\delta_0)$,

$$\begin{aligned}
\|r_{\tilde{\alpha}(\delta)}(T^*T)x^*\|^2 &= \int_0^{\|T\|^2+} r^2_{\tilde{\alpha}(\delta)}(\lambda)\, d\|E_\lambda x^*\|^2 \\
&\geq \int_0^{\|T\|^2+} r^2_{\frac{\alpha^*}{2}}(\lambda)\, d\|E_\lambda x^*\|^2 \\
&= \|r_{\frac{\alpha^*}{2}}(T^*T)x^*\|^2.
\end{aligned} \tag{3.13}$$

Now, for all $\delta \in (0,\delta_0)$,

$$\begin{aligned}
\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \|R_{\tilde{\alpha}(\delta)}y^\delta - x^*\| &\geq \|R_{\tilde{\alpha}(\delta)}Tx^* - x^*\| \\
&= \left\|\left(I - g_{\tilde{\alpha}(\delta)}(T^*T)T^*T\right)x^*\right\| \\
&= \|r_{\tilde{\alpha}(\delta)}(T^*T)x^*\| \geq \|r_{\frac{\alpha^*}{2}}(T^*T)x^*\|,
\end{aligned}$$

where the last inequality follows from (3.13). Dividing through by $\varphi(\delta)$, taking limit for $\delta \to 0^+$, and using the definition of $f(\alpha,\delta)$ and (3.12) we conclude that

$$\|r_{\frac{\alpha^*}{2}}(T^*T)x^*\| = 0.$$

Now since $\frac{\alpha^*}{2} < \alpha_0$, (ρ, s) is an order-source pair for $\{g_\alpha\}$ and hypothesis (b.ii) of Theorem 3.3 holds, it follows from Lemma 3.4 that $r_{\frac{\alpha^*}{2}}(T^*T)$ is invertible. Therefore $x^* = 0$, contradicting the hypothesis that $x^* \neq 0$. Hence, α^* must be equal to zero, as wanted.

We proceed now to prove the second part of the lemma. Suppose that there exists $x^* \in X$, $x^* \neq 0$ such that $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta) = O(\varphi(\delta))$ as $\delta \to 0^+$. Then there exist positive constants k and d such that $\inf_{\alpha\in(0,\alpha_0)} f(\alpha,\delta) \leq k$ for every $\delta \in (0,d)$, where $f(\alpha,\delta)$ is as previously defined. Let $\{\delta_n\}_{n\in\mathbb{N}} \subset (0,d)$ be such that $\delta_n \downarrow 0$ and $\alpha_n = \alpha_n(\delta_n) \in (0,\alpha_0)$ such that

$$f(\alpha_n,\delta) \leq k + \delta_n, \quad \forall\, \delta \in (0,d),\ \forall\, n \in \mathbb{N}.$$

Without loss of generality we assume that the sequence $\{\alpha_n\}$ converges (since if that is not the case, we can take a subsequence which does). Now, like in the previously case, by defining $\tilde{\alpha}(\delta) = \alpha_n$ for $\delta \in (\delta_{n+1}, \delta_n]$, $n = 1, 2, \dots,$ and $\tilde{\alpha}(\delta) = \alpha(\delta_1)$ for $\delta > \delta_1$, since $\delta_n \downarrow 0$ it follows that $f(\tilde{\alpha}(\delta), \delta) \le k + \delta_1$ for every $\delta \in (0, d)$ and therefore

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \|R_{\tilde{\alpha}(\delta)} y^\delta - x^*\| = O(\varphi(\delta)) \quad \text{as } \delta \to 0^+.$$

Following the same steps as in the proof of part (I) we obtain that

$$\lim_{\delta \to 0^+} \tilde{\alpha}(\delta) = 0,$$

i.e. $\tilde{\alpha}(\delta)$ is an admissible parameter choice rule. This concludes the proof of the lemma. □

Lemma 3.6. *Let $\{g_\alpha\}_{\alpha \in (0,\alpha_0)}$ be an SRM, $r_\alpha(\lambda) \doteq 1 - \lambda g_\alpha(\lambda)$, $R_\alpha = g_\alpha(T^*T)T^*$, (ρ, s) an order-source pair for $\{g_\alpha\}$, $\Theta(t) \doteq \sqrt{t}\,\rho(t)$ for $t > 0$, and suppose that:*

(a) *The function ρ is of local upper type β, for some $\beta \ge 0$.*

(b) *There exist positive constants $\gamma_1, \gamma_2, \lambda^*, c_1$, with $\lambda^* \le \|T\|^2$ and $c_1 > 1$ such that*

  (i) $0 \le r_\alpha(\lambda) \le 1$, *for $\alpha > 0$, $0 \le \lambda \le \lambda^*$,*

  (ii) $r_\alpha(\lambda) \ge \gamma_1$, *for $0 \le \lambda < h(\alpha) \le \lambda^*$, $\alpha \in (0, \alpha_0)$ where h is as in* (2.3),

  (iii) $|r_\alpha(\lambda)|$ *is non-decreasing with respect to α for each $\lambda \in (0, \|T\|^2]$,*

  (iv) $g_\alpha(c_1\alpha) \ge \frac{\gamma_2}{\alpha}$ *for $0 < c_1\alpha \le \lambda^*$,*

  (v) $g_\alpha(\lambda) \ge g_\alpha(\tilde{\lambda})$, *for $0 < \alpha \le \lambda \le \tilde{\lambda} \le \lambda^*$.*

(c) *There exist $\{\lambda_n\}_{n=1}^\infty \subset \sigma(TT^*)$ and $c \ge 1$ such that $\lambda_n \downarrow 0$ and $\frac{\lambda_n}{\lambda_{n+1}} \le c$ for every $n \in \mathbb{N}$.*

If for some $x \in X$ we have that

$$\sup_{y^\delta \in \overline{B_\delta(Tx)}} \inf_{\alpha \in (0,\alpha_0)} \|R_\alpha y^\delta - x\| = O(\rho(\Theta^{-1}(\delta))) \quad \textit{when } \delta \to 0^+, \tag{3.14}$$

then $x \in \mathcal{R}(s(T^*T))$. In particular, if ρ is optimal qualification of $\{g_\alpha\}$ and $s_\rho \in \mathcal{S}$, then $x \in \mathcal{R}(s_\rho(T^*T))$.

Proof. Without loss of generality we may assume that $\alpha_0 \le \frac{\lambda^*}{c_1}$ and $x \ne 0$ (this is so because hypotheses (a) and (c) are independent of α_0 and if (b) holds for $\alpha \in (0, \alpha_0)$, then it holds for $\alpha \in (0, \hat{\alpha}_0)$ for every $\hat{\alpha}_0 < \alpha_0$ with the same constants, while if $x = 0$, the result of the lemma is trivial).

Let $\bar\alpha \in \sigma(TT^*)$ be such that $0 < c_1\,\bar\alpha \le \alpha_0$ (hypothesis (c) guarantees the existence of such $\bar\alpha$), and define

$$\bar\delta = \bar\delta(\bar\alpha) \doteq \frac{\bar\alpha^{1/2}}{\gamma_2}\|R_{\bar\alpha}Tx - x\| = \frac{\bar\alpha^{1/2}}{\gamma_2}\|r_{\bar\alpha}(T^*T)x\|.$$

Then, clearly the equation

$$\|R_\alpha Tx - x\|^2 = \frac{(\gamma_2\,\bar\delta)^2}{\alpha} \tag{3.15}$$

in the unknown α has $\alpha = \bar\alpha$ as a solution. Moreover, since

$$\|R_\alpha Tx - x\|^2 = \int_0^{\|T\|^2+} r_\alpha^2(\lambda)\, d\,\|E_\lambda x\|^2$$

and $x \ne 0$, hypotheses (b.ii), (b.iii) imply that the function $\mu(\alpha) \doteq \alpha\|R_\alpha Tx - x\|^2$ is strictly increasing for α in $(0,\alpha_0)$. Hence, $\alpha = \bar\alpha \doteq \eta(\bar\delta)$ (where $\eta(\delta) = \mu^{-1}(\delta)$) is the unique solution of (3.15). Note that if $\bar\alpha \to 0^+$, then $\bar\delta \to 0^+$. Moreover, by hypothesis (b.iii) and Lemma 3.4, it follows immediately by Fatou's lemma that $\bar\delta \to 0^+$ only if $\bar\alpha \to 0^+$.

Now, for $\delta > 0$ define

$$\bar y^{\,\delta} \doteq Tx - \delta G_{\bar\alpha} z, \tag{3.16}$$

where $G_{\bar\alpha} \doteq F_{c_1\bar\alpha} - F_{\bar\alpha}$ with $\{F_\lambda\}$ being the spectral family associated to TT^* and

$$z \doteq \begin{cases} \|G_{\bar\alpha}Tx\|^{-1}Tx, & \text{if } G_{\bar\alpha}Tx \ne 0,\\ \text{arbitrary with } \|G_{\bar\alpha}z\| = 1, & \text{in other case.}\end{cases}$$

Note that since $\bar\alpha \in \sigma(TT^*)$ and $c_1 > 1$ it follows that $G_{\bar\alpha}$ is not the null operator and therefore the definition makes sense. Note also that $\|\bar y^\delta - Tx\| = \delta$, which implies that $\bar y^\delta \in \overline{B_\delta(Tx)}$.

Now, by using (3.16) and the fact that $g_\alpha(T^*T)T^* = T^*g_\alpha(TT^*)$ it follows that for every $\alpha \in (0,\alpha_0)$ and $\delta > 0$,

$$\begin{aligned}
\left\langle R_\alpha Tx - x, R_\alpha(\bar y^{\,\delta} - Tx)\right\rangle &= \left\langle g_\alpha(T^*T)T^*Tx - x, -g_\alpha(T^*T)T^*\,\delta G_{\bar\alpha}z\right\rangle\\
&= \delta\left\langle g_\alpha(T^*T)T^*Tx - x, -T^*g_\alpha(TT^*)G_{\bar\alpha}z\right\rangle\\
&= \delta\left\langle Tg_\alpha(T^*T)T^*Tx - Tx, -g_\alpha(TT^*)G_{\bar\alpha}z\right\rangle\\
&= \delta\left\langle (TT^*g_\alpha(TT^*) - I)Tx, -g_\alpha(TT^*)G_{\bar\alpha}z\right\rangle\\
&= \delta\left\langle -r_\alpha(TT^*)Tx, -g_\alpha(TT^*)G_{\bar\alpha}z\right\rangle\\
&= \delta\int_0^{\|T\|^2+} r_\alpha(\lambda)g_\alpha(\lambda)\, d\,\langle F_\lambda Tx, G_{\bar\alpha}z\rangle\,.
\end{aligned} \tag{3.17}$$

Now by hypothesis (b.i) and since $c_1\bar{\alpha} \le \lambda^*$ one has that both $g_\alpha(\lambda)$ and $r_\alpha(\lambda)$ are nonnegative for all $\lambda \in [0, c_1\bar{\alpha}]$. Also, from the definitions of $G_{\bar{\alpha}}$ and z it follows immediately that the function $m(\lambda) \doteq \langle F_\lambda Tx, G_{\bar{\alpha}} z\rangle$ for $\lambda \in [0, c_1\bar{\alpha}]$ is real and non-decreasing and therefore

$$\int_0^{c_1\bar{\alpha}} r_\alpha(\lambda) g_\alpha(\lambda)\, d\langle F_\lambda Tx, G_{\bar{\alpha}} z\rangle \ge 0. \tag{3.18}$$

On the other hand, since $m(\lambda) = \langle Tx, F_\lambda G_{\bar{\alpha}} z\rangle$, $F_\lambda G_{\bar{\alpha}} = G_{\bar{\alpha}}$ for every $\lambda \ge c_1\bar{\alpha}$, it follows that $m(\lambda)$ is constant for $\lambda \ge c_1\bar{\alpha}$ and therefore

$$\int_{c_1\bar{\alpha}}^{\|T\|^2+} r_\alpha(\lambda) g_\alpha(\lambda)\, d\langle F_\lambda Tx, G_{\bar{\alpha}} z\rangle = 0. \tag{3.19}$$

From (3.18) and (3.19) we conclude that

$$\int_0^{\|T\|^2+} r_\alpha(\lambda) g_\alpha(\lambda)\, d\langle F_\lambda Tx, G_{\bar{\alpha}} z\rangle \ge 0,$$

which, by virtue of (3.17), implies that

$$\langle R_\alpha Tx - x, R_\alpha(\bar{y}^\delta - Tx)\rangle \ge 0. \tag{3.20}$$

Hence, for every $\alpha \in (0, \alpha_0)$, $\delta > 0$ and $\bar{\alpha} \in \sigma(TT^*)$ such that $c_1\bar{\alpha} \le \lambda^*$ we obtain the following estimate:

$$\begin{aligned}
\|R_\alpha \bar{y}^\delta - x\|^2 &= \|R_\alpha Tx - x\|^2 + \|R_\alpha(\bar{y}^\delta - Tx)\|^2 \\
&\qquad + 2\langle R_\alpha Tx - x, R_\alpha(\bar{y}^\delta - Tx)\rangle \\
&= \|R_\alpha Tx - x\|^2 + \delta^2 \|g_\alpha(T^*T)T^* G_{\bar{\alpha}} z\|^2 \\
&\qquad + 2\langle R_\alpha Tx - x, R_\alpha(\bar{y}^\delta - Tx)\rangle \quad \text{(by (3.16))} \\
&\ge \|R_\alpha Tx - x\|^2 + \delta^2 \|T^* g_\alpha(TT^*) G_{\bar{\alpha}} z\|^2 \quad \text{(by (3.20))} \\
&= \|R_\alpha Tx - x\|^2 + \delta^2 \|(TT^*)^{\frac{1}{2}} g_\alpha(TT^*) G_{\bar{\alpha}} z\|^2 \\
&= \|R_\alpha Tx - x\|^2 + \delta^2 \int_0^{\|T\|^2+} \lambda\, g_\alpha^2(\lambda)\, d\|F_\lambda G_{\bar{\alpha}} z\|^2 \\
&\ge \|R_\alpha Tx - x\|^2 + \delta^2 \int_{\bar{\alpha}}^{c_1\bar{\alpha}} \lambda\, g_\alpha^2(\lambda)\, d\|F_\lambda G_{\bar{\alpha}} z\|^2.
\end{aligned} \tag{3.21}$$

We now consider two different cases for $\alpha \in (0, \alpha_0)$.

Case I: $\alpha \le \bar{\alpha}$. Since $c_1\bar{\alpha} \le \lambda^*$ and $c_1 > 1$, it follows from hypothesis (b.v) that

$$g_\alpha(\lambda) \ge g_\alpha(c_1\bar{\alpha}) \ge g_\alpha(\lambda^*) \quad \text{for every } \lambda \in [\bar{\alpha}, c_1\,\bar{\alpha}]. \tag{3.22}$$

On the other hand, from hypothesis (b.i) it follows that $r_\alpha(\lambda^*) \le 1$, which implies that $\lambda^*\, g_\alpha(\lambda^*) \ge 0$ and therefore $g_\alpha(\lambda^*) \ge 0$. It then follows from (3.22) that $g_\alpha^2(\lambda) \ge g_\alpha^2(c_1\,\bar{\alpha})$ for every $\lambda \in [\bar{\alpha}, c_1\,\bar{\alpha}]$. Then,

$$\begin{aligned}\int_{\bar{\alpha}}^{c_1\,\bar{\alpha}} \lambda\, g_\alpha^2(\lambda)\, d\|F_\lambda G_{\bar{\alpha}} z\|^2 &\ge \bar{\alpha}\, g_\alpha^2(c_1\,\bar{\alpha}) \int_{\bar{\alpha}}^{c_1\,\bar{\alpha}} d\|F_\lambda G_{\bar{\alpha}} z\|^2 \\ &= \bar{\alpha}\, g_\alpha^2(c_1\,\bar{\alpha}) \left(\|F_{c_1\,\bar{\alpha}} G_{\bar{\alpha}} z\|^2 - \|F_{\bar{\alpha}} G_{\bar{\alpha}} z\|^2\right) \\ &= \bar{\alpha}\, g_\alpha^2(c_1\,\bar{\alpha}) \|G_{\bar{\alpha}} z\|^2 \\ &= \bar{\alpha}\, g_\alpha^2(c_1\,\bar{\alpha}),\end{aligned} \tag{3.23}$$

where the second to last equality follows from the definition of $G_{\bar{\alpha}}$ and the spectral property $F_\lambda F_\mu = F_{\min\{\lambda,\mu\}}$.

At the same time, the hypotheses (b.i) and (b.iii) imply that $g_\alpha(\lambda)$ is non-increasing as a function of α for each fixed $\lambda \in [0, \lambda^*]$. Since $\alpha \le \bar{\alpha}$ and $c_1\,\bar{\alpha} \le \lambda^*$, we then have that

$$g_\alpha(c_1\,\bar{\alpha}) \ge g_{\bar{\alpha}}(c_1\,\bar{\alpha}), \tag{3.24}$$

and from hypothesis (b.iv) we also have that

$$g_{\bar{\alpha}}(c_1\,\bar{\alpha}) \ge \frac{\gamma_2}{\bar{\alpha}} > 0. \tag{3.25}$$

From (3.24) and (3.25) we conclude that

$$g_\alpha^2(c_1\,\bar{\alpha}) \ge \left(\frac{\gamma_2}{\bar{\alpha}}\right)^2. \tag{3.26}$$

Substituting (3.26) into (3.23), we obtain

$$\int_{\bar{\alpha}}^{c_1\,\bar{\alpha}} \lambda\, g_\alpha^2(\lambda)\, d\|F_\lambda G_{\bar{\alpha}} z\|^2 \ge \frac{\gamma_2^2}{\bar{\alpha}},$$

which, by virtue of (3.21) implies that if $\alpha \le \bar{\alpha}$, then $\|R_\alpha \bar{y}^{\,\delta} - x\|^2 \ge \frac{(\gamma_2\,\delta)^2}{\bar{\alpha}}$.

Case II: $\alpha > \bar{\alpha}$. In this case, it follows from hypothesis (b.iii) that $r_\alpha^2(\lambda) \ge r_{\bar{\alpha}}^2(\lambda)$ for every $\lambda \in (0, \|T\|^2]$. Then,

$$\begin{aligned}\|R_\alpha T x - x\|^2 &= \int_0^{\|T\|^2+} r_\alpha^2(\lambda)\, d\|E_\lambda x\|^2 \\ &\ge \int_0^{\|T\|^2+} r_{\bar{\alpha}}^2(\lambda)\, d\|E_\lambda x\|^2 = \|R_{\bar{\alpha}} T x - x\|^2,\end{aligned}$$

which, together with (3.21) imply that $\|R_\alpha \bar{y}^{\,\delta} - x\|^2 \ge \|R_{\bar{\alpha}} T x - x\|^2$.

Summarizing the results of cases I and II, we obtain that for every $\alpha \in (0, \alpha_0)$, $\delta > 0$, $\bar{\alpha} \in \sigma(TT^*)$ with $c_1\bar{\alpha} \leq \alpha_0$ and $\bar{y}^\delta$ as in (3.16), there holds

$$\|R_\alpha \bar{y}^\delta - x\|^2 \geq \begin{cases} \frac{(\gamma_2\,\delta)^2}{\bar{\alpha}}, & \text{if } 0 < \alpha \leq \bar{\alpha}, \\ \|R_{\bar{\alpha}} Tx - x\|^2, & \text{if } \bar{\alpha} < \alpha < \alpha_0 \end{cases}$$
$$\geq \min\left\{ \|R_{\bar{\alpha}} Tx - x\|^2, \frac{(\gamma_2\,\delta)^2}{\bar{\alpha}} \right\}. \tag{3.27}$$

Then

$$\begin{aligned} \min\left\{ \|R_{\bar{\alpha}} Tx - x\|, \frac{\gamma_2\,\delta}{\sqrt{\bar{\alpha}}} \right\} &= \left(\min\left\{ \|R_{\bar{\alpha}} Tx - x\|^2, \frac{(\gamma_2\,\delta)^2}{\bar{\alpha}} \right\} \right)^{1/2} \\ &\leq \inf_{\alpha \in (0,\alpha_0)} \|R_\alpha \bar{y}^\delta - x\| \quad \text{(by (3.27))} \\ &\leq \sup_{y^\delta \in \overline{B_\delta(Tx)}} \inf_{\alpha \in (0,\alpha_0)} \|R_\alpha y^\delta - x\| \quad (\text{as } \bar{y}^\delta \in \overline{B_\delta(Tx)}) \\ &= O(\rho(\Theta^{-1}(\delta))) \quad \text{for } \delta \to 0^+ \quad \text{(by hypothesis)}, \end{aligned}$$

and since $\bar{\alpha} = \eta(\bar{\delta})$ solves equation (3.15), the previous inequality implies that

$$\|R_{\eta(\bar{\delta})} Tx - x\| = \frac{\gamma_2\,\bar{\delta}}{\sqrt{\bar{\alpha}}} = O(\rho(\Theta^{-1}(\bar{\delta}))) \quad \text{for } \bar{\delta} \to 0^+, \tag{3.28}$$

and therefore

$$\frac{\bar{\delta}}{\rho(\Theta^{-1}(\bar{\delta}))} = O\left(\sqrt{\eta(\bar{\delta})}\right) \quad \text{for } \bar{\delta} \to 0^+. \tag{3.29}$$

Now, since for every $\delta > 0$ one has $\delta = \Theta(\Theta^{-1}(\delta))$, it follows from the definition of Θ that $\delta = \sqrt{\Theta^{-1}(\delta)}\,\rho(\Theta^{-1}(\delta))$. Then, from (3.29) we obtain that

$$\sqrt{\Theta^{-1}(\bar{\delta})} = O\left(\sqrt{\eta(\bar{\delta})}\right) \quad \text{for } \bar{\delta} \to 0^+.$$

From this and (3.28) we then deduce that

$$\|R_{\eta(\bar{\delta})} Tx - x\| = O(\rho(\eta(\bar{\delta}))) \quad \text{for } \bar{\delta} \to 0^+, \tag{3.30}$$

where

$$\bar{\delta} = \mu(\bar{\alpha}), \quad \bar{\alpha} \in \sigma(TT^*), \quad c_1\,\bar{\alpha} \leq \alpha_0.$$

Now let $L \doteq \max\{\lambda_j : \lambda_j \leq \frac{\alpha_0}{c_1}\}$. Then, since $\lambda_n \downarrow 0$, for any $\alpha \in (0, L]$ there exists a unique $n = n(\alpha) \in \mathbb{N}$ such that $\lambda_{n+1} < \alpha \leq \lambda_n$ (note that $n(\alpha) \to \infty$ if and only if $\alpha \to 0^+$). Then for $\alpha \in (0, L]$ and $n = n(\alpha)$ so defined we have that

$$\begin{aligned}
\|R_\alpha T x - x\|^2 &= \int_0^{\|T\|^2+} r_\alpha^2(\lambda)\, d\|E_\lambda x\|^2 \\
&\le \int_0^{\|T\|^2+} r_{\lambda_n}^2(\lambda)\, d\|E_\lambda x\|^2 \quad \text{(by hypothesis (b.iii))} \\
&= \|R_{\lambda_n} T x - x\|^2 \\
&= O(\rho^2(\lambda_n)) \quad \text{(by virtue of (3.30), with } \bar{\delta} = \mu(\lambda_n)). \qquad (3.31)
\end{aligned}$$

Also, from hypothesis (c) we have that $\lambda_n \le c\, \lambda_{n+1}$ for all $n \in \mathbb{N}$, and since ρ is non-decreasing and positive (since $\rho \in \mathcal{O}$), it follows that

$$\rho^2(\lambda_n) \le \rho^2(c\, \lambda_{n+1}) \quad \text{for all } n \in \mathbb{N}. \qquad (3.32)$$

Now since $c \ge 1$ and by hypothesis (a) ρ is of local upper type β, there exists a positive constant d such that

$$\rho(c\, \lambda_{n+1}) \le d\, c^\beta \rho\left(\frac{1}{c}\, c\, \lambda_{n+1}\right) = d\, c^\beta \rho(\lambda_{n+1}) \quad \text{for all } n \in \mathbb{N}. \qquad (3.33)$$

Hence

$$\begin{aligned}
\rho(\lambda_{n(\alpha)}) &\le \rho(c\lambda_{n(\alpha)+1}) && \text{(by (3.32))} \\
&\le dc^\beta \rho(\lambda_{n(\alpha)+1}) && \text{(by (3.33))} \\
&\le dc^\beta \rho(\alpha) && \text{(since } \lambda_{n(\alpha)+1} < \alpha \text{ and } \rho \in \mathcal{O}). \qquad (3.34)
\end{aligned}$$

From (3.31) and (3.34) it follows that $\|R_\alpha T x - x\| = O(\rho(\alpha))$ for $\alpha \to 0^+$. Since $T^\dagger T$ is the projection on $\mathcal{N}(T)^\perp = X$ (since T is invertible), we have that $x = T^\dagger T x = T^\dagger y$. Then $\|(R_\alpha - T^\dagger) y\| = O(\rho(\alpha))$ for $\alpha \to 0^+$. Finally, Theorem 2.13 implies that $T^\dagger y = x \in \mathcal{R}(s(T^*T))$. This concludes the proof of the lemma. □

Remark 3.7. Note that since

$$\sup_{y^\delta \in \overline{B_\delta(Tx)}} \inf_{\alpha \in (0,\alpha_0)} \|R_\alpha y^\delta - x\| \le \mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x, \delta),$$

hypothesis (3.14) of the preceding lemma holds if $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x, \delta) = O(\rho(\Theta^{-1}(\delta)))$ for $\delta \to 0^+$.

Having stated and proved the previous three lemmas, we are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. As in Lemma 3.6, without loss of generality we may assume that $\alpha_0 \le \frac{\lambda^*}{c_1}$. We will show that $\psi(x,\delta) \doteq \rho \circ \Theta^{-1}(\delta)$ for $x \in X^{s_\rho}$ and $\delta \in (0, \Theta(\alpha_0))$ is saturation function of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on X^{s_ρ} (see Definition 2.6).

First we note that since $\{g_\alpha\}$ satisfies (H4) and ρ is continuous (ρ being of local upper type), by virtue of Lemma 3.1 one has that $\psi \in \mathcal{U}_{X^{s_\rho}}(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$. Next we will show that ψ satisfies the (S1) condition for saturation on X^{s_ρ}. Suppose that it is not true, i.e. that there exist $x^* \in X$, $x^* \neq 0$ and $x \in X^{s_\rho}$ such that

$$\limsup_{\delta\to 0^+} \frac{\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta)}{\psi(x,\delta)} = 0.$$

Then $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*,\delta) = o(\psi(x,\delta))$ as $\delta \to 0^+$ and from Lemma 3.5 (I) it follows that there exists an *a priori* parameter choice rule $\hat{\alpha}(\delta)$ such that

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \|R_{\hat{\alpha}(\delta)} y^\delta - x^*\| = o(\psi(x,\delta)) \quad \text{for } \delta \to 0^+. \tag{3.35}$$

On the other hand, from hypothesis (c) it follows that there exists $\bar{\alpha} \in \sigma(TT^*)$ such that $0 < c_1\bar{\alpha} \le \alpha_0$ and $h(\bar{\alpha}) < \|T\|^2$. Now define

$$\bar{\delta} = \bar{\delta}(\bar{\alpha}) \doteq \frac{\bar{\alpha}^{1/2}}{\gamma_2}\|R_{\bar{\alpha}}Tx^* - x^*\| = \frac{\bar{\alpha}^{1/2}}{\gamma_2}\|r_{\bar{\alpha}}(T^*T)x^*\|$$

and for $\delta > 0$,

$$\bar{y}^\delta \doteq Tx^* - \delta\, G_{\bar{\alpha}} z, \tag{3.36}$$

as in (3.16). Following the same steps as in the proof of Lemma 3.6 we obtain as in (3.27) that for every $\alpha \in (0,\alpha_0)$, $\delta > 0$, $\bar{\alpha} \in \sigma(TT^*)$ with $c_1\bar{\alpha} \le \alpha_0$ and $h(\bar{\alpha}) < \|T\|^2$, and $\bar{y}^\delta$ as in (3.36), there holds

$$\|R_\alpha \bar{y}^\delta - x^*\|^2 \ge \min\left\{\|R_{\bar{\alpha}}Tx^* - x^*\|^2, \frac{(\gamma_2\,\delta)^2}{\bar{\alpha}}\right\}. \tag{3.37}$$

Then for $\delta > 0$ such that $\hat{\alpha}(\delta) \in (0,\alpha_0)$,

$$\begin{aligned}
\min\left\{\|R_{\bar{\alpha}}Tx^* - x^*\|, \frac{\gamma_2\,\delta}{\sqrt{\bar{\alpha}}}\right\} &= \left(\min\left\{\|R_{\bar{\alpha}}Tx^* - x^*\|^2, \frac{(\gamma_2\,\delta)^2}{\bar{\alpha}}\right\}\right)^{1/2} \\
&\le \inf_{\alpha\in(0,\alpha_0)} \|R_\alpha \bar{y}^\delta - x^*\| \;\; \text{(by (3.37))} \\
&\le \|R_{\hat{\alpha}(\delta)}\bar{y}^\delta - x^*\| \;\; (\text{as } \hat{\alpha}(\delta) \in (0,\alpha_0)) \\
&\le \sup_{y^\delta\in\overline{B_\delta(Tx^*)}} \|R_{\hat{\alpha}(\delta)} y^\delta - x^*\| \;\; (\text{as } \bar{y}^\delta \in \overline{B_\delta(Tx^*)}) \\
&= o(\rho(\Theta^{-1}(\delta))) \;\; \text{for } \delta \to 0^+ \;\; \text{(by (3.35))},
\end{aligned}$$

and since $\bar{\alpha} = \eta(\bar{\delta})$ solves equation (3.15) with $x = x^*$, the previous inequality implies that

$$\|R_{\eta(\bar{\delta})} T x^* - x^*\| = \frac{\gamma_2 \, \bar{\delta}}{\sqrt{\bar{\alpha}}} = o(\rho(\Theta^{-1}(\bar{\delta}))) \quad \text{for } \bar{\delta} \to 0^+.$$

Following analogous steps as in the proof of Lemma 3.6, we obtain, as in (3.30), that

$$\|R_{\eta(\bar{\delta})} T x^* - x^*\| = o(\rho(\eta(\bar{\delta}))) \quad \text{for } \bar{\delta} \to 0^+, \tag{3.38}$$

where

$$\bar{\delta} = \mu(\bar{\alpha}), \quad \bar{\alpha} \in \sigma(TT^*), \quad c_1 \, \bar{\alpha} \le \alpha_0, \quad h(\bar{\alpha}) < \|T\|^2.$$

Now,

$$\begin{aligned}
\|R_{\bar{\alpha}} T x^* - x^*\|^2 &= \int_0^{\|T\|^2+} r_{\bar{\alpha}}^2(\lambda) \, d\|E_\lambda x^*\|^2 \\
&\ge \int_{h(\bar{\alpha})}^{\|T\|^2+} r_{\bar{\alpha}}^2(\lambda) \, d\|E_\lambda x^*\|^2 \quad (\text{since } 0 < h(\bar{\alpha}) < \|T\|^2\,) \\
&\ge \gamma^2 \rho^2(\bar{\alpha}) \int_{h(\bar{\alpha})}^{\|T\|^2+} s_\rho^{-2}(\lambda) \, d\|E_\lambda x^*\|^2,
\end{aligned} \tag{3.39}$$

where the last inequality follows from the fact that (ρ, s_ρ) is an order-source pair for $\{g_\alpha\}$ (with γ the constant in (2.3)). Since $\eta(\bar{\delta}) = \bar{\alpha}$ and $\bar{\delta} \to 0^+$ if and only if $\bar{\alpha} \to 0^+$, (3.38) and (3.39) imply that

$$\int_{h(\bar{\alpha})}^{\|T\|^2+} s_\rho^{-2}(\lambda) \, d\|E_\lambda x^*\|^2 = o(1) \quad \text{for } \bar{\alpha} \to 0^+$$

and therefore we get $\|s_\rho^{-1}(T^*T)x^*\| = 0$. Then $x^* = 0$, contradicting the fact that $x^* \neq 0$. Hence, $\psi(x, \delta) = \rho(\Theta^{-1}(\delta))$ satisfies condition (S1) on X^{s_ρ}.

Also, since ψ is trivially invariant over X^{s_ρ}, ψ does not depend on x. Thus, it satisfies condition (S2).

It only remains to be proved that ψ satisfies (S3) on X^{s_ρ}. Suppose that is not the case. Then there must exist a set M, $X^{s_\rho} \subsetneqq M \subset X \setminus \{0\}$ and $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$ such that $\tilde{\psi}|_{X^{s_\rho}} = \psi$ and $\tilde{\psi}$ satisfies (S1) and (S2) on M. Let $x^* \in M \setminus X^{s_\rho}$. As $\tilde{\psi} \in \mathcal{U}_M(\mathcal{E}^{\text{tot}}_{\{R_\alpha\}})$, we have that

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{\{x^*\}}{\preceq} \tilde{\psi}. \tag{3.40}$$

Also, since $\tilde{\psi}$ is invariant over M we have that

$$\tilde{\psi} \overset{\{x^*\}, X^{s_\rho}}{\preceq} \tilde{\psi},$$

and since $\tilde{\psi}$ coincides with ψ on X^{s_ρ}, it follows that

$$\tilde{\psi} \overset{\{x^*\},X^{s_\rho}}{\preceq} \psi.$$

This, together with (3.40) implies that

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}} \overset{\{x^*\},X^{s_\rho}}{\preceq} \psi$$

and therefore for every $x \in X^{s_\rho}$, $\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*, \delta) = O(\psi(x, \delta))$ as $\delta \to 0^+$, that is,

$$\mathcal{E}^{\text{tot}}_{\{R_\alpha\}}(x^*, \delta) = O(\rho(\Theta^{-1}(\delta))) \quad \text{as } \delta \to 0^+.$$

Lemma 3.5 then implies that there exists an admissible *a priori* parameter choice rule $\tilde{\alpha}(\delta)$ such that

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \|R_{\tilde{\alpha}(\delta)} y^\delta - x^*\| = O(\rho(\Theta^{-1}(\delta))) \quad \text{as } \delta \to 0^+$$

and therefore

$$\sup_{y^\delta \in \overline{B_\delta(Tx^*)}} \inf_{\alpha \in (0,\alpha_0)} \|R_\alpha y^\delta - x^*\| = O(\rho(\Theta^{-1}(\delta))) \quad \text{as } \delta \to 0^+.$$

Hence, by virtue of Lemma 3.6, $x^* \in \mathcal{R}(s_\rho(T^*T))$ and since $x^* \neq 0$, we have that $x^* \in \mathcal{R}(s_\rho(T^*T)) \setminus \{0\} = X^{s_\rho}$ which contradicts our original assumption. This concludes the proof of Theorem 3.3. □

4 Examples

Although the main results of this article are very theoretical in nature, we provide below two examples of regularization methods with optimal qualification which do possess saturation. In both cases the saturation function and saturation set are found. Example 4.1, for the Tikhonov–Phillips case, is introduced mainly to present a classical method possessing saturation, according to the theory developed in the previous sections, which is consistent with already well-known results. On the other hand Example 4.2, although somewhat artificial, is presented with the objective of showing that Theorem 3.3 is not vacuous, in the sense that there exist non trivial regularization methods satisfying the hypothesis of the theorem.

Example 4.1. Consider $R_\alpha = g_\alpha(T^*T)T^*$ with $g_\alpha(\lambda) = \frac{1}{\lambda+\alpha}$. The family of Tikhonov–Phillips regularization operators $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ has optimal qualification $\rho(\alpha) = \alpha$ (see [4]). It can be easily checked that ρ is of local upper type 1,

$s_\rho(\lambda) = \lambda \in \mathcal{S}$ and $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ satisfies all hypotheses of Theorem 3.3. Therefore, the function

$$\psi(x,\delta) = \rho(\Theta^{-1}(\delta)) = \delta^{\frac{2}{3}}$$

defined for $x \in X^{s_\rho} \doteq \mathcal{R}(T^*T) \setminus \{0\}$ and $\delta \in (0, \alpha_0^{\frac{3}{2}})$ is saturation function of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on X^{s_ρ}.

Example 4.2. Given $k \in \mathbb{R}^+$, for $\alpha > 0$ let

$$u_\alpha^k(\lambda) \doteq \begin{cases} \frac{e^{-\frac{\lambda}{\sqrt{\alpha}}}}{\lambda}, & \text{for } 0 < \lambda < \alpha,\\ \frac{e^{-\sqrt{\frac{\lambda}{\alpha}}}}{\lambda}, & \text{for } \alpha \le \lambda < 3\alpha,\\ \frac{e^{-\sqrt{\frac{\lambda}{\alpha}}}}{\lambda} + \frac{\alpha^k}{\lambda^{k+1}}, & \text{for } \lambda \ge 3\alpha, \end{cases}$$

and

$$g_\alpha^k(\lambda) \doteq \begin{cases} \frac{1}{\lambda} - \alpha^k\sqrt{\lambda} - u_\alpha^k(\lambda), & \text{for } \lambda > 0,\\ \lim_{\lambda\to 0^+} g_\alpha^k(\lambda) = \frac{1}{\sqrt{\alpha}}, & \text{for } \lambda = 0. \end{cases}$$

The family $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ is an SRM ([5]).

Now, defining

$$v_\alpha^k(\lambda) \doteq \begin{cases} e^{-\frac{\lambda}{\sqrt{\alpha}}}, & \text{for } 0 \le \lambda < \alpha,\\ e^{-\sqrt{\frac{\lambda}{\alpha}}}, & \text{for } \alpha \le \lambda < 3\alpha,\\ e^{-\sqrt{\frac{\lambda}{\alpha}}} + \left(\frac{\alpha}{\lambda}\right)^k, & \text{for } \lambda \ge 3\alpha, \end{cases}$$

we get $r_\alpha^k(\lambda) = 1 - \lambda g_\alpha^k(\lambda) = \alpha^k\lambda^{\frac{3}{2}} + v_\alpha^k(\lambda)$. If $\rho(\alpha) = \alpha^k$, then $s_\rho(\lambda) = \lambda^k$, then $s_\rho \in \mathcal{S}$ and s_ρ satisfies (2.5). Also, it can be shown that ρ and s_ρ verify (2.3) with $\gamma = 1$ and $h(\alpha) = 3\alpha$. From Theorem 2.11 it then follows that $\rho(\alpha) = \alpha^k$ is optimal qualification of $\{g_\alpha\}$.

On the other hand, for $k \ge 1$ and $\alpha > 0$, the function $g_\alpha^k(\lambda)$ is non-increasing. Thus, hypothesis (b.v) of Theorem 3.3 holds and

$$G_\alpha^k \doteq \|g_\alpha^k(\cdot)\|_\infty = g_\alpha^k(0) = \frac{1}{\sqrt{\alpha}},$$

which implies immediately that also hypothesis (H4) is verified. From now on we shall assume $k \ge 1$.

It can be easily checked that ρ is of local upper type k and $\{g_\alpha\}_{\alpha\in(0,\alpha_0)}$ satisfies all hypotheses of Theorem 3.3. Therefore, the function

$$\psi(x,\delta) = \rho(\Theta^{-1}(\delta)) = \delta^{\frac{2k}{2k+1}}$$

defined for $x \in X^{s_\rho} \doteq \mathcal{R}((T^*T)^k) \setminus \{0\}$ and $\delta \in (0, \alpha_0^{k+\frac{1}{2}})$ is saturation function of $\{R_\alpha\}_{\alpha\in(0,\alpha_0)}$ on X^{s_ρ}.

5 Conclusions

In this article families of real functions $\{g_\alpha\}$ defining a spectral regularization methods with optimal qualification were considered. Sufficient conditions on the family and on the optimal qualification guaranteeing the existence of saturation were found. Appropriate characterizations of both the saturation function and the saturation set were given and two examples were provided.

Bibliography

[1] A. B. Bakushinskii and A. V. Goncharskii, *Ill-Posed Problems: Theory and Applications*, Kluwer, Dordrecht, 1994.

[2] H. W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Mathematics and its Applications 375, Kluwer, Dordrecht, 1996.

[3] C. W. Groetsch, *The Theory of Tikhonov Regularization for Fredholm Equations of the First Kind*, Research Notes in Mathematics 105, Pitman, Boston, 1984.

[4] T. Herdman, R. D. Spies and K. G. Temperini, Generalized Qualification and qualification levels for spectral regularization methods, *J. Optim. Theory Appl.* **141** (2009), no. 3, 547–567.

[5] T. Herdman, R. D. Spies and K. G. Temperini, Global saturation of regularization methods for inverse ill-posed problems, *J. Optim. Theory Appl.* **148** (2011), no. 1, 164–196.

[6] B. Iaffei, Generalized Bessel potentials on Lipschitz type spaces, *Math. Nachr.* **278** (2005), no. 4, 421–436.

[7] A. S. Leonov, Elimination of accuracy saturation in regularizing algorithms, *Numer. Anal. Appl.* **1** (2008), no. 2, 135–150.

[8] A. S. Leonov and A. G. Yagola, Special regularizing methods for ill-posed problems with sourcewise represented solutions, *Inverse Problems* **14** (1998), no. 6, 1539–1550.

[9] P. Mathé, Saturation of regularization methods for linear ill-posed problems in Hilbert spaces, *SIAM J. Numer. Anal.* **42** (2004), no. 3, 968–973 (electronic).

[10] P. Mathé and S. V. Pereverzev, Geometry of linear ill-posed problems in variable Hilbert scales, *Inverse Problems* **19** (2003), no. 3, 789–803.

[11] A. Neubauer, On converse and saturation results for regularization methods, in: *Beiträge zur angewandten Analysis und Informatik*, Shaker, Aachen (1994), 262–270.

[12] A. Neubauer, On converse and saturation results for Tikhonov regularization of linear ill-posed problems, *SIAM J. Numer. Anal.* **34** (1997), no. 2, 517–527.

[13] T. I. Seidman, Nonconvergence results for the application of least-squares estimation to ill-posed problems, *J. Optim. Theory Appl.* **30** (1980), no. 4, 535–547.

[14] R. D. Spies and K. G. Temperini, Arbitrary divergence speed of the least-squares method in infinite-dimensional inverse ill-posed problems, *Inverse Problems* **22** (2006), no. 2, 611–626.

[15] A. N. Tikhonov, A. V. Goncharsky, V. V. Stepanov and A. G. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems*, Kluwer, Dordrecht, 1995.

[16] A. N. Tikhonov, A. S. Leonov and A. G. Yagola, *Nonlinear Ill-Posed Problems*, Volume 1 and 2, Chapman and Hall, London, 1998.

[17] G. M. Vainikko and A. Y. Veretennikov, *Iteration Procedures in Ill-Posed Problems* (in Russian), Nauka, Moscow, 1986.

[18] V. A. Vinokurov, Regularizability of functions, in: *Ill-Posed Problems in the Natural Sciences*, Advances in Science and Technology in the USSR: Mathematics and Mechanics Series, “Mir”, Moscow (1987), 53–69.

[19] A. Yagola and K. Dorofeev, Sourcewise representation and a posteriori error estimates for ill-posed problems, in: *Operator Theory and Its Applications*, Fields Institute Communications 25, American Mathematical Society, Providence (2000), 543–550.

Received May 24, 2012.

Author information

Gisela L. Mazzieri, Instituto de Matemática Aplicada del Litoral, IMAL,
CONICET-UNL, Güemes 3450, S3000GLN, Santa Fe, Argentina;
and Departamento de Matemática, Facultad de Bioquímica y Ciencias Biológicas,
Universidad Nacional del Litoral, Santa Fe, Argentina.
E-mail: glmazzieri@santafe-conicet.gov.ar

Ruben D. Spies, Instituto de Matemática Aplicada del Litoral, IMAL,
CONICET-UNL, Güemes 3450, S3000GLN, Santa Fe, Argentina;
and Departamento de Matemática, Facultad de Ingeniería Química,
Universidad Nacional del Litoral, Santa Fe, Argentina.
E-mail: rspies@santafe-conicet.gov.ar

Karina G. Temperini, Instituto de Matemática Aplicada del Litoral, IMAL,
CONICET-UNL, Güemes 3450, S3000GLN, Santa Fe, Argentina;
and Departamento de Matemática, Facultad de Humanidades y Ciencias,
Universidad Nacional del Litoral, Santa Fe, Argentina.
E-mail: ktemperini@santafe-conicet.gov.ar