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Technical Note

Forced vibrations of a clamped-free beam with a mass at the free end with an external periodic disturbance acting on the mass with applications in ships' structures

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Abstract

An exact, analytical solution is obtained for the title problem which constitutes a classical one although no solution is available in well known textbooks and handbooks normally used by the structural engineer in several fields of technology: ocean and naval engineering, aerospace applications, etc. The authors performed this study motivated by a situation where excessive displacements were noticed in a structural element carrying a relatively small motor at the free end and placed at the engine room of a naval vessel. The Bernoulli-Euler model has been employed.

Keywords: Analytical solution; Ocean and naval engineering; Aerospace applications; Motor; Engine room; Bernoulli-Euler

1. Introduction

Consider the structural system shown in Fig. 1. The knowledge of its dynamic parameters is of practical interest in order to avoid excessive displacements and dynamic stress resultants and certainly: resonance conditions. The problem is a

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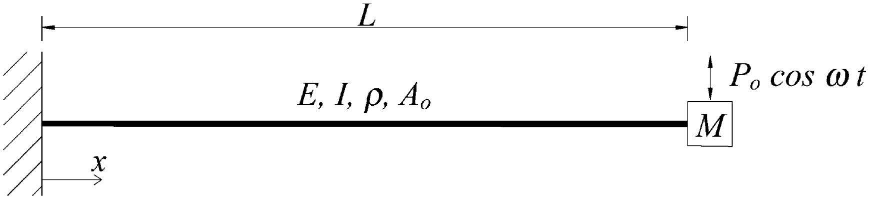

Fig. 1. Beam-mass system subjected to transverse, forced sinusoidal excitation at the free end.

classical one within the realm of the general theory of vibrations of continuous systems but no results are available in well known textbooks and handbooks which, in general, only contain information regarding mode shapes and natural frequencies (Blevins, 1979; Laura et al., 1974). The present problem arose in an engine room of a small naval vessel where severe and trouble some displacements were observed in a system similar to the one depicted in Fig. 1.

Displacements and bending moment amplitudes are obtained in this study as a function of the ratio *external frequency/fundamental natural frequency* using the classical Euler-Bernoulli theory. Accordingly shear and rotatory inertia effects are not taken into account. Structural damping has not been considered either.

As correctly stated by Watson (Watson, 1998) “any vibratory forces on couples that may emanate from a main engine must be carefully assessed before it is accepted as suitable”. This world renowned author expresses the view that if a moderate couple is generated, it could only be accepted if it is demonstrated that it is within acceptable limits at all locations of the ship where it could affect personnel or equipment. However the one considered in this note could cause considerable trouble to personnel or equipment, even though the motor or engine in operation may belong to a secondary classification type.

On the other hand minimization of the under-water noise signature is a preponderant factor in the overall mechanical installation, especially in the case of warships and fishery and oceanographic vessels (Watson, 1998).

2. Analysis and solution of the problem

The problem under consideration is described by the following differential system:

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} + \rho A_o \frac{\partial^2 w(x,t)}{\partial t^2} = 0 \quad (1a)$$

$$w(0,t) = 0 \quad (1b)$$

$$\frac{\partial w(0,t)}{\partial x} = 0 \quad (1c)$$

$$\frac{\partial^2 w(L,t)}{\partial x^2} = 0 \quad (1d)$$

$$-\left[-EI\frac{\partial^3 w(L,t)}{\partial x^3}\right] = M\frac{\partial^2 w(L,t)}{\partial t^2} - P_0 \cos \omega t, \tag{1e}$$

where E is Young's modulus, I the constant moment of inertia, A_o the constant cross-sectional area, ρ the mass density and M the concentrated mass at the free end.

Using the standard method of separation of variables, one assumes

$$w(x,t) = W(x)T(t) = W(x) cos\ wt. \tag{2}$$

Substituting this form in Eq. (1) results in the equality

$$\frac{W(x)^{(IV)}}{W(x)} = \frac{\rho A_o}{EI}\omega^2 = k^4 \tag{3}$$

The solution of the ordinary differential equation

$$W(x)^{(IV)} - k^4 W(x) = 0 \tag{4}$$

or $\dfrac{d^4W(x)}{dx^4} - \dfrac{\rho A_o}{EI}\ \omega^2\ W(x) = 0$ is simply

$$W(x) = A\ cos\ kx + B\ sin\ kx + C\ cosh\ kx + D\ sinh\ kx \tag{5}$$

with

$$k^4 = \frac{\rho A_o}{EI}\omega^2 \tag{6}$$

or $k^2 = \omega\sqrt{\dfrac{\rho A_o}{EI}}$.

The fundamental circular frequency is:

$$\omega_1 = \frac{\alpha_1^2}{L^2}\sqrt{\frac{EI}{\rho A_0}} \Rightarrow \omega_1^2 = \frac{\alpha_1^4\ EI}{L^4 \rho A_0}. \tag{7}$$

Accordingly

$$k^2L^2 = \frac{\omega}{\omega_1}\alpha_1^2\ \Rightarrow\ k\ L = \alpha_1\sqrt{\frac{\omega}{\omega_1}} \tag{8}$$

where α_1=fundamental eigenvalue (Laura et al., 1974).

Applying the boundary conditions one obtains:

- $W(0) = 0 \qquad \mathrm{A} + \mathrm{C} = 0 \Rightarrow \mathrm{C} = -\mathrm{A}$ (9a)
- $\left.\dfrac{dW(x)}{dx}\right|_{x=0} = 0 \qquad \mathrm{B} + \mathrm{D} = 0 \Rightarrow \mathrm{D} = -\mathrm{B}$ (9b)

Substituting C and D in Eq. (5) one obtains:

$$W(x) = A(\cos kx - \cosh kx) + B(\sin kx - \sinh kx) \tag{10a}$$

then

$$\frac{dW(x)}{dx} = kA(-\sin kx - \sinh kx) + Bk(\cos kx - \cosh kx) \tag{10b}$$

$$\frac{d^2W(x)}{dx^2} = k^2A(-\cos kx - \cosh kx) + k^2B(-\sin kx - \sinh kx) \tag{10c}$$

$$\frac{d^3W(x)}{dx^3} = k^3A(\sin kx - \sinh kx) + k^3B(-\cos kx - \cosh kx). \tag{10d}$$

From $\left.\frac{d^2W(x)}{dx^2}\right|_{x=L} = 0$ one obtains

$$A(-\cos kL - \cosh kL) + B(-\sin kL - \sinh kL) = 0,$$

$$A = -B\frac{\sin KL + \sinh KL}{\cos KL + \cosh KL}$$

Substituting A in Eq. (10d) results in:

$$\frac{d^3W(x)}{dx^3} = k^3B\left(-\frac{\sin kL + \sinh kL}{\cos kL + \cosh kL}(\sin kx - \sinh kx) - \cos kx - \cosh kx\right).$$

Using now:

$$-EI\left.\frac{d^3W(x)}{dx^3}\right|_{x=L} = P_0 + M\omega^2W(L), \tag{11}$$

where

$$-EI\left.\frac{d^3W(x)}{dx^3}\right|_{x=L} = -EIk^3B\left(-\frac{\sin kL + \sinh kL}{\cos kL + \cosh kL}(\sin kL - \sinh kL) - \cos kL - \cosh kL\right). \tag{12a}$$

Defining

$$\beta_1 = \left(-\frac{\sin kL + \sinh kL}{\cos kL + \cosh kL}(\sin kL - \sinh kL) - \cos kL - \cosh kL\right) \tag{12b}$$

one obtains

$$-EI\left.\frac{d^3W}{dx^3}\right|_{x=L} = -EIk^3B\beta_1. \tag{12c}$$

On the other hand:

$$W(L) = A(cos\ kL - cosh\ kL) + B(sin\ kL - sinh\ kL). \tag{13a}$$

Substituting A in $W(L)$ results in:

$$W(L) = B\left(sin\ kL - sinh\ kL - \frac{sin\ kL + sinh\ kL}{cos\ kL + cosh\ kL}(cos\ kL - cosh\ kL)\right).$$

Defining

$$\beta_2 = \left(sin\ kL - sinh\ kL - \frac{sin\ kL + sinh\ kL}{cos\ kL + cosh\ kL}(cos\ kL - cosh\ kL)\right) \tag{13b}$$

Accordingly

$$W(L) = B\beta_2 \tag{13c}$$

Substituting (12c) and (13c) in Eq. (11) one obtains:

$$-EIk^3B\beta_1 = P_0 + M\omega^2B\beta_2,$$

Then

$$-B\beta_1 = \frac{P_0}{EI\ k^3} + B\,\frac{M\omega^2}{EI\ k^3}\beta_2$$

and

$$-B\left(\beta_1 + \beta_2\,\frac{M\omega^2}{EI\ k^3}\right) = \frac{P_0}{EI\ k^3}$$

or

$$-B\left(\beta_1\ k^3L^3 + \beta_2\,\frac{M\omega^2L^3}{EI}\right) = \frac{P_0L^3}{EI}.$$

Finally

$$B = -\frac{\frac{P_0L^3}{EI}}{\beta_1\ k^3L^3 + \beta_2\,\frac{M\omega^2L^3}{EI}} = -\frac{\frac{P_0L^3}{EI}}{\beta_1\ (k\ L)^3 + \beta_2\,\frac{M}{\rho\ A_0L}\omega^2\frac{\rho\ A_0L^4}{EI}}. \tag{14}$$

As the beam mass M_b is $\rho\ A_0\ L$, one is able to use the mass ratio $m = M/M_b$ Substituting (6) and (8) in (14) one obtains

$$B = -\frac{\frac{P_0L^3}{EI}}{\beta_1\ \alpha_1^3\left(\frac{\omega}{\omega_1}\right)^{3/2} + \beta_2\ m\ \alpha_1^4\left(\frac{\omega}{\omega_1}\right)^2}.$$

Substituting B in the deflection expression yields the following functional relation for the deflection amplitude.

$$W(x) = -\frac{\frac{P_0L^3}{EI}}{\beta_1\,\alpha_1^3\left(\frac{\omega}{\omega_1}\right)^{3/2} + \beta_2\,m\,\alpha_1^4\left(\frac{\omega}{\omega_1}\right)^2}\left(\sin kx - \sinh kx - \frac{\sin kL + \sinh kL}{\cos kL + \cosh kL}(\cos kx - \cosh kx)\right) \tag{15}$$

And the bending moment amplitude expression becomes $M(x) = -EI\frac{d^2W(x)}{dx^2}$.

Accordingly the bending moment amplitude is given by:

$$M(x) = -\frac{P_0L}{\beta_1\,\alpha_1\sqrt{\frac{\omega}{\omega_1}} + \beta_2\,m\,\alpha_1^2\frac{\omega}{\omega_1}}\left(-\sin kx - \sinh kx + \frac{\sin kL + \sinh kL}{\cos kL + \cosh kL}(\cos kx + \cosh kx)\right). \tag{16}$$

3. Numerical results

Table 1 depicts deflection and bending moment amplitudes for $M/M_b = 0$ as a function of ω/ω_1 when this parameter varies between 0 (static case) and 0.95. Tables 2, 3, 4, 5 and 6 present similar information for M/M_b= 0.2, 0.4, 0.6, 0.8 and 1, respectively. As expected the amplitudes of displacement and bending moments increase quite drastically as ω/ω_1 approach unity.

It is interesting to point out that the static case ($\omega/\omega_1 = 0$) has been computed making $\omega/\omega_1 < < 1$. The corresponding results of displacement and bending moment amplitudes agree admirably well with those obtained using the classical “strength of materials” solution for the static situation.

No claim of originality is made by the authors but it is expected that naval designers will find present results useful in their professional work.

Table 1
Values of deflections and bending moments amplitude parameters (M/M_b=0; α_1=1.87510407)

$\frac{|W(x)|}{P_o L^3/EI}$

$\frac{\omega}{\omega_1}$	$k\,L$	x/L: 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	0	0.00483	0.01867	0.0405	0.06933	0.10417	0.144	0.18783	0.23467	0.2835	0.33333
0.10	0.59296	0	0.00489	0.01887	0.04094	0.07008	0.10527	0.14551	0.18976	0.23704	0.28632	0.3366
0.20	0.83857	0	0.00506	0.01953	0.04234	0.07243	0.10874	0.15021	0.1958	0.24445	0.29513	0.34682
0.30	1.02704	0	0.00537	0.02071	0.04486	0.07668	0.11502	0.15875	0.20674	0.25788	0.31111	0.36535
0.40	1.18592	0	0.00586	0.02259	0.04889	0.08348	0.12507	0.1724	0.22424	0.27938	0.33667	0.395
0.50	1.32590	0	0.00664	0.02554	0.05519	0.09409	0.14075	0.1937	0.25154	0.31292	0.37654	0.44124
0.60	1.45245	0	0.00788	0.03027	0.0653	0.11112	0.16591	0.22788	0.29535	0.36671	0.4405	0.51541
0.70	1.56882	0	0.01004	0.03849	0.08288	0.14073	0.20964	0.28729	0.37149	0.46021	0.55165	0.64431
0.80	1.67714	0	0.01447	0.05537	0.11894	0.20148	0.29937	0.40917	0.52767	0.65199	0.77964	0.90869
0.90	1.77888	0	0.02796	0.10672	0.22868	0.38632	0.57238	0.77999	1.00283	1.2354	1.47318	1.71291
0.95	1.82763	0	0.05505	0.20991	0.44917	0.75771	1.12091	1.525	1.95746	2.40751	2.86651	3.32856

$\frac{|M(x)|}{P_o L}$

$\frac{\omega}{\omega_1}$	$k\,L$	x/L: 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.10	0.59296	1.01145	0.90989	0.80833	0.7068	0.60533	0.50393	0.40267	0.30159	0.20075	0.1002	0
0.20	0.83857	1.04722	0.94078	0.83437	0.72806	0.62196	0.51623	0.41103	0.30657	0.20309	0.10081	0
0.30	1.02704	1.11211	0.99682	0.88159	0.7666	0.65212	0.5385	0.42616	0.31559	0.20733	0.10193	0
0.40	1.18592	1.21598	1.0865	0.95714	0.82826	0.70035	0.5741	0.45034	0.32999	0.21409	0.10371	0
0.50	1.32590	1.3781	1.22644	1.07502	0.92441	0.77553	0.62958	0.488	0.35241	0.22461	0.10648	0
0.60	1.45245	1.63832	1.45101	1.26411	1.07861	0.89605	0.71846	0.54829	0.38828	0.24143	0.11091	0
0.70	1.56882	2.09084	1.84146	1.59278	1.34651	1.10534	0.87274	0.65289	0.45047	0.27058	0.11858	0
0.80	1.67714	3.0196	2.64265	2.26702	1.89591	1.53435	1.18882	0.86706	0.57773	0.33018	0.13424	0
0.90	1.77888	5.84632	5.08066	4.31823	3.56685	2.83867	2.1494	1.51763	0.96409	0.51105	0.18176	0
0.95	1.82763	11.5265	9.97937	8.43933	6.9235	5.45843	4.07841	2.82382	1.73964	0.87402	0.27709	0

Table 2
Values of deflections and bending moments amplitude parameters (M/M_b=0.2; α_1=1.61639966)

$\frac{|W(x)|}{P_o L^3/EI}$

$\frac{\omega}{\omega_1}$	kL	x/L 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	0	0.00483	0.01867	0.0405	0.06933	0.10417	0.144	0.18783	0.23467	0.2835	0.33333
0.10	0.51115	0	0.00489	0.01887	0.04093	0.07007	0.10526	0.14549	0.18976	0.23705	0.28636	0.33667
0.20	0.72288	0	0.00505	0.01949	0.04228	0.07236	0.10867	0.15016	0.19579	0.24452	0.2953	0.3471
0.30	0.88534	0	0.00535	0.02063	0.04473	0.07651	0.11485	0.15862	0.20673	0.25805	0.31151	0.36602
0.40	1.02230	0	0.00582	0.02245	0.04865	0.08315	0.12474	0.17217	0.22422	0.2797	0.33744	0.39628
0.50	1.14297	0	0.00656	0.02529	0.05476	0.09352	0.14017	0.19329	0.25151	0.31348	0.37789	0.44349
0.60	1.25206	0	0.00775	0.02985	0.06456	0.11016	0.16493	0.22718	0.29529	0.36766	0.44278	0.51922
0.70	1.35238	0	0.00982	0.03777	0.08161	0.13907	0.20796	0.28609	0.37139	0.46183	0.55556	0.65083
0.80	1.44575	0	0.01406	0.05403	0.11657	0.19839	0.29624	0.40694	0.52749	0.65501	0.78689	0.9208
0.90	1.53345	0	0.02697	0.10349	0.22296	0.37887	0.56481	0.77461	1.00238	1.24269	1.49065	1.74206
0.95	1.57547	0	0.05291	0.20286	0.43672	0.74148	1.10442	1.51328	1.95647	2.42336	2.90451	3.39199

$\frac{|M(x)|}{P_o L}$

$\frac{\omega}{\omega_1}$	kL	x/L 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.10	0.51115	1.01092	0.9096	0.80828	0.70697	0.6057	0.50447	0.40332	0.30226	0.20133	0.10057	0
0.20	0.72288	1.04504	0.93959	0.83415	0.72877	0.62351	0.51844	0.41367	0.30932	0.2055	0.10235	0
0.30	0.88534	1.10692	0.99397	0.88106	0.76829	0.65579	0.54377	0.43245	0.32211	0.21305	0.10557	0
0.40	1.02230	1.20594	1.081	0.95613	0.83151	0.70744	0.58427	0.46248	0.34257	0.22511	0.11071	0
0.50	1.14297	1.36045	1.21678	1.07323	0.93013	0.78798	0.64744	0.5093	0.37446	0.24392	0.11873	0
0.60	1.25206	1.60839	1.43463	1.2611	1.08832	0.91716	0.74872	0.58434	0.42557	0.27406	0.13159	0
0.70	1.35238	2.03942	1.81333	1.58763	1.36323	1.1416	0.92465	0.71469	0.51431	0.32637	0.15389	0
0.80	1.44575	2.92382	2.59029	2.25747	1.92709	1.60187	1.28538	0.98188	0.69619	0.43358	0.1996	0
0.90	1.53345	5.61486	4.95422	4.29529	3.64229	3.00178	2.38236	1.79428	1.24911	0.75943	0.33849	0
0.95	1.57547	11.0218	9.70376	8.38947	7.0881	5.814	4.58588	3.42604	2.35961	1.41383	0.61743	0

Table 3
Values of deflections and bending moments amplitude parameters (M/M_b=0.4; α_1=1.47240849)

$\frac{|W(x)|}{P_o L^3/EI}$

$\frac{\omega}{\omega_1}$	$k\,L$	x/L 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	0	0.00483	0.01867	0.0405	0.06933	0.10417	0.144	0.18783	0.23467	0.2835	0.33333
0.10	0.46562	0	0.00488	0.01886	0.04093	0.07006	0.10525	0.14548	0.18976	0.23705	0.28637	0.33669
0.20	0.65848	0	0.00505	0.01948	0.04226	0.07232	0.10862	0.15012	0.19577	0.24451	0.29532	0.34717
0.30	0.80647	0	0.00534	0.0206	0.04467	0.07642	0.11475	0.15853	0.20666	0.25804	0.31157	0.36617
0.40	0.93123	0	0.0058	0.02239	0.04853	0.08299	0.12455	0.17199	0.2241	0.27968	0.33756	0.39657
0.50	1.04115	0	0.00653	0.02518	0.05455	0.09323	0.13983	0.19298	0.2513	0.31344	0.3781	0.444
0.60	1.14052	0	0.00769	0.02966	0.0642	0.10966	0.16436	0.22665	0.29493	0.3676	0.44314	0.52008
0.70	1.23191	0	0.00972	0.03744	0.08099	0.13822	0.20698	0.28518	0.37076	0.46173	0.55617	0.6523
0.80	1.31696	0	0.01388	0.05341	0.11543	0.1968	0.29443	0.40525	0.52632	0.65481	0.78804	0.92352
0.90	1.39685	0	0.02653	0.10199	0.22021	0.37504	0.56045	0.77052	0.99957	1.2422	1.49339	1.74861
0.95	1.43513	0	0.05194	0.1996	0.43071	0.73314	1.09491	1.50438	1.95035	2.42229	2.91049	3.40625

$\frac{|M(x)|}{P_o L}$

$\frac{\omega}{\omega_1}$	$k\,L$	x/L 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.10	0.46562	1.01068	0.90946	0.80823	0.70702	0.60582	0.50466	0.40355	0.30251	0.20155	0.10071	0
0.20	0.65848	1.04407	0.93901	0.83396	0.72895	0.62402	0.51923	0.41464	0.31033	0.2064	0.10292	0
0.30	0.80647	1.1046	0.99259	0.88061	0.76871	0.65701	0.54563	0.43474	0.32452	0.21518	0.10693	0
0.40	0.93123	1.20147	1.07833	0.95524	0.83233	0.70979	0.58787	0.4669	0.34722	0.22923	0.11335	0
0.50	1.04115	1.3526	1.2121	1.07168	0.93157	0.79211	0.65376	0.51705	0.38262	0.25114	0.12335	0
0.60	1.14052	1.59508	1.42671	1.25848	1.09077	0.92416	0.75942	0.59748	0.43937	0.28627	0.13938	0
0.70	1.23191	2.01658	1.79973	1.58313	1.36743	1.15362	0.94302	0.7372	0.53797	0.34728	0.16723	0
0.80	1.31696	2.88131	2.56497	2.24912	1.93493	1.62426	1.31955	1.02373	0.74013	0.47238	0.22433	0
0.90	1.39685	5.51222	4.89311	4.27517	3.66125	3.05583	2.46478	1.89515	1.35491	0.85277	0.39795	0
0.95	1.43513	10.7981	9.57063	8.34567	7.12947	5.9318	4.76543	3.64567	2.58986	1.61687	0.74669	0

Table 4
Values of deflections and bending moments amplitude parameters (M/M_b=0.6; α_1=1. 37566854)

$\frac{|W(x)|}{P_o L^3/EI}$

$\frac{\omega}{\omega_1}$	$k\,L$	x/L 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	0	0.00483	0.01867	0.0405	0.06933	0.10417	0.144	0.18783	0.23467	0.2835	0.33333
0.10	0.43502	0	0.00488	0.01886	0.04092	0.07005	0.10524	0.14548	0.18975	0.23705	0.28637	0.33669
0.20	0.61522	0	0.00504	0.01947	0.04224	0.0723	0.1086	0.1501	0.19575	0.2445	0.29533	0.34719
0.30	0.75348	0	0.00533	0.02058	0.04463	0.07637	0.11469	0.15847	0.20662	0.25802	0.31158	0.36622
0.40	0.87005	0	0.00579	0.02235	0.04846	0.08289	0.12443	0.17187	0.22401	0.27964	0.33758	0.39668
0.50	0.97274	0	0.00651	0.02511	0.05443	0.09306	0.13963	0.19278	0.25114	0.31337	0.37815	0.44419
0.60	1.06559	0	0.00766	0.02955	0.064	0.10937	0.16402	0.22632	0.29466	0.36747	0.44322	0.5204
0.70	1.15097	0	0.00967	0.03725	0.08064	0.13772	0.2064	0.28461	0.37031	0.46151	0.5563	0.65285
0.80	1.23044	0	0.01378	0.05306	0.11478	0.19589	0.29335	0.40418	0.52548	0.65441	0.78828	0.92454
0.90	1.30507	0	0.02628	0.10115	0.21864	0.37283	0.55785	0.76795	0.99753	1.24124	1.49399	1.75106
0.95	1.34084	0	0.05141	0.19777	0.42731	0.72833	1.08926	1.49878	1.94592	2.42021	2.91178	3.41157

$\frac{|M(x)|}{P_o L}$

$\frac{\omega}{\omega_1}$	$k\,L$	x/L 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.10	0.43502	1.01055	0.90938	0.8082	0.70704	0.60588	0.50476	0.40367	0.30263	0.20166	0.10078	0
0.20	0.61522	1.04353	0.93868	0.83383	0.72902	0.62427	0.51962	0.41513	0.31086	0.20686	0.10322	0
0.30	0.75348	1.10332	0.9918	0.88031	0.76889	0.65761	0.54658	0.43592	0.32577	0.21629	0.10764	0
0.40	0.87005	1.19899	1.07681	0.95467	0.83267	0.71095	0.5897	0.46917	0.34963	0.23137	0.11472	0
0.50	0.97274	1.34825	1.20943	1.07068	0.93217	0.79415	0.65696	0.52103	0.38684	0.25489	0.12575	0
0.60	1.06559	1.58771	1.42219	1.25679	1.09178	0.9276	0.76485	0.60422	0.44651	0.29261	0.14345	0
0.70	1.15097	2.00393	1.79199	1.58024	1.36916	1.15953	0.95232	0.74876	0.55019	0.35814	0.17419	0
0.80	1.23044	2.8578	2.55059	2.24374	1.93815	1.63523	1.33684	1.04519	0.76282	0.49252	0.23722	0
0.90	1.30507	5.45554	4.85844	4.26223	3.66905	3.0823	2.50644	1.94683	1.40954	0.90122	0.42894	0
0.95	1.34084	10.6747	9.49515	8.31751	7.14648	5.98946	4.85614	3.75815	2.70872	1.72225	0.81406	0

Table 5
Values of deflections and bending moments amplitude parameters (M/M_b=0.8; α_1=1.30408675)

$\frac{|W(x)|}{P_o L^3/EI}$

$\frac{\omega}{\omega_1}$	kL	x/L										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	0	0.00483	0.01867	0.0405	0.06933	0.10417	0.144	0.18783	0.23467	0.2835	0.33333
0.10	0.41239	0	0.00488	0.01886	0.04092	0.07005	0.10524	0.14547	0.18975	0.23705	0.28637	0.3367
0.20	0.58321	0	0.00504	0.01947	0.04223	0.07229	0.10858	0.15008	0.19573	0.2445	0.29533	0.3472
0.30	0.71428	0	0.00533	0.02057	0.04461	0.07634	0.11465	0.15843	0.20658	0.258	0.31158	0.36625
0.40	0.82478	0	0.00578	0.02233	0.04841	0.08283	0.12436	0.1718	0.22394	0.2796	0.33759	0.39673
0.50	0.92213	0	0.0065	0.02507	0.05435	0.09295	0.1395	0.19265	0.25103	0.3133	0.37815	0.44428
0.60	1.01014	0	0.00764	0.02948	0.06387	0.10919	0.1638	0.22609	0.29447	0.36737	0.44323	0.52055
0.70	1.09108	0	0.00963	0.03713	0.08042	0.13741	0.20602	0.28422	0.36999	0.46133	0.55632	0.65311
0.80	1.16641	0	0.01371	0.05284	0.11437	0.19529	0.29264	0.40347	0.52488	0.65407	0.78832	0.92503
0.90	1.23717	0	0.02613	0.10062	0.21764	0.3714	0.55615	0.76623	0.9961	1.24042	1.49407	1.75223
0.95	1.27107	0	0.05107	0.1966	0.42513	0.72522	1.08556	1.49503	1.9428	2.41842	2.91195	3.41411

$\frac{|M(x)|}{P_o L}$

$\frac{\omega}{\omega_1}$	kL	x/L										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.10	0.41239	1.01047	0.90932	0.80818	0.70704	0.60592	0.50482	0.40374	0.30271	0.20173	0.10083	0
0.20	0.58321	1.04318	0.93846	0.83375	0.72906	0.62442	0.51986	0.41543	0.31118	0.20715	0.1034	0
0.30	0.71428	1.1025	0.9913	0.88011	0.76898	0.65796	0.54715	0.43663	0.32653	0.21697	0.10808	0
0.40	0.82478	1.19742	1.07584	0.95429	0.83285	0.71163	0.5908	0.47055	0.3511	0.23268	0.11556	0
0.50	0.92213	1.34549	1.20772	1.07001	0.93248	0.79535	0.6589	0.52346	0.38941	0.25718	0.12722	0
0.60	1.01014	1.58305	1.4193	1.25564	1.0923	0.92964	0.76812	0.60831	0.45086	0.29649	0.14593	0
0.70	1.09108	1.99594	1.78703	1.57828	1.37007	1.16302	0.95793	0.75577	0.55765	0.36477	0.17844	0
0.80	1.16641	2.84295	2.54138	2.2401	1.93984	1.64172	1.34725	1.05821	0.77665	0.50483	0.24512	0
0.90	1.23717	5.41977	4.83627	4.25347	3.67312	3.09793	2.53151	1.97817	1.44281	0.93082	0.44791	0
0.95	1.27107	10.5968	9.44688	8.29846	7.15534	6.02349	4.91069	3.82633	2.78108	1.7866	0.85529	0

Table 6
Values of deflections and bending moments amplitude parameters(M/M_b=1; α_1=1.24791741)

$\frac{|W(x)|}{P_o L^3/EI}$

$\frac{\omega}{\omega_1}$	kL	x/L										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	0	0.00483	0.01867	0.0405	0.06933	0.10417	0.144	0.18783	0.23467	0.2835	0.33333
0.10	0.39463	0	0.00488	0.01886	0.04092	0.07005	0.10523	0.14547	0.18975	0.23705	0.28637	0.3367
0.20	0.55809	0	0.00504	0.01946	0.04222	0.07228	0.10857	0.15007	0.19572	0.24449	0.29533	0.34721
0.30	0.68351	0	0.00532	0.02056	0.04459	0.07632	0.11462	0.15841	0.20656	0.25798	0.31158	0.36627
0.40	0.78925	0	0.00578	0.02231	0.04838	0.08278	0.12431	0.17174	0.2239	0.27957	0.33758	0.39676
0.50	0.88241	0	0.00649	0.02504	0.05429	0.09287	0.13941	0.19255	0.25095	0.31325	0.37815	0.44433
0.60	0.96663	0	0.00763	0.02943	0.06378	0.10906	0.16364	0.22593	0.29434	0.36728	0.44322	0.52063
0.70	1.04408	0	0.00961	0.03705	0.08026	0.13718	0.20576	0.28395	0.36975	0.46118	0.55631	0.65325
0.80	1.11617	0	0.01367	0.05268	0.11408	0.19488	0.29215	0.40296	0.52444	0.6538	0.78829	0.9253
0.90	1.18388	0	0.02602	0.10025	0.21695	0.37041	0.55496	0.765	0.99504	1.23976	1.494	1.75287
0.95	1.21632	0	0.05083	0.1958	0.42362	0.72306	1.08297	1.49236	1.94051	2.41699	2.91181	3.41551

$\frac{|M(x)|}{P_o L}$

$\frac{\omega}{\omega_1}$	kL	x/L										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.00	0	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0
0.10	0.39463	1.01041	0.90929	0.80817	0.70705	0.60594	0.50485	0.40379	0.30276	0.20178	0.10086	0
0.20	0.55809	1.04295	0.93832	0.83369	0.72908	0.62452	0.52003	0.41564	0.31139	0.20734	0.10353	0
0.30	0.68351	1.10194	0.99095	0.87997	0.76903	0.6582	0.54753	0.43712	0.32705	0.21743	0.10837	0
0.40	0.78925	1.19633	1.07516	0.95401	0.83295	0.71208	0.59154	0.47148	0.35209	0.23356	0.11613	0
0.50	0.88241	1.34359	1.20654	1.06952	0.93267	0.79614	0.66019	0.52508	0.39114	0.25873	0.12822	0
0.60	0.96663	1.57984	1.41729	1.25482	1.09262	0.93098	0.7703	0.61106	0.45379	0.2991	0.14761	0
0.70	1.04408	1.99043	1.78359	1.57687	1.37061	1.16532	0.96167	0.76048	0.56266	0.36925	0.18132	0
0.80	1.11617	2.83273	2.53499	2.2375	1.94085	1.646	1.35419	1.06694	0.78596	0.51312	0.25044	0
0.90	1.18388	5.39517	4.82089	4.24721	3.67556	3.10823	2.54823	1.99918	1.46518	0.95076	0.4607	0
0.95	1.21632	10.5433	9.41344	8.28483	7.16066	6.04589	4.94706	3.87203	2.82972	1.82994	0.8831	0

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