Pergamon

Ocean Engineering 27 (2000) 385–391

OCEAN ENGINEERING

www.elsevier.com/locate/oceaneng

Unsteady thermoelastic analysis of a circular disk with a hot central core and subjected to adiabatic conditions

Carlos A. Rossit*, Patricio A.A. Laura

Institute of Applied Mechanics (Conicet) and Department of Engineering, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina

Abstract

The unsteady thermoelastic analysis of a cooling circular disk or cylinder which is originally at uniform temperature is a classical problem of the theory of thermal stresses. More recent studies consider the case of composite structural configurations. The present paper deals with a situation which, apparently, has not been previously considered: unsteady thermal stresses caused by the presence of a hot, central nucleus. The temperature field is obtained in terms of a Fourier–Bessel expansion and then, radial and tangential stresses are evaluated analytically. The problem is of basic interest in mechanical and naval engineering systems. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Cooling; Circular disk; Cylinder; Fourier–Bessel; Radial stresses; Tangential stresses

1. Introduction

Consider first the case of a circular cylinder or disk at a constant initial temperature T_0. If beginning from an instant $t = 0$, the lateral surface is kept at zero temperature, the temperature distribution as a function of the radial and temporal variables is given by (Timoshenko and Goodier, 1951)

* Corresponding author.

PII: S0029-8018(98)00052-3

$$T = T_0 \sum_{n=1}^{\infty} A_n J_0\left(\alpha_n \frac{r}{a}\right) e^{-p_n t} \tag{1}$$

where the α_ns are the roots of the equation

$$J_0(x) = 0, A_n = \frac{2}{\alpha_n J_1(\alpha_n)} \text{ and } p_n = \frac{k}{c_p \rho} \frac{\alpha_n^2}{a^2}.$$

From well known field and thermoelastic relations one obtains:

$$\sigma_r = \frac{2\alpha E T_0}{1-\nu} \sum_{n=1}^{\infty} e^{-p_n t} \left[\frac{1}{\alpha_n^2} - \frac{1}{\alpha_n^2} \frac{a}{r} \frac{J_1\left(\frac{\alpha_n r}{a}\right)}{J_1(\alpha_n)} \right] \tag{2}$$

$$\sigma_\theta = \frac{2\alpha E T_0}{1-\nu} \sum_{n=1}^{\infty} e^{-p_n t} \cdot \left[\frac{1}{\alpha_n^2} + \frac{1}{\alpha_n^2} \frac{a}{r} \frac{J_1\left(\frac{\alpha_n r}{a}\right)}{J_1(\alpha_n)} - \frac{J_0\left(\frac{\alpha_n r}{a}\right)}{\alpha_n J_1(\alpha_n)} \right]. \tag{3}$$

The solutions and numerical examples quoted in the previously mentioned reference were obtained by Dinnik (1915) and Lees (1922). Stodola (1924) has studied the problem thoroughly in connection with the heating of shafts and rotors making operational recommendations in order to reduce maximum stresses.

The present paper deals with the case where the subdomain $0 \leq r \leq r_0$ is initially at temperature T_0 while the remaining is kept at zero temperature. For $t > 0$ the boundary is insulated. It is felt that the problem is of basic academic and also technological interest since it corresponds, approximately to several practical situations, e.g. when an additional structural element is welded to the disk at a central position (Fig. 1). The present analysis constitutes a first order approximation in the determination of the thermal stress field caused by the cooling of the "hot spot".

The first part of the paper deals with the determination of the unsteady thermal field and the second step constitutes the evaluation of the stresses. It may be of unterest to point out that an approximate solution for the temperature problem in the case of a non-circular outer shape is available (Laura et al., 1985).

2. The unsteady thermal field

Following previous works (Laura et al., 1985) one expresses the situation of Fourier's equation

$$k \nabla^2 T = c_p \rho \frac{\partial T}{\partial t} \tag{4}$$

in terms of

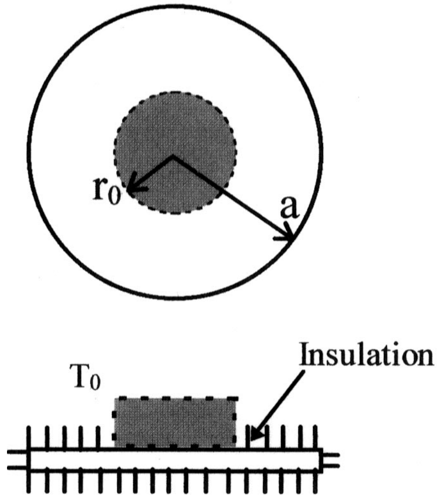

Fig. 1. Thermal system under study.

$$T = \sum_{n=0}^{\infty} B_n J_0(\beta_n r)\, e^{-\frac{k}{c_p \rho} \beta_n^2 t} \tag{5}$$

where β_n is the separation constant and where in order to satisfy the adiabatic condition:

$$\left. \frac{\mathrm{d}J_0(\beta_n r)}{\mathrm{d}r} \right|_{r=a} = 0. \tag{6}$$

Accordingly

$$J_1(\beta_n a) = 0 \tag{7}$$

and then

$$\beta_n a = \alpha_n \tag{8}$$

where the α_ns are the roots of $J_1(x) = 0$. The B_ns are determined using the initial boundary condition

$$T(r,0) = T_0 \text{ if } 0 \leq r \leq r_0 \tag{9a}$$

$$T(r,0) = 0 \text{ if } r_0 < r \leq a. \tag{9b}$$

Then one obtains

$$B_0 = T_0 r_0^2,\ B_n = \frac{2r_0}{\alpha_n} T_0 \frac{J_1\left(\frac{\alpha_n r_0}{a}\right)}{J_0^2(\alpha_n)}. \tag{10}$$

3. The thermoelastic solution

Since the thermomechanical behavior of the system depends only upon the temporal variable t, and the distance to the geometric center of the configuration r, one only needs to determine the radial displacement u_r to describe the state of deformation of the disk and to determine the non-zero components of the stress tensor: σ_r and σ_θ. The corresponding expressions are (Timoshenko and Goodier, 1951; Boley and Weiner, 1960):

$$u_r(r,t) = \alpha \frac{1 + \nu}{r} \int_0^r T.r \, \mathrm{d}r + C_1(1 - \nu)r + \frac{C_2(1 + \nu)}{r} \tag{11}$$

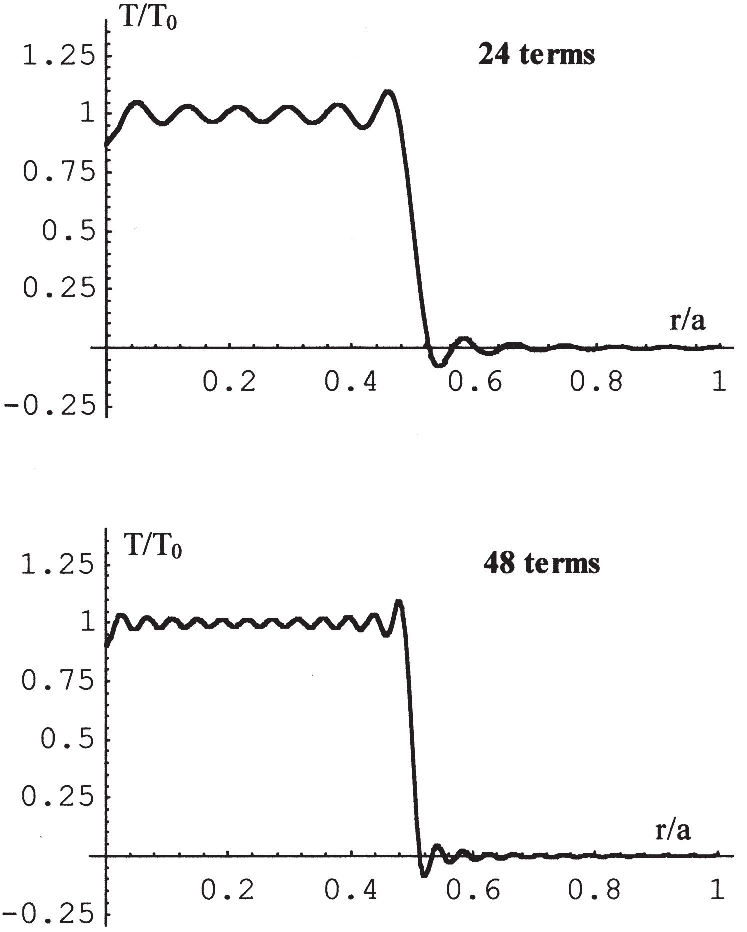

Fig. 2. Initial dimensionless temperature distribution ($r/a = 0.5$).

$$\sigma_r(r,t) = -\frac{E\alpha}{r^2}\int_0^r Tr\,\mathrm{d}r + C_1 - \frac{C_2}{r^2} \tag{12}$$

$$\sigma_\theta(r,t) = \frac{E\alpha}{r^2}\int_0^r Tr\,\mathrm{d}r - T + C_1 + \frac{C_2}{r^2}. \tag{13}$$

The boundary conditions being:

$$u_r(0,t)\text{: Finite} \tag{14}$$

$$\sigma_r(a,t) = 0 \tag{15}$$

4. Numerical results and conclusions

In order to have an idea of the convergence of the Fourier–Bessel expansion at $t = 0$ the initial temperature distribution was plotted as a function of r/a and $kt/c_p\rho a^2 = 0$ using 24 and 48 terms of the expansion. As it was to be expected the convergence is rather slow (Fig. 2). However this point does not have practical significance since

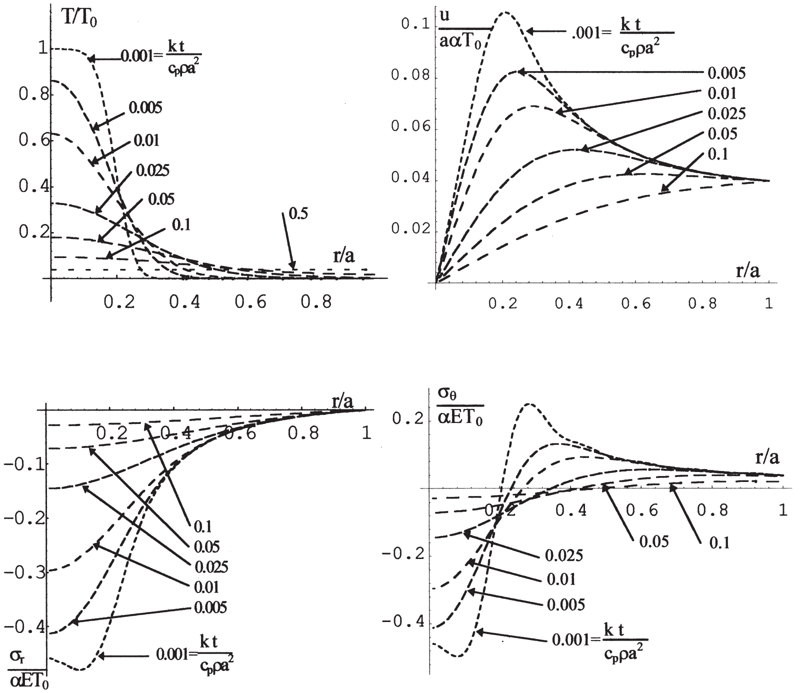

Fig. 3. Dimensionless temperature, radial displacement and radial and tangential stresses as a function of r/a and $kt/c_p\rho a^2 \cdot (r_0/a = 0.2)$.

as soon as $kt/c_p\rho a^2 > 0$ the higher eigenvalues carry enough weight in the exponential function with negative exponent (specially in view of the fact that the squares of the eigenvalues appear in the exponent) and only a few terms yield the desired accuracy. This fact can be observed inmediately in Fig. 3, where for $kt/c_p\rho a^2 = 0.001$ and using 24 terms one obtains a very smooth plot of the dimensionless temperature distribution.[1]

One observes that the maximum values of $\sigma_r/E\alpha T_0$ and $\sigma_\theta/E\alpha T_0$ are approximately equal (-0.479 and -0.496) and take place at $r/a = 0.10$ approximately. Fig. 4 depicts the same dimensionless variables. As it was to be expected the tangential stress is not zero at $r = a$.

As $t \rightarrow \infty$ one has a uniform temperature in the disk ($T_f = T_0(r/a)^2$), the radial displacement function follows a linear variation and the radial and tangential stresses approach zero. It is important to point out that the analytical determinations have been greatly facilitated by the use of MATHEMATICA (Wolfram, 1993).

Future works will also take into account the presence of a concentric, circular inhomogeneity as could be caused by the welding process.

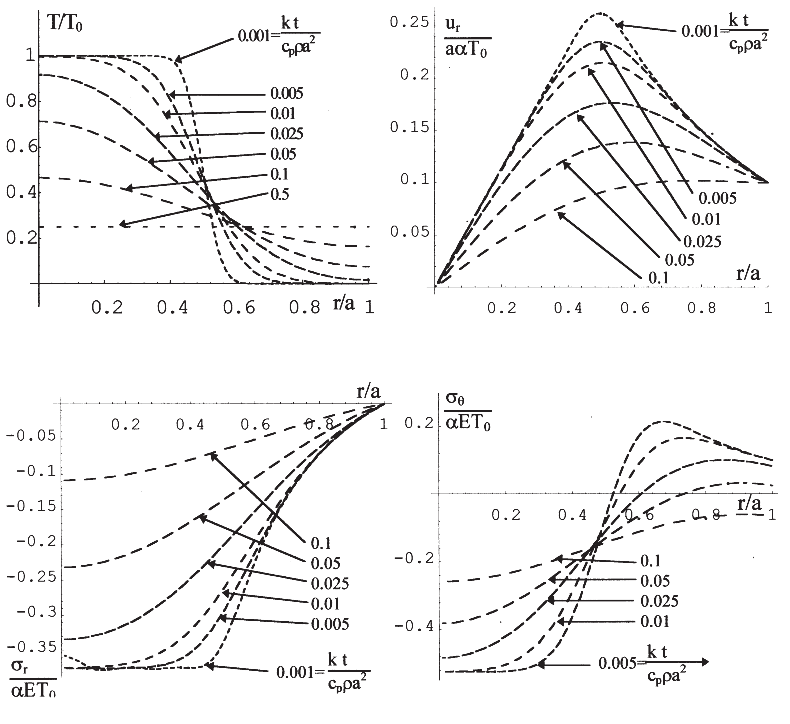

Fig. 4. Dimensionless temperature, radial displacement and radial and tangential stresses as a function of r/a and $kt/c_p\rho a^2 \cdot (r_0/a = 0.5)$.

[1] The remaining curves where obtained using 12 terms.

Acknowledgements

The present study has been sponsored by Secretaría General de Ciencia y Tecnología of Universidad Nacional del Sur and by CONICET Research and Development Program.

References

Boley, B.A., Weiner, J.H., 1960. Theory of Thermal Stresses. Wiley, New York.
Dinnik, A., 1915. Applications of Bessel function to elasticity problems. Ekaterinoslav 2, 15.
Laura, P.A.A., Grossi, R.O., Ercoli, L., Sanchez Sarmiento, G., 1985. Unsteady thermal field in a long, prismatic rod with a complicated initial condition and adiabatic boundary. Int. J. Heat Mass Transfer 28, 1953–1965.
Lees, C.A., 1922. Proc. Royal Soc. (London) 101, 411.
Stodola, A., 1924. Dampf und Gasturbinen, 6th ed., p. 961.
Timoshenko, S., Goodier, J.N., 1951. Theory of Elasticity. McGraw–Hill, New York.
Wolfram, S., 1993. Mathematica. A system for doing mathematics by computer. Addison–Wesley, New York.