Annals of Physics 342 (2014) 195–204

Contents lists available at ScienceDirect

Annals of Physics

journal homepage: www.elsevier.com/locate/aop

Non-Hermitian Hamiltonians with unitary and antiunitary symmetries

Francisco M. Fernández *, Javier Garcia

INIFTA (UNLP, CCT La Plata-CONICET), División Química Teórica, Blvd. 113 S/N, Sucursal 4, Casilla de Correo 16, 1900 La Plata, Argentina

HIGHLIGHTS

- PT-symmetric Hamiltonians exhibit real eigenvalues when PT symmetry is unbroken.
- PT-symmetric multidimensional oscillators appear to show PT phase transitions.
- This transition was conjectured to be a high-energy phenomenon.
- We show that point group symmetry is useful for predicting broken PT symmetry in multidimensional oscillators.
- PT-symmetric oscillators with C_{2v} symmetry exhibit phase transitions at the trivial Hermitian limit.

ARTICLE INFO

Article history:
Received 8 September 2013
Accepted 1 January 2014
Available online 10 January 2014

Keywords:
PT-symmetry
Multidimensional oscillator
Point-group symmetry
PT phase transition
Broken PT symmetry

ABSTRACT

We analyse several non-Hermitian Hamiltonians with antiunitary symmetry from the point of view of their point-group symmetry. It enables us to predict the degeneracy of the energy levels and to reduce the dimension of the matrices necessary for the diagonalization of the Hamiltonian in a given basis set. We can also classify the solutions according to the irreducible representations of the point group and thus analyse their properties separately. One of the main results of this paper is that some PT-symmetric Hamiltonians with point-group symmetry C_{2v} exhibit complex eigenvalues for all values of a potential parameter. In such cases the PT phase transition takes place at the trivial Hermitian limit which suggests that the phenomenon is not robust. Point-group symmetry enables us to explain such anomalous behaviour and to choose a suitable antiunitary operator for the PT symmetry.

* Corresponding author.
E-mail addresses: fernande@quimica.unlp.edu.ar, framfer@gmail.com (F.M. Fernández).

0003-4916/$ – see front matter © 2014 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.aop.2014.01.002

1. Introduction

It was shown some time ago that some complex non-Hermitian Hamiltonians may exhibit real eigenvalues [1,2]. The conjecture that such intriguing feature may be due to unbroken PT-symmetry [3] gave rise to a very active field of research [4] (and references therein). The first studied PT-symmetric models were mainly one-dimensional anharmonic oscillators [3–6] and lately the focus shifted towards multidimensional problems [7–15]. Among the most widely studied multidimensional PT-symmetric models we mention the complex versions of the Barbanis [7,8,10–12,14,15] and Hénon–Heiles [7,12] Hamiltonians. Several methods have been applied to the calculation of their spectra: the diagonalization method [7–10,12,14], perturbation theory [7,9,10,12], classical and semiclassical approaches [7,8], among others [12,15]. Typically, those models depend on a potential parameter g so that the Hamiltonian is Hermitian when $g = 0$ and non-Hermitian when $g \neq 0$. Bender and Weir [14] conjectured that the models studied so far may exhibit PT phase transitions so that their spectra are entirely real for sufficiently small but nonzero values of $|g|$. Such phase transition appears to be a high-energy phenomenon.

Multidimensional oscillators exhibit point-group symmetry (PGS) [16,17]. As far as we know such a property has not been taken into consideration in those earlier studies of the PT-symmetric models, except for the occasional parity in one of the variables. It is to be expected that PGS may be relevant to the study of the spectra of multidimensional PT-symmetric anharmonic oscillators. One of the purposes of this paper is to start such research.

The main interest in the study of PT-symmetric oscillators has been to enlarge the class of such models that exhibit real spectra, at least for some values of the potential parameter. In such cases PT-symmetry is broken at particular values $g = g_c$ of the parameter that are known as exceptional points [18–21] and can be easily calculated as critical parameters by means of the diagonalization method [22]. The PT phase transition is determined by the smallest $|g_c|$. Another goal of this paper is to test that conjecture about PT phase transitions by trying to find PT-symmetric models that do not exhibit real spectra, except at the trivial Hermitian limit $g = 0$.

In Section 2 we outline the main ideas of unitary (point-group) and antiunitary symmetries. In Section 3 we show that two exactly solvable PT-symmetric oscillators with different PGS exhibit quite different spectra. One of them shows a phase transition at the trivial Hermitian limit. In Section 4 we discuss some non-Hermitian operators, already studied earlier by other authors, from the point of view of PGS. All of them have been shown to exhibit nontrivial phase transitions. In Section 5 we show a PT-symmetric anharmonic oscillator with complex eigenvalues for all values of the potential parameter. In Section 6 we explain why the PT symmetry is broken for the models in Sections 3 and 5. Finally, in Section 7 we summarize the main results of the paper and draw conclusions.

2. Unitary and antiunitary symmetries

We assume that there is a group of unitary transformations $G = \{U_1, U_2, \ldots, U_n\}$ and a set of antiunitary transformations $S = \{A_1, A_2, \ldots, A_m\}$ that leave the non-Hermitian Hamiltonian operator invariant

$$U_j H U_j^{-1} = H, \qquad A_k H A_k^{-1} = H, \quad j = 1, 2, \ldots, n, \; k = 1, 2, \ldots, m. \tag{1}$$

Therefore, if ψ is an eigenvector of H with eigenvalue E we have

$$H U_j \psi = E U_j \psi, \quad j = 1, 2, \ldots, n \tag{2}$$

and

$$H A_k \psi = E^* A_k \psi, \quad k = 1, 2, \ldots, m. \tag{3}$$

The latter equation tells us that the eigenvalues of H are either real or appear as pairs of conjugate complex numbers.

It is well known that a product of antiunitary operators is a unitary one [23]. Therefore, since $A_i A_j$ leaves the Hamiltonian invariant then $A_i A_j = U_k \in G$, provided that G is the actual symmetry point group for H [24,25].

If $A_j\psi = \lambda\psi$ then the antiunitary symmetry is said to be unbroken and $E = E^*$. For some non-Hermitian Hamiltonians with degenerate states the eigenvalue can be real even though $A_j\psi \neq \lambda\psi$ [22].

3. Exactly solvable examples

In this section we discuss exactly solvable PT-symmetric models similar to those studied earlier by Nanayakkara [9] and Cannata et al. [13]. In the present case we focus on the PGS of the Hamiltonian operators that was not considered by those authors. The first simple model is the Hamiltonian operator

$$H = p_x^2 + p_y^2 + x^2 + y^2 + iaxy \tag{4}$$

where a is a real parameter. It is exactly solvable and invariant under the operations of the symmetry point group $C_{2v} : \{E, C_2, \sigma_{v1}, \sigma_{v2}\}$ that transform the variables according to

$$\begin{aligned} &E : (x, y) \to (x, y), \\ &C_2 : (x, y) \to (-x, -y), \\ &\sigma_{v1} : (x, y) \to (y, x), \\ &\sigma_{v2} : (x, y) \to (-y, -x). \end{aligned} \tag{5}$$

Note that C_2 is a rotation by an angle π around the z axis and σ_v are vertical reflection planes [24,25]. It should be assumed that the same transformations apply to the momenta (p_x, p_y). In the case of a two-dimensional model the effect of the symmetry operations on the z variable is irrelevant and for this reason there may be more than one point group suitable for the description of the problem. For example, here we can also choose the symmetry point groups C_{2h} or D_2 [24,25]. For concreteness we restrict ourselves to the C_{2v} point group with irreducible representations $\{A_1, B_1, A_2, B_2\}$.

To the PGS discussed above we can also add the antiunitary operations

$$A(x) = C_2(x)T, \qquad A(y) = C_2(y)T, \tag{6}$$

where T is the time reversal operation [26] and

$$\begin{aligned} &C_2(x) : (x, y) \to (x, -y), \\ &C_2(y) : (x, y) \to (-x, y), \end{aligned} \tag{7}$$

are rotations by π about the x and y axis, respectively. Note that $A(x)A(y) = C_2$ is an example of the product of two antiunitary operators that results in one of the elements of the symmetry point group for H.

This model is separable into two harmonic oscillators by means of the change of variables

$$\begin{aligned} x &= \frac{1}{\sqrt{2}}(s + t), \\ y &= \frac{1}{\sqrt{2}}(s - t), \end{aligned} \tag{8}$$

that leads to

$$\begin{aligned} H &= p_s^2 + p_t^2 + ks^2 + k^*t^2, \\ k &= 1 + i\frac{a}{2}. \end{aligned} \tag{9}$$

If we write $\omega = \sqrt{k} = \omega_R + i\omega_I$ then the eigenvalues are given by

$$E_{mn} = 2(m + n + 1)\omega_R + 2(m - n)i\omega_I, \tag{10}$$

where $m, n = 0, 1, \ldots$ and

$$\omega_R = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{a^2}{4}}}, \qquad \omega_I = \frac{a}{4\omega_R}. \tag{11}$$

We see that all the eigenvalues with $m = n$ are real and those with $m \neq n$ are complex when $a \neq 0$ (more precisely: $E_{mn} = E_{nm}^*$). In this case the PT phase transition [14] takes place at the trivial Hermitian limit $a = 0$. It is also obvious that the perturbation series for this model exhibits only powers of a^2 when $m = n$ and all powers of a when $m \neq n$.

The eigenfunctions can be written as

$$\psi_{mn}(s, t) = \phi_m(k, s)\phi_n(k^*, t), \tag{12}$$

where $\phi_m(k, s)$ is an eigenfunction of $p_s^2 + ks^2$. Therefore

$$\begin{aligned} A(x)\psi_{mn}(s, t) &= \psi_{mn}^*(t, s) = \psi_{nm}(s, t), \\ A(y)\psi_{mn}(s, t) &= \psi_{mn}^*(-t, -s) = (-1)^{m+n}\psi_{nm}(s, t) \end{aligned} \tag{13}$$

that are consistent with Eq. (3).

The states $\psi_{2m\,2n}$, $\psi_{2m+1\,2n+1}$, $\psi_{2m+1\,2n}$ and $\psi_{2m\,2n+1}$ are bases for the irreducible representations A_1, A_2, B_1 and B_2, respectively. It is clear that only some of the states with symmetries A_1 and A_2 have real eigenvalues and that those with symmetries B_1 and B_2 exhibit only complex ones. Moreover, the antiunitary operators $A(x)$ and $A(y)$ transform functions of symmetry B_1 into functions of symmetry B_2 and vice versa, which shows that PT symmetry is broken for all $a \neq 0$. More precisely, the eigenvalue of $\psi_{2m+1\,2n}$ (B_1) is the complex conjugate of the one for $\psi_{2n\,2m+1}$ (B_2).

We also appreciate that the eigenfunctions of the non-Hermitian Hamiltonian retain their symmetry in the Hermitian limit: $\lim_{a\to 0} \psi_{mn}(s, t) = \phi_m(1, s)\phi_n(1, t)$.

In order to test the effect of symmetry on the spectra of the non-Hermitian Hamiltonians we next consider the less symmetric operator

$$H = p_x^2 + p_y^2 + 2x^2 + y^2 + iaxy, \tag{14}$$

that is invariant under the operations of the point group $C_2 : \{E, C_2\}$. In this case the eigenfunctions are bases for the irreducible representations $\{A, B\}$ and all the eigenvalues

$$E_{mn} = (2m + 1)\omega_1 + (2n + 1)\omega_2, \tag{15}$$

where

$$\omega_1 = \sqrt{\frac{3}{2} + \frac{\sqrt{1 - a^2}}{2}}, \qquad \omega_2 = \sqrt{\frac{3}{2} - \frac{\sqrt{1 - a^2}}{2}}, \tag{16}$$

are real provided that $|a| < 1$. In this less symmetric example we find a PT phase transition at $a = 1$ for all the states. This particular value of the potential parameter exhibits all the properties of an exceptional point [18–21] and also makes singular the Jacobian of the variable transformation that separates the two-dimensional Schrödinger equation into two one-dimensional eigenvalue equations [13].

The results of this section suggest that PGS determines whether the PT symmetry is broken or unbroken. In order to confirm such conjecture we should find other examples (preferably non exactly solvable) with PT phase transitions at the trivial Hermitian limit. Before doing so we first discuss the non-Hermitian Hamiltonians studied so far from the point of view of PGS.

4. Earlier two- and three-dimensional models

The PT-symmetric version of the Barbanis Hamiltonian

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + iaxy^2, \tag{17}$$

is one of the simplest nontrivial two-dimensional models chosen by several authors as a suitable illustrative example [7,8,10–12,14,15]. Most of them have exploited the fact that it is invariant under y parity: $P_y : (x, y) \to (x, -y)$. If we take into account that the effect of P_y is equivalent to a rotation by an angle π about the x axis then we realize that the appropriate symmetry point group for this model is C_2 already discussed in the preceding section. This model with a rather low symmetry appears to exhibit a PT phase transition at $a \approx 0.1$ [14].

The slightly modified Hamiltonian [14]

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}x^2 + y^2 + iax^2y, \tag{18}$$

exhibits the same symmetry and in this case the phase transition occurs approximately at $a \approx 0.08$.

A more interesting non-Hermitian anharmonic oscillator is the PT-symmetric version of the Hénon–Heiles one [7,12]

$$H = p_x^2 + p_y^2 + x^2 + y^2 + ia\left(xy^2 - \frac{1}{3}x^3\right). \tag{19}$$

Earlier treatments of this problem have taken into account the y parity already discussed above. This symmetry is insufficient to account for the existence of two-fold degenerate eigenvalues already mentioned by Wang [12]. The fact is that this Hamiltonian is invariant under rotations around the z axis by angles $2\pi/3$ and $4\pi/3$ as well as under three vertical and equivalent reflection planes σ_v [17]. The appropriate symmetry point group is thus C_{3v} and the eigenfunctions are bases for the irreducible representations $\{A_1, A_2, E\}$ [24,25]. This PGS already shows that the degeneracy just mentioned is not accidental and comes from the irreducible representation E.

If instead of the three vertical planes σ_v we choose three equivalent axes C_2 perpendicular to the principal C_3 one the suitable point group results to be D_3. The results coming from any of these choices are equivalent. In Section 3 we already explained why we can choose more than one symmetry point group for the two-dimensional models discussed here.

The eigenvalues and eigenfunctions of the Hermitian operator $H(a = 0)$ are

$$E_{mn}(a = 0) = 2(m + n + 1), \quad m, n = 0, 1, \ldots, \tag{20}$$

and

$$\varphi_{mn}(x, y) = \phi_m(x)\phi_n(y), \tag{21}$$

respectively, where $\phi_j(q)$ is a normalized eigenfunction of the harmonic oscillator $H = p_q^2 + q^2$. It is convenient for the discussion below to label the eigenfunctions as $\psi_{M,j}(x, y)$, where $M = m + n,\ j = 0, 1, \ldots, M$ and $E_{M,0} \leq E_{M,1} \leq \cdots \leq E_{M,M}$ so that (as outlined in Section 3)

$$\lim_{a\to 0} \psi_{M,j}(x, y) = \sum_{i=0}^{M} c_{M-i,i,j}\varphi_{M-i,i}(x, y), \tag{22}$$

where the coefficients c_{ij} are determined by the symmetry of the eigenfunction. For example, the first eigenfunctions in this limit and their corresponding symmetries are

$$\begin{aligned}
&M = 0 : \{\varphi_{00}\},\ A_1,\\
&M = 1 : \{\varphi_{10}, \varphi_{01}\},\ E,\\
&M = 2 : \begin{cases} \left\{\dfrac{1}{\sqrt{2}}\left(\varphi_{20} + \varphi_{02}\right)\right\},\ A_1 \\ \left\{\dfrac{1}{\sqrt{2}}\left(\varphi_{20} - \varphi_{02}\right), \varphi_{11}\right\},\ E. \end{cases}
\end{aligned} \tag{23}$$

The projection operators P^S are suitable for a systematic construction of symmetry-adapted functions [24,25]. For example, for $M = 3$ we have

$$P^{A_1}\varphi_{30} = \frac{1}{4}\varphi_{30} - \frac{\sqrt{3}}{4}\varphi_{12},$$

$$P^{A_2}\varphi_{21} = \frac{3}{4}\varphi_{21} - \frac{\sqrt{3}}{4}\varphi_{03},$$

$$P^E\varphi_{30} = \frac{3}{4}\varphi_{30} + \frac{\sqrt{3}}{4}\varphi_{12},$$
$$P^E\varphi_{21} = \frac{1}{4}\varphi_{21} + \frac{\sqrt{3}}{4}\varphi_{03}. \quad (24)$$

These functions are not normalized to unity because $\langle P^S\varphi|P^S\varphi\rangle \leq \langle\varphi|\varphi\rangle$ for any projection operator P^S. Note that the functions with symmetries A_1 and A_2 exhibit even and odd parity, respectively, with respect to the operation P_y discussed above. On the other hand, one of the functions of the basis for the irreducible representation E is even and the other odd.

The spectrum of this model also appears to be real for all $0 \leq a < a_c$ and the perturbation series exhibits only even powers of $g = ia$ [7,12]. The order of the first energy levels for sufficiently small values of a (say $a = 0.1$) is: $1A_1$, $1E$, $2E$, $2A_1$, $3A_1$, $1A_2$, $3E$, $4E$, $5E$, $4A_1$, $6E$, $5A_1$, $2A_2$, $7E$, where the number before the symbol of the irreducible representation just indicates the order of appearance of the eigenvalue. Present discussion of the Hénon–Heiles Hamiltonian provides a PGS explanation of the results obtained by Wang [12] by means of perturbation theory and diagonalization.

Bender et al. [7] and Bender and Weir [14] also discussed some PT-symmetric Hamiltonians in three dimensions. One of them is

$$H = p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2 + iaxyz. \quad (25)$$

In order to analyse its PGS it is convenient to transform it into

$$H = p_x^2 + p_y^2 + p_z^2 + x^2 + y^2 + z^2 + i\frac{a}{2}(x^2 - y^2)z, \quad (26)$$

by means of a rotation about the z axis by an angle of $\pi/4$.

The corresponding symmetry point group is D_{2d} with operations $\{E, S_4, S_4^3, C_2, C_2', C_2'', \sigma_d', \sigma_d''\}$ where S_4 is an improper rotation of order 4 about the z axis [24,25]. From the irreducible representations $\{A_1, A_2, B_1, B_2, E\}$ we conclude that some energy levels are two-fold degenerate.

The eigenfunctions of the Hamiltonian with $a = 0$ are products of harmonic-oscillator eigenfunctions $\varphi_{mnj}(x, y, z) = \phi_m(x)\phi_n(y)\phi_j(z)$ and the eigenvalues $E_{mnj}(a = 0) = 2(m + n + j) + 3$ are $\frac{(M+1)(M+2)}{2}$-fold degenerate, where $M = m + n + j$. When $a \to 0$ the eigenfunctions of the Hamiltonian operator (26) become linear combinations of degenerate eigenfunctions $\varphi_{mnj}(x, y, z)$ with the appropriate symmetry. For example, for the first values of M we have

$$M = 0 : \{\varphi_{000}\} \quad A_1$$
$$M = 1 : \begin{cases} \{\varphi_{001}\} & B_2 \\ \{\varphi_{100}, \varphi_{010}\} & E \end{cases}$$
$$M = 2 : \begin{cases} \left\{\frac{1}{\sqrt{2}}(\varphi_{200} + \varphi_{020})\right\} & A_1 \\ \varphi_{002} & A_1 \\ \varphi_{110} & B_1 \\ \left\{\frac{1}{\sqrt{2}}(\varphi_{200} - \varphi_{020})\right\} & B_2 \\ \{\varphi_{101}, \varphi_{011}\} & E. \end{cases} \quad (27)$$

In this case it also appears to be a PT phase transition at a finite nonzero value of the potential parameter a.

5. Non-Hermitian oscillator with C_{2v} point-group symmetry

In Section 3 we saw that the phase transition for the exactly solvable example with symmetry point group C_{2v} occurs at $a = 0$. The purpose of this section is to show that at least one family of PT-symmetric anharmonic oscillators with that symmetry exhibits the same behaviour. A suitable example is the non-Hermitian modification of the Pullen–Edmonds Hamiltonian [16]

$$H = p_x^2 + p_y^2 + \alpha\left(x^2 + y^2\right) + \beta x^2 y^2 + iaxy. \tag{28}$$

Note that both the unitary and antiunitary transformations that leave this Hamiltonian invariant are exactly those already introduced in Section 3. In fact, when $\alpha = 1$ and $\beta = 0$ we obtain the first exactly solvable example discussed there. When $a = 0$ we recover the Pullen–Edmonds Hamiltonian with C_{4v} PGS [16].

In order to discuss the results from the point of view of PGS we apply the diagonalization method with symmetry-adapted products $\varphi_{mn}(x, y)$ of eigenfunctions $\phi_n(q)$ of the harmonic oscillator $H = p_q^2 + q^2$. We thus obtain basis sets with the following functions

$$\begin{aligned}
\varphi^{+}_{2m\,2n} &= \begin{cases} \varphi_{2n\,2n}(x, y), & m = n \\ \dfrac{1}{\sqrt{2}}\left[\varphi_{2m\,2n}(x, y) + \varphi_{2n\,2m}(x, y)\right], & m \neq n, \end{cases} \\
\varphi^{-}_{2m\,2n} &= \frac{1}{\sqrt{2}}\left[\varphi_{2m\,2n}(x, y) - \varphi_{2n\,2m}(x, y)\right], \quad m \neq n, \\
\varphi^{+}_{2m+1\,2n+1} &= \begin{cases} \varphi_{2n+1\,2n+1}(x, y), & m = n \\ \dfrac{1}{\sqrt{2}}\left[\varphi_{2m+1\,2n+1}(x, y) + \varphi_{2n+1\,2m+1}(x, y)\right], & m \neq n, \end{cases} \\
\varphi^{-}_{2m+1\,2n+1} &= \frac{1}{\sqrt{2}}\left[\varphi_{2m+1\,2n+1}(x, y) - \varphi_{2n+1\,2m+1}(x, y)\right], \quad m \neq n, \\
\varphi^{+}_{2m\,2n+1} &= \frac{1}{\sqrt{2}}\left[\varphi_{2m\,2n+1}(x, y) + \varphi_{2n+1\,2m}(x, y)\right], \\
\varphi^{-}_{2m\,2n+1} &= \frac{1}{\sqrt{2}}\left[\varphi_{2m\,2n+1}(x, y) - \varphi_{2n+1\,2m}(x, y)\right],
\end{aligned} \tag{29}$$

with symmetry

$$\begin{aligned}
&\varphi^{+}_{2m\,2n},\ \varphi^{+}_{2m+1\,2n+1} : A_1, \\
&\varphi^{-}_{2m\,2n},\ \varphi^{-}_{2m+1\,2n+1} : A_2, \\
&\varphi^{+}_{2m\,2n+1} : B_1, \\
&\varphi^{-}_{2m\,2n+1} : B_2.
\end{aligned} \tag{30}$$

Since basis functions of different symmetries do not mix then we can carry out four independent diagonalizations, one for each irreducible representation. This fact not only reduces the dimension of the matrices to be diagonalized (which may be crucial for greater space dimension) but also enables us to analyse the behaviour of the eigenfunctions according to their symmetry.

Because $A(x)\varphi^{+}_{2m\,2n+1} = -\varphi^{-}_{2m\,2n+1}$ then $A(x)\psi_{B_1} = \lambda_{B_1B_2}\psi_{B_2}$ and $E_{B_1} = E^*_{B_2}$ according to Eq. (3). Therefore, the eigenvalues for B eigenfunctions are expected to be complex for any $a > 0$ as in the case of the exactly solvable model discussed in Section 3. We have verified this conclusion by numerical calculation (see below).

Straightforward application of the diagonalization method with those symmetry-adapted basis sets shows that there are no real eigenvalues with eigenfunctions of symmetry B. More precisely, the characteristic polynomials for the bases with symmetries B_1 and B_2 exhibit odd powers of $g = ia$ which do not appear in those for the other two irreducible representations A_1 and A_2. The characteristic polynomials for the entire basis set $\{\varphi_{mn}\}$ are only functions of g^2 and the complex eigenvalues appear as pairs of complex conjugate numbers. In other words, the coefficients of the characteristic polynomials are real for the full basis set as argued elsewhere [27]. On the other hand, the coefficients of the characteristic polynomials for B_1 and B_2 are complex and every complex root E_{B_1} of the former has its counterpart $E^*_{B_2}$ as a root of the latter.

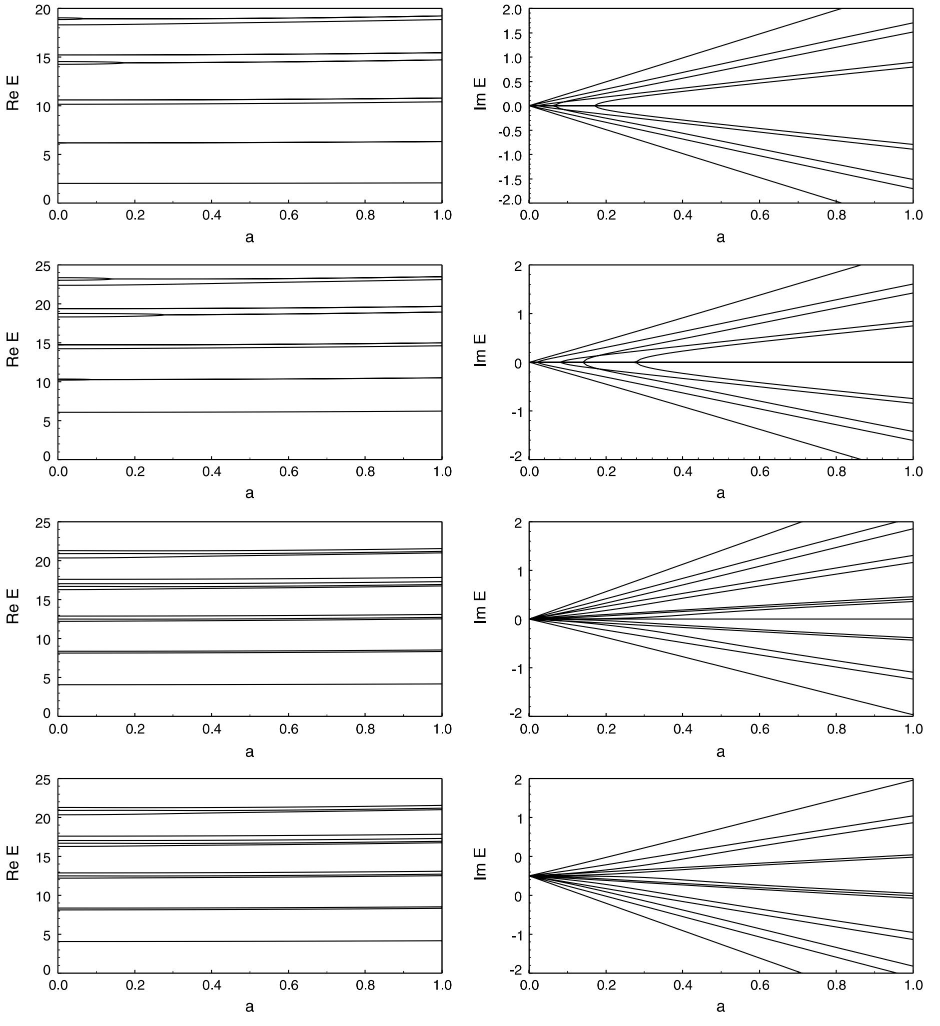

Fig. 1. Lowest eigenvalues with symmetries A_1, A_2, B_1 and B_2 (top to bottom) of the Hamiltonian operator (28) with $\alpha = 1$ and $\beta = 0.1$.

Fig. 1 shows results for $\alpha = 1, \beta = 0.1$ and $0 \leq a \leq 1$. We appreciate that the A states exhibit phase transitions at nonzero values of a but the eigenvalues of symmetry B are complex for all $a > 0$ as argued above. We clearly see that in this case the PT-symmetry is broken when $a > 0$ and the phase transition takes place at the trivial Hermitian limit. In other words, the PT phase transition does not appear to be such a robust phenomenon as it was believed [14].

6. Broken PT symmetry

All the Hamiltonian operators discussed here are of the form $H = H_0 + gH_1$ in such a way that they are Hermitian when $g = 0$ and PT symmetric when g is imaginary. All of them exhibit unbroken PT symmetry for sufficiently small $|g|$ except the two cases of symmetry C_{2v} discussed in Sections 3 and 5. The perturbation series for the models with unbroken PT symmetry exhibit only even powers of the perturbation parameter g. If any of the perturbation corrections of odd order were nonzero then we would expect a complex eigenvalue even for vanishing small values of $|g|$.

In all the cases discussed so far the Hermitian part of the Hamiltonian is invariant under inversion [24,25]: $\hat{\imath}H_0\hat{\imath} = H_0$ so that the unperturbed eigenfunctions exhibit definite parity u or g (note that C_2 may appear instead of $\hat{\imath}$ in some of the examples above but the argument remains unchanged, except for notation). In all the cases with unbroken PT symmetry the non-Hermitian part changes sign under inversion $\hat{\imath}H_1\hat{\imath} = -H_1$ and the whole Hamiltonian is invariant under the antiunitary symmetry given by $A = \hat{\imath}T$. On the other hand, the non-Hermitian parts of the two Hamiltonians with broken PT symmetry are invariant under inversion. These Hamiltonians are not invariant under $A = \hat{\imath}T$ but under other antiunitary operators that we have called $A(x)$ and $A(y)$.

Let us focus on the perturbation correction of first order that is determined by matrix elements of the form $\langle\varphi_i|H_1|\varphi_j\rangle$ where φ_i and φ_j are two degenerate eigenfunctions of H_0. The product $\varphi_i\varphi_j$ is invariant under inversion [24,25] so that $\langle\varphi_i|H_1|\varphi_j\rangle = 0$ if H_1 changes sign under such operation, and as a result the perturbation correction of first order vanishes. On the other hand, in the examples with broken PT symmetry some of those matrix elements do not vanish because the whole integrand is invariant under inversion and the perturbation corrections of first order are nonzero. As a result the corresponding eigenvalues are complex even for vanishing small values of $|g|$. Throughout this discussion we have been tacitly assuming that the symmetry of H_0 is greater than that of H_1 so that the non-Hermitian perturbation removes the symmetry of the unperturbed Hermitian operator.

For example, the eigenfunctions of symmetry E of the Hamiltonian operator (28) with $a = 0$ are linear combinations of the harmonic-oscillator eigenfunctions $\varphi_{2m\,2n+1}$ and $\varphi_{2m+1\,2n}$ that are odd under inversion (or under C_2). When $a \neq 0$ the perturbation, which is invariant under inversion, removes the degeneracy at first order and the eigenfunctions of symmetries B_1 and B_2 emerge. The corresponding eigenvalues are complex conjugate of each other as described above.

The two dimensional isotropic harmonic oscillator is invariant under the two-dimensional rotation group (we can choose the $C_{\infty v}$ point group [24,25]). However, in this case there is additional dynamical symmetry (which we do not discuss in detail here) and the degeneracy is larger than the one predicted by that point group [28]. The degenerate eigenfunctions $\varphi_{M-j\,j}(x, y)$, $j = 0, 1, \ldots, M$ exhibit the same behaviour with respect to inversion: $\varphi_{M-j\,j}(-x, -y) = (-1)^M\varphi_{M-j\,j}(x, y)$. Since the off-diagonal matrix elements $\langle\varphi_{M-i\,i}|xy|\varphi_{M-j\,j}\rangle$ are nonzero when $|i - j| = 1$ the perturbation correction of first order is also nonzero and the corresponding eigenvalues are complex even for vanishing small values of $|a|$ as shown in Section 3. The greater symmetry of H_0 in this case accounts for the fact that not only the functions of symmetries B_1 and B_2 but also some of symmetries A_1 and A_2 have complex eigenvalues.

The general argument given above does not apply to the second exactly solvable example discussed in Section 3 because in this case there are no degenerate states when $a = 0$ and the perturbation corrections of first order are given solely by diagonal matrix elements that vanish because of the symmetry of the eigenfunctions with respect to either the variable x or y.

7. Conclusions

Throughout this paper we have discussed Hamiltonians that are Hermitian when a potential parameter a is zero and non-Hermitian but PT symmetric when $a \neq 0$. Those in Section 4 discussed earlier by several authors exhibit different kinds of PGS but they share the property of having phase transitions at nonzero values of a [14]. On the other hand, the exactly solvable PT-symmetric harmonic oscillator of Section 3 exhibits a phase transition at $a = 0$; that is to say, some of its eigenvalues are complex for all values of $a > 0$. This operator exhibits C_{2v} PGS and some of the eigenvalues for the A eigenfunctions and all of those for the B ones are complex. For such eigenfunctions the PT symmetry is broken for all values of a and the phase transition occurs at the Hermitian limit.

In order to verify if the broken PT symmetry was actually due to PGS and not to the particular form of the Hamiltonian (an exactly solvable two-dimensional harmonic oscillator) we constructed a family of simple but nontrivial examples with the same PGS and found that the eigenvalues with eigenfunctions of symmetry B are complex for all nonzero values of the model parameter a.

Upon comparing the models with broken and unbroken PT symmetry we concluded that the relevant difference is the behaviour of the non-Hermitian part of the Hamiltonian with respect to inversion. We have shown that the spectrum may not be real for some values of the potential parameter unless the Hamiltonian is invariant under the antiunitary operator $A = \hat{\imath}T$. The most important conclusion of this paper is that the existence of a phase transition as a high-energy phenomenon [14] is not a general property of multidimensional oscillators with arbitrary antiunitary symmetries (like that given by the operators $A(x)$ and $A(y)$ introduced in Section 3). It does not appear to be a robust phenomenon unless we restrict PT symmetry to the antiunitary operator $A = \hat{\imath}T$.

References

[1] E. Caliceti, S. Graffi, M. Maioli, Comm. Math. Phys. 75 (1980) 51–66.
[2] G. Alvarez, J. Phys. A 28 (1995) 4589–4598.
[3] C.M. Bender, S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243–5246.
[4] C.M. Bender, Rep. Progr. Phys. 70 (2007) 947–1018.
[5] F.M. Fernández, R. Guardiola, M. Znojil, J. Phys. A 31 (1998) 10105–10112.
[6] F.M. Fernández, R. Guardiola, J. Ros, M. Znojil, J. Phys. A 32 (1999) 3105–3116.
[7] C.M. Bender, G.V. Dunne, P.N. Meisinger, M. Simsek, Phys. Lett. A 281 (2001) 311–316.
[8] A. Nanayakkara, C. Abayaratne, Phys. Lett. A 303 (2002) 243–248.
[9] A. Nanayakkara, Phys. Lett. A 304 (2002) 67–72.
[10] A. Nanayakkara, Phys. Lett. A 334 (2005) 144–153.
[11] H. Bíla, M. Tater, M. Znojil, Phys. Lett. A 351 (2006) 452–456.
[12] Q.-H. Wang, Pramana J. Phys. 73 (2009) 315–322.
[13] F. Cannata, M.V. Ioffe, D.N. Nishnianidze, J. Math. Phys. 51 (2010) 022108.
[14] C.M. Bender, D.J. Weir, J. Phys. A 45 (2012) 425303.
[15] C.R. Handy, D. Vrincenau, J. Phys. A 46 (2013) 135202.
[16] R.A. Pullen, A.R. Edmonds, J. Phys. A 14 (1981) L477–L484.
[17] R.A. Pullen, A.R. Edmonds, J. Phys. A 14 (1981) L319–L327.
[18] W.D. Heiss, A.L. Sannino, J. Phys. A 23 (1990) 1167–1178.
[19] W.D. Heiss, Phys. Rev. E 61 (2000) 929–932.
[20] W.D. Heiss, H.L. Harney, Eur. Phys. J. D 17 (2001) 149–151.
[21] W.D. Heiss, Czech. J. Phys. 54 (2004) 1091–1099.
[22] F.M. Fernández, J. Garcia, arXiv:1305.5164 [math-ph].
[23] E. Wigner, J. Math. Phys. 1 (1960) 409–413.
[24] M. Tinkham, Group Theory and Quantum Mechanics, McGraw-Hill Book Company, New York, 1964.
[25] F.A. Cotton, Chemical Applications of Group Theory, John Wiley & Sons, New York, 1990.
[26] C.E. Porter, in: C.E. Porter (Ed.), Statistical Theories of Spectra: Fluctuations, Academic Press Inc., New York, London, 1965.
[27] F.M. Fernández, arXiv:1301.7639v3 [quant-ph].
[28] H.V. McIntosh, Amer. J. Phys. 27 (1959) 620–625.