N-Complexes as Functors, Amplitude Cohomology and Fusion Rules[⋆]

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Abstract: We consider N-complexes as functors over an appropriate linear category in order to show first that the Krull-Schmidt Theorem holds, then to prove that amplitude cohomology (called generalized cohomology by M. Dubois-Violette) only vanishes on injective functors providing a well defined functor on the stable category. For left truncated N-complexes, we show that amplitude cohomology discriminates the isomorphism class up to a projective functor summand. Moreover amplitude cohomology of positive N-complexes is proved to be isomorphic to an Ext functor of an indecomposable N-complex inside the abelian functor category. Finally we show that for the monoidal structure of N-complexes a Clebsch-Gordan formula holds, in other words the fusion rules for N-complexes can be determined.

1. Introduction

Let N be a positive integer and let k be a field. In this paper we will consider N-complexes of vector spaces as linear functors (or modules) over a k-category, see the definitions at the beginning of Sect. 2.

Recall first that a usual k-algebra is deduced from any finite object k-category through the direct sum of its vector spaces of morphisms. Modules over this algebra are precisely k-functors from the starting category, with values in the category of k-vector spaces. Consequently if the starting category has an infinite number of objects, linear functors with values in vector spaces are called modules over the category, as much as modules over an algebra are appropriate algebra morphisms.

An N-complex as considered by M. Kapranov in [19] is a $\mathbb{Z}$-graded vector space equipped with linear maps d of degree 1 verifying $d^N = 0$. The amplitude (or generalized) cohomology are the vector spaces $\operatorname{Ker} d^a / \operatorname{Im} d^{N-a}$ for each amplitude a between

1 and $\mathsf{N} - 1$. Note that we use the terminology *amplitude cohomology* in order to give a graphic idea of this theory and in order to clearly distinguish it from classical cohomology theories.

M. Dubois-Violette has shown in [9] a key result, namely that for N-complexes arising from cosimplicial modules through the choice of an element $q \in k$ such that $1 + q + \cdots + q^{\mathsf{N}-1} = 0$, amplitude cohomology can be computed using the classical cohomology provided the truncated sums $1 + q + \cdots + q^n$ are invertible for $1 \leq n \leq \mathsf{N} - 1$. As a consequence he obtains in a unified way that Hochschild cohomology at roots of unity or in non-zero characteristic is zero or isomorphic to classical Hochschild cohomology (see also [20]) and the result proven in 1947 by Spanier [26], namely that Mayer [22] amplitude cohomology can be computed by means of classical simplicial cohomology.

Note that N-complexes are useful for different approaches, as Yang-Mills algebras [8], Young symmetry of tensor fields [13, 14] as well as for studying homogeneous algebras and Koszul properties, see [1, 2, 16, 23, 24] or for analysing cyclic homology at roots of unity [28]. A comprehensive description of the use of N-complexes in these various settings is given in the course by M. Dubois-Violette at the Institut Henri Poincaré, [12].

We first make clear an obvious fact, namely that an N-complex is a module over a specific k-category presented as a free k-category modulo the N-truncation ideal. This way we obtain a Krull-Schmidt theorem for N-complexes. The list of indecomposables is well-known, in particular projective and injective N-complexes coincide. This fact enables us to enlarge Kapranov's acyclicity Theorem in terms of injectives. More precisely, for each amplitude a verifying $1 \leq a \leq \mathsf{N} - 1$ a classic 2-complex is associated to each N-complex. We prove first in this paper that an N-complex is acyclic for a given amplitude if and only if the N-complex is projective (injective), which in turn is equivalent to acyclicity for any amplitude.

In [15, 9] a basic result is obtained for amplitude cohomology for $\mathsf{N} \geq 3$ which has no counterpart in the classical situation $\mathsf{N} = 2$, namely hexagons arising from amplitude cohomologies are exact. This gap between the classical and the new theory is confirmed by a result we obtain in this paper: amplitude cohomology does not discriminate arbitrary N-complexes without projective summands for $\mathsf{N} \geq 3$, despite the fact that for $\mathsf{N} = 2$ it is well known that usual cohomology is a complete invariant up to a projective direct summand. Nevertheless we prove that left truncated N-complexes sharing the same amplitude cohomology are isomorphic up to a projective (or equivalently injective) direct summand.

We also prove that amplitude cohomology for *positive* N-complexes coincides with an Ext functor in the category of N-complexes. In other words, for each given amplitude there exists an indecomposable module such that the amplitude cohomologies of a positive N-complex are actually extensions of a particular degree between the indecomposable and the given positive N-complex. We use the characterisation of Ext functors and the description of injective positive N-complexes. In this process the fact that for positive N-complexes, projectives no longer coincide with injectives requires special care.

We underline the fact that various indecomposable modules are used in order to show that amplitude cohomology of positive N-complexes is an Ext functor. This variability makes the result compatible with the non-classical exact hexagons [15, 9] of amplitude cohomologies quoted above.

M. Dubois-Violette has studied in [11] (see Appendix A) the monoidal structure of N-complexes in terms of the coproduct of the Taft algebra, see also [12]. J. Bichon in [3] has studied the monoidal structure of N-complexes, considering them as comodules,

see also the work by R. Boltje [4] and A. Tikaradze [27]. We recall in this paper that the k-category we consider is the universal cover of the Taft Hopf algebra $\mathcal{U}_q^+(sl_2)$. As such, there exists a tensor product of modules (i.e. N-complexes) for each non-trivial N^{th}-root of unity (see also [4, 5]). Using Gunnlaugsdottir's axiomatisation of Clebsch-Gordan's formula [18] and amplitude cohomology we show that this formula is valid for N-complexes, determining this way the corresponding fusion rules.

2. N-Complexes and Categories

Let $\mathcal{C}$ be a small category over a field k. The set of objects is denoted $\mathcal{C}_0$. Given x, y in $\mathcal{C}_0$, the k-vector space of morphisms from x to y in $\mathcal{C}$ is denoted $_y\mathcal{C}_x$. Recall that composition of morphisms is k-bilinear. In this way, each $_x\mathcal{C}_x$ is a k-algebra and each $_y\mathcal{C}_x$ is a $_y\mathcal{C}_y$-$_x\mathcal{C}_x$ – bimodule.

For instance let Λ be a k-algebra and let E be a complete finite system of orthogonal idempotents in Λ, that is $\sum_{e \in E} e = 1$, $ef = fe = 0$ if $f \neq e$ and $e^2 = e$, for all e, $f \in E$. The associated category $\mathcal{C}_{\Lambda, E}$ has a set of objects E and morphisms $_f(\mathcal{C}_{\Lambda, E})_e = f\Lambda e$. Conversely any finite object set category $\mathcal{C}$ provides an associative algebra Λ through the matrix construction. Both procedures are mutually inverse.

In this context linear functors $F : \mathcal{C}_{\Lambda, E} \to Mod_k$ coincide with left Λ-modules. Consequently for any arbitrary linear category $\mathcal{C}$, left modules are defined as k-functors $F : \mathcal{C} \to Mod_k$. In other words, a left $\mathcal{C}$-module is a set of k-vector spaces $\{_x M\}_{x \in \mathcal{C}_0}$ equipped with "left oriented" actions that is, linear maps

$$_y\mathcal{C}_x \otimes_k {}_x M \to {}_y M$$

verifying the usual associativity constraint.

Notice that right modules are similar; they are given by a collection of k-vector spaces $\{M_x\}_{x \in \mathcal{C}_0}$ and "right oriented"' actions. From now on a module will mean a left module.

Free k-categories are defined as follows: let E be an arbitrary set and let $V = \{_y V_x\}_{x, y \in E}$ be a set of k-vector spaces. The free category $\mathcal{F}_E(V)$ has a set of objects E and a set of morphisms from x to y, the direct sum of tensor products of vector spaces relying x to y:

$$_y(\mathcal{F}_E(V))_x = \bigoplus_{n \geq 0} \bigoplus_{x_1, \ldots, x_n \in E} ({}_y V_{x_n} \otimes \cdots \otimes {}_{x_2} V_{x_1} \otimes {}_{x_1} V_x).$$

For instance, let $E = \mathbb{Z}$ and let $_{i+1}V_i = k$ while $_j V_i = 0$ otherwise. This data can be presented by the double infinite quiver having $\mathbb{Z}$ as a set of vertices and an arrow from i to $i + 1$ for each $i \in \mathbb{Z}$. The corresponding free category $\mathcal{L}$ has one dimensional vector space morphisms from i to j if and only if $i \leq j$, namely

$$_j\mathcal{L}_i = {}_j V_{j-1} \otimes \cdots \otimes {}_{i+2}V_{i+1} \otimes {}_{i+1}V_i.$$

Otherwise $_j\mathcal{L}_i = 0$.

A module over $\mathcal{L}$ is precisely a graded vector space $\{_i M\}_{i \in \mathbb{Z}}$ together with linear maps $d_i : {}_i M \to {}_{i+1}M$. This fact makes use of the evident universal property characterizing free linear categories.

On the other hand we recall from [19] the definition of an N-complex: it consists of a graded vector space $\{_i M\}_{i \in \mathbb{Z}}$ and linear maps $d_i : {}_i M \to {}_{i+1}M$ verifying that $d_{i+N-1} \circ \cdots \circ d_i = 0$ for each $i \in \mathbb{Z}$.

In order to view an N-complex as a module over a k-linear category we have to consider a quotient of $\mathcal{L}$. Recall that an ideal I of a k-category $\mathcal{C}$ is a collection of sub-vector spaces $_yI_x$ of each morphism space $_y\mathcal{C}_x$, such that the image of the composition map $_z\mathcal{C}_y \otimes {_yI_x}$ is contained in $_zI_x$ and $_yI_x \otimes {_x\mathcal{C}_u}$ is contained in $_yI_u$ for each choice of objects. Quotient k-categories exist in the same way that algebra quotients exist.

Returning to the free category $\mathcal{L}$, consider the truncation ideal I_N given by the entire $_j\mathcal{L}_i$ in case $j \geq i+N$ and 0 otherwise. Then $\mathcal{L}_N := \mathcal{L}/I_N$ has one dimensional morphisms from i to j if and only if $i \leq j \leq i + N - 1$.

Clearly N-complexes coincide with $\mathcal{L}_N$-modules. We have obtained the following

Theorem 2.1. *The categories of N-complexes and of $\mathcal{L}_N$-modules are isomorphic.*

An important point is that $\mathcal{L}_N$ is a locally bounded category, which means that the direct sum of morphism spaces starting (or ending) at each given object is finite dimensional. More precisely:

$$\forall x_0, y_0 \in (\mathcal{L}_N)_0, \quad dim_k \left[\bigoplus_{y \in \mathbb{Z}} {_y(\mathcal{L}_N)_{x_0}} \right] = N = dim_k \left[\bigoplus_{x \in \mathbb{Z}} {_{y_0}(\mathcal{L}_N)_x} \right].$$

It is known that for locally bounded categories the Krull-Schmidt theorem holds, for instance see the work by C. Sáenz [25]. We infer that each N-complex of finite dimensional vector spaces is isomorphic to a direct sum of indecomposable ones in an essentially unique way, meaning that given two decompositions, the multiplicities of isomorphic indecomposable N-complexes coincide.

Moreover, indecomposable N-complexes are well known, they correspond to "short segments" in the quiver: the complete list of indecomposable modules is given by $\{M_i^l\}_{i \in \mathbb{Z}, 0 \leq l \leq N-1}$, where i denotes the beginning of the module, $i + l$ its end and l its length. More precisely, $_i(M_i^l) = {_{i+1}(M_i^l)} = \cdots = {_{i+l}(M_i^l)} = k$ while $_j(M_i^l) = 0$ for other indices j. The action of $d_i, d_{i+1}, \ldots, d_{i+l-1}$ is the identity and d_j acts as zero if the index j is different. The corresponding N-complex is concentrated in the segment $[i, i + l]$.

Note that the simple N-complexes are $\{M_i^0\}_{i \in \mathbb{Z}}$ and that each M_i^l is uniserial, which means that M_i^l has a unique filtration

$$0 \subset M_{i+l}^0 \subset \cdots \subset M_{i+2}^{l-2} \subset M_{i+1}^{l-1} \subset M_i^l$$

such that each submodule is maximal in the following one.

Summarizing the preceding discussion, we have the following

Proposition 2.2. *Let M be an N-complex of finite dimensional vector spaces. Then*

$$M \simeq \bigoplus_{i \in \mathbb{Z}, \, 0 \leq l \leq N-1} n_i^l M_i^l$$

for a unique finite set of positive integers n_i^l.

Indecomposable projective and injective $\mathcal{L}_N$-modules are also well known, we now recall them briefly. Note from [17] that projective functors are direct sums of representable functors. Clearly $_-(\mathcal{L}_N)_i = M_i^{N-1}$.

In order to study injectives notice first that for a locally bounded k-category, right and left modules are in duality: the dual of a left module is a right module which has the dual vector spaces at each object, the right actions are obtained by dualising the left actions. Projectives and injectives correspond under this duality. Right projective modules are direct sums of $_i(\mathcal{L}_N)_-$ as above, clearly $(_i(\mathcal{L}_N)_-)^* \simeq M_i^{N-1}$.

This way we have provided the main steps of the proof of the following

Proposition 2.3. *Let M_i^l be an indecomposable N-complex, $i \in \mathbb{Z}$ and $l \leq N - 1$. Then M_i^l is projective if and only if $l = N - 1$, which in turn is equivalent for M_i^l to be injective.*

Corollary 2.4. *Let $M = \bigoplus_{i \in \mathbb{Z}, 0 \leq l \leq N-1} n_i^l M_i^l$ be an N-complex. Then M is projective if and only if $n_i^l = 0$ for $l \leq N - 2$, which in turn is equivalent for M to be injective.*
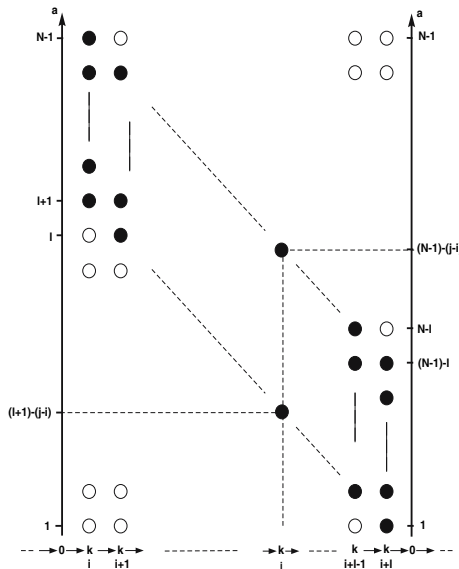
3. Amplitude Cohomology

Let M be an N-complex. For each amplitude a between 1 and $N - 1$, at each object i we have $\operatorname{Im} d^{N-a} \subseteq \operatorname{Ker} d^a$. More precisely we define as in [19]

$$(AH)_a^i(M) := \operatorname{Ker}(d_{i+a-1} \circ \cdots \circ d_i)/\operatorname{Im}(d_{i-1} \circ \cdots \circ d_{i-N+a}),$$

and we call this bi-graded vector space the amplitude cohomology of the N-complex. As remarked in the Introduction, M. Dubois-Violette in [9] has shown the depth of this theory; he calls it generalised cohomology.

As a fundamental example we compute amplitude cohomology for indecomposable N-complexes M_i^l. In the following picture the amplitude is to be read vertically while the degree of the cohomology is to be read horizontally. A black dot means one dimensional cohomology, while an empty dot stands for zero cohomology.

From this easy computation we notice that for a non-projective (equivalently non-injective) indecomposable module M_i^l ($0 \leq l \leq N-2$) and any amplitude a there exists a degree j such that $(AH)_a^j(M_i^l) \neq 0$. Concerning projective or injective indecomposable modules M_i^{N-1} we notice that $(AH)_a^j(M_i^{N-1}) = 0$ for any degree j and any amplitude a. These facts are summarized as follows:

Proposition 3.1. *Let M be an indecomposable N-complex. Then M is projective (or equivalently injective) in the category of N-complexes if and only if its amplitude cohomology vanishes at some amplitude a which in turn is equivalent to its vanishing at any amplitude.*

Remark 3.2. From the very definition of amplitude cohomology one can check that for a fixed amplitude a we obtain a linear functor $(AH)_a^*$ from $\mathsf{mod}\,\mathcal{L}_N$ to the category of graded vector spaces.

Moreover $(AH)_a^*$ is additive, in particular:

$$(AH)_a^*(M \oplus M') = (AH)_a^*(M) \oplus (AH)_a^*(M').$$

This leads to the following result, which provides a larger frame to the acyclicity result of M. Kapranov [19]. See also the short proof of Kapranov's acyclicity result by M. Dubois-Violette in Lemma 3 of [9] obtained as a direct consequence of a key result of this paper, namely the exactitude of amplitude cohomology hexagons.

Theorem 3.3. *Let M be an N-complex of finite dimensional vector spaces. Then $(AH)_a^*$ $(M) = 0$ for some a if and only if M is projective (or equivalently injective). Moreover, in this case $(AH)_a^*(M) = 0$ for any amplitude $a \in [1, N-1]$.*

In order to understand the preceding result in a more conceptual framework we will consider the stable category of N-complexes, $\underline{\mathsf{mod}\,\mathcal{L}_N}$. More precisely, let I be the ideal of $\mathsf{mod}\,\mathcal{L}_N$ consisting of morphisms which factor through a projective N-complex. The quotient category $\mathsf{mod}\,\mathcal{L}_N/I$ is denoted $\underline{\mathsf{mod}\,\mathcal{L}_N}$. Clearly all projectives become isomorphic to zero in $\underline{\mathsf{mod}\,\mathcal{L}_N}$. Of course this construction is well known and applies for any module category. We have in fact proven the following

Theorem 3.4. *For any amplitude a there is a well-defined functor*

$$(AH)_a^* : \underline{\mathsf{mod}\,\mathcal{L}_N} \to gr(k),$$

where $gr(k)$ is the category of graded k-vector spaces.

Our next purpose is to investigate how far amplitude cohomology distinguishes N-complexes. First we recall that in the classical case ($N = 2$), cohomology is a complete invariant of the stable category.

Proposition 3.5. *Let M and M' be 2-complexes of finite dimensional vector spaces without projective direct summands. If $H^*(M) \simeq H^*(M')$, then $M \simeq M'$.*

Proof. Indecomposable 2-complexes are either simple or projective. We assume that M has no projective direct summands; this is equivalent for M to be semisimple, in other words M is a graded vector space with zero differentials. Consequently $H^i(M) = {}_iM$ for all i. $\square$

The following example shows that the favorable situation for $N = 2$ is no longer valid for $N \geq 3$.

Example 3.6. Consider M the 3-complex which is the direct sum of all simple modules, in other words, $_iM = k$ and $d_i = 0$. Then for any degree i we have

$$(AH)_1^i(M) = k \quad \text{and} \quad (AH)_2^i(M) = k.$$

Let M' be the direct sum of all the length one indecomposable 3-complexes,

$$M' = \bigoplus_{i \in \mathbb{Z}} M_i^1.$$

Recall that the amplitude cohomology of M_i^1 is given by

$$(AH)_2^i(M_i^1) = k \text{ and } (AH)_1^{i+1}(M_i^1) = k,$$

while all other amplitude cohomologies vanish. Summing up provides $(AH)_2^i(M') = k$ and $(AH)_1^i(M') = k$, for all i. However it is clear that M and M' are not isomorphic. Notice that both M and M' are free of projective direct summands.

As quoted in the introduction the preceding example confirms that amplitude cohomology is a theory with different behaviour than the classical one. This fact has been previously noticed by M. Dubois-Violette in [9], for instance when dealing with non-classical exact hexagons of amplitude cohomologies.

At the opposite, we will obtain in the following that for either left or right truncated N-complexes amplitude cohomology is a complete invariant up to projectives. More precisely, let M be an N-complex which is zero at small enough objects, namely $_iM = 0$ for $i \leq b$, for some b which may depend on M. Of course this is equivalent to the fact that for the Krull-Schmidt decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{l=0}^{N-1} n_i^l M_i^l$$

there exists a minimal i_0, in the sense that $n_i^l = 0$ if $i < i_0$ and $n_{i_0}^l \neq 0$ for some l.

Proposition 3.7. *Let M be a non-projective N-complex which is zero at small enough objects. Let l_0 be the smallest length of an indecomposable factor of M starting at the minimal starting object i_0. Then $(AH)_a^i(M) = 0$ for all $i \leq i_0 - 1$ and $(AH)_a^{i_0}(M) = 0$ for $a \leq l_0$. Moreover $dim_k(AH)_{l_0+1}^{i_0}(M) = n_{i_0}^{l_0}$.*

Proof. The fundamental computation we made of amplitude cohomology for indecomposable N-complexes shows the following: the smallest degree affording non-vanishing amplitude cohomology provides the starting vertex of an indecomposable non-projective module. Moreover, at this degree the smallest value of the amplitude affording non-zero cohomology is $l + 1$, where l is the length of the indecomposable.

In other words amplitude cohomology determines the multiplicity of the *smallest* indecomposable direct summand of a left-truncated N-complex. Of course *smallest* concerns the lexicographical order between indecomposables, namely $M_i^l \leq M_j^r$ in case $i < j$ or in case $i = j$ and $l \leq r$. $\square$

Theorem 3.8. *Let M be an* N*-complex which is zero at small enough objects and which does not have projective direct summands. The dimensions of its amplitude cohomology determine the multiplicities of each indecomposable direct summand.*

Proof. The proposition above shows that the multiplicity of the smallest indecomposable direct summand is determined by the amplitude cohomology (essentially this multiplicity is provided by the smallest non-zero amplitude cohomology, where amplitude cohomology is also ordered by lexicographical order).

We factor out this smallest direct summand X from M and we notice that the multiplicities of other indecomposable factors remain unchanged. Moreover, factoring out the amplitude cohomology of X provides the amplitude cohomology of the new module. It's smallest indecomposable summand comes strictly after X in the lexicographical order. Through this inductive procedure, multiplicities of indecomposable summands can be determined completely. In other words: if two left-truncated N-complexes of finite dimensional vector spaces share the same amplitude cohomology, then the multiplicities of their indecomposable direct factors coincide for each couple (i, n). □

Remark 3.9. Clearly the above theorem is also true for N-complexes which are zero for large enough objects, that is right-truncated N-complexes.

4. Amplitude Cohomology is Ext

An N-complex M is called positive in case $_iM = 0$ for $i \leq -1$. In this section we will prove that amplitude cohomology of positive N-complexes of finite dimensional vector spaces coincides with an Ext functor in this category.

First we provide a description of injective positive N-complexes as modules. Notice that positive N-complexes are functors on the full subcategory $\mathcal{L}_N^{\geq 0}$ of $\mathcal{L}_N$ provided by the positive integer objects. Alternatively, $\mathcal{L}_N^{\geq 0}$ is the quotient of the free k-category generated by the quiver having positive integer vertices and an arrow from i to $i+1$ for each object, by the truncation ideal given by morphisms of length greater than N.

Theorem 4.1. *The complete list up to isomorphism of injective positive indecomposable* N*-complexes is*

$$\left\{M_0^l\right\}_{l=0,\ldots,N-1} \sqcup \left\{M_i^{N-1}\right\}_{i \geq 1}.$$

Proof. As we stated before, injective modules are duals of projective right modules. The indecomposable ones are representable functors $_{i_0}(\mathcal{L}_N^{\geq 0})_-$, for $i_0 \geq 0$.

Clearly for each i_0 we have $(_{i_0}\mathcal{L}_N^{\geq 0}\_)^* = M_0^{i_0}$ if $i_0 \leq N-1$ while $(_{i_0}\mathcal{L}_N^{\geq 0}\_)^* = M_{i_0-(N-1)}^{N-1}$ otherwise.

In order to show that amplitude cohomology is an instance of an Ext, we need to have functors sending short exact sequences of positive N-complexes into long exact sequences: this will enable to use the axiomatic characterization of Ext. For this purpose we recall the following standard consideration about N-complexes which provides several classical 2-complexes associated to a given N-complex, by contraction. More precisely fix an integer e as an initial condition and an amplitude of contraction a (which provides also a coamplitude of contraction $b = N - a$).

The contraction $C_{e,a}M$ of an N-complex is the following 2-complex, which has $_eM$ in degree 0 and alternating a^{th} and b^{th} composition differentials:

$$\cdots \to {}_{e-b}M \xrightarrow{d^b} {}_eM \xrightarrow{d^a} {}_{e+a}M \xrightarrow{d^b} {}_{e+\mathsf{N}}M \xrightarrow{d^a} \cdots.$$

Of course the usual cohomology of this complex provides amplitude cohomology:

Lemma 4.2. *In the above situation,*

$$H^{2i}(C_{e,a}M) = (AH)_a^{e+iN}(M) \quad \text{and} \quad H^{2i+1}(C_{e,a}M) = (AH)_b^{e+iN+a}(M).$$

Notice that in order to avoid repetitions and in order to set H^0 as the first positive degree amplitude cohomology, we must restrict the range of the initial condition. More precisely, for a given amplitude contraction a the initial condition e verifies $0 \le e < b$, where b is the coamplitude verifying $a + b = \mathsf{N}$. Indeed, if $e \ge b$, set $e' = e - b$ and $a' = b$. Then $b' = a$ and $0 \le e' < b'$.

Remark 4.3. An exact sequence of N-complexes provides an exact sequence of contracted complexes at any initial condition e and any amplitude a.

We focus now on the functor $H^*(C_{e,a}-)$, which for simplicity we shall denote $H^*_{e,a}$ from now on. We already know that $H^*_{e,a}$ sends a short exact sequence of N-complexes into a long exact sequence, since $H^*_{e,a}$ is the usual cohomology. Our next purpose is twofold. First we assert that $H^*_{e,a}$ vanishes in positive degrees when evaluated on injectives of the category of positive N-complexes. Then we will show that it is representable in degree 0.

Proposition 4.4. *In positive degrees we have:*

$$H^*_{e,a}(M_0^l) = 0 \text{ for } l \le \mathsf{N} - 1, \quad \text{and} \quad H^*_{e,a}(M_i^{\mathsf{N}-1}) = 0 \text{ for } i \ge 1.$$

Proof. Concerning indecomposable modules of length $\mathsf{N} - 1$, they are already injective in the entire category of N-complexes. We have noticed that all their amplitude cohomologies vanish.

Consider now M_0^l, with $l \le \mathsf{N} - 1$. In non-zero even degree $2i$ the amplitude cohomology to be considered is in degree $e + iN$, which is larger than l since $i \ne 0$ and $\mathsf{N} > l$. Hence $H^{2i}_{e,a}(M_0^l) = 0$.

In odd degree $2i+1$ the amplitude cohomology to be considered is in degree $e+iN+a$. As before, in case $i \ne 0$ this degree is larger than l, then $H^{2i+1}_{e,a}(M_0^l) = 0$ for $i \ne 0$. It remains to consider the case $i = 0$, namely $H^1_{e,a}(M_0^l) = (AH)_{\mathsf{N}-a}^{e+a}(M_0^l)$. From the picture we have drawn for amplitude cohomology in the previous section, we infer that in degree $e + a$ the cohomology is not zero only for amplitudes inside the closed interval $[l + 1 - (e + a), \ \mathsf{N} - 1 - (e + a)]$. We are concerned by the amplitude $\mathsf{N} - a$ which is larger than $\mathsf{N} - a - e - 1$, hence $H^1_{e,a}(M_0^l) = 0$. $\square$

Proposition 4.5. *Let $a \in [1, \mathsf{N} - 1]$ be an amplitude and let $e \in [0, \mathsf{N} - 1 - a]$ be an initial condition. Then $H^0_{e,a}(-) = (AH)_a^e(-)$ is a representable functor given by the indecomposable N-complex M_e^{a-1}. More precisely,*

$$(AH)_a^e(X) = \mathsf{Hom}_{\mathcal{L}_{\mathsf{N}}^{\ge 0}}(M_e^{a-1}, X).$$

Proof. We will verify this formula for an arbitrary indecomposable positive N-complex $X = M_i^l$. The morphism spaces between indecomposable N-complexes are easy to determine using diagrams through the defining quiver of $\mathcal{L}_N^{\geq 0}$. Non-zero morphisms from an indecomposable M to an indecomposable M' exist if and only if M starts during M' and M ends together with or after M'. Then we have:

$$\mathsf{Hom}_{\mathcal{L}_N^{\geq 0}}(M_e^{a-1}, M_i^l) = \begin{cases} k & \text{if } e \in [i, i+l] \text{ and } e+a-1 \geq i+l \\ 0 & \text{otherwise} \end{cases}.$$

Considering amplitude cohomology and the fundamental computation we have made, we first notice that $(AH)_a^e(M_i^l)$ has a chance to be non-zero only when the degree e belongs to the indecomposable, namely $e \in [i, i+l]$. This situation already coincides with the first condition for non-vanishing of Hom. Next, for a given e as before, the precise conditions that the amplitude a must verify in order to obtain k as amplitude cohomology is

$$(l+1) - (e-i) \leq a \leq (N-1) - (e-i).$$

The second inequality holds since the initial condition e belongs to $[0, N-1-a]$ and $i \geq 0$. The first inequality is precisely $e+a-1 \geq i+l$. $\square$

As we wrote before it is well known (see for instance [21]) that a functor sending naturally short exact sequences into long exact sequences, vanishing on injectives and being representable in degree 0 is isomorphic to the corresponding Ext functor. Then we have the following:

Theorem 4.6. *Let $\mathcal{L}_N^{\geq 0}$ be the category of positive N-complexes of finite dimensional vector spaces and let $AH_a^j(M)$ be the amplitude cohomology of an N-module M with amplitude a in degree j. Let $b = N - a$ be the coamplitude.*
Let $j = qN + e$ be the euclidean division with $0 \leq e \leq N - 1$.
Then for $e < b$ we have:

$$AH_a^j(M) = \mathsf{Ext}^{2q}_{\mathcal{L}_N^{\geq 0}}(M_e^{a-1}, M),$$

and for $e \geq b$ we have:

$$AH_a^j(M) = \mathsf{Ext}^{2q+1}_{\mathcal{L}_N^{\geq 0}}(M_{e-b}^{b-1}, M).$$

5. Monoidal Structure and Clebsch-Gordan Formula

The k-category $\mathcal{L}_N$ is the universal cover of the associative algebra $\mathcal{U}_q^+(sl_2)$, where q is a non-trivial N^{th} root of unity, see [5] and also [7]. More precisely, let $C = <t>$ be the infinite cyclic group and let C act on $(\mathcal{L}_N)_0 = \mathbb{Z}$ by $t.i = i + N$. This is a free action on the objects while the action on morphisms is obtained by translation: namely the action of t on the generator of $_{i+1}V_i$ is the generator of $_{i+1+N}V_{i+N}$.

Since the action of C is free on the objects, the categorical quotient exists, see for instance [6]. The category $\mathcal{L}_N/C$ has a set of objects $\mathbb{Z}/N$. This category $\mathcal{L}_N/C$ has a finite number of objects, hence we may consider its matrix algebra $a(\mathcal{L}_N/C)$ obtained as

the direct sum of all its morphism spaces equipped with matrix multiplication. In other words, $a(\mathcal{L}_N/C)$ is the path algebra of the crown quiver having $\mathbb{Z}/N$ as a set of vertices and an arrow from $\bar{i}$ to $\overline{i+1}$ for each $\bar{i} \in \mathbb{Z}/N$, truncated by the two-sided ideal of paths of length greater or equal to N.

As described in [5] this truncated path algebra bears a comultiplication, an antipode and a counit providing a Hopf algebra isomorphic to the Taft algebra, also known as the positive part $\mathcal{U}_q^+(sl_2)$ of the quantum group $\mathcal{U}_q(sl_2)$. The monoidal structure obtained for the $\mathcal{U}_q^+(sl_2)$-modules can be lifted to $\mathcal{L}_N$-modules providing the monoidal structure on N-complexes introduced by M. Kapranov [19] and studied by J. Bichon [3] and A. Tikaradze [27].

We recall the formula: let M and M' be N-complexes. Then $M \otimes M'$ is the N-complex given by

$$_i(M \otimes M') = \bigoplus_{j+r=i} (_jM \otimes _rM')$$

and

$$d_i(m_j \otimes m'_r) = m_j \otimes d_r m'_r + q^r d_j m_j \otimes m'_r.$$

Notice that in general $_i(M \otimes M')$ is not finite dimensional.

Proposition 5.1. *Let M and M' be N-complexes of finite dimensional vector spaces. Then $M \otimes M'$ is a direct sum of indecomposable N-complexes of finite dimensional vector spaces, each indecomposable appearing a finite number of times.*

Proof. Using the Krull-Schmidt Theorem we have

$$M = \bigoplus_{i\in\mathbb{Z},\ 0\leq l\leq N-1} n_i^l M_i^l \quad \text{and} \quad M' = \bigoplus_{i\in\mathbb{Z},\ 0\leq l\leq N-1} n_i^{\prime l} M_i^l.$$

The tensor product $M_i^l \otimes M_j^r$ consists of a finite number of non-zero vector spaces which are finite dimensional. It follows from the Clebsch-Gordan formula that we prove below that for a given indecomposable N-complex M_l^u, there is only a finite number of couples of indecomposable modules sharing M_l^u as an indecomposable factor. Then each indecomposable module appears a finite number of times in $M \otimes M'$. $\square$

The following result is a Clebsch-Gordan formula for indecomposable N-complexes, see also the work by R. Boltje, Chap. III [4]. The fusion rules, *i.e.* the positive coefficients arising from the decomposition of the tensor product of two indecomposables, can be determined as follows.

Theorem 5.2. *Let q be a non-trivial N^{th} root of unity and M_i^u and let M_j^v be indecomposable N-complexes. Then,*
if $u + v \leq N - 1$ we have

$$M_i^u \otimes M_j^v = \bigoplus_{l=0}^{min(u,v)} M_{i+j+l}^{u+v-2l},$$

if $u + v = e + N - 1$ with $e \geq 0$ we have

$$M_i^u \otimes M_j^v = \bigoplus_{l=0}^{e} M_{i+j+l}^{N-1} \oplus \bigoplus_{l=e+1}^{min(u,v)} M_{i+j+l}^{u+v-2l}.$$

Proof. Using Gunnlaugsdottir's axiomatization [18], p.188, it is enough to prove the following:

$$M_i^0 \otimes M_j^0 = M_{i+j}^0,$$
$$M_0^1 \otimes M_j^u = M_j^{u+1} \oplus M_{j+1}^{u-1} \text{ for } u < \mathsf{N} - 1,$$
$$M_0^1 \otimes M_j^{\mathsf{N}-1} = M_j^{\mathsf{N}-1} \oplus M_{j+1}^{\mathsf{N}-1}.$$

The first fact is trivial. The second can be worked out using amplitude cohomology, which characterizes truncated N-complexes. Indeed the algorithm we have described in 3.8 enables us first to determine the fusion rule for $M_0^1 \otimes M_j^u$ ($u < \mathsf{N} - 1$), that is to determine the non-projective indecomposable direct summands. More precisely, since $u < \mathsf{N} - 1$, the smallest non-vanishing amplitude cohomology degree is j, with smallest amplitude $u + 2$, providing M_j^{u+1} as a direct factor. The remaining amplitude cohomology corresponds to M_{j+1}^{u-1}. A dimension computation shows that in this case there are no remaining projective summands.

On the converse, the third case is an example of vanishing cohomology. In fact, since $M_j^{\mathsf{N}-1}$ is projective, it is known at the Hopf algebra level that $X \otimes M_j^{\mathsf{N}-1}$ is projective. A direct dimension computation between projectives shows that the formula holds. □

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