Memory Effects in Superfluid Vortex Dynamics

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The dissipative dynamics of a vortex line in a superfluid is investigated within the frame of a non-Markovian quantal Brownian motion model. Our starting point is a recently proposed interaction Hamiltonian between the vortex and the superfluid quasiparticle excitations, which is generalized to incorporate the effect of scattering from fermion impurities (^3He atoms). Thus, a non-Markovian equation of motion for the mean value of the vortex position operator is derived within a weak-coupling approximation. Such an equation is shown to yield, in the Markovian and elastic scattering limits, a ^3He contribution to the longitudinal friction coefficient equivalent to that arising from the Rayfield–Reif formula. Simultaneous Markov and elastic scattering limits are found, however, to be incompatible, since an unexpected breakdown of the Markovian approximation is detected at low cyclotron frequencies. Then, a non-Markovian expression for the longitudinal friction coefficient is derived and computed as a function of temperature and ^3He concentration. Such calculations show that cyclotron frequencies within the range 0.01–0.03 ps^{-1} yield a very good agreement to the longitudinal friction figures computed from the Iordanskii and Rayfield–Reif formulas for pure ^4He, up to temperatures near 1 K. A similar performance is found for nonvanishing ^3He concentrations, where the comparison is also shown to be very favourable with respect to the available experimental data. Memory effects are shown to be weak and increasing with temperature and concentration.

KEY WORDS: Vortex dynamics; mutual friction; ^{3}He–^{4}He mixtures.

1. INTRODUCTION

It is widely accepted that the superfluid vortex dynamics at zero temperature is ruled by the Magnus force:[1]

$$m_v \dot{\mathbf{v}} = \rho_s \kappa \hat{\mathbf{z}} \times (\mathbf{v} - \mathbf{v}_s). \tag{1}$$

Here we are assuming a straight vortex line parallel to the z axis, moving with a velocity $\mathbf{v}$. $\mathbf{v}_s$ denotes a uniform background superfluid velocity which, in the simplest case, may be assumed to be time-independent and then dropped from Eq. (1), if $\mathbf{v}$ is reinterpreted by Galilean invariance as a vortex velocity relative to a background superfluid at rest. ρ_s denotes the superfluid mass density and κ the quantized circulation of the vortex velocity field (*e.g.*, $\kappa = h/m_4$ for one quantum of counterclockwise circulation, being m_4 the mass of a ^{4}He atom and h the Planck's constant). Then, the right-hand side of Eq. (1) represents the Magnus force per unit length acting on the vortex and, accordingly, m_v on the left-hand side represents an effective vortex mass per unit length. However, there is no consensus in the literature as regards the value of m_v. Most of the treatments so far, have neglected m_v by assuming that it must be equivalent to the hydrodynamic mass of a core of atomic dimensions.[1] Then, from Eq. (1) we get the well-known law of zero temperature vortex dynamics, which states that a vortex must move at the velocity which the superfluid possesses at the location of the vortex itself.[2] On the other hand, more recent theories[3–5] claim that the vortex mass should not be ignored, since it is shown to be logarithmically divergent with the system size, thus exceeding by far the core mass. Such a large mass, however, can be shown to be consistent with the above law of vortex dynamics, if the dissipative mechanisms acting at zero temperature are taken into account. In fact, one must consider the effect of the vortex coupling to the superfluid which should give rise to dissipation in the form of phonon emission, in close analogy to the photon radiation mechanism stemming from an accelerated charge in electrodynamics.[4,6] Another dissipative mechanism should arise in ordinary helium from the vortex coupling to the Fermi gas of ^{3}He impurities. Whatever the case, it is remarkable that even for an unbounded system leading to a divergent vortex mass, such dissipative mechanisms should make the vortex reach the superfluid velocity, in accordance with the fundamental law of zero temperature vortex dynamics (see Section 4.3).

At nonvanishing temperatures, in addition to phonon radiation and ^{3}He scattering, there exists a third dissipative mechanism stemming from the vortex scattering of superfluid quasiparticle excitations (phonons and/or rotons).

Now, let us return to Eq. (1) and note that for a background superfluid at rest ($\mathbf{v}_s = 0$), the vortex dynamics turns out to be identical to the two-dimensional one of an electron moving in a uniform magnetic field subjected to the Lorentz force. Then, expressing the two-dimensional vector $\mathbf{v}$ in complex notation as $V = v_x + i v_y$, we can rewrite Eq. (1) in a more

compact form

$$\dot{V} = i\Omega V, \tag{2}$$

which clearly shows that the vortex will move in a circle at the angular frequency (cyclotron frequency)

$$\Omega = \frac{\rho_s \kappa}{m_v}. \tag{3}$$

The above mechanisms of dissipation lead to a complex shift of Ω, according to which (2) becomes

$$\dot{V} = (i\Omega_{\text{eff}} - \nu_d)V, \tag{4}$$

where Ω_{eff} denotes the effective angular frequency into which the unperturbed cyclotron frequency is shifted, and $\nu_d > 0$ represents a damping frequency that sets the time scale at which V tends to zero. The above equation, however, can be written in a more familiar form if we return to the vector notation of Eq. (1):

$$m_v \dot{\mathbf{v}} = (\rho_s \kappa - D')\hat{\mathbf{z}} \times \mathbf{v} - D\mathbf{v}, \tag{5}$$

where

$$D' = \rho_s \kappa (1 - \Omega_{\text{eff}}/\Omega), \tag{6}$$

$$D = \rho_s \kappa \nu_d / \Omega, \tag{7}$$

respectively denote *transverse* and *longitudinal friction* coefficients,[1,7] and we have assumed that the normal fluid remains at rest. Actually, a vortex in motion may drag the normal fluid in its vicinity, but this effect should be negligible below 1 K and we shall restrict our study to such a situation.[7]

At this point, it is important, to notice that the cyclotron motion represented by Eq. (2) is also characteristic of the helical waves on vortex lines and rings, usually known as Kelvin waves.[1,8,9] In fact, each vortex line element in such waves executes motion about the undisturbed line in a circle of radius d and with a frequency ω, which approximately fulfil

$$m_v \omega^2 d = \rho_s \kappa v_i + \rho_s \kappa \omega d, \tag{8}$$

where the amplitude of the deformation d is assumed to be much less than the wavelength λ. The above equation corresponds to the centripetal component of an expression of the form Eq. (1), where now $\mathbf{v}_s = -v_i \hat{\theta}$ denotes the local self-induced velocity,[10] which points in a direction opposite to

the one of the superfluid velocity field generated by the undisturbed vortex line. Therefore, the line velocity in Eq. (1) may be expressed as $\mathbf{v} = \omega d\hat{\theta}$, where the frequency ω will be nonpositive if $\mathbf{v}$ points in the same sense as the self-induced velocity. In fact, the solution of the quadratic Eq. (8) yields two frequency branches of opposite sign:

$$\omega_\pm = \frac{\rho_s \kappa}{2m_v} \left[1 \pm \sqrt{1 + \frac{4v_i m_v}{\rho_s \kappa d}} \; \right], \tag{9}$$

whose physical meaning can be easily understood in the limit of long wavelengths $(v_i m_v / (\rho_s \kappa d) \ll 1, v_i \sim \kappa d / \lambda^2)$. That is, the positive fast branch $\omega_+ \simeq \Omega$ corresponds to the cyclotron motion previously described, while the negative slow branch $\omega_- \simeq -v_i / d \sim -\kappa/\lambda^2$ corresponds to the motion of the vortex element with its local, self-induced velocity. Then we may see that for a massless vortex line only the slow branch would exist, this being the common assumption in the literature of helium vortex waves. As regards experimental studies, only the slow branch has been detected by means of a resonant coupling to transverse radio-frequency electric fields acting on vortex lines charged with ions.[11] On the other hand, the thermal excitation of Kelvin waves has been theoretically investigated with rather surprising results.[8] In fact, it was found that the entropy of such waves increases above temperatures about 1.85 K, so that the free energy of the vortices is driven negative, with the consequence that superfluidity would be destroyed. This phenomenon has been called the "free energy catastrophe" and the authors suggest that it could arise from their neglect of the effect of the vortex line on the neighboring phonons and rotons in the system. Now, given that the major contribution to the free energy comes from the slow branch, such a "catastrophe" would apparently be shifted toward temperatures above the lambda transition if only the excitation of the cyclotron branch were taken into account. Actually, this has implicitly been assumed by most of the studies on thermal excitations of vortices through phonon and roton scattering, since the authors have only considered straight vortex lines. In particular, the phonon scattering excitation of the slow branch was analyzed by Fetter[12] and Sonin,[13] who concluded that it yields only a small correction to the friction force calculated for a rectilinear vortex. In fact, the former restriction to considering only straight vortex lines in calculations of the friction coefficients, arises naturally if we accept the *basic premise* that it is only the relative motion of an *element* of line with respect to the normal fluid that matters in such type of calculations. That is, any relative motion of a vortex line element should be subjected to the same kind of friction force per unit length, i.e., the same value of the friction coefficients D and D'. Following these

considerations, we have focused our calculations on the simplest situation of a damped cyclotron motion of a straight vortex line.

There are, to our knowledge, no experimental results on the transverse friction coefficient below 1.3 K. On the other hand, as regards the longitudinal friction coefficient, we must refer to the pioneering experiments performed by Rayfield and Reif (R–R) in the early sixties.[14] In fact, they studied the temperature dependence of the rate of energy loss of charge-carrying vortex rings moving through helium II. The radii of such rings are large(> 500 Å) compared to the distance over which a vortex line is expected to interact appreciably with a quasiparticle. Hence the frictional forces on these vortex rings must be the same as those on vortex lines bent into circles. So, R–R, were able to measure what they called the *attenuation coefficient* α, which turns out to be simply proportional to the longitudinal friction coefficient ($\alpha = \kappa D/2$). Here it is important to notice that the energy losses in the R–R experiment are consistent with a friction owing to the axial displacement of rings, i.e., the main relative motion of each line element with respect to the normal fluid will not correspond to the cyclotron motion. We shall see, however, that in accordance with the above basic premise, the longitudinal friction coefficient arising from our theory shows an excellent agreement with the one arising from the R–R attenuation coefficient α. We notice also that for a cyclotron motion, the radiation damping should be at least comparable to the scattering one for temperatures below 1 K (see Section 4.3), even though we shall focus exclusively on the scattering processes, since phonon emission is supposed to be negligible for the axial displacements in the R–R experiment.

Using kinetic-theory arguments, R–R showed that α comprises three terms, one for each class of quasiparticle interacting with the vortex, namely phonons, rotons and ^{3}He impurities. Each of such contributions was shown to be proportional to a corresponding averaged cross section over all momenta, and R–R could determine by fitting to their experimental results, that the roton and ^{3}He cross sections are approximately temperature-independent, with respective values 9.5 and 18.3 Å. The R–R experiments were performed in the range of temperatures between 0.28 and 0.7 K, and ^{3}He concentrations between 1.4×10^{-7} (ordinary helium) and 2.84×10^{-5}. Then, at the lowest temperatures and highest ^{3}He concentrations, only the ^{3}He contribution to α is appreciable, allowing its separate study. Analogously, in the opposite limit of high temperatures and low ^{3}He concentrations, only the roton contribution to α survives. Unfortunately, only scant information as regards the phonon contribution to α could be derived from such experiments, since even though phonon scattering is dominant at the lowest temperatures in pure ^{4}He, the scattering from ^{3}He impurities becomes the most important contribution in

ordinary helium. R–R employed Pitaevskii's[15] calculation of the phonon-scattering cross section to evaluate the phonon contribution to α, but soon after the publication of R–R's paper, Iordanskii[16] reported an improved theory of the frictional force due to phonons, which seems to be so far the most reliable one. Both, Iordanskii's theory and the above kinetic-theory analysis of R–R are based upon an *elastic scattering assumption*, by which the energy of any quasiparticle that collides with the vortex is conserved after the collision. This amounts to ignoring any energy the vortex could exchange in such a process, in particular the cyclotron energy quantum $\hbar\Omega$, which then should be negligible with respect to the energy of any quasiparticle colliding with the vortex. In conclusion, one should expect a cyclotron frequency of finite value, most likely compatible with an elastic scattering approximation. Such a hypothesis, has been recently put forward in Ref. 17 (henceforth to be designated as *I*), where we have studied the friction arising from the scattering of superfluid quasiparticle excitations in the form of a translationally invariant interaction potential. Then, the first order expansion, in the vortex velocity of such a potential was shown to yield vortex transitions between nearest Landau levels, mediated by one-quasiparticle transitions. Thus, in the frame of such a model of quantal Brownian motion for the vortex dynamics, the longitudinal friction coefficient was computed by making use of weak-coupling and Markov approximations. The result was shown to be equivalent, in the limit of elastic scattering, to that arising from the Iordanskii formula and, proposing a simple functional form for the scattering amplitude, with a single adjustable parameter whose value was set to get agreement to the Iordanskii result for phonons, an excellent agreement with experimental data was found, up to temperatures about $1.5\,\mathrm{K}$. Finite values of the cyclotron frequency of order $0.01\,\mathrm{ps}^{-1}$ were also shown to yield practically the same results.

In the present article, we pursue such an investigation in order to analyze the incidence of vortex-^{3}He scattering, which, as mentioned for ordinary helium, turns out to be the most important contribution to the friction at low temperatures. But more importantly, we report an unexpected breakdown of the Markov approximation at low cyclotron frequencies, unnoticed in previous treatments within the elastic scattering limit. Actually, the interaction of the vortex with the remaining degrees of freedom of helium, leads to integrodifferential equations of motion for the vortex observables, according to which the present vortex motion turns out to be influenced by its whole previous history. In the Markov approximation, such a memory is assumed to be negligible and the vortex equations of motion are approximated by differential equations like (4).[18] We shall show that for low enough cyclotron frequencies,

the Markov approximation fails and non-Markovian or memory effects must be taken into account. Such effects can be of importance in diverse quantum Brownian motion problems[19–22] and, particularly, in physical situations which involve Brownian models of the dynamics of charged particles. For example, a fully non-Markovian reformulation of the Abraham-Lorentz theory of radiation reaction in electrodynamics, has been shown to lead to the elimination of "runaway solutions" and causality violations occurring in the original theory.[23] In transport theory, the phenomenological Drude–Lorentz result for the ac conductivity has been shown to be affected by important memory effects, especially away from resonance[24] and, in the context of two-dimensional magnetotransport, the classical magnetoresistance appears as a consequence of memory effects which are beyond the Boltzmann–Drude approach.[25,26] It may be useful to expand on the last problem, since it corresponds just to a classical two-dimensional Brownian motion of an electron, subjected to a uniform magnetic field perpendicular to the plane. In fact, the electron is supposed to move through a random array of stationary scatterers (background impurities) with short range forces, and there are memory effects of two types: (i) the electron may recollide with the same impurity, or (ii) its trajectory may repeatedly pass through a space region which is free of impurities. It has recently been shown that backscattering processes of the type (ii) are responsible, at low cyclotron frequencies, of additional memory effects leading to unexpected features of the magnetoresistance.[26] Even though there are obviously important differences with the vortex problem, it is instructive to compare with this simpler problem, where the source of memory effects at low cyclotron frequencies has been fully recognized.

Microscopic approaches to quantal Brownian motion also show that memory effects are often important when the weak-coupling approximation becomes poorer.[19] This point will be analyzed for our model in Section 4.2.

This paper is organized as follows, in the next section, starting from a straightforward generalization to include ^{3}He of the Hamiltonian utilized in I, a non-Markovian equation of motion for the vortex dynamics is derived within a weak-coupling approximation. Next, we analyze the Markov approximation and the limit of elastic scattering, showing that under such approximations, the longitudinal friction coefficient stemming from ^{3}He scattering, can be shown to be equivalent to that arising from, the corresponding R–R formula. In Section 3 we analyze the breakdown of the Markov approximation at low cyclotron frequencies and develop a non-Markovian treatment, from which expressions for the longitudinal and transverse friction coefficients are derived. In Section 4 we focus on the simpler case of a pure ^{4}He system. We compare in Section 4.1 our

results for the longitudinal friction with the Iordanskii (phonon) plus the R–R (roton) results, finding a very good agreement within the cyclotron frequency range 0.01–0.03 ps^{-1}, up to temperatures near 1 K. In Section 4.2 we study the frequency ratio $\Omega_{\text{eff}}/\Omega$, which provides a measure of the memory introduced into the vortex dynamics. We explain also the difficulties involved in the calculation of the transverse friction coefficient, due to which only its order of magnitude could be estimated. In Section 4.3 following the theory developed by Arovas and Freire,[4] we discuss the memory effects related to phonon radiation at zero temperature. Finally, Sec. 5 deals with dilute solutions of ^{3}He in ^{4}He, where we compare our results to the available experimental data, and extend our study of the memory parameter $\Omega_{\text{eff}}/\Omega$ in the presence of ^{3}He.

2. VORTEX EQUATION OF MOTION, MARKOV APPROXIMATION AND THE LIMIT OF ELASTIC SCATTERING

Our starting point is the following Hamiltonian, which arises as a straightforward generalization of the Hamiltonian proposed in I, in order to take into account the presence of ^{3}He impurities:

$$H = H_0 + H_{int}, \tag{10}$$

where

$$H_0 = \hbar\Omega\left(a^\dagger a + \frac{1}{2}\right) + \sum_{\mathbf{k}} \hbar\omega_k b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{q},\sigma} \epsilon_q c_{\mathbf{q},\sigma}^\dagger c_{\mathbf{q},\sigma}, \tag{11}$$

and

$$H_{int} = \frac{2i}{\Omega} \sum_{\mathbf{k},\mathbf{q},\sigma} \delta_{k_z q_z}[\Lambda(k,q)b_{\mathbf{k}}^\dagger b_{\mathbf{q}} + \Gamma(k,q)c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{q},\sigma}](\mathbf{k} - \mathbf{q}) \times \hat{\mathbf{z}} \cdot \mathbf{v}. \tag{12}$$

H_0 gives the noninteracting part of the Hamiltonian and it comprises three terms, the first of which corresponds to the cyclotron motion of the vortex line, the second one to helium II excitations, and the last one to ^{3}He quasiparticles, i.e., $a^\dagger$, $b_{\mathbf{k}}^\dagger$ and $c_{\mathbf{q},\sigma}^\dagger$ respectively denote, a creation operator of right circular quanta, a creation operator of helium II quasiparticle excitations of momentum $\hbar\mathbf{k}$ and frequency ω_k and a creation operator of ^{3}He quasiparticles of momentum $\hbar\mathbf{q}$, energy ϵ_q and spin 1/2 projection σ. The interaction Hamiltonian H_{int} arises as a straightforward generalization of the form given in I, to include the effect of vortex-^{3}He scattering processes. In fact, if we replace in Eq. (12) the vortex velocity operator $\mathbf{v}$ as a linear combination of creation and annihilation operators of right circular quanta,[17] it becomes clear that the interaction

consists of vortex-quasiparticle scattering events that make the vortex to raise or lower one Landau level. Then, in addition to the scattering amplitude $\Lambda(k, q)$ related to the vortex interactions with phonons and rotons discussed in *I*, now we are including a scattering amplitude $\Gamma(k, q)$, which takes into account vortex-^{3}He interactions.

In previous works[27–29] we derived, by means of a standard reduction-projection procedure and a weak-coupling Markov approximation, a generalized master equation for the density operator of the vortex. Our aim was to obtain an equation of motion for the mean value of the complex vortex position operator $R = x + iy$. Now we are interested in re-deriving such an equation of motion from the more general Hamiltonian Eqs (10–12). This time we have employed a simpler and more direct procedure (see the Appendix), which leads to the following integrodifferential equation of motion for $v(t) \equiv e^{-i\Omega t} \langle \dot{R}(t) \rangle$:

$$\dot{v}(t) + \int_0^t d\tau \mathcal{D}(\tau) v(t - \tau) = 0, \tag{13}$$

where

$$\mathcal{D}(\tau) = \frac{1}{\hbar^2 \pi \Omega \rho_s L / m_4} \sum_{\mathbf{k}, \mathbf{q}} \delta_{k_z q_z} (\mathbf{k} - \mathbf{q})^2 \Big[|\Lambda(k, q)|^2 (\omega_k - \omega_q)(n_q - n_k)$$
$$\times e^{i(\omega_k - \omega_q - \Omega)\tau} + \frac{2}{\hbar} |\Gamma(k, q)|^2 (\epsilon_k - \epsilon_q)(f_q - f_k) \, e^{i(\epsilon_k/\hbar - \epsilon_q/\hbar - \Omega)\tau} \Big], \tag{14}$$

being $n_k = [\exp(\hbar\omega_k/k_B T) - 1]^{-1}$ and $f_k = \{\exp[(\epsilon_k - \mu)/k_B T] + 1\}^{-1}$ the thermal equilibrium Bose and Fermi occupation numbers for the corresponding quasiparticle excitations, respectively.

The dynamics behind Eq. (13) can be understood by noting that the present vortex motion is actually influenced by its whole previous history, each time being weighted by a memory kernel $\mathcal{D}(\tau)$ such that $\tau = 0$ weights the present time, while $\tau = t$ weights the initial condition. Then, if $\mathcal{D}(\tau)$ possesses a microscopic lifetime τ_m compared to the characteristic times that rule the motion of $v(t)$, only the present time t will have a non-negligible influence upon the vortex motion, provided $t \gg \tau_m$. This constitutes the so-called *Markov* or *long time, limit* approximation,[18–22] under which Eq. (13) becomes a differential equation:

$$\dot{v}(t) + \nu v(t) = 0, \tag{15}$$

where

$$\nu = \int_0^\infty d\tau \mathcal{D}(\tau). \tag{16}$$

Recalling that $\langle \dot{R}(t) \rangle = e^{i\Omega t} v(t)$, we may realize that the imaginary part of v yields the shift of the cyclotron frequency previously mentioned in Eq. (4), i.e., $\Omega_{\text{eff}} = \Omega - \text{Im}\,v$, while the real part, which must be nonnegative, corresponds to the damping frequency v_d defined in the same equation. Then, the transverse and longitudinal friction coefficients arise from Eqs. (6) and (7) as

$$D'_M = (\rho_s \kappa / \Omega) \, \text{Im}\, v, \tag{17}$$

$$D_M = (\rho_s \kappa / \Omega) \, \text{Re}\, v, \tag{18}$$

where the subscript M indicates Markov approximation. The real and imaginary parts of the frequency v, when considered as functions of Ω, obey Kramers-Krönig relations[30] which lead to the following expression for the transverse friction coefficient:

$$D'_M(\Omega) = \frac{1}{\pi \Omega} \text{P} \int_{-\infty}^{\infty} d\omega \, \frac{\omega D_M(\omega)}{\omega - \Omega}, \tag{19}$$

where P denotes the Cauchy principal part and,

$$D_M(\omega) = \frac{2\pi}{L\hbar\omega} \sum_{\mathbf{k},\mathbf{q}} \delta_{k_z q_z} (\mathbf{k} - \mathbf{q})^2 \Big[|\Lambda(k,q)|^2 (n_q - n_k) \delta(\omega_k - \omega_q - \omega)$$

$$+ 2|\Gamma(k,q)|^2 (f_q - f_k) \delta(\epsilon_k/\hbar - \epsilon_q/\hbar - \omega) \Big]. \tag{20}$$

The above even function of ω when evaluated at $\omega = \Omega$ gives the longitudinal friction coefficient, and it is easy to verify that $D_M(\Omega) > 0$, since $\omega_k > \omega_q \Rightarrow n_q > n_k$, and the same for the terms containing the fermion occupation numbers. Note that only the scattering events that conserve energy will contribute to the longitudinal friction coefficient (see the arguments of the Dirac deltas in Eq. (20) for $\omega = \Omega$). This consequence of the Markov approximation can be physically understood in terms of the time-energy uncertainty principle. In fact, in the long time limit only the microscopic states with the longest lifetimes are expected to remain with a nonnegligible probability of undergoing a scattering transition, and according to the time-energy uncertainty principle, energy should be practically conserved at the end of such transitions.[18]

We have studied in I the phonon-roton contribution to the longitudinal friction coefficient which arises from the first term inside the square brackets in Eq. (20). We showed that the limit of elastic scattering $\Omega \to 0$ yields an excellent agreement with the values derived from experimental data for the roton temperature range, provided the scattering amplitude Λ is set to get agreement with the Iordanskii results for the low-temperature phonon dominated regime. Let us now examine the ^3He contribution to

the longitudinal friction in Eq. (20) in the limit $\Omega \to 0$. Denoting such, a term by $D_3(\Omega)$, we have

$$
\begin{aligned}
D_3(0) = \lim_{\Omega \to 0} D_3(\Omega) &= -\frac{4\pi\hbar}{L} \sum_{\mathbf{k},\mathbf{q}} \delta_{k_z q_z} (\mathbf{k} - \mathbf{q})^2 |\Gamma(k,q)|^2 \frac{\partial f_k}{\partial \epsilon_k} \delta(\epsilon_k - \epsilon_q) \\
&= -\frac{2A^2\hbar}{(2\pi)^4} \int d^3\mathbf{k} \int d^3\mathbf{q} \, \frac{\partial f_k}{\partial \epsilon_k} \delta(\epsilon_k - \epsilon_q)(\mathbf{k} - \mathbf{q})^2 \delta(k_z - q_z)|\Gamma(q,k)|^2,
\end{aligned}
\tag{21}
$$

where the continuum limit was explicitly considered in the last expression, A being the area of the system in the $x - y$ plane. Here most of the integration can be analytically performed in spherical coordinates, leading to the following one-dimensional integral:

$$
D_3(0) = -\frac{4A^2 m^*}{3\pi^2\hbar} \int_0^\infty dk \, k^4 |\Gamma(k,k)|^2 \frac{\partial f_k}{\partial \epsilon_k},
\tag{22}
$$

where a Landau–Pomeranchuk dispersion relation $\epsilon_k = \hbar^2 k^2 / 2m^*$ was utilized. The above expression can be shown to be equivalent to the R–R formula,[14] provided the scattering amplitude fulfills

$$
|\Gamma(k,k)|^2 = \frac{9\pi\hbar^3}{128m^* A^2} u(k)\sigma(k),
\tag{23}
$$

where $u(k) = \hbar k / m^*$ denotes the group velocity of ^{3}He quasiparticles and $\sigma(k)$ corresponds to a total momentum-transfer cross section for vortex-^{3}He scattering, which, in a low ^{3}He concentration regime,[14] can be approximated by the constant value $\sigma_0 = (18.3 \pm 0.7)$ Å.

3. BREAKDOWN OF THE MARKOV APPROXIMATION NON-MARKOVIAN TREATMENT

We have shown in *I*, that the finite values of the cyclotron frequency extracted from recent theories, yield values of the longitudinal friction coefficient of the order of that obtained in the elastic limit $\Omega \to 0$. We shall henceforth work under such an assumption, i.e., $D_M(\Omega) \sim D_M(0)$. Thus it can be shown that Eq. (19) can be approximated as follows:

$$
D_M'(\Omega) \simeq \frac{2}{\pi\Omega} \int_0^\infty d\omega D_M(\omega),
\tag{24}
$$

where we have also assumed $D_M'(\Omega) \gg D_M(\Omega)$. The quasiparticle frequency cutoff in Eq. (20) (roughly two times the roton frequency) yields the same cutoff to the frequency ω in Eq. (24). This shows that the integral in Eq. (24) possesses a finite value and thus $D_M'(\Omega)$ would diverge

as Ω^{-1} in the limit $\Omega \to 0$. Later we shall see that this unphysical result arises from a breakdown of the Markov approximation. In fact, one could expect the effective frequency Ω_{eff} to be lower than the cyclotron frequency by the effect of friction, but the existence of a critical cyclotron frequency below which the effective frequency becomes negative, seems to be quite unphysical, i.e., one would expect effective frequency values bounded as $0 < \Omega_{\text{eff}} = \Omega - \Omega D'_M(\Omega)/\rho_s \kappa < \Omega$. We shall see in the following that a non-Markovian treatment yields in fact such bounds. To see this, let us return to the integrodifferential equation (13) and take its Laplace transform according to the definition $\tilde{v}(z) = \int_0^\infty \exp(izt)v(t)dt \, (\text{Im} z > 0)$:

$$\tilde{\mathcal{D}}(z)\tilde{v}(z) = iz\tilde{v}(z) + v(0), \tag{25}$$

where the Laplace transform of the memory kernel in Eq. (14) reads as,

$$\tilde{\mathcal{D}}(z) = \frac{i(\Omega - z)}{\rho_s \kappa \Omega \pi} \int_{-\infty}^{\infty} d\omega \frac{\omega D_M(\omega)}{\omega + \Omega - z}. \tag{26}$$

Then from Eq. (25) we have,

$$\tilde{v}(z) = \frac{v(0)}{-iz + \tilde{\mathcal{D}}(z)} \tag{27}$$

and $v(t)$ arises from the singularities of $\tilde{v}(z)$ in the lower half-plane, $\text{Im} z < 0$. For instance, if the expression (27) has a unique simple pole located at $z_0 = -i\tilde{\mathcal{D}}(z_0)$, we get

$$v(t) = v(0)e^{-iz_0 t}, \tag{28}$$

and the Markov approximation would be valid provided $\tilde{\mathcal{D}}(z_0) \simeq \tilde{\mathcal{D}}(0)$ (cf. Eq. 16). That is, taking the limit $z \to i0^+$ in the Cauchy integral of Eq. (26) we get,[31]

$$\tilde{\mathcal{D}}(0) = \frac{i}{\rho_s \kappa \pi} P \int_{-\infty}^{\infty} d\omega \frac{\omega D_M(\omega)}{\omega + \Omega} + \frac{\Omega D_M(\Omega)}{\rho_s \kappa} = \nu. \tag{29}$$

Therefore, from Eq. (26) we may realize that for such an approximation to be valid, it necessarily should be $|z_0| = |\nu| \ll \Omega$, i.e., $\Omega D_M(\Omega)/\rho_s \kappa \ll \Omega$ and $\Omega D'_M(\Omega)/\rho_s \kappa \ll \Omega$. Now, according to the low-cyclotron frequency approximation Eq. (24), the last condition will not be fulfilled for low enough frequencies, that is, the real part of z_0 will remain finite for $\Omega \to 0$. This suggests the following approximation to find the poles of Eq. (27):

$$iz_0 = \tilde{\mathcal{D}}(z_0) \simeq \tilde{\mathcal{D}}(\text{Re} z_0) = \frac{i(\Omega - \text{Re} z_0)}{\rho_s \kappa \pi \Omega} P \int_{-\infty}^{\infty} \frac{\omega D_M(\omega)d\omega}{\omega + \Omega - \text{Re} z_0}$$
$$+ \frac{(\Omega - \text{Re} z_0)^2}{\rho_s \kappa \Omega} D_M(\Omega - \text{Re} z_0), \tag{30}$$

or, equivalently,

$$\mathrm{Im}z_0 = -\frac{(\Omega - \mathrm{Re}z_0)^2}{\rho_s \kappa \Omega} D_M(\Omega - \mathrm{Re}z_0) \tag{31}$$

$$\mathrm{Re}z_0 = \frac{(\Omega - \mathrm{Re}z_0)}{\rho_s \kappa \pi \Omega} \mathrm{P} \int_{-\infty}^{\infty} \frac{\omega D_M(\omega) d\omega}{\omega + \Omega - \mathrm{Re}z_0}$$

$$\simeq \frac{2(\Omega - \mathrm{Re}z_0)}{\rho_s \kappa \pi \Omega} \int_0^{\infty} D_M(\omega) d\omega, \tag{32}$$

where the last equality arises from the approximation Eq. (24), i.e., assuming $D_M(0) \sim D_M(\Omega_{\mathrm{eff}}) \ll \int_0^{\infty} d\omega D_M(\omega)/\Omega_{\mathrm{eff}}$, $(\Omega_{\mathrm{eff}} = \Omega - \mathrm{Re}z_0)$. Then from Eq. (32) we get the solution

$$\Omega_{\mathrm{eff}} = \Omega - \mathrm{Re}z_0 = \Omega/\{1 + [2/(\rho_s \kappa \pi \Omega)] \int_0^{\infty} D_M(\omega) d\omega\}, \tag{33}$$

where, in fact, the effective frequency turns out to be bounded according to our previous discussion, $0 < \Omega_{\mathrm{eff}} < \Omega$. Note also that $\Omega_{\mathrm{eff}} \to 0$ corresponds to the limit of a vanishing cyclotron frequency, as expected. Finally, the friction coefficients $D = -(\rho_s \kappa/\Omega)\mathrm{Im}z_0$ and $D' = (\rho_s \kappa/\Omega)\mathrm{Re}z_0$ reads as,

$$D = (\Omega_{\mathrm{eff}}/\Omega)^2 D_M(\Omega_{\mathrm{eff}}), \tag{34}$$

$$D' = \rho_s \kappa (1 - \Omega_{\mathrm{eff}}/\Omega), \tag{35}$$

which generalize the previous Markovian expressions Eq. (20) and Eq. (24). Here it is expedient to recall that Eqs. (34) and (35) were extracted under the approximations of Eqs. (24) and (30), both being equivalent to $D \ll D'$. If, in addition, we have $D' \ll \rho_s \kappa$, then $\Omega_{\mathrm{eff}} \simeq \Omega$ and Eqs. (34) and (35) tend to the Markovian expressions. In other words, the frequency ratio $\Omega_{\mathrm{eff}}/\Omega$ can be thought of as a measure of the proximity to the Markovian limit. Note that the limit of a vanishing cyclotron frequency is a strongly non-Markovian one, with $D \sim \mathcal{O}(\Omega^2)$ and $\rho_s \kappa - D' \sim \mathcal{O}(\Omega)$. Such a behavior of the transverse coefficient corresponds to the lower limit of $\Omega_{\mathrm{eff}} \sim \mathcal{O}(\Omega)^2$ as given by Eq. (33). As regards the longitudinal coefficient, since the effective frequency was absent from our previous analysis in *I*, the limiting values for elastic scattering $(\Omega \to 0)$ there reported are now drastically changed to vanishing values. However, we shall next see that there is a range of cyclotron frequency values that keep the previous agreement with the experimental values up to temperatures near $1\,\mathrm{K}$.

4. ANALYSIS OF RESULTS FOR A PURE ^{4}He SYSTEM

4.1. Longitudinal Friction Coefficient and the Cyclotron Frequency Range

In Table I we may compare values of the longitudinal friction coefficient computed from Eq. (34) for $\Omega = 0.01 - 0.03\,\text{ps}^{-1}$, with the corresponding values arising from the Iordanskii formula[16] (phonon range), plus the R–R formula[14,32] (roton range). Such calculations were performed taking into account only the phonon-roton contribution in Eq. (20), i.e., for a pure ^{4}He system. In the phonon dominated regime($T < 0.4\,\text{K}$), the lack of experimental data leads us to compare with the results stemming from the Iordanskii formula, i.e., those arising from the limit $\Omega \to 0$ of Eq. (20).[17] Actually, it will be enough to have $\hbar\Omega / k_B T < 1$ in order to keep $D_M(\Omega) \simeq D_M(0)$ (cf. the figures for $T = 0.1\,\text{K}$ in Table I), but too small values of Ω would affect the Markovian approximation $\Omega_{\text{eff}} \simeq \Omega$, leading to appreciable discrepancies between the Iordanskii result, $D_M(0)$, and our expression (34). The phonon influence becomes negligible for $T \gtrsim 0.65\,\text{K}$, allowing the comparison of our results with those arising from the R–R formula,[14] which constitutes a good fit to experimentally derived values up to temperatures about 1.3 K.[32] Taking into account an estimated uncertainty of order 10%, we may see from Table I that our results lie within such error bounds for temperatures up to 0.8 K (0.9 K) for $\Omega = 0.01\,\text{ps}^{-1}(0.03\,\text{ps}^{-1})$. Note that for $T = 1\,\text{K}$, our result for $\Omega = 0.01\,\text{ps}^{-1}(0.03\,\text{ps}^{-1})$ falls 30%(4%) below the lower error

TABLE I

Longitudinal friction coefficient [10^{-6}g cm^{-1}s^{-1}] versus temperature for a pure ^{4}He system. The values in the third and fifth columns were calculated from Eq. (34) and have to be compared with the corresponding values in the second column, which arise from Refs. 14, 16, 32. The values in the third and fourth columns were calculated for $\Omega = 0.01\text{ps}^{-1}$, while the values in the fifth and sixth ones correspond to $\Omega = 0.03\text{ps}^{-1}$. Powers of 10 are enclosed in brackets

T[K]	D_{Refs}	$D_{0.01}$	$\Omega_{\text{eff}}/\Omega$	$D_{0.03}$	$\Omega_{\text{eff}}/\Omega$
0.1	2.61[−8]	2.63[−8]	1.000	2.97[−8]	1.000
0.2	8.34[−7]	8.15[−7]	0.999	8.45[−7]	1.000
0.3	6.33[−6]	5.98[−6]	0.996	6.11[−6]	0.999
0.4	2.84[−5]	2.59[−5]	0.989	2.63[−5]	0.996
0.5	2.06[−4]	2.07[−4]	0.979	1.95[−4]	0.993
0.6	2.43[−3]	2.53[−3]	0.962	2.39[−3]	0.987
0.7	1.79[−2]	1.80[−2]	0.938	1.77[−2]	0.979
0.8	8.27[−2]	7.67[−2]	0.900	8.07[−2]	0.964
0.9	0.274	0.221	0.840	0.256	0.940
1.0	0.714	0.460	0.749	0.618	0.900

boundary. To summarize, we have identified a narrow range of cyclotron frequency values ($0.01\,\mathrm{ps}^{-1} \lesssim \Omega \lesssim 0.03\,\mathrm{ps}^{-1}$) yielding longitudinal friction figures which are in agreement with the Iordanskii formula and experimental data. However, it is important to note that according to recent theories,[3-5] which show that the cyclotron frequency scales down *logarithmically* with the vortex size, such a range of Ω would be consistent with a relatively wide range of macroscopic sizes of the system.

4.2. Effective Frequency, Memory Effects and Transverse Friction Coefficient

Another feature of Table I shows that the frequency ratio $\Omega_{\mathrm{eff}}/\Omega$ decreases with increasing temperature, which reflects a corresponding increase of D' (Eq. 35). Particularly, at the lowest temperatures, the Markov approximation $\Omega_{\mathrm{eff}}/\Omega \simeq 1$ shows to be excellent, becoming gradually less adequate as the temperature increases. The highest temperature range (0.9–1.0 K) displays the largest differences with the Markov approximation, as well as the highest discrepancy with the experimental results. In other words, the longitudinal friction phenomenon appears to be consistent with weakly, at most, non-Markovian processes ($\Omega_{\mathrm{eff}}/\Omega > 0.9$). There is, however, another possible interpretation of such a discrepancy with the experimental results for $\Omega_{\mathrm{eff}}/\Omega < 0.9$, which is related to an eventual failure of the weak-coupling approximation. In fact, according to Eqs. (33) and (20), we may see that for fixed Ω, the parameter $\Omega_{\mathrm{eff}}/\Omega$ will behave as a decreasing function of the coupling strengths Λ and Γ, such that $\Omega_{\mathrm{eff}}/\Omega \rightarrow 1$ for a vanishing coupling ($D_M(\omega) \rightarrow 0$), while $\Omega_{\mathrm{eff}}/\Omega \rightarrow 0$ for an infinite coupling ($D_M(\omega) \rightarrow \infty$). Then, only a higher portion of the interval $0 < \Omega_{\mathrm{eff}}/\Omega < 1$ should be expected to be consistent with a weak-coupling approximation. In conclusion, the above discrepancy with the experimental results for $\Omega_{\mathrm{eff}}/\Omega < 0.9$ may also be regarded as an indication of a possible failure of the weak-coupling approximation.

The transverse friction coefficient, in contrast to the longitudinal one, possesses a strong dependence on Ω. In fact, Eq. (24) shows that in the Markovian limit, the bounds $0.01\,\mathrm{ps}^{-1} < \Omega < 0.03\,\mathrm{ps}^{-1}$ lead to a factor 3 of spreading [$D'_M(0.01\,\mathrm{ps}^{-1}) = 3D'_M(0.03\,\mathrm{ps}^{-1})$], while the non-Markovian figures of Table I can reduce such a factor somewhat (> 2.5). Another important difference between both friction coefficients stems from the degree of dependence on the quasiparticle dispersion relation cutoff features.[33,34] On the one hand, the longitudinal coefficient, which depends mainly on $D_M(\Omega_{\mathrm{eff}})$, turns out to be almost independent of such a cutoff, since Ω_{eff} and Ω are two orders of magnitude lower than the roton frequency. The transverse coefficient, on the other hand, being mainly

dependent on the integral $\int_0^\infty D_M(\omega)d\omega$, has therefore an important dependence on the quasiparticle cutoff through the corresponding dependence of the scattering amplitudes Λ and Γ, which is mostly uncertain. In summary, due to the above uncertainties in the calculation of the transverse coefficient, only its order of magnitude should be reliable which, nevertheless, turns out to be quite useful to ensure that the condition $D \ll D'$ is fulfilled. Recall that such a condition was assumed in the derivation of Eqs. (34) and (35), and in fact, taking into account that, $\rho_s\kappa \simeq 145 \times 10^{-6}\,\mathrm{g\,cm^{-1}\,s^{-1}}$, all the values of Table I can be shown to be consistent with $D \ll D'$. It is worthwhile recalling also the lack of experimental results on D' for temperatures below $1.3\,\mathrm{K}$. Taking into account only the vortex drag due to the scattering of rotons, the transverse coefficient can be written in terms of a transverse scattering length $\sigma_\perp$, viz. $D' = \rho_n v_G \sigma_\perp$, where ρ_n denotes the normal fluid density and v_G the average group velocity of thermal rotons[1,7]. However, only speculative assumptions about the form of $\sigma_\perp$ for temperatures below $1.3\,\mathrm{K}$ were reported.[7] In addition, it has been argued that the so called Iordanskii force[16] gives rise to an additional transverse coefficient to be substracted from D', yielding a total transverse coefficient[1,7] given by $D_t = D' - \rho_n\kappa$. However, the sign, amplitude, and existence of this Iordanskii force are still subject to debate.[35−38] Recently, Fortin[37] has applied the formalism of Thouless, Ao, and Niu[39] to compute the transverse and longitudinal coefficients due to the scattering of noninteracting phonons in two dimensions. He finds a transverse coefficient which turns out to be of opposite sign to ours and to that of Refs. 7, 1, which is interpreted in terms of a negative vortex mass due to phonons. Such a discrepancy in the sign stems from the fact that, according to his equations, the transverse and longitudinal coefficients would be related, as functions of the cyclotron frequency, by Kramers–Krönig relations, while in our case such relations are connecting instead the real and imaginary parts of the Markovian frequency ν (cf. Eq. 19).

4.3. Phonon Emission and Memory Effects at Zero Temperature

At this point, as a useful complement to our study, it will be instructive to discuss in some detail the memory effects related to phonon emission at zero temperature. We will base our analysis on the theory developed by Arovas and Freire[4] for vortex dynamics in superfluid films. In fact, suppose that the vortex is set in motion at positive times by the action of a homogeneous time-dependent superfluid flow, i.e., it is assumed that both, the superfluid velocity $\mathbf{v}_s$ and the vortex velocity $\mathbf{v}$, are zero for negative times. Then, the vortex equation of motion can be written[4]

$$\int_0^t M(\tau)\dot{V}(t-\tau)d\tau = i\rho_s\kappa[V(t)-V_s(t)]. \tag{36}$$

The right-hand side of this equation corresponds to the usual Magnus force $\rho_s\kappa\hat{\mathbf{z}}\times(\mathbf{v}-\mathbf{v_s})$ (cf. Eq. 1) expressed in complex notation ($V = v_x + iv_y$), while the left-hand side will differ from the familiar Newtonian product of mass times acceleration, unless the memory or causal[4] kernel $M(\tau)$ has a negligible lifetime. The memory, which actually plays an important role in this case, stems from the vortex coupling to the low lying excitations of the superfluid (phonons), in close analogy to the retardation and radiation effects stemming from electron-photon coupling in electrodynamics[4]. Now, we focus upon the long time limit of Eq. (36). That is, for $t \gg$ lifetime of $M(\tau)$, the left-hand side could be approximated as $\tilde{M}\dot{V}(t)$, where the effective vortex mass $\tilde{M}$ is given by the Fourier (Laplace) transform of the memory kernel $M(\tau)$ at zero frequency, $\tilde{M} = \int_0^\infty d\tau M(\tau) = M' + iM''$, the imaginary part M'' being related to the dissipation of vortex energy in the form of phonon emission. Then the solution of Eq. (36) in the long time limit could be easily obtained for constant V_s:

$$V(t) = V_s\{1 - \exp[i(\Omega + i\nu_r)t]\}, \tag{37}$$

where the cyclotron Ω and radiation damping ν_r frequencies respectively read as,

$$\Omega = \frac{\rho_s\kappa M'}{M'^2 + M''^2} \tag{38}$$

$$\nu_r = \frac{-\rho_s\kappa M''}{M'^2 + M''^2}. \tag{39}$$

M', however, is shown to diverge for an unbounded two-dimensional system[4] ($M'(\omega \to 0) \sim -\ln\omega$), while for a finite macroscopic system one should expect a corresponding finite value of $M'(\gg |M''|)$, leading to the familiar expression Eq. (3) for the cyclotron frequency in Eq. (38), with $M' = m_v$ the vortex mass per unit length. As regards the radiation damping frequency, from Eq. (14) of Ref. 4 we have $|M''(\omega \to 0)| = \kappa^2\rho_s/(8c_s^2)$, ($c_s =$ sound velocity) and then, $\nu_r = \kappa(\Omega/c_s)^2/8$. This result derives as well from the expression for the mean power radiated by unit length of a vortex performing cyclotron motion (see Eq. 2.11 of Ref. 6). Therefore, the radiation damping should be weak $\nu_r \sim 10^{-3}\Omega$ for our range of cyclotron frequency values ($\Omega \sim 0.01\,\text{ps}^{-1}$), whereas it would be relatively strong, $\nu_r \sim \Omega$, for cyclotron frequencies arising from a hydrodynamical model for the vortex mass,[1] $\Omega \approx 3\text{ps}^{-1}$. It is interesting to notice that a vanishing damping frequency for an infinite system in Eq. (39) does not preclude the approach of the vortex velocity to the superfluid velocity at long times. Actually, it

simply means that such an approach will be slower than the exponential one of Eq. (37). To see this, let us integrate by parts the left-hand side of Eq. (36) getting $M(0)V(t) + \int_0^t \dot{M}(\tau)V(t-\tau)d\tau$. Then, approximating in the long time limit the last integral as $\int_0^t \dot{M}(\tau)V(t)d\tau$, the left-hand side of Eq. (36) turns out to be simply $M(t)V(t)$, from which we get

$$V(t) = V_s \left[\frac{1 - iM(t)/\rho_s\kappa}{1 + (M(t)/\rho_s\kappa)^2} \right], \tag{40}$$

where[4] $M(t)/\rho_s\kappa \simeq \xi/(2c_s t)$, $\xi = \kappa/(2\pi c_s)$ being the coherence length. Thus, the above expression shows that in the case of an infinite system, the approach of the vortex velocity to the superfluid velocity turns out to be, in contrast to Eq. (37), a slow nonexponential one.

5. ANALYSIS OF RESULTS FOR DILUTE SOLUTIONS OF ^{3}He IN ^{4}He

In case of a ^{3}He–^{4}He mixture we have to take into account both terms in the expression (20) for $D_M(\omega)$. The calculation of the fermion term $D_3(\omega)$ (cf. Eq. 21) turns out to be similar to that leading to $D_3(0)$ in Eq. (22), namely

$$D_3(\omega) = \frac{m^* A^2}{\pi^2 \hbar^2 \omega} \int_0^\infty dk |\Gamma(k,q)|^2 (f_k - f_q) k^2 (q^2 + k^2/3), \tag{41}$$

where the value of the momentum q arises from the argument of the second Dirac delta in Eq. (20), i.e., $q^2 = k^2 + 2m^*\omega/\hbar$. Thus, the cutoff of the Landau–Pomeranchuk dispersion relation imposes the same cutoff ($\simeq 1\,\text{ps}^{-1}$, see Ref. 33) to the frequency ω in Eq. (41). Recall that the cutoff uncertainties will be reflected in the evaluation of the transverse friction coefficient, as was already mentioned in Section 4.2. To compute Eq. (41) we utilized the following simple generalization of the expression Eq. (23) for $k \neq q$:

$$|\Gamma(k,q)|^2 = \frac{9\pi}{128} \frac{\hbar^4}{(m^* A)^2} \sigma_0 \sqrt{kq}. \tag{42}$$

In Table II we may compare some experimental results for the longitudinal friction coefficient (third column), with the corresponding results arising from our approach, along with the values computed from the Iordanskii[16] and R–R[14] formulas. Given the low ^{3}He concentrations of Table II, the Fermi temperatures turn out to be at most two orders of magnitude below the experimental ones, and so the Fermi occupation numbers in Eqs. (22) and (41) can be very well approximated by the

Maxwell–Boltzmann statistics. At $T = 0.28$ K the phonon contribution to the friction, stemming from the Iordanskii formula, turns out to be negligible in comparison to the ^{3}He term given by Eqs. (22) and (23) (cf. the values of D_{Refs} in Tables I and II). Then, the experimental value D_{exp} for $T = 0.28$ K and $C = 2.84 \times 10^{-5}$ in Table II (being $C = n_3/(n_3 + n_4)$, $n_i =$ number density of ^{i}He atoms), which was measured within an error less than 1%, was utilized by R–R to calculate the effective cross section $\sigma_0 = 18.3$ Å in Eq. (23), assuming for the effective mass the value $m^* = 2.5m_3$ ($m_3 =$ actual mass of a ^{3}He atom). A similar procedure was followed in our case, since the value of σ_0 in Eq. (42) was set to get agreement with the experimental value $D_{\text{exp}} = 4.69 \times 10^{-3}$ for $\Omega = 0.02$ ps^{-1}, i.e., the center of the cyclotron frequency range discussed in Section 4.1. Then, equating (34) to 4.69×10^{-3}, we extracted the value $\sigma_0 = 18.54$ Å, which turns out to be slightly greater than the R–R result. Although the phonon contribution to D_M in Eq. (34) is in fact negligible for $T = 0.28$ K and $C = 2.84 \times 10^{-5}$, this is not the case for $\Omega_{\text{eff}}/\Omega$ (Eq. 33), since the phonon contribution to the integral $\int_0^\infty D_M(\omega)d\omega$ turns out to be greater than the ^{3}He one. Nevertheless, as seen from Table II, the factor $(\Omega_{\text{eff}}/\Omega)^2$ in Eq. (34) remains close to unity and then, in practice, almost all the longitudinal friction should be ascribed to ^{3}He scattering.

In a second experiment, to prove the proportionality of the friction coefficient to the ^{3}He number density in the dilute limit, R–R performed a measure for the same temperature $T = 0.28$ K and a smaller concentration of $C = 7.55 \times 10^{-6}$. They obtained the value $D_{\text{exp}} = (1.35 \pm 0.06) \times 10^{-3}$, which turns out to be in agreement with their theoretical calculation arising from Eq. (22), within the limits of estimated error. As regards our cal-

TABLE II

Longitudinal friction coefficient [10^{-6} g cm^{-1} s^{-1}] vs. temperature and ^{3}He concentration for mixtures. The third column corresponds to experimental results, while the fourth one corresponds to values arising from Iordanskii formula[16] (phonon contribution) plus R–R formulas[14] (roton+^{3}He contribution). The remaining notation is the same as in Table I.

T[K]	C	D_{exp}	D_{Refs}	$D_{0.01}$	$\Omega_{\text{eff}}/\Omega$	$D_{0.02}$	$\Omega_{\text{eff}}/\Omega$	$D_{0.03}$	$\Omega_{\text{eff}}/\Omega$
0.28	2.84[-5]	4.69[-3]	4.69[-3]	4.69[-3]	0.995	4.69[-3]	0.997	4.64[-3]	0.998
0.28	7.55[-6]	1.35[-3]	1.25[-3]	1.25[-3]	0.996	1.25[-3]	0.998	1.24[-3]	0.999
0.61	7.55[-6]	4.56[-3]	5.01[-3]	4.87[-3]	0.960	4.90[-3]	0.980	4.80[-3]	0.986
0.615	1.4[-7]	3.34[-3]	3.43[-3]	3.56[-3]	0.959	3.52[-3]	0.979	3.39[-3]	0.986
0.643	1.4[-7]	6.04[-3]	6.17[-3]	6.36[-3]	0.953	6.32[-3]	0.976	6-11[-3]	0.984
0.67	1.4[-7]	1.01[-2]	1.04[-2]	1.07[-2]	0.947	1.07[-2]	0.973	1.04[-2]	0.982

culation from Eq. (34), it yields practically the same figures as the R–R formula (Table II).

The diluted sample was also used to verify the additivity of ^{3}He and roton scattering, by performing an experiment at the relatively high temperature of 0.61 K. The directly measured value $D_{exp} = 4.56 \times 10^{-3}$ was then contrasted with that arising from the addition of R–R formulas for ^{3}He and roton scattering contributions, namely 4.79×10^{-3}. The phonon contribution, on the other hand, was ignored, presumably because of some discrepancies arising from Pitaevskii's[15] calculation of the phonon-scattering cross section. Actually, taking into account such a contribution, the value of the friction coefficient should have been increased to 5.15×10^{-3}. Pitaevskii's result was later modified by Iordanskii[16] in that the coefficient of proportionality to T^5 of the friction coefficient due to phonons was shown to be smaller by a factor ~ 0.62. That is, taking into account the phonon contribution stemming from the Iordanskii formula, the corrected value 5.01×10^{-3} (Table II) is in fact closer to the experimental one. Finally, we compare with our results computed from Eq. (34). From Table II we see that such results are closer to the observed one than the previous estimation of Iordanskii+R–R, and in this better agreement it is important to remark on the role played by the memory effect, which is embodied in the factor $(\Omega_{eff}/\Omega)^2 < 1$ in Eq. (34).

Next we analyze a set of measures performed for ordinary helium ($C = 1.4 \times 10^{-7}$) at temperatures of 0.615, 0.643 and 0.67 K. The second measure ($T = 0.643$ K) was reported in Ref. 40, while the remaining two are included in Ref. 14. Under such conditions, the incidence of ^{3}He scattering is almost negligible in both, the Iordanskii+R–R results and our figures computed from Eq. (34). From Table II we see that, analogous to the above results for $T = 0.61$ K, theoretical calculations again overestimate the experimental data, and the best agreement is also obtained for our result at $\Omega = 0.03 \, ps^{-1}$.

Finally, from Table II we notice that the memory effect remains small ($\Omega_{eff}/\Omega \gtrsim 0.95$), showing the same increase with temperature as in Table I. On the other hand, the dependence of Ω_{eff}/Ω on concentration is not clear from Table II except for $T = 0.28$ K, where we find a slight reduction for a higher concentration. It is not difficult, however, to generalize such a result taking into account that in the dilute limit, the Maxwell–Boltzmann approximation to f_k and f_q in Eq. (41) yields a $D_3(\omega)$ proportional to the ^{3}He number density, which in turn implies a growing of $\int_0^\infty D_M(\omega)d\omega$ with concentration and, accordingly, a decreasing behavior for Ω_{eff}/Ω in Eq. (33). In other words, Ω_{eff}/Ω is shown to be a decreasing function of the number of quasiparticles, i.e., phonons, rotons, and ^{3}He impurities, interacting with the vortex.

We hope, in concluding this report, that the present theoretical results will stimulate an experimental investigation of degenerate-^{3}He richer vortex friction regimes.

APPENDIX A: DERIVATION OF THE VORTEX EQUATION OF MOTION

Our starting point is the Hamilton equation of motion for the creation operator of right circular quanta, whose time derivative turns out to be proportional to the vortex velocity:

$$\dot{a}^\dagger = \sqrt{\pi \, \rho_s L/m_4}\, \dot{R}, \tag{A.1}$$

where L denotes the vortex line length. Thus we have,

$$\dot{a}^\dagger = \frac{i}{\hbar}[H, a^\dagger] = i\Omega a^\dagger + \hat{O}, \tag{A.2}$$

where $\hat{O}$ denotes the quasiparticle operator:

$$\hat{O} = \frac{i}{\hbar\sqrt{\pi\rho_s L/m_4}} \sum_{\mathbf{k},\mathbf{q},\sigma} \delta_{k_z q_z}[(k_y - q_y) + i(q_x - k_x)]$$
$$[\Lambda(k,q)b_\mathbf{k}^\dagger b_\mathbf{q} + \Gamma(k,q)c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{q},\sigma}]. \tag{A.3}$$

Next we consider the Hamilton equation for $\dot{a}^\dagger$:

$$\ddot{a}^\dagger = \frac{i}{\hbar}[H, \dot{a}^\dagger] = i\Omega\dot{a}^\dagger + \frac{i}{\hbar}[H, \hat{O}]$$
$$= i\Omega\dot{a}^\dagger - \frac{1}{\hbar\sqrt{\pi\rho_s L/m_4}} \sum_{\mathbf{k},\mathbf{q},\sigma} \delta_{k_z q_z}[(k_y - q_y) + i(q_x - k_x)]$$
$$x[\Lambda(k,q)(\omega_k - \omega_q)b_\mathbf{k}^\dagger b_\mathbf{q} + \frac{1}{\hbar}\Gamma(k,q)(\epsilon_k - \epsilon_q)c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{q},\sigma}]$$
$$- \frac{2i}{\hbar^2\pi\rho_s L/m_4} \sum_{\mathbf{k}',\mathbf{k}} \sum_{\mathbf{q},\sigma} \delta_{k_z k_z'} \delta_{k_z' q_z}[(q_x - k_x')(k_y' - k_y)$$
$$+ (k_y' - q_y)(k_x' - k_x)][\Lambda(k,k')\Lambda(k',q)b_\mathbf{k}^\dagger b_\mathbf{q} + \Gamma(k,k')\Gamma(k',q)c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{q},\sigma}]a^\dagger. \tag{A.4}$$

As in I, we shall make use of a weak-coupling approximation, which consists in retaining only second order terms in the scattering amplitudes Λ and Γ. This approximation is discussed in Section 4.2. Then to approximate the above expression, we note that to the zeroth order in such parameters we have

$$a^\dagger = -i\dot{a}^\dagger/\Omega \tag{A.5}$$

$$b_\mathbf{k}^\dagger(t)b_\mathbf{q}(t) = e^{i(\omega_k - \omega_q)t}b_\mathbf{k}^\dagger(0)b_\mathbf{q}(0) \tag{A.6}$$

$$c_{\mathbf{k},\sigma}^\dagger(t)c_{\mathbf{q},\sigma}(t) = e^{i(\epsilon_k - \epsilon_q)t/\hbar}c_{\mathbf{k},\sigma}^\dagger(0)c_{\mathbf{q},\sigma}(0), \tag{A.7}$$

while the first order arises from the Hamilton equations for $b_\mathbf{k}^\dagger b_\mathbf{q}$ and $c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{q},\sigma}$. We have, for instance,

$$
\begin{aligned}
\frac{d}{dt}(b_\mathbf{k}^\dagger b_\mathbf{q}) &= \frac{i}{\hbar}[H, b_\mathbf{k}^\dagger b_\mathbf{q}] \\
&= i(\omega_k - \omega_q)b_\mathbf{k}^\dagger b_\mathbf{q} + \frac{i}{\hbar\sqrt{\pi\rho_s L/m_4}}\Bigg\{ \sum_{\mathbf{k}'}\delta_{k_z' k_z}\Lambda(k', k)[[(k_y - k_y') \\
&\quad + i(k_x - k_x')]a^\dagger + [(k_y' - k_y) + i(k_x - k_x')]a]b_{\mathbf{k}'}^\dagger b_\mathbf{q} \\
&\quad - \sum_{\mathbf{q}'}\delta_{q_z' q_z}\Lambda(q, q')[[(q_y' - q_y) + i(q_x' - q_x)]a^\dagger \\
&\quad + [(q_y - q_y') + i(q_x' - q_x)]a]b_\mathbf{k}^\dagger b_{\mathbf{q}'}\Bigg\},
\end{aligned} \tag{A.8}
$$

and we may find a formal solution to this equation in $b_\mathbf{k}^\dagger(t)b_\mathbf{q}(t)$ by noting that the only dependence on $b_\mathbf{k}^\dagger b_\mathbf{q}$ on the right-hand side comes from the first term. Thus we have to the first order in Λ:

$$
\begin{aligned}
b_\mathbf{k}^\dagger(t)b_\mathbf{q}(t) &= e^{i(\omega_k - \omega_q)t}b_\mathbf{k}^\dagger(0)b_\mathbf{q}(0) + \int_0^t d\tau\, e^{i(\omega_k - \omega_q)\tau}\frac{1}{\hbar\sqrt{\pi\rho_s L/m_4}} \\
&\quad \Bigg\{ \sum_{\mathbf{k}'}\delta_{k_z' k_z}\Lambda(k', k)[[(k_y - k_y') + i(k_x - k_x')]\dot{a}^\dagger(t - \tau)/\Omega \\
&\quad + [(k_y - k_y') + i(k_x' - k_x)]\dot{a}(t - \tau)/\Omega]e^{i(\omega_{k'} - \omega_q)(t - \tau)}b_{\mathbf{k}'}^\dagger(0)b_\mathbf{q}(0) \\
&\quad - \sum_{\mathbf{q}'}\delta_{q_z' q_z}\Lambda(q, q')[[(q_y' - q_y) + i(q_x' - q_x)]\dot{a}^\dagger(t - \tau)/\Omega \\
&\quad + [(q_y' - q_y) + i(q_x - q_x')]\dot{a}(t - \tau)/\Omega]e^{i(\omega_k - \omega_{q'})(t - \tau)}b_\mathbf{k}^\dagger(0)b_{\mathbf{q}'}(0)\Bigg\},
\end{aligned} \tag{A.9}
$$

and analogously we may find a similar expression for $c_{\mathbf{k},\sigma}^\dagger(t)c_{\mathbf{q},\sigma}(t)$. Finally, replacing Eq. (A.9) (and the corresponding expression for $c_{\mathbf{k},\sigma}^\dagger(t)c_{\mathbf{q},\sigma}(t)$) in Eq. (A.4) and, taking mean values according to $\langle b_\mathbf{k}^\dagger(0)b_\mathbf{q}(0)\rangle = \delta_{\mathbf{kq}}n_k$ and $\langle c_{\mathbf{k},\sigma}^\dagger(0)c_{\mathbf{q},\sigma}(0)\rangle = \delta_{\mathbf{kq}}f_k$, we obtain, Eq. (13).

REFERENCES

1. R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, 1991).
2. P. Nozieres and D. Pines, *The Theory of Quantum Liquids* (Addison-Wesley, New York, 1990), Vol. II, Chap. 8.
3. J. M. Duan, Phys. Rev. B **49**, 12381 (1994).
4. D. P. Arovas and J. A. Freire, Phys. Rev. B **55**, 1068 (1997).
5. J.-M. Tang, Intl. J. Mod. Phys. B **15**, 1601 (2001).
6. W. F. Vinen, Phys. Rev. B **64**, 134520 (2001).
7. C. F. Barenghi, R. J. Donnelly, and W. F. Vinen, J. Low. Temp. Phys. **52**, 189 (1983).
8. C. F. Barenghi, R. J. Donnelly, and, W. F. Vinen, Phys. Fluids **28**, 498 (1985).
9. W. I. Glaberson and R. J. Donnelly, in *Progress in Low Temperature Physics IX*. ed. D. F. Brewer (North-Holland, Amsterdam, 1986). Chap. 1, pp. 1-142.
10. F. R. Hama, Phys. Fluids **6**, 526 (1963); R. J. Arms and F. R. Hama, Phys. Fluids **8**, 553 (1965).
11. R. A. Ashton and W. I. Glaberson, Phys. Rev. Lett. **42**, 1062 (1979).
12. A. L. Fetter, Phys. Rev. **186**, 128 (1969).
13. E. B. Sonin, Zh. Éksp. Teor. Fiz. **69**, 921 (1975) [Sov. Phys. JETP **42**, 469 (1976)].
14. G. W. Rayfield and F. Reif, Phys. Rev. **136**, A 1194 (1964).
15. L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **35**, 1271 (1958) [Sov. Phys. JETP **8**, 888 (1959)].
16. S. V. Iordanskii, Zh. Éksp. Teor. Fiz. **49**, 225 (1965) [Sov. Phys. JETP **22**, 160 (1966)].
17. H. M. Cataldo and D. M. Jezek, Phys. Rev. B **65**, 184523 (2002).
18. A very simple quantum mechanical model which exhibits the basic features of a non-Markovian dynamics corresponds to the decay of a discrete state, resonantly coupled to a continuum of final states, see C. Cohen Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (Wiley, New York, 1977), Vol. II, Chap, XIII.
19. F. Haake and R. Reibold, Phys, Rev. A **32**, 2462 (1985).
20. G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. A **37**, 4419 (1988).
21. H. M. Cataldo, Physica A **165**, 249 (1990).
22. H. M. Cataldo, Phys. Lett, A, **148**, 246 (1990).
23. G. W. Ford and R. F. O'Connell, Phys, Lett. A **157**, 217 (1991).
24. G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys, Rev. A **36**, 1466 (1987).
25. A. V. Bobylev, F. A. Maaø, A. Hansen, and E. H. Hauge, Phys. Rev. Lett. **75**, 197 (1995).
26. A. Dmitriev, M. Dyakonov, and R. Jullien, Phys. Rev. Lett. **89**, 266804 (2002).
27. H. M. Cataldo, M. A. Despósito, E. S. Hernández, and D. M. Jezek, Phys. Rev. B **55**, 3792 (1997).
28. H. M. Cataldo, M. A. Despósito, E. S. Hernández, and D. M. Jezek, Phys. Rev. B **56**, 8282 (1997).
29. H. M. Cataldo, M. A. Despósito, and D. M. Jezek, J. Phys. Condens. Matter **11**, 10277 (1999).
30. R. Kubo, M. Toda, and N. Hashitsume, *Statistical physics II. Non-equilibrium statistical mechanics* (Springer, Berlin, 1985).
31. See for instance, R. Balescu, *Statistical mechanics of charged particles* (Wiley, New York, 1963) App. 2.
32. J. D. P. Van Dijk, G. M. Postma, J. Wiebes, and H. C. Kramers, Physica B & C **85B**, 85 (1977).
33. W. Hsu, D. Pines, and C. H. Aldrich, III, Phys. Rev. B **32**, 7179 (1985).
34. F. Pistolesi, Phys. Rev. Lett. **81**, 397 (1998).
35. E. B. Sonin, Phys. Rev. B **55**, 485 (1997).
36. C. Wexler and D. J. Thouless, Phys. Rev, B **58**, R8897 (1998).
37. J.-Y. Fortin, Phys. Rev. B **63**, 174502 (2001).
38. D. J. Thouless, M. R. Geller, W. F. Vinen, J.-Y. Fortin, and S. W. Rhee, Phys. Rev. B **63**, 224504 (2001).
39. D. J. Thouless, P. Ao, and Q. Niu, Phys. Rev. Lett. **76**, 3758 (1996).
40. G. W. Rayfield, Phys. Rev. **168**, 222 (1968).