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Sufficient Conditions for Generic Feedback Stabilizability of Switching Systems via Lie-Algebraic Solvability

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***Abstract*—We address the stabilization of switching linear systems (SLSs) with control inputs under arbitrary switching. A sufficient condition for the stability of autonomous (without control inputs) SLSs is that the individual subsystems are stable and the Lie algebra generated by their evolution matrices is solvable. This sufficient condition for stability is known to be extremely restrictive and therefore of very limited applicability. Our main contribution is to show that, in contrast to the autonomous case, when control inputs are present the existence of feedback matrices for each subsystem so that the corresponding closed-loop matrices satisfy the aforementioned Lie-algebraic stability condition can become a generic property, hence substantially improving the applicability of such Lie-algebraic techniques in some cases. Since the validity of this Lie-algebraic stability condition implies the existence of a common quadratic Lyapunov function (CQLF) for the SLS, our results yield an analytic sufficient condition for the generic existence of a control CQLF for the SLS.**

***Index Terms*—Common quadratic Lyapunov function (CQLF), switching linear systems (SLSs), uniform global exponential stability (UGES).**

I. Introduction

Switched systems are dynamical systems that combine a finite number of subsystems by means of a switching signal [1], [2]. In recent years, considerable research effort has been devoted to studying the stability and stabilizability of switched systems [1], [3]–[5]. In this technical note, we focus on the case where each subsystem is linear and also on stability under "arbitrary switching", where stability holds for every possible switching signal. We refer to the switched systems under consideration as *switching linear systems* (SLSs).

A SLS may either be *autonomous* or have *control inputs*. For *autonomous* SLSs, it is known that the uniform global exponential stability (UGES, where 'uniform' means 'over all switching signals') is equivalent to the existence of a Lyapunov function common to all subsystems [6]. A computationally appealing stability condition, though more restrictive, is the existence of a common *quadratic* Lyapunov function (CQLF) [5, § 4.2]. A CQLF may be efficiently numerically sought for by solving linear matrix inequalities [7]. An *analytical* stability condition, even more restrictive, states that a SLS admits a CQLF (and is hence UGES) if every individual subsystem is stable and the Lie algebra generated by their evolution matrices is solvable. The solvability of a matrix Lie algebra is equivalent to the existence of a single similarity transformation that transforms each matrix into upper triangular form. This Lie-algebraic stability condition is simple to check numerically and holds both for discrete-time SLSs [8], [9] and continuous-time SLSs [10], [11]. These Lie-algebraic stability conditions, although mathematically elegant and possibly computationally advantageous (cf. [12]), have had very limited applicability due to their restrictiveness.

The situation can be radically different for SLSs with control inputs, where feedback may be employed to stabilize the SLS. Indeed, the main contribution of the current technical note is to establish that the existence of feedback matrices for each subsystem so that the closed-loop SLS satisfies the aforementioned Lie-algebraic stability condition can become a *generic* property, namely, a property that is valid for almost every set of system parameters. We give conditions that ensure the genericity of this property and thus may enhance the applicability of such Lie-algebraic stabilization techniques when control inputs are present. These conditions depend on the number of states, subsystems and control inputs of each subsystem. In order to be satisfied, the given conditions require each subsystem to have a "substantial" number of inputs, although possibly fewer inputs than states.

Feedback control design based on Lie-algebraic solvability has been previously pursued by the authors [13]–[16]. A central contribution in [13], [14] is an iterative design algorithm that searches for a set of stabilizing feedback matrices that attain the target simultaneously triangularizable closed-loop structure via the application of a common eigenvector assignment (CEA) procedure and the reduction of state dimension at each iteration. The main theoretical result in [13], [14] establishes that the proposed algorithm will iterate successfully until the state dimension is reduced to 1 *if and only if* feedback matrices exist so that the corresponding closed-loop subsystem matrices are stable and simultaneously triangularizable, i.e. if and only if feedback matrices exist so that the closed-loop system satisfies the aforementioned Lie-algebraic stability condition. A numerical implementation for the proposed iterative design algorithm and the CEA procedure are also provided in [14], together with a key structural condition, which, when satisfied, guarantees a directly computable solution for the CEA procedure. If this structural condition is not satisfied, then the required quantities are sought by means of an optimization problem.

In addition to its limited applicability, the aforementioned Lie-algebraic stability condition is also non-robust, in the sense that even if it is satisfied for a given autonomous SLS, it is almost surely not satisfied by SLSs with parameters arbitrarily close to the given one. The work in [15] then provides a robust result by relaxing, for single input systems, the simultaneous triangularization requirement to *approximate*

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(in a specific sense) simultaneous triangularization. The main theoretical contribution in [15] establishes that if a single-control-input SLS satisfying the aforementioned Lie-algebraic condition exists in a suitably small neighborhood of the given SLS, then the proposed algorithm is guaranteed to find feedback matrices so that the corresponding closed-loop SLS admits a CQLF even if the Lie-algebraic condition is not met by the given system. (Agrachev *et al.* [17] have recently derived, for autonomous SLSs, robust stability conditions related to Lie-algebraic solvability and formulated directly in terms of Lie brackets.)

Our current main results build upon the key structural condition provided in [14]: if such structural condition is satisfied at every iteration of the algorithm, then the considered feedback control design via Lie-algebraic solvability problem may be not restrictive *at all* for systems with the given dimensions. In this regard, the main result in [16] is the identification of the situation that prevents the structural condition from being satisfied at every iteration of the algorithm.

In the present technical note, we build upon the results of [16] by providing sufficient conditions for the structural condition to hold at every iteration of the algorithm *for almost every set of system parameters with the given dimensions*. We thus provide sufficient conditions for the *genericity* of the property of existence of feedback matrices so that the closed-loop subsystem matrices are stable and generate a solvable Lie algebra, a property which implies the existence of a CQLF for the closed-loop system. Consequently, a side contribution of the current technical note is the derivation of an analytic condition that ensures the genericity of the property of existence of feedback matrices so that the corresponding closed-loop SLS admits a CQLF. Preliminary results on this topic have been previously presented by the authors in [18]. Even though our previous results [13]–[16], [18] focus on discrete-time SLSs, the current results are valid for both discrete- and continuous-time SLSs.

Notation: The index set $\{1, 2, \ldots, N\}$ is denoted $\underline{N}$. The kernel (null space) of a matrix or linear map $A : \mathcal{X} \to \mathcal{Y}$ is denoted $\ker A$ and its image (range), $\mathrm{img} A$. Given a subspace $\mathcal{B} \subset \mathcal{Y}$, the subspace $\{v \in \mathcal{X} : Av \in \mathcal{B}\}$ is denoted $(A)^{-1}\mathcal{B}$. For $x \in \mathbb{C}^{n\times m}$, its transpose is denoted x', its conjugate transpose x^* and its Moore-Penrose generalized inverse $x^\dagger$. If $\mathcal{S}, \mathcal{T}$ are vector spaces, then $\mathcal{S} \subset \mathcal{T}$ means that $\mathcal{S}$ is a subspace of $\mathcal{T}$, $\mathcal{S} \oplus \mathcal{T}$ denotes the direct sum of $\mathcal{S}$ and $\mathcal{T}$ (which implies that $\mathcal{S} \cap \mathcal{T} = 0$), and $\mathrm{d}(\mathcal{S})$ denotes the dimension of $\mathcal{S}$. If I is a finite set, then $\#I$ denotes the number of elements in I.

II. Problem Formulation

Consider the discrete- or continuous-time SLSs

$$x_{k+1} = A_{\sigma(k)} x_k + B_{\sigma(k)} u_k^{\sigma(k)} \tag{1}$$
$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u^{\sigma(t)}(t) \tag{2}$$

where the switching function $\sigma(\cdot)$ takes values in $\underline{N}$, $x \in \mathbb{R}^n$ and for all $i \in \underline{N}$, $u^i \in \mathbb{R}^{m_i}$, the matrices $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m_i}$ are known, and B_i have full column rank. We are interested in state-feedback control design of the form

$$u_k^{\sigma(k)} = K_{\sigma(k)} x_k, \quad \text{or} \quad u^{\sigma(t)}(t) = K_{\sigma(t)} x(t) \tag{3}$$

so that the resulting closed-loop system

$$x_{k+1} = A_{\sigma(k)}^{\mathrm{CL}} x_k, \quad \text{or} \quad \dot{x}(t) = A_{\sigma(t)}^{\mathrm{CL}} x(t), \text{ where} \tag{4}$$
$$A_i^{\mathrm{CL}} = A_i + B_i K_i, \quad \text{for } i \in \underline{N} \tag{5}$$

admit a CQLF and hence be stable under arbitrary switching. Note that at every time instant, the control law (3) requires knowledge of the "active" subsystem given by $\sigma(k)$ or $\sigma(t)$.

Algorithm ITF: Iterative triangularisation by feedback

Data: $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m_i}$ for $i \in \underline{N}$
Output: K_i for $i \in \underline{N}$
Initialisation: $A_i^1 \doteq A_i$, $B_i^1 \doteq B_i$, $K_i^0 \doteq 0$, $U_1 \doteq \mathrm{I}$, $\ell \leftarrow 0$;
repeat

$$\ell \leftarrow \ell + 1, \quad n_\ell \leftarrow n - \ell + 1, \tag{6}$$
$$[v_1^\ell, \{F_i^\ell\}_{i=1}^N] \leftarrow \mathrm{CEA}(\{A_i^\ell\}_{i=1}^N, \{B_i^\ell\}_{i=1}^N), \tag{7}$$
$$A_i^{\ell,\mathrm{CL}} \doteq A_i^\ell + B_i^\ell F_i^\ell, \tag{8}$$
$$K_i^\ell \leftarrow K_i^{\ell-1} + F_i^\ell \left(\prod_{r=1}^{\ell} U_r^*\right). \tag{9}$$

if $\ell < n$ **then**
Construct a unitary matrix (10) and assign (11)–(13):

$$[v_1^\ell | v_2^\ell | \cdots | v_{n_\ell}^\ell] \in \mathbb{C}^{n_\ell \times n_\ell}, \tag{10}$$
$$U_{\ell+1} \leftarrow [v_2^\ell | \cdots | v_{n_\ell}^\ell], \tag{11}$$
$$A_i^{\ell+1} \leftarrow U_{\ell+1}^* A_i^{\ell,\mathrm{CL}} U_{\ell+1}, \tag{12}$$
$$B_i^{\ell+1} \leftarrow U_{\ell+1}^* B_i^\ell. \tag{13}$$

end if
until $\ell = n$;
$K_i \leftarrow K_i^n$;

Fig. 1. Algorithm for iterative triangularization by feedback.

As is well-known, ensuring that each A_i^{CL} be stable is necessary but not sufficient to ensure the stability of the autonomous SLS (4) under arbitrary switching. A sufficient condition is given by the following result [9, Theorem 6.18], [10, Theorem 2].

Lemma 1 (Lie-Algebraic-Solvability Stability Condition): If every A_i^{CL} is stable and the Lie algebra generated by $\{A_i^{\mathrm{CL}} : i \in \underline{N}\}$ is solvable, then (4) admits a CQLF and hence is UGES. □

In this technical note, we specifically consider stabilizing state feedback design for the SLSs (1) and (2) based on the Lie-algebraic-solvability condition of Lemma 1. We thus focus on the SLS class defined next.

Definition 1 (SLASF): A set $\mathcal{Z} = \{(A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times m_i}) : i \in \underline{N}\}$ is said to be *SLASF (Solvable Lie Algebra with Stability by Feedback)* if there exist $K_i \in \mathbb{R}^{m_i\times n}$ such that A_i^{CL} as in (5) are stable (Schur-stable for a discrete-time SLS; Hurwitz-stable for a continuous-time SLS) and generate a solvable Lie algebra. □

In matrix terms, the fact that the Lie algebra generated by the matrices A_i^{CL} is solvable is equivalent to the existence of an invertible matrix $T \in \mathbb{C}^{n\times n}$ such that $T^{-1} A_i^{\mathrm{CL}} T$ is upper triangular for all $i \in \underline{N}$. Note that even if the matrices A_i^{CL} have real entries, those of T may be complex [19].

III. Previous Results

Control design that causes the closed-loop system to be stable by satisfying the conditions of Lemma 1 can be performed iteratively by seeking feedback matrices that assign a common eigenvector with stable corresponding eigenvalues at every iteration [13], [14]. Although the latter references deal exclusively with discrete-time SLSs, the only difference between the discrete- and continuous-time cases is the stability region considered (the open unit disk or the open left half-plane). The control design method of [13], [14] is represented by Algorithm ITF (Iterative Triangularization by Feedback), shown in Fig. 1. Algorithm ITF seeks feedback matrices K_i so that the closed-loop matrices A_i^{CL} given by (5) are stable and simultaneously triangularizable.

A. Algorithm for Iterative Triangularization by Feedback

Algorithm ITF begins by setting internal matrices equal to the subsystem matrices of the SLS to be stabilized ($A_i^1 = A_i$ and $B_i^1 = B_i$ at the Initialization step). At every iteration [ℓ indicates iteration number, see (6)], the algorithm executes Procedure CEA [see (7)] on its internal system matrices A_i^ℓ and B_i^ℓ. Procedure CEA aims to compute a vector, v_1^ℓ, and corresponding feedback matrices, F_i^ℓ, so that v_1^ℓ is a feedback-assignable unit eigenvector common to all internal subsystems, with corresponding stable eigenvalues. That is, if Procedure CEA is successful, then v_1^ℓ will satisfy $\|v_1^\ell\| = 1$ and $(A_i^\ell + B_i^\ell F_i^\ell)v_1^\ell = \lambda_i^\ell v_1^\ell$ for some scalars λ_i^ℓ satisfying $|\lambda_i^\ell| < 1$ for discrete-time or $\mathfrak{Re}\{\lambda_i\} < 0$ for continuous-time, for all $i \in \underline{N}$.

The algorithm then computes internal closed-loop matrices [$A_i^{\ell,\mathrm{CL}}$ in (8)], updates internal feedback matrices [K_i^ℓ in (9)] and then reduces the internal state dimension by 1. This reduction occurs at (10)–(13) [n_ℓ is the internal state dimension, see (6)]. Note that v_1^ℓ is the first column of the unitary matrix (10), and considering (11) then $U_{\ell+1}^* U_{\ell+1} = \mathrm{I}$ and $U_{\ell+1}^* v_1^\ell = 0$.

Iterations run until the internal state reaches dimension 1. If the given system matrices form a SLASF set (recall Definition 1), the matrices K_i computed by Algorithm ITF will be the required feedback matrices. In addition, for each subsystem $i \in \underline{N}$, the eigenvalues of $A_i^{\mathrm{CL}} = A_i + B_i K_i$ will be equal to λ_i^ℓ for $\ell = 1, \ldots, n$.

If the given system matrices A_i, B_i, for $i \in \underline{N}$, form a SLASF set, then at every iteration of Algorithm ITF a stable feedback-assignable common eigenvector v_1^ℓ is ensured to exist for the internal system with matrices A_i^ℓ, B_i^ℓ, for $i \in \underline{N}$. Conversely, if a (stable) feedback-assignable common eigenvector v_1^ℓ exists at every iteration of Algorithm ITF, then the given system matrices form a SLASF set. The latter constitutes the main theoretical result that underpins our iterative control design algorithm [13], [14].

B. Procedure for Common Eigenvector Assignment

As expressed in the previous paragraph, the existence of a feedback-assignable common eigenvector with corresponding stable eigenvalues is central to our development. This section recalls the structural condition introduced in [14] which, when satisfied, ensures that such a vector exists and allows its computation in a straightforward and numerically efficient way.

We introduce some notation required to state this structural condition. Define $m_i^\ell \doteq \mathrm{rank}(B_i^\ell) = \mathrm{d}(\mathrm{img}B_i^\ell)$, and factor $B_i^\ell = b_i^\ell r_i^\ell$, where $r_i^\ell : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i^\ell}$ has full row rank and $b_i^\ell : \mathbb{R}^{m_i^\ell} \to \mathbb{R}^{n_\ell}$ has full column rank. We adopt the convention that b_i^ℓ is an empty matrix if $m_i^\ell = 0$. Note that $\mathrm{img}B_i^\ell = \mathrm{img}b_i^\ell$. Let Λ^ℓ be the vector with components λ_i^ℓ, $i \in \underline{N}$, i.e.

$$\Lambda^\ell \doteq \left[\lambda_1^\ell, \lambda_2^\ell, \ldots, \lambda_N^\ell\right]' \tag{14}$$

and build the matrix

$$Q_\ell(\Lambda^\ell) \doteq \left[R_\ell(\Lambda^\ell), -B_\ell\right], \text{ where}$$
$$R_\ell(\Lambda^\ell) \doteq \begin{bmatrix} \lambda_1^\ell I - A_1^\ell \\ \vdots \\ \lambda_N^\ell I - A_N^\ell \end{bmatrix}, \text{ and } B_\ell \doteq \mathrm{blkdiag}\left[b_1^\ell, \ldots, b_N^\ell\right] \tag{15}$$

where blkdiag denotes block diagonal concatenation.

Lemma 2 (Structural Condition [14], [16]): Let

$$p_\ell \doteq n_\ell + \sum_{i=1}^{N} m_i^\ell - N n_\ell. \tag{16}$$

Then,

(a) A vector that can be assigned by feedback as a common eigenvector with corresponding eigenvalues λ_i^ℓ for $i \in \underline{N}$ exists if and only if $\mathrm{d}(\ker Q_\ell(\Lambda^\ell)) > 0$.

(b) If $Q_\ell(\Lambda^\ell)w = 0$ with $w \neq 0$ partitioned as

$$w \doteq \left[v', u_1', \ldots, u_N'\right]', \qquad \text{then } v \neq 0, \text{ and} \tag{17}$$
$$\left(A_i^\ell + B_i^\ell F_i^\ell\right) v = \lambda_i^\ell v, \quad \text{for } i \in \underline{N} \tag{18}$$

for every F_i^ℓ satisfying $r_i^\ell F_i^\ell v = u_i$. For each $i \in \underline{N}$ one such F_i^ℓ is $F_i^\ell = (r_i^\ell)^\dagger u_i v^\dagger$.

(c) $\mathrm{d}(\ker Q_\ell(\Lambda^\ell)) \geq p_\ell$ for every choice of Λ^ℓ as in (14). Consequently, if $p_\ell > 0$, then a feedback-assignable common eigenvector exists for every choice of corresponding eigenvalues.

Lemma 2 gives a structural condition, namely $p_\ell > 0$, for a feedback-assignable common eigenvector v to exist for each choice of corresponding eigenvalues λ_i^ℓ. This condition is *structural* because the quantities involved in the computation of p_ℓ are only matrix ranks and dimensions. If the structural condition $p_\ell > 0$ is satisfied, a feedback-assignable common eigenvector v_1^ℓ, as required at iteration ℓ of Algorithm ITF, can be computed as follows:

1) Select the corresponding (stable) closed-loop eigenvalues λ_i^ℓ for each subsystem $i \in \underline{N}$ and build Λ^ℓ as in (14);
2) Find a vector $w \neq 0$ with components partitioned as in (17) so that $Q_\ell(\Lambda^\ell)w = 0$ (namely, so that $w \in \ker Q_\ell(\Lambda^\ell)$);
3) Select the first n_ℓ components of w to construct the subvector v in (17). The feedback-assignable common eigenvector sought is finally computed as $v_1^\ell = v/\|v\|$.

Feedback matrices to assign the eigenvector v_1^ℓ with corresponding eigenvalues λ_i^ℓ can be obtained as $F_i^\ell = (r_i^\ell)^\dagger u_i v^\dagger$. Procedure CEA is thus summarized in Fig. 2 for the case when the structural condition of Lemma 2 is satisfied.

Procedure CEA (Common Eigenvector Assignment)

Input: $A_i^\ell \in \mathbb{R}^{n_\ell \times n_\ell}$, $B_i^\ell \in \mathbb{R}^{n_\ell \times m_i}$, for $i \in \underline{N}$
Output: v_1^ℓ, F_i^ℓ for $i \in \underline{N}$
Factor $B_i^\ell = b_i^\ell r_i^\ell$ with $b_i^\ell \in \mathbb{R}^{n_\ell \times m_i^\ell}$ and $m_i^\ell = \mathrm{rank}(B_i^\ell)$;
if $p_\ell = n_\ell + \sum_{i=1}^N m_i^\ell - N n_\ell > 0$ **then**
  Select $\lambda_i^\ell \in \mathbb{R}$ stable and construct Λ^ℓ as in (14);
  Find $w \neq 0$ such that $Q_\ell(\Lambda^\ell)w = 0$;
  Partition w as in (17) ;
  $v_1^\ell = v/\|v\|$;
  $F_i^\ell = (r_i^\ell)^\dagger u_i v^\dagger$, for $i \in \underline{N}$;
end if

Fig. 2. Procedure CEA when the structural condition is satisfied.

Even if the SLS matrices A_i, B_i have real entries, those of the matrices A_i^ℓ, B_i^ℓ internal to Algorithm ITF can be complex at some iteration ℓ. This is so because the vector v_1^ℓ returned by Procedure CEA (a feedback-assignable common eigenvector) can have complex components even if A_i^ℓ, B_i^ℓ have real entries, causing $A_i^{\ell+1}, B_i^{\ell+1}$ to have complex entries. However, when the structural condition $p_\ell > 0$ is satisfied, the closed-loop eigenvalues λ_i^ℓ, i.e. the components of Λ^ℓ, can be arbitrarily selected. Hence, selecting $\lambda_i^\ell \in \mathbb{R}$ will cause the vector v_1^ℓ to be real. In the sequel, we assume that real eigenvalues will be selected and hence all matrices internal to Algorithm ITF will have real entries.

C. Structural Condition

If the structural condition given by Lemma 2, namely $p_\ell > 0$, holds at iteration ℓ of Algorithm ITF, then Procedure CEA can compute a feedback-assignable common eigenvector and the corresponding feedback matrices, for every choice of corresponding (stable) closed-loop eigenvalues. In addition, if $p_\ell > 0$ the closed-loop eigenvalues λ_i^ℓ for every $i \in \underline{N}$ can be freely chosen. The quantity p_ℓ depends on m_i^ℓ, the rank of B_i^ℓ. At the first iteration of Algorithm ITF, i.e. when $\ell = 1$, the internal matrices $B_i^1 = B_i$ have $n = n_1$ rows, m_i columns, and since by assumption they have full column rank, then $m_i^1 = m_i$. At subsequent iterations, the matrices B_i^ℓ have $n_\ell = n - \ell + 1$ rows and m_i columns. Since the matrix (10) is unitary by construction, then according to (11) and (13) we have

$$m_i^\ell - 1 \leq m_i^{\ell+1} \leq m_i^\ell \tag{19}$$

and moreover, $m_i^{\ell+1}$ depends on the vector v_1^ℓ returned by Procedure CEA as

$$m_i^{\ell+1} = \begin{cases} m_i^\ell & \text{if } v_1^\ell \notin \text{img } B_i^\ell, \\ m_i^\ell - 1 & \text{if } v_1^\ell \in \text{img } B_i^\ell. \end{cases} \tag{20}$$

From (20), then $m_i^{\ell+1} = m_i^\ell - 1$ when $m_i^\ell = n_\ell$, because $v_1^\ell \in \mathbb{R}^{n_\ell} = \text{img} B_i^\ell$. The following theorem and corollary follow from (6), (16), and (20).

Theorem 1 ([16]): Consider Algorithm ITF at iteration ℓ and p_ℓ as in (16), with $m_i^\ell = \text{rank}(B_i^\ell)$. Then, $p_{\ell+1} \geq p_\ell - 1$, with equality if and only if

$$v_1^\ell \in \mathcal{B}^\ell, \quad \text{with } \mathcal{B}^\ell \doteq \bigcap_{i \in \underline{N}} \mathcal{B}_i^\ell \quad \text{and } \mathcal{B}_i^\ell \doteq \text{img } B_i^\ell. \tag{21}$$

Corollary 1: Let $p_\ell > 0$. Then,

(a) $p_q > 0$ for $q = \ell, \ldots, \ell + p_\ell - 1$.
(b) $p_{\ell+1} > 0$ if $v_1^\ell \notin \text{img} B_k^\ell$ for some $k \in \underline{N}$.
(c) $p_{\ell+1} \ngtr 0$ if and only if $p_\ell = 1$ and (21) holds.

Corollary 1(c) identifies the condition that prevents the *inductivity* of the structural condition $p_\ell > 0$ from iteration ℓ to iteration $\ell + 1$ of the algorithm. The following section builds on these results.

IV. Main Results

In this section, we derive conditions to ensure that the structural condition $p_\ell > 0$ will hold at every iteration of Algorithm ITF, i.e. for $\ell = 1, \ldots, n$. Subsequently, we will establish that, for some state vector dimensions, n, number of subsystems, N, and number of control inputs, m_i for $i \in \underline{N}$, these conditions are valid for almost every set of system parameters—the entries of A_i and B_i for $i \in \underline{N}$.

In Section IV-A, we recall and extend the property of transversality of subspaces (see, e.g., Chapter 0 of [20]), which is required for the derivation of our main results. In Section IV-B, we derive conditions that ensure the validity of the structural condition at every iteration of Algorithm ITF. In Section IV-C, we analyse the conditions derived in Section IV-B and relate them to the genericity of the SLASF property (recall Definition 1). A brief numerical example is given in Section IV-D. Proofs are provided in the Appendix.

A. Transversality of Subspaces

Definition 2 (Transverse): Two subspaces $\mathcal{S}, \mathcal{T}$ of an ambient space $\mathcal{X}$ are said to be transverse when the dimension of their intersection is minimal, given the dimensions of $\mathcal{S}$ and $\mathcal{T}$, i.e., when

$$\text{d}(\mathcal{S} \cap \mathcal{T}) = \max\{0, \text{d}(\mathcal{S}) + \text{d}(\mathcal{T}) - \text{d}(\mathcal{X})\}. \tag{22}$$

Equivalently, $\mathcal{S}$ and $\mathcal{T}$ are transverse when the dimension of their sum is maximal. We extend this definition to sets of subspaces as follows. Let $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ be a set of subspaces of an ambient space $\mathcal{X}$. We say that S is transverse when both the intersection of the subspaces in every subset of S has minimal dimension *and* the sum of the subspaces in every subset of S has maximal dimension.

It is well-known [20, Ch.0] that transversality of two subspaces $\mathcal{S}$ and $\mathcal{T}$ is a generic property, i.e. it is satisfied by almost every $\mathcal{S}$ and $\mathcal{T}$ selected "randomly" among all subspaces of $\mathcal{X}$. Also, it is evident that the extension of this property to sets of subspaces according to Definition 2 preserves genericity, in the sense that almost every set containing a finite number of subspaces taken "randomly" among all subspaces of $\mathcal{X}$ will be transverse according to Definition 2.

We will require the following properties related to transversality.

Lemma 3: Let $S = \{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ be a set of subspaces of the ambient space $\mathcal{X}$, and define

$$p \doteq \text{d}(\mathcal{X}) + \sum_{i \in \underline{N}} \text{d}(\mathcal{S}_i) - N\text{d}(\mathcal{X}).$$

Then,

(a) $\text{d}(\mathcal{S}_i \cap \mathcal{S}_j) = \text{d}(\mathcal{S}_i) + \text{d}(\mathcal{S}_j) - \text{d}(\mathcal{S}_i + \mathcal{S}_j)$.
(b) If S is transverse, then $\text{d}(\bigcap_{i \in \underline{N}} \mathcal{S}_i) = \max\{0, p\}$.
(c) If S is transverse and $p \geq 0$, then $\text{d}(\mathcal{S}_i + \mathcal{S}_j) = \text{d}(\mathcal{X})$ for all $i, j \in \underline{N}$ with $i \neq j$.
(d) Let $J = I \cup \{j\}$, with $J \subset \underline{N}$ and $\#J = \#I + 1$. Suppose that $p \geq 0$ and that $\{\mathcal{S}_i : i \in I\}$ is transverse. Then, $\{\mathcal{S}_i : i \in J\}$ is transverse if and only if $\bigcap_{i \in I} \mathcal{S}_i + \mathcal{S}_j = \mathcal{X}$.

B. Validity of the Structural Condition at every Iteration

The derivations of this section require deep analysis of the condition (21). In the sequel, let $\mathcal{S}_i^\ell$ denote the set of vectors $v \in \mathcal{B}_i^\ell = \text{img} B_i^\ell$ for which there exist a matrix F_i^ℓ and a stable scalar λ so that

$$\left(A_i^\ell + B_i^\ell F_i^\ell\right) v = \lambda v. \tag{23}$$

By definition, $\mathcal{S}_i^\ell$ is the set of feedback-assignable stable eigenvectors for the internal subsystem (A_i^ℓ, B_i^ℓ) that are contained in $\mathcal{B}_i^\ell$. Consequently, if v_1^ℓ is a stable feedback-assignable common eigenvector, then $v_1^\ell \in \mathcal{B}_i^\ell$ if and only if $v_1^\ell \in \mathcal{S}_i^\ell$. The following result is straightforward.

Lemma 4:

(a) The set $\mathcal{S}_i^\ell$ is a subspace.
(b) $v \in \mathcal{S}_i^\ell$ if and only if $v \in \mathcal{B}_i^\ell$ and $A_i^\ell v \in \mathcal{B}_i^\ell$.

Define the following quantities:

$$\rho_i^\ell \doteq \text{d}\left(\mathcal{S}_i^\ell\right), \qquad q_\ell \doteq n_\ell + \sum_{i \in \underline{N}} \rho_i^\ell - N n_\ell \tag{24}$$

$$\mathcal{S}^\ell \doteq \bigcap_{i \in \underline{N}} \mathcal{S}_i^\ell, \qquad \rho^\ell \doteq \text{d}(\mathcal{S}^\ell). \tag{25}$$

The core technical result of the technical note is given below as Theorem 2. This result gives conditions under which the structural condition of Lemma 2 will hold at every iteration of Algorithm ITF, irrespective of the choice of closed-loop eigenvalues λ_i^ℓ performed in Procedure CEA.

Theorem 2: Let $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ be transverse, $q_1 \geq 0$, and (A_i, B_i) be controllable for all $i \in \underline{N}$. Then,

(a) $p_\ell > 0$ for $\ell = 1, \ldots, n$.
(b) The set $\mathcal{Z} = \{(A_i, B_i) : i \in \underline{N}\}$, which identifies the given SLS, is SLASF.

C. Genericity of the SLASF Property

Theorem 2 gives sufficient conditions under which a given SLS will be SLASF. We next show that for some state vector dimensions, n, number of subsystems, N, and number of control inputs, m_i for each $i \in \underline{N}$, these conditions are satisfied for almost every set of matrices $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m_i}$ with $i \in \underline{N}$.

The three conditions required by Theorem 2 are that the set of subspaces $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ be transverse, that the quantity q_1 be nonnegative and that the pairs (A_i, B_i) be controllable. It is well-known that controllability is a generic property [20] and hence we next focus on the first two conditions.

We show first that transversality of $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ is generic in the space of parameters of the matrices A_i, B_i. From Lemma 5(b), it follows that $\mathcal{S}_i^1 = \mathcal{B}_i^1 \cap (A_i^1)^{-1}\mathcal{B}_i^1$. Note that arbitrary choices for the entries of $B_i = B_i^1$ yield arbitrary $\mathcal{B}_i^1 = \mathrm{img} B_i^1$, although generically of dimension $m_i = m_i^1$. In addition, arbitrary choices for the entries of $A_i = A_i^1$ yield arbitrary $(A_i^1)^{-1}\mathcal{B}_i^1$, also generically of dimension m_i. Therefore, the subspaces $\mathcal{B}_i^1$ and $(A_i^1)^{-1}\mathcal{B}_i^1$ will be transverse generically and from (22), then $\mathrm{d}(\mathcal{S}_i^1) = \mathrm{d}(\mathcal{B}_i^1 \cap (A_i^1)^{-1}\mathcal{B}_i^1) = \max\{0, 2m_i - n\}$ generically. We conclude that arbitrary choices for the entries of A_i and B_i produce arbitrary $\mathcal{S}_i^1$ even though subject to the constraint that

$$\mathrm{d}\left(\mathcal{S}_i^1\right) = \max\{0, 2m_i - n\}, \text{ generically.} \tag{26}$$

Due to the fact that arbitrary $\mathcal{S}_i^1$ can be produced, then the set $\{\mathcal{S}_i^1 : i \in \underline{N}\}$ is transverse generically in the space of parameters of A_i, B_i.

Consider next the quantity q_1. From (24) and (26) follows that $q_1 = n + \sum_{i\in\underline{N}} \max\{0, 2m_i - n\} - Nn$ generically. The condition $q_1 \geq 0$ imposes a restriction on n, N, and m_i for each $i \in \underline{N}$. We summarize our main result as Theorem 3 below, and subsequently show that $q_1 \geq 0$ may hold in some non-trivial cases.

Theorem 3 (Genericity of the SLASF Property): If the state dimension n, the number of subsystems N, and the number of control inputs m_i for each $i \in \underline{N}$, are such that

$$n + \sum_{i\in\underline{N}} \max\{0, 2m_i - n\} - Nn \geq 0 \tag{27}$$

then the SLASF property holds for almost every set of system parameters $A_i \in \mathbb{R}^{n\times n}$ and $B_i \in \mathbb{R}^{n\times m_i}$ for all $i \in \underline{N}$.

Corollary 2: Under the same conditions for n, N and m_i as in Theorem 3, the property of existence of feedback matrices so that the closed-loop SLS admits a CQLF is generic.

We next analyse the condition (27). Note that non-trivial cases are those for which $n \geq 2$, $N \geq 2$ and $1 \leq m_i \leq n-1$ for all $i \in \underline{N}$. Combining these conditions with (27) leads to the following:

- In order for (27) to hold in a non-trivial case, then it is necessary that $N \leq n/2$ and $m_i > n/2$.
- Under the condition $N \leq n/2$, then it is sufficient (but not necessary) that $m_i = n-1$ for all $i \in \underline{N}$ for (27) to hold.

D. Numerical Example

Consider a discrete-time system of the form (1), with $n = 4$, $N = 2$, $m_1 = m_2 = 3$

$$A_1 = \begin{bmatrix} 3.2 & -4.0 & -3.4 & -3.6 \\ 4.1 & -2.2 & 4.7 & -0.8 \\ -3.7 & 0.5 & 4.6 & 4.2 \\ 4.1 & 4.6 & -0.2 & 2.9 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2.6 & 2.1 & 3.2 & -0.6 \\ 2.4 & -4.7 & 2.0 & -1.2 \\ -1.1 & -2.2 & -1.8 & 2.7 \\ 1.6 & -4.5 & 4.5 & 3.0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -2.2 & 4.6 & 0.1 \\ 1.8 & -1.6 & 2.0 \\ 1.6 & 0.9 & 3.9 \\ -3.4 & -2.8 & 4.6 \end{bmatrix} \quad B_2 = \begin{bmatrix} -0.3 & 0.3 & 2.5 \\ -4.9 & -3.3 & -0.5 \\ -1.6 & 1.0 & -4.2 \\ -3.4 & -2.4 & -2.7 \end{bmatrix}.$$

Each of the entries of A_1, A_2, B_1, B_2 has been generated by rounding a random value uniformly distributed in the interval $[-5, 5]$. By direct computation, it can be verified that all the eigenvalues of both A_1 and A_2 are unstable and that the set $\{\mathcal{S}_1^1, \mathcal{S}_2^1\}$ is transverse. Also, B_1 and B_2 have full column rank and (A_1, B_1), (A_2, B_2) are controllable. According to (16), we have $p_1 = 2$ and from (24), $q_1 = 0$. According to Theorem 2, the given system is SLASF and $p_\ell > 0$ at every iteration of Algorithm ITF, irrespective of the choice of eigenvalues performed in Procedure CEA. Choosing the stable eigenvalues $\lambda_i^\ell = 0$ for $i = 1$, 2 at every iteration ℓ, Algorithm ITF yields

$$K_1 = \begin{bmatrix} 6.0399 & -2.4801 & -0.7205 & -3.4911 \\ 1.3398 & 0.1776 & 0.3202 & -0.3436 \\ 0.5672 & -0.5322 & -0.7014 & -1.0545 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 1.1774 & -0.1109 & 1.4596 & 0.4449 \\ -0.6460 & -1.0743 & -0.6828 & -0.6902 \\ -0.7242 & -0.6951 & -0.7428 & 0.5276 \end{bmatrix}.$$

It can be verified that all the eigenvalues of A_i^{CL} [recall (5)] are zero (within rounding inaccuracy) and that $T^{-1}A_i^{\mathrm{CL}}T$ is upper triangular for $i = 1, 2$, with

$$T = \begin{bmatrix} 0.1570 & -0.9876 & 0 & 0 \\ -0.5668 & -0.0901 & -0.8189 & 0 \\ -0.3808 & -0.0605 & 0.2702 & 0.8822 \\ 0.7135 & 0.1134 & -0.5063 & 0.4709 \end{bmatrix}.$$

See [16] for more numerical examples on cases where genericity conditions hold at all or just some iterations of Algorithm ITF.

V. Conclusion

We have considered both continuous- and discrete-time SLSs with control inputs and under arbitrary switching. A stability result for SLSs with no control inputs states that the SLS is stable if the subsystem A matrices are stable and generate a solvable Lie algebra. This stability result encounters very limited applicability due to its restrictiveness and non-robustness. However, we have established that when control inputs are present, the property of existence of feedback matrices so that the closed-loop SLS subsystem matrices are stable and generate a solvable Lie algebra can become generic, i.e. valid for almost every set of system parameters. We have derived sufficient conditions that ensure the genericity of this property. In order for these conditions to hold in non-trivial cases, the number of subsystems of the SLS has to be not greater than half the number of system states and every subsystem is required to have more control inputs than half the number of states.

Since the aforementioned Lie-algebraic stability condition implies the existence of a CQLF for the SLS, our results also provide an analytic sufficient condition for the genericity of the existence of feedback matrices so that the closed-loop SLS admits a CQLF.

Appendix

Proof of Lemma 3: The proof of (a)–(c) is a direct application of subspace algebra.

(d) ($\Rightarrow$) For a set $K \subset \underline{N}$, define $p_K = \mathrm{d}(\mathcal{X}) + \sum_{i\in K} \mathrm{d}(\mathcal{S}_i) - \#K\mathrm{d}(\mathcal{X})$. Since $p_{\underline{N}} = p \geq 0$, then $p_I \geq 0$ and $p_J \geq 0$ because $\mathrm{d}(\mathcal{S}_i) \leq \mathrm{d}(\mathcal{X})$ for all $i \in \underline{N}$. By Lemma 3(b) and since $p_I \geq 0$ and $p_J \geq 0$, then $\mathrm{d}(\bigcap_{i\in I} \mathcal{S}_i) = p_I$ and $\mathrm{d}(\bigcap_{i\in J} \mathcal{S}_i) = p_J$. By Lemma 3(a), we have

$$\begin{aligned} \mathrm{d}\left(\bigcap_{i\in J} \mathcal{S}_i\right) &= \mathrm{d}\left(\bigcap_{i\in I} \mathcal{S}_i\right) + \mathrm{d}(\mathcal{S}_j) - \mathrm{d}\left(\bigcap_{i\in I} \mathcal{S}_i + \mathcal{S}_j\right) \\ &= p_J = p_I + \mathrm{d}(\mathcal{S}_j) - \mathrm{d}\left(\bigcap_{i\in I} \mathcal{S}_i + \mathcal{S}_j\right). \end{aligned} \tag{28}$$

Necessity is established by substituting the expressions for p_I and p_J into (28) and recalling that $\#J = \#I + 1$.

($\Leftarrow$) Let $K \subset I$. We have $\mathrm{d}(\mathcal{X}) = \mathrm{d}(\bigcap_{i\in I}\mathcal{S}_i + \mathcal{S}_j) \le \mathrm{d}(\bigcap_{i\in K}\mathcal{S}_i + \mathcal{S}_j) \le \mathrm{d}(\mathcal{X})$. Taking $K = \{k\}$, jointly with the fact that $\{\mathcal{S}_i : i \in I\}$ is transverse, establishes that the dimension of the sum of the subspaces in every subset of $\{\mathcal{S}_i : i \in J\}$ has maximum dimension. Also, we have

$$\mathrm{d}\left(\bigcap_{i\in K}\mathcal{S}_i \cap \mathcal{S}_j\right) = \mathrm{d}\left(\bigcap_{i\in K}\mathcal{S}_i\right) + \mathrm{d}(\mathcal{S}_j) - \underbrace{\mathrm{d}\left(\bigcap_{i\in K}\mathcal{S}_i + \mathcal{S}_j\right)}_{\mathrm{d}(\mathcal{X})}$$

which, jointly with the fact that $\{\mathcal{S}_i : i \in I\}$ is transverse, establishes that the dimension of the intersection of the subspaces in every subset of $\{\mathcal{S}_i : i \in J\}$ has minimum dimension. ∎

Proof of Theorem 2: The proof of Theorem 2 requires the following two lemmas.

Lemma 5: Let c_i^ℓ denote the number of controllability indices of (A_i^ℓ, B_i^ℓ) that are equal to 1. Then, $\rho_i^\ell = c_i^\ell$.

Proof: According to the standard construction for the controllability indices of a system (see, e.g. [20]), it follows that $c_i^\ell = 2m_i^\ell - \mathrm{rank}[\beta_i^\ell, A_i^\ell\beta_i^\ell]$, where β_i^ℓ is any matrix satisfying $\mathrm{img}\beta_i^\ell = \mathrm{img}B_i^\ell = \mathcal{B}_i^\ell$. Since $\mathcal{S}_i^\ell \subset \mathcal{B}_i^\ell$, write $\mathcal{B}_i^\ell = \hat{\mathcal{B}}_i^\ell \oplus \mathcal{S}_i^\ell$ and let $\alpha = \mathrm{d}(\hat{\mathcal{B}}_i^\ell)$. Then, $\rho_i^\ell = m_i^\ell - \alpha$.

Let $\{b_1, \ldots, b_\alpha\}$ be a basis for $\hat{\mathcal{B}}_i^\ell$, $\{b_{\alpha+1}, \ldots, b_{m_i^\ell}\}$ be a basis for $\mathcal{S}_i^\ell$, and $\beta_i^\ell = [b_1, \ldots, b_{m_i^\ell}]$. By Lemma 4(b), we have that $A_i^\ell b_k \notin \mathcal{B}_i^\ell$ for $k = 1, \ldots, \alpha$ and $A_i^\ell b_k \in \mathcal{B}_i^\ell$ for $k = \alpha+1, \ldots, m_i^\ell$. Therefore, $\mathrm{rank}[\beta_i^\ell, A_i^\ell\beta_i^\ell] \le m_i^\ell + \alpha$.

If $\mathrm{rank}[\beta_i^\ell, A_i^\ell\beta_i^\ell] < m_i^\ell + \alpha$, then $\sum_{j=1}^{m_i^\ell} c_j b_j + \sum_{k=1}^{\alpha} d_k A_i^\ell b_k = 0$ for some scalars c_j and d_k, where not all the d_k are zero. Then, $A_i^\ell \sum_{k=1}^{\alpha} d_k b_k \in \mathcal{B}_i^\ell$ and $A_i^\ell \sum_{k=1}^{\alpha} d_k b_k \notin \mathcal{S}_i^\ell$, which leads to a contradiction. Therefore, $\mathrm{rank}[\beta_i^\ell A_i^\ell \beta_i^\ell] = m_i^\ell + \alpha$ and $\rho_i^\ell = m_i^\ell - \alpha = 2m_i^\ell - \mathrm{rank}[\beta_i^\ell A_i^\ell \beta_i^\ell]$. ∎

Lemma 6: Consider Algorithm ITF at iteration ℓ. Suppose that (A_i^ℓ, B_i^ℓ) is controllable and $A_i^{\ell,\mathrm{CL}} v_1^\ell = \lambda_i^\ell v_1^\ell$ with $v_1^\ell \neq 0$ and scalar λ_i^ℓ. Then, $\mathcal{S}_i^{\ell+1} \supset U_{\ell+1}^* \mathcal{S}_i^\ell$, $(A_i^{\ell+1}, B_i^{\ell+1})$ is controllable, and

$$\rho_i^{\ell+1} \begin{cases} = \rho_i^\ell - 1 & \text{if } v_1^\ell \in \mathcal{S}_i^\ell, \\ \ge \rho_i^\ell & \text{otherwise.} \end{cases} \tag{29}$$

Proof: Let $\{t_j : j = 1, \ldots, m_i^\ell\}$ be a basis for $\mathcal{B}_i^\ell$ and let $\kappa_{i,j}^\ell$, for $j = 1, \ldots, m_i^\ell$ be the controllability indices of (A_i^ℓ, B_i^ℓ). By (8) and the feedback invariance of controllability indices, $\kappa_{i,j}^\ell$ also are the controllability indices of the pair $(A_i^{\ell,\mathrm{CL}}, B_i^\ell)$.

Since (A_i^ℓ, B_i^ℓ) is controllable, then $(A_i^{\ell,\mathrm{CL}}, B_i^\ell)$ also is controllable, and $D = \{(A_i^{\ell,\mathrm{CL}})^k t_j : j = 1, \ldots, m_i^\ell : k = 0, \ldots, \kappa_{i,j}^\ell - 1\}$ is a basis for $\mathbb{R}^{n_\ell}$. Write v_1^ℓ with respect to the basis D: $v_1^\ell = \sum_{j,k} c_{j,k} (A_i^{\ell,\mathrm{CL}})^k t_j$, where not all the $c_{j,k}$ are zero. Combining the latter with $A_i^{\ell,\mathrm{CL}} v_1^\ell = \lambda_i^\ell v_1^\ell$ yields

$$\sum_{j,k} c_{j,k} \left(A_i^{\ell,\mathrm{CL}}\right)^{k+1} t_j = \sum_{j,k} \lambda_i^\ell c_{j,k} \left(A_i^{\ell,\mathrm{CL}}\right)^k t_j. \tag{30}$$

From (30), it follows that $c_{j,k} \neq 0$ for at least one pair of indices (j,k) such that $k = \kappa_{i,j}^\ell - 1$, or otherwise the vectors in D would be linearly dependent, a contradiction.

Let $\bar{\kappa} = \max_j\{\kappa_{i,j}^\ell : c_{j,k} \neq 0 \text{ with } k = \kappa_{i,j}^\ell - 1\}$, and let $\bar{\imath}$ be such that $c_{\bar{\imath},\bar{\kappa}-1} \neq 0$. From the basis D, construct another basis, $\bar{D}$, by replacing the basis vector $(A_i^{\ell,\mathrm{CL}})^{\bar{\kappa}-1} t_{\bar{\imath}}$ by v_1^ℓ. Note that $\mathrm{Span}\{U_{\ell+1}^* t : t \in \bar{D}\} = \mathbb{R}^{n_\ell - 1}$ and $U_{\ell+1}^* v_1^\ell = 0$ (cf. Section III-A). By (10)–(12) and the fact that $A_i^{\ell,\mathrm{CL}} v_1^\ell = \lambda_i^\ell v_1^\ell$, then $U_{\ell+1}^* A_i^{\ell,\mathrm{CL}} = A_i^{\ell+1} U_{\ell+1}^*$. Hence $U_{\ell+1}^* (A_i^{\ell,\mathrm{CL}})^k t_j = (A_i^{\ell+1})^k U_{\ell+1}^* t_j$ and

$$\left\{ \left(A_i^{\ell+1}\right)^k U_{\ell+1}^* t_j : j = 1, \ldots, m_i^\ell; \quad k = 0, \ldots, \kappa_{i,j}^\ell - 1, (j,k) \neq (\bar{\imath}, \bar{\kappa}-1) \right\} \tag{31}$$

is a basis for $\mathbb{R}^{n_\ell - 1}$ [recall that, by (6), $n_{\ell+1} = n_\ell - 1$]. From (13), it follows that $\mathcal{B}_i^{\ell+1} = U_{\ell+1}^* \mathcal{B}_i^\ell$. We have that a basis for $\mathcal{B}_i^{\ell+1}$, is $E = \{U_{\ell+1}^* t_j : j = 1, \ldots, m_i^\ell\}$ if $\bar{\kappa} > 1$ or $E = \{U_{\ell+1}^* t_j : j = 1, \ldots, m_i^\ell; j \neq \bar{\imath}\}$ if $\bar{\kappa} = 1$. The condition $\bar{\kappa} = 1$ hence happens if and only if $v_1^\ell \in \mathcal{S}_i^\ell$.

The preceding derivations show that the controllability indices of $(A_i^{\ell+1}, B_i^{\ell+1})$ are given by $\kappa_{i,j}^{\ell+1} = \kappa_{i,j}^\ell$ for $j = 1, \ldots, m_i^\ell$ with $j \neq \bar{\imath}$ and $\kappa_{i,\bar{\imath}}^{\ell+1} = \kappa_{i,\bar{\imath}}^\ell - 1$ whenever $\bar{\kappa} = \kappa_{i,\bar{\imath}}^\ell > 1$. From the latter expressions, and recalling Lemma 5, (29) and the controllability of $(A_i^{\ell+1}, B_i^{\ell+1})$ are established.

Note that $\rho_i^{\ell+1} = \rho_i^\ell + 1$ whenever $\bar{\kappa} = 2$, since then $\kappa_{i,\bar{\imath}}^{\ell+1} = \bar{\kappa} - 1 = 1$ and hence $(A_i^{\ell+1}, B_i^{\ell+1})$ has one controllability index equal to one more than (A_i^ℓ, B_i^ℓ). The fact that $\mathcal{S}_i^{\ell+1} \supset U_{\ell+1}^* \mathcal{S}_i^\ell$ follows from the latter consideration and the basis E. ∎

Proof of Theorem 2: (a) First, we prove that the conditions

$$\left\{\mathcal{S}_i^\ell : i \in \underline{N}\right\} \text{ transverse}, \quad q_\ell \ge 0, \left(A_i^\ell, B_i^\ell\right) \text{ controllable} \tag{32}$$

imply that $p_\ell > 0$. Since $\mathcal{S}_i^\ell \subset \mathcal{B}_i^\ell$, then $\rho_i^\ell \le m_i^\ell$ and $q_\ell \le p_\ell$. From controllability of (A_i^ℓ, B_i^ℓ) and Lemma 5, then $\rho_i^\ell = m_i^\ell$ if and only if $m_i^\ell = n_\ell$. Hence, if $q_\ell = p_\ell$, then $p_\ell = n_\ell > 0$. Otherwise, $0 \le q_\ell < p_\ell$.

Next, we establish the validity of (32) for $\ell = 1, \ldots, n$. Note that (32) hold at $\ell = 1$ by assumption and because $A_i^1 = A_i$ and $B_i^1 = B_i$. Next, suppose that (32) hold at some $1 \le \ell \le n-1$. By the argument in the previous paragraph, then $p_\ell > 0$, which ensures the existence and computation of $v_1^\ell \neq 0$ such that $A_i^{\ell,\mathrm{CL}} v_1^\ell = \lambda_i^\ell v_1^\ell$ with scalar λ_i^ℓ for all $i \in \underline{N}$. Hence, $(A_i^{\ell+1}, B_i^{\ell+1})$ is controllable by Lemma 6.

Also by Lemma 6, we have $\mathcal{S}_i^{\ell+1} \supset U_{\ell+1}^* \mathcal{S}_i^\ell$ for all $i \in \underline{N}$. Since $q_\ell \ge 0$, by Lemma 3(b) we have that $\rho^\ell = q_\ell$, and from Lemma 3(c) we have $\mathrm{d}(\mathcal{S}_i^\ell + \mathcal{S}_j^\ell) = n_\ell$ for all $i, j \in \underline{N}$ with $i \neq j$. It follows that $\mathrm{d}(\mathcal{S}_i^{\ell+1} + \mathcal{S}_j^{\ell+1}) \ge \mathrm{d}(U_{\ell+1}^*(\mathcal{S}_i^\ell + \mathcal{S}_j^\ell)) = n_\ell - 1 = n_{\ell+1}$ for all $i, j \in \underline{N}$ with $i \neq j$. The latter fact establishes that the sum of the sets in every subset of $\{\mathcal{S}_i^{\ell+1} : i \in \underline{N}\}$ has maximal dimension and also that $\{\mathcal{S}_i^{\ell+1}, \mathcal{S}_j^{\ell+1}\}$ is transverse for all $i, j \in \underline{N}$ with $i \neq j$.

Let T be a subset of $\{\mathcal{S}_i^{\ell+1} : i \in \underline{N}\}$. We proceed by induction on the number of subspaces in T. We have already established that T is transverse if $\#T = 2$. Suppose next that T is transverse whenever $\#T = 2, \ldots, \alpha$, with $\alpha \le N-1$. Let $T = \{\mathcal{S}_i^{\ell+1} : i \in I\}$, with $I \subset \underline{N}$ and $\#I = \alpha$, and let $R = T \cup \{\mathcal{S}_j^{\ell+1}\}$ so that $\#R = \alpha + 1$. By Lemma 6 and properties of maps and subspaces, we have

$$\bigcap_{i\in I}\mathcal{S}_i^{\ell+1} + \mathcal{S}_j^{\ell+1} \supset \bigcap_{i\in I} U_{\ell+1}^*\mathcal{S}_i^\ell + U_{\ell+1}^*\mathcal{S}_j^\ell \supset U_{\ell+1}^*\left(\bigcap_{i\in I}\mathcal{S}_i^\ell + \mathcal{S}_j^\ell\right). \tag{33}$$

By (32) and since $I \subset \underline{N}$, then $\{\mathcal{S}_i^\ell : i \in I\}$ is transverse. By Lemma 3(d), then $\mathrm{d}(\bigcap_{i\in I}\mathcal{S}_i^\ell + \mathcal{S}_j^\ell) = n_\ell$. Combining the latter equality with (33), then $\mathrm{d}(\bigcap_{i\in I}\mathcal{S}_i^{\ell+1} + \mathcal{S}_j^{\ell+1}) = n_\ell - 1 = n_{\ell+1}$. By Lemma 3(d) then R is transverse. We have thus established that our induction hypothesis is valid for $\alpha + 1$ and we conclude that $\{\mathcal{S}_i^{\ell+1} : i \in \underline{N}\}$ is transverse. By Lemma 3(b), then $\rho^{\ell+1} = \max\{0, q_{\ell+1}\}$.

From (24) and (29), it follows that the minimum value for $q_{\ell+1}$ is $q_\ell - 1$, and this happens only if $\rho_i^{\ell+1} = \rho_i^\ell - 1$ for all $i \in \underline{N}$. However, if $q_\ell = 0$, then $\rho_i^{\ell+1} \ge \rho_i^\ell$ for at least one $i \in \underline{N}$ because, since $\rho^\ell = \mathrm{d}(\mathcal{S}^\ell) = q_\ell = 0$, then $v_1^\ell \notin \mathcal{S}^\ell$. Consequently $q_{\ell+1} \ge 0$ and hence we have established (32) for $\ell = 1, \ldots, n$.

(b) By Theorem 2(a), the structural condition $p_\ell > 0$ will hold at every iteration of Algorithm ITF. Consequently, a stable feedback-assignable common eigenvector exists and can be computed at every iteration of the algorithm, and thus the algorithm will finish successfully, yielding feedback matrices $K_i \in \mathbb{R}^{m_i \times n}$ so that the system (1) or (2) is SLASF. ∎

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