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RESEARCH ARTICLES

The Lagrange–D'Alembert–Poincaré Equations and Integrability for the Euler's Disk

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Abstract—Nonholonomic systems are described by the Lagrange–D'Alembert's principle. The presence of symmetry leads, upon the choice of an arbitrary principal connection, to a reduced D'Alembert's principle and to the Lagrange–D'Alembert–Poincaré reduced equations. The case of rolling constraints has a long history and it has been the purpose of many works in recent times. In this paper we find reduced equations for the case of a thick disk rolling on a rough surface, sometimes called *Euler's disk*, using a 3-dimensional abelian group of symmetry. We also show how the reduced system can be transformed into a single second order equation, which is an hypergeometric equation.

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1. INTRODUCTION

The study of the motion of a disk rolling on a horizontal rough plane has been the purpose of many works since at least the middle of the XIX century, as part of a more general study of systems with rolling constraints, see for instance [1–27] for historical remarks concerning nonholonomic systems and, in particular, the rolling disk. The recent literature on the rolling disk includes important advances in the study of the dynamics from a qualitative point of view, see for instance, [5, 12, 15]. In [5, 28] a qualitative analysis of the motion of the point of contact of the disk with the plane is done, including a computer analysis of the trajectories. In some of the previous works a reduction by a 4-dimensional group of symmetry is performed. In [7] a 3-dimensional abelian group of symmetry is used to reduce the planar disk and the geometric aspects of the problem are emphasized by using a reduced D'Alembert's principle and obtaining Lagrange–D'Alembert–Poincaré reduced equations, see [8] for the general theory of Lagrange–D'Alembert–Poincaré equations. As a consequence, equations of motion in terms of variables which include Euler's angles are naturally obtained.

The present paper generalizes [7] in the sense that we carry out a similar program, but this time for the so called *Euler's disk*. This is a *thick disk*, that is, a cylinder, rolling without sliding on its rim on a horizontal rough plane. It is assumed that there is only one point of contact between the disk and the plane, in other words, we study only the part of the motion satisfying this condition, and chose the configuration space accordingly. However, the equations of motion are analytic and can be extended naturally to a bigger manifold which includes the vertical position, with no physical meaning. The distribution of mass of the disk is assumed to have circular symmetry with respect to the axis perpendicular to the disk and passing through the center, which is assumed to coincide with the center of mass, so two of the three principal moments of inertia are equal. The case of a planar disk is the case of zero thickness, studied in [7]. Equations of motion for this system in terms of Euler's angles have been written in [15], using balance of momentum arguments and mentioning

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previous results on integrability due to Appell [29, 30]. While it is perhaps in some aspects easier to work with a system reduced by a 4-dimensional group of symmetry, it is also of interest to have reduced equations of motion which involve more variables. The main point of the present paper is that we apply the geometrically inspired method described in [8], which involves *reducing D'Alembert's principle* rather than using balance of momentum, to obtain the reduced Lagrange–D'Alembert–Poincaré equations (Eqs. (1)–(4) in Section 2), using a 3-dimensional group of symmetry. We also derive an hypergeometric equation involving some of the variables. One advantage of this approach is that it opens the door for the application of new ideas in numerical analysis that explicitly make use of the D'Alembertian approach, [31, 32] and references therein. By further transforming the Lagrange–D'Alembert–Poincaré equations we obtain eight explicit equations of motion in terms of our variables. The first six of those equations, or equivalently Eqs. (13)–(16), are consistent with the four equations obtained in [15] (equations (5.3) of [15]); while the last two of them give the acceleration of the point of contact between the disk and the plane. In [15] also equations for the sliding disk were obtained.

Although in the present paper we only consider the idealized model of the Euler's disk described above, we must at least mention that different types of dissipative phenomena have been recently studied both from a theoretical and also an experimental perspective, see for instance [15, 22, 33–37]. Having general equations for the idealized model of the Euler's disk is a starting point to understand more realistic models which include dissipation.

In Section 2, we apply the techniques of reduction given in [8] to describe the Lagrange–d'Alembert–Poincaré equations for the example of the Euler's disk and reduce it to an hypergeometric equation.

2. THE MOTION OF THE EULER'S DISK

2.1. Kinematics of the Rigid Body

The configuration space for the rigid body is the group $SO(3)$, see for instance [38, 39]. The motion of the rigid body is given by a curve $A(t)$ on $SO(3)$. The *spatial angular velocity* $\hat{\omega}$ and the *body angular velocity* $\hat{\Omega}$ are elements of the Lie algebra $\mathfrak{so}(3)$ and they are defined by the conditions $\dot{A} = A\,\hat{\Omega} = \hat{\omega}\,A$.

We recall that there is a natural identification $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$ given by

$$\hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix},$$

where $x = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3$.

We have the well known formulas

$$\widehat{x \times y} = [\hat{x}, \hat{y}], \quad x \cdot y = -\frac{1}{2}\mathrm{tr}\hat{x}\hat{y} \quad \text{and} \quad \hat{x}\,y = x \times y.$$

Besides, if A is any element of $SO(3)$ and x is any element of $\mathbb{R}^3$ we have

$$\widehat{A\,x} = A\hat{x}A^{-1}.$$

For any motion $A(t)$, define $z(t) = A(t)\,\mathbf{e}_3$. Then

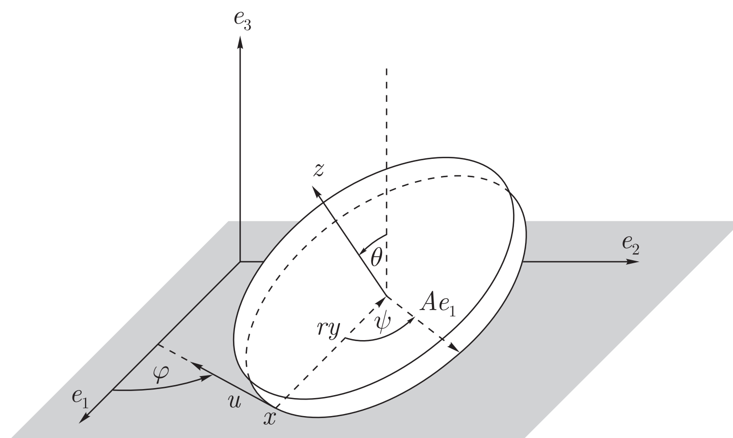
$$\dot{z} = \dot{A}\,\mathbf{e}_3 = \hat{\omega}z = \omega \times z.$$

We have that $\langle \omega, z\rangle = \langle \Omega, \mathbf{e}_3\rangle = \Omega^3$, and that $A(\Omega^1\hat{\mathbf{e}}_1 + \Omega^2\hat{\mathbf{e}}_2)A^{-1} = \widehat{(z \times \dot{z})}$. The spatial velocity ω can be written $\omega = A\Omega$ and then $\omega = \Omega^3\,z + z \times \dot{z}$. This gives a decomposition of ω as a sum of its component parallel to z plus its component normal to z.

Now we are going to apply the reduction theory described in Ref. [8] to the Euler's disk. See also appendix in Ref. [7].

2.2. Kinematics of the Euler's Disk

Let us consider an Euler's disk of radius r and thickness re rolling on a rough horizontal plane and having only one point of contact with it, as described in the introduction. We choose a fixed reference frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ such that the axis $\mathbf{e}_3$ is directed upwards and the axis $\mathbf{e}_1$ and $\mathbf{e}_2$ lie on the horizontal plane, as indicated in the figure below. For each $A \in SO(3)$ the ortonormal frame $(A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3)$ is rigidly attached to the side of the disk on which the point of contact with the surface lies and has its origin at the center of this side, in such a way that the axis $z = A\mathbf{e}_3$ is directed to the other side. The point of contact of the disk with the plane is $x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = (x_1, x_2, 0)$.



Euler's disk.

We are interested only in the motion of the disk satisfying the condition $0 < \langle A(t)\mathbf{e}_3, \mathbf{e}_3\rangle < 1$. Then, the configuration space for the Euler's disk is

$$Q = \left(0, \frac{\pi}{2}\right) \times S^1 \times S^1 \times \mathbb{R}^2,$$

and a point $q \in Q$ is written as $q = (\theta, \varphi, \psi, x)$, where the meaning of the variables $\theta,\ \varphi,\ \psi$ is the following. The angle θ is the angle from the axis $\mathbf{e}_3$ to the vector $z = A\mathbf{e}_3$. The vector y is the unit vector directed from the point of contact with the plane to the origin of the system $(A\mathbf{e}_1, A\mathbf{e}_2, A\mathbf{e}_3)$. The vector u is defined by $u = z \times y$, so u is tangent to the disk at the point of contact x and has the direction of the motion of this point on the plane. The unit vector u has the expression $u = (-\cos\varphi, -\sin\varphi, 0)$ which defines the angle φ. The angle ψ is the angle from the vector $-y$ to the vector $A\mathbf{e}_1$ where the positive sense for measuring the angle ψ on the plane of the disk is the counterclockwise sense, as viewed from z.

We have the vector $z = A\mathbf{e}_3$ where the matrix A is

$$A = \begin{pmatrix} -\cos\theta\cos\psi\sin\varphi - \cos\varphi\sin\psi & \cos\theta\sin\psi\sin\varphi - \cos\varphi\cos\psi & \sin\theta\sin\varphi \\ \cos\theta\cos\psi\cos\varphi - \sin\varphi\sin\psi & -\cos\theta\sin\psi\cos\varphi - \cos\psi\sin\varphi & -\sin\theta\cos\varphi \\ \sin\theta\cos\psi & -\sin\theta\sin\psi & \cos\theta \end{pmatrix}.$$

We are going to assume that the initial position of the disk corresponds to the disk lying on the plane with the vector z directed upwards, in which case we have $\theta = 0,\ \varphi = \pi/2,\ \ \psi = \pi$ and $x = (r, 0, 0)$.

Notice that in this case we have $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The spatial angular velocity ω and the body angular velocity Ω, given by the formulas $\hat{\omega} = \dot{A}A^{-1}$ and $\hat{\Omega} = A^{-1}\dot{A}$, and the identification $\hat{}\colon \mathbb{R}^3 \to \mathfrak{so}(3)$, are

$$\omega = (\dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi, \dot{\theta}\sin\varphi - \dot{\psi}\cos\varphi\sin\theta, \dot{\varphi} + \dot{\psi}\cos\theta)$$

and

$$\Omega = (-\dot{\theta}\sin\psi + \dot{\varphi}\cos\psi\sin\theta, -\dot{\theta}\cos\psi - \dot{\varphi}\sin\theta\sin\psi, \dot{\varphi}\cos\theta + \dot{\psi}).$$

The nonholonomic constraint is given by the distribution

$$\mathcal{D}_{(\theta,\varphi,\psi,x)} = \left\{(\theta,\varphi,\psi,x,\dot{\theta},\dot{\varphi},\dot{\psi},\dot{x}) | \dot{x} = \dot{\psi}ru\right\}.$$

The symmetry group is $G = SO(2) \times \mathbb{R}^2$, where the group $SO(2)$ is identified with the set of elements of $SO(3)$ of the type

$$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the factor $\mathbb{R}^2$ is identified with the subspace of $\mathbb{R}^3$ defined by $x_3 = 0$.

The group $SO(2) \times \mathbb{R}^2$ acts on the right on Q by the action

$$(\theta,\varphi,\psi,x)(\alpha,a) = (\theta,\varphi,\psi+\alpha,x+a),$$

where the sum $\psi + \alpha$ is defined modulo 2π. With this action Q becomes a right principal bundle with structure group $SO(2) \times \mathbb{R}^2$. The map $\pi\colon Q \to C$, where $C = (0,\pi/2) \times S^1$, given by $\pi(\theta,\varphi,\psi,x) = (\theta,\varphi)$ is a submersion and we have an identification $Q/\left(SO(2)\times\mathbb{R}^2\right) \equiv C$ given by

$$[(\theta,\varphi,\psi,x)]_{SO(2)\times\mathbb{R}^2} \equiv (\theta,\varphi).$$

The vertical distribution $\mathcal{V}$ is given by

$$\mathcal{V}_{(\theta,\varphi,\psi,x)} = \left\{(\theta,\varphi,\psi,x,\dot{\theta},\dot{\varphi},\dot{\psi},\dot{x}) | \dot{\theta} = 0, \dot{\varphi} = 0\right\}.$$

The vector bundle $\mathcal{S} = \mathcal{D} \cap \mathcal{V}$ is given by

$$\mathcal{S}_{(\theta,\varphi,\psi,x)} = \{(\theta,\varphi,\psi,x,0,0,\xi,\xi ru)\}.$$

Since $dim\mathcal{D}_{(\theta,\varphi,\psi,x)} = 3, dim\mathcal{V}_{(\theta,\varphi,\psi,x)} = 3$ and $dim\mathcal{S}_{(\theta,\varphi,\psi,x)} = 1$, we have

$$\mathcal{D}_{(\theta,\varphi,\psi,x)} + \mathcal{V}_{(\theta,\varphi,\psi,x)} = T_{(\theta,\varphi,\psi,x)}Q$$

that is, the dimension assumption (see appendix in Ref. [7]) is satisfied.

We choose the horizontal spaces

$$\mathcal{H}_{(\theta,\varphi,\psi,x)} = \{(\theta,\varphi,\psi,x,\alpha,\beta,0,0)\}$$

satisfying

$$\mathcal{H}_{(\theta,\varphi,\psi,x)} \oplus \mathcal{S}_{(\theta,\varphi,\psi,x)} = \mathcal{D}_{(\theta,\varphi,\psi,x)}.$$

It is also clear that the distribution $\mathcal{H}$ is $SO(2)\times\mathbb{R}^2$-invariant. The connection 1-form $\mathcal{A}$ whose horizontal spaces are $\mathcal{H}_{(\theta,\varphi,\psi,x)}$ is

$$\mathcal{A}(\theta,\varphi,\psi,x,\dot{\theta},\dot{\varphi},\dot{\psi},\dot{x}) = (\dot{\psi},\dot{x}).$$

The adjoint bundle $\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$ is a trivial bundle and we have an identification

$$\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2} \equiv C \times (\mathfrak{so}(2) \times \mathbb{R}^2)$$

given by

$$[(\theta, \varphi, \psi, x, 0, 0, \xi, a)]_{SO(2)\times\mathbb{R}^2} \equiv (\theta, \varphi, \xi, a).$$

The vector bundle isomorphism

$$\alpha_{\mathcal{A}} : TQ/G \to T(Q/G) \oplus \tilde{\mathfrak{g}},$$

is described as follows.

Since G is abelian and $Q/\left(SO(2) \times \mathbb{R}^2\right) \equiv C$ we obtain $\tilde{\mathfrak{g}} \equiv C \times \left(\mathfrak{so}(2) \times \mathbb{R}^2\right)$ and

$$TC \oplus \tilde{\mathfrak{g}} \equiv C \times \mathbb{R}^2 \oplus C \times \left(\mathfrak{so}(2) \times \mathbb{R}^2\right) \equiv C \times \mathbb{R}^2 \oplus \left(\mathfrak{so}(2) \times \mathbb{R}^2\right).$$

Then

$$\alpha_{\mathcal{A}}\left([(\theta, \varphi, \psi, x, \dot\theta, \dot\varphi, \dot\psi, \dot x)]_{SO(2)\times\mathbb{R}^2}\right) = (\theta, \varphi, \dot\theta, \dot\varphi) \oplus (\theta, \varphi, \dot\psi, \dot x).$$

The subbundle $\tilde{\mathfrak{s}} \subset \widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$ is given by

$$\tilde{\mathfrak{s}} = \{(\theta, \varphi, 0, 0) \oplus (\theta, \varphi, \xi, \xi r u)\}.$$

About the structure of the bundle $\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$ we have, since the Lie algebra $\mathfrak{so}(2) \times \mathbb{R}^2$ is abelian, that the Lie algebra structure on each fiber of $\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$ is abelian.

Let $(\theta, \varphi, \xi, a)$ be a curve on $\widetilde{\mathfrak{so}(2) \times \mathbb{R}^2}$. Since the group $SO(2) \times \mathbb{R}^2$ is abelian using the formula in appendix in Ref. [7] for the covariant derivative, we have

$$\frac{D(\theta, \varphi, \xi, a)}{Dt} = (\theta, \varphi, \dot\xi, \dot a).$$

The $\tilde{\mathfrak{g}}$-valued 2-form $\tilde{\mathcal{B}}$ is equal to zero because the distribution of horizontal spaces is integrable.

2.3. Dynamical Equations for the Euler's Disk

As we have explained in the introduction, the center of mass of the Euler's disk coincides with the geometric center of the disk, its moments of inertia with respect to the axis that are parallel to $A\mathbf{e}_1$ and $A\mathbf{e}_2$ and pass through the center of mass are equal, say $I_1 = I_2$. The moment of inertia I_3 with respect to the axis $z = A\mathbf{e}_3$ is arbitrary.

Let $w = x + ry$ and let $\tilde w = x + r\tilde y$ be the center of mass, where $\tilde y = y + (1/2)ez$ and $y = u \times z$.

Then the Lagrangian $L\colon TQ \to \mathbb{R}$ for such a system, given by the kinetic minus the potential energy, is

$$\begin{aligned} L &= \frac{1}{2}I_1\dot z^2 + \frac{1}{2}I_3(\Omega^3)^2 + \frac{1}{2}M\dot{\tilde w}^2 - Mg\tilde w_3 \\ &= \frac{1}{2}I_1\dot z^2 + \frac{1}{2}I_3(\Omega^3)^2 + \frac{1}{2}M\dot w^2 - Mgry_3 + \frac{1}{2}Mre\langle \dot w, \dot z\rangle + \frac{1}{8}Mr^2e^2\dot z^2 - \frac{1}{2}Mgrez_3, \end{aligned}$$

where g is the acceleration of gravity and M is the mass of the disk.

This Lagrangian may be written as

$$\begin{aligned} &L(\theta, \varphi, \psi, x, \dot\theta, \dot\varphi, \dot\psi, \dot x) = -Mgr\sin\theta + \frac{1}{2}\left(I_1 + \frac{1}{4}Mr^2e^2\right)(\dot\theta^2 + \dot\varphi^2\sin^2\theta) + \frac{1}{2}M\dot x^2 + \frac{1}{2}Mr^2\dot\theta^2 \\ &+ \frac{1}{2}Mr^2\dot\varphi^2\cos^2\theta + Mr\dot x_1(\dot\theta\sin\theta\sin\varphi - \dot\varphi\cos\theta\cos\varphi) - Mr\dot x_2(\dot\theta\cos\varphi\sin\theta + \dot\varphi\cos\theta\sin\varphi) \\ &+ \frac{1}{2}I_3(\dot\varphi\cos\theta + \dot\psi)^2 + \frac{1}{2}Mre\Big[\dot x_1(\dot\varphi\cos\varphi\sin\theta + \dot\theta\sin\varphi\cos\theta) \\ &+ \dot x_2(\dot\varphi\sin\varphi\sin\theta - \dot\theta\cos\theta\cos\varphi) - \dot\varphi^2 r\cos\theta\sin\theta\Big] - \frac{1}{2}Mgre\cos\theta. \end{aligned}$$

The Lagrangian L and the constraint $\mathcal{D}$ are invariant under the right action of the 3-dimensional abelian group $SO(2) \times \mathbb{R}^2$.

Using the isomorphism

$$\alpha_{\mathcal{A}}\left([(\theta, \varphi, \psi, x, \dot{\theta}, \dot{\varphi}, \dot{\psi}, \dot{x})]_{SO(2)\times\mathbb{R}^2}\right) = (\theta, \varphi, \dot{\theta}, \dot{\varphi}) \oplus (\theta, \varphi, \overline{v})$$

where $\bar{v} = (v_0, v_1) = (\dot{\psi}, \dot{x})$, the reduced Lagrangian $\ell(\theta, \varphi, \dot{\theta}, \dot{\varphi}, \bar{v})$ is given by

$$\begin{aligned}\ell(\theta, \varphi, \dot{\theta}, \dot{\varphi}, \bar{v}) &= -Mgr\sin\theta + \frac{1}{2}\left(I_1 + \frac{1}{4}Mr^2e^2\right)(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta) + \frac{1}{2}M{v_1}^2 + \frac{1}{2}Mr^2\dot{\theta}^2\\
&+ \frac{1}{2}Mr^2\dot{\varphi}^2\cos^2\theta + Mrv_{11}(\dot{\theta}\sin\theta\sin\varphi - \dot{\varphi}\cos\theta\cos\varphi) - Mrv_{12}(\dot{\theta}\cos\varphi\sin\theta + \dot{\varphi}\cos\theta\sin\varphi)\\
&+ \frac{1}{2}I_3(\dot{\varphi}\cos\theta + v_0)^2 + \frac{1}{2}Mre\Big[v_{11}(\dot{\varphi}\cos\varphi\sin\theta + \dot{\theta}\sin\varphi\cos\theta)\\
&+ v_{12}(\dot{\varphi}\sin\varphi\sin\theta - \dot{\theta}\cos\theta\cos\varphi) - r\cos\theta\sin\theta\dot{\varphi}^2\Big] - \frac{1}{2}Mgre\cos\theta.\end{aligned}$$

Now we shall write the Lagrange–D'Alembert–Poincaré equations. Since the procedure is similar to the one explained in [7] we shall omit some technical details.

Since the group is abelian the vertical Lagrange–D'Alembert–Poincaré equations become

$$\frac{D}{Dt}\frac{\partial \ell}{\partial \bar{v}}\bigg|_{\tilde{\mathfrak{s}}} = 0.$$

Since $(\theta, \varphi, \bar{v}) \in \tilde{\mathfrak{s}}$ we have $v_1 = v_0 ru$. A generator of $\tilde{\mathfrak{s}}$ is

$$(\theta, \varphi, 1, ru),$$

and we have

$$\begin{aligned}\frac{d}{dt}\frac{\partial \ell}{\partial \bar{v}}(\theta, \varphi, 1, ru) =& I_3(\dot{v_0} + \ddot{\varphi}\cos\theta - \dot{\theta}\dot{\varphi}\sin\theta) + Mr^2(-2\dot{\theta}\dot{\varphi}\sin\theta + \ddot{\varphi}\cos\theta)\\
&+ Mr\langle\dot{v_1}, u\rangle - \frac{1}{2}Mer^2(2\dot{\theta}\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta).\end{aligned}$$

Since the covariant derivative coincides with the ordinary derivative in this case, we have the following vertical *Lagrange–D'Alembert–Poincaré equation* (see [7] for details)

$$I_3(\dot{v_0} + \ddot{\varphi}\cos\theta - \dot{\theta}\dot{\varphi}\sin\theta) + Mr\langle\dot{v_1}, u\rangle + Mr^2(\ddot{\varphi}\cos\theta - 2\dot{\theta}\dot{\varphi}\sin\theta) - \frac{1}{2}Mer^2(\ddot{\varphi}\sin\theta + 2\dot{\theta}\dot{\varphi}\cos\theta) = 0.$$

The reduced nonholonomic restriction is

$$v_0 ru = v_1.$$

Similarly, we can calculate the horizontal Lagrange–D'Alembert–Poincaré equations

$$\left(\frac{\partial^C \ell}{\partial\theta} - \frac{D}{Dt}\frac{\partial \ell}{\partial\dot{\theta}}\right).\delta\theta = \frac{\partial \ell}{\partial \bar{v}}\left(\tilde{\mathcal{B}}(\dot{\theta}, \delta\theta)\right) = 0$$

and

$$\left(\frac{\partial^C \ell}{\partial\varphi} - \frac{D}{Dt}\frac{\partial \ell}{\partial\dot{\varphi}}\right).\delta\varphi = \frac{\partial \ell}{\partial \bar{v}}\left(\tilde{\mathcal{B}}(\dot{\varphi}, \delta\varphi)\right) = 0.$$

Proceeding as in [7] we obtain the horizontal *Lagrange–D'Alembert–Poincaré equations* as follows

$$
\begin{aligned}
2(Mr^2+I_3-I_1)\dot\theta\dot\varphi\sin\theta\cos\theta-(I_1\sin^2\theta+(I_3+Mr^2)\cos^2\theta)\ddot\varphi \\
+Mr\cos\theta(\dot{v_{11}}\cos\varphi+\dot{v_{12}}\sin\varphi)+I_3(\dot\theta v_0\sin\theta-\dot{v_0}\cos\theta) \\
-\frac{1}{2}Mre\Big(\frac{1}{2}re(2\dot\theta\dot\varphi\sin\theta\cos\theta+\ddot\varphi\sin^2\theta) \\
+\sin\theta(\dot{v_{11}}\cos\varphi+\dot{v_{12}}\sin\varphi)+2r\dot\theta\dot\varphi(\sin^2\theta-\cos^2\theta)-2r\ddot\varphi\sin\theta\cos\theta\Big)&=0, \\
(I_1-I_3-Mr^2)\dot\varphi^2\sin\theta\cos\theta-I_3v_0\dot\varphi\sin\theta-(I_1+Mr^2)\ddot\theta \\
+Mr\sin\theta(\dot{v_{12}}\cos\varphi-\dot{v_{11}}\sin\varphi) \\
-Mgr\cos\theta+\frac{1}{2}Mre\Big(\frac{1}{2}re\dot\varphi^2\sin\theta\cos\theta+r\dot\varphi^2(\sin^2\theta-\cos^2\theta) \\
+\cos\theta(\dot{v_{12}}\cos\varphi-\dot{v_{11}}\sin\varphi)+g\sin\theta-\frac{1}{2}re\ddot\theta\Big)&=0.
\end{aligned}
$$

So we have the following system of reduced equations for the Euler's disk

$$
\begin{aligned}
I_3(\dot{v_0}+\ddot\varphi\cos\theta-\dot\theta\dot\varphi\sin\theta)+Mr\langle\dot{v_1},u\rangle+Mr^2(\ddot\varphi\cos\theta-2\dot\theta\dot\varphi\sin\theta) \\
-\frac{1}{2}Mer^2(\ddot\varphi\sin\theta+2\dot\theta\dot\varphi\cos\theta)&=0, \qquad (1)\\
v_0ru&=v_1, \qquad (2)\\
2(Mr^2+I_3-I_1)\dot\theta\dot\varphi\sin\theta\cos\theta-(I_1\sin^2\theta+(I_3+Mr^2)\cos^2\theta)\ddot\varphi \\
+Mr\cos\theta(\dot{v_{11}}\cos\varphi+\dot{v_{12}}\sin\varphi)+I_3(\dot\theta v_0\sin\theta-\dot{v_0}\cos\theta) \\
-\frac{1}{2}Mre\left(\frac{1}{2}re(2\dot\theta\dot\varphi\sin\theta\cos\theta+\ddot\varphi\sin^2\theta)\right. \\
\left.+\sin\theta(\dot{v_{11}}\cos\varphi+\dot{v_{12}}\sin\varphi)+2r\dot\theta\dot\varphi(\sin^2\theta-\cos^2\theta)-2r\ddot\varphi\sin\theta\cos\theta\right)&=0, \qquad (3)\\
(I_1-I_3-Mr^2)\dot\varphi^2\sin\theta\cos\theta-I_3v_0\dot\varphi\sin\theta-(I_1+Mr^2)\ddot\theta \\
+Mr\sin\theta(\dot{v_{12}}\cos\varphi-\dot{v_{11}}\sin\varphi)-Mgr\cos\theta+\frac{1}{2}Mre\left(\frac{1}{2}re\dot\varphi^2\sin\theta\cos\theta\right. \\
\left.+r\dot\varphi^2(\sin^2\theta-\cos^2\theta)+\cos\theta(\dot{v_{12}}\cos\varphi-\dot{v_{11}}\sin\varphi)+g\sin\theta-\frac{1}{2}re\ddot\theta\right)&=0, \qquad (4)
\end{aligned}
$$

where the two first equations are the vertical Lagrange–D'Alembert–Poincaré equation and the reduced nonholonomic restriction, and the two last equations are the horizontal Lagrange–D'Alembert–Poincaré equations.

2.4. Explicit equations for the Euler's disk.

Differentiating the constraint (Eq. (2)) and solving the system (1), (3)–(4) for $(\dot a,\dot b,\dot{v_0},\dot{v_{11}},\dot{v_{12}})$, we have the following explicit equations for the Euler's disk, where the two first equations define a and b.

$$\dot\theta = a \qquad (5)$$

$$\dot\varphi = b \qquad (6)$$

$$\dot\psi = v_0 \qquad (7)$$

$$\dot a = \frac{1}{4I_1+(4+e^2)Mr^2}\left(-2eb^2Mr^2\cos(2\theta)+2(Mgre-2b(I_3+Mr^2)v_0)\sin\theta\right.$$

$$+\cos\theta(-2Mr(2g+berv_0)+b^2(4I_1-4I_3+(-4+e^2)Mr^2)\sin\theta)\big), \tag{8}$$

$$\dot b = -\frac{2a}{4I_1I_3+(4I_1+I_3e^2)Mr^2}\big(ebI_3Mr^2+b(4I_1(I_3+Mr^2)$$
$$+I_3(-2I_3+(-2+e^2)Mr^2))\cot\theta-2I_3(I_3+Mr^2)v_0\csc\theta\big), \tag{9}$$

$$\dot{v_0} = \frac{1}{8I_1I_3+2(4I_1+I_3e^2)Mr^2}\big(a\csc\theta\big(4b(3I_1-I_3)I_3+b(16I_1+(-4+3e^2)I_3)Mr^2$$
$$-8I_3(I_3+Mr^2)v_0\cos\theta+bI_3(4I_1-4I_3+(-4+e^2)Mr^2)\cos(2\theta)$$
$$+4eI_3Mr^2(v_0+2b\cos\theta)\sin\theta\big)\big), \tag{10}$$

$$\dot{v_{11}} = \frac{1}{4I_1I_3+(4I_1+I_3e^2)Mr^2}\big(ar\cos\varphi(-2I_3Mr^2e(v_0+2b\cos\theta)$$
$$+4I_3(I_3+Mr^2)v_0\cot\theta-2b(4I_1(I_3+Mr^2)+I_3(-2I_3+(-2+e^2)Mr^2))\csc\theta$$
$$+I_3b(4I_1-4I_3+(-4+e^2)Mr^2)\sin\theta)+br(4I_1I_3+(4I_1+I_3e^2)Mr^2)v_0\sin\varphi\big), \tag{11}$$

$$\dot{v_{12}} = -\frac{1}{8I_1I_3+2(4I_1+I_3e^2)Mr^2}\big(2br(4I_1I_3+(4I_1+I_3e^2)Mr^2)v_0\cos\varphi$$
$$+ar\csc\theta(4b(3I_1-I_3)I_3+b(16I_1+(-4+3e^2)I_3)Mr^2-8I_3(I_3+Mr^2)v_0\cos\theta$$
$$+bI_3(4I_1-4I_3+(-4+e^2)Mr^2)\cos(2\theta)+4I_3eMr^2(v_0+2b\cos\theta)\sin\theta)\sin\varphi\big). \tag{12}$$

The first six equations form an analytic ODE in the variables $(\theta,\varphi,\psi,a,b,v_0)$. The last two equations are the derivative of the rolling constraint with respect to time and give the acceleration of the point of contact of the disk with the plane. For each initial condition compatible with the rolling constraint there is uniqueness of solution of the system of eight equations.

2.5. An Hypergeometric Equation Relating θ and b

By introducing the derivative of the rolling constraint (2) in Eqs. (1), (3) and (4), we obtain the following system of equations which is equivalent to the system (1)–(4) and, of course, also equivalent to the system (5)–(12) for initial conditions satisfying the constraint Eq. (2):

$$v_0ru = v_1, \tag{13}$$

$$(I_3+Mr^2)\dot{v_0}+\left((I_3+Mr^2)\cos\theta-\frac{1}{2}Mr^2e\sin\theta\right)\ddot\varphi$$
$$-\big((I_3+2Mr^2)\sin\theta+Mr^2e\cos\theta\big)\dot\theta\dot\varphi = 0, \tag{14}$$

$$2\left(\left(Mr^2+I_3-I_1-\frac{1}{4}Mr^2e^2\right)\sin\theta\cos\theta-\frac{1}{2}Mr^2e(\sin^2\theta-\cos^2\theta)\right)\dot\theta\dot\varphi$$
$$-\left(I_1\sin^2\theta+(I_3+Mr^2)\cos^2\theta+\frac{1}{4}Mr^2e^2\sin^2\theta-Mr^2e\sin\theta\cos\theta\right)\ddot\varphi$$
$$-\left((I_3+Mr^2)\cos\theta-\frac{1}{2}Mr^2e\sin\theta\right)\dot{v_0}+I_3\dot\theta v_0\sin\theta = 0, \tag{15}$$

$$\left(\left(I_1-I_3-Mr^2+\frac{1}{4}Mr^2e^2\right)\sin\theta\cos\theta+\frac{1}{2}Mr^2e(\sin^2\theta-\cos^2\theta)\right)\dot\varphi^2$$
$$-\left(I_1+Mr^2+\frac{1}{4}Mr^2e^2\right)\ddot\theta-(I_3+Mr^2)\dot\varphi v_0\sin\theta-Mgr\cos\theta$$
$$-\frac{1}{2}Mr^2ev_0\dot\varphi\cos\theta+\frac{1}{2}Mgre\sin\theta = 0. \tag{16}$$

By dividing by $a = \dot{\theta}$ Eqs. (14) and (15), we obtain the following system of two linear differential equations in two variables b and v_0 :

$$(\beta+1)\frac{dv_0}{d\theta} + \left((\beta+1)\cos\theta - \frac{1}{2}e\sin\theta\right)\frac{db}{d\theta} - ((\beta+2)\sin\theta + e\cos\theta)b = 0, \quad (17)$$

$$\left((\beta+1)\cos\theta - \frac{1}{2}e\sin\theta\right)\frac{dv_0}{d\theta} + \left(\alpha\sin^2\theta + (\beta+1)\cos^2\theta + \frac{1}{4}e^2\sin^2\theta - e\sin\theta\cos\theta\right)\frac{db}{d\theta}$$
$$-2\left(\left(1+\beta-\alpha-\frac{1}{4}e^2\right)\sin\theta\cos\theta - \frac{1}{2}e(\sin^2\theta - \cos^2\theta)\right)b - \beta v_0\sin\theta = 0, \quad (18)$$

where $\alpha = \frac{I_1}{Mr^2}$ and $\beta = \frac{I_3}{Mr^2}$.

This system leads to a second order differential equation as follows. The system (17)–(18) can be solved for $db/d\theta$ and $dv_0/d\theta$ obtaining

$$\frac{db}{d\theta} = \frac{((\beta+2)\sin\theta + e\cos\theta)}{(\beta+1)\cos\theta - 1/2e\sin\theta}\, b + \frac{\beta+1}{(\beta+1)\cos\theta - 1/2e\sin\theta}\,\frac{dv_0}{d\theta}, \quad (19)$$

$$\frac{dv_0}{d\theta} = \frac{2\left(\beta-\alpha+1-1/4e^2\right)\sin\theta\cos\theta - e(\sin^2\theta-\cos^2\theta)}{(\beta+1)\cos\theta - 1/2e\sin\theta}\, b$$
$$- \frac{\alpha\sin^2\theta + (\beta+1)\cos^2\theta + 1/4e^2\sin^2\theta - e\sin\theta\cos\theta}{(\beta+1)\cos\theta - 1/2e\sin\theta}\,\frac{db}{d\theta}$$
$$+ \frac{\beta\sin\theta}{(\beta+1)\cos\theta - 1/2e\sin\theta}\, v_0. \quad (20)$$

Of course one can also obtain equations (19) and (20) from (8), (9) and (10).

Thinking of b and v_0 as being functions of θ, we obtain from (19) and (20) the following second order differential equation for $b = b(\theta)$:

$$b'' = -3\cot\theta\, b' + \frac{2((2+e^2)\beta + 4\alpha(\beta+1) + e\beta\cot\theta)}{\beta e^2 + 4\alpha(\beta+1)}\, b. \quad (21)$$

By setting $u = \text{cotg}\theta$ in Eq. (21) we obtain the equation

$$\frac{d^2b}{du^2} - \frac{u}{1+u^2}\frac{db}{du} - \frac{2((2+e^2)\beta + 4\alpha(\beta+1) + e\beta u)}{\beta e^2 + 4\alpha(\beta+1)}\frac{1}{(1+u^2)^2}b = 0.$$

With the change of variable $z = \dfrac{1+iu}{2}$, the last equation becomes the equation

$$\frac{d^2b}{dz^2} - \frac{1-2z}{2z(1-z)}\frac{db}{dz} + \frac{\gamma + \delta(1-2z)i}{4z^2(1-z)^2}b = 0, \quad (22)$$

where

$$\gamma = \frac{2(2+e^2)\beta + 8\alpha(\beta+1)}{\beta e^2 + 4\alpha(\beta+1)} \quad \text{and} \quad \delta = \frac{2\beta e}{\beta e^2 + 4\alpha(\beta+1)}.$$

Eq. (22) can be transformed (see [40]) in the following equation:

$$z^2(z-1)^2\frac{d^2b}{dz^2} + ((1-a_1-a_2)z(z-1)^2 + (1-b_1-b_2)z^2(z-1))\frac{db}{dz}$$
$$+ (a_1a_2(1-z) + b_1b_2z + c_1c_2z(z-1))b = 0, \quad (23)$$

where $a_1 = \frac{3}{4} + \sqrt{\frac{9}{16} - \frac{\gamma+\delta i}{4}}$, $a_2 = \frac{3}{4} - \sqrt{\frac{9}{16} - \frac{\gamma+\delta i}{4}}$, $b_1 = \frac{3}{4} + \sqrt{\frac{9}{16} - \frac{\gamma-\delta i}{4}}$,
$b_2 = \frac{3}{4} - \sqrt{\frac{9}{16} - \frac{\gamma-\delta i}{4}}$, $c_1 = 0$ y $c_2 = -2$.

This equation has the solutions $b(z) = z^{a_1}(1-z)^{b_1} f(z)$, where $f(z) = {}_2F_1(a, b; c; x)$ or $f(z) = k\ x^{1-c}\ {}_2F_1(a+1-c, b+1-c; 2-c; x)$ (being k a constant) are the hypergeometric functions that satisfy the equation $z(1-z)\dfrac{d^2 f}{dz^2} + (c - (a+b+1)z)\dfrac{df}{dz} - abf = 0$ where $a = a_1 + b_1 + c_1,\ b = a_1 + b_1 + c_2$ y $c = 1 + a_1 - a_2$.

This equation is also valid for the planar disk, which corresponds to the case $\delta = 0$.

Erratum: If we make in (21) the change of variable $z = \tan(\theta/2)$ we obtain the following Heun's equation

$$b'' + \frac{3-u^2}{u(1+u^2)}\, b' - \frac{4\gamma u - 2\delta u^2 + 2\delta}{u(1+u^2)^2}\, b = 0. \tag{24}$$

The Frobenius expansions of the linearly independent solutions of Eq. (24) are

$$b_1(u) = a_0\left(1 + \frac{2}{3}\,\delta\, u + \left(\frac{1}{2}\,\gamma + \frac{1}{6}\,\delta^2\right)u^2 + \left(-\frac{2}{9}\,\delta + \frac{11}{45}\,\gamma\,\delta + \frac{1}{45}\,\delta^3\right)u^3 + \ldots\right)$$

and

$$\begin{aligned} b_2(u) =& C a_0\left(1 + \frac{2}{3}\,\delta\, u + \left(\frac{1}{2}\,\gamma + \frac{1}{6}\,\delta^2\right)u^2 + \left(-\frac{2}{9}\,\delta + \frac{11}{45}\,\gamma\,\delta + \frac{1}{45}\,\delta^3\right)u^3 + \ldots\right)\ln(u) + \\ &+ \frac{C\,a_0}{2(\gamma - 2 - \delta^2)}\,\frac{1}{u^2} - \frac{C\,a_0\,\delta}{\gamma - 2 - \delta^2}\,\frac{1}{u} + \ c_2 + \left(\frac{2}{3}\,\delta\, c_2 - \frac{C\,a_0\,\delta(20\gamma - 19 - 8\delta^2)}{9(\gamma - 2 - \delta^2)}\right)u + \\ &+ \left(-\frac{3}{8}\,C\,a_0\left(\gamma + \frac{1}{3}\delta^2\right) + \frac{1}{2}\left(\gamma + \frac{1}{3}\,\delta^2\right)c_2 - \frac{C\,a_0(9 - 14\,\delta^2 + 10\,\delta^2\,\gamma - 4\,\delta^4)}{18(\gamma - 2 - \delta^2)}\right)u^2 + \ldots \end{aligned}$$

For zero thickness we have $\delta = 0$ and we obtain the corresponding Frobenius solution for the planar disk. We must say that there is an obvious mistake in our Frobenius solutions for the planar disk in [7]. This does not affects the main results in that paper.

3. CONCLUSION

We have written the Lagrange–D'Alembert–Poincaré equations for the Euler's disk. These equations are consistent with equations obtained previously by other authors. One advantage of having Lagrange–D'Alembert–Poincaré equations is that it opens the door to apply numerical integrators specially designed for nonholonomic systems. We also find an hypergeometric equation relating two of the variables.

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