

SHIMURA CORRESPONDENCE FOR LEVEL p^2 AND THE CENTRAL VALUES OF L-SERIES II

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ABSTRACT. Given a Hecke eigenform f of weight 2 and square-free level N , by the work of Kohnen, there is a unique weight $3/2$ modular form of level $4N$ mapping to f under the Shimura correspondence. Furthermore, by the work of Waldspurger the Fourier coefficients of such a form are related to the quadratic twists of the form f . Gross gave a construction of the half integral weight form when N is prime, and such construction was later generalized to square-free levels. However, in the non-square free case, the situation is more complicated since the natural construction is vacuous. The problem being that there are too many special points so that there is cancellation while trying to encode the information as a linear combination of theta series.

In this paper, we concentrate in the case of level p^2 , for $p > 2$ a prime number, and show how the set of special points can be split into subsets (indexed by bilateral ideals for an order of reduced discriminant p^2) which gives two weight $3/2$ modular forms mapping to f under the Shimura correspondence. Moreover, the splitting has a geometric interpretation which allows to prove that the forms are indeed a linear combination of theta series associated to ternary quadratic forms.

Once such interpretation is given, we extend the method of Gross-Zagier to the case where the level and the discriminant are not prime to each other to prove a Gross-type formula in this situation.

INTRODUCTION

The theory of modular forms of half-integral weight was developed by Shimura in [11]. There he defined a map known as the “Shimura correspondence” that associates to a modular form g of half-integral weight $k/2$ (k odd), level $4N$ and character ψ , such that g is an eigenform for the Hecke operators, a modular form f of weight $k - 1$, level $2N$ and character ψ^2 . In the same work, he noted that the Fourier coefficients of the half-integral weight modular form have more information than the Fourier coefficients of the integral weight modular form and raised the question of the meaning of these Fourier coefficients. In 1981, Waldspurger answered the question by relating the Fourier coefficients of g to the central values of twisted L -series of f (see [14]).

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In [4], Gross gave an explicit method to construct, when f has weight 2, prime level p and trivial character, a weight $3/2$ modular form (of level $4p$ and trivial character) mapping to f via the Shimura correspondence. A formula for the central values of twists of the L -series of f by imaginary quadratic characters was proved by Gross. His result was later generalized (by a different method) to odd *square-free* levels N in [1], where the authors construct one modular form whose Fourier coefficients are related to central values of twists of the L -series of f by imaginary quadratic characters with discriminants satisfying 2^t quadratic conditions (where t is the number of prime factors of N).

Generalizing Gross method in a different direction, the case of level p^2 , for a prime $p > 2$, was studied in [9]. In that work, given a modular form of weight 2 and level p^2 , *two* weight $3/2$ modular forms were constructed, and a “Gross formula” was conjectured (see [9, Conjecture 2]). Some examples as well as an application to computing central values of twists by *real* quadratic characters were presented in [8] and [12]. The main purpose of this paper is to explain the nature of such formula and also prove it.

Let $f = \sum a(n)q^n$ be a cusp form of weight 2 and level N . Let K be an imaginary quadratic field of discriminant $D < 0$ and let \mathcal{O}_K be its ring of integers. We denote $\mathcal{J}(\mathcal{O}_K)$ the class group of \mathcal{O}_K .

We define the twisted L -series

$$L(f, D, s) := \sum \frac{a(n)}{n^s} \left(\frac{D}{n} \right).$$

Note that $L(f, D, s)$ as defined here may not be a primitive L -series when N and D have a common factor. Our main result, Theorem 4.11, deals with the case $N = p^2$, for $p > 2$ an odd prime, and $D = -pd$ (with $p \nmid d$) a fundamental discriminant, but Section 1 and the appendix on Rankin’s method are more general, including the case where N and D have a common factor as required for Section 4. Note that the results in Section 3 assume that D is odd, but the main result in Section 4 is proved for odd and even discriminants altogether.

The proof of our formula consists of two parts. First, we need to compute the Fourier coefficients of the half-integral weight modular forms constructed in [9], and give an interpretation of such construction in terms of special points. In Section 1, we give an adèlic definition of the special points of discriminant D and in Proposition 1.4 we prove that the adèlic definition coincides with Eichler’s original formulation. This interpretation is crucial to split the special points of discriminant D into subsets indexed by bilateral ideals, as defined in (6).

To compute the Fourier coefficients of the weight $3/2$ modular forms attached to such splitting, we fix a quaternion algebra over \mathbb{Q} ramified at p and ∞ , and an order $\tilde{\mathcal{O}}$ with reduced discriminant p^2 . We consider the algebra \mathbb{T}_0 of Hecke operators with index prime to p acting on $\mathcal{M}(\tilde{\mathcal{O}})$, a vector space (over \mathbb{R}) spanned

by representatives of $\tilde{\mathcal{O}}$ -ideal classes. Since the eigenspaces of \mathbb{T}_0 do not have multiplicity one (in general), we need to add some extra operators.

The group of bilateral $\tilde{\mathcal{O}}$ -ideals modulo \mathbb{Q}^\times -equivalence is a dihedral group of order $2(p+1)$. The norm p bilateral $\tilde{\mathcal{O}}$ -ideals can be taken as representatives for the symmetries. Each bilateral $\tilde{\mathcal{O}}$ -ideal defines an operator in $\mathcal{M}(\tilde{\mathcal{O}})$. The operator $W_{\mathfrak{p}}$ associated to a norm p bilateral $\tilde{\mathcal{O}}$ -ideal \mathfrak{p} commutes with \mathbb{T}_0 and is self-adjoint for the natural inner product of $\mathcal{M}(\tilde{\mathcal{O}})$ (defined in (1) below), although operators related to different norm p bilateral $\tilde{\mathcal{O}}$ -ideals clearly do not commute. Considering the algebra generated by the Hecke operators and one $W_{\mathfrak{p}}$, a multiplicity one theorem does hold (see Theorem 2.7).

We show that there is a connection between bilateral $\tilde{\mathcal{O}}$ -ideals of norm p and suborders of $\tilde{\mathcal{O}}$ (in the sense of [9]) which associates to \mathfrak{p} the order $\mathbb{Z} + \mathfrak{p}$. This relation allows to define for each ideal \mathfrak{p} , a map

$$\Theta_{\mathfrak{p}} : \mathcal{M}(\tilde{\mathcal{O}}) \mapsto M_{3/2}(4p^2, \varkappa_{\mathfrak{p}}),$$

where $\varkappa_{\mathfrak{p}}(n) := \binom{p}{n}$ (see Section 2.4). If \mathbf{e} is an eigenvector for the Hecke operators such that $\Theta_{\mathfrak{p}}(\mathbf{e}) \neq 0$, then $W_{\mathfrak{p}}(\mathbf{e}) = \mathbf{e}$ (see Proposition 2.17). This implies that the only eigenvectors to be considered for $\Theta_{\mathfrak{p}}$ are those where $W_{\mathfrak{p}}$ acts trivially. This explains the orthogonality condition on the eigenvector $\mathbf{e}_{f, \tilde{\mathcal{O}}}$ needed for Conjecture 2 in [9].

In Section 3 we relate the Fourier coefficients of a modular form coming from Rankin's method (see Theorem A.14) to the height of special points. This allows, given a character φ of the class group $\mathcal{J}(\mathcal{O}_K)$, to relate the central value of

$$L_{\varphi}(f, s) := \sum_{\mathcal{A}} \varphi(\mathcal{A}) L_{\mathcal{A}}(f, s)$$

to the height of a sum of special points of discriminant D (Proposition 3.10). The special case $\varphi = 1_D$ (the trivial character on $\mathcal{J}(\mathcal{O}_K)$) relates to central values of twists of the L -series by the factorization

$$L_{1_D}(f, s) = L(f, s) L(f, D, s).$$

The second part of the proof is the well known Rankin's method. This part is a little more technical hence it is left to the appendix. Following [4] and [5] we define a Rankin convolution L -series and using Rankin's method we compute its central value. This is done in a very similar way than that of [5]. The difficulty in our case comes from the fact that the level of f and the discriminant of the imaginary quadratic field are not prime to each other. The formula for the central value of $L_{\mathcal{A}}(f, s)$ (for an ideal class $\mathcal{A} \in \mathcal{J}(\mathcal{O}_K)$) proved in Theorem A.14 is similar to [5, Proposition 4.4] with the condition $\gcd(N, D) = 1$ removed. In addition to giving a more complete result in general, this lifts an important restriction when N is not squarefree; for instance, when N is a perfect square, both sides in the formula in Theorem A.14 vanish trivially for $\gcd(N, D) = 1$.

In the last section, we relate the heights of special points to the Fourier coefficients of ternary theta series. The key idea is to split the set of special points of discriminant D into $p + 1$ subsets, indexed by the norm p bilateral $\tilde{\mathcal{O}}$ -ideals. Note that depending on whether $\left(\frac{D/p}{p}\right)$ is a square or not, half of these subsets will be empty, while the other half will have the same number of elements. By counting the number of special points in each set, we conclude the proof of our main result.

Theorem A (Theorem 4.11). *Let f be a new eigenform of weight 2, level p^2 with $p > 2$ an odd prime. Fix a norm p bilateral $\tilde{\mathcal{O}}$ -ideal \mathfrak{p} , and let \mathbf{e}_f be an eigenvector in the f -isotypical component of $\mathcal{M}(\tilde{\mathcal{O}})$ such that $W_{\mathfrak{p}}(\mathbf{e}_f) = \mathbf{e}_f$.*

If d is an integer such that $D = -pd < 0$ is a fundamental discriminant, and such that $\left(\frac{d}{p}\right) = \chi(\mathfrak{p})$, then

$$L(f, 1) L(f, D, 1) = 4\pi^2 \frac{\langle f, f \rangle}{\langle \mathbf{e}_f, \mathbf{e}_f \rangle} \frac{c_d^2}{\sqrt{pd}},$$

where the c_d are the Fourier coefficients of $\Theta_{\mathfrak{p}}(\mathbf{e}_f) = \sum_{d \geq 1} c_d q^d$.

This formula is the same as Conjecture 2 in [9], with the difference of a factor of $8\pi^2$ coming from a different normalization in the Petersson inner product. The extra factor of $\frac{p}{p-1}$ which shows up in [9], when f is the quadratic twist of a level p form, is due to the fact that $L(f, D, s)$ we use here is not primitive at p in this case.

The case of odd discriminant D is proved using the results in Section 3. We avoid the technical difficulties of the case of even discriminants by resorting to a theorem of Waldspurger, which allows us to recover the formula for even discriminants from the case of odd discriminants.

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1. QUATERNION ALGEBRAS, BILATERAL IDEALS AND SPECIAL POINTS

Let B be a quaternion algebra over \mathbb{Q} , i.e. a central simple algebra of dimension 4 over \mathbb{Q} . Given a place v of \mathbb{Q} , let $B_v := B \otimes \mathbb{Q}_v$ be its completion at v . The algebra B_v is either a division algebra or isomorphic to the algebra $M_2(\mathbb{Q}_v)$ of 2×2 matrices with coefficients in \mathbb{Q}_v . The place v is said to be *ramified* if B_v is a division algebra and *split* if not. The number of ramified places is finite and even. The discriminant of B is the product of all ramified primes of \mathbb{Q} .

We require B to be definite or ramified at ∞ , meaning $B_{\infty} = B \otimes \mathbb{R}$ is a division algebra (the Hamilton quaternions). The discriminant of B is thus the product of an *odd* number of primes.

It is well known that B has an antiautomorphism called *conjugation* and for $x \in B$ we denote its action by \bar{x} . For $x \in B$ we define $Nx = x\bar{x}$ and $\text{Tr } x = x + \bar{x}$ the *reduced norm* and *reduced trace* of x , respectively. The *norm* of a lattice

\mathfrak{a} is defined as $\mathcal{N}\mathfrak{a} := \gcd\{\mathcal{N}x : x \in \mathfrak{a}\}$. We equip \mathfrak{a} with the quadratic form $\mathcal{N}_{\mathfrak{a}}(x) := \mathcal{N}x/\mathcal{N}\mathfrak{a}$, which is primitive; its determinant is a square, and we denote its positive square root by $\text{disc}(\mathfrak{a})$. In particular, when $R \subseteq B$ is an order, $\text{disc}(R)$ is its (*reduced*) *discriminant*. The subscript p will denote localization at p , namely $\mathfrak{a}_p := \mathfrak{a} \otimes \mathbb{Z}_p$.

Given a lattice \mathfrak{a} in B , its right order is given by

$$R_r(\mathfrak{a}) := \{x \in B : \mathfrak{a}x \subseteq \mathfrak{a}\}.$$

The left order of \mathfrak{a} is defined similarly. If R is an order in B , we let $\tilde{\mathcal{J}}(R)$ be the set of *left R -ideals*, i.e. the set of lattices $\mathfrak{a} \subseteq B$ such that $\mathfrak{a}_p = R_p x_p$ for every prime p , with $x_p \in B_p^\times$. Note that we do not define what an ideal in a quaternion algebra is since we will only deal with left R -ideals; we recommend the reader to look at [13] for such definition and the theory of ideals in general. We just want to remark that the condition of \mathfrak{a} being locally principal is equivalent to \mathfrak{a} being a projective R -module, which is the standard definition of an ideal for a non-maximal order of a number field. It is clear from the definition that if \mathfrak{a} is a left R -ideal, its left order is just R .

Let $\tilde{\mathcal{M}}(R)$ be the vector space over \mathbb{R} with basis $\tilde{\mathcal{J}}(R)$. Consider the *height pairing*,

$$(1) \quad \langle \mathfrak{a}, \mathfrak{b} \rangle := \frac{1}{2} \#\{x \in B^\times : \mathfrak{a}x = \mathfrak{b}\} = \begin{cases} \frac{1}{2} \#R_r(\mathfrak{a})^\times & \text{if } \mathfrak{a}x = \mathfrak{b}, x \in B^\times, \\ 0 & \text{otherwise,} \end{cases}$$

as an inner product on it. For a geometric interpretation of the height pairing see [4, §4], where it is introduced as a pairing on the Picard group of certain curves of genus zero. In this geometric context, special points (that will be defined in Section 1.1) should be regarded as analogues of Heegner points on modular curves.

More generally, given R -ideals \mathfrak{a} and \mathfrak{b} , define

$$\text{Hom}(\mathfrak{a}, \mathfrak{b}) := \{u \in B^\times : \mathfrak{a}u \subseteq \mathfrak{b}\}.$$

Then $\langle \mathfrak{a}, \mathfrak{b} \rangle = \frac{1}{2} \#\{u \in \text{Hom}(\mathfrak{a}, \mathfrak{b}) : \mathcal{N}u = \frac{\mathcal{N}\mathfrak{b}}{\mathcal{N}\mathfrak{a}}\}$.

Let $\mathfrak{a} \in \tilde{\mathcal{J}}(R)$, and $m \geq 1$ an integer. We set

$$\mathcal{T}_m(\mathfrak{a}) := \left\{ \mathfrak{b} \in \tilde{\mathcal{J}}(R) : \mathfrak{b} \subseteq \mathfrak{a}, \quad \mathcal{N}\mathfrak{b} = m \mathcal{N}\mathfrak{a} \right\}.$$

The *Hecke operators* $t_m : \tilde{\mathcal{M}}(R) \rightarrow \tilde{\mathcal{M}}(R)$ are then defined by

$$t_m \mathfrak{a} := \sum_{\mathfrak{b} \in \mathcal{T}_m(\mathfrak{a})} \mathfrak{b}$$

for $m \geq 1$ and $\mathfrak{a} \in \tilde{\mathcal{J}}(R)$, and extended by linearity to all of $\tilde{\mathcal{M}}(R)$. Moreover, we have

$$\langle \mathfrak{a}, t_m \mathfrak{b} \rangle = \frac{1}{2} \#\left\{ u \in \text{Hom}(\mathfrak{a}, \mathfrak{b}) : \mathcal{N}u = m \frac{\mathcal{N}\mathfrak{b}}{\mathcal{N}\mathfrak{a}} \right\},$$

and the Hecke operators are self-adjoint with respect to the height pairing [9, Proposition 1.3].

We define an equivalence relation on the set of left R -ideals by $\mathfrak{a}, \mathfrak{b} \in \widetilde{\mathcal{J}}(R)$ are in the same class if $\mathfrak{a} = \mathfrak{b}x$, for some $x \in B^\times$; we write $[\mathfrak{a}]$ for the class of \mathfrak{a} . The set of all left R -ideal classes, which we denote by $\mathcal{J}(R)$, is known to be finite. If we denote by $\mathcal{M}(R)$ the vector space over \mathbb{R} with basis $\mathcal{J}(R)$, it has an inner product and an action of Hecke operators by considering the quotient map

$$\widetilde{\mathcal{M}}(R) \twoheadrightarrow \mathcal{M}(R).$$

Note that $\mathcal{J}(R)$ is an orthogonal basis of $\mathcal{M}(R)$, and it is clear that $\langle \cdot, \cdot \rangle$ is positive definite on $\mathcal{M}(R)$.

The Hecke operators t_m with $(m, \text{disc}(R)) = 1$ generate a commutative algebra \mathbb{T}_0 , which is indeed generated by the t_p with $p \nmid \text{disc}(R)$ prime. Since the Hecke operators are self-adjoint it follows, by the spectral theorem, that $\mathcal{M}(R)$ has an orthogonal basis of common eigenvectors for \mathbb{T}_0 .

We remark that $\mathcal{M}(R)$ has a natural integral structure as the free \mathbb{Z} -module spanned by $\mathcal{J}(R)$ which is quite important; the height pairing and the Hecke operators are defined over \mathbb{Z} . However, the eigenvectors may not be defined over \mathbb{Z} , but only over a totally real number field. For our purposes, using \mathbb{R} as the field of coefficients will suffice.

The left R -ideal \mathfrak{a} is *bilateral* if its right order is also R . The set of bilateral R -ideals forms a group under ideal multiplication. This group contains the principal ideals generated by non-zero rational elements, which we denote by \mathbb{Q}^\times , in its center, so we can consider bilateral R -ideals modulo \mathbb{Q}^\times -equivalence, namely two bilateral ideals \mathfrak{a} and \mathfrak{b} are \mathbb{Q}^\times -equivalent if $\mathfrak{a} = \mathfrak{b}x$ with $x \in \mathbb{Q}^\times$. The group of bilateral R -ideals modulo \mathbb{Q}^\times -equivalence acts on $\mathcal{M}(R)$ by left multiplication on the basis elements. We denote $W_{\mathfrak{m}}$ the operator corresponding to the bilateral R -ideal \mathfrak{m} .

Remark 1.1. Note that this action commutes with the action of \mathbb{T}_0 . Furthermore, these operators are unitary; hence, those of order 2 are self-adjoint. On the other hand, the group of bilateral R -ideals modulo \mathbb{Q}^\times needs not to be commutative (see Section 2).

We now give an adèlic reinterpretation of the ideal theory which will allow us to work locally. Let $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ be the profinite completion of \mathbb{Z} , and let $\widehat{\mathbb{Q}} := \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ the ring of finite adèles of \mathbb{Q} . Let also $\widehat{B} := B \otimes \widehat{\mathbb{Q}}$ and $\widehat{R} := R \otimes \widehat{\mathbb{Z}}$. By the Eichler local-global principle for lattices, we have a bijection between global lattices \mathfrak{a} in B and collections $\{\mathfrak{a}_p\}$ of local lattices such that $\mathfrak{a}_p = R_p$ for all p except finitely many. Hence, if \mathfrak{a} is a left R -ideal with $\mathfrak{a}_p = R_p x_p$, we have $(x_p) \in \widehat{B}^\times$, and any $(x_p) \in \widehat{B}^\times$ determines an ideal in B in that way. Furthermore, \widehat{R}^\times acts on \widehat{B}^\times by left multiplication, and each ideal corresponds to a unique orbit for this action,

i.e.

$$\text{left } R\text{-ideals} \quad \longleftrightarrow \quad \widehat{R}^\times \backslash \widehat{B}^\times.$$

Note that if $\mathfrak{a} \leftrightarrow (x_p)$, its right order $R_r(\mathfrak{a})$ is given locally by $x_p^{-1}R_p x_p$. Hence, if we set

$$N(\widehat{R}^\times) := \left\{ (x_p) \in \widehat{B}^\times : x_p R_p = R_p x_p \quad \forall p \right\},$$

we have the correspondence

$$\text{bilateral } R\text{-ideals} \quad \longleftrightarrow \quad \widehat{R}^\times \backslash N(\widehat{R}^\times).$$

Moreover, if $p \nmid \text{disc}(R)$, we have that R_p is a maximal order in the split quaternion algebra over \mathbb{Q}_p and its local normalizer $N(R_p^\times) = R_p^\times \mathbb{Q}_p^\times$ (this is just a statement about the ring of 2 by 2 matrices with coefficients in \mathbb{Z}_p , see [3, Proposition 1]). Hence the group of bilateral R -ideals modulo \mathbb{Q}^\times -equivalence, $\widehat{R}^\times \backslash N(\widehat{R}^\times) / \mathbb{Q}^\times$, equals the finite local product

$$\prod_{p|\text{disc}(R)} R_p^\times \backslash N(R_p^\times) / \mathbb{Q}_p^\times.$$

1.1. Special points. Let B be a quaternion algebra (over \mathbb{Q}) of discriminant N , and let $K \subset B$ be a quadratic subfield of discriminant D_0 . We note that

- If B is definite, then $D_0 < 0$; and
- if $p \mid N$, then $\left(\frac{D_0}{p}\right) \neq 1$.

These are all the *local* obstructions for such an embedding to exist; by Hasse's principle, there are no additional *global* obstructions, i.e. if K is an imaginary quadratic field of discriminant D_0 and for each prime number $p \mid N$, $\left(\frac{D_0}{p}\right) \neq 1$ then there exists an embedding of K into B .

The situation for orders is quite different. If K embeds into B , let R be an order in B , and let $O := R \cap K$. Then O is an order in K of discriminant $D = D_0 s^2$, for some $s \in \mathbb{Z}$. Conversely, given an order O in K , there might be new local obstructions for the existence of an embedding of O into R , for example:

- If R is an order of discriminant p^2 (which are defined in [10] and called "orders of level p^2 "), then $p \mid D$.
- If R is an order of level p^2 with character sign σ (see Remark 2.14 for the definition), then $p \mid D$ and $\left(\frac{D/p}{p}\right) = \sigma$.

Also, there are of course *global* obstructions. However, if O can be locally embedded in R at every place, it follows that O can be embedded in some order R' that is locally conjugate to R (i.e. in the same *genus*). From now on we fix an order R in the quaternion algebra B .

Definition 1.2. If O is an order in K of discriminant D , a special point for O is a pair (\mathfrak{a}, ψ) , where \mathfrak{a} is a left R -ideal and $\psi : K \hookrightarrow B$ is an embedding such that $R_r(\mathfrak{a}) \cap \psi(K) = \psi(O)$.

This means that a special point of discriminant D is an optimal embedding of O into the right order of an ideal \mathfrak{a} . Note that this definition is slightly different from the one given in [3]. There a special point is just a pair (ψ, R') , where R' is in the same genus of R and ψ is an optimal embedding of O into R' . Clearly given two orders R_1, R_2 in the same genus, there exists an ideal \mathfrak{a} whose left order is R_1 and whose right order is R_2 , but there might be more than one. The advantage of our definition will become clear while counting classes of special points and their adèlic description.

The group B^\times has a right action on special points given for $\alpha \in B^\times$ by $(\mathfrak{a}, \psi) \cdot \alpha = (\mathfrak{a}\alpha, \alpha^{-1}\psi\alpha)$. If we fix a set of representatives $\mathcal{J}(R) = \{\mathfrak{a}_1, \dots, \mathfrak{a}_h\}$ for the ideal classes, any special point is equivalent to (\mathfrak{a}_i, ψ) , where ψ is an optimal embedding into $R_r(\mathfrak{a}_i)$. Furthermore, $(\mathfrak{a}_i, \psi_1) \sim (\mathfrak{a}_i, \psi_2)$ if and only if there exists $\alpha \in R_r(\mathfrak{a}_i)^\times$ such that $\psi_1 = \alpha^{-1}\psi_2\alpha$. Noting that if (\mathfrak{a}_i, ψ) is a special point, $\psi(O)^\times$ acts trivially, Eichler deduces the formula for the number of non-equivalent special points of discriminant D where R is an Eichler order (see [3, Proposition 5] for Eichler orders of square free level and [6] for the general case). If D is a fundamental discriminant, the formula reads

$$(2) \quad \prod_{p|N} \left(1 - \left(\frac{D}{p}\right)\right) \prod_{p|H} \left(1 + \left(\frac{D}{p}\right)\right) h(O_D),$$

where B is the quaternion algebra with discriminant N and R is an Eichler order of discriminant HN . The Kronecker symbols represent the condition for such an embedding to exist.

We want to give an adèlic definition for the classes of special points. Following the previous notation, let $\widehat{O} := O \otimes \widehat{\mathbb{Z}}$ and $\widehat{K} := K \otimes \widehat{\mathbb{Q}}$. We will assume that $\widehat{K} \cap \widehat{R} = \widehat{O}$.

Denote by $\widetilde{\mathcal{J}}(O)$ the set of O -(fractional) ideals and by $\mathcal{J}(O)$ the set of O -ideal classes. We have the following adèlic interpretation:

R -ideals:	$\widetilde{\mathcal{J}}(R)$	\longleftrightarrow	$\widehat{R}^\times \backslash \widehat{B}^\times$
R -ideals modulo scalars:	$\widetilde{\mathcal{J}}(R)/\mathbb{Q}^\times$	\longleftrightarrow	$\widehat{R}^\times \backslash \widehat{B}^\times / \mathbb{Q}^\times$
K -points:	$\widetilde{\mathcal{J}}(R)/K^\times$	\longleftrightarrow	$\widehat{R}^\times \backslash \widehat{B}^\times / K^\times$
R -classes:	$\mathcal{J}(R)$	\longleftrightarrow	$\widehat{R}^\times \backslash \widehat{B}^\times / B^\times$
O -ideals:	$\widetilde{\mathcal{J}}(O)$	\longleftrightarrow	$\widehat{O}^\times \backslash \widehat{K}^\times$
O -ideals modulo scalars:	$\widetilde{\mathcal{J}}(O)/\mathbb{Q}^\times$	\longleftrightarrow	$\widehat{O}^\times \backslash \widehat{K}^\times / \mathbb{Q}^\times$
O -classes:	$\mathcal{J}(O)$	\longleftrightarrow	$\widehat{O}^\times \backslash \widehat{K}^\times / K^\times$

Note that the height pairing defined at the beginning of this section induces an inner product on the \mathbb{Z} -module spanned by K -points and R -classes as well.

Lemma 1.3. *The embedding $K \subset B$ induces an injective map (by right multiplication)*

$$\tilde{\mathcal{J}}(O) \hookrightarrow \tilde{\mathcal{J}}(R).$$

Proof. The inclusion $K \subseteq B$, composed with a projection, induces a map $\widehat{K}^\times \rightarrow \widehat{B}^\times \rightarrow \widehat{R}^\times \backslash \widehat{B}^\times$, and it is clear that the kernel of this composite map is $\widehat{K}^\times \cap \widehat{R}^\times = \widehat{O}^\times$. \square

This induces the following diagram:

$$\begin{array}{ccc} O\text{-ideals} & \hookrightarrow & R\text{-ideals} \\ \downarrow & & \downarrow \\ O\text{-classes} & \hookrightarrow & K\text{-points} \\ & \searrow \text{dotted} & \downarrow \\ & & R\text{-classes} \end{array}$$

where the horizontal arrows are injective, and the vertical arrows surjective. Note that, despite the O -classes and the R -classes being independent of the embedding $K \subset B$, the dotted map does indeed depend on the choice of embedding, as do the two horizontal maps.

1.2. O -points. From now on, we fix an embedding $i : K \hookrightarrow B$. For $x \in \widehat{B}^\times$, we define $\widehat{R}_x := x^{-1}\widehat{R}x$, $R_x := B \cap \widehat{R}_x$, and $O_x := K \cap \widehat{R}_x$. Note that R_x is an order in B (the *right* order of $\widehat{R}x$) locally conjugate to R , and that O_x is an order in K . Note also that R_x depends only on the class of x as an R -ideal modulo scalars, and that O_x depends only on the class of x as a K -point.

If we set $\widehat{N}_O := \{x \in B^\times : O_x = O\}$, then we define the O -points as the elements of $\widehat{R}^\times \backslash \widehat{N}_O / K^\times$.

Proposition 1.4. *If O is an order in K of discriminant D , then the classes of special points for O are in one-to-one correspondence with the O -points.*

Proof. Recall that we fixed an embedding $i : K \hookrightarrow B$. If (\mathfrak{a}, ψ) is a special point of discriminant D , there exists $\alpha \in B^\times$ such that $\alpha^{-1}\psi\alpha = i$. Furthermore, α is determined up to multiplication on the right by K^\times . Then the point $(\mathfrak{a}, \psi) \sim (\alpha\mathfrak{a}, i)$. Since $\alpha\mathfrak{a}$ is a left R -ideal, there exists $x \in \widehat{B}^\times$ such that $\widehat{R}x = \alpha\mathfrak{a}$. Then we associate to the pair (\mathfrak{a}, ψ) the O -point x . It is immediate that x is in \widehat{N}_O and that it is defined up to multiplication on the right by K^\times and multiplication on the left by \widehat{R}^\times , i.e. it is a point in the double quotient $\widehat{R}^\times \backslash \widehat{N}_O / K^\times$.

Conversely, to an element $x \in \widehat{N}_O$, we associate the equivalence class of $(\widehat{R}x, i)$. The condition of the embedding being optimal is clear, and multiplication on the right by \widehat{R}^\times give the same special point. We are left to prove that right

multiplication by K^\times gives equivalent special points, but this is clear since if $k \in K^\times$, conjugation by k acts trivially on i . \square

1.3. Groups acting on O -points.

Proposition 1.5. (1) \widehat{K}^\times acts on \widehat{N}_O by right multiplication, i.e. $\widehat{N}_O \widehat{K}^\times \subseteq \widehat{N}_O$.
 (2) $\widehat{O}^\times \backslash \widehat{K}^\times$ acts on $\widehat{R}^\times \backslash \widehat{N}_O^\times$.
 (3) The action in (2) is free.

This action induces a free action of the group of O -classes on the set of O -points, and the space of orbits is $\widehat{R}^\times \backslash \widehat{N}_O / \widehat{K}^\times$. The *canonical O -orbit* is the image of the map O -classes $\leftrightarrow K$ -points.

Proof of Proposition 1.5. (1) indeed, if $a \in \widehat{K}^\times$, $x \in \widehat{B}^\times$, then $O_{xa} = K \cap a^{-1}x^{-1}\widehat{R}xa = a^{-1}(K \cap x^{-1}\widehat{R}x)a = a^{-1}O_xa = O_x$.
 (2) if $a \in \widehat{O}^\times = K \cap x^{-1}\widehat{R}x$ then $xa \in \widehat{R}x$ and since a is a unit, $\widehat{R}x = \widehat{R}ax$, which implies that \widehat{O}^\times acts trivially on $\widehat{R}^\times \backslash \widehat{N}_O$.
 (3) if $xa \in \widehat{R}x$, then $a \in \widehat{K}^\times \cap x^{-1}\widehat{R}x = \widehat{O}^\times$. \square

Recall that $N(\widehat{R}^\times) = \{(x_p) : x_p^{-1}R_p x_p = R\}$, modulo left multiplication by \widehat{R}^\times , corresponds to the group of bilateral R -ideals. It clearly acts on \widehat{N}_O by left multiplication. Hence the group

$$G := \widehat{R}^\times \backslash N(\widehat{R}^\times) / \mathbb{Q}^\times \times \widehat{O}^\times \backslash \widehat{K}^\times / K^\times$$

(i.e. the group of bilateral ideals modulo \mathbb{Q}^\times -equivalence times the class group of O) acts on the set of O -points.

Proposition 1.6. *If R is an Eichler order and O is an order in K of discriminant D , the action of G on the classes of O -points is transitive. Furthermore, if $\gcd(\text{disc}(R), D) = 1$ the action is free, while if p is a prime that divides $\gcd(\text{disc}(R), D)$, the pair $(\mathfrak{p}_R, \mathfrak{p}_O^{-1})$ acts trivially, where \mathfrak{p}_R is the bilateral R -ideal of norm p and \mathfrak{p}_O is the O -ideal of norm p (in other words, the actions of \mathfrak{p}_R and \mathfrak{p}_O are the same.)*

Proof. That the action is transitive is a consequence of [3, Proposition 3] in the case where R has square free discriminant, and by the work of [6] for general levels. Once we know that the action is transitive, the second statement follows from (2) (which gives the number of non-equivalent O -points) and from [3, Proposition 1] (which gives the number of bilateral ideals). \square

Note that although in [3] only Eichler orders of square free level are studied, most of the results proven there are also true for all Eichler orders by the work of Hijikata ([6]). Furthermore, Proposition 1.6 holds also for orders of level p^2M (that will be studied in the next section), by [10, Theorem 2.7 and Theorem 4.8].

2. ORDERS OF LEVEL p^2

Fix a prime $p > 2$ and let B be the quaternion algebra over \mathbb{Q} which is ramified at p and ∞ . Let \mathcal{O} be a maximal order in B and let $\tilde{\mathcal{O}}$ be the unique order of index p in \mathcal{O} . We note that

$$\tilde{\mathcal{O}} = \{x \in \mathcal{O} : p \mid \Delta x\},$$

where $\Delta x := (\text{Tr } x)^2 - 4\mathcal{N}x$ is the discriminant of the characteristic polynomial of x .

There is a quadratic character $\chi : \tilde{\mathcal{J}}(\tilde{\mathcal{O}}) \mapsto \{\pm 1\}$, given on $\mathfrak{a} \in \tilde{\mathcal{J}}(\tilde{\mathcal{O}})$, by $\chi(\mathfrak{a}) = \left(\frac{\mathcal{N}_{\mathfrak{a}}(x)}{p}\right)$, where $x \in \mathfrak{a}$ is any element such that $p \nmid \mathcal{N}_{\mathfrak{a}}(x)$. Furthermore, the character depends only on the equivalence class of \mathfrak{a} , i.e. it is a character on $\mathcal{J}(\tilde{\mathcal{O}})$ (see [10, Proposition 5.1]).

There is a \mathbb{T}_0 -equivariant bilinear map in $\mathcal{M}(\tilde{\mathcal{O}})$ with values in the space $M_2(p^2)$ of modular forms of weight 2 and level p^2 defined on the basis by

$$\phi([\mathfrak{a}], [\mathfrak{b}]) := \vartheta(\mathfrak{a}^{-1}\mathfrak{b}) = \frac{1}{2} \sum_{x \in \mathfrak{a}^{-1}\mathfrak{b}} q^{\mathcal{N}x/\mathcal{N}\mathfrak{a}^{-1}\mathfrak{b}}$$

This map induces a correspondence between eigenvectors in $\mathcal{M}(\tilde{\mathcal{O}})$ and eigenforms of weight 2 and level p^2 . For an eigenform f of weight 2 and level p^2 we denote by $\mathcal{M}(\tilde{\mathcal{O}})^f$ the f -isotypical component of $\mathcal{M}(\tilde{\mathcal{O}})$, i.e. the eigenspace for the action of \mathbb{T}_0 with the same eigenvalues as f . We have the following result due to Pizer ([10, Theorem 8.2]) :

$$\dim \mathcal{M}(\tilde{\mathcal{O}})^f = \begin{cases} 1 & \text{if } f \text{ is an oldform,} \\ 1 & \text{if } f \text{ is the quadratic twist of a level } p \text{ form,} \\ 0 & \text{if } f \text{ is the non-quadratic twist of a level } p \text{ form,} \\ 2 & \text{if } f \text{ is not the twist of a level } p \text{ form.} \end{cases}$$

There is a natural inclusion $\mathcal{M}(\mathcal{O}) \hookrightarrow \mathcal{M}(\tilde{\mathcal{O}})$ which is \mathbb{T}_0 -equivariant (see [9, Proposition 1.14]).

Definition 2.1. The space of old forms $\mathcal{M}(\tilde{\mathcal{O}})^{\text{old}}$ is the image of $\mathcal{M}(\mathcal{O})$ under the natural inclusion. Its orthogonal complement is denoted $\mathcal{M}(\tilde{\mathcal{O}})^{\text{new}}$ and is called the new space.

Proposition 2.2. *The eigenvectors in $\mathcal{M}(\tilde{\mathcal{O}})^{\text{old}}$ correspond to eigenforms in $M_2^{\text{old}}(p^2)$, and the eigenvectors in $\mathcal{M}(\tilde{\mathcal{O}})^{\text{new}}$ correspond to eigenforms in $M_2^{\text{new}}(p^2)$.*

Proof. The first assertion is clear. For the second assertion, it is clear that if $f \in M_2^{\text{new}}(p^2)$ and $\mathbf{e}_f \in \mathcal{M}(\tilde{\mathcal{O}})^f$ then \mathbf{e}_f is orthogonal to $\mathcal{M}(\tilde{\mathcal{O}})^{\text{old}}$. It might happen that there exists $\mathbf{e} \in \mathcal{M}(\tilde{\mathcal{O}})^{\text{new}}$ such that $f = \phi(\mathbf{e}, \mathbf{e})$ is old. But this is not the case, since if f is an old form, the f -isotypical component in $\mathcal{M}(\tilde{\mathcal{O}})$ has dimension 1. Then, since $\mathcal{M}(\mathcal{O})^f$ is non-empty, we have $\mathcal{M}(\mathcal{O})^f = \mathcal{M}(\tilde{\mathcal{O}})^f$. \square

2.1. Bilateral $\tilde{\mathcal{O}}$ -ideals and its action on $\mathcal{M}(\tilde{\mathcal{O}})$.

Proposition 2.3. *The group of bilateral $\tilde{\mathcal{O}}$ -ideals modulo \mathbb{Q}^\times -equivalence is isomorphic to the dihedral group D_{p+1} of $2(p+1)$ elements. Furthermore the rotations correspond to bilateral $\tilde{\mathcal{O}}$ -ideals of norm 1 while the symmetries correspond to bilateral $\tilde{\mathcal{O}}$ -ideals of norm p .*

Proof. See Proposition 9.26 of [10]. □

By the remark in page 6, we are interested in the operators corresponding to bilateral $\tilde{\mathcal{O}}$ -ideals of order 2. By Proposition 2.3, any norm p bilateral $\tilde{\mathcal{O}}$ -ideal \mathfrak{p} has order 2. There is also the one corresponding to the unique norm 1 bilateral $\tilde{\mathcal{O}}$ -ideal $\mathfrak{m}_{\tilde{\mathcal{O}}}$ of order 2, which will be denoted by \widetilde{W} . This ideal commutes with any norm p bilateral $\tilde{\mathcal{O}}$ -ideal.

Proposition 2.4. *The operator \widetilde{W} acts on $\mathcal{M}(\tilde{\mathcal{O}})$ as the Atkin-Lehner involution W_{p^2} , i.e.*

$$\phi(\mathbf{u}, \widetilde{W}\mathbf{v}) = \phi(\mathbf{u}, \mathbf{v})|_{W_{p^2}}.$$

Proof. In the basis of $\tilde{\mathcal{O}}$ -ideal classes we have

$$\phi([\mathbf{a}], \widetilde{W}[\mathbf{b}]) = \vartheta(\mathbf{a}^{-1} \mathfrak{m}_{\tilde{\mathcal{O}}} \mathbf{b}) = \vartheta((\mathbf{a}^{-1} \mathfrak{m}_{\tilde{\mathcal{O}}} \mathbf{a}) \mathbf{a}^{-1} \mathbf{b}).$$

Clearly, $\mathbf{a}^{-1} \mathfrak{m}_{\tilde{\mathcal{O}}} \mathbf{a}$ is the unique norm 1 bilateral $R_r(\mathbf{a})$ -ideal of order 2, and it thus follows from [10, Theorem 9.20] applied to the order $R_r(\mathbf{a})$ that $\vartheta((\mathbf{a}^{-1} \mathfrak{m}_{\tilde{\mathcal{O}}} \mathbf{a}) \mathbf{a}^{-1} \mathbf{b}) = \vartheta(\mathbf{a}^{-1} \mathbf{b})|_{W_{p^2}}$. □

Corollary 2.5. *Let f be an eigenform of level p , and let \mathbf{e}_f be a non-zero eigenvector in $\mathcal{M}(\tilde{\mathcal{O}})^f$. Then $\phi(\mathbf{e}_f, \mathbf{e}_f)$ is a non-zero multiple of*

$$f(z) - \epsilon_p p f(pz),$$

where ϵ_p is the sign of the Atkin-Lehner involution at p , i.e. $f|_{W_p} = \epsilon_p f$.

Proof. Since f is an old form, $\mathbf{e}_f \in \mathcal{M}(\tilde{\mathcal{O}})^{\text{old}}$. Locally, \widetilde{W} amounts to left multiplication by the element of order 2 in $\tilde{\mathcal{O}}_p^\times \backslash \mathcal{O}_p^\times$, which is clearly trivial in $\mathcal{M}(\tilde{\mathcal{O}})^{\text{old}}$. Thus $\widetilde{W}\mathbf{e}_f = \mathbf{e}_f$, and by the Proposition

$$\phi(\mathbf{e}_f, \mathbf{e}_f)|_{W_{p^2}} = \phi(\mathbf{e}_f, \mathbf{e}_f).$$

The statement follows from the fact that W_{p^2} acts on the oldspace generated by $f(z)$ and $pf(pz)$ by the matrix $\begin{pmatrix} 0 & \epsilon_p \\ \epsilon_p & 0 \end{pmatrix}$. □

2.2. Multiplicity one. Let R, R' be orders and \mathfrak{a} be a left R -ideal. We define

$$\Psi_{R'}^R(\mathfrak{a}) := \left\{ \mathfrak{b} \in \tilde{\mathcal{J}}(R') : \mathfrak{b} \subseteq \mathfrak{a}, \quad \mathcal{N}\mathfrak{b} = \mathcal{N}\mathfrak{a} \right\}.$$

If $\mathfrak{a} \in \tilde{\mathcal{J}}(\mathcal{O})$, then the subgroup of rotations acts transitively (although not necessarily faithfully) on the set $\Psi_{\mathcal{O}}^{\mathcal{O}}(\mathfrak{a})$ (see Section 3.3 of [7]).

Let ρ denote a generator of the norm 1 bilateral $\tilde{\mathcal{O}}$ -ideals, then we have $\chi(\rho) = -1$ since the bilateral $\tilde{\mathcal{O}}$ -ideals of norm 1 are exactly $\Psi_{\mathcal{O}}^{\mathcal{O}}(\mathcal{O})$ and half of these ideals have positive character while the other half have negative character. Also W_{ρ^2} preserves characters.

Lemma 2.6. *If $\mathbf{e} \in \mathcal{M}(\tilde{\mathcal{O}})$ is an eigenvector for W_{ρ} with eigenvalue $\epsilon = \pm 1$, then $\phi(\mathbf{e}, \mathbf{e})$ is*

- (1) *an oldform, if $\epsilon = 1$;*
- (2) *the quadratic twist of a level p form, if $\epsilon = -1$.*

Proof. Let \mathbf{e}_f be an element in $\mathcal{M}(\tilde{\mathcal{O}})^f$. Then \mathbf{e}_f can be written as

$$\sum_{[\mathfrak{a}] \in \mathcal{J}(\mathcal{O})} \left(\sum_{\{[\mathfrak{b}] \in \Psi_{\mathcal{O}}^{\mathcal{O}}(\mathfrak{a}) : \chi([\mathfrak{b}])=1\}} n_{[\mathfrak{b}]}[\mathfrak{b}] \right. \\ \left. + \sum_{\{[\mathfrak{b}] \in \Psi_{\mathcal{O}}^{\mathcal{O}}(\mathfrak{a}) : \chi([\mathfrak{b}])=-1\}} n_{[\mathfrak{b}]}[\mathfrak{b}] \right)$$

Since $(W_{\rho})^2 = W_{\rho^2} = 1$ and ρ^2 acts transitively on both sets of the previous sum, we get that

$$\mathbf{e}_f = \sum_{[\mathfrak{a}] \in \mathcal{J}(\mathcal{O})} n_{[\mathfrak{a}]} \sum_{\{[\mathfrak{b}] \in \Psi_{\mathcal{O}}^{\mathcal{O}}(\mathfrak{a}) : \chi([\mathfrak{b}])=1\}} [\mathfrak{b}] + \\ \sum_{[\mathfrak{a}] \in \mathcal{J}(\mathcal{O})} m_{[\mathfrak{a}]} \sum_{\{[\mathfrak{b}] \in \Psi_{\mathcal{O}}^{\mathcal{O}}(\mathfrak{a}) : \chi([\mathfrak{b}])=-1\}} [\mathfrak{b}].$$

Since $\chi(\rho) = -1$, W_{ρ} permutes the set on the first sum with the set on the second one; since \mathbf{e}_f is an eigenvector of W_{ρ} with eigenvalue $\epsilon = \pm 1$, then $n_{[\mathfrak{a}]} = \epsilon m_{[\mathfrak{a}]}$ for all ideals $[\mathfrak{a}] \in \mathcal{J}(\mathcal{O})$. Clearly if $\epsilon = 1$, then \mathbf{e}_f is in $\mathcal{M}(\tilde{\mathcal{O}})^{\text{old}}$ and \mathbf{e}_f is as in Corollary 2.5 i.e. it has the same eigenvalues as a weight 2 and level p form. On the other hand, if $\epsilon = -1$, then \mathbf{e}_f is the twist of an eigenvector in $\mathcal{M}(\tilde{\mathcal{O}})^{\text{old}}$ (see [8]) therefore it corresponds to a quadratic twist of a level p form, as claimed. \square

Theorem 2.7. *Let \mathfrak{p} be a norm p bilateral $\tilde{\mathcal{O}}$ -ideal. The algebra $\mathbb{T}_{0,\mathfrak{p}}$ generated by \mathbb{T}_0 and $W_{\mathfrak{p}}$ is commutative, its action on the space $\mathcal{M}(\tilde{\mathcal{O}})$ is semisimple, and its eigenspaces have multiplicity one.*

Proof. By the remark in page 6, we have that $\mathbb{T}_{0,\mathfrak{p}}$ is commutative and its elements are self-adjoint with respect to the height pairing.

We will now prove multiplicity one. Let f be a modular form of weight 2 and level p^2 , and consider the eigenspace $\mathcal{M}(\tilde{\mathcal{O}})^f$, which we know has dimension at most 2. Since $W_{\mathfrak{p}}^2 = 1$, we will assume $W_{\mathfrak{p}} = \pm 1$ on $\mathcal{M}(\tilde{\mathcal{O}})^f$, as otherwise the statement is clearly true.

In this case, we have the identity (on $\mathcal{M}(\tilde{\mathcal{O}})^f$)

$$W_{\rho} = W_{\rho\mathfrak{p}}W_{\mathfrak{p}} = W_{\mathfrak{p}}W_{\rho\mathfrak{p}} = W_{\rho^{-1}}$$

(the middle equation comes from the fact that $W_{\mathfrak{p}} = \pm 1$.) Hence $W_{\rho}^2 = 1$ on $\mathcal{M}(\tilde{\mathcal{O}})^f$, and there is an eigenvector $\mathbf{e} \in \mathcal{M}(\tilde{\mathcal{O}})^f$ for W_{ρ} with eigenvalue ± 1 . By the Lemma, $\phi(\mathbf{e}, \mathbf{e})$ is either an oldform or the quadratic twist of a level p form, whose eigenspace for \mathbb{T}_0 is already one-dimensional. \square

Corollary 2.8. *The action of $W_{\mathfrak{p}}$ on $\mathcal{M}(\tilde{\mathcal{O}})$ gives an orthogonal decomposition*

$$\mathcal{M}(\tilde{\mathcal{O}}) = \ker(W_{\mathfrak{p}} - 1) \oplus \ker(W_{\mathfrak{p}} + 1),$$

and the action of \mathbb{T}_0 on each of the components has multiplicity one.

Proof. The claimed decomposition is clear because $W_{\mathfrak{p}}^2 = 1$ on $\mathcal{M}(\tilde{\mathcal{O}})$, and the multiplicity one on each of the components follows from the theorem. \square

Remark 2.9.

- (1) The action of $W_{\mathfrak{p}}$ on $\mathcal{M}(\tilde{\mathcal{O}})^{\text{old}}$ is given by left multiplication by \mathfrak{p} acting on $\mathcal{M}(\mathcal{O})$. When f is an eigenform of level p , this action on $\mathcal{M}(\mathcal{O})^f$ is known to be $-\epsilon_p(f)$, where $\epsilon_p(f)$ is the eigenvalue of the Atkin-Lehner involution $|_{W_p}$. Hence, $\mathcal{M}(\tilde{\mathcal{O}})^f \subseteq \ker(W_{\mathfrak{p}} + \epsilon_p(f))$.
- (2) When f is the quadratic twist of an eigenform g of level p , one can see that the operator $W_{\mathfrak{p}} = -\epsilon_p(g)\chi(\mathfrak{p})$ where $\chi(\mathfrak{p})$ is the character of the left $\tilde{\mathcal{O}}$ -ideal \mathfrak{p} . Basically, if Φ denotes the twisting operator in $\mathcal{M}(\mathcal{O})$, given on a basis element $[\mathfrak{b}] \in \mathcal{J}(\tilde{\mathcal{O}})$ by $\Phi([\mathfrak{b}]) = \chi(\mathfrak{b})[\mathfrak{b}]$, it amounts to see that $W_{\mathfrak{p}}\Phi = \chi(\mathfrak{p})\Phi W_{\mathfrak{p}}$.

Hence, $\mathcal{M}(\tilde{\mathcal{O}})^f \subseteq \ker(W_{\mathfrak{p}} + \epsilon_p(g)\chi(\mathfrak{p}))$.

Recall from [9] there is a \mathbb{T}_0 -linear map

$$\Theta : \mathcal{M}(\tilde{\mathcal{O}}) \rightarrow M_{3/2}(4p, \varkappa_p),$$

where $\varkappa_p(n) := \left(\frac{p}{n}\right)$ is the quadratic character modulo p or $4p$ (according to whether $p \equiv 1$ or $3 \pmod{4}$, respectively). This map can be defined as

$$\Theta([\tilde{\mathcal{O}}]) := \frac{1}{2} \sum_{x \in \tilde{\mathcal{O}}/\mathbb{Z}} q^{-\Delta x/p},$$

and extended to all of $\mathcal{M}(\tilde{\mathcal{O}})$ by conjugation, i.e.:

$$\Theta([\mathbf{b}]) := \Theta([\mathbf{b}^{-1}\tilde{\mathcal{O}}\mathbf{b}]).$$

Note that $\Theta([\rho\mathbf{b}]) = \Theta([\mathbf{b}])$, since $\rho^{-1}\tilde{\mathcal{O}}\rho = \tilde{\mathcal{O}}$. In other words the diagram

$$\begin{array}{ccc} \mathcal{M}(\tilde{\mathcal{O}}) & \xrightarrow{\Theta} & \\ W_\rho \downarrow & & M_{3/2}(4p, \varkappa_p) \\ \mathcal{M}(\tilde{\mathcal{O}}) & \xrightarrow{\Theta} & \end{array}$$

is commutative.

Proposition 2.10. *If $\mathbf{e} \in \mathcal{M}(\tilde{\mathcal{O}})$ is an eigenvector for \mathbb{T}_0 , and $\Theta(\mathbf{e}) \neq 0$, then \mathbf{e} is old.*

Proof. If $\Theta(\mathbf{e}) \neq 0$, we may assume that $\mathbf{e} \in \ker \Theta^\perp$, since Θ is \mathbb{T}_0 -linear.

Since $\Theta W_\rho = \Theta$, it follows that $\ker \Theta$ is invariant by W_ρ . Moreover, $\ker \Theta^\perp$ is also invariant by W_ρ , because W_ρ is unitary.

Hence, $W_\rho(\mathbf{e}) \in \ker \Theta^\perp$, and the vector

$$\mathbf{e} - W_\rho(\mathbf{e})$$

is both in $\ker \Theta^\perp$ and in $\ker \Theta$. Thus \mathbf{e} is an eigenvector for W_ρ with eigenvalue 1, and the result follows from Lemma 2.6. \square

2.3. Suborders and symmetries. In view of Proposition 2.10, the map Θ is trivial on $\mathcal{M}(\tilde{\mathcal{O}})^{\text{new}}$. In order to obtain non-zero half-integral weight modular forms corresponding to new vectors in $\mathcal{M}(\tilde{\mathcal{O}})$, we need to work with the orders of index p in $\tilde{\mathcal{O}}$, which we call *orders of level p^2* (see [9]). It is known that they play a very important role in the theory of Shimura correspondence for level p^2 .

Remark 2.11. As suggested by the referee, we want to clarify the terminology here. In [10], the author uses the name “orders of level p^2 ” for $\tilde{\mathcal{O}}$. In his context, this was the natural term to use, since the main achievement was to construct bases of integral weight modular forms and, using the trace formula, he proves that weight 2 modular forms of level p^2 appear in $\mathcal{M}(\tilde{\mathcal{O}})$. However, since orders in quaternion algebras are in correspondence with ternary quadratic forms, it is more natural to index orders by the level of the corresponding ternary form, which was the definition we used in [9] and we maintain in this article. We hope this will make no confusion to the reader.

We recall the following

Proposition 2.12. *Let $L \subset \tilde{\mathcal{O}}$ be a lattice such that $[\tilde{\mathcal{O}} : L] = p$. Then L is an order if and only if $\mathbb{Z} + p\mathcal{O} \subset L$.*

Proof. This is Proposition 2.2 of [9]. \square

Proposition 2.13. *If \mathfrak{p} is a norm p bilateral $\tilde{\mathcal{O}}$ -ideal then $\mathbb{Z} + \mathfrak{p}$ is an order of index p in $\tilde{\mathcal{O}}$. Conversely, if R_p is an order with $[\tilde{\mathcal{O}} : R_p] = p$ then $L = \{x \in R_p : p \mid \text{Tr } x\}$ is a norm p bilateral $\tilde{\mathcal{O}}$ -ideal.*

Proof. Since \mathfrak{p} has norm p in $\tilde{\mathcal{O}}$, by a local computation it follows that $\mathfrak{p} \subseteq \tilde{\mathcal{O}}$, and $[\tilde{\mathcal{O}} : \mathfrak{p}] = p^2$. Also it is easy to check that $p\mathcal{O} \subset \mathfrak{p}$, hence $\mathbb{Z} + p\mathcal{O} \subset \mathbb{Z} + \mathfrak{p}$ and $[\tilde{\mathcal{O}} : \mathbb{Z} + \mathfrak{p}] = p$. By Proposition 2.12 it follows that $\mathbb{Z} + \mathfrak{p}$ is an order.

For the converse, note that for $x \in \tilde{\mathcal{O}}$ we have $p \mid \text{Tr } x$ if and only if $p \mid \mathcal{N}x$. Hence $L = \{x \in R_p : p \mid \mathcal{N}x\}$, and it follows that $R_p L \subseteq L$. Therefore, the left order of L is either \mathcal{O} , $\tilde{\mathcal{O}}$ or R_p . From the fact that all the lattices with one of them as left order are locally principal (see [2]) and $[\tilde{\mathcal{O}} : L] = p^2$, it follows that L is a left $\tilde{\mathcal{O}}$ -ideal of norm p , hence bilateral. \square

Remark 2.14. Recall the definition of the character χ in an order R_p of index p in $\tilde{\mathcal{O}}$:

$$\chi(R_p) := \left(\frac{-\Delta x/p}{p} \right),$$

where $x \in R_p$ such that $p \parallel \Delta x$. This is well defined by [9, Lemma 2.3], and it's clear that

$$\chi(\mathbb{Z} + \mathfrak{p}) = \chi(\mathfrak{p}),$$

i.e. the correspondence preserves the character.

2.4. Theta series of weight 3/2. Using the correspondence given in the previous section we define, for \mathfrak{p} a bilateral $\tilde{\mathcal{O}}$ -ideal of norm p , a map

$$\Theta_{\mathfrak{p}} : \mathcal{M}(\tilde{\mathcal{O}}) \rightarrow M_{3/2}(4p^2, \varkappa_p)$$

by

$$\Theta_{\mathfrak{p}}([\tilde{\mathcal{O}}]) := \Theta([\mathbb{Z} + \mathfrak{p}]) = \frac{1}{2} \sum_{x \in \mathbb{Z} + \mathfrak{p}/\mathbb{Z}} q^{-\Delta x/p}.$$

This definition extends to all of $\mathcal{M}(\tilde{\mathcal{O}})$ by conjugation, namely:

$$\Theta_{\mathfrak{p}}([\mathfrak{b}]) := \Theta_{\mathfrak{b}^{-1}\mathfrak{p}\mathfrak{b}}([\mathfrak{b}^{-1}\tilde{\mathcal{O}}\mathfrak{b}]).$$

In [9], for $\mathfrak{b} \in \mathcal{J}(\tilde{\mathcal{O}})$ and $R_p = \mathbb{Z} + \mathfrak{p}$, it is defined

$$\Theta_{R_p}([\mathfrak{b}]) := \Theta(R_r(\mathfrak{c})),$$

where \mathfrak{c} is any left R_p -ideal with index p in \mathfrak{b} .

Lemma 2.15. *If $R_p = \mathbb{Z} + \mathfrak{p}$ then $\Theta_{\mathfrak{p}} = \Theta_{R_p}$.*

Proof. If $\mathfrak{b} \in \mathcal{J}(\tilde{\mathcal{O}})$ and \mathfrak{c} is any left R_p -ideal with index p in \mathfrak{b} , we claim that $R_r(\mathfrak{c}) = \mathbb{Z} + \mathfrak{b}^{-1}\mathfrak{p}\mathfrak{b}$, which implies the assertion.

To see it, we prove equality of both orders at the different completions. For primes $q \neq p$, the statement is trivial, since $\mathfrak{c}_q = \mathfrak{b}_q$ and $R_r(\mathfrak{c}_q) = R_r(\mathfrak{b}_q) = \mathfrak{b}_q^{-1}\mathfrak{b}_q = (\mathbb{Z} + \mathfrak{b}^{-1}\mathfrak{p}\mathfrak{b})_q$.

At p , if $\mathfrak{b}_p = \tilde{\mathcal{O}}_p x_p$, we can take $\mathfrak{c}_p := (\mathbb{Z}_p + \mathfrak{p}_p)x_p$ (the global lattice \mathfrak{c} with this local completions satisfies the hypothesis). Then $R_r(\mathfrak{c}_p) = \mathbb{Z}_p + x_p^{-1}\mathfrak{p}_p x_p = \mathbb{Z}_p + \mathfrak{b}_p^{-1}\mathfrak{p}_p \mathfrak{b}_p$ as claimed. \square

Note that

$$\Theta_{\mathfrak{p}}([\rho\mathfrak{b}]) = \Theta_{\rho^{-1}\mathfrak{p}\rho}([\mathfrak{b}]) = \Theta_{\mathfrak{p}\rho^2}([\mathfrak{b}]),$$

hence for any rotation ρ^k we have:

$$\begin{array}{ccc} \mathcal{M}(\tilde{\mathcal{O}}) & \xrightarrow{\Theta_{\mathfrak{p}\rho^{2k}}} & M_{3/2}(4p^2) \\ W_{\rho^k} \downarrow & & \\ \mathcal{M}(\tilde{\mathcal{O}}) & \xrightarrow{\Theta_{\mathfrak{p}}} & \end{array}$$

Proposition 2.16. *The image of the map $\Theta_{\mathfrak{p}}(\mathcal{M}(\mathcal{O})^f)$ depends only on the character of \mathfrak{p} for any eigenform f .*

Proof. If \mathfrak{p} and \mathfrak{p}' are two bilateral ideals of norm p with the same character, then they differ by the square of a rotation, say $\mathfrak{p} = \mathfrak{p}'\rho^{2k}$. Since the operator W_{ρ^k} commutes with the Hecke operators, the space $\mathcal{M}(\tilde{\mathcal{O}})^f$ is invariant under W_{ρ^k} , hence the statement follows from the previous observation. \square

Using the same argument for \tilde{W} , since $\tilde{W}^2 = 1$, we have the commuting triangle

$$\begin{array}{ccc} \mathcal{M}(\tilde{\mathcal{O}}) & \xrightarrow{\Theta_{\mathfrak{p}}} & M_{3/2}(4p^2) \\ \tilde{W} \downarrow & & \\ \mathcal{M}(\tilde{\mathcal{O}}) & \xrightarrow{\Theta_{\mathfrak{p}}} & \end{array}$$

hence $\ker(\tilde{W} + 1) \subseteq \ker \Theta_{\mathfrak{p}}$; in other words, eigenvectors corresponding to modular forms f with $\epsilon(f) = -1$ have trivial image under $\Theta_{\mathfrak{p}}$.

A similar computation shows that

$$\Theta_{\mathfrak{p}}([\mathfrak{p}\mathfrak{b}]) = \Theta_{\mathfrak{p}^{-1}\mathfrak{p}\mathfrak{p}}([\mathfrak{b}]) = \Theta_{\mathfrak{p}}([\mathfrak{b}]),$$

thus

$$\begin{array}{ccc} \mathcal{M}(\tilde{\mathcal{O}}) & \xrightarrow{\Theta_{\mathfrak{p}}} & M_{3/2}(4p^2) \\ W_{\mathfrak{p}} \downarrow & & \\ \mathcal{M}(\tilde{\mathcal{O}}) & \xrightarrow{\Theta_{\mathfrak{p}}} & \end{array}$$

and again we have $\ker(W_{\mathfrak{p}} + 1) \subseteq \ker \Theta_{\mathfrak{p}}$.

Summarizing, we have

Proposition 2.17. *In the irreducible components of $\mathcal{M}(\tilde{\mathcal{O}})$ where $\Theta_{\mathfrak{p}}$ is non-zero, we have $W_{\mathfrak{p}} = \tilde{W} = 1$. In particular, the image $\Theta_{\mathfrak{p}}(\mathcal{M}(\mathcal{O})^f)$, for an eigenform f , is at most 1-dimensional.*

3. SPECIAL POINTS FOR LEVEL p^2

Following Section 1, we fix $D < 0$ an odd fundamental discriminant. We require $p \mid D$, since there are no special points of discriminant D for $\tilde{\mathcal{O}}$ otherwise. Let K be the imaginary quadratic field of discriminant D , and \mathcal{O}_K its ring of integers. Let u_D be half the number of units in \mathcal{O}_K , i.e. $u_D = \frac{1}{2}\#\mathcal{O}_K^\times$. Write $D = p^*D_0$, where $p^* = \left(\frac{-1}{p}\right)p$.

Fix a rational prime $q > 0$ satisfying the conditions:

- $q \nmid 2D$.
- $\left(\frac{-q}{p}\right) = -1$.
- $q \equiv -1 \pmod{D_0}$.

By quadratic reciprocity, these conditions imply that q is split in K . We fix an ideal \mathfrak{q} of \mathcal{O}_K of norm q , and note that its genus $\text{gen}[\mathfrak{q}]$ is the only element of the set

$$Q = \{\text{gen}[\mathfrak{q}] : \mathcal{N}\mathfrak{q} \equiv -p^2 \pmod{D_0}\}$$

appearing in Theorem A.14 below.

Let $B = K + Kj$ with

$$j^2 = -q,$$

and $jk = \bar{k}j$ for all $k \in K$, where \bar{k} is the complex conjugate of k .

Proposition 3.1. *B is a quaternion algebra ramified precisely at p and ∞ .*

Proof. Clearly, B is a quaternion algebra over \mathbb{Q} , so we just need to find the set of ramified primes. In the basis $\{1, \sqrt{D}, j, \sqrt{D}j\}$, the norm form is

$$\mathcal{N}(x_0 + x_1\sqrt{D} + x_2j + x_3\sqrt{D}j) = x_0^2 - Dx_1^2 + qx_2^2 - Dqx_3^2.$$

Since the norm form in B is positive definite, B ramifies at infinity. To check whether B is ramified at a prime l or not, we need to see if the norm form represents 0 in \mathbb{Q}_l for each prime l . Consider the different cases:

- If $l \nmid 2Dq$ then it is clear that B is split at l , since the discriminant of the norm form is an l -adic unit in this case.
- If $l = p$, the norm form represents zero if and only if $\left(\frac{-q}{p}\right) = 1$. The second condition on q assures that this is not the case, hence B ramifies at p .
- If $l \mid D$ but $l \neq p$ then the norm form represents zero if and only if $\left(\frac{-q}{l}\right) = 1$, which is clearly the case since $-q \equiv 1 \pmod{l}$.
- If $l = q$, the norm form represents zero if and only if $\left(\frac{D}{q}\right) = 1$ which is the case by quadratic reciprocity. In fact, the conditions in q imply $\varepsilon_{D_0}(-q) = +1$ and $\varepsilon_{p^*}(-q) = -1$, hence $\varepsilon_D(-q) = -1$. Since $D < 0$ it follows that $\left(\frac{D}{q}\right) = 1$ as claimed.

Since the number of ramified primes for a quaternion algebra is even, we do not need to consider the prime 2. \square

Let \mathfrak{D}_0 be the ideal in \mathcal{O}_K of norm D_0 , and \mathfrak{p}_K the ideal of \mathcal{O}_K of norm p . To simplify the notation, in this section only we will omit the subscript K writing $\mathfrak{p} = \mathfrak{p}_K$. Define

$$\mathcal{O} := \{\alpha + \beta j : \alpha \in \mathfrak{D}_0^{-1}, \beta \in \mathfrak{D}_0^{-1}\mathfrak{q}^{-1}, \alpha - q\beta \in \mathcal{O}_K\},$$

and

$$\tilde{\mathcal{O}} := \{\alpha + \beta j : \alpha \in \mathfrak{D}_0^{-1}, \beta \in \mathfrak{D}_0^{-1}\mathfrak{p}\mathfrak{q}^{-1}, \alpha - q\beta \in \mathcal{O}_K\},$$

This is consistent with the notation of the previous section by the following theorem.

Theorem 3.2. \mathcal{O} is a maximal order in B and $\tilde{\mathcal{O}}$ is the unique order of index p in \mathcal{O} .

Proof. To prove that \mathcal{O} is an order, since $1 \in \mathcal{O}$ and \mathcal{O} is closed under addition, we just need to check it is closed under multiplication. Let $a_1 + b_1j, a_2 + b_2j \in \mathcal{O}$, then

$$(a_1 + b_1j)(a_2 + b_2j) = (a_1a_2 - qb_1\bar{b}_2) + (a_1b_2 + \bar{a}_2b_1)j.$$

To prove that this is in \mathcal{O} , we claim that it belongs to $\mathcal{O}_l := \mathcal{O} \otimes \mathbb{Z}_l$ for all primes l . Consider the cases:

- If $l \nmid D_0$ the claim is clear, since in this case

$$\mathcal{O}_l = (\mathcal{O}_K + \mathfrak{q}^{-1}j) \otimes \mathbb{Z}_l,$$

with $j^2 = -q$.

- If $l \mid D_0$, then

$$a_1a_2 - qb_1\bar{b}_2 = a_1(a_2 - qb_2) + qb_2(a_1 - qb_1) + qb_1(qb_2 - \bar{b}_2).$$

The first two terms clearly belong to $\mathfrak{D}_0^{-1} \otimes \mathbb{Z}_l$. The last also belongs to $\mathfrak{D}_0^{-1} \otimes \mathbb{Z}_l$ since $q \equiv -1 \pmod{l}$, and $b + \bar{b} \in \mathcal{O}_K \otimes \mathbb{Z}_l$ for all $b \in \mathfrak{D}_0^{-1} \otimes \mathbb{Z}_l$.

Analogously,

$$(a_1b_2 + \bar{a}_2b_1) = (a_1 - b_1q)b_2 + b_1(b_2q - a_2) + b_1(a_2 + \bar{a}_2),$$

and the same reasoning applies.

Finally, the proof that

$$(a_1a_2 - qb_1\bar{b}_2) - q(a_1b_2 + \bar{a}_2b_1)j \in \mathcal{O}_K \otimes \mathbb{Z}_l,$$

follows from a similar computation.

The proof that $\tilde{\mathcal{O}}$ is an order is the same, except for $l = p$, where $\tilde{\mathcal{O}}_p = (\mathcal{O}_K + \mathfrak{p}j) \otimes \mathbb{Z}_p$ and the claim is clear. Also this shows that $\tilde{\mathcal{O}}$ has index p in \mathcal{O} .

It remains to prove that \mathcal{O} is maximal, or equivalently, that its reduced discriminant is p . We compute the l -valuation of the discriminant for each prime l :

- If $l \nmid Dq$ then $\mathcal{O}_l = (\mathcal{O}_k + \mathcal{O}_K j) \otimes \mathbb{Z}_l$, and the discriminant of the norm form in \mathcal{O}_l is an l -adic unit.
- If $l \mid D_0$, $\mathcal{O}_l \subset (\mathfrak{D}_0^{-1} + \mathfrak{D}_0^{-1} j) \otimes \mathbb{Z}_l$ with index l . The discriminant of the norm form in the latter is l^{-2} since $l \nmid \mathcal{N}j = q$ hence the discriminant of the norm form in \mathcal{O}_l is an l -adic unit.
- If $l = q$, $\mathcal{O}_q = (\mathcal{O}_K + \mathfrak{q}^{-1} j) \otimes \mathbb{Z}_q$. Since $\mathcal{N}j = q$, the discriminant of the norm form in \mathcal{O}_q is a q -adic unit.
- If $l = p$, $\mathcal{O}_p = (\mathcal{O}_K + \mathcal{O}_K j) \otimes \mathbb{Z}_p$ hence the discriminant of the norm form in \mathcal{O}_p is p^2 since $p \mid D$.

□

3.1. **Counting special points.** Recall that

$$\left\langle \tilde{\mathcal{O}}\mathfrak{b}, t_m \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b} \right\rangle = \frac{1}{2} \# \text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b})[m],$$

where

$$\text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b})[m] := \left\{ u \in \text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}) : \mathcal{N}u = m\mathcal{N}\mathfrak{a} \right\}.$$

Let \mathcal{D} be the set of ideals,

$$\mathcal{D} := \{ \mathfrak{d} : \mathcal{N}\mathfrak{d} \mid D_0 \}.$$

Note that the elements of \mathcal{D} are in one to one correspondence with the elements of order 1 or 2 of the class group $\mathcal{J}(\mathcal{O}_K)$, since D is odd and hence squarefree.

Lemma 3.3. *Let $\mathfrak{a}, \mathfrak{b}$ ideals of \mathcal{O}_K of norm prime to D , and let $\mathfrak{d} \in \mathcal{D}$. Then*

$$\begin{aligned} \text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}\mathfrak{d}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}\mathfrak{d}) &= \left\{ \alpha + \beta j : \alpha \in \mathfrak{D}_0^{-1}\mathfrak{a}, \beta \in \mathfrak{D}_0^{-1}\mathfrak{p}\mathfrak{q}^{-1}\mathfrak{b}^{-1}\bar{\mathfrak{a}}\bar{\mathfrak{b}}, \right. \\ &\quad \alpha + q\beta \in (\mathcal{O}_K)_l \quad \forall l \mid \mathcal{N}\mathfrak{d}, \\ &\quad \left. \alpha - q\beta \in (\mathcal{O}_K)_l \quad \forall l \mid D_0 \text{ and } l \nmid \mathcal{N}\mathfrak{d} \right\}. \end{aligned}$$

Proof. By definition, $\text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}\mathfrak{d}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}\mathfrak{d}) = (\mathfrak{b}\mathfrak{d})^{-1}\tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}\mathfrak{d}$, i.e.

$$\begin{aligned} \text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}\mathfrak{d}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}\mathfrak{d}) &= \\ &= \left\{ b_0(\alpha + \beta j)ab_1 : (\alpha + \beta j) \in \tilde{\mathcal{O}}, b_0 \in (\mathfrak{b}\mathfrak{d})^{-1}, b_1 \in \mathfrak{b}\mathfrak{d} \text{ and } a \in \mathfrak{a} \right\}. \end{aligned}$$

For $\alpha \in K$, $\alpha j = j\bar{\alpha}$ thus $b_0(\alpha + \beta j)ab_1 = ab_0b_1\alpha + \bar{a}b_0\bar{b}_1\beta j$. The first term lies in $\mathfrak{D}_0^{-1}\mathfrak{a}$ while the second one lies in $\mathfrak{D}_0^{-1}\mathfrak{q}^{-1}\mathfrak{b}^{-1}\bar{\mathfrak{a}}\bar{\mathfrak{b}}$ since $\mathfrak{d}^{-1}\bar{\mathfrak{d}} = \mathcal{O}_K$.

We claim that $ab_0b_1\alpha - q\bar{a}b_0\bar{b}_1\beta \in \mathcal{O}_K \otimes \mathbb{Z}_l$ for all primes $l \mid D_0/d$. Indeed

$$ab_0b_1\alpha - q\bar{a}b_0\bar{b}_1\beta = ab_0b_1(\alpha - q\beta) + q\beta b_0(ab_1 - \overline{ab_1}),$$

so the claim follows from the condition on the norms of \mathfrak{a} and \mathfrak{b} , and the definition of $\tilde{\mathcal{O}}$.

On the other hand, if $l \mid d$, then

$$ab_0b_1\alpha + q\bar{a}b_0\bar{b}_1\beta = ab_0b_1(\alpha - q\beta) + q\beta b_0(ab_1 + \overline{ab_1}).$$

The first term is in \mathcal{O}_K as before, and since $b_1 \in \mathfrak{b}\mathfrak{d}$, $l \mid ab_1 + \overline{ab_1}$ so the second term lies in $\mathcal{O}_K \otimes \mathbb{Z}_l$ finishing the proof. \square

We denote $\delta(n) := 2^t$, where t is the number of prime factors of $\gcd(n, D_0)$. This is relevant because of the following computation.

Lemma 3.4. *With the same notation as above, let $\alpha \in \mathfrak{D}_0^{-1}\mathfrak{a}$ and $\beta \in \mathfrak{D}_0^{-1}\mathfrak{p}q^{-1}\mathfrak{b}^{-1}\overline{\mathfrak{a}}\overline{\mathfrak{b}}$ such that $\mathcal{N}(\alpha + \beta j) \in \mathbb{Z}$, and set $n = \frac{q\mathcal{N}\beta|D_0|}{p\mathcal{N}\mathfrak{a}} \in \mathbb{Z}$. Then*

$$\#\{\mathfrak{d} : \mathfrak{d} \in \mathcal{D}, \alpha + \beta j \in \text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}\mathfrak{d}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}\mathfrak{d})\} = \delta(n).$$

Proof. Take a prime $l \mid D_0$. When $l \mid n$, it follows that $\mathcal{N}\beta \in \mathbb{Z}_l$, hence $\beta \in \mathcal{O}_K \otimes \mathbb{Z}_l$ (since l is ramified). Since $\mathcal{N}(\alpha + \beta j) = \mathcal{N}\alpha + q\mathcal{N}\beta \in \mathbb{Z}$, it follows that $\alpha \in \mathcal{O}_K \otimes \mathbb{Z}_l$, and the condition at l in the previous lemma is trivially satisfied for all \mathfrak{d} .

If $l \nmid n$, then neither α nor β are in $\mathcal{O}_K \otimes \mathbb{Z}_l$, but $l\mathcal{N}\alpha \in \mathbb{Z}_l$ and $l\mathcal{N}\beta \in \mathbb{Z}_l$. We claim that this implies $\alpha + \bar{\alpha} \in \mathbb{Z}_l$ and $\beta + \bar{\beta} \in \mathbb{Z}_l$. In fact, $l\alpha \in \mathcal{O}_K \otimes \mathbb{Z}_l$, hence $D \mid \Delta(l\alpha) = (l\text{Tr}(\alpha))^2 - 4l(l\mathcal{N}\alpha)$. Since $l \mid D$, it follows that $(l\text{Tr}(\alpha))^2 \in l\mathbb{Z}_l$, thus $\text{Tr}(\alpha) \in \mathbb{Z}_l$.

Then, the condition $\mathcal{N}\alpha + q\mathcal{N}\beta \in \mathcal{O}_K \otimes \mathbb{Z}_l$ is equivalent to $\alpha^2 - q^2\beta^2 \in \mathcal{O}_K \otimes \mathbb{Z}_l$ (here we have used that $q \equiv -q^2 \pmod{l}$). Therefore, either $\alpha - q\beta \in \mathcal{O}_K \otimes \mathbb{Z}_l$ or $\alpha + q\beta \in \mathcal{O}_K \otimes \mathbb{Z}_l$, but not both. Therefore, the condition at l in the previous lemma is satisfied for exactly half of the possible \mathfrak{d} . Namely, when $\alpha - q\beta \in \mathcal{O}_K \otimes \mathbb{Z}_l$, the condition holds for all \mathfrak{d} with $l \nmid \mathcal{N}\mathfrak{d}$, and when $\alpha + q\beta \in \mathcal{O}_K \otimes \mathbb{Z}_l$, the condition holds for all \mathfrak{d} with $l \mid \mathcal{N}\mathfrak{d}$.

This implies the lemma, since the conditions on \mathfrak{d} for each $l \nmid n$ are independent. \square

Let \mathfrak{a} and \mathfrak{b} be ideals of \mathcal{O}_K of norm prime to D as in the lemma, and consider the map

$$\Psi_{\mathfrak{b}} : \text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}) \rightarrow \tilde{\mathcal{I}}(\mathcal{O}_K) \times \tilde{\mathcal{I}}(\mathcal{O}_K)$$

given by

$$u = \alpha + \beta j \mapsto (\alpha\mathfrak{D}_0\mathfrak{a}^{-1}, \beta\mathfrak{D}_0\mathfrak{q}\mathfrak{p}^{-1}\mathfrak{b}\overline{\mathfrak{b}}^{-1}\overline{\mathfrak{a}}^{-1}).$$

Note that $\Psi_{\mathfrak{b}}$ is well defined by Lemma 3.3 (i.e. it maps to a pair of *integral* lattices). Furthermore, its image is contained in

$$\Lambda := \{(\mathfrak{L}_1, \mathfrak{L}_2) : \mathfrak{L}_1 \sim \mathfrak{p}\mathfrak{a}^{-1}, \text{gen}[\mathfrak{L}_2] = \text{gen}[\mathfrak{a}\mathfrak{q}]\},$$

where by abuse of notation we allow \mathfrak{L}_1 or \mathfrak{L}_2 to be the zero ideal.

Moreover, if $\Psi_{\mathfrak{b}}(\alpha + \beta j) = (\mathfrak{L}_1, \mathfrak{L}_2)$ then clearly

$$\mathcal{N}\mathfrak{L}_1 = \frac{\mathcal{N}\alpha|D_0|}{\mathcal{N}\mathfrak{a}}, \quad \mathcal{N}\mathfrak{L}_2 = \frac{\mathcal{N}\beta|D_0|q}{p\mathcal{N}\mathfrak{a}},$$

and so

$$\frac{\mathcal{N}\mathfrak{L}_1 + p\mathcal{N}\mathfrak{L}_2}{|D_0|} = \frac{\mathcal{N}\alpha + q\mathcal{N}\beta}{\mathcal{N}\mathfrak{a}} = \frac{\mathcal{N}(\alpha + \beta j)}{\mathcal{N}\mathfrak{a}}.$$

Therefore, if we set

$$\Lambda[m] := \{(\mathfrak{L}_1, \mathfrak{L}_2) \in \Lambda : \mathcal{N} \mathfrak{L}_1 + p\mathcal{N} \mathfrak{L}_2 = m|D_0|\},$$

the maps $\Psi_{\mathfrak{b}}$ restrict to

$$\Psi_{\mathfrak{b}} : \text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b})[m] \rightarrow \Lambda[m].$$

Lemma 3.5. *The number of pairs $(\mathfrak{L}_1, \mathfrak{L}_2) \in \Lambda[m]$ with $\mathcal{N} \mathfrak{L}_2 = n$ is*

$$\begin{cases} 1 & \text{if } m = n = 0, \\ r_{\mathfrak{a}}(m) & \text{if } m > 0, n = 0, \\ R_{\text{gen}[\mathfrak{a}\mathfrak{q}]}(n) & \text{if } m|D| = p^2n \neq 0, \\ r_{\mathfrak{a}}(m|D| - p^2n)R_{\text{gen}[\mathfrak{a}\mathfrak{q}]}(n) & \text{if } m|D| > p^2n \neq 0. \end{cases}$$

Proof. Since $\mathcal{N} \mathfrak{L}_2 = n$, then the number of choices for this ideal is $R_{\text{gen}[\mathfrak{a}\mathfrak{q}]}(n)$ if $n \neq 0$, and 1 otherwise. Similarly, $\mathcal{N} \mathfrak{L}_1 = m|D_0| - pn$, and the number of choices for \mathfrak{L}_1 is either $r_{\mathfrak{p}\mathfrak{a}^{-1}}(m|D_0| - pn)$ (for $m|D| > p^2n$) or 1 otherwise.

The result follows by noting that $r_{\mathfrak{p}\mathfrak{a}^{-1}}(m|D_0| - pn) = r_{\mathfrak{a}}(m|D| - p^2n)$, because \mathfrak{p} is ramified; and when $n = 0$, since $\sqrt{D} \in \mathcal{O}_K$, this number is just $r_{\mathfrak{a}}(m|D|) = r_{\mathfrak{a}}(m)$. \square

Lemma 3.6. *Let $(\mathfrak{L}_1, \mathfrak{L}_2) \in \Lambda[m]$, with $\mathcal{N} \mathfrak{L}_2 = n$. Then*

$$\sum_{\mathfrak{b} \in \mathcal{J}(\mathcal{O}_K)} \#\Psi_{\mathfrak{b}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2) = \begin{cases} h_D & \text{if } m = n = 0 \\ 2u_D h_D & \text{if } m > 0, n = 0, \\ 2u_D \delta(n) & \text{if } m|D| = p^2n \neq 0, \\ 4u_D^2 \delta(n) & \text{if } m|D| > p^2n \neq 0. \end{cases}$$

Proof. First note that $\#\Psi_{\mathfrak{b}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2)$ depends only on the class of \mathfrak{b} , since the following diagram commutes

$$\begin{array}{ccc} \text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}) & \xrightarrow{\Psi_{\mathfrak{b}}} & \Lambda \\ \text{conjugation by } \gamma \downarrow & & \nearrow \\ \text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}\gamma, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}\gamma) & \xrightarrow{\Psi_{\mathfrak{b}\gamma}} & \Lambda \end{array}$$

for any $\gamma \in \mathcal{O}_K$.

Suppose $n \neq 0$. If $\mathfrak{b}^2\mathfrak{a}\mathfrak{q} \approx \mathfrak{L}_2$, then $\Psi_{\mathfrak{b}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2) = \emptyset$. Fix an ideal \mathfrak{b}_0 such that $\mathfrak{b}_0^2\mathfrak{a}\mathfrak{q} \sim \mathfrak{L}_2$. Then the set of classes \mathfrak{b} such that $\mathfrak{b}^2\mathfrak{a}\mathfrak{q} \sim \mathfrak{L}_2$ equals $\{\mathfrak{b}_0\mathfrak{d} : \mathfrak{d} \in \mathcal{D}\}$, hence

$$\sum_{\mathfrak{b} \in \mathcal{J}(\mathcal{O}_K)} \#\Psi_{\mathfrak{b}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2) = \sum_{\mathfrak{d}} \#\Psi_{\mathfrak{b}_0\mathfrak{d}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2).$$

Let α be a generator of $\mathfrak{L}_1\mathfrak{D}_0^{-1}\mathfrak{a}$ (which is principal), and let β be a generator of $\mathfrak{L}_2\mathfrak{D}_0^{-1}\bar{\mathfrak{a}}\mathfrak{q}^{-1}\mathfrak{p}\mathfrak{b}_0^{-1}\bar{\mathfrak{b}}_0$ (which is also principal by the choice of \mathfrak{b}_0). Given such a pair (α, β) , we have

$$\#\{\mathfrak{d} : \alpha + \beta\mathfrak{j} \in \Psi_{\mathfrak{b}_0\mathfrak{d}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2)\} = \delta(n)$$

by Lemma 3.4.

Now suppose $n = 0$: in this case $\mathfrak{L}_2 = 0$ and it follows from Lemma 3.4 that $\#\Psi_{\mathfrak{b}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2) = 1$ for all \mathfrak{b} , thus

$$\sum_{\mathfrak{b} \in \mathcal{J}(\mathcal{O}_K)} \#\Psi_{\mathfrak{b}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2) = h_D.$$

The statement follows by counting the number of choices for α and β , which can be $2u_D$ or 1 in each case (when the norm is non-zero or zero, respectively). \square

The following formula extends [4, Proposition 10.8] to level p^2 .

Theorem 3.7. *Let \mathfrak{a} be an ideal for \mathcal{O}_K . Then*

$$(3) \quad \sum_{\mathfrak{b} \in \mathcal{J}(\mathcal{O}_K)} \left\langle \tilde{\mathcal{O}}\mathfrak{b}, t_m \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b} \right\rangle = u_D h_D r_{\mathfrak{a}}(m) \\ + 2u_D^2 \sum_{n=1}^{|D|m/p^2} \delta(n) r_{\mathfrak{a}}(m|D| - p^2 n) R_{\text{gen}[\mathfrak{a}q]}(n),$$

where $\delta(n) := 2^t$, with t being the number of prime factors of $\gcd(n, D_0)$.

Proof. We have

$$\sum_{\mathfrak{b} \in \mathcal{J}(\mathcal{O}_K)} \left\langle \tilde{\mathcal{O}}\mathfrak{b}, t_m \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b} \right\rangle = \frac{1}{2} \sum_{\mathfrak{b} \in \mathcal{J}(\mathcal{O}_K)} \#\text{Hom}(\tilde{\mathcal{O}}\mathfrak{b}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b})[m] \\ = \frac{1}{2} \sum_{\mathfrak{b} \in \mathcal{J}(\mathcal{O}_K)} \sum_{(\mathfrak{L}_1, \mathfrak{L}_2) \in \Lambda[m]} \#\Psi_{\mathfrak{b}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2).$$

In order to evaluate this we split the inner sum by the norm of the ideal \mathfrak{L}_2 . This is suitable to apply Lemmas 3.5 and 3.6, which together give

$$\frac{1}{2} \sum_{\mathfrak{b} \in \mathcal{J}(\mathcal{O}_K)} \sum_{\substack{(\mathfrak{L}_1, \mathfrak{L}_2) \in \Lambda[m] \\ N\mathfrak{L}_2 = n}} \#\Psi_{\mathfrak{b}}^{-1}(\mathfrak{L}_1, \mathfrak{L}_2) \\ = \begin{cases} u_D h_D r_{\mathfrak{a}}(m) & \text{if } n = 0, \\ 2u_D^2 \delta(n) r_{\mathfrak{a}}(m|D| - p^2 n) R_{\text{gen}[\mathfrak{a}q]}(n) & \text{if } n \neq 0. \end{cases}$$

Note that the four different cases of the lemmas become just two cases by use of the convention $r_{\mathfrak{a}}(0) = \frac{1}{2u_D}$.

The statement follows by adding this expression over $n \geq 0$. \square

3.2. Special points and central values of L -series. Let f be a cusp form in $S_2^{\text{new}}(\Gamma_0(N))$. In the appendix we recall the definition of an L -series $L_{\mathcal{A}}(f, s)$, a Rankin convolution of $L(f, s)$ and a partial zeta function associated to an ideal

class \mathcal{A} of $\mathbb{Q}(\sqrt{D})$. These L -series are interesting due to their relation to the L -series of f and its twists; for instance we have the factorization

$$\sum_{\mathcal{A}} L_{\mathcal{A}}(f, s) = L(f, s) L(f, D, s),$$

where the sum is over all ideal classes of $\mathbb{Q}(\sqrt{D})$. The main result of the appendix is the following generalization of [5, (4.4) p.283] regarding the central values of this L -series.

Theorem 3.8 (Theorem A.14). *Let $D < 0$ be an odd fundamental discriminant, \mathcal{A} be an ideal in $\mathbb{Q}[\sqrt{D}]$ and $f(z)$ be a cusp form in $S_2^{\text{new}}(\Gamma_0(N))$. Then,*

$$L_{\mathcal{A}}(f, 1) = \frac{8\pi^2}{\sqrt{|D|}} \langle f, g_{\mathcal{A}} \rangle,$$

with $g_{\mathcal{A}} = g_{\mathcal{A}}^{(N)} = \sum b_{\mathcal{A}}(m)q^m$, where

$$(4) \quad b_{\mathcal{A}}(m) := \frac{1 - \varepsilon_D(N\eta)}{2} \cdot \frac{h(D)}{u_D} r_{\mathcal{A}}(m) \\ + \sum_{\text{gen}[\mathfrak{q}] \in Q} \sum_{n=1}^{|D|m/N} \delta(n) r_{\mathcal{A}}(m|D| - nN) R_{\text{gen}[\mathcal{A}\mathfrak{q}]}(n),$$

where the first sum is over the set of genera

$$Q := \{\text{gen}[\mathfrak{q}] : \mathcal{N}\mathfrak{q} \equiv -N \pmod{D_0}\},$$

and where $\delta(n) := 2^t$, with t the number of prime factors of $\text{gcd}(n, D_0)$.

Comparing the right hand side of (3) in Theorem 3.7 with the formula (4) for the Fourier coefficients of the form $g_{\mathcal{A}}^{(p^2)}$ in Theorem A.14, we obtain an explicit formula for $g_{\mathcal{A}}^{(p^2)}$ in terms of special points.

Corollary 3.9. *On the above notation,*

$$(5) \quad g_{\mathcal{A}}^{(p^2)} = \frac{1}{2u_D^2} \sum_{\mathfrak{b} \in \mathcal{J}(\mathcal{O}_K)} \phi(\tilde{\mathcal{O}}\mathfrak{b}, \tilde{\mathcal{O}}\mathfrak{a}\mathfrak{b}),$$

Proof. In the formula for $b_{\mathcal{A}}(m)$ of Theorem A.14, note that $\varepsilon_D(N\eta) = 0$ (since $N = p^2$ and $p \mid D$ in our case), and the set Q consists of a unique element $\text{gen}[\mathfrak{q}]$. Then the statement follows immediately from Theorem 3.7. \square

Assume now f is a normalized eigenform in $S_2^{\text{new}}(p^2)$. Fix a character φ of $\mathcal{J}(\mathcal{O}_K)$, and define

$$L_{\varphi}(f, s) := \sum_{\mathcal{A}} \varphi(\mathcal{A}) L_{\mathcal{A}}(f, s).$$

Consider $\mathfrak{c}_{\varphi} = \sum_{\mathfrak{a}} \varphi^{-1}(\mathfrak{a}) \tilde{\mathcal{O}}\mathfrak{a} \in \mathcal{M}(\tilde{\mathcal{O}})$, and denote $\mathfrak{c}_{f, \varphi}$ its projection to the f -isotypical component of $\mathcal{M}(\tilde{\mathcal{O}})$.

Proposition 3.10.

$$L_\varphi(f, 1) = \frac{4\pi^2}{u_D^2} \cdot \frac{\langle f, f \rangle}{\sqrt{|D|}} \langle \mathbf{c}_{f,\varphi}, \mathbf{c}_{f,\varphi} \rangle.$$

Proof. The proof is similar to Proposition 11.2 of [4], given Theorem A.14 and Corollary 3.9. In our case, the f -isotypical component in $\mathcal{M}(\tilde{\mathcal{O}})$ may have dimension 2; however, since f is new, the f -isotypical component of $S_2^{\text{new}}(p^2)$ has dimension 1, and the same reasoning as given by Gross still applies. \square

Remark 3.11. If the f -isotypical component $\mathcal{M}(\tilde{\mathcal{O}})^f$ is zero, the proposition implies that $L_\varphi(f, 1) = 0$ for all characters φ . Equivalently,

$$L_{\mathcal{A}}(f, 1) = 0$$

for all ideal classes \mathcal{A} , and for all discriminants, whenever f is a twist of a form of level p by a non-quadratic character.

4. PROOF OF THE MAIN THEOREM

In this section we want to relate the central value of the L -series $L_\varphi(f, 1)$ to coefficients of half-integral weight modular forms. We will assume from now on that $\varphi = 1_D$. This case of the Rankin convolution L -series is related to our main formula because of the factorization

$$L_{1_D}(f, s) = L(f, s) L(f, D, s).$$

We will start with the case of odd discriminants D , which follows from the results in Section 3. The case of even discriminants could be proved by a similar calculation, but we avoid the technical difficulties of this case by resorting to a theorem of Waldspurger. This step is done in the proof of Theorem 4.11; until then we will assume that the discriminant D is odd, just so that we can use the results in previous sections.

Let \mathcal{P} denote the set of norm p bilateral $\tilde{\mathcal{O}}$ -ideals. If $\mathfrak{b} \in \mathcal{M}(\tilde{\mathcal{O}})$, $\mathfrak{p} \in \mathcal{P}$, and D is a negative fundamental discriminant, with $D = -pd$, the coefficient of q^d in the q -expansion of $\Theta_{\mathfrak{p}}([\mathfrak{b}])$ is

$$(6) \quad c_{d,\mathfrak{p}}(\mathfrak{b}) = \frac{1}{2} \#\{x \in \mathbb{Z} + \mathfrak{b}^{-1}\mathfrak{p}\mathfrak{b}/\mathbb{Z} : \Delta x = D\}.$$

Let $\mathbf{c}_{d,\mathfrak{p}} := \sum_{[\mathfrak{b}]} c_{d,\mathfrak{p}}(\mathfrak{b})[\mathfrak{b}]$. Then if $\mathfrak{e} \in \mathcal{M}(\tilde{\mathcal{O}})$, the coefficient of q^d in the q -expansion of $\Theta_{\mathfrak{p}}(\mathfrak{e})$ is $\langle \mathbf{c}_{d,\mathfrak{p}}, \mathfrak{e} \rangle$.

We want to give an adèlic description of this set. Let $\omega_D \in \mathcal{O}_K$ an element of discriminant D ; adding an integer we will assume that $\text{Tr } \omega_D \equiv 0 \pmod{p}$. It's easy to check that then $\mathcal{O}_K = \langle 1, \omega_D \rangle$ and $\mathfrak{p}_K = \langle p, \omega_D \rangle$. Moreover,

$$4 \mathcal{N}\omega_D = (\text{Tr } \omega_D)^2 - D \equiv -D \pmod{p^2}.$$

Fix an embedding $i : K \hookrightarrow B$, and let $\mathbf{x} = i(\omega_D)$. Once such embedding is fixed, by Proposition 1.4 the special points of discriminant D correspond to some elements

in the double coset $\tilde{\mathcal{J}}(\tilde{\mathcal{O}})/K^\times$. Explicitly, let $x \in \mathbb{Z} + \mathfrak{b}^{-1}\mathfrak{p}\mathfrak{b}$ of discriminant D ; adding an integer we may assume that $\text{Tr } x = \text{Tr } \mathbf{x}$. Hence $\mathcal{N}x = \mathcal{N}\mathbf{x}$ as well, and so there exists $\alpha \in B^\times/K^\times$ with

$$\alpha^{-1}x\alpha = \mathbf{x}.$$

The correspondence associates to x the \mathcal{O}_K -point $\mathfrak{b}\alpha$. The condition $x \in \mathbb{Z} + \mathfrak{b}^{-1}\mathfrak{p}\mathfrak{b}$ translates to the condition $\mathbf{x} \in \mathbb{Z} + (\mathfrak{b}\alpha)^{-1}\mathfrak{p}(\mathfrak{b}\alpha)$.

For $\mathfrak{p} \in \mathcal{P}$, define

$$\mathcal{C}_{\mathfrak{p}} := \left\{ \mathfrak{a} \in \tilde{\mathcal{J}}(\tilde{\mathcal{O}}) : (\mathfrak{a}, i) \text{ is a special point for } \mathcal{O}_K \text{ and } \mathbf{x} \in \mathfrak{a}^{-1}\mathfrak{p}\mathfrak{a} \right\}.$$

Recall from Section 1 that (\mathfrak{a}, i) is a special point for \mathcal{O}_K if $R_r(\mathfrak{a}) \cap K = \mathcal{O}_K$.

Lemma 4.1. $\mathcal{C}_{\mathfrak{p}}$ is closed under the action of \widehat{K}^\times by right multiplication, i.e. $\mathcal{C}_{\mathfrak{p}}\widehat{K}^\times = \mathcal{C}_{\mathfrak{p}}$.

Proof. Let $\mathfrak{a} \in \mathcal{C}_{\mathfrak{p}}$ and $\hat{\alpha} \in \widehat{K}^\times$. By Proposition 1.5, $\mathfrak{a}\hat{\alpha}$ is an \mathcal{O}_K -point. Since $\mathbf{x} \in \mathfrak{a}^{-1}\mathfrak{p}\mathfrak{a}$,

$$\hat{\alpha}^{-1}\mathbf{x}\hat{\alpha} \in (\mathfrak{a}\hat{\alpha})^{-1}\mathfrak{p}(\mathfrak{a}\hat{\alpha}).$$

But $\hat{\alpha}^{-1}\mathbf{x}\hat{\alpha} = \mathbf{x}$, because all elements are in \widehat{K} , which is commutative. Then $\mathfrak{a}\hat{\alpha} \in \mathcal{C}_{\mathfrak{p}}$ as claimed. \square

Lemma 4.2. $\mathcal{C}_{\mathfrak{p}}$ is closed under the action of \widetilde{W} by left multiplication.

Proof. Recall from Section 1 that the bilateral ideals act on the \mathcal{O}_K -points by left multiplication. Let $\mathfrak{a} \in \mathcal{C}_{\mathfrak{p}}$; then $\mathbf{x} \in \mathfrak{a}^{-1}\mathfrak{p}\mathfrak{a}$, and we must show that $\widetilde{W}\mathfrak{a} \in \mathcal{C}_{\mathfrak{p}}$. But \widetilde{W} is the order two rotation of the group of bilateral $\tilde{\mathcal{O}}$ -ideals (which is a dihedral group), hence it commutes with \mathfrak{p} for any $\mathfrak{p} \in \mathcal{P}$. Thus,

$$\mathbf{x} \in \mathfrak{a}^{-1}\mathfrak{p}\mathfrak{a} = (\widetilde{W}\mathfrak{a})^{-1}\mathfrak{p}(\widetilde{W}\mathfrak{a}),$$

hence $\widetilde{W}\mathfrak{a} \in \mathcal{C}_{\mathfrak{p}}$ as claimed. \square

The last two lemmas imply that $\mathcal{C}_{\mathfrak{p}}$ is closed under the action of $\{1, \widetilde{W}\} \times \mathcal{J}(\mathcal{O}_K)$. Moreover, $c_{d,p}(\mathfrak{b}) = \frac{1}{2}\#\{\mathfrak{a} \in \mathcal{C}_{\mathfrak{p}}/K^\times : \mathfrak{a} \sim \mathfrak{b}\}$.

Let R be an order in the same genus of $\tilde{\mathcal{O}}$ such that $i(\mathcal{O}_K) \subseteq R$. In particular, $\mathbf{x} \in R$. Such an order R exists because $p \mid D$. Indeed, $i(\mathcal{O}_K)$ is contained in some maximal order R_0 of B ; but the condition $p \mid D$ implies that $p \mid \Delta x$ for all $x \in i(\mathcal{O}_K)$, hence $i(\mathcal{O}_K)$ is actually contained in the unique order of index p in R_0 .

Lemma 4.3. Assume that $\chi(\mathfrak{p}) = \left(\frac{d}{p}\right)$. Then, there is an $\hat{\alpha} \in \widehat{B}^\times$ such that

- (1) $\hat{\alpha}^{-1}\tilde{\mathcal{O}}\hat{\alpha} = R$,
- (2) $\hat{\alpha}^{-1}\mathfrak{p}\hat{\alpha} = Ri(\mathfrak{p}_K)$.

Moreover, $\hat{\alpha}$ modulo multiplication by \mathbb{Q}^\times , is unique up to left multiplication by the group generated by \widetilde{W} and $W_{\mathfrak{p}}$.

Proof. Since R is in the same genus as $\tilde{\mathcal{O}}$, there exists $\hat{\alpha}$ satisfying the first condition. Two such elements differ by a bilateral $\tilde{\mathcal{O}}$ -ideal.

We claim that the left R -ideal $Ri(\mathfrak{p}_K)$ is a bilateral ideal. It is enough to prove that all the localizations are bilateral ideals. Since $Ri(\mathcal{O}_K) = R$, and $(\mathcal{O}_K)_q = (\mathfrak{p}_K)_q$ at all primes $q \neq p$, the localizations at primes $q \neq p$ give bilateral ideals. At the ramified prime p , there is a unique maximal order, and a unique order of index p in it, hence the localization at p must also give a bilateral ideal.

Recall that the group of bilateral $\tilde{\mathcal{O}}$ -ideals modulo \mathbb{Q}^\times is a dihedral group. We claim that $\chi(Ri(\mathfrak{p}_K)) = \left(\frac{d}{p}\right)$. Indeed, $\mathbf{x} \in Ri(\mathfrak{p}_K)$ and $\mathcal{N}\mathbf{x} = \mathcal{N}\omega_D \equiv \frac{-D}{4} \pmod{p^2}$, hence $\left(\frac{\mathcal{N}\mathbf{x}/p}{p}\right) = \left(\frac{-D/(4p)}{p}\right) = \left(\frac{d}{p}\right)$. Then by assumption $\chi(Ri(\mathfrak{p}_K)) = \chi(\mathfrak{p})$, hence $\hat{\alpha}^{-1}\mathfrak{p}\hat{\alpha}$ and $Ri(\mathfrak{p}_K)$, being bilateral, are in the same orbit by conjugation in this dihedral group. Thus the second condition holds by changing $\hat{\alpha}$ accordingly.

The last statement follows from the fact that the stabilizer of \mathfrak{p} in the dihedral group is the group generated by \widetilde{W} and $W_{\mathfrak{p}}$. \square

Note that, for an ideal class $[\mathfrak{a}]$ in $\mathcal{J}(\mathcal{O}_K)$, the $\tilde{\mathcal{O}}$ -ideal $\hat{\alpha}Ri(\mathfrak{a})$ with $\hat{\alpha}$ as in the lemma is an \mathcal{O}_K -point, since $\mathbf{x} \in i(\mathfrak{p}_K)$. Then we define

$$\mathcal{C}_{D,\mathfrak{p}} := \left\{ \hat{\alpha}Ri(\mathfrak{a}) : [\mathfrak{a}] \in \mathcal{J}(\mathcal{O}_K) \right\} \cup \left\{ \widetilde{W}\hat{\alpha}Ri(\mathfrak{a}) : [\mathfrak{a}] \in \mathcal{J}(\mathcal{O}_K) \right\}.$$

Lemma 4.4.

- (1) $W_{\mathfrak{p}}\mathcal{C}_{D,\mathfrak{p}} = \mathcal{C}_{D,\mathfrak{p}}$.
- (2) $\mathcal{C}_{D,\mathfrak{p}}$ is independent of the choice of $\hat{\alpha}$.

Proof. The first statement follows from

$$\mathfrak{p}\hat{\alpha}Ri(\mathfrak{a}) = \hat{\alpha}Ri(\mathfrak{p}_K)i(\mathfrak{a}) = \hat{\alpha}Ri(\mathfrak{p}_K\mathfrak{a}),$$

since multiplication by \mathfrak{p}_K is a permutation of $\mathcal{J}(\mathcal{O}_K)$. For the second statement note that clearly $\widetilde{W}\mathcal{C}_{D,\mathfrak{p}} = \mathcal{C}_{D,\mathfrak{p}}$, and use the final statement of the previous lemma. \square

Theorem 4.5. *With the previous notation,*

$$\mathcal{C}_{D,\mathfrak{p}} = \mathcal{C}_{\mathfrak{p}}/K^\times.$$

Proof. Let $\hat{\alpha}$ be as in Lemma 4.3, so that $\hat{\alpha}R \in \mathcal{C}_{D,\mathfrak{p}}$. Then $\hat{\alpha}R = \tilde{\mathcal{O}}\hat{\alpha}$ by the first condition in the lemma, and the second condition implies that it is in $\mathcal{C}_{\mathfrak{p}}$.

Then, by Lemma 4.1 and Lemma 4.2, $\mathcal{C}_{D,\mathfrak{p}} \subseteq \mathcal{C}_{\mathfrak{p}}/K^\times$. The theorem follows from the fact that $\mathcal{C}_{\mathfrak{p}}/K^\times$ has exactly $2h_D$ elements, where h_D is the class number of \mathcal{O}_K . This will be proved in the following lemmas.

Lemma 4.6. *We have*

$$\{\mathcal{O}_K\text{-points for } \tilde{\mathcal{O}}\} = \bigcup_{\mathfrak{p} \in \mathcal{P}} \mathcal{C}_{\mathfrak{p}}/K^\times.$$

Furthermore, the union is disjoint.

Proof. The fact that the set of \mathcal{O}_K -points are the union over the norm p bilateral $\tilde{\mathcal{O}}$ -ideals follows from the fact that there is a bijection between bilateral ideals of norm p and orders of index p in $\tilde{\mathcal{O}}$ (given by $\mathfrak{p} \leftrightarrow \mathbb{Z} + \mathfrak{p}$) and the union of such orders is $\tilde{\mathcal{O}}$. The second claim comes from the fact that the intersection of two different index p suborders of $\tilde{\mathcal{O}}$ gives $\mathbb{Z} + p\mathcal{O}$, hence the discriminant of such elements is divisible by p^2 . \square

Lemma 4.7. *If $\chi(\mathfrak{p}) \neq \left(\frac{d}{p}\right)$ then $\mathcal{C}_{\mathfrak{p}} = \emptyset$.*

Proof. If $\mathcal{C}_{\mathfrak{p}} \neq \emptyset$, there exists $\hat{\alpha}$ such that $\mathbf{x} \in \hat{\alpha}^{-1}\mathfrak{p}\hat{\alpha}$. By definition, $\chi(\hat{\alpha}^{-1}\mathfrak{p}\hat{\alpha})$ is computed by choosing an element in this ideal of norm divisible by p but not by p^2 . Since \mathbf{x} is such an element,

$$\chi(\mathfrak{p}) = \chi(\hat{\alpha}^{-1}\mathfrak{p}\hat{\alpha}) = \left(\frac{\mathcal{N}\mathbf{x}/p}{p}\right) = \left(\frac{-D/p}{p}\right) = \left(\frac{d}{p}\right).$$

\square

Proposition 4.8. *If $\mathfrak{p}_1, \mathfrak{p}_2 \in \mathcal{P}$ with $\chi(\mathfrak{p}_1) = \chi(\mathfrak{p}_2)$ then $\mathcal{C}_{\mathfrak{p}_1}$ and $\mathcal{C}_{\mathfrak{p}_2}$ have the same number of elements.*

Proof. If $\chi(\mathfrak{p}_1) = \chi(\mathfrak{p}_2)$ there exists a rotation \mathfrak{o} in the group of bilateral ideals such that $\mathfrak{o}^{-1}\mathfrak{p}_2\mathfrak{o} = \mathfrak{p}_1$. Then $\mathfrak{o}\mathcal{C}_{\mathfrak{p}_1} = \mathcal{C}_{\mathfrak{p}_2}$, where the action is given by left multiplication. \square

Lemma 4.9. *The number of \mathcal{O}_K -points for \mathcal{O} is h_D .*

Proof. The set of \mathcal{O}_K -points is an homogeneous space for $\text{Bil}(\mathcal{O})/\mathbb{Q}^\times \times \mathcal{J}(\mathcal{O}_K)$. The results follows from the fact that $\text{Bil}(\mathcal{O})/\mathbb{Q}^\times$ has two elements, and the norm p ideal in $\text{Bil}(\mathcal{O})$ acts as the ideal $\mathfrak{p}_K \in \tilde{\mathcal{J}}(\mathcal{O}_K)$. \square

Proposition 4.10. *The number of \mathcal{O}_K -points for $\tilde{\mathcal{O}}$ is $(p+1)h_D$.*

Proof. See Theorem 2.7 and Theorem 4.8 of [10]. \square

The last proposition asserts that the total number of \mathcal{O}_K -points for $\tilde{\mathcal{O}}$ is $(p+1)h_D$. By Lemma 4.6, this equals $\sum_{\mathfrak{p} \in \mathcal{P}} \#\mathcal{C}_{\mathfrak{p}}/K^\times$. By Lemma 4.7, half of this numbers are zero and by Proposition 4.8 all the non-empty sets have the same number of elements. This implies that

$$\#\mathcal{C}_{\mathfrak{p}}/K^\times = 2h_D,$$

which finishes the proof of Theorem 4.5. \square

In particular, $2\mathbf{c}_{d,\mathfrak{p}} = \sum_{\mathfrak{b} \in \mathcal{C}_{D,\mathfrak{p}}} [\mathfrak{b}]$ in $\mathcal{M}(\tilde{\mathcal{O}})$. Since $W_{\mathfrak{p}}$ commutes with \mathbb{T}_0 and acts trivially on $\mathbf{c}_{d,\mathfrak{p}}$, then the projection $\mathbf{c}_{f,\mathfrak{p}}$ to the f -isotypical component is a vector on which $W_{\mathfrak{p}}$ acts trivially.

Assume there is a non-zero eigenvector $\mathbf{e}_f \in \mathcal{M}(\tilde{\mathcal{O}})^f$ with $W_{\mathfrak{p}}\mathbf{e}_f = \mathbf{e}_f$, as otherwise $\mathbf{c}_{f,\mathfrak{p}} = 0$ and $L(f, 1) = 0$ by Proposition 3.10. When $\dim \mathcal{M}(\tilde{\mathcal{O}})^f = 2$, this vector always exists by Theorem 2.7 (multiplicity one). In any case \mathbf{e}_f is unique up to a constant, and therefore

$$\mathbf{c}_{f,\mathfrak{p}} = \frac{\langle \mathbf{c}_{d,\mathfrak{p}}, \mathbf{e}_f \rangle}{\langle \mathbf{e}_f, \mathbf{e}_f \rangle} \mathbf{e}_f,$$

and

$$(7) \quad \langle \mathbf{c}_{f,\mathfrak{p}}, \mathbf{c}_{f,\mathfrak{p}} \rangle = \frac{|\langle \mathbf{c}_{d,\mathfrak{p}}, \mathbf{e}_f \rangle|^2}{\langle \mathbf{e}_f, \mathbf{e}_f \rangle}.$$

Theorem 4.11 (Main Theorem). *Let f be a new eigenform of weight 2, level p^2 with $p > 2$ an odd prime. Fix a norm p bilateral $\tilde{\mathcal{O}}$ -ideal \mathfrak{p} , and let \mathbf{e}_f be an eigenvector in the f -isotypical component of $\mathcal{M}(\tilde{\mathcal{O}})$ such that $W_{\mathfrak{p}}(\mathbf{e}_f) = \mathbf{e}_f$.*

If d is an integer such that $D = -pd < 0$ is a fundamental discriminant, and such that $\left(\frac{d}{p}\right) = \chi(\mathfrak{p})$, then

$$L(f, 1) L(f, D, 1) = 4\pi^2 \frac{\langle f, f \rangle}{\langle \mathbf{e}_f, \mathbf{e}_f \rangle} \frac{c_d^2}{\sqrt{pd}},$$

where the c_d are the Fourier coefficients of $\Theta_{\mathfrak{p}}(\mathbf{e}_f) = \sum_{d \geq 1} c_d q^d$.

Proof. • *First case: odd discriminants.*

By Proposition 3.10,

$$L(f, 1) L(f, D, 1) = L_{1_D}(f, 1) = \frac{4\pi^2}{u_D^2} \cdot \frac{\langle f, f \rangle}{\sqrt{|D|}} \langle \mathbf{c}_f, \mathbf{c}_f \rangle,$$

where \mathbf{c}_f is the projection of $\mathbf{c}_1 = \sum_{\mathfrak{a}} R\mathfrak{a}$ to the f -isotypical component in $\mathcal{M}(\tilde{\mathcal{O}})$. Since $W_{\mathfrak{p}}$ acts trivially in \mathbf{c}_1 , it acts trivially in its projection \mathbf{c}_f so we just need to project it to the eigenspace where $W_{\mathfrak{p}}$ acts trivially.

Since $2\mathbf{c}_{d,\mathfrak{p}} = \sum_{\mathfrak{b} \in \mathcal{C}_{D,\mathfrak{p}}} [\mathfrak{b}]$, from the definition of $\mathcal{C}_{D,\mathfrak{p}}$ we get

$$\langle \mathbf{c}_{d,\mathfrak{p}}, \mathbf{e}_f \rangle = \frac{1}{2} \left(\langle \mathbf{c}_1, \mathbf{e}_f \rangle + \langle \widetilde{W}\mathbf{c}_1, \mathbf{e}_f \rangle \right).$$

If the operator \widetilde{W} acts as -1 in \mathbf{e}_f , then $c_d = \langle \mathbf{c}_{d,\mathfrak{p}}, \mathbf{e}_f \rangle = 0$ and the left hand side of the main formula also vanishes (since the sign of the functional equation is -1 in this case). Otherwise, \widetilde{W} acts trivially in \mathbf{e}_f and so $\langle \mathbf{c}_{d,\mathfrak{p}}, \mathbf{e}_f \rangle = \langle \mathbf{c}_1, \mathbf{e}_f \rangle$, hence $\mathbf{c}_{f,\mathfrak{p}} = \mathbf{c}_f$. Therefore

$$\langle \mathbf{c}_f, \mathbf{c}_f \rangle = \langle \mathbf{c}_{f,\mathfrak{p}}, \mathbf{c}_{f,\mathfrak{p}} \rangle = \frac{|\langle \mathbf{c}_{d,\mathfrak{p}}, \mathbf{e}_f \rangle|^2}{\langle \mathbf{e}_f, \mathbf{e}_f \rangle} = \frac{c_d^2}{\langle \mathbf{e}_f, \mathbf{e}_f \rangle}.$$

• *Second case: even discriminants.* The modular form $\Theta_p(\mathbf{e}_f)$ lies in the space $S_{3/2}(4p^2, \mathfrak{z}_p)$ and maps to f via the Shimura correspondence. In this situation, Corollary 2 of [14] states:

Theorem 4.12 (Waldspurger). *Let $pd_1, pd_2 \in \mathbb{N}$ be square-free integers, and suppose that $d_1/d_2 \in \mathbb{Q}_q^{\times 2}$ for $q = p$ and $q = 2$. Then one has the equality*

$$c_{d_1}^2 L(f, -pd_2^\dagger, 1) \sqrt{pd_2} = c_{d_2}^2 L(f, -pd_1^\dagger, 1) \sqrt{pd_1},$$

where $-pd_i^\dagger$ is the discriminant of the quadratic field $\mathbb{Q}[\sqrt{-pd_i}]$.

For the particular case of $\Theta_p(\mathbf{e}_f)$, one actually has

Proposition 4.13. *Let $-pd_1$ and $-pd_2$ be fundamental discriminants and suppose that $d_1/d_2 \in \mathbb{Q}_p^{\times 2}$. Then one has the equality*

$$c_{d_1}^2 L(f, -pd_1, 1) \sqrt{pd_2} = c_{d_2}^2 L(f, -pd_2, 1) \sqrt{pd_1}.$$

We show that the case of even discriminants of the Main Theorem follows from Proposition 4.13. By Theorem 4 of [15], there exists an odd fundamental discriminant $-pd_0$ such that $L(f, -pd_0, 1) \neq 0$. The Main Theorem for odd fundamental discriminants implies that the coefficient $c_{d_0} \neq 0$. Let $-pd$ be an even fundamental discriminant, then

$$\begin{aligned} L(f, -pd, 1)L(f, 1) &= \frac{L(f, -pd, 1)}{L(f, -pd_0, 1)} L(f, -pd_0, 1) L(f, 1) \\ &= \frac{c_d^2}{\sqrt{pd}} \frac{\sqrt{pd_0}}{c_{d_0}^2} L(f, -pd_0, 1) L(f, 1) \\ &= 4\pi^2 \frac{\langle f, f \rangle}{\langle \mathbf{e}_f, \mathbf{e}_f \rangle} \frac{c_d^2}{\sqrt{pd}} \end{aligned}$$

where the second equality follows from Proposition 4.13 and the last one follows from the Main Theorem for odd fundamental discriminants. \square

Proof of Proposition 4.13. To prove the result, we follow the proof of the Corollary 2 in [14]. The same reasoning implies the result once we prove that the local factor at 2 of the weight 3/2 modular form $\Theta_p(\mathbf{e}_f)$ is the same for pd_1 and pd_2 . Let λ_2 be the eigenvalue of the Hecke operator T_2 acting on f . Let α, α' denote the roots of the polynomial $x^2 - \frac{\lambda_2}{\sqrt{2}}x + 1$.

The space $S_{3/2}(4p^2, \mathfrak{z}_p)$ is 4-dimensional. Generators for this space are obtained by choosing 2 local functions at the prime p and 2 local functions at the prime 2. Following the notation of [14] (p.453), define the functions $c'_2[\delta]$ and $c''_2[\delta]$ on d (where $-pd$ is a fundamental discriminant) by:

$$c'_2[\delta](d) := \begin{cases} \delta - (2, d)_2 \mathfrak{z}_{p,2}(2)/\sqrt{2} & \text{if } 2 \nmid d. \\ \delta & \text{if } 2 \mid d, \end{cases}$$

where $(*, *)_2$ denotes the Hilbert symbol at 2, and $\varkappa_{p,2}$ is the character on \mathbb{Q}_2^\times associated to \varkappa_p . And the function

$$c_2''[\delta](d) := \delta.$$

Then the set of local functions at the prime 2 is given by

$$\begin{cases} \{c_2'[\alpha], c_2'[\alpha']\} & \text{if } \alpha \neq \alpha', \\ \{c_2'[\alpha], c_2''[\alpha]\} & \text{if } \alpha = \alpha'. \end{cases}$$

Moreover, we have $c_2'[\alpha](d) = 1$ for $2 \parallel d$. But since the coefficients c_d of $\Theta_p(\mathbf{e}_f)$ vanish in this case, its local function at 2 (up to a global constant) must be, in the first case,

$$c_2'[\alpha] - c_2'[\alpha'].$$

This function clearly attains the same value at odd and even values of d (namely $\alpha - \alpha'$).

In the second case, the local function at 2 must be $c_2''[\alpha]$, since $c_2''[\alpha](d) = 0$ when $2 \parallel d$. This function also clearly attains the same value for odd and even values of d . The rest of the proof is exactly the same as Waldspurger. \square

APPENDIX A. RANKIN'S METHOD

Notation A.1. If n, m are integers, we write $n \mid m^\infty$ if every prime factor of n divides m . We denote by $\gcd(n, m^\infty)$ the unique positive integer M that satisfies

- $M \mid n$,
- $M \mid m^\infty$,
- $\gcd(\frac{n}{M}, m) = 1$.

Let $D < 0$ be a fundamental discriminant. If \mathcal{A} is an ideal class of $\mathbb{Q}(\sqrt{D})$, we denote $\Theta_{\mathcal{A}}$ the theta series

$$\Theta_{\mathcal{A}}(z) := \sum_{n=0}^{\infty} r_{\mathcal{A}}(n)q^n = \frac{1}{2} \sum_{x \in \mathfrak{a}} q^{N(x)/N\mathfrak{a}},$$

where \mathfrak{a} is any ideal in the class \mathcal{A} . It is well known that $\Theta_{\mathcal{A}}$ is a weight 1 modular form of level $|D|$ and nebentypus ε_D , where $\varepsilon_D : (\mathbb{Z}/D\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ denotes the character $\varepsilon_D(n) = \left(\frac{D}{n}\right)$ of the field $\mathbb{Q}(\sqrt{D})$.

Definition A.2. Let $f(z) = \sum a(n)q^n$ be a cusp form in $S_2^{\text{new}}(\Gamma_0(N))$. Define

$$L_{\mathcal{A}}(f, s) := \left(\sum_{(m, N)=1} \frac{\varepsilon_D(m)}{m^{2s-1}} \right) \cdot \left(\sum_{m=1}^{\infty} \frac{a(m)r_{\mathcal{A}}(m)}{m^s} \right)$$

which converges for $\Re(s) > 3/2$.

A.1. Rankin's Method. For each decomposition $D = D_1 D_2$ of D as the product of two fundamental discriminants, define the Eisenstein series

$$E_s^{(D_1, D_2)}(z) := \frac{1}{2} \sum_{\substack{m, n \in \mathbb{Z} \\ D_2 | m}} \frac{\varepsilon_{D_1}(m) \varepsilon_{D_2}(n)}{(mz + n)} \frac{y^s}{|mz + n|^{2s}}$$

The series $E_s^{(D_1, D_2)}(z)$ is a non-holomorphic weight 1 modular form of level $|D|$ and Nebentypus ε_D .

Let $\eta = \gcd(N, D)$ and $N_0 = N/\eta$. In [5], they work in the case $\eta = 1$; when N is a perfect square, this restriction makes their formula for the central value vanish trivially on both sides (see the remark after Proposition A.12).

Proposition A.3.

$$(4\pi)^{-s} \Gamma(s) L_{\mathcal{A}}(f, s) = \langle f, G_{\bar{s}-1, \mathcal{A}} \rangle_{\Gamma_0(N)}$$

where

$$G_{s, \mathcal{A}}(z) := \mathrm{Tr}_N^{N_0 | D|} (\Theta_{\mathcal{A}}(z) E_s^{(1, D)}(N_0 z))$$

Proof. Similar to [5, (1.2) p. 272]. □

A.2. Computation of the trace. For D a fundamental discriminant, let

$$\kappa(D) := \begin{cases} 1 & \text{if } D > 0 \\ i & \text{if } D < 0. \end{cases}$$

If $D = D_1 D_2$ is a decomposition of D as the product of two fundamental discriminants, χ_{D_1, D_2} denotes the corresponding genus character, i.e. for ideals \mathcal{A} of norm prime to D , $\chi_{D_1, D_2}(\mathcal{A}) = \varepsilon_{D_1}(N\mathcal{A}) = \varepsilon_{D_2}(N\mathcal{A})$.

Recall the usual operator

$$U_{|D|}(f) := \frac{1}{|D|} \sum_{j \bmod |D|} f\left(\frac{z+j}{|D|}\right)$$

on spaces of modular forms.

Proposition A.4. *Assume D is odd. Then the function $G_{s, \mathcal{A}}(z)$ defined in the last proposition is given by*

$$G_{s, \mathcal{A}}(z) = (\mathcal{E}_s(N_0 z) \Theta_{\mathcal{A}}(z))_{|U_{|D|}},$$

where

$$(8) \quad \mathcal{E}_s(z) := \sum_{D=D_1 D_2} \frac{\varepsilon_{D_1}(N\eta) \chi_{D_1, D_2}(\mathcal{A})}{\kappa(D_1) |D_1|^{s+1/2}} E_s^{(D_1, D_2)}(|D_2|z)$$

The sum is over all decompositions of D as a product of two fundamental discriminants D_1 and D_2 .

Proof. As in [5, (2.4) p.276], we can prove the result with

$$\mathcal{E}_s(z) = \sum_{\substack{D=D_1D_2 \\ \gcd(D_1, \eta)=1}} \frac{\varepsilon_{D_1}(N_0) \chi_{D_1, D_2}(\mathcal{A})}{\kappa(D_1) |D_1|^{s+1/2}} E_s^{(D_1, D_2)}(|D_2|z)$$

and the statement follows since

$$\varepsilon_{D_1}(N\eta) = \begin{cases} \varepsilon_{D_1}(N_0) & \text{when } \gcd(D_1, \eta) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

□

A.3. Fourier expansions. From now on we will assume D is odd as in Proposition A.4. This implies that η is odd and squarefree.

We first give an explicit description of the Fourier coefficients of the function $\mathcal{E}_s(z)$ defined in (8) above.

Proposition A.5. *We have*

$$\mathcal{E}_s(z) = \sum_{n \in \mathbb{Z}} e_s(n, y) e^{2\pi i n x}.$$

where the coefficients are given by

$$e_s(0, y) = L(\varepsilon_D, 2s+1) (|D|y)^s + \frac{\varepsilon_D(N\eta)}{i\sqrt{|D|}} V_s(0) L(\varepsilon_D, 2s) (|D|y)^{-s}$$

if $n = 0$ and by

$$e_s(n, y) = \frac{i}{\sqrt{|D|}} (|D|y)^{-s} V_s(ny) \sum_{\substack{d|n \\ d>0}} \frac{\varepsilon_{\mathcal{A}}(n, d)}{d^{2s}}$$

if $n \neq 0$, with

$$\varepsilon_{\mathcal{A}}(n, d) := \varepsilon_{D_1}(-N\eta d) \varepsilon_{D_2}(n/d) \chi_{D_1, D_2}(\mathcal{A})$$

for the unique decomposition $D = D_1 D_2$ as a product of fundamental discriminants such that $|D_2| = \gcd(D, d)$, and where

$$V_0(t) := \begin{cases} 0 & \text{if } t < 0 \\ -\pi i & \text{if } t = 0 \\ -2\pi i e^{-2\pi t} & \text{if } t > 0 \end{cases}$$

Proof. The proof from [5, (3.2) p.277] works, with the following differences:

- Replace $\varepsilon_{D_1}(N)$ by $\varepsilon_{D_1}(N\eta)$, as in (8).
- In the last step of the computation of $e_s(n, y)$, in [5] they use the identity

$$i\varepsilon_{D_1}(-N) = \frac{\varepsilon_D(N)}{i} \varepsilon_{D_2}(-N),$$

which is only true in the case $(D_2, N) = 1$. Thus a formula like theirs is good only so far as $\eta = 1$, but ours is always true. \square

Note that $\varepsilon_{\mathcal{A}}(n, d) = 0$ unless $\eta \mid d$. In particular, $e_s(n, y) = 0$ unless $\eta \mid n$. To ease notation, write $\eta^* = \left(\frac{-1}{\eta}\right) \eta$ and $D_0 = D/\eta^*$, so that $D = \eta^* D_0$ is a discriminant decomposition, and N is prime to D_0 (because we are assuming D is odd, hence squarefree).

We let

$$(9) \quad \begin{aligned} \tilde{\varepsilon}_{\mathcal{A}}(n, d) &:= \varepsilon_{\mathcal{A}}(\eta n, \eta d) \\ &= \varepsilon_{D_1}(-Nd) \varepsilon_{\eta^* D_2}(n/d) \chi_{D_1, \eta^* D_2}(\mathcal{A}), \end{aligned}$$

for the unique decomposition $D_0 = D_1 D_2$ as a product of fundamental discriminants such that $|D_2| = \gcd(D_0, d)$.

Corollary A.6. *The Fourier coefficients of*

$$G_{0, \mathcal{A}}(z) = \frac{2\pi}{\sqrt{|D|}} \sum_{m=0}^{\infty} b_{\mathcal{A}}(m) q^m,$$

are given by

$$b_{\mathcal{A}}(m) = \sum_{n=0}^{|D|m/N} \sigma_{\mathcal{A}}(n) r_{\mathcal{A}}(m|D| - nN),$$

where

$$\begin{aligned} \sigma_{\mathcal{A}}(n) &= \frac{\sqrt{|D|}}{2\pi} e_0(\eta n, y) e^{2\pi \eta n y} \\ &= \begin{cases} \frac{\sqrt{|D|}}{2\pi} L(\varepsilon_D, 1) - \frac{\varepsilon_D(N\eta)}{2} L(\varepsilon_D, 0) & \text{for } n = 0 \\ \sum_{d>0} d|n| \tilde{\varepsilon}_{\mathcal{A}}(n, d) & \text{for } n > 0 \end{cases} \end{aligned}$$

Proof. Similar to [5, (3.4) p.281]. \square

Proposition A.7.

$$\sigma_{\mathcal{A}}(0) = \frac{1 - \varepsilon_D(N\eta)}{2} \cdot \frac{h(D)}{u_D}$$

Proof.

$$\sigma_{\mathcal{A}}(0) = \frac{\sqrt{|D|}}{2\pi} L(\varepsilon_D, 1) - \frac{\varepsilon_D(N\eta)}{2} L(\varepsilon_D, 0).$$

But

$$\frac{\sqrt{|D|}}{2\pi} L(\varepsilon_D, 1) = L(\varepsilon_D, 0) = \frac{h(D)}{u_D},$$

by the class number formula and the functional equation for $L(\varepsilon_D, s)$. \square

Denote:

$$\sigma_{\mathcal{A}}(n_0, n_1) := \sum_{\substack{d_0|n_0 \\ d_0>0}} \tilde{\varepsilon}_{\mathcal{A}}(n_0 n_1, d_0).$$

Recall that the number of integral ideals of norm n in $\mathbb{Q}(\sqrt{D})$ is

$$R_D(n) = \sum_{\substack{d|n \\ d>0}} \varepsilon_D(d).$$

Lemma A.8. *For $n > 0$*

(1) *If $d_0 d_1 \mid n$ with $\gcd(d_1, D) = 1$, then*

$$\tilde{\varepsilon}_{\mathcal{A}}(n, d_0 d_1) = \tilde{\varepsilon}_{\mathcal{A}}(n, d_0) \varepsilon_D(d_1).$$

(2) *If $n = n_0 n_1$ with $\gcd(n_1, n_0 D) = 1$, then*

$$\sigma_{\mathcal{A}}(n) = \sigma_{\mathcal{A}}(n_0, n_1) R_D(n_1).$$

(3) *Let $n = n_0 n_1$, where $n_0 = \gcd(n, D^\infty)$. Then*

$$\sigma_{\mathcal{A}}(n) = \sigma_{\mathcal{A}}(n_0, n_1) R_D(n)$$

Proof.

(1) Follows from the definition of $\tilde{\varepsilon}_{\mathcal{A}}(n, d)$ in (9), because the hypothesis implies $\gcd(D_0, d_0 d_1) = \gcd(D_0, d_0)$.

(2) if $\gcd(n_1, n_0 D) = 1$ then

$$\begin{aligned} \sigma_{\mathcal{A}}(n_0 n_1) &= \sum_{d_0|n_0} \sum_{d_1|n_1} \tilde{\varepsilon}_{\mathcal{A}}(n_0 n_1, d_0 d_1) && \text{(since } \gcd(n_0, n_1) = 1) \\ &= \sum_{d_0|n_0} \sum_{d_1|n_1} \tilde{\varepsilon}_{\mathcal{A}}(n_0 n_1, d_0) \varepsilon_D(d_1) && \text{(by part (1))} \\ &= \sigma_{\mathcal{A}}(n_0, n_1) \sum_{\substack{d_1|n_1 \\ d_1>0}} \varepsilon_D(d_1) \\ &= \sigma_{\mathcal{A}}(n_0, n_1) R_D(n_1). \end{aligned}$$

(3) Since $n/n_1 = n_0 \mid D^\infty$, there is a unique ideal of norm n/n_1 , hence $R_D(n_1) = R_D(n)$. Thus the statement follows directly from (2). □

Lemma A.9. *Let $\tilde{\eta} = \gcd(n, \eta^\infty)$, and $n' = \gcd(n, D_0^\infty)$. Then*

$$\sigma_{\mathcal{A}}(n) = \tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) \cdot \sum_{d' \parallel n'} \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta} d')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} \cdot R_D(n)$$

Proof. First note that

$$\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) = \varepsilon_{D_0}(-N \tilde{\eta}) \varepsilon_{\eta^*}(n/\tilde{\eta}) \chi_{D_0, \eta^*}(\mathcal{A}).$$

Since $\gcd(\eta, n/\tilde{\eta}) = 1$, and $\gcd(D_0, N\tilde{\eta}) = 1$, it follows that

$$\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) \neq 0.$$

Let $n = n_0 n_1$ with $n_0 = \gcd(n, D_0^\infty)$ as in the previous lemma, and note that $n_0 = \tilde{\eta} n'$, since η and D_0 are relatively prime.

Suppose $d_0 \mid n_0$ is such that $\tilde{\varepsilon}_{\mathcal{A}}(n, d_0) \neq 0$. By the definition of $\tilde{\varepsilon}_{\mathcal{A}}$, it follows that

$$\varepsilon_{\eta^*}(n/d_0) \neq 0 \quad \text{and} \quad \varepsilon_{D_2}(n/d_0) \neq 0,$$

where $|D_2| = \gcd(D_0, d_0)$. The first inequality implies that $\tilde{\eta} \mid d_0$, so we can write $d_0 = \tilde{\eta} d'$, where $d' \mid n'$. Since $d' \mid n' \mid D_0^\infty$, and $|D_2| = \gcd(D_0, d_0)$, it follows that $d' \mid D_2^\infty$. The second inequality implies $\varepsilon_{D_2}(n'/d') \neq 0$, and so we finally conclude that $d' \parallel n'$.

It follows from the above discussion that

$$\begin{aligned} \sigma_{\mathcal{A}}(n_0, n_1) &= \sum_{d' \parallel n'} \tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta} d') \\ &= \tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) \cdot \sum_{d' \parallel n'} \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta} d')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} \end{aligned}$$

This finishes the proof by Lemma A.8 (3). □

Lemma A.10. *The function*

$$d' \mapsto \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta} d')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})}$$

is multiplicative. In particular,

$$\sum_{d' \parallel n'} \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta} d')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} = \begin{cases} \delta(n) & \text{if all terms are 1,} \\ 0 & \text{otherwise} \end{cases}$$

where $\delta(n) := 2^t$, with t the number of prime factors of $\gcd(n, D_0)$.

Proof. Let $D_0 = D_1 D_2$ where $|D_2| = \gcd(D_0, d')$. We have

$$\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta} d') = \varepsilon_{D_1}(-N \tilde{\eta} d') \varepsilon_{\eta^* D_2} \left(\frac{n}{\tilde{\eta} d'} \right) \chi_{D_1, \eta^* D_2}(\mathcal{A})$$

and

$$\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) = \varepsilon_{D_0}(-N \tilde{\eta}) \varepsilon_{\eta^*} \left(\frac{n}{\tilde{\eta}} \right) \chi_{D_0, \eta^*}(\mathcal{A})$$

Hence

$$\begin{aligned} \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}d')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} &= \varepsilon_{D_1}(d') \varepsilon_{D_2}(-N\tilde{\eta}) \varepsilon_{\eta^*}(d') \varepsilon_{D_2} \left(\frac{n}{\tilde{\eta}d'} \right) \chi_{\eta^*D_1, D_2}(\mathcal{A}) \\ &= \varepsilon_{\eta^*D_1}(d') \varepsilon_{D_2} \left(-N \frac{n}{d'} \right) \chi_{\eta^*D_1, D_2}(\mathcal{A}) \end{aligned}$$

(we have used $\chi_{D_1, \eta^*D_2} = \chi_{D_0, \eta^*} \cdot \chi_{\eta^*D_1, D_2}$).

To check multiplicativity, let $d' = d''d'''$ with $\gcd(d'', d''') = 1$, and let $D_0 = D_1''D_2'' = D_1'''D_2'''$ be the discriminant decompositions corresponding to d'' and d''' , i.e. $|D_2''| = \gcd(D_0, d'')$ and $|D_2'''| = \gcd(D_0, d''')$. Note that $D_2 = D_2''D_2'''$, and so $D_1'' = D_1D_2'''$ and $D_1''' = D_1D_2''$.

Then

$$\frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}d'')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} = \varepsilon_{\eta^*D_1''}(d'') \varepsilon_{D_2''}(d''') \varepsilon_{D_2'''} \left(-N \frac{n}{d'} \right) \chi_{\eta^*D_1'', D_2''}(\mathcal{A})$$

and

$$\frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}d''')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} = \varepsilon_{\eta^*D_1'''}(d''') \varepsilon_{D_2'''}(d'') \varepsilon_{D_2''} \left(-N \frac{n}{d'} \right) \chi_{\eta^*D_1''', D_2'''}(\mathcal{A}).$$

Hence the product

$$\begin{aligned} \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}d'')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} \cdot \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}d''')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} &= \varepsilon_{\eta^*D_1''D_2'''}(d'') \varepsilon_{\eta^*D_1'''D_2'''}(d''') \varepsilon_{D_2} \left(-N \frac{n}{d'} \right) \chi_{\eta^*D_1, D_2}(\mathcal{A}) \\ &= \varepsilon_{\eta^*D_1(D_2''')^2}(d'') \varepsilon_{\eta^*D_1(D_2'')^2}(d''') \varepsilon_{D_2} \left(-N \frac{n}{d'} \right) \chi_{\eta^*D_1, D_2}(\mathcal{A}) \\ &= \varepsilon_{\eta^*D_1}(d') \varepsilon_{D_2} \left(-N \frac{n}{d'} \right) \chi_{\eta^*D_1, D_2}(\mathcal{A}) \\ &= \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}d')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})}. \end{aligned}$$

□

Lemma A.11. *Suppose there is an ideal $\mathfrak{a} \in \mathcal{A}$ such that $N\mathfrak{a} \equiv -nN \pmod{D}$, and let \mathfrak{b} be an ideal of norm n . Then the following conditions are equivalent:*

- (1) $\frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}d')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} = 1$ for all $d' \parallel n'$.
- (2) $\chi_{l^*, D/l^*}(\mathfrak{a}\mathfrak{b}) = \varepsilon_{l^*}(-N)$ for all prime discriminants $l^* \mid D_0$.
- (3) There is an ideal \mathfrak{q} in the same genus as $\mathfrak{a}\mathfrak{b}$ such that

$$N\mathfrak{q} \equiv -N \pmod{D_0}.$$

Moreover, this also implies

$$\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) = 1$$

Proof. We first prove that (1) implies (2). Let l^* be a prime discriminant with $l \mid D_0$, and consider the following two cases:

- $l \nmid n$: then $\gcd(\mathcal{N}\mathbf{ab}, l) = 1$, so $\chi_{l^*, D/l^*}(\mathbf{ab}) = \varepsilon_{l^*}(\mathcal{N}\mathbf{ab}) = \varepsilon_{l^*}(-N)$, by hypothesis.
- $l \mid n$: use (1) with $d' = \gcd(n, l^\infty)$, thus $|D_2| = l$, and we have

$$\begin{aligned} \chi_{D/l^*, l^*}(\mathbf{a}) &= \varepsilon_{D/l^*}(d') \varepsilon_{l^*}\left(-N \frac{n}{d'}\right) \\ &= \varepsilon_{l^*}(-N) \varepsilon_{D/l^*}(d') \varepsilon_{l^*}(n/d') \\ &= \varepsilon_{l^*}(-N) \chi_{D/l^*, l^*}(\mathbf{b}), \end{aligned}$$

where the last equality holds since $d' \mid D$.

To prove that (2) implies (1), note that since the expression in (1) is multiplicative, it is enough to check it for $d' = \gcd(n, l^\infty)$, where l is any prime dividing n' . Since $l \mid n$, a computation similar to the second case above applies.

Clearly (3) implies (2); to see the converse take an ideal \mathbf{c} in the same genus as \mathbf{ab} , with $\gcd(\mathcal{N}\mathbf{c}, D_0) = 1$. By (2), we know that

$$\mathcal{N}\mathbf{c} \equiv -Nr^2 \pmod{D_0}$$

for some $r \in (\mathbb{Z}/D_0)^\times$, and we can take $\mathbf{q} = \tilde{r}\mathbf{c}$ where $\tilde{r} \in \mathbb{Z}$ is such that $\tilde{r}r \equiv 1 \pmod{D_0}$.

For the final assertion, we use the definition of $\tilde{\varepsilon}_{\mathcal{A}}$ in (9) with $d = \tilde{\eta}$, so that $D_1 = D_0$ and $D_2 = \eta^*$, and thus

$$\begin{aligned} \tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) &= \varepsilon_{D_0}(-N\tilde{\eta}) \varepsilon_{\eta^*}(n/\tilde{\eta}) \chi_{D_0, \eta^*}(\mathcal{A}) \\ &= \varepsilon_{D_0}(-N) \chi_{D_0, \eta^*}(\mathbf{b}) \chi_{D_0, \eta^*}(\mathcal{A}), \end{aligned}$$

since $\chi_{D_0, \eta^*}(\mathbf{b}) = \varepsilon_{D_0}(\tilde{\eta}) \varepsilon_{\eta^*}(n/\tilde{\eta})$. Hence,

$$\begin{aligned} \tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) &= \varepsilon_{D_0}(-N) \chi_{D_0, \eta^*}(\mathbf{q}) \\ &= \varepsilon_{D_0}(-N) \varepsilon_{D_0}(-N) = 1. \end{aligned}$$

□

Denote by $R_{\text{gen}[\mathbf{b}]}(n)$ the number of integral ideals of norm n in a given genus $\text{gen}[\mathbf{b}]$. We can finally obtain a closed formula for $\sigma_{\mathcal{A}}(n)$ when $n > 0$:

Proposition A.12. *For $n > 0$, suppose there is an ideal $\mathbf{a} \in \mathcal{A}$ such that $\mathcal{N}\mathbf{a} \equiv -nN \pmod{D}$. Then*

$$\sigma_{\mathcal{A}}(n) = \delta(n) \sum_{\text{gen}[\mathbf{q}] \in Q} R_{\text{gen}[\mathcal{A}\mathbf{q}]}(n)$$

where the sum is over the set of genera

$$Q := \{\text{gen}[\mathbf{q}] : \mathcal{N}\mathbf{q} \equiv -N \pmod{D_0}\}.$$

Proof. We have

$$\begin{aligned}
\sigma_{\mathcal{A}}(n) &= \tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) \cdot \sum_{d' \parallel n'} \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta} d')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} \cdot R_D(n) && \text{(by Lemma A.9)} \\
&= \tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta}) \cdot \sum_{d' \parallel n'} \frac{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta} d')}{\tilde{\varepsilon}_{\mathcal{A}}(n, \tilde{\eta})} \cdot \sum_{\substack{\mathfrak{b} \\ \mathcal{N} \mathfrak{b} = n}} 1 \\
&= \sum_{\substack{\mathfrak{b} \\ \mathcal{N} \mathfrak{b} = n}} 1 \cdot \begin{cases} \delta(n) & \text{if } \text{gen}[\mathfrak{a}\mathfrak{b}] \in Q, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

where the last equality follows from Lemma A.10 and Lemma A.11.

Now we note that $\text{gen}[\mathfrak{a}\mathfrak{b}] \in Q$ is equivalent to $\text{gen}[\mathfrak{b}] = \text{gen}[\mathfrak{a}\mathfrak{q}]$ for some $\text{gen}[\mathfrak{q}] \in Q$, hence we can rewrite the last summation as

$$\begin{aligned}
\sigma_{\mathcal{A}}(n) &= \delta(n) \sum_{\text{gen}[\mathfrak{q}] \in Q} \sum_{\substack{\mathfrak{b} \\ \mathcal{N} \mathfrak{b} = n \\ \text{gen}[\mathfrak{b}] = \text{gen}[\mathfrak{a}\mathfrak{q}]}} 1 \\
&= \delta(n) \sum_{\text{gen}[\mathfrak{q}] \in Q} R_{\text{gen}[\mathfrak{a}\mathfrak{q}]}(n)
\end{aligned}$$

□

Remark A.13.

- (1) In the case $\eta = 1$, we have $D_0 = D$, and the condition on $\mathcal{N}\mathfrak{q}$ in the definition of Q determines its genus $\text{gen}[\mathfrak{q}]$, in case it exists. This depends on the sign of $\varepsilon_D(\mathcal{N}\mathfrak{q})$, so we have

$$\#Q = \begin{cases} 1 & \text{if } \varepsilon_D(N) = -1, \\ 0 & \text{if } \varepsilon_D(N) = 1. \end{cases}$$

This is the case of [5], and in this case the proposition is part (a) of Proposition 4.6 in [5, p.285]. As noted before, when N is a perfect square we have $\#Q = 0$ for all D prime to N .

- (2) When $\eta \neq 1$, we have $Q \neq \emptyset$. Indeed, for any $\alpha \in \mathbb{Z}$ such that $\varepsilon_{\eta^*}(\alpha) = \varepsilon_{D_0}(-N)$, by genus theory there is an ideal \mathfrak{q} with

$$\mathcal{N}\mathfrak{q} \equiv \begin{cases} -N & \pmod{D_0}, \\ \alpha & \pmod{\eta}. \end{cases}$$

The number of such $\alpha \pmod{\eta}$, up to squares is 2^{t-1} , where t is the number of prime factors of η . Each one results in an ideal lying in a different genus, hence

$$\#Q = 2^{t-1}.$$

(3) In the particular case $N = p^2$, it follows that

$$\#Q = \begin{cases} 0 & \text{if } p \nmid D, \\ 1 & \text{if } p \mid D. \end{cases}$$

A.4. The central value of L -series. We conclude this section with a formula for the central value of L -series which is similar to [5, (4.4) p.283], but not requiring $\gcd(D, N) = 1$ as in the original formulation. As remarked above, this generalization is essential to obtain a non-trivial result in the case of level p^2 , which is the main interest of this paper.

Theorem A.14. *Let $D < 0$ be an odd fundamental discriminant, \mathcal{A} be an ideal in $\mathbb{Q}[\sqrt{D}]$ and $f(z)$ be a cusp form in $S_2^{\text{new}}(\Gamma_0(N))$. Then,*

$$L_{\mathcal{A}}(f, 1) = \frac{8\pi^2}{\sqrt{|D|}} \langle f, g_{\mathcal{A}} \rangle,$$

with $g_{\mathcal{A}} = g_{\mathcal{A}}^{(N)} = \sum b_{\mathcal{A}}(m)q^m$, where

$$b_{\mathcal{A}}(m) := \frac{1 - \varepsilon_D(N\eta)}{2} \cdot \frac{h(D)}{u_D} r_{\mathcal{A}}(m) + \sum_{\text{gen}[\mathfrak{q}] \in Q} \sum_{n=1}^{|D|m/N} \delta(n) r_{\mathcal{A}}(m|D| - nN) R_{\text{gen}[\mathcal{A}\mathfrak{q}]}(n),$$

where the first sum is over the set of genera

$$Q := \{\text{gen}[\mathfrak{q}] : \mathcal{N}\mathfrak{q} \equiv -N \pmod{D_0}\},$$

and where $\delta(n) := 2^t$, with t the number of prime factors of $\gcd(n, D_0)$.

Proof. This follows by combining Proposition A.3, Corollary A.6, and Proposition A.12, using the renormalization

$$g_{\mathcal{A}}(z) = \frac{\sqrt{|D|}}{2\pi} G_{0,\mathcal{A}}(z).$$

Note that when evaluating each term $\sigma_{\mathcal{A}}(n) r_{\mathcal{A}}(m|D| - nN)$ in the sum of Corollary A.6, the hypothesis of Proposition A.12 holds whenever $r_{\mathcal{A}}(m|D| - nN) \neq 0$, so we can indeed substitute the value of $\sigma_{\mathcal{A}}(n)$ without restriction. \square

Remark A.15. We expect a similar formula to hold for any fundamental discriminant $D < 0$. The case of even discriminants is harder since the discriminant is not square free in this case. Nevertheless, the result for odd discriminants will be enough for our purposes.

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