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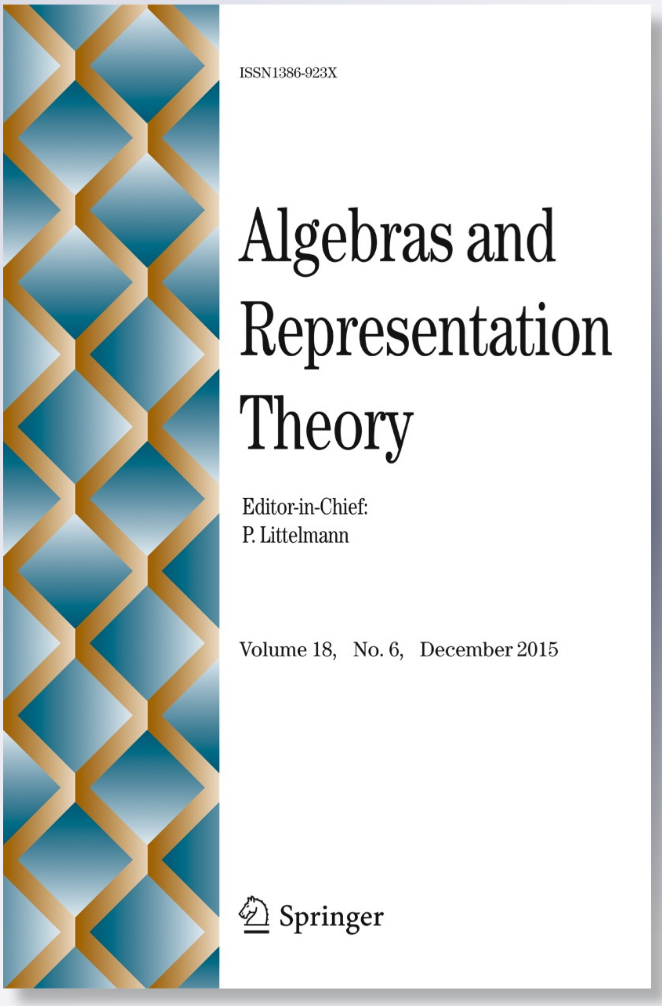

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On the First Hochschild Cohomology Group of a Cluster-Tilted Algebra

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Abstract Given a cluster-tilted algebra B, we study its first Hochschild cohomology group $\mathrm{HH}^1(B)$ with coefficients in the B-B-bimodule B. If C is a tilted algebra such that B is the relation-extension of C, then we show that if B is tame, then $\mathrm{HH}^1(B)$ is isomorphic, as a k-vector space, to the direct sum of $\mathrm{HH}^1(C)$ with $k^{n_{B,C}}$, where $n_{B,C}$ is an invariant linking the bound quivers of B and C. In the representation-finite case, $\mathrm{HH}^1(B)$ can be read off simply by looking at the quiver of B.

Keywords Cluster-tilted algebra · Hochschild cohomology

Mathematics Subject Classifications (2010) 13F60 · 16E40 · 16G20

Presented by Peter Littelmann.

To the memory of Dieter Happel

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1 Introduction

This paper is devoted to the study of the first Hochschild cohomology group $\mathrm{HH}^1(B)$ with coefficients in the B-B-bimodule B, see [24].

Cluster-tilted algebras were defined in [16] and in [22] for the type $\mathbb{A}$, as a by-product of the extensive theory of cluster algebras of Fomin and Zelevinsky [26]. Now, it has been shown in [2] that every cluster-tilted algebra B is the relation-extension of a tilted algebra C. Our goal is to relate the Hochschild cohomologies of the two algebras B and C. The main step in our argument consists in defining an equivalence relation between the arrows in the quiver of B which are not in the quiver of C. The number of equivalence classes is then denoted by $n_{B,C}$. The first two authors have shown in [9] that, if the cluster-tilted algebra B is schurian, then there is a short exact sequence of vector spaces

$$0 \longrightarrow k^{n_{B,C}} \longrightarrow \mathrm{HH}^1(B) \xrightarrow{\varphi} \mathrm{HH}^1(C) \longrightarrow 0.$$

This holds true, for instance, when B is representation-finite. In a further paper, [6], it is actually proven that if B is a cluster-tilted algebra (but generally not schurian) then φ is still surjective and the kernel of the morphism φ is equal to the first cohomology group $\mathrm{HH}^1(B, E)$ of B with coefficients in the bimodule E.

Our objective in the present paper is to show that the result of [9] also holds true in case the cluster-tilted algebra B is tame (that is, is of Dynkin or euclidean type).

Theorem 1.1 *Let k be an algebraically closed field and let B be a tame cluster-tilted algebra which is the relation-extension of the tilted algebra C. Then there is a short exact sequence of vector spaces*

$$0 \longrightarrow k^{n_{B,C}} \longrightarrow \mathrm{HH}^1(B) \longrightarrow \mathrm{HH}^1(C) \longrightarrow 0.$$

We next show that, for any cluster-tilted algebra B, we have $\mathrm{HH}^1(B) = 0$ if and only if B is hereditary and its quiver is a tree, that is, B is simply connected. This answers positively for all cluster-tilted algebras Skowroński's question in [38, Problem 1]: For which algebras is simple connectedness equivalent to the vanishing of the first Hochschild cohomology group?

Finally, we consider the case where the cluster-tilted algebra B is representation-finite and show that the k-dimension of $\mathrm{HH}^1(B)$ can be computed simply by looking at the quiver of B: indeed, in this case, for any tilted algebra C such that B is the relation-extension of C, we have $\mathrm{HH}^1(C) = 0$ and moreover the invariant $n_{B,C}$ does not depend on the particular choice of C (and thus is denoted simply by n_B). Recalling that an arrow in the quiver of B is called *inner* if it belongs to two chordless cycles, our theorem may be stated as follows.

Theorem 1.2 *Let B be a representation-finite cluster-tilted algebra. Then the dimension n_B of $\mathrm{HH}^1(B)$ equals the number of chordless cycles minus the number of inner arrows in the quiver of B.*

The paper is organised as follows. In Section 2, after briefly setting the notation and recalling the necessary notions, we present results on systems of relations in cluster-tilted algebras. We then introduce the arrow equivalence relation in Section 3. In Section 4 we describe the tilted algebras C that are associated to the cluster-tilted algebra B and Section 5 is devoted to the proof of Theorem 1.1. Section 6 contains the proof of Theorem 1.2.

2 Systems of Relations

Let k be an algebraically closed field, then it is well-known that any basic and connected finite dimensional k-algebra C can be written in the form $C = kQ/I$, where Q is a connected quiver, kQ its path algebra and I an admissible ideal of kQ. The pair (Q, I) is then called a *bound quiver*. We recall that finitely generated C-modules can be identified with representations of the bound quiver (Q, I), thus any such module M can be written as $M = (M(x), M(\alpha))_{x\in Q_0,\alpha\in Q_1}$ (see, for instance, [10]).

A *relation* from $x \in Q_0$ to $y \in Q_0$ is a linear combination $\rho = \sum_{i=1}^{m} a_i w_i$ where each w_i is a path of length at least two from x to y and $a_i \in k$ for each i. If $m = 1$ then ρ is *monomial*. The relation ρ is *minimal* if each scalar a_i is non-zero and $\sum_J a_i w_i \notin I$ for any non-empty proper subset J of the set $\{1, \dots, m\}$, and it is *strongly minimal* if each scalar a_i is nonzero and $\sum_J b_i w_i \notin I$ for any non-empty proper subset J of the set $\{1, \dots, m\}$, where each b_i is a nonzero scalar. Two paths u, v will be called *parallel* if they have the same source and target. They are called *antiparallel* if the source (or the target) of u equals the target (or the source, respectively) of v.

We sometimes consider an algebra C as a category, in which the object class C_0 is a complete set $\{e_1, \dots, e_n\}$ of primitive orthogonal idempotents of C and $C(x, y) = e_x C e_y$ is the set of morphisms from e_x to e_y. An algebra C is *constricted* if, for any arrow from x to y in Q_1, we have $\dim_k e_x C e_y = 1$, see [14].

For a general background on the cluster category and cluster-tilting, we refer the reader to [15]. It is shown in [2] that, if T is a tilting module over a hereditary algebra A, so that $C = \mathrm{End}_A(T)$ is a tilted algebra, then the trivial extension $\tilde{C} = C \ltimes \mathrm{Ext}^2_C(DC, C)$ (the *relation-extension* of C) is cluster-tilted and, conversely, any cluster-tilted algebra is of this form (but in general, not uniquely: see [3]). As a consequence, we have a description of the quiver of $\tilde{C}$. Let R be a *system of relations* for the tilted algebra $C = kQ/I$, that is, R is a subset of $\cup_{x,y\in Q_0} e_x I e_y$ such that R, but no proper subset of R, generates I as an ideal of kQ. It is shown in [2] that the quiver $\tilde{Q}$ of $\tilde{C}$ is as follows:

(a) $\tilde{Q}_0 = Q_0$;
(b) For $x, y \in Q_0$, the set of arrows in $\tilde{Q}$ from x to y equals the set of arrows in Q from x to y (which we call *old arrows*) plus $|R \cap I(y, x)|$ additional arrows (which we call *new arrows*).

The relations in $\tilde{I}$ are given by the partial derivatives of the potential $W = \sum_{\rho\in R} \alpha_\rho \rho$, with α_ρ the new arrow associated to the relation ρ, see [31].

Now we show that R can be chosen as a set of strongly minimal relations.

Lemma 2.1 *If $\rho = \sum_{i=1}^{m} \lambda_i w_i \in R$, with $\lambda_i \neq 0$, is not strongly minimal, there exists $\rho' = \sum_{i=1}^{m} \mu_i w_i \in I$ with $\mu_1 = \lambda_1$ which is strongly minimal.*

Proof We proceed by induction on m. If $m = 2$ and $\rho = \lambda_1 w_1 + \lambda_2 w_2$ is not strongly minimal, then it is clear that w_1, w_2 are relations in I and hence we may take $\rho' = \lambda_1 w_1$. Assume now $m > 2$ and ρ is not strongly minimal. Then there is a relation $\rho_1 = \sum_J \beta_j w_j \in I$, with J a proper non-empty subset of $\{1, \dots, m\}$, $\beta_j \neq 0$. By induction on m we may assume that ρ_1 is strongly minimal. If $1 \in J$, we take $\rho' = \frac{\lambda_1}{\beta_1}\rho_1$ and we are done. If $1 \notin J$, let s be the first element in J. We apply the inductive hypothesis to the relation $\rho - \frac{\lambda_s}{\beta_s}\rho_1$. □

A relation ρ is called *triangular* if it is a linear combination of paths that do not contain oriented cycles.

Lemma 2.2 *Any system of triangular relations $R = \{\rho_1, \cdots, \rho_t\}$ can be replaced by a system of strongly minimal relations $R' = \{\rho'_1, \cdots, \rho'_t\}$. Moreover, each ρ'_i is a linear combination of the paths which occur in the relation ρ_i.*

Proof We proceed by induction on t. If $t = 1$, then $\rho = \sum_{i=1}^{m} \lambda_i w_i$ is already strongly minimal, since if it is not, then, by the previous lemma we get a relation $\rho' = \sum_{i=1}^{m} \mu_i w_i \in I$ with $\mu_1 = \lambda_1$. Without loss of generality we may assume that w_1 has maximal length, and hence the relation $\rho - \rho' = \sum_{i=2}^{m} (\lambda_i - \mu_i) w_i$ belonging to the ideal generated by ρ yields a contradiction: in its triangular expression as an element in $< \rho >$, there should be a summand of the form $\mu u_1 w_1 u_2$, with μ a nonzero scalar, u_1, u_2 paths in Q, and then $u_1 w_1 u_2$ is w_1 or a path of greater length, so this term cannot appear in $\rho - \rho'$.
Let $t > 1$, let $\{w_1, \ldots, w_s\}$ be a complete set of paths appearing in the relations ρ_i, that is,

$$\rho_i = \lambda_{1i} w_1 + \cdots + \lambda_{si} w_s.$$

Without loss of generality, we may assume that w_1 has maximal length and that $\lambda_{11} \neq 0$. Now, the ideal generated by the set $\{\rho_1, \cdots, \rho_t\}$ is equal to the ideal generated by the set

$$\{\rho_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_t\}$$

with

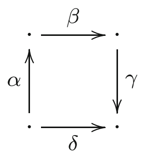
$$\tilde{\rho}_j = \rho_j - \frac{\lambda_{1j}}{\lambda_{11}} \rho_1.$$

If we apply the previous lemma to ρ_1 we get a strongly minimal relation ρ'_1 with λ_{11} as the first coefficient. Following an argument similar to what we did in the case $t = 1$, using the maximality of w_1 we get that the relation $\rho_1 - \rho'_1$ belongs to the ideal $< \tilde{\rho}_2, \cdots, \tilde{\rho}_t >$, and so we get a system of relations $\{\rho'_1, \tilde{\rho}_2, \cdots, \tilde{\rho}_t\}$, with ρ'_1 strongly minimal. Now we proceed by induction on the set $\{\tilde{\rho}_2, \cdots, \tilde{\rho}_t\}$, and we get a system of relations $\{\rho'_2, \cdots, \rho'_t\}$ which are strongly minimal with respect to the ideal $I' =< \rho'_2, \cdots, \rho'_t >$. Assume that one of these relations is not strongly minimal with respect to I, say $\rho'_i = \sum_{i=2}^{s} \beta_i w_i$ and $\rho'' = \sum_J \mu_i w_i \in I$, where J is a proper subset of $\{2, \cdots, s\}$. So $\rho'' \notin I'$ says that if we write it as an element in I, the relation ρ'_1 should appear. Again we get a contradiction when considering the summands that contain w_1 as a subpath. □

Let w be a nontrivial walk in a bound quiver (Q, I). Assume that one writes $w = uw'v$ where each of u, w', v is a subwalk of w. We say that u, v *point to the same direction* in w if u and v, or u^{-1} and v^{-1}, are paths in Q.

A reduced walk $w = uw'v$ having u and v pointing to the same direction is called a *sequential walk* if there is a relation $\rho = \sum_i \lambda_i u_i$ such that $u = u_1$ or $u = u_1^{-1}$, there is a relation $\sigma = \sum_j \mu_j v_j$ such that $v = v_1$ or $v = v_1^{-1}$ and no subpath w_1 of w', or of $(w')^{-1}$, is involved in a relation of the form $\sum \lambda_i w_i$.

Example 2.3 The quiver



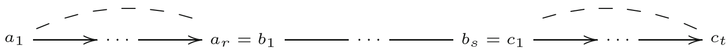
bound by the relation $\alpha\beta = 0$ contains a sequential walk $w = \alpha\beta\gamma\delta^{-1}\alpha\beta$.

The following lemma generalises [1, 9, 30].

Lemma 2.4 *Let $C = kQ/I$ be a tilted algebra. Then the bound quiver (Q, I) contains no sequential walk.*

Proof Suppose that $w = uw'v$ is a sequential walk, with u, v the (only) two subwalks of w which are involved in relations pointing to the same direction. Clearly, w' may have self-intersections and may also intersect the paths u and v. Then (Q, I) contains a bound subquiver (Q', I'), maybe not full, consisting of the points and arrows on w as well as all the points and arrows which lie on a path parallel to u or to v and bound to it by a relation. This may be visualised as in the following pictures (drawn under the assumption that w', the walk from b_1 to b_s, has no self-intersections and intersects neither u nor v):

(a)



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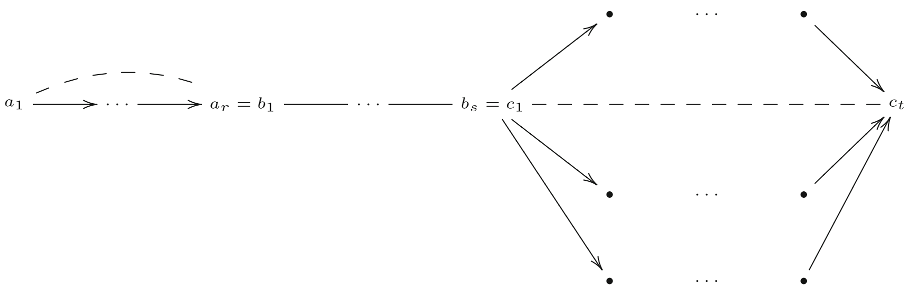

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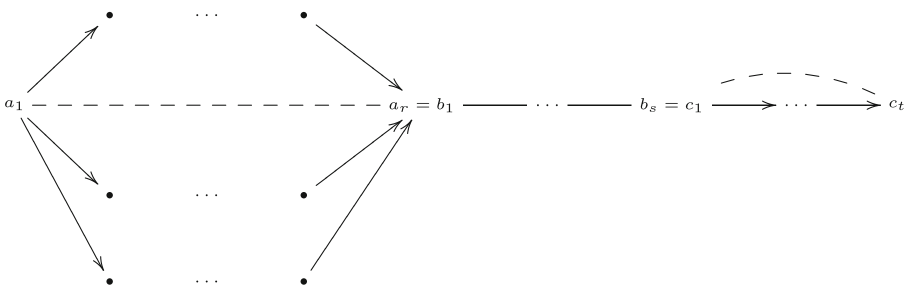

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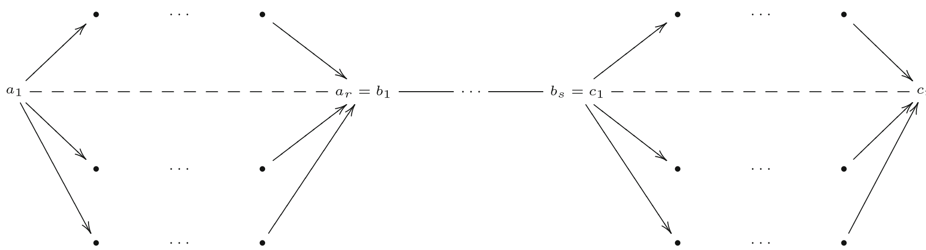

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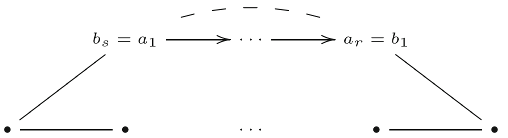
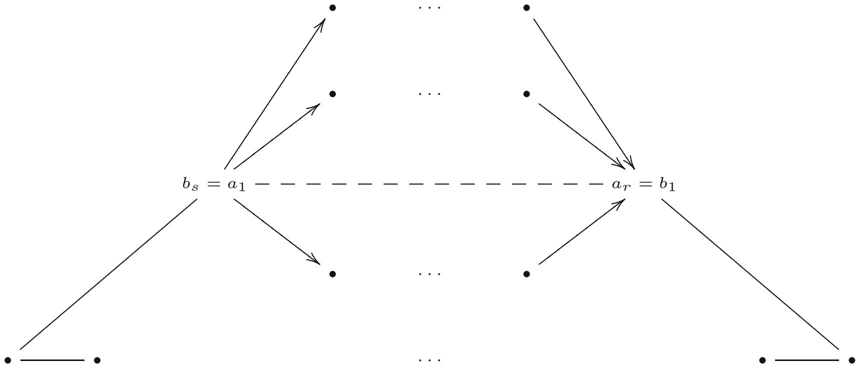


(f)



where we have represented relations by dotted lines. The last two cases occur when $u = v$. Moreover, the minimality of the length of w' implies that there is no additional arrow between two b_j's. Let $C' = kQ'/I'$. Because of [28, III.6.5, p. 146], C' is also a tilted algebra. In each of these cases above, let M be the C'-module defined as a representation by

$$M(x) = \begin{cases} k, & \text{if } x \text{ is a point of the walk } w', \\ 0, & \text{otherwise,} \end{cases}$$

and

$$M(\alpha) = \begin{cases} \text{id}, & \text{if } s(\alpha), t(\alpha) \text{ are points of the walk } w', \\ 0, & \text{otherwise,} \end{cases}$$

for every point x and arrow α in the quiver of C'. Since there is no subpath w_1 of w', or of $(w')^{-1}$, involved in a relation of the form $\sum \lambda_i w_i$, then M is indeed a module. It is actually a tree module, and therefore it is indecomposable, see [32]. On the other hand, it can be seen that both of its projective and its injective dimensions equal two, a contradiction because C' is tilted. □

Let now $\tilde{C} = k\tilde{Q}/\tilde{I}$ be the relation-extension of a tilted algebra C. A walk $w = \alpha w' \beta$ in $(\tilde{Q}, \tilde{I})$ is called a C-*sequential walk* if:

(i) w' consists entirely of old arrows,
(ii) α, β are two new arrows corresponding respectively to old relations $\rho = \sum_i \lambda_i u_i$ and $\sigma = \sum_j \mu_j v_j$, and
(iii) $w = u_i w' v_j$ is a sequential walk in (Q, I) for any i, j.

Corollary 2.5 *Let $C = kQ/I$ be a tilted algebra. Then the bound quiver of its relation-extension $\tilde{C}$ contains no C-sequential walk.*

3 Arrow Equivalence

The following lemma is an easy consequence of the main result in [8]. For the benefit of the reader, we give an independent proof.

Recall from [25] that for a given arrow β, the *cyclic partial derivative* ∂_β in β is defined on each cyclic path $\beta_1\beta_2\cdots\beta_s$ by $\partial_\beta(\beta_1\beta_2\cdots\beta_s) = \sum_{i:\beta=\beta_i}\beta_{i+1}\cdots\beta_s\beta_1\cdots\beta_{i-1}$. Note that $\partial_\beta(\beta_1\beta_2\cdots\beta_s) = \partial_\beta(\beta_j\cdots\beta_s\beta_1\cdots\beta_{j-1})$ for every j such that $1 \le j \le s$.

From now on, we consider a given presentation of a tilted algebra $C = kQ/I$ with minimal system of relations R and consider its relation-extension $B = k\tilde{Q}/\tilde{I}$ together with the presentation having as relations the cyclic partial derivatives of the potential $W = \sum_{\rho\in R}\alpha_\rho\rho$, with α_ρ the new arrow associated to the relation ρ. All these relations are triangular. Then we reduce this system to an equivalent system of strongly minimal relations each of which is a linear combination of the relations obtained from the partial derivatives (see Lemma 2.2.)

Lemma 3.1 *Let $C = kQ/I$ be a tilted algebra and $B = k\tilde{Q}/\tilde{I}$ be such that $B = \tilde{C}$ and $\tilde{I}$ is generated by the partial derivatives of the potential. Let $\rho = \sum_{i=1}^m a_i w_i$ be a minimal relation in $\tilde{I}$. Then either ρ is a relation in I, or there exist m new arrows $\alpha_1, \ldots, \alpha_m$ such that $w_i = u_i\alpha_i v_i$ (with u_i, v_i paths consisting entirely of old arrows).*

Proof Let $\rho_1, \ldots, \rho_s$ be a system of minimal relations for the tilted algebra C. Then each relation ρ_i induces a new arrow α_i and the product $\rho_i\alpha_i$ is a linear combination of cyclic paths in the quiver of the cluster-tilted algebra B. The potential of B can be given as $W = \sum_{i=1}^s \rho_i\alpha_i$ and the ideal of B is generated by all partial derivatives $\partial_\beta W$ of the potential W with respect to the arrows β. If β is one of the new arrows α_i then $\partial_\beta W$ is just the "old" relation $\rho_i \in I$.

If β is an old arrow then $\partial_\beta W$ is a sum of terms which are cyclic permutations of $(\partial_\beta\rho_i)\alpha_i$. Now, each of the summands contains exactly one new arrow α_i. □

The previous lemma asserts that if $\rho = \sum_i a_i w_i$ is a strongly minimal relation lying in $\tilde{I}$ but not in I, then on each w_i lies exactly one new arrow α_i and each new arrow appears in this way. Clearly, the α_i are not necessarily distinct as arrows of $\tilde{Q}$.

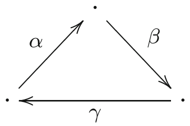
Lemma 3.1 above brings us to our main definition. Let $B = k\tilde{Q}/\tilde{I}$ be a cluster-tilted algebra and $C = kQ/I$ a tilted algebra such that $B = \tilde{C}$. We define a relation $\sim$ on the set $\tilde{Q}_1 \setminus Q_1$ of new arrows as follows. For every $\alpha \in \tilde{Q}_1 \setminus Q_1$, we set $\alpha \sim \alpha$. If $\rho = \sum_{i=1}^m a_i w_i$ is a strongly minimal relation in $\tilde{I}$ and α_i are as in Lemma 3.1 above, then we set $\alpha_i \sim \alpha_j$ for any i, j such that $1 \le i, j \le m$.

By Corollary 2.5, the relation $\sim$ is unambiguously defined. It is clearly reflexive and symmetric. We let $\approx$ be the least equivalence relation defined on the set $\tilde{Q}_1 \setminus Q_1$ such that $\alpha \sim \beta$ implies $\alpha \approx \beta$ (that is, $\approx$ is the transitive closure of $\sim$).

We define the *relation invariant* of B to be the number $n_{B,C}$ of equivalence classes under the relation $\approx$.

Observe that the equivalence relation $\approx$ is related to the direct sum decomposition of the C-C- bimodule E. Indeed, E is generated as C-C-bimodule by the new arrows. If two new arrows occur in a strongly minimal relation, this means that they are somehow yoked together in E. It is proven in [6, Lemma 4.3] that E decomposes, as C-C-bimodule, into the direct sum of $n_{B,C}$ summands.

The following two lemmata will be useful in Section 4. They use essentially the fact that cluster-tilted algebras of type $\tilde{\mathbb{A}}$ are gentle (because of [5, Lemma 2.5]) and in particular all relations are monomial of length 2 contained inside 3-cycles that is, cycles of the form



bound by $\alpha\beta = \beta\gamma = \gamma\alpha = 0$.

Lemma 3.2 *Let B be a cluster-tilted algebra of type $\tilde{\mathbb{A}}$ and let C_1, C_2 be tilted algebras such that $B = \tilde{C}_1 = \tilde{C}_2$. Let R_1, R_2 be systems of relations for C_1, C_2 respectively. Then $|R_1| = |R_2|$.*

Proof Indeed, in order to obtain C_1 and C_2 from B, we have to delete exactly one arrow from each chordless cycle (for the notion of chordless cycle, see [13] or Section 6 below). Because B is of type $\tilde{\mathbb{A}}$, then the chordless cycles are 3-cycles, and no arrow belongs to two distinct 3-cycles. Deleting exactly one arrow from each 3-cycle leaves a path of length 2. The system of relations for the tilted algebra consists in exactly these paths of length 2. This implies the statement. □

Lemma 3.3 *Let $B = \tilde{C}$, where C is a tilted algebra of type $\tilde{\mathbb{A}}$. Let R be a system of relations for C. Then $n_{B,C} = |R|$. In particular, $n_{B,C}$ does not depend on the choice of C.*

Proof Let α_i, α_j be two equivalent new arrows, then there exists a sequence of new arrows

$$\alpha_i = \beta_1 \sim \beta_2 \sim \cdots \sim \beta_t = \alpha_j$$

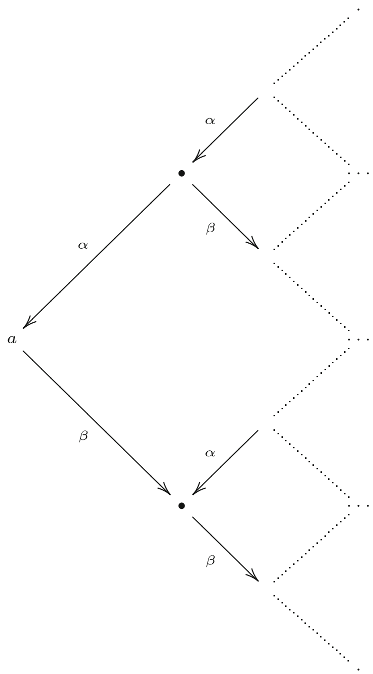
where β_ℓ, $\beta_{\ell+1}$ appear in the same strongly minimal relation in $(\tilde{Q}, \tilde{I})$. Now, B is gentle. Hence strongly minimal relations contain just one monomial. Therefore $\beta_\ell = \beta_{\ell+1}$ for each ℓ, and $\alpha_i = \alpha_j$. This shows that the relation invariant $n_{B,C}$ is equal to the number of new arrows, and the latter is equal to $|R|$ because of [2, Theorem 2.6]. □

4 The Tame Cluster-Tilted Algebras

Our objective in this section is to describe the tilted algebras C that are associated to the tame cluster-tilted algebra B. Because of [16, Theorem A], the tame representation-infinite cluster-tilted algebras are just the cluster-tilted algebras of euclidean type, that is, the relation-extensions of the tilted algebras of euclidean type. Our strategy will consist of reducing the proof to the case where C is a constricted algebra.

An algebra K is *tame concealed* if there exists a tame hereditary algebra A and a postprojective tilting A-module T such that $K = \mathrm{End}_A(T)$. Then $\Gamma(\mathrm{mod} K)$ consists of a postprojective component $\mathcal{P}_K$, a preinjective component $\mathcal{Q}_K$ and a family $\mathcal{T}_K = (\mathcal{T}_\lambda)_{\lambda\in\mathbb{P}_1(k)}$ of stable tubes separating $\mathcal{P}_K$ from $\mathcal{Q}_K$, see [37, 4.3].

We now define *tubular extensions* of a tame concealed algebra. A *branch* L with a *root* a is a finite connected full bound subquiver, containing a, of the following infinite tree, where all compositions $\alpha\beta$ of two arrows labeled as α and β are zero.



Let now K be a tame concealed algebra, and $(E_i)_{i=1}^n$ be a family of simple regular K-modules. For each i, let L_i be a branch with root a_i. The tubular extension $B = K[E_i, L_i]_{i=1}^n$ has as objects those of $K, L_1, \cdots, L_n$ and as morphism spaces

$$B(x,y) = \begin{cases} K(x,y) & \text{if } x, y \in K_0 \\ L_i(x,y) & \text{if } x, y \in (L_i)_0 \\ L_i(x,a_i) \otimes_k E_i(y) & \text{if } x \in (L_i)_0,\ y \in K_0 \\ 0 & \text{otherwise.} \end{cases}$$

The *tubular coextension* ${}_{i=1}^{n}[E_i, L_i]K$ is defined dually.
For each $\lambda \in \mathbb{P}_1(k)$, let r_λ denote the rank of the stable tube $\mathcal{T}_\lambda$ of $\Gamma(\text{mod} K)$. The *tubular type* $n_B = (n_\lambda)_{\lambda \in \mathbb{P}_1(k)}$ of B is defined by

$$n_\lambda = r_\lambda + \sum_{E_i \in \mathcal{T}_\lambda} |(L_i)_0|.$$

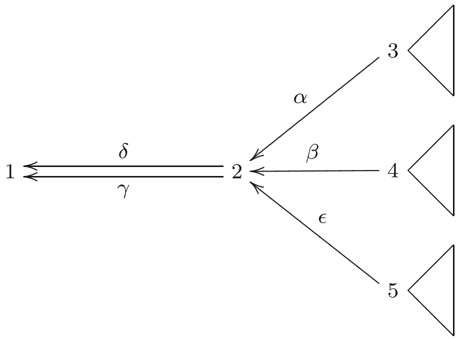
Since all but at most finitely many n_λ equal 1, we write for n_B the finite sequence containing at least two n_λ, including all those larger than 1, in non-decreasing order. We say that n_B is *domestic* if it is one of the forms (p, q), $(2, 2, r)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. The following structure theorem is due to Ringel, see [37, Theorem 4.9, p. 241].

Theorem 4.1 *Let C be a representation-infinite tilted algebra of euclidean type. Then C contains a unique tame concealed full convex subcategory K and C is a domestic tubular extension or a domestic tubular coextension of K.*

As a consequence of Ringel's theorem, we obtain the following.

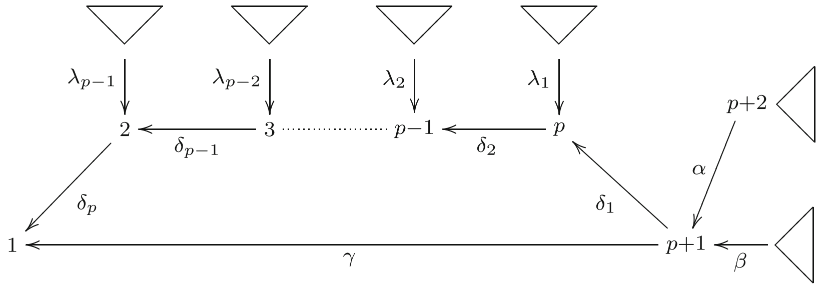
Lemma 4.2 *Let C be a tilted algebra of euclidean type which is not constricted. Then C is given by one of the following two bound quivers, or their duals.*

(1)



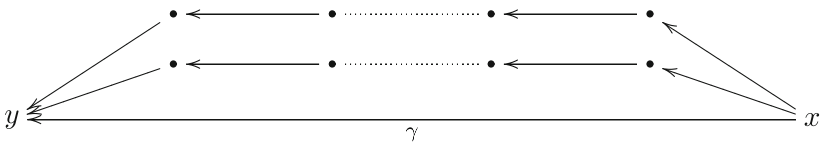
where the triangles are branches, possibly empty, bound by $\alpha\delta = 0$, $\beta\delta = \beta\gamma$, $\epsilon\gamma = 0$, *and the branch relations.*

(2)

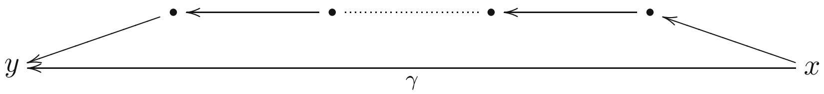


where the triangles are branches, possibly empty, bound by $\alpha\delta_1 \cdots \delta_p = \alpha\gamma$, $\beta\gamma = 0$, $\lambda_i\delta_{i+1} = 0$ *for all* i *such that* $1 \leq i < p$, *and the branch relations.*

Proof Assume C is a tilted algebra of euclidean type which is not constricted. Then there exists an arrow $\gamma : x \to y$ such that $\dim_k C(x, y) \geq 2$. Since C is tame, we actually have $\dim_k C(x, y) = 2$. In particular, C is representation-infinite. Applying Ringel's theorem, we get that C is, up to duality, a domestic tubular extension of a unique tame concealed full convex subcategory K of C. On the other hand, let K' be the convex envelope of the points x, y in C. Then K' is of the form



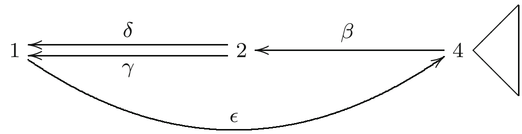
with $\dim_k K'(x, y) = 2$. Note that K' is a full convex subcategory of C, hence it is tilted (because of [28, III.6.5 p.146]). Applying Lemma 2.4 to K', we deduce that K' is of the form



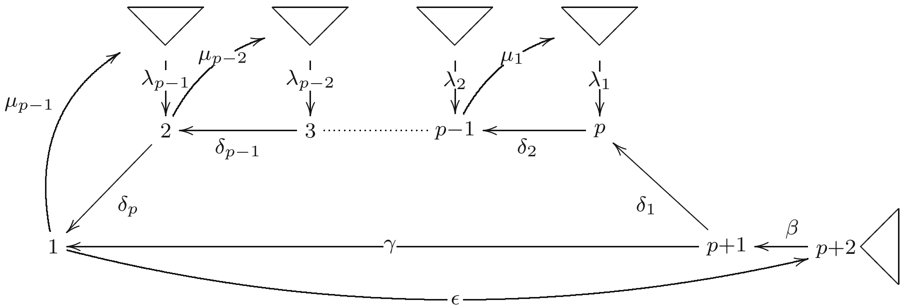
Since K' is hereditary, we get that $K' = K$. The statement now follows by considering the possible branch extensions of K. □

Lemma 4.3 *Let B be a cluster-tilted algebra of euclidean type. Assume that there exists no constricted tilted algebra C such that $B = \tilde{C}$. Then B is a cluster-tilted algebra of type $\tilde{\mathbb{A}}$ of one of the following forms or their duals:*

(i)



(ii)



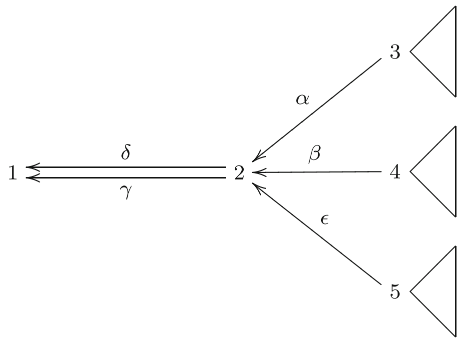
where the triangles are cluster-tilted algebras of type $\mathbb{A}$, possibly empty, bound by $\beta\gamma = 0$, $\gamma\epsilon = 0$, $\epsilon\beta = 0$, and, in the case (ii), by the additional relations $\lambda_i\delta_{i+1} = 0$, $\delta_{i+1}\mu_i = 0$, $\mu_i\lambda_i = 0$.

Proof Let B be cluster-tilted of euclidean type. Because of [2], there exists a tilted algebra C such that

$$B = \tilde{C} = C \ltimes \mathrm{Ext}^2(DC, C).$$

The hypothesis says that C is not constricted. Because of Lemma 4.2, C is given by one of the bound quivers in (1) or (2) above. We examine these cases separately.

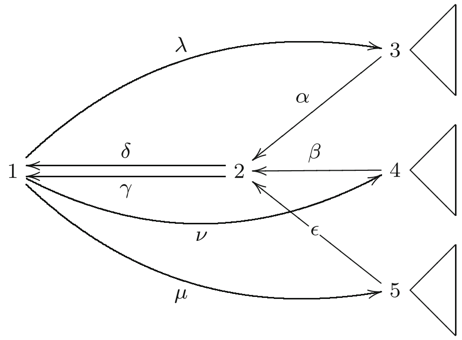
(1) Assume C is given by the quiver



where the triangles are branches, possibly empty, bound by $\alpha\delta = 0$, $\beta\delta = \beta\gamma$, $\epsilon\gamma = 0$ and the branch relations. Observe that, if one of the branches is empty, then it has no root and consequently, the arrow from that root to the point 2 does not exist.

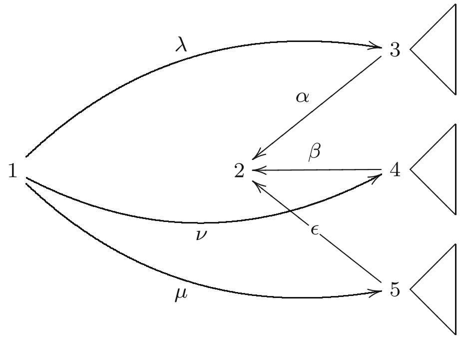
We consider the following subcases:

(1a) Assume none of the branches rooted at 3,4,5 is empty. In this case, we refer to C as C_1. Then the corresponding cluster-tilted algebra B is of the form



where the triangles are cluster-tilted algebras of type $\mathbb{A}$, and there are, additionally, the relations of C_1 and the relations $\lambda\alpha = -\nu\beta$, $\nu\beta = \mu\epsilon$, $\delta\lambda = 0$, $\delta\nu = \gamma\nu$ and $\gamma\mu = 0$.

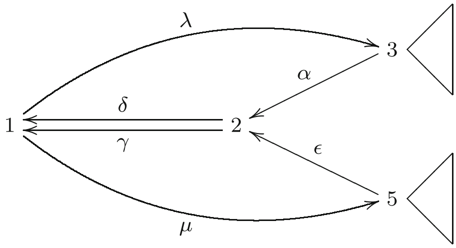
Now, this algebra B can be written as $B=\tilde{C}'_1$, where C'_1 is given by the quiver



where the triangles are again branches, bound by relations $\lambda\alpha = -\nu\beta$, $\nu\beta = \mu\epsilon$.

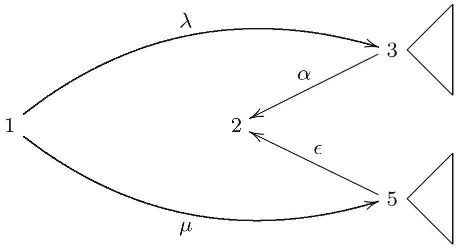
This is easily seen to be a representation-finite tilted algebra of type $\tilde{\mathbb{D}}$ (indeed, one can simply construct the Auslander-Reiten quiver of the algebra and identify a complete slice). In particular, C_1' is constricted, a contradiction.

(1b) Assume that the branch rooted, say at 4, is empty while the other two are not. In this case, we refer to C as C_2. Then the cluster-tilted algebra B is of the form



where the triangles are cluster-tilted algebras of type $\mathbb{A}$, bound by the relations of C_2 and the additional relations $\lambda\alpha = 0$, $\delta\lambda = 0$, $\gamma\mu = 0$, $\mu\epsilon = 0$.

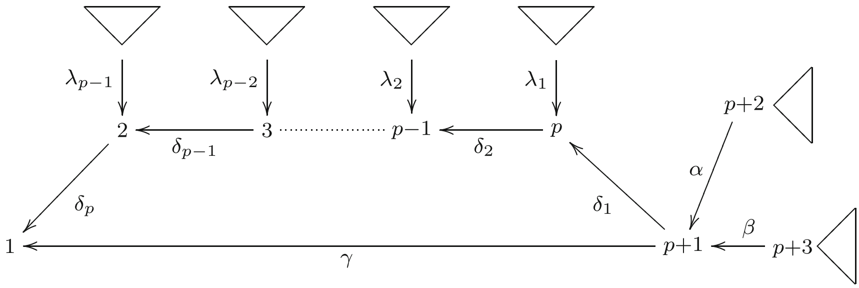
Again, the algebra B can be written as $B = \tilde{C}_2'$, where C_2' is given by the quiver



where the triangles are branches, bound by $\lambda\alpha = 0$, $\mu\epsilon = 0$ and the branch relations. This is easily seen to be a representation-finite tilted algebra of type $\tilde{\mathbb{A}}$ (see, for instance, [11]), thus C_2' is constricted, another contradiction.

(1c) If at least two of the branches, say at 4 and 5, are empty, then we are left with the quiver (i) of the statement.

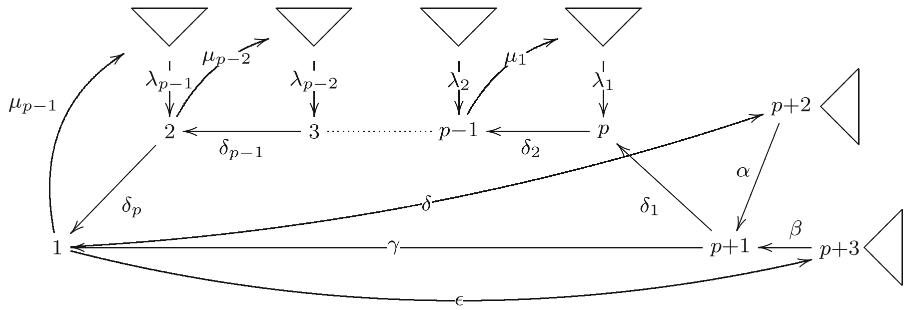
(2) Assume C is given by the quiver



where the triangles are branches, possibly empty, bound by $\alpha\delta_1 \cdots \delta_p = \alpha\gamma$, $\beta\gamma = 0$, $\lambda_i\delta_{i+1} = 0$ for all $1 \le i < p$, and the branch relations.

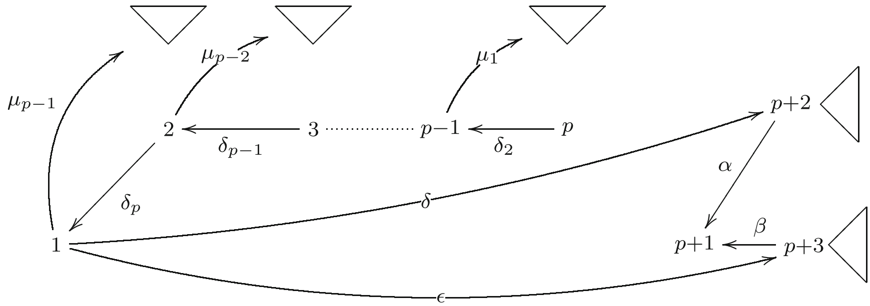
We consider the following subcases.

(2a) Assume that none of the branches rooted at $p+1$, $p+2$ is empty. In this case, we refer to C as C_3. Then the corresponding cluster-tilted algebra is of the form



where the triangles are cluster-tilted algebras of type $\mathbb{A}$, bound by the relations of C_3 and the additional relations $\epsilon\beta = \delta\alpha$, $\gamma\epsilon = 0$, $\delta_1 \cdots \delta_p \delta = \gamma\delta$ and $\mu_i\lambda_i + \delta_{i+2} \cdots \delta_p \delta\alpha\delta_1 \cdots \delta_i = 0$, $\delta_{i+1}\mu_i = 0$, for all i.

Now, this algebra can be written as $B = \tilde{C}'_3$, where C'_3 is given by the quiver



with the inherited relations. This is again seen to be a representation-finite tilted algebra of type $\tilde{\mathbb{D}}$. In particular, it is constricted, a contradiction.

(2b) If at least one of the branches, say at $p+2$ is empty, then we are left with the quiver (ii) of the statement. □

Observe that in the proof of Lemma 4.3, in each of the cases (1a), (1b) and (2a), we have replaced the original non-constricted tilted algebra C_1, C_2 and C_3 by a constricted one C'_1, C'_2 and C'_3, respectively.

Lemma 4.4 *With the above notation, for each* $i \in \{1, 2, 3\}$, *we have* $n_{B,C_i} = n_{B,C'_i}$ *and* $\mathrm{HH}^1(C_i) \cong \mathrm{HH}^1(C'_i)$.

Proof The first statement follows immediately from the description of the relations in the respective algebras. Thus $n_{B,C_1} = n_{B,C'_1} = 1$, $n_{B,C_2} = n_{B,C'_2} = 2$ and $n_{B,C_3} = n_{B,C'_3} = 1$.

It suffices to show the second statement. We consider each of the cases as in the proof of Lemma 4.3.

(1a) Let D_1 be the full convex subcategory of C_1 (and C'_1) generated by all points except the point 1. Then D_1 is a representation-finite tilted algebra and C_1 (or C'_1) is a one-point coextension (or extension, respectively) of D_1 by an indecomposable module. This module being a rigid brick, we deduce immediately from Happel's sequence [29, 5.3] that

$$\mathrm{HH}^1(C_1) \cong \mathrm{HH}^1(D_1) \cong \mathrm{HH}^1(C_1').$$

(1b) Let D_2 be the full convex subcategory of C_2 (and C_2') generated by all points except the point 1. Then D_2 is a representation-finite tilted algebra and C_2 (or C_2') is a one-point coextension (or extension, respectively) of D_2 by the direct sum of two Hom-orthogonal, rigid bricks X, Y such that $\mathrm{Ext}^1_{D_2}(X, Y) = 0$ and $\mathrm{Ext}^1_{D_2}(Y, X) = 0$. Again Happel's sequence yields

$$\mathrm{HH}^1(C_2) \cong \mathrm{HH}^1(D_2) \cong \mathrm{HH}^1(C_2').$$

(2a) Let D_3 be the full convex subcategory of C_3 (and C_3') generated by all points except the points $1, 2, \cdots, p$. Then there is a sequence

$$C_3 = E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_p = D_3,$$

where E_i is a one-point coextension of E_{i+1}. Moreover, each E_i is a direct product of representation-finite tilted algebras and the coextension module is a direct sum of rigid bricks with supports in distinct connected components of E_i. Similarly, there is a sequence

$$C_3' = F_p \supsetneq F_{p-1} \supsetneq \cdots \supsetneq F_0 = D_3,$$

where F_{i+1} is a one-point extension of F_i. Moreover, each F_i is a direct product of representation-finite tilted algebras and the extension module is a direct sum of rigid bricks with supports in distinct connected components of F_i. Therefore easy inductions yield

$$\mathrm{HH}^1(C_3) \cong \mathrm{HH}^1(D_3) \cong \mathrm{HH}^1(C_3').$$

□

Lemma 4.5 *Let $B = \tilde{C}$ be a non-hereditary cluster-tilted algebra of type $\tilde{\mathbb{A}}$ of one of the forms of Lemma 4.3 and R a system of relations for C. Then*

(i) *If B is of the form (i), then* $\mathrm{HH}^1(B) = k^{|R|+2}$

(ii) *If B is of the form (ii), then* $\mathrm{HH}^1(B) = k^{|R|+1}$

Proof (i) We use the formula of [21], as applied to our special situation in [9, Proposition 5.1]

$$\dim_k \mathrm{HH}^1(B) = \dim_k Z(B) - |\tilde{Q}_0 \,//\, N| + |\tilde{Q}_1 \,//\, N| - |(\tilde{Q}_1 \,//\, N)_e| - \dim_k \operatorname{Im} R_g.$$

Here, $Z(B)$ is the centre of B, so $\dim_k Z(B) = 1$. Next, $\tilde{Q}_0 \,//\, N$ is the set of nonzero oriented cycles in $(\tilde{Q}, \tilde{I})$ (where, as usual, $B = k\tilde{Q}/\tilde{I}$) including points. Then

$$|\tilde{Q}_0 \,//\, N| = |\tilde{Q}_0| = |Q_0|.$$

Thirdly, $\tilde{Q}_1 \,//\, N$ is the set of pairs (α, w), where $\alpha \in \tilde{Q}_1$ and w is a nonzero path (of length ≥ 0) parallel to α. This consists of all pairs (α, α), with $\alpha \in \tilde{Q}_1$ and the two pairs (δ, γ), (γ, δ) arising from the double arrow

$$1 \underset{\gamma}{\overset{\delta}{\leftleftarrows}} 2\,.$$

Thus, $|\tilde{Q}_1 \,//\, N| = |\tilde{Q}_1| + 2$.

Since it is shown in [9, Proof of Proposition 5.1] that $R_g = 0$, there remains to compute

$$(\tilde{Q}_1 \,//\, N)_e = (\tilde{Q}_1 \,//\, N) \setminus \left((\tilde{Q}_1 \,//\, N)_g \cup (\tilde{Q}_1 \,//\, N)_a\right).$$

Here:

1. $(\tilde{Q}_1 // N)_g$ is the set of all pairs $(\alpha, w) \in \tilde{Q}_1 // N$ where w is either a point or a path starting or ending with the arrow α. Therefore
$$(\tilde{Q}_1 // N)_g = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}_1\}.$$
2. $(\tilde{Q}_1 // N)_a$ is the set of all pairs $(\alpha, w) \in \tilde{Q}_1 // N$ where, in each relation where α appears, replacing α by w yields a zero path. Therefore
$$(\tilde{Q}_1 // N)_a = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}_1\} \cup \{(\delta, \gamma)\}.$$

This implies that
$$|(\tilde{Q}_1 // N)| - |(\tilde{Q}_1 // N)_e| = |(\tilde{Q}_1 // N)_g \cup (\tilde{Q}_1 // N)_a| = |\tilde{Q}_1| + 1.$$
Therefore
$$\begin{aligned}\dim_k \mathrm{HH}^1(B) &= 1 - |\tilde{Q}_0| + |\tilde{Q}_1| + 1\\ &= 1 - |Q_0| + |Q_1| + |R| + 1\\ &= |R| + 2,\end{aligned}$$
because $|\tilde{Q}_1| = |Q_1| + |R|$ and $|Q_0| = |Q_1|$.

(ii) For this case again $\dim_k Z(B) = 1$ and $|(\tilde{Q}_0 // N)| = |\tilde{Q}_0| = |Q_0|$. Here
$$\tilde{Q}_1 // N = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}\} \cup \{(\gamma, \delta_1 \cdots \delta_p)\}.$$
Now, as before
$$(\tilde{Q}_1 // N)_g = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}_1\},$$
while
$$(\tilde{Q}_1 // N)_a = \{(\alpha, \alpha) \mid \alpha \in \tilde{Q}_1\},$$
so that
$$|(\tilde{Q}_1 // N)| - |(\tilde{Q}_1 // N)_e| = |(\tilde{Q}_1 // N)_g \cup (\tilde{Q}_1 // N)_a| = |\tilde{Q}_1|.$$
Therefore
$$\begin{aligned}\dim_k \mathrm{HH}^1(B) &= 1 - |\tilde{Q}_0| + |\tilde{Q}_1|\\ &= 1 - |Q_0| + |Q_1| + |R|\\ &= |R| + 1,\end{aligned}$$
because $|\tilde{Q}_1| = |Q_1| + |R|$ and $|Q_0| = |Q_1|$. □

Note that the previous proof can also be done applying the formula in [36].

5 Hochschild Cohomology of Tame Cluster-Tilted Algebras

We need a few results from [9] and [6] which we now recall. Let B be a split extension of a subalgebra C by a two-sided bimodule ${}_C E_C$, that is, let B have the k-vector space structure of $C \oplus E$ with the multiplication given by
$$(c, x)(c', x') = (cc', cx' + xc' + xx')$$
for $(c, x), (c', x') \in B$. Then there exists a short exact sequence of $C - C$-bimodules
$$0 \longrightarrow E \longrightarrow B \overset{p}{\longrightarrow} C \longrightarrow 0$$
where $p : (c, x) \mapsto c$ is an algebra morphism, and there is a morphism $q : c \mapsto (c, 0)$ of $C - C$- bimodules (but also of algebras) such that $pq = id_C$. Thus, in particular, a trivial

extension $B = C \ltimes E$ is a split extension such that $E^2 = 0$. We recall the following result from [9].

Lemma 5.1 [9, Lemma 4.1] *If B is a split extension of C, then there exists a morphism $\varphi : \mathrm{HH}^1(B) \to \mathrm{HH}^1(C)$ given by $[\delta] \mapsto [p\delta q]$.*

The morphism φ is shown in [6] to be surjective in case C is a tilted algebra and B is its relation-extension. In fact we have

Theorem 5.2 [6, Theorem 3.5] *Let B be the trivial extension of a tilted algebra C by the relation bimodule $E = \mathrm{Ext}^2_C(DC, C)$, then there exists a short exact sequence of vector spaces*

$$0 \longrightarrow \mathrm{HH}^1(B, E) \longrightarrow \mathrm{HH}^1(B) \xrightarrow{\varphi} \mathrm{HH}^1(C) \longrightarrow 0.$$

In the sequel we always write $E = \mathrm{Ext}^2_C(DC, C)$. Our objective in this section is to prove the following theorem.

Theorem 5.3 *Let B be a tame cluster-tilted algebra, and C a tilted algebra such that $B = C \ltimes E$. Then there exists a short exact sequence of k-vector spaces*

$$0 \longrightarrow k^{n_{B,C}} \longrightarrow \mathrm{HH}^1(B) \xrightarrow{\varphi} \mathrm{HH}^1(C) \longrightarrow 0.$$

Applying Theorem 5.2, it suffices to prove that $\mathrm{HH}^1(B, E) = k^{n_{B,C}}$. We set $n = n_{B,C}$ for simplicity. Our first task is to prove that $\mathrm{HH}^1(B, E)$ can be written in a simpler form. This is achieved in the following two statements.

Lemma 5.4 *Let $B = C \ltimes E$ with C a tilted algebra, then $Der_0(B, E) = Der_0(C, E) \oplus \mathrm{End}_{C^e} E$.*

Proof Let $\delta \in \mathrm{Der}_0(B, E)$, then we can define two k-linear maps $d : C \to E$ and $f : E \to E$ by

$$d(c) = \delta(c, 0) \quad \text{for all } c \in C,$$
$$f(x) = \delta(0, x) \quad \text{for all } x \in E,$$

that is, $d = \delta|_C$ and $f = \delta|_E$.

We first prove that $d : C \to E$ is a normalised derivation. Indeed, let $c, c' \in C$ then

$$\begin{aligned} d(cc') &= \delta(cc', 0) = \delta((c, 0)(c', 0)) \\ &= (c, 0)\delta(c', 0) + \delta(c, 0)(c', 0) \\ &= cd(c') + d(c)c'. \end{aligned}$$

On the other hand, $d(e_i) = \delta(e_i, 0) = 0$ for every i.

Next, we prove that $f : E \to E$ is a morphism of $C - C$-bimodules. Let $c \in C$ and $x \in E$, then

$$\begin{aligned} f(cx) &= \delta(0, cx) = \delta((c, 0)(0, x)) \\ &= (c, 0)\delta(0, x) + \delta(c, 0)(0, x) \\ &= cf(x) + d(c)x = cf(x) \end{aligned}$$

because $d(c), x \in E$ and $E^2 = 0$. Similarly, $f(xc) = f(x)c$.

We have shown that $\mathrm{Der}_0(B, E) = \mathrm{Der}_0(C, E) + \mathrm{End}_{C^e} E$. But now $\mathrm{Der}_0(C, E) \subseteq \mathrm{Hom}_k(C, E)$ while $\mathrm{End}_{C^e} E \subseteq \mathrm{Hom}_k(E, E)$ and we have an obvious direct sum decomposition

$$\mathrm{Hom}_k(B, E) = \mathrm{Hom}_k(C, E) \oplus \mathrm{Hom}_k(E, E).$$

Therefore $\mathrm{Der}_0(B, E) = \mathrm{Der}_0(C, E) \oplus \mathrm{End}_{C^e} E$. □

Proposition 5.5 *Let $B = C \ltimes E$ with C a tilted algebra, then* $\mathrm{HH}^1(B, E) = \mathrm{HH}^1(C, E) \oplus \mathrm{End}_{C^e} E$.

Proof Because of Lemma 5.4, we have a direct sum decomposition $\mathrm{Der}_0(B, E) = \mathrm{Der}_0(C, E) \oplus \mathrm{End}_{C^e} E$. We prove that it induces a direct sum decomposition $\mathrm{Int}_0(B, E) = \mathrm{Int}_0(C, E) \oplus 0$ on the level of the inner derivations, that is, if $\delta \in \mathrm{Int}_0(B, E)$ then $d = \delta|_C \in \mathrm{Int}_0(C, E)$ while $f = \delta|_E = 0$.

Assume $\delta \in \mathrm{Int}_0(B, E)$ then there exists $(c, x) \in B = C \oplus E$ such that $\delta = \delta_{(c,x)}$, that is, for all $(c', x') \in B$, we have

$$\begin{aligned}\delta(c', x') &= (c, x)(c', x') - (c', x')(c, x)\\ &= (cc', xc' + cx') - (c'c, x'c + c'x)\\ &= (cc' - c'c, cx' - x'c + xc' - c'x).\end{aligned}$$

But $\delta(c', x') \in E$. Therefore $cc' - c'c = 0$, or $cc' = c'c$. Since c' is an arbitrary element of C, this means that c is in the centre of C. Now C is a tilted algebra, so it is triangular and therefore $c = \lambda$ is a scalar. But then

$$\delta(c', x') = xc' - c'x \in E.$$

Now, let $d = \delta|_C$ and $f = \delta|_E$. Then, for all $x' \in E$, we have

$$f(x') = \delta(0, x') = 0$$

so that $f = 0$, as desired. On the other hand, $d(c') = \delta(c', 0) = xc' - c'x = [x, c']$, that is, $d = [x, -] \in \mathrm{Int}_0(C, E)$. This establishes our claim. But then we have

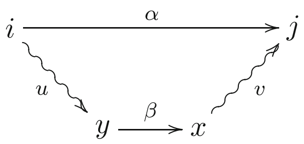
$$\begin{aligned}\mathrm{HH}^1(B, E) &= \frac{\mathrm{Der}_0(B,E)}{\mathrm{Int}_0(B,E)} \simeq \frac{\mathrm{Der}_0(C,E)\oplus\mathrm{End}_{C^e} E}{\mathrm{Int}_0(C,E)\oplus 0}\\ &\simeq \frac{\mathrm{Der}_0(C,E)}{\mathrm{Int}_0(C,E)} \oplus \mathrm{End}_{C^e} E\\ &= \mathrm{HH}^1(C, E) \oplus \mathrm{End}_{C^e} E.\end{aligned}$$

□

We shall now prove that if C is constricted then $\mathrm{HH}^1(C, E) = 0$. Our proof will use essentially the tameness of B, which implies that C is a tilted algebra of Dynkin or euclidean type.

Lemma 5.6 *Assume that C is tilted and constricted, and that $B = C \ltimes E$ is tame. Then* $\mathrm{HH}^1(C, E) = 0$.

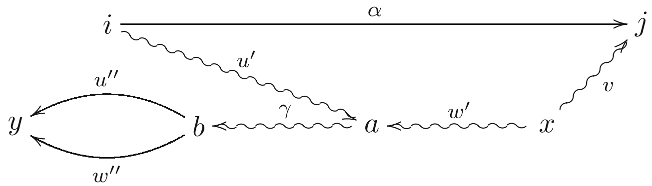
Proof It suffices to prove that $\mathrm{Der}_0(C, E) = 0$. Assume thus that $\alpha : i \to j$ is an old arrow, then $\delta \in \mathrm{Der}_0(C, E)$ implies $\delta(\alpha) = e_i\delta(\alpha)e_j \in e_iEe_j$. Now, assume $w \in E$ is nonzero, then e_iwe_j contains a new arrow β and we have the following situation



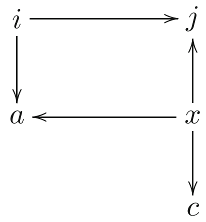
with β a new arrow and u, α, v old. Let $\rho = \sum_i \lambda_i w_i$ be the relation in C corresponding to β. Because C is triangular, at least one of the paths u or v is nontrivial. We have four cases to consider.

(i) If the paths u and v share no arrows with any of the w_i then the bound quiver of the full subcategory of C generated by i, j and all the points lying on the summands of ρ, contains a sequential walk $w_1 u^{-1} \alpha v^{-1} w_1$, a contradiction to Lemma 2.4.

(ii) The paths u, v cannot be in the situation of intersecting one of the w_i but sharing neither the last arrow of u nor the first arrow of v. Indeed, assume that this happens, say, with u (the case where it happens with v is treated similarly). Then we have $u = u'\gamma u''$ with γ a subpath of w_1, that is, $w_1 = w'\gamma w''$

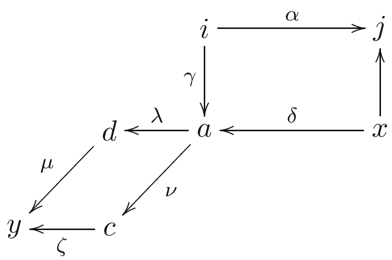


Then ρ is monomial because, otherwise, letting c be any point of w_2 distinct from x, y, we have a wild full subcategory of C

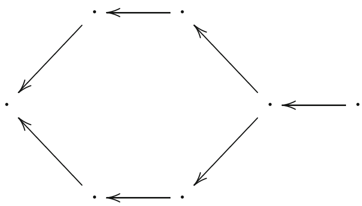


a contradiction.

Next, $u \neq 0$ implies $u'' \neq 0$ and the fact that ρ is a relation implies $w'' \neq 0$. But C cannot contain two tame concealed full subcategories (namely, corresponding to the cycles $(u'')^{-1}w''$ and $u'(w')^{-1}v\alpha^{-1}$). Therefore, there is a relation linking u'' and w''. In particular, both have length at least two. Because of [12, 4.7] we must have $b = a$. On the other hand, we have $a \neq x$, because otherwise α is parallel to $u'v$ which contradicts our assumption that C is constricted. This shows that C contains a full subcategory D of the form

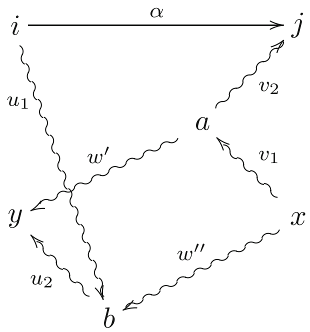


where we have $\gamma\lambda\mu \neq 0$ and $\delta\nu\zeta = 0$ (thus $\delta\nu \neq 0$, $\nu\zeta \neq 0$) and there is a relation linking $\lambda\mu$ and $\nu\zeta$. Because D contains no wild full subcategory and must satisfy [12, 4.7] we must have $\delta\lambda = 0$ and $\gamma\nu = 0$. Let D' be the full convex subcategory of D consisting of all points except y. Then D' is the one-point coextension of a tame hereditary algebra of type $\tilde{\mathbb{A}}_{2,2}$ by two indecomposable regular modules lying on two different tubes of rank 2. Therefore, applying [37, 4.7], we get that D' is a tilted algebra of type $\tilde{\mathbb{A}}_{3,3}$ having a complete slice in the postprojective component. Now D is the one-point coextension of D' by an indecomposable D'-module M such that $\mathrm{Hom}_{D'}(M, I_d) \neq 0$ and $\mathrm{Hom}_{D'}(M, I_c) \neq 0$. Then M maps nontrivially into two different tubes of D' and so M is postprojective. Therefore D is a tilted algebra of wild type

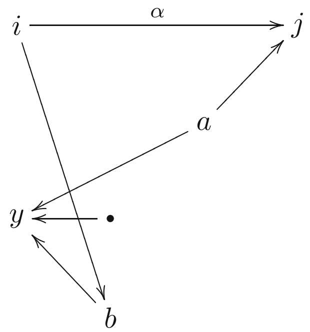


a contradiction.

(iii) If the paths u and v share the last arrow of u and the first arrow of v with two different w_i, say w_1 and w_2, then, in particular, ρ is not a monomial relation. We then have the following situation



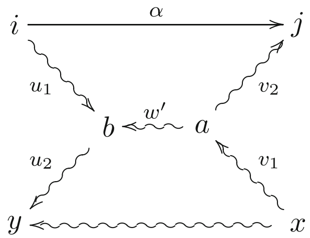
with $u = u_1 u_2$, $v = v_1 v_2$, $w_1 = w'' u_2$ and $w_2 = v_1 w'$. Assume first that ρ is not a binomial relation. Then C contains a full subcategory C' of the form



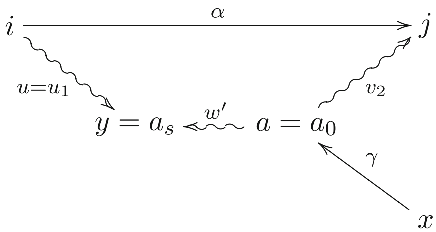
and, since C' is wild, we get a contradiction to the tameness of B. Therefore ρ is binomial. Consider the full subcategory H of C generated by the points i, j, a, y, b.

Then H is hereditary of type $\tilde{\mathbb{A}}$ and C contains a full subcategory $C' = H[M]$, where M is the indecomposable H-module with semisimple socle $S_y \oplus S_j$ (or, $S_y \oplus S_j \oplus S_j$ if u_1 is trivial) and such that $M/\operatorname{soc}(M) = S_a \oplus S_b$. Then $M \cong (P_a \oplus P_b)/P_y \cong \tau^{-1}P_y$ is postprojective, and hence $C' = H[M]$ is a tilted algebra of wild type, a contradiction.

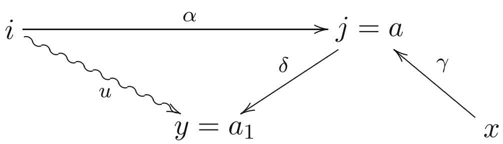
(iv) If the paths u and v share the last arrow of u and the first arrow of v with the same w_i, say w_1, then we have the following situation



with $u = u_1u_2$, $v = v_1v_2$, $w = v_1w'u_2$. If both u_2 and v_1 are non-trivial, then C contains a wild full subcategory. Thus u_2 or v_1 is trivial. Assume that u_2 is trivial, then $y = b$ and $u = u_1$ (the case where v_1 is trivial is entirely similar). If ρ is not a monomial relation, or the path v_1 contains more than one arrow, then C contains a wild full subcategory. Therefore, ρ is monomial and v_1 consists of one arrow γ. We have the following situation



Let H denote the full subcategory of C generated by the points i, j, $a = a_0$, $a_1, \ldots, a_s = y$. Then H is hereditary of type $\tilde{\mathbb{A}}$, and C contains a full subcategory $C' = H[M]$, where M is the H-module with top S_a and socle $S_{a_{s-1}} \oplus S_j$ (or, $S_{a_{s-1}}$ if v_2 is trivial). Then M is regular lying in an exceptional tube. In this case, $C' = H[M]$ is wild, except for $s = 1$ in which case it is tilted of type $\tilde{\mathbb{A}}$, see [11]. Then we have the following situation



In particular, the potential has a summand of the form $w_1 = \gamma\delta\beta$, hence B has a relation of the form $\beta\gamma = 0$. But then $e_i w e_j = 0$ as required. □

We next study the $C - C$-bimodule endomorphisms of E. The following notation will be useful. If u is a subpath of a path v we shall say that u *divides* v and write $u|v$. In particular, if an arrow α (or a point x) lies on the path v, then we write $\alpha|v$ (or $x|v$, respectively). We use the symbol $\not|$ in the obvious way.

We recall from [6] that, if $\mathcal{S}_1, \cdots, \mathcal{S}_n$ are the distinct equivalence classes of new arrows, and E_j is the $C-C$-bimodule generated by the elements of $\mathcal{S}_j$, then we have $E = \oplus_{j=1}^n E_j$ (see [6, 4.3]).

Lemma 5.7 *Let C be constricted and $E = \oplus_{i=1}^n E_i$ be the decomposition induced from the arrow equivalence relation then, for every new arrow α in E_i and every $\delta \in \mathrm{Hom}_{C^e}(E_i, E)$, we have*

$$\delta(\alpha) = \lambda_\alpha \alpha$$

for some scalar $\lambda_\alpha \in k$.

Proof Let us denote by $\{\alpha_1, \cdots, \alpha_n\}$ the set of all new arrows and by $\{\rho_{\alpha_1}, \cdots, \rho_{\alpha_n}\}$ the set of corresponding relations in C so that the potential is

$$W = \sum_{i=1}^{n} \rho_{\alpha_i} \alpha_i .$$

We may assume that the equivalence class $[\alpha_1]$ of α_1 is $\{\alpha_1, \cdots, \alpha_r\}$ with $r \leq n$. For any i with $1 \leq i \leq r$ and δ as in the statement, we clearly have

$$\delta(\alpha_i) = \sum_j \lambda_{ij} u_{ij} \alpha_j v_{ij}$$

where $\lambda_{ij} \in k$, and u_{ij}, v_{ij} are paths in C such that α_i and $u_{ij}\alpha_j v_{ij}$ are parallel.

We claim that $\lambda_{ik} = 0$ when $k \neq i$, and this implies $\delta(\alpha_i) = \lambda_{ii}\alpha_i$ because C is triangular. Without loss of generality, let $i = 1$ and assume that $k \neq 1$, then $\rho_{\alpha_k} \neq \rho_{\alpha_1}$ and hence there exists an arrow γ_k such that $\gamma_k|\rho_{\alpha_1}$ but $\gamma_k \not| \rho_{\alpha_k}$. Deriving the potential W yields

$$\partial_{\gamma_k}(W) = \sum_{s \in S, t \in T, t \neq k} w_{st} \alpha_t w'_{st} \in \tilde{I}$$

that is, the arrow α_k does not appear in the above sum. There exists a summand of $\partial_{\gamma_k}(W)$ which is a minimal relation in $\tilde{I}$ which contains α_1 but not α_k. Further, because $[\alpha_1] = \{\alpha_1, \cdots, \alpha_r\}$, those α_t which appear in this minimal relation are such that $t \leq r$. Let thus this minimal relation be

$$\rho = \sum_{s \in S', t \in T', t \neq k, 1 \in T'} w_{st} \alpha_t w'_{st} .$$

Applying our morphism $\delta : E_i \to E$ yields

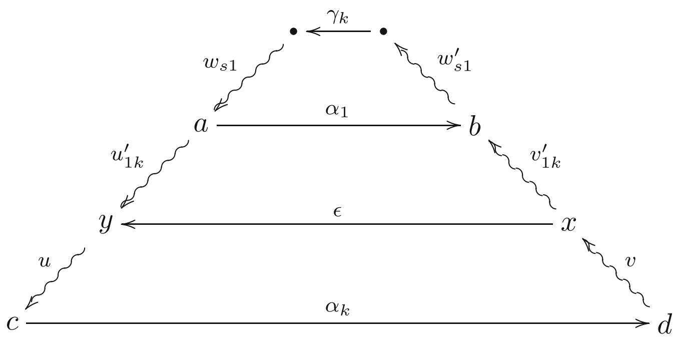
$$0 = \delta(\rho) = \sum_{s,t,j} \lambda_{tj} w_{st} u_{tj} \alpha_j v_{tj} w'_{st} .$$

The summands for which $t = 1$, $j = k$ are of the form

$$\lambda_{1k} w_{s1} u_{1k} \alpha_k v_{1k} w'_{s1}.$$

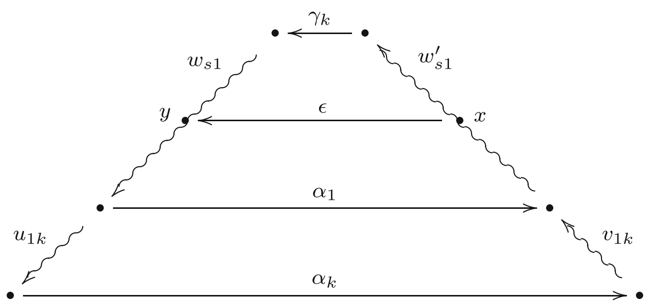
The arrow α_k belongs to the cycle $\alpha_k \rho_{\alpha_k}$ in the potential W and therefore the above summands contain each a subpath of the form $u\alpha_k v$, where $v\epsilon u$ (for some arrow $\epsilon : x \to y$) is a subpath dividing ρ_{α_k}. We split the proof into several cases.

(i) $x|v_{1k}$ and $y|u_{1k}$. Then we have the following situation



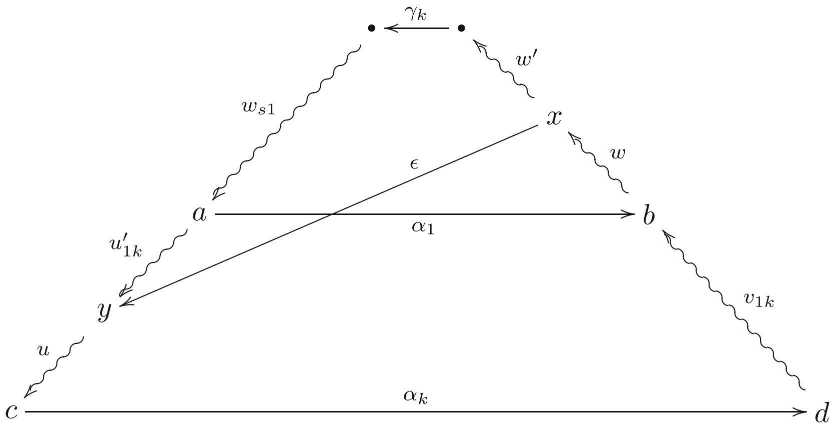
where $u_{1k} = u'_{1k}u$ and $v_{1k} = vv'_{1k}$. In this case, we have the relation ρ_{α_1} which involves the path $w'_{s1}\gamma_k w_{s1}$ and perhaps other paths from b to a in C. Now, the paths v'_{1k}, u'_{1k} cannot both be trivial, because otherwise B would contain a 2-cycle formed by the arrows α_1 and ϵ. On the other hand, v'_{1k} and u'_{1k} do not share arrows with the summands of ρ_{α_1} (because of their directions). We consider the full subcategory of C generated by all the points lying on the summands of ρ_{α_1} and the points x, y. We thus get a contradiction to Lemma 2.4 since $(w'_{s1}\gamma_k w_{s1})u'_{1k}\epsilon^{-1}v'_{1k}(w'_{s1}\gamma_k w_{s1})$ is a sequential walk.

(ii) $x|w'_{s1}$ and $y|w_{s1}$. Then we have the following situation



a contradiction to the hypothesis that C is constricted. Notice that $\epsilon \neq \gamma_k$ because $\epsilon|\rho_{\alpha_k}$ while $\gamma_k \not| \rho_{\alpha_k}$.

(iii) $x|w'_{s1}$ and $y|u_{1k}$. Then we have the following situation



where $u_{1k} = u'_{1k}u$, $w'_{s1} = ww'$ and $v = v_{1k}w$. Because C is constricted, there must be a relation in C on the longer path from x to y. This relation together with the arrow ϵ yields a contradiction to Lemma 2.4.

(iv) $x|v_{1k}$ and $y|w_{s1}$. This case is symmetric to the previous one.

This shows that we have no such terms in the sum and therefore $\lambda_{1k} = 0$. This completes the proof of the lemma. □

Corollary 5.8 *With the above notation,* $\mathrm{Hom}_{C^e}(E_i, E_j) = 0$ *whenever* $i \neq j$.

Lemma 5.9 *For every* i*, we have that* $\mathrm{End}_{C^e} E_i = k$.

Proof Assume $E_i =< \alpha_1 >$ and $[\alpha_1] = \{\alpha_1, \cdots, \alpha_r\}$. Because of Lemma 5.7, we have, for $\delta \in \mathrm{End}_{C^e} E_i$

$$\delta(\alpha_i) = \lambda_i \alpha_i$$

for some scalar $\lambda_i \in k$. Now there exists a strongly minimal relation containing the arrow α_1, let it be

$$\rho = \sum_{j \in J} \mu_j w_j \alpha_j w'_j.$$

Applying δ yields

$$0 = \delta(\rho) = \sum_{j \in J} \mu_j \lambda_j w_j \alpha_j w'_j.$$

Subtracting from this the relation $\lambda_1 \rho$ we get

$$\sum_{j \in J \setminus \{1\}} \mu_j (\lambda_j - \lambda_1) w_j \alpha_j w'_j = 0.$$

Because ρ is strongly minimal, we get $\lambda_j = \lambda_1$, for every $j \in J \setminus \{1\}$. Because the arrow equivalence is transitive, we get $\lambda_j = \lambda_1$ for every $j \in \{1, \cdots, r\}$. □

Now we are ready to prove Theorem 5.3.

Proof If C is hereditary, then $B = C$, $n_{B,C} = 0$ and $\mathrm{HH}^1(B) = \mathrm{HH}^1(C)$. If not, assume first that C is constricted. Because $\mathrm{HH}^1(C, E) = 0$, it suffices to prove that $\mathrm{End}_{C^e}\, E = k^n$. But we have also shown that

$$\mathrm{Hom}_{C^e}(E_i, E_j) = \begin{cases} 0 & \text{if } i \neq j \\ k & \text{if } i = j. \end{cases}$$

This implies immediately the statement.

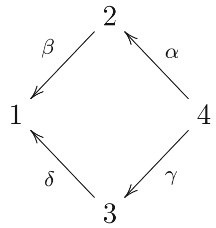
Otherwise, C is, up to duality, of one of the forms (i) (ii) of Lemma 4.2. As observed in the proof of Lemma 4.3, we have two distinct cases:

(a) Either one can replace the non-constricted algebra C by a constricted algebra C' such that $n_{B,C} = n_{B,C'}$ and $\mathrm{HH}^1(C) \cong \mathrm{HH}^1(C')$ because of Lemma 4.4. The statement then follows from the previous argument applied to B and C'.

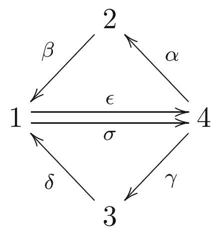
(b) Otherwise B is, up to duality, of one of the forms (i) (ii) of Lemma 4.3. Note that there exist several tilted algebras C having B as a relation-extension. However, because of Lemma 3.2, the cardinality $|R|$ of a system of relations R for each such tilted algebra C is independent of the choice of C. Moreover, in this case, $n_{B,C} = |R|$, by Lemma 3.3.

Using Lemma 4.5, it suffices to prove that, if C is of the form (i), then $\mathrm{HH}^1(C) = k^2$ and, if C is of the form (ii), then $\mathrm{HH}^1(C) = k$. This follows from another straightforward application of Happel's sequence. □

Example 5.10 Let C be the tilted algebra of euclidean type $\tilde{\mathbb{A}}_{2,2}$ given by the quiver



bound by the relations $\alpha\beta = 0$ and $\gamma\delta = 0$. The corresponding cluster-tilted algebra B is given by the quiver



bound by the relations $\alpha\beta = \beta\epsilon = \epsilon\alpha = 0$ and $\gamma\delta = \delta\sigma = \sigma\gamma = 0$. Note that B is not schurian so the results from [9] cannot be used. The arrow equivalence class in this example consists of the two new arrows ϵ and σ, and therefore the relation invariant $n_{B,C}$ is equal to 2. Now Theorem 5.3 implies that $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(C) \oplus k^2 \cong k^3$.

The following result has been proved for schurian cluster-tilted algebras in [9, Corollary 3.4]. The statement is inspired from Skowroński's famous question [38, Problem 1]: For which algebras is simple connectedness equivalent to the vanishing of the first Hochschild cohomology group?

Theorem 5.11 *Let* $B = k\tilde{Q}/\tilde{I}$ *be a cluster-tilted algebra. Then* $\mathrm{HH}^1(B) = 0$ *if and only if* B *is hereditary with ordinary quiver a tree.*

Proof By [4], the cluster repetitive algebra is a Galois covering of B with infinite cyclic group $\mathbb{Z}$. Moreover it is connected if and only if B is not hereditary (because of [4, 1.4, Lemma 5]). Assume thus that B is not hereditary. Because of the universal property of the Galois covering, there exists a group epimorphism

$$\pi_1(\tilde{Q}, \tilde{I}) \to \mathbb{Z}.$$

Let k^+ denote the additive group of the field k. The previous epimorphism induces a monomorphism of abelian groups

$$\mathrm{Hom}(\mathbb{Z}, k^+) \to \mathrm{Hom}(\pi_1(\tilde{Q}, \tilde{I}), k^+)$$

which, composed with the canonical monomorphism $\mathrm{Hom}(\pi_1(\tilde{Q}, \tilde{I}), k^+) \to \mathrm{HH}^1(B)$ of [35, Corollary 3], yields a monomorphism $\mathrm{Hom}(\mathbb{Z}, k^+) \to \mathrm{HH}^1(B)$.

Therefore, if B is not hereditary, we have $\mathrm{HH}^1(B) \neq 0$. On the other hand, if B is hereditary, then, because of [29, 1.6], we have $\mathrm{HH}^1(B) = 0$ if and only if the quiver $\tilde{Q}$ of B is a tree. □

6 The Representation-Finite Case

Throughout this section, let B be a representation-finite cluster-tilted algebra. We present easy methods to compute the relation invariant $n_{B,C}$ and thus $\mathrm{HH}^1(B)$ in this case. Let $\tilde{Q}$ be the quiver of B and let n be the number of points in $\tilde{Q}$.

Choose a tilted algebra C such that $B = C \ltimes \mathrm{Ext}^2_C(DC, C)$. The *number of relations* in C is the dimension of $\mathrm{Ext}^2_C(S_C, S_C)$, where S_C is the sum of a complete set of representatives of the isomorphism classes of simple C-modules. We say that a relation r in B is a *new relation* if it is not a relation in C. It has been shown in [9, Corollary 3.3] that in this case $n_{B,C}$ is equal to the number of relations in C minus the number of new commutativity relations in B, and, moreover,

$$\mathrm{HH}^1(B) = k^{n_{B,C}}.$$

In particular, the integer $n_{B,C}$ does not depend on the choice of the tilted algebra C, and therefore we shall denote it in the rest of this section by n_B. The objective of this section is to show that one can read off the integer n_B from the quiver $\tilde{Q}$ of B.

Recall from [13] that a *chordless cycle* in $\tilde{Q}$ is a full subquiver induced by a set of points $\{x_1, x_2, \ldots, x_p\}$ which is topologically a cycle, that is, the edges in the chordless cycle are precisely the edges $x_i — x_{i+1}$.

Lemma 6.1 *The number of chordless cycles in $\tilde{Q}$ is equal to the number of zero relations in C plus twice the number of commutativity relations in C.*

Proof Consider the map {relations in C} $\to$ {new arrows in B} that associates to a relation $\rho \in \mathrm{Ext}^2_C(S_i, S_j)$ the new arrow $\alpha(\rho) : j \to i$. By [19, Corollary 3.7], every chordless cycle contains exactly one new arrow, and therefore it suffices to show that if ρ is a commutativity relation, then $\alpha(\rho)$ lies in precisely two chordless cycles in $\tilde{Q}$, and if ρ is a zero relation, then $\alpha(\rho)$ lies in precisely one chordless cycle in $\tilde{Q}$.

If ρ is a commutativity relation, say $\rho = c_1 - c_2$ where c_1, c_2 are paths from i to j in Q, then the concatenations $\alpha(\rho)c_1$ and $\alpha(\rho)c_2$ are two chordless cycles. Then it follows from the fact that $\widetilde{Q}$ is a planar quiver (see [23, Theorem A1]), that $\alpha(\rho)$ lies in precisely two chordless cycles.

Otherwise, ρ is a zero relation in C, and $\alpha(\rho)\rho$ is a chordless cycle in $\tilde{Q}$. We have to show that $\alpha(\rho)$ does not lie in two chordless cycles. Suppose the contrary. Because of [27, Proposition 9.7], every chordless cycle in $\tilde{Q}$ is oriented. Therefore there exists another path ρ' from i to j in $\tilde{Q}$ such that $\alpha(\rho)\rho'$ is a chordless cycle. If ρ' is also a path in Q, then ρ and ρ' are two parallel paths whose difference $\rho - \rho'$ is not a relation in C. This implies that the fundamental group of C is non-trivial, and this contradicts the well-known fact that tilted algebras of Dynkin type are simply connected (see, for instance, [33]). On the other hand, if ρ' is a path in $\tilde{Q}$ but not in Q then it must contain at least one new arrow. But then the chordless cycle $\alpha(\rho)\rho'$ contains two new arrows, a contradiction to [19, Corollary 3.7]. □

An arrow in $\tilde{Q}$ is called *inner arrow* if it is contained in two chordless cycles. Arrows which are not inner arrows are called *outer arrows*.

Lemma 6.2 *The number of new inner arrows in B is equal to the number of commutativity relations in C.*

Proof Each commutavity relation in C gives a new inner arrow in B. Conversely, suppose that α is a new inner arrow in B and let ρ, ρ' be the two paths in $\tilde{Q}$ such that $\alpha\rho$ and $\alpha\rho'$ are the chordless cycles. By [19, Corollary 3.7], ρ and ρ' contain no new arrows, and hence ρ and ρ' are paths in Q. Since the algebra C is simply connected, it follows that $\rho - \rho'$ is a relation in C. □

Lemma 6.3 *The number of old inner arrows in B is equal to the number of new commutativity relations in B.*

Proof We recall from [17, 23] the description of B as a bound quiver algebra: For any arrow α in $\tilde{Q}$ let $\mathcal{S}_\alpha$ be the set of paths ρ in $\tilde{Q}$ such that $\rho\alpha$ is a chordless cycle and define

$$\rho_\alpha = \begin{cases} \rho & \text{if } \mathcal{S}_\alpha = \{\rho\} \\ \rho - \rho' & \text{if } \mathcal{S}_\alpha = \{\rho, \rho'\}. \end{cases}$$

Let I be the ideal in $k\tilde{Q}$ generated by the relations $\cup_{\alpha \in (\tilde{Q})_1} \{\rho_\alpha\}$. Then

$$B = k\tilde{Q}/I.$$

Because of the previous remarks, commutativity relations are in bijection with inner arrows. If the relation is new, then the arrow is old and if the arrow is new then the relation is old. □

We are now able to prove the main theorem of this section.

Theorem 6.4 *Let B be a representation-finite cluster-tilted algebra and $\tilde{Q}$ the quiver of B. Then n_B equals the number of chordless cycles in $\tilde{Q}$ minus the number of inner arrows in $\tilde{Q}$.*

Proof By definition, n_B is the number of relations in C minus the number of new commutativity relations in B. By Lemmata 6.1 and 6.2, the number of relations in C is equal to the number of chordless cycles minus the number of new inner arrows in $\tilde{Q}$. On the other hand, the number of new commutativity relations in B is equal to the number of old inner arrows in $\tilde{Q}$, because of Lemma 6.3. Therefore

$$n_B = \text{ number of chordless cycles in } \tilde{Q} - \text{ number of inner arrows in } \tilde{Q}. \qquad \square$$

Corollary 6.5 *If $\tilde{Q}$ is connected then*

$$n_B = 1 + \text{ number of outer arrows in } \tilde{Q} - n.$$

Proof Because of [23, Theorem A1] the quiver $\widetilde{Q}$ is planar. In particular, every arrow lies in at most two chordless cycles. Hence one can associate a simplicial complex on the 2-dimensional sphere to the quiver $\tilde{Q}$, in such a way that Q_0 is the set of points, Q_1 the set of edges and the set of chordless cycles is the set of faces of the simplicial complex except the face coming from the "outside" of the quiver (the unbounded component of the complement when embedded in the plane). Using Euler's formula, we see that the number of chordless cycles in $\tilde{Q}$ is equal to $1 + |(\tilde{Q})_1| - |(\tilde{Q})_0|$, and then Theorem 6.4 yields

$$n_B = 1 + (|(\tilde{Q})_1| - \text{ number of inner arrows in } \tilde{Q}) - |(\tilde{Q})_0|,$$

and the statement follows. □

Remark 6.6 If $\tilde{Q}$ is not connected then

$$\begin{aligned} n_B = &\text{ number of connected components of } \tilde{Q} \\ &+ \text{ number of outer arrows in } \tilde{Q} - n. \end{aligned}$$

As an application, we show the following corollary on deleting points. Let $x \in (\tilde{Q})_0$, and $e_x \in B$ the associated idempotent. Then B/Be_xB is cluster-tilted and the quiver of B/Be_xB is obtained from $\tilde{Q}$ by deleting the point x and all arrows adjacent to x, see [18, Section 2]. Define the *Hochschild degree* of x to be the integer

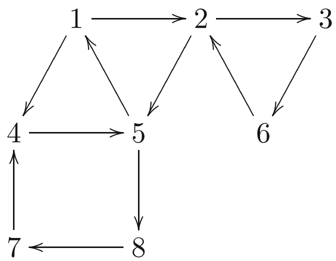
$$\deg_{\mathrm{HH}}(x) = n_B - n_{B/Be_xB}.$$

Corollary 6.7

$$\begin{aligned} \deg_{\mathrm{HH}}(x) = &\text{ number of chordless cycles going through } x \\ &- \text{ number of inner arrows on the chordless cycles going through } x. \end{aligned}$$

Proof Using Theorem 6.4, we get that $\deg_{\mathrm{HH}}(x)$ is equal to the number of chordless cycles that are adjacent to x minus the number of inner arrows in $\tilde{Q}$ plus the number of inner arrows in Q_{B/Be_xB}. Now α is an inner arrow in $\tilde{Q}$ which is not an inner arrow in Q_{B/Be_xB}, precisely if α lies on two chordless cycles in $\tilde{Q}$ at least one of which goes through x. □

Example 6.8 The following quiver is the quiver of a cluster-tilted algebra of type $\mathbb{E}_8$.



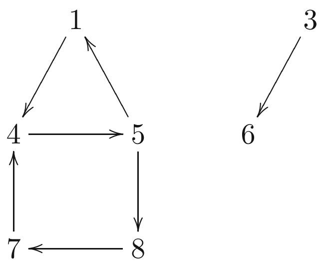
The quiver has 4 chordless cycles and 2 inner arrows, so Theorem 6.4 yields

$$\mathrm{HH}^1(B) = k^{4-2} = k^2.$$

On the other hand, the quiver has 9 outer arrows, so, using Corollary 6.5, we also get

$$\mathrm{HH}^1(B) = k^{1+9-8} = k^2.$$

The point 2 has Hochschild degree $2 - 1 = 1$, by Corollary 6.7. So $\mathrm{HH}^1(B/Be_2B) = k$. The quiver of B/Be_2B is the following.



Observe that Remark 6.6 applies here: the number of connected components is 2, the number of outer arrows is 6, and $n = 7$. Thus we get $n_B = 2 + 6 - 7 = 1$.

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