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Symmetric simple exclusion process with free boundaries

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Abstract We consider the one dimensional symmetric simple exclusion process with additional births and deaths restricted to a subset of configurations where there is a leftmost hole and a rightmost particle. At a fixed rate birth of particles occur at the position of the leftmost hole and at the same rate, independently, the rightmost particle dies. We prove convergence to a hydrodynamic limit and discuss its relation with a free boundary problem.

Keywords Hydrodynamic limit · Free boundary problems · Stochastic inequalities

Mathematics Subject Classification 60K35 · 82C20

1 Introduction

A free boundary problem in its simplest version is given by the linear heat equation in a domain Ω which itself changes in time with a law which depends on the same solution. In the Stefan problem for instance the heat equation is complemented by

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Dirichlet boundary conditions while the local velocity of the points of the boundary are specified in terms of the local gradient of the solution.

The purpose of this paper is to study a particle version of such free boundary problems. The linear heat equation is in our case replaced by the one dimensional symmetric simple exclusion process, SSEP, in the set of configurations having a rightmost particle and a leftmost hole. Furthermore, at a given rate the rightmost particle dies and at the same rate, independently, a birth of a particle occurs at the position of the leftmost hole. Since the leftmost hole and the rightmost particle move, we call them free boundaries.

More precisely, Call $\eta \in \{0,1\}^{\mathbb{Z}}$ a particle configuration, think of η as the subset of $\mathbb{Z}$ occupied by particles and consider those η having a rightmost particle located at $\mathrm{R}(\eta) := \max(\eta)$ and a leftmost hole located at $\mathrm{L}(\eta) := \min(\mathbb{Z}\setminus\eta)$. Let (η_t) be the Markov process performing symmetric simple exclusion process at rate $\frac{1}{2}$ and such that the rightmost particle and the leftmost hole are killed at rate J:

$$\eta \to \eta\setminus\{\mathrm{R}(\eta)\} \text{ and } \eta \to \eta \cup \{\mathrm{L}(\eta)\} \text{ at rate } \mathrm{J} \text{ each.}$$

Since particles are injected to the left and extracted from the right, J can be seen as the average current of particles through the system. It is well known that under a diffusive space and time scaling the collective behavior of the SSEP is ruled by the linear heat equation [4]. We perform the same scaling with a parameter ε such that time is $\varepsilon^{-2}t$, space $\varepsilon^{-1}r$ and the killing is $\mathrm{J} = \mathrm{J}(\varepsilon) = \varepsilon j$, where j is the macroscopic current.

Consider a function $\rho : \mathbb{R} \to [0,1]$ identically zero to the right of $\mathrm{R}(\rho) := \sup\{r : \rho(r) > 0\} < \infty$ and identically one to the left of $\mathrm{L}(\rho) := \inf\{r : \rho(r) < 1\} > -\infty$ and continuous in $(\mathrm{L}(\rho), \mathrm{R}(\rho))$. Call $\mathcal{R}$ the set of functions with those properties. We consider a macroscopic density $\rho \in \mathcal{R}$ and ask the initial configuration $\eta^{(\varepsilon)}$ indexed by ε to approach the density ρ as follows:

$$\lim_{\varepsilon\to 0} \sup_{a\le b} \left| \varepsilon \sum_{\varepsilon x \in [a,b]} \eta^{(\varepsilon)}(x) - \int_a^b \rho(r)\, dr \right| = 0 \tag{1.1}$$

Our main result is

Theorem 1 *Let $(\eta_t^{(\varepsilon)})$ be the process with killing at rate $j\varepsilon$ and with initial configuration $\eta^{(\varepsilon)}$ satisfying (1.1). Then for $t \ge 0$ there exists a function $\rho_t \in \mathcal{R}$ such that $\rho_0 = \rho$ and*

$$\lim_{\varepsilon\to 0} P\left(\sup_{a\le b} \left| \varepsilon \sum_{\varepsilon x\in[a,b]} \eta^{(\varepsilon)}_{t\varepsilon^{-2}}(x) - \int_a^b \rho_t(r)\, dr \right| > \gamma \right) = 0 \quad \textit{for all } \gamma > 0. \tag{1.2}$$

We characterize the limit ρ_t in terms of "lower and upper barriers", the inequalities being in the sense of mass transport. The notion is defined by introducing first a map from density functions $\rho \in \mathcal{R}$ to functions $\phi(r|\rho)$, $r \in \mathbb{R}$, that we call "interfaces" and then by saying that $\rho \leq \rho'$ if $\phi(\cdot|\rho) \leq \phi(\cdot|\rho')$. See Sect. 4 where the notion is first defined at the particles level and then for densities in $\mathcal{R}$.

The barriers are defined by functions $\rho_t^{\delta,\pm}$, $\delta > 0$, which satisfy "discretized free boundary problems". More specifically $\rho_t^{\delta,-}$ evolves according to the heat equation in the intervals $[n\delta, (n+1)\delta)$ and at times $n\delta$ takes the rightmost portion $j\delta$ of mass from the right and puts it on the left. Namely, define the δ-quantiles $\mathrm{R}^\delta(\rho)$ and $\mathrm{L}^\delta(\rho)$ by

$$\int_{\mathrm{R}^\delta(\rho)}^{\infty} \rho(r)\, dr = \delta, \qquad \int_{-\infty}^{\mathrm{L}^\delta(\rho)} (1-\rho(r))\, dr = \delta, \tag{1.3}$$

Define also

$$(\Gamma^\delta \rho)(r) := \begin{cases} 1 & \text{if } r \leq \mathrm{L}^\delta(\rho) \\ \rho(r) & \text{if } \mathrm{L}^\delta(\rho) < r < \mathrm{R}^\delta(\rho) \\ 0 & \text{if } r \geq \mathrm{R}^\delta(\rho), \end{cases} \tag{1.4}$$

and let $G_t(r, r')$ be the Gaussian kernel (see (5.11)). Set $\rho_0^{\delta,-} = \rho$ and iteratively

$$\rho_t^{\delta,-} := \begin{cases} G_{t-n\delta}\rho_{n\delta}^{\delta,-}, & \text{if } t \in [n\delta, (n+1)\delta), \quad n = 0, 1, \ldots \\ \Gamma^{j\delta}\rho_{n\delta-}^{\delta,-}, & \text{if } t = n\delta, \quad n = 1, 2, \ldots. \end{cases} \tag{1.5}$$

which is well defined for δ small enough. Define $\rho_t^{\delta,+}$ with the same evolution but with initial profile $\rho_0^{\delta,+} = \Gamma^{j\delta}\rho$. We prove that

$$\rho_t^{\delta,-} \leq \rho_t \leq \rho_t^{\delta,+}$$

for any δ and any $t \in \delta\mathbb{N}$. We also prove that any function $\tilde{\rho}_t$ which satisfies the above inequality (for all δ and t as above) must necessarily be equal to ρ_t (uniqueness of separating elements). The precise statement is in Theorem 5. In particular this allows to show that:

Theorem 2 *Let $\rho \in \mathcal{R}$ and ρ_t be the evolution of Theorem 1 with initial datum ρ. Let $\rho_t^{\delta,-}$ be the evolution (1.5) with the same initial datum. Then for any $a < b$ real numbers and for any $\delta > 0$,*

$$\left| \int_a^b \rho_t^{\delta,-}(r)\, dr - \int_a^b \rho_t(r)\, dr \right| \leq 2j\delta, \quad \forall t \geq 0 \tag{1.6}$$

Theorems 1 and 2 are proved at the end of Sect. 6.1.

A formal limit of our particle system leads to conjecture that ρ_t solves

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}, \quad r \in (\mathrm{L}_t, \mathrm{R}_t), \\
&\mathrm{R}_0, \mathrm{L}_0, \rho(r, 0) \text{ given} \\
&\rho(\mathrm{L}_t, t) = 1, \quad \rho(\mathrm{R}_t, t) = 0; \quad \frac{\partial \rho}{\partial r}(\mathrm{L}_t, t) = \frac{\partial \rho}{\partial r}(\mathrm{R}_t, t) = -2j
\end{aligned} \tag{1.7}$$

where $\mathrm{R}_t := \mathrm{R}(\rho(\cdot, t))$ and $\mathrm{L}_t := \mathrm{L}(\rho(\cdot, t))$. Indeed the density flux $J(r)$ associated to the equation $\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}$ is equal to $J(r) := -\frac{1}{2}\frac{\partial \rho}{\partial r}$ so that the last two conditions in (1.7) just state that the outgoing flux at R_t is equal to the killing rate j and that the incoming flux at L_t is equal to the birth rate j.

The free boundary problem (1.7) is not of Stefan type. In fact in (1.7) we impose both Dirichlet and Neumann conditions as we prescribe the values of the function and of its derivative at the boundaries, while in the classical Stefan problem the Dirichlet boundary conditions are complemented by assigning the speed of the boundary (in terms of the derivative of the solution at the boundaries). We can obviously recover the velocity of the boundaries from a smooth solution $\rho(r, t)$ of (1.7) by differentiating the identities $\rho(\mathrm{L}_t, t) = 1$, $\rho(\mathrm{R}_t, t) = 0$, thus obtaining:

$$\frac{d\mathrm{L}_t}{dt} = \frac{1}{2j}\frac{\partial^2 \rho}{\partial r^2}(\mathrm{L}_t, t), \quad \frac{d\mathrm{R}_t}{dt} = \frac{1}{2j}\frac{\partial^2 \rho}{\partial r^2}(\mathrm{R}_t, t) \tag{1.8}$$

The traditional way to study the hydrodynamic limit of particle systems is to prove that the limit law of the system is supported by weak solutions of a PDE for which existence and uniqueness of weak solutions holds true. In our case this approach is problematic. While we know closeness to the heat equation away from the boundaries, we do not control the motion of the boundaries: we only know that they do not escape to infinity. We are not aware of existence and uniqueness theorems for (1.7), however in general in free boundary problems some assumptions of regularity on the motion of the boundaries is required, for instance Lipschitz continuity.

Our proof of hydrodynamic limit avoids this pattern as we prove directly existence of the limit by squeezing the particles density between lower and upper barriers which have a unique separating element. This suggests a variational approach to the analysis of (1.7) based on a proof that its classical solutions are also squeezed by the barriers, or more generally that any limit of "approximate solutions" of (1.7) lies in between the barriers. Such an approach has been carried in [1] for a simpler version of (1.7) where particles are in the semi-infinite line with reflections at the origin, so that there is a single free boundary R_t where particles are killed at rate j, while births occur at the origin, at same rate j. An extension to our case is in preparation.

A similar equation has been derived in [9] for a different interface process. In that case an existence and uniqueness theorem for the limit equation is proved to hold (see also reference therein). Regularity of the free boundary motions in [9] follows from some monotonicity properties intrinsic to the model and absent in our case.

Free boundary problems have also been derived in [7] for particles evolutions via branching and in [10] for a variant of the simple exclusion process.

We have a proof that the limit evolution ρ_t in Theorem 1 satisfies (1.7) in the very special case when ρ_t is a time-independent linear profile with slope $-2j$. Stationary solutions for free boundary problems have also been studied in [5] for non local evolutions in systems which undergo a phase transition.

The model we study in this paper is inspired by previous works on Fourier law with current reservoirs [3–6]; its actual formulation came out from discussions with Stefano Olla to whom we are indebted. We are also indebted to F. Comets and H. Lacoin for helpful comments and discussions and to a referee of PTRF for useful comments.

In Sect. 2 we define the particle process and in Sect. 3 we prove the existence of a unique invariant measure for the process as seen from the median.

Due to the non local nature of the birth-death process the usual techniques for hydrodynamic limit fail. To overcome this problem we use inequalities based on imbedding the particle process in an interface process. The relationship between the one dimensional nearest neighbors simple exclusion process and the interface process is known since the seminal paper by Rost [12], where he established the hydrodynamics of the asymmetric simple exclusion process. In Sect. 4 we define the interface dynamics and show that our particle process can be realized in terms of the interface dynamics. We also introduce the delta interface processes which correspond to the delta particle processes where the killing of particles and holes are grouped together and occur only a finite number of times (uniformly in the hydrodynamic limit) and give as limit the delta macroscopic evolution defined in (1.5).

In Sect. 4 we also establish basic inequalities between the true and the delta interface dynamics by giving a simultaneous explicit graphical construction of all of them; that is, a coupling. In Sect. 5 we prove convergence in the hydrodynamic limit. The proof uses that any limit point of the true interface dynamics is squeezed in between two approximate evolutions which depend on an approximating parameter $\delta > 0$. In Sect. 6 we establish basic properties of the macroscopic evolution. In particular, we prove the existence of a stationary solution for the limit evolution and use this result to prove that at any positive time the particle density (in the limit evolution) is identically 0 and 1 outside of a compact. In the last section we summarize the results.

2 The free boundary SSEP

The space of particle configurations is

$$\mathcal{X} := \left\{ \eta \in \{0,1\}^{\mathbb{Z}} : \sum_{x \ge 0} \eta(x) < \infty,\ \sum_{x \le 0} (1 - \eta(x)) < \infty \right\}$$

that is, for any $\eta \in \mathcal{X}$ the number of particles to the right of the origin and the number of holes to its left are both finite. A configuration $\eta \in \mathcal{X}$ has a rightmost particle located at $\mathrm{R}(\eta)$ and a leftmost hole located at $\mathrm{L}(\eta)$, where

$$\mathrm{R}(\eta) := \max\{x \in \mathbb{Z} : \eta(x) = 1\}; \qquad \mathrm{L}(\eta) := \min\{x \in \mathbb{Z} : \eta(x) = 0\}. \tag{2.1}$$

We define the *median* $\mathrm{M}(\eta)$ of a configuration $\eta \in \mathcal{X}$ as the unique $m \in \mathbb{Z} + \frac{1}{2}$ such that

$$\sum_{x > m} \eta(x) - \sum_{x < m} (1 - \eta(x)) = 0, \tag{2.2}$$

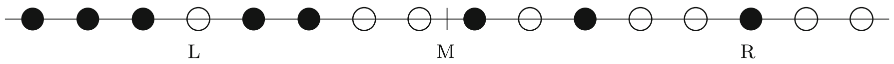

Fig. 1 A typical configuration in $\mathcal{X}$, *black* and *white circles* represent respectively particles and holes. R is the position of the rightmost particle, L the position of the leftmost hole and M is the median

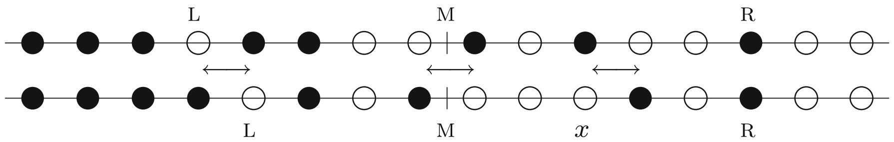

Fig. 2 The effect of three SSEP jumps: the *upper line* is before and the *bottom line* is after the jumps. The jumps are between the positions L and L + 1, $\mathrm{M} - \frac{1}{2}$ and $\mathrm{M} + \frac{1}{2}$ and x and $x + 1$. None of these jumps change the position of M

that is, the number of particles to the right of $\mathrm{M}(\eta)$ is the same as the number of holes to its left (Fig. 1). The definition is well-posed because for $\eta \in \mathcal{X}$, as m increases by one (2.2) increases by one and goes to $\pm\infty$ as m goes to $\mp\infty$.

We next define a family of dynamics indexed by a *current* $\mathrm{J} > 0$. The particle dynamics is a (countable state) Markov process on $\mathcal{X}$ whose generator is

$$L^{\mathrm{J}}_{\text{part}} = L_0 + L^{\mathrm{J}}_{\mathrm{R}} + L^{\mathrm{J}}_{\mathrm{L}}, \tag{2.3}$$

where

$$L_0 f(\eta) := \sum_{x \in \mathbb{Z}} \frac{1}{2} \Big[f(\eta^{x,x+1}) - f(\eta) \Big] \tag{2.4}$$

with $\eta^{x,x+1}(y) = \eta(y)$ if $y \neq x, x+1$, $\eta^{x,x+1}(x) = \eta(x+1)$, $\eta^{x,x+1}(x+1) = \eta(x)$; and

$$L^{\mathrm{J}}_{\mathrm{L}} f(\eta) := \mathrm{J}(f(\eta \cup \{\mathrm{L}(\eta)\}) - f(\eta)); \quad L^{\mathrm{J}}_{\mathrm{R}} f(\eta) := \mathrm{J}\left(f(\eta \backslash \{\mathrm{R}(\eta)\}) - f(\eta)\right), \tag{2.5}$$

where η is identified with the set of occupied sites $\{x \in \mathbb{Z} : \eta(x) = 1\}$.

In other words, particles perform symmetric simple exclusion with generator L_0 and, at rate J, the rightmost particle is "killed" and replaced by a hole and at the same rate independently the leftmost hole is killed and a particle is born at its place (Fig. 2). We omit the proof that the process is well defined at all times. Denote by (η_t) the process with generator $L^{\mathrm{J}}_{\text{part}}$.

Denote by B_t and A_t the number of particles killed, respectively born, in the time interval $[0, t]$. By definition, A_t and B_t are independent Poisson processes with rate J.

Lemma 1 *For any* $\eta_0 \in \mathcal{X}$,

$$\mathrm{M}(\eta_t) = \mathrm{M}(\eta_0) + A_t - B_t \tag{2.6}$$

That is, the marginal distribution of the median of η_t *is a continuous time symmetric nearest neighbor random walk on* $\mathbb{Z} + \frac{1}{2}$ *with rate* J *to jump right and* J *to jump left.*

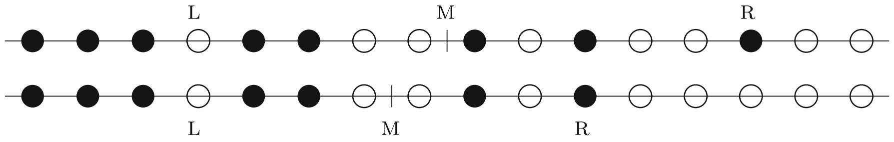

Fig. 3 The effect of killing the rightmost particle: the *upperline* is before the killing and the *bottom line* is after the killing. The position R of the rightmost particle moves to the *left* by 3 (for this configuration) and M moves by 1 to the *left* (for any configuration). Analogously, the killing of the leftmost hole moves M one unit to the *right* (not in the picture)

Proof Note that (a) jumps due to the exclusion dynamics do not change $\mathrm{M}(\cdot)$, (b) when the rightmost particle dies, $\mathrm{M}(\cdot)$ decreases by 1 and (c) when the leftmost hole dies (and a particle appears in its place), $\mathrm{M}(\cdot)$ increases by 1 (Fig. 3). This shows (2.6). □

3 The process as seen from the median

Since $\mathrm{M}(\eta_t)$ is a symmetric simple random walk the law of η_t cannot be tight but we show below that the process as seen from the median has a unique invariant measure. Let

$$\tilde{\eta}_t := \theta_{\mathrm{M}(\eta_t)-1/2}\eta_t,$$

where for $y \in \mathbb{Z}$, the translation $\theta_y : \mathcal{X} \to \mathcal{X}$ is the map $(\theta_y\eta)(x) = \eta(x-y)$. Clearly $\tilde{\eta}_t \in \mathcal{X}^0 := \{\eta \in \mathcal{X} : \mathrm{M}(\eta) = 1/2\}$ and we have that $(\tilde{\eta}_t)$ is a Markov process on $\mathcal{X}^0$ with generator

$$\tilde{L}^{\mathrm{J}}_{\text{part}} = L_0 + \tilde{L}^{\mathrm{J}}_{\mathrm{L}} + \tilde{L}^{\mathrm{J}}_{\mathrm{R}} \tag{3.1}$$

where for any $\eta \in \mathcal{X}^0$,

$$\tilde{L}^{\mathrm{J}}_{\mathrm{R}} f(\eta) := \mathrm{J}\,(f(\theta_{-1}(\eta\backslash\{\mathrm{R}(\eta)\})) - f(\eta))\,; \quad \tilde{L}^{\mathrm{J}}_{\mathrm{L}} f(\eta) := \mathrm{J}\,(f(\theta_1(\eta \cup \{\mathrm{L}(\eta)\})) - f(\eta)])$$

Before stating the result we introduce some notation. For $\eta \in \mathcal{X}$, define

$$N^0(\eta) := \sum_{x<\mathrm{R}(\eta)} (1-\eta(x)), \qquad N^1(\eta) := \sum_{x>\mathrm{L}(\eta)} \eta(x) \tag{3.2}$$

the number of holes in η to the left of the rightmost particle and the number of particles in η to the right of leftmost hole, respectively. Clearly

$$N^0 + N^1 = \mathrm{R} - \mathrm{L} + 1. \tag{3.3}$$

Let $\eta^0 \in \mathcal{X}^0$ be the Heaviside configuration given by

$$\eta^0(x) = \mathbf{1}\{x \le 0\}. \tag{3.4}$$

Define ψ on $\mathcal{X}$ by

$$\psi(\eta) := \sum_{x<y} (1 - \eta(x))\eta(y) \tag{3.5}$$

Observe that $\psi(\eta) < \infty$ for all $\eta \in \mathcal{X}$. If $\eta \in \mathcal{X}^0$, $\psi(\eta)$ is the number of jumps $10 \to 01$ needed to get from η^0 to η. In particular, $\psi(\eta^0) = 0$.

Theorem 3 *For any* $\mathrm{J} > 0$ *the process* $(\tilde{\eta}_t)$ *has a unique invariant measure* μ_{J} *on* $\mathcal{X}^0$ *and*

$$\mu_{\mathrm{J}}[\mathrm{R} - \mathrm{L} + 1] = \frac{1}{2\mathrm{J}} \tag{3.6}$$

Proof We show that ψ is a Lyapunov function for $(\tilde{\eta}_t)$ by computing $L^{\mathrm{J}}_{\mathrm{part}}\psi$. Since $\psi(\theta_x\eta) = \psi(\eta)$ for all x and η,

$$(\tilde{L}^{\mathrm{J}}_{\mathrm{L}} + \tilde{L}^{\mathrm{J}}_{\mathrm{R}})\psi(\eta) = (L^{\mathrm{J}}_{\mathrm{L}} + L^{\mathrm{J}}_{\mathrm{R}})\psi(\eta) = -\mathrm{J}(N^0(\eta) - N^1(\eta)) = -\mathrm{J}\,(\mathrm{R}(\eta) - \mathrm{L}(\eta) + 1) \tag{3.7}$$

the second identity holds because when the rightmost particle disappears, ψ decreases by N^0, the number of holes to the left of R. Analogously, when the leftmost hole disappears, ψ decreases by N^1, the number of particles to the right of L. The third identity follows from (3.3).

Since ψ increases by one when there is a transition $10 \to 01$ while it decreases by one due to the opposite transition:

$$L_0\psi(\eta) = \frac{1}{2}\sum_x \{\eta(x)(1 - \eta(x+1)) - (1 - \eta(x))\eta(x+1)\} = \frac{1}{2}, \tag{3.8}$$

because for any configuration $\eta \in \mathcal{X}$, the number of pairs 10 exceeds by one the number of pairs 01. The sums in (3.8) are finite for $\eta \in \mathcal{X}$.

Call ν_t the law of $\tilde{\eta}_t$ starting from a configuration $\eta \in \mathcal{X}^0$. For any $t \geq 0$ we have $\nu_t\psi < \infty$ and

$$\frac{1}{t}(\nu_t\psi - \nu_0\psi) = \frac{1}{t}\int_0^t \nu_s[L^{\mathrm{J}}_{\mathrm{part}}\psi]\,ds = \frac{1}{2} - \frac{\mathrm{J}}{t}\int_0^t \nu_s[\mathrm{R} - \mathrm{L} + 1]\,ds. \tag{3.9}$$

Hence, calling $\mu_t := \frac{1}{t}\int_0^t \nu_s\,ds$,

$$\mu_t[\mathrm{R} - \mathrm{L} + 1] \leq \frac{1}{2\mathrm{J}} + \frac{1}{\mathrm{J}t}\nu_0\psi. \tag{3.10}$$

Since for any c we have $\#\{\eta \in \mathcal{X}^0 : \mathrm{R} - \mathrm{L} + 1 \leq c\} < \infty$, then the family $\{\mu_t, t \geq 0\}$ of probabilities on $\mathcal{X}^0$ is tight on $\mathcal{X}^0$ and it has therefore a limit point μ. μ is stationary by construction and unique because the process is irreducible.

Identity (3.6) follows from (3.9) by replacing ν_t by μ, but we need first to show that $\mu\psi < \infty$. Introduce a new Lyapunov function ψ_2 on $\mathcal{X}$ bysetting

$$\psi_2(\eta) := \sum_{x<y<z} [1-\eta(x)][1-\eta(y)]\eta(z)$$

Then,

$$L_0\psi_2(\eta) = \frac{1}{2}\left(\sum_{x<y} [1-\eta(x)]\eta(y)[1-\eta(y+1)] - \sum_{x<y} [1-\eta(x)][1-\eta(y)]\eta(y+1)\right)$$
$$= \frac{1}{2}\sum_{x<y} [1-\eta(x)][\eta(y)-\eta(y+1)] = \frac{1}{2}\sum_{x} [1-\eta(x)]\eta(x+1)$$

and

$$L^{\mathrm{J}}_{\mathrm{R}}\psi_2(\eta) = -\mathrm{J}\sum_{x<y<\mathrm{R}} [1-\eta(x)][1-\eta(y)] = -\frac{\mathrm{J}}{2}N^0(\eta)[N^0(\eta)-1]$$
$$L^{\mathrm{J}}_{\mathrm{L}}\psi_2(\eta) = -\mathrm{J}\sum_{\mathrm{L}<y<z} [1-\eta(y)]\eta(z) = -\mathrm{J}\sum_{y<z} [1-\eta(y)]\eta(z) + \mathrm{J}\sum_{\mathrm{L}<z} \eta(z)$$
$$= -\mathrm{J}\psi(\eta) + \mathrm{J}N^1(\eta)$$

Let ν_t as defined before (3.9) and μ_t after it. Then $\nu_t\psi_2 < \infty$ and since $\sum_x [1-\eta(x)]\eta(x+1) \le N^0(\eta)$,

$$\frac{1}{t}(\nu_t\psi_2 - \nu_0\psi_2) \le -\frac{\mathrm{J}}{t}\int_0^t \nu_s\left[-\frac{N^0}{2\mathrm{J}} + \psi - N^1 + \frac{N^0}{2}(N^0-1)\right] ds$$

so that, since $N^0(N^0-1) \ge 0$,

$$\mu_t\psi \le \frac{1}{\mathrm{J}t}\nu_0\psi_2 + \mu_t\left[\frac{N^0}{2\mathrm{J}} + N^1\right] \le \text{Constant},$$

by (3.3) and (3.10). Then $\mu\psi < \infty$ and setting $\nu_0 = \mu$ in (3.9) we get (3.6). □

The theorem says that under the invariant measure the average distance between the rightmost particle and the leftmost hole is $(2\mathrm{J})^{-1}$. This is in agreement with the Fick's law (the analogue of the Fourier law for mass densities). In fact Fick's law states that the stationary current J flowing in a system of length ℓ when at the endpoints the densities are $\rho_\pm$ is:

$$J = -\frac{1}{2}\frac{\rho_+ - \rho_-}{\ell}$$

$1/2$ being the particle mobility. In our case $J = \mathrm{J}$, $\rho_+ = 1$ and $\rho_- = 0$ hence $\ell = (2\mathrm{J})^{-1}$. The validity of Fick's law in our case is however not completely obvious as the endpoints $\mathrm{R}(\eta_t)$ and $\mathrm{L}(\eta_t)$ depend on time.

4 The interface process

The interfaces Let the set of *vertices* be

$$\mathcal{V} := \{v = (v_1, v_2) \in \mathbb{Z} \times \mathbb{Z}^+ \text{ with } v_1 + v_2 \text{ even}\}. \tag{4.1}$$

and for a vertex $v = (v_1, v_2) \in \mathcal{V}$ let $V_v : \mathbb{Z} \to \mathbb{Z}$ be the *cone* with vertex v defined by

$$V_v(x) := |x - v_1| + v_2. \tag{4.2}$$

Define the space of *interfaces* as

$$\begin{aligned} &\mathcal{Y}_v := \left\{\xi \in \mathbb{Z}^{\mathbb{Z}} : |\xi(x) - \xi(x+1)| = 1,\ x \in \mathbb{Z};\ \#\{x : \xi(x) \neq V_v(x)\} < \infty\right\} \\ &\mathcal{Y} := \cup_{v \in \mathcal{V}} \mathcal{Y}_v \end{aligned} \tag{4.3}$$

That is, an interface in $\mathcal{Y}_v$ coincides with the cone V_v for all but a finite number of sites. Interfaces share the property "$x + \xi(x)$ even" which is conserved by the dynamics defined later. For an interface $\xi \in \mathcal{Y}$ define

$$\text{L}(\xi) := \sup\{x \in \mathbb{Z} : \xi(x - y) = \xi(x) + y \ \text{ for all} y \geq 0\} \tag{4.4}$$

$$\text{R}(\xi) := \inf\{x \in \mathbb{Z} : \xi(x + y) = \xi(x) + y \ \text{ for all} y \geq 0\}. \tag{4.5}$$

Any interface ξ coincides with a cone V_v outside the finite interval $(\text{L}(\xi), \text{R}(\xi))$. The vertex $v = v(\xi)$ is the following function of $\text{R} = \text{R}(\xi))$, $\text{L} = \text{L}(\xi)$, $\xi(\text{L})$ and $\xi(\text{R})$:

$$v_1 = v_1(\xi) := \frac{\xi(\text{L}) - \xi(\text{R})}{2} + \frac{\text{L} + \text{R}}{2}, \quad v_2 = v_2(\xi) := \frac{\xi(\text{L}) + \xi(\text{R})}{2} + \frac{\text{L} - \text{R}}{2}. \tag{4.6}$$

Correspondence between interfaces and particle configurations Given an interface ξ we say that there is a particle at x if $\xi(x+1) < \xi(x)$ and that there is a hole at x if $\xi(x+1) > \xi(x)$. This defines the map $D : \mathcal{Y} \to \mathcal{X}$ given by

$$\eta(x) \equiv D(\xi)(x) = \frac{1}{2} - \frac{\xi(x+1) - \xi(x)}{2} \tag{4.7}$$

The map is clearly surjective but it is not injective as D is invariant under uniform vertical shifts: $D(\xi + n) = D(\xi)$ for all $n \in \mathbb{Z}$. However the extremes of the non conic part of ξ correspond to the leftmost hole and the rightmost particle of $D(\xi)$ and the absise of the vertex of the cone containing ξ corresponds to the median of $D(\xi)$. More precisely:

Lemma 2 *For any $\xi \in \mathcal{Y}$,*

$$\text{L}(D(\xi)) = \text{L}(\xi), \quad \text{R}(D(\xi)) = \text{R}(\xi) - 1, \quad \text{M}(D(\xi)) = v_1(\xi) - 1/2.$$

Proof The first and second identities follows directly from the definitions. The third identity is trivially satisfied by any cone and its mapped particle configuration: $v_1(V_{(a,b)}) = a$ and $\text{M}(D(V_{(a,b)})) = a - \frac{1}{2}$. Any interface ξ in the cone $V_{(a,b)}$ can be attained from $V_{(a,b)}$ by a finite number of moves of the type:

$$\xi = (\ldots, z, z-1, z, \ldots) \to (\ldots, z, z+1, z, \ldots) = \xi' \tag{4.8}$$

The corresponding particle moves are

$$\eta = (\ldots, 1, 0, \ldots) \to (\ldots, 0, 1, \ldots) = \eta' \tag{4.9}$$

and

$$D(\xi) = \eta \quad \text{if and only if} \quad D(\xi') = \eta'. \tag{4.10}$$

Observing that $v(\xi) = v(\xi')$ and that $\text{M}(D(\xi)) = \text{M}(D(\xi'))$, we conclude that $D(\xi)$ has median $a - \frac{1}{2} = v_1(\xi)$ for all $\xi \in \mathcal{Y}_{(a,b)}$. □

Interface dynamics Let (ξ_t) be the Markov process on $\mathcal{Y}$ with generator

$$\begin{aligned}
\mathcal{L}^{\text{J}}_{\text{inter}} &:= \mathcal{L}_\Delta + \mathcal{L}^{\text{J}}_{\text{R}} + \mathcal{L}^{\text{J}}_{\text{L}}, \quad \text{with} \\
\mathcal{L}_\Delta f(\xi) &:= \frac{1}{2} \sum_{x \in \mathbb{Z}} \{f(\xi + \Delta_x \xi) - f(\xi)\}, \\
\mathcal{L}^{\text{J}}_{\text{R}} f(\xi) &:= \text{J} \left[f(\max\{\xi, V_{v(\xi)+(-1,1)}\}) - f(\xi) \right], \\
\mathcal{L}^{\text{J}}_{\text{L}} f(\xi) &:= \text{J} \left[f(\max\{\xi, V_{v(\xi)+(1,1)}\}) - f(\xi) \right],
\end{aligned} \tag{4.11}$$

where $\Delta_x \xi(y) := (\xi(x+1) + \xi(x-1) - 2\xi(x)) \mathbf{1}\{y = x\}$. The jumps of $\xi(x)$ due to $\mathcal{L}_\Delta$ occur only when $\xi(x)$ has the two neighbors at equal height. The jump is up by 2 if the neighboring heights are both above $\xi(x)$ and down by 2 if they are both below $\xi(x)$. $\mathcal{L}^{\text{J}}_{\text{R}}$ acts by changing the rightmost downward variation of ξ into an upward one with the interface to its right being a straight line with slope 1. The cone containing the updated interface is obtained from the previous cone by a translation up by 1 and left by 1; see Fig. 4. A symmetrical picture describes the action of $\mathcal{L}^{\text{J}}_{\text{L}}$. We denote by

$$A_t = \#(\text{jumps due to} \mathcal{L}^{\text{J}}_{\text{L}} \text{in}[0,t]); \qquad B_t = \#(\text{jumps due to} \mathcal{L}^{\text{J}}_{\text{R}} \text{in}[0,t]) \tag{4.12}$$

$A = (A_t)$ and $B = (B_t)$ are independent Poisson processes of intensity J.

The interface evolution induces via the map D the particle evolution described in Sect. 2.

Let (ξ_t) be the interface process with generator $\mathcal{L}^{\text{J}}_{\text{inter}}$ and starting interface ξ and define

$$\eta_t := D(\xi_t) \tag{4.13}$$

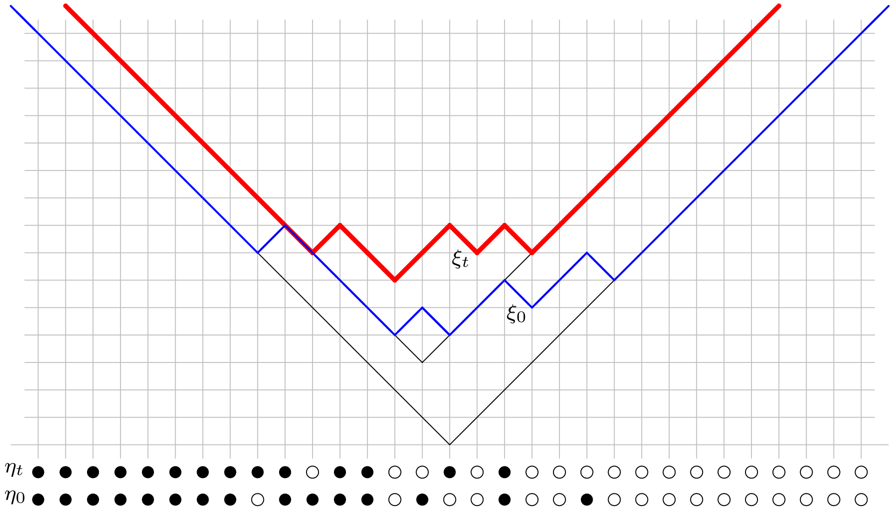

Fig. 4 The *thick red line* represents the interface ξ_t; its corresponding particle configuration is $\eta_t = D(\xi_t)$. The *narrow blue line* represents ξ_0, the interface at time 0; $\eta_0 = D(\xi_0)$. Two particles and one hole have been killed and several particles have moved due to the exclusion dynamics; in particular the third particle moved to the place originally occupied by the second one and the second hole has moved three units to the *left*. Due to the killings the vertex of the cone containing ξ_t has moved by $2(-1, 1) + 1(1, 1)$ which amounts to three units up and one unit *left* so that $\xi_0 \in \mathcal{Y}_{(0,0)}$ while $\xi_t \in \mathcal{Y}_{(-1,3)}$ (color figure online)

Lemma 3 *The particle process* (η_t) *defined by* (4.13) *is Markov with generator* $L^{\mathrm{J}}_{\text{part}}$, *defined in* (2.3). *Moreover, if* $\xi_0 \in \mathcal{Y}_{(0,0)}$, *then*

$$\xi_t(0) = 2B_t + 2\sum_{x\geq 0} \eta_t(x) \tag{4.14}$$

Proof It just follows from the definitions of (ξ_t) and D that $(D(\xi_t))$ is Markov with generator $L^{\mathrm{J}}_{\text{part}}$. Calling $\mathrm{R} \equiv \mathrm{R}(\xi_t) = \mathrm{R}(\eta_t) + 1$ we get from (4.7)

$$\xi_t(0) = \xi_t(\mathrm{R}) - \mathrm{R} + 2\sum_{x\geq 0} \eta_t(x)$$

From (4.6), $\xi_t(\mathrm{R}) = \mathrm{R} - (v_1 - v_2)$ and since $v_1 = A_t - B_t$ and $v_2 = A_t + B_t$, we get (4.14). □

Harris graphical construction We construct explicitly the interface process (ξ_t) as a function of the initial interface and of the Poisson processes governing the different jumps.

We construct first the process with generator $\mathcal{L}_\Delta$ and later use it to define the process with the moving boundary conditions. The probability space (Ω, P) is the product of independent rate-$\frac{1}{2}$ Poisson processes on $\mathbb{R}_+$ indexed by $\mathbb{Z} \times \{\uparrow, \downarrow\}$. A typical element of Ω is $\omega = (\omega_x^{\uparrow},\ \omega_x^{\downarrow},\ x \in \mathbb{Z})$. The Poisson points in $\omega_x^{\uparrow}$ and respectively $\omega_x^{\downarrow}$ are called up-arrows and down-arrows, respectively.

We define operators $T_t : \Omega \times \mathcal{Y} \to \mathcal{Y}$ with $t \geq 0$ where $(T_t(\omega, \xi))$ is the process with initial interface ξ using the arrows of ω, as follows. We drop the dependence on ω and write just $T_t\xi$, instead. We can take ω such that at most one arrow occurs at any given time.

Set $T_0\xi = \xi$. Assume that $\xi_{t'} := T_{t'}\xi$ is defined for all $t' \in [0, s]$.

Let t be the first arrow after s belonging to $\omega_x^{\uparrow} \cup \omega_x^{\downarrow}$ for some x such that $\xi_s(x+1) = \xi_s(x-1)$. Since $\xi \in \mathcal{Y}$, there are a finite number of such x and $t - s > 0$ a.s. These are the arrows *involved* in the evolution at time s.

Set

$$T_{t'}\xi = T_s\xi \text{ for } t' \in [s, t)$$

and (1) If t is an up-arrow, then the interface at x is set to $\xi_t(x-1) + 1$ no matter its value at $t-$ and does not change at the other sites:

$$T_t\xi(x) := T_{t-}\xi(x-1) + 1; \quad T_t\xi(y) := T_{t-}\xi(y),\ \ y \neq x.$$

(2) Analogously, if t is a down-arrow,

$$T_t\xi(x) := T_{t-}\xi(x-1) - 1; \quad T_t\xi(y) := T_{t-}\xi(y),\ \ y \neq x.$$

The reader can show that the process $T_t\xi$ so defined is Markov and evolves with the generator $\mathcal{L}_\Delta$ with initial interface ξ at time $t = 0$.

In the next definition we need to use the operator T_t in different time intervals. With this in mind we define

$$T_{[s,t]}(\omega, \xi) := T_{t-s}(\theta_{-s}\omega, \xi), \tag{4.15}$$

where $\theta_s\omega$ is the translation by s of the arrows in ω. That is, $T_{[s,t]}$ has the same distribution as T_{t-s} but uses the arrows in ω belonging to the interval $[s, t]$. We drop the dependence on ω in the notation and write simply $T_{[s,t]}\xi$.

Generalizing the boundary conditions Consider the partial order in the vertex space $\mathcal{V}$ given by

$$v \leq v' \quad \text{if } V_v(x) \leq V_{v'}(x) \text{ for all } x \in \mathbb{Z}, \tag{4.16}$$

so that vertex order corresponds to cone order. Let $z = (z_t)$ with $z_t \in \mathcal{V}$ be a non decreasing path of vertices with finite number of finite jumps in finite time intervals: $\|z_t - z_{t'}\| < \infty$ for all $t < t'$. Let $T_{[s,t]}$ be the family of random operators governing the $\mathcal{L}_\Delta$ motion, defined in (4.15). Typically z will be a function of the Poisson processes A and B which are independent of the arrows ω used to define the operators $T_{[s,t]}$. We abuse notation and call P the probability associated to ω, A and B.

Let $0 = s_0 < s_1 < \cdots$ be the times of jumps of z. Define iteratively $T_t^z\xi := \xi$ for $t < 0$ and

$$T_t^z\xi := \begin{cases} \max\{T_{t-}^z\xi, V_{z_t}\} & \text{if } t = s_n, \\ T_{[s_n,t]}T_{s_n}^z\xi & \text{if } t \in (s_n, s_{n+1}). \end{cases} \tag{4.17}$$

So (ξ_t^z) evolves with $\mathcal{L}_\Delta$ in the intervals (s_n, s_{n+1}) and at times s_n is updated to the maximum between the interface at time s_n- and the cone with vertex z_{s_n}.

Given an initial interface $\xi \in \mathcal{Y}$ and a number $\delta > 0$, we will consider the following choices for z_t, denoted by O_t, R_t and $z_t^{\delta,-}$ and $z_t^{\delta,+}$:

$$O_t := v(\xi), \qquad \text{for all } t; \tag{4.18}$$

$$R_t := v(\xi) + (A_t - B_t, A_t + B_t) \tag{4.19}$$

$$z_t^{\delta,-} := R_{n\delta}, \qquad z_t^{\delta,+} := R_{(n+1)\delta}, \qquad t \in [n\delta, (n+1)\delta), \quad n \geq 0. \tag{4.20}$$

When the path is (O_t), the cone does not move and the resulting process has generator $\mathcal{L}_\Delta$. When the path is (R_t) the process has generator $\mathcal{L}_{\text{inter}}^{\text{J}}$. The process with path $(z_t^{\delta,-})$ records the increments of (R_t) in the intervals $[n\delta, (n+1)\delta)$ and takes the maximum of the interface at the end of this interval and the cone with vertex $R_{n\delta}$. The process with path $(z_t^{\delta,+})$ records the increments of (R_t) in the intervals $[n\delta, (n+1)\delta)$ but takes the maximum between the interface at the beginning of this interval and the cone with center $R_{(n+1)\delta}$. The processes with paths (4.20) will be used later.

Monotonicity and attractivity Consider the natural partial order in $\mathcal{Y}$ given by: $\xi \leq \xi'$ if and only if $\xi(x) \leq \xi'(x)$ for all $x \in \mathbb{Z}$. Use this order to define stochastic order for random interfaces in $\mathcal{Y}$. If ξ, ξ' are random, ξ is stochastically dominated by ξ' if $Ef(\xi) \leq Ef(\xi')$ for non decreasing $f : \mathcal{Y} \to \mathbb{R}$. This is equivalent to the existence of a coupling $(\hat{\xi}, \hat{\xi}')$ whose marginals have the same distribution as ξ and ξ', respectively, such that $P(\hat{\xi} \leq \hat{\xi}') = 1$.

Let (ξ_t^1) and (ξ_t^2) be two realizations of a stochastic process (ξ_t) on $\mathcal{Y}$ with initial random interfaces ξ^1 and ξ^2, respectively. We say that the process (ξ_t) is *attractive* if the following holds:

$$\text{If } \xi^1 \leq \xi^2 \text{ stochastically, then } \xi_t^1 \leq \xi_t^2 \text{ stochastically, for all } t \geq 0. \tag{4.21}$$

Lemma 4 *If $\xi \leq \xi'$ then $T_t\xi \leq T_t\xi'$ almost surely. As a consequence, the process with generator $\mathcal{L}_\Delta$ is attractive.*

Proof Consider $\xi \leq \xi'$ and call $\xi_t = T_t\xi$, $\xi_t' = T_t\xi'$, assume that $\xi_{t-} \leq \xi_{t-}'$, $\xi_{t-}(x \pm 1) = \xi_{t-}'(x \pm 1)$ and that $t \in \omega_x^{\uparrow}$. Then $\xi_t(x) = \xi_t'(x) = \xi_{t-}(x-1) + 1$ no matter the values $\xi_{t-}(x)$ and $\xi_{t-}'(x)$. Analogous argument applies when $t \in \omega_x^{\downarrow}$. Hence an arrow does not change the order if the two interfaces coincide at $x \pm 1$. If at least one of the neighbors of x in ξ_{t-} is different of the corresponding neighbor in ξ_{t-}', a similar argument shows that no jump can break the domination. We have proven that if the ξ process is dominated by the ξ' process just before an arrow, then the domination persist after the jump(s) produced by the arrow. Since the set of involved arrows is finite in any time interval, an iterative argument concludes the proof. $\square$

Remark It is usual to realize the process with generator $\mathcal{L}_\Delta$ by introducing only one rate-$\frac{1}{2}$ Poisson process of marks ω_x, $x \in \mathbb{Z}$ associated to each x and updating $\xi(x) \to \xi(x-1) \pm 1 \neq \xi(x)$ whenever a ω_x-mark appears provided $\xi(x-1) = \xi(x+1)$. This is indeed a realization of the process, but order is not preserved: the jumps cross if two interfaces coincide at $x \pm 1$ and differ by 2 at x at the updating time of x. Using the up-arrows and down-arrows, only one of the interfaces jumps, and order is preserved.

Lemma 5 *If $\xi \in \mathcal{Y}$, then $v(T_t\xi) = v(\xi)$ a.s. As a consequence, $V_{v(\xi)} \leq T_t\xi$. Furthermore, if z is a non decreasing path with $v(\xi) = z_0$, then $v(T_t^z\xi) = z_t$, which implies $V_{z_t} \leq T_t^z\xi$.*

Proof By the definition of $(T_t\xi)$, any time L jumps due to an arrow, the opposite jump is performed by $\xi(\text{L})$ and analogously, any jump of R is replicated by $\xi(\text{R})$. This implies

$$\begin{aligned} \text{L}(T_t\xi) + \xi(\text{L}(T_t\xi)) &= \text{L}(\xi) + \xi(\text{L}(\xi)), \\ \text{R}(T_t\xi) - \xi(\text{R}(T_t\xi)) &= \text{R}(\xi) - \xi(\text{R}(\xi)) \end{aligned} \tag{4.22}$$

Putting these identities in the definition (4.6) of $v(\xi)$ and noting that the total number of jumps of L and R is a.s. finite (it is dominated by a Poisson process of rate 2), we get $v(T_t\xi) = v(\xi)$.

The fact that $v(T_t^z\xi) = z_t$ is true by definition at the z-events and by the first part of the lemma, it is true for $t \in [s, s')$ where s and s' are successive z-events. $\square$

Proposition 1 *Let z and z' be non decreasing paths in $\mathcal{V}$.*

$$\textit{If } z_t \leq z'_t \textit{ for all } t \geq 0 \textit{ and } \xi \leq \xi', \textit{ then } T_t^z\xi \leq T_t^{z'}\xi' \textit{ for all } t \geq 0 \quad \textit{a.s.}. \tag{4.23}$$

In particular, by taking $z = z'$, Proposition 1 says that for any non decreasing path z, the process $(T_t^z\xi)$ is attractive.

Proof Since the domination is preserved in intervals with no z or z' events by Lemma 4, we only need to check that the inequality $T_t^z\xi \leq T_t^{z'}\xi$ is preserved when z and z' events occur. We thus suppose the inequality is satisfied for all $s < t$ and this is evidently still true if t is a z'-event. If instead t is a z-event and not a z'-event, $T_{t-}^z\xi \leq T_{t-}^{z'}\xi'$ implies

$$T_t^z\xi = \max\{T_{t-}^z\xi, V_{z_t}\} \leq \max\{T_{t-}^{z'}\xi', V_{z_t}\} \leq \max\{T_{t-}^{z'}\xi', V_{z'_t}\} = T_t^{z'}\xi,$$

where the last inequality follows from $V_{z_t} \leq V_{z'_t}$ and the last identity from the last inequality in Lemma 5. $\square$

Let $\xi \in \mathcal{Y}$ and $v = (v_1, v_2) \in \mathcal{V}$. Define $\theta_v\xi$, the translation by v of ξ, by

$$\theta_v\xi(x) := \xi(x - v_1) - v_2.$$

Recall the order (4.16) in $\mathcal{V}$. Taking $v, v' \in \mathcal{V}$ and $\xi, \xi' \in \mathcal{Y}$, the following statement is immediate.

$$\text{If } v \geq v' \quad \text{and} \quad \xi \leq \xi', \text{ then } \theta_v \xi \leq \theta_{v'} \xi'. \tag{4.24}$$

Call $o := (0, 0)$ and take $v \in \mathcal{V}$ satisfying $v \geq o$ and $\xi \in \mathcal{Y}_o$, then

$$\theta_v \xi \leq \xi \quad \text{and} \quad \max\{\theta_v \xi, V_o\} \leq \xi. \tag{4.25}$$

The interface process as seen from the vertex Take $\xi \in \mathcal{Y}_o$ and let z be a non decreasing path of vertices. Define $(\tilde{T}_t^z \xi)$, the interface process as seen from the vertex, by

$$\tilde{T}_t^z \xi(x) := \theta_{z_t} T_t^z \xi(x), \tag{4.26}$$

Of course $\tilde{T}_t^z \xi \in \mathcal{Y}_o$.

Monotonicity We show that if the initial interface ξ' dominates ξ and any jump of z' is dominated by a jump of z, then the interface process as seen from z' dominates the one as seen from z. More precisely,

Proposition 2 *Let z and z' be non decreasing paths on $\mathcal{V}$ and $\xi \leq \xi'$ be interfaces in $\mathcal{Y}_o$.*

$$\textit{If } z_t - z_{t-} \geq z'_t - z'_{t-} \textit{ for all } t \geq 0, \textit{ then } \tilde{T}_t^z \xi \leq \tilde{T}_t^{z'} \xi' \textit{ for all } t \geq 0 \quad \textit{a.s.}. \tag{4.27}$$

Proof Since by Lemma 4, the domination is preserved in intervals with no z or z' events, it suffices to take care of those events. Assume that $\tilde{T}_{t-}^z \xi \leq \tilde{T}_{t-}^{z'} \xi'$ and that t is a z-event, then

$$\tilde{T}_t^z \xi = \theta_{z_t - z_{t-}} \tilde{T}_{t-}^z \xi \leq \theta_{z'_t - z'_{t-}} \tilde{T}_{t-}^{z'} \xi' = \tilde{T}_t^{z'} \xi', \tag{4.28}$$

where the inequality holds by (4.24). □

Let

$$z_t(\mathrm{J}) := R_t = (A_t - B_t, A_t + B_t), \tag{4.29}$$

recalling that A and B are independent Poisson processes of rate J. For each $\mathrm{J} > 0$ define the interface process $(\tilde{\xi}_t)$ by

$$\tilde{\xi}_t := \tilde{T}_t^{z(\mathrm{J})} \xi \tag{4.30}$$

Then, $(\tilde{\xi}_t)$ has generator is $\tilde{\mathcal{L}}^{\mathrm{J}}_{\text{inter}}$ given by

$$\tilde{\mathcal{L}}^{\mathrm{J}}_{\text{inter}} = \mathcal{L}_\Delta + \tilde{\mathcal{L}}^{\mathrm{J}}_{\mathrm{R}} + \tilde{\mathcal{L}}^{\mathrm{J}}_{\mathrm{L}} \tag{4.31}$$

where $\mathcal{L}_\Delta$ was defined in (4.11) and the other generators govern the updating with the maximum of the cone and the corresponding translation of the origin:

$$\begin{aligned}\tilde{\mathcal{L}}^{\mathrm{J}}_{\mathrm{R}} f(\xi) &:= \mathrm{J}\left[f(\max\{\theta_{(-1,1)}\xi, V_o\}) - f(\xi)\right],\\ \tilde{\mathcal{L}}^{\mathrm{J}}_{\mathrm{L}} f(\xi) &:= \mathrm{J}\left[f(\max\{\theta_{(1,1)}\xi, V_o\}) - f(\xi)\right].\end{aligned} \tag{4.32}$$

For $\xi \in \mathcal{Y}_o$, the process $(D(\tilde{\xi}_t))$ has the same law as the particle process $(\tilde{\eta}_t)$ defined in Sect. 3 with initial particle configuration $D(\xi) \in \mathcal{X}_0$. The map $D : \mathcal{Y}_0 \to \mathcal{X}_0$ (interfaces with vertex in the origin to particle configurations with median $-\frac{1}{2}$) is bijective. Since the process $(\tilde{\eta}_t)$ has a unique invariant measure on $\mathcal{X}_0$, $(\tilde{\xi}^z_t)$ has a unique invariant measure $\tilde{\mu}^{\mathrm{J}}$ on $\mathcal{Y}_0$.

Corollary 1 *The process $(\tilde{\xi}_t)$ is attractive. Furthermore, if $\tilde{\xi}_0 = V_o$ the law of $\tilde{\xi}_t$ is non decreasing in t, is stochastically dominated by the invariant measure $\tilde{\mu}^{\mathrm{J}}$ and converges to $\tilde{\mu}^{\mathrm{J}}$ as $t \to \infty$.*

Proof Attractivity follows by taking $z = z'$ in (4.27). As before use the notation $\tilde{T}^z_{[s,t]}\xi$ to indicate that the evolution uses the Poisson processes of the Harris construction in the time interval $[s, t]$ for both the interface evolution and the vertex evolution z. So that, for $s, t \geq 0$, $\tilde{T}^z_{t+s}\xi$ has the same law as $\tilde{T}^z_{[-s,t]}\xi$. Since V_o is minimal in $\mathcal{Y}_o$, using (4.27) we get almost surely

$$\tilde{T}^z_{[0,t]} V_o \leq \tilde{T}^z_{[0,t]} \tilde{T}^z_{[-s,0]} V_o = \tilde{T}^z_{[-s,t]} V_o.$$

This shows that the law of $\tilde{\xi}_t$ is stochastically non decreasing. Take a random ξ with law $\tilde{\mu}^{\mathrm{J}}$. Then,

$$\tilde{T}^z_{[0,t]} V_o \leq \tilde{T}^z_{[0,t]}\xi \sim \tilde{\mu}^{\mathrm{J}}.$$

by invariance of $\tilde{\mu}^{\mathrm{J}}$. The convergence of the law of $\tilde{\xi}_t$ to the unique invariant measure $\tilde{\mu}^{\mathrm{J}}$ is routine for countable state irreducible Markov processes. □

Corollary 2 *The invariant measures $\tilde{\mu}^{\mathrm{J}}$ for the interface processes $(\tilde{\xi}_t)$ are stochastically ordered:*

$$\textit{If } \mathrm{J} \geq \mathrm{J}',\ \textit{then } \tilde{\mu}^{\mathrm{J}} \leq \tilde{\mu}^{\mathrm{J}'}$$

Proof Take Poisson processes (A, A', B, B') such that (A, A') and (B, B') are independent. A and B have rate J while A' and B' have rate J' and $A \supset A'$, $B \supset B'$. In this way the vertex paths z and z' defined by (4.29) with (A, B) and (A', B'), respectively, satisfy the conditions of Proposition 2. This implies that $\tilde{T}^z_t V_o \leq \tilde{T}^{z'}_t V_o$ almost surely for all t. Like in Corollary 1, the coupled process $(\tilde{T}^z_t V_o, \tilde{T}^{z'}_t V_o)$ is stochastically non decreasing for the (partial) coordinatewise order and each coordinate is dominated by the respective invariant measure. This implies the existence of $\lim_{t\to\infty}(\tilde{T}^z_t V_o, \tilde{T}^{z'}_t V_o)$ in distribution. The coordinates of the limit are ordered and its marginals have distributions $\tilde{\mu}^{\mathrm{J}}, \tilde{\mu}^{\mathrm{J}'}$, respectively. □

5 Hydrodynamic limit

It is well known that in the diffusive scaling limit (space scaled as ε^{-1} and time as ε^{-2}) the hydrodynamic limit of the SSEP process alone converges to the linear heat equation:

$$\frac{\partial \rho_t}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho_t}{\partial r^2}, \quad \rho_0 = \rho \tag{5.1}$$

Since the distance between the first hole and last particle is random, it is not clear a-priori that our model should be scaled diffusively as well. From Theorem 3 we know that at equilibrium the mean distance between the rightmost particle and the leftmost hole is of order J^{-1} and this suggest that together with the above diffusive scaling we should scale J proportionally to ε. Indeed we will prove that under such scaling limit the density of the process converges as $\varepsilon \to 0$ to a deterministic evolution.

5.1 Results

Hydrodynamics of interfaces Initial configurations For each $\varepsilon > 0$ the interface evolution starts from an interface $\xi^{(\varepsilon)}$ such that:

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{Z}} |\varepsilon\xi^{(\varepsilon)}(x) - \phi_0(\varepsilon x)| = 0 \tag{5.2}$$

where $\phi_0(r) = r$ for all $r \ge \mathrm{R}(\phi) > 0$; $\phi_0(r) = -r$ for all $r \le \mathrm{L}(\phi) < 0$ and ϕ_0 is differentiable in $(\mathrm{L}(\phi), \mathrm{R}(\phi))$ with derivative ϕ_0' such that $\sup_{r\in(\mathrm{L}(\phi),\mathrm{R}(\phi))} |\phi_0'(r)| < 1$.

Fix the macroscopic current $j > 0$, define $\mathrm{J}(\varepsilon) := j\varepsilon$ and call

$$(\xi_t^{(\varepsilon)}) := \text{ interface process with generator } \mathcal{L}_\Delta + \mathcal{L}_{\mathrm{R}}^{j\varepsilon} + \mathcal{L}_{\mathrm{L}}^{j\varepsilon}, \text{ starting from } \xi^{(\varepsilon)} \tag{5.3}$$

The following is the hydrodynamic limit for the interface process.

Theorem 4 *There is a function $\phi_t(r)$, $t \ge 0$, $r \in \mathbb{R}$, so that for any $\gamma > 0$ and $t > 0$:*

$$\lim_{\varepsilon\to 0} P\left[\sup_{x\in\mathbb{Z}} |\varepsilon\xi^{(\varepsilon)}_{\varepsilon^{-2}t}(x) - \phi_t(\varepsilon x)| \ge \gamma\right] = 0 \tag{5.4}$$

The proof will be given in Sect. 5.5 and properties of $\phi_t(r)$ will be discussed in Sect. 6.

Hydrodynamics of particles As we shall prove at the end of Sect. 6.1, the hydrodynamic limit for the particle process Theorem 1 is a corollary of Theorem 4.

The proof that (5.1) is the hydrodynamic limit for the SSEP is quite simple because the correlation functions obey closed equations. The addition of birth-death processes spoils such a property but if the rates are cylinder functions and are "small" (births and deaths happen at rate ε^2) the proofs carry over and the limit is a reaction diffusion equation as in [2]. In our case the birth-death rates are not as small (because the killing rates are of order ε) but the main difficulty is that the killings are highly non local functions of the configuration, since births and deaths occur at the position of the leftmost hole and the rightmost particle, respectively. This spoils completely an analysis based on the study of the hierarchy of the correlation functions.

The way out is to use inequalities namely to sandwich the interface process between two *delta processes*, using Proposition 2. The corresponding delta particle process behave as the exclusion dynamics in macroscopic time intervals of length δ and the (accumulated) killings occur at the extremes of those intervals.

5.2 The delta processes

The delta interface process $(\xi_t^{\delta,\pm})$ is defined via (4.17) and (4.20) by

$$\xi_t^{\delta,\pm} := T_t^{z\pm}\xi, \text{ with } z\pm := z_t^{\pm\delta} \tag{5.5}$$

This process is obtained by patching together finitely many pieces of the evolution with generator $\mathcal{L}_\Delta$ as explained after (4.20). By Proposition 1, as

$$\begin{aligned} z_t^{\delta,-} \leq R_t \leq z_t^{\delta,+}, &\quad \text{for all } t \geq 0, \\ \text{then, } \xi_t^{\delta,-} \leq \xi_t \leq \xi_t^{\delta,+}, &\quad \text{for all } t \geq 0. \end{aligned} \tag{5.6}$$

The *delta particle processes* are defined by using the map (4.7), setting $\eta_t^{\delta,\pm} = D(\xi_t^{\delta,\pm})$. In other words, $(\eta_t^{\delta,-})$ evolves with the generator L_0 in the intervals $[n\delta,(n+1)\delta)$, and $\eta_{n\delta}^{\delta,-}$ is obtained from $\eta_{n\delta-}$ by removing its $B_{n\delta} - B_{(n-1)\delta}$ rightmost particles and its $A_{n\delta} - A_{(n-1)\delta}$ leftmost holes, where A and B are the independent Poisson processes with intensity J. The interpretation of the process $(\eta_t^{\delta,+})$ is analogous but the removal of particles and holes is done at the beginning rather than at the end of each time interval. For a particle configuration $\eta \in \mathcal{X}$ and positive integers a and b define the quantiles $\mathrm{L}^a(\eta)$ and $\mathrm{R}^b(\eta)$ as the lattice points satisfying

$$\sum_{x \geq \mathrm{R}^b(\eta)} \eta(x) = b, \qquad \sum_{x \leq \mathrm{L}^a(\eta)} (1-\eta(x)) = a. \tag{5.7}$$

and define

$$\Gamma^{a,b}(\eta)(x) = \eta(x)(1 - \mathbf{1}_{x \geq \mathrm{R}^b}) + (1-\eta(x))\mathbf{1}_{x \leq \mathrm{L}^a}; \tag{5.8}$$

this is the configuration obtained from η by erasing particles and holes as explained above.

We abuse notation and write $T_{[s,t]}\eta$ the evolution $D(T_{[s,t]}\xi)$, with $\eta = D\xi$. Then the delta particle processes satisfy the following: for $n = 0, 1, \ldots,$

$$
\begin{aligned}
\eta_t^{\delta,\pm} &= T_{[n\delta,t]}\eta_{n\delta}^{\delta,\pm}, && \text{for } t \in [n\delta,(n+1)\delta)\\
\eta_t^{\delta,-} &= \Gamma^{A_{n\delta},B_{n\delta}}\eta_{t-}^{\delta,-}, && \text{for } t = n\delta\\
\eta_t^{\delta,+} &= \Gamma^{A_{(n+1)\delta},B_{(n+1)\delta}}\eta_{t-}^{\delta,+}, && \text{for } t = n\delta
\end{aligned}
\tag{5.9}
$$

where recall A and B are independent Poisson processes of parameter J.

The rescaled delta processes We consider a family of processes indexed by ε by considering $\varepsilon^{-2}\delta$ instead of δ and εj instead of J. We call $(\xi_t^{\varepsilon,\delta,\pm})$ and $(\eta_t^{\varepsilon,\delta,\pm})$ the interface and particle processes so obtained. Later we consider those processes with time rescaled by a factor ε^{-2} and space by a factor ε^{-1}.

For any fixed (macroscopic) $\delta > 0$, we will prove the existence of $\phi_t^{\delta,\pm}$, the limit as $\varepsilon \to 0$ of $\varepsilon\xi_{t\varepsilon^{-2}}^{\varepsilon,\delta,\pm}$. We also prove that $\phi_t^{\delta,\pm}$ are close to each other and that their difference vanishes as $\delta \to 0$. Taking δ to zero, their common limit ϕ_t is then the hydrodynamic limit of the rescaled evolution $\xi_t^{(\varepsilon)}$ —as this is squeezed between $\xi_t^{\varepsilon,\delta,-}$ and $\xi_t^{\varepsilon,\delta,+}$.

While the above outline involves the interface process alone (which then implies convergence for the particle process as well), yet the analysis of the limit as $\varepsilon \to 0$ of $(\xi_t^{\varepsilon,\delta,\pm})$ is more conveniently studied by looking at the delta particle process $(\eta_t^{\varepsilon,\delta,\pm})$ and then translating the results to the delta interface process. We start from the particle model defining first the corresponding approximate macroscopic density delta evolutions and then prove existence of the hydrodynamic limit for the delta particle process.

5.3 The macroscopic delta evolutions

In this subsection we fix $\delta > 0$ and define the macroscopic delta evolutions of densities $\rho_t^{\delta,\pm}$ and interfaces $\phi_t^{\delta,\pm}$.

Preliminary results For any density $\rho : \mathbb{R} \to [0,1]$ define the $\mathbb{R}_+ \cup \{+\infty\}$ valued functions

$$
F(r;\rho) := \int_r^\infty \rho(r')\,dr', \quad \hat{F}(r;\rho) := \int_{-\infty}^r (1-\rho(r'))\,dr'
$$

representing the *mass* of ρ to the right of r and the *antimass* to its left. These are the macroscopic analogues of the number of particles, respectively holes, to the right, respectively left, of r. We introduce two disjoint subsets of densities called $\mathcal{R}$ and $\mathcal{U}$.

Let $\mathcal{R}$ be the set of densities $\rho \in L^\infty(\mathbb{R},[0,1])$ satisfying the following conditions:

(i) $F(0,\rho) = \hat{F}(0,\rho) < \infty$, that is mass to the right and antimass to the left are finite and the origin is the *median* of ρ.
(ii) ρ has *finite boundaries*. That is, there exist $-\infty < \mathrm{L}(\rho) \le \mathrm{R}(\rho) < \infty$ such that ρ has *finite support* $[\mathrm{L}(\rho),\mathrm{R}(\rho)]$: ρ put no mass to the right of $\mathrm{R}(\rho)$ and no antimass to the left of $\mathrm{L}(\rho)$.
(iii) ρ is *continuous in the interior of the support*. That is, if $\mathrm{L}(\rho) < \mathrm{R}(\rho)$ then $\rho(r)$ is continuous in $(\mathrm{L}(\rho),\mathrm{R}(\rho))$ with values in $(0,1)$.

We call $\mathcal{R}^*$ the set of $\rho \in \mathcal{R}$ such that $\mathrm{R}(\rho) > 0$.

Let $\mathcal{U} = \{u \in C(\mathbb{R}, (0, 1)) : F(0, u) = \hat{F}(0, u) < \infty\}$. This is the set of continuous densities with median 0.

Let $h \in \mathcal{R}$ be the *Heaviside density* defined by $h(r) = \mathbf{1}_{\{r \le 0\}}$. Clearly $\mathrm{L}(h) = \mathrm{R}(h) = 0$.

Lemma 6 *$F(r; u) < \infty$ and $\hat{F}(r; u) < \infty$ for all $r \in \mathbb{R}$ and $u \in \mathcal{U} \cup \mathcal{R}$.*

If $\rho \in \mathcal{R}$ then $\mathrm{R}(\rho) \ge 0$ and $\mathrm{L}(\rho) \le 0$; the two inequalities are strict unless $\rho = h$.

If $\mathrm{R}(\rho) > 0$ then $\mathrm{L}(\rho) < 0$ and the derivatives $F'(r; \rho)$ and $\hat{F}'(r; \rho)$ exist in $(\mathrm{L}(\rho), \mathrm{R}(\rho))$ where they are respectively strictly negative and positive.

If $u \in \mathcal{U}$ then F, $\hat{F} \in C^1(\mathbb{R})$ and $F'(r; u) < 0$ and $\hat{F}'(r; u) \ge 0$ for all $r \in \mathbb{R}$.

For any $u \in \mathcal{U} \cup \mathcal{R}$ and for any $\delta > 0$ there are unique points $\mathrm{R}^\delta(u)$, $\mathrm{L}^\delta(u)$ such that

$$\hat{F}(\mathrm{L}^\delta; u) = \delta; \qquad F(\mathrm{R}^\delta; u) = \delta. \tag{5.10}$$

If $F(0; u) \gtreqless \delta$ then $\mathrm{R}^\delta(u) \gtreqless 0$ and $\mathrm{L}^\delta(u) \lesseqgtr 0$.

Proof Let $u \in \mathcal{U} \cup \mathcal{R}$ then for any $r \in \mathbb{R}$

$$F(r; u) = \int_0^\infty u(r')\, dr' - \int_0^r u(r')\, dr' \le F(0; u) + |r| < \infty,$$

with an analogous argument showing that also $\hat{F}(r; u) < \infty$. If $\rho \in \mathcal{R}$ then $\mathrm{R}(\rho) \ge 0$ because if $\mathrm{R}(\rho) < 0$ then $F(0; \rho) = 0$. Since $F(0; \rho) = \hat{F}(0; \rho)$, then $\hat{F}(0; \rho) = 0$ and this gives a contradiction since $\mathrm{R}(\rho) < 0$ and $\rho \equiv 1$ is not allowed. Moreover $\mathrm{R}(\rho) = 0$ if and only if ρ is the Heaviside density because $F(0; \rho) = \hat{F}(0; \rho)$.

If $u \in \mathcal{U}$ then $F'(r; u) = -u(r) < 0$ and $\hat{F}'(r; u) = 1 - u(r) > 0$, by the definition of $\mathcal{U}$. If $\rho \in \mathcal{R}$ and $\mathrm{R}(\rho) > 0$ then $\mathrm{L}(\rho) < 0$ and by the definition of $\mathcal{R}$, ρ is continuous in $(\mathrm{L}(\rho), \mathrm{R}(\rho))$ and away from 0 and 1. Hence $F'(r; \rho) = -\rho(r) < 0$ and $\hat{F}'(r; \rho) = 1 - \rho(r) > 0$ for all $r \in (\mathrm{L}(\rho), \mathrm{R}(\rho))$.

Let $u \in \mathcal{U} \cup \mathcal{R}$. By the monotonicity of $F(r; u)$ if $F(0; u) > \delta$ then $\mathrm{R}^\delta(u) > 0$, while if $F(0; u) < \delta$ then $\mathrm{R}^\delta(u) < 0$ with the analogous property for $\mathrm{L}^\delta(u)$. □

Definition of Γ^δ and G_t. We call $\Gamma^\delta : \mathcal{U} \cup \mathcal{R} \to \mathcal{R}$ the following map. If $\mathrm{R}^\delta(u) \le 0$ and therefore $\mathrm{L}^\delta(u) \ge 0$ we set $\Gamma^\delta(u) = h$, the Heaviside density. If instead $\mathrm{R}^\delta(u) > 0$ and hence $\mathrm{L}^\delta(u) < 0$ we set $\rho = \Gamma^\delta(u)$ equal to 0 for $r > \mathrm{R}^\delta(u)$, equal to 1 for $r < \mathrm{L}^\delta(u)$ and equal to u elsewhere. In this latter case $\rho = \Gamma^\delta(u) \in \mathcal{R}^*$ (that is, $\mathrm{R}(\rho) = \mathrm{R}^\delta(u) > 0$). Thus Γ^δ acts by removing a portion δ of mass from the right of $\mathrm{R}^\delta(u)$ and put it back to the left of $\mathrm{L}^\delta(u)$.

Denote by G_t the Gaussian kernel:

$$G_t(r, r') := \frac{1}{\sqrt{2\pi t}} e^{-(r-r')^2/2t} \tag{5.11}$$

And write $G_t\rho(r) = \int dr' G_t(r, r')\rho(r')$. Recall that $G_t\rho$ is the solution of the heat equation (5.1) with initial data ρ.

Lemma 7 *Let $\rho \in \mathcal{R} \cup \mathcal{U}$. Then $G_t\rho \in \mathcal{U}$ for any $t > 0$. Moreover, calling*

$$\mathcal{U}_\delta = \left\{u \in \mathcal{U} : \mathrm{L}^\delta(u) < 0 < \mathrm{R}^\delta(u)\right\} \tag{5.12}$$

for any $j > 0$ there is $\delta(j) > 0$ so that for any $u \in \mathcal{R} \cup \mathcal{U}$ and $\delta < \delta(j)$, $G_\delta u \in \mathcal{U}_{j\delta}$ and therefore $\Gamma^{j\delta}(G_\delta u) \in \mathcal{R}^*$.

Proof Since $\rho \in \mathcal{R} \cup \mathcal{U}$, we have $F(0; \rho) < \infty$. Then for any $t > 0$:

$$\begin{aligned} F(0; G_t\rho) &= \int_0^\infty dr \int_{-\infty}^{+\infty} dr' G_t(r, r')\rho(r') \\ &\le \int_0^\infty dr \int_0^{+\infty} dr' G_t(r, r')\rho(r') + \int_0^\infty dr \int_{-\infty}^0 dr' G_t(r, r') \le F(0; \rho) + c, \end{aligned}$$

where we used Fubini and $\int G_t(r, r')\, dr' = 1$ to bound the first term. Since $G_t\rho \in [0, 1]$, $F(r; G_t\rho) < \infty$ for all r (see the beginning of the proof of Lemma 6). An analogous argument shows that $\hat{F}(r; G_t\rho) < \infty$ for all r and $t > 0$ and, being the solution of the heat equation, $G_t\rho \in C^\infty$ for all $t > 0$. To prove that $G_t\rho \in \mathcal{U}$ it remains to show that $F(0; G_t\rho) = \hat{F}(0; G_t\rho)$ for all $t > 0$. Using the symmetry properties of G_t and Fubini, for any $t \ge 0$ we have

$$F(0; \rho_t) - \hat{F}(0; \rho_t) = \int_0^\infty dr\, G_t\rho(r) - \int_{-\infty}^0 dr\, [1 - G_t\rho(r)] = 0.$$

The last statement in Lemma 7 follows from the following inequalities

$$\int_{\sqrt{\delta}}^\infty dr\, G_\delta u(r) \ge \int_{\sqrt{\delta}}^\infty dr\, G_\delta h(r) \ge C\sqrt{\delta}, \quad C > 0$$

which is larger than $j\delta$ for δ small enough. The first inequality follows from the fact that u is stochastically larger than the Heaviside density h, in the sense that u is obtained from h by moving mass to the right. The last inequality follows from direct computations. □

Delta density evolutions We are finally ready to define the delta density evolutions $\rho_t^{\delta,\pm}$. Restrict to $\delta \le \delta(j)$ as defined in Lemma 7, take the initial density $\rho \in \mathcal{R} \cup \mathcal{U}$ and define iteratively $\rho_0^{\delta,-} = \rho$ and

$$\rho_t^{\delta,-} := \begin{cases} G_{t-n\delta}\rho_{n\delta}^{\delta,-}, & t \in [n\delta, (n+1)\delta) \\ \Gamma^{j\delta}\rho_{t-}^{\delta,-}, & t = n\delta. \end{cases} \tag{5.13}$$

The evolution $\rho_t^{\delta,+}$ is defined as $\rho_t^{\delta,-}$ but with initial datum $\rho_0^{\delta,+} := \Gamma^{j\delta}(\rho)$.

Delta interface evolutions The delta interface evolutions are defined as follows. Fix an initial interface ϕ belonging to the cone with vertex at the origin and for $t \geq 0$ define iteratively $\phi_0^{\delta,-} = \phi$, $\phi_0^{\delta,+} = \max\{\phi, V_{(0,\delta j)}\}$ and for $n \geq 0$,

$$\begin{aligned} \phi_t^{\delta,\pm} &:= G_{t-n\delta}\phi_{n\delta}^{\delta,-}, && \text{if } t \in [n\delta, (n+1)\delta) \\ \phi_t^{\delta,-} &:= \max\{\phi_{t-}^{\delta,-}, V_{(0,n\delta j)}\}, && \text{if } t = n\delta. \\ \phi_t^{\delta,+} &:= \max\{\phi_{t-}^{\delta,+}, V_{(0,(n+1)\delta j)}\}, && \text{if } t = n\delta. \end{aligned} \tag{5.14}$$

We leave the proof of the following Lemma to the reader. It relates both definitions.

Lemma 8 *The delta density evolutions $\rho_t^{\delta,\pm}$ defined in* (5.13) *and the delta interface evolutions $\phi_t^{\delta,\pm}$ defined in* (5.14) *are related by*

$$\phi_t^{\delta,\pm}(r) - \phi_t^{\delta,\pm}(r') = 2(r - r') - 2\int_{r'}^{r} \rho_t^{\delta,\pm}(r'')\, dr''. \tag{5.15}$$

The initial data are related by $\phi(0) = \int_{-\infty}^{0}(1 - \rho(r))\, dr$ (this is the same as $\int_0^\infty \rho(r)dr$ as $\rho \in \mathcal{R}$); this fixes the vertex of the cone of ϕ at the origin.

5.4 Hydrodynamic limit of the delta particle process

We now study the hydrodynamic limit for the delta particle process defined in Sect. 5.2.

We introduce partitions $\mathcal{D}^{(\ell)}$ of $\mathbb{Z}$ into intervals $I^{(\ell)}$ of length ℓ where, denoting by $I_x^{(\ell)}$ the interval which contains x, $I_0^{(\ell)} = [0, \ell - 1]$ ($\mathcal{D}^{(\ell)}$ is now completely specified). We take ℓ equal to the integer part of $\varepsilon^{-\beta}$ with $\beta \in (0, 1)$, $\rho \in L^\infty(\mathbb{R}, [0, 1])$ and (by an abuse of notation) we write

$$\mathcal{A}_x^{(\ell)}(\eta) = \frac{1}{\ell}\sum_{y \in I_x^{(\ell)}} \eta(y), \quad \mathcal{A}_x^{(\ell)}(\rho) = \frac{1}{\varepsilon\ell}\int_{\varepsilon I_x^{(\ell)}} \rho(r)\, dr \tag{5.16}$$

not making explicit the dependence on ε.

Introduce an accuracy parameter of the form ε^α, $0 < \alpha < \beta$; the parameter $\beta \in (0, 1)$ is fixed while α will change at each step of the iteration scheme used in the sequel. Let $\mathcal{G}_{\varepsilon,\alpha,\beta}(\rho)$be the set of particle configurations which $(\varepsilon, \alpha, \beta)$-*recognize* the macroscopic density $\rho \in \mathcal{R}$ defined by:

$$\mathcal{G}_{\varepsilon,\alpha,\beta}(\rho) = \Bigg\{ \eta \in \mathcal{X} : |\varepsilon \mathrm{L}(\eta) - \mathrm{L}(\rho)| + |\varepsilon \mathrm{R}(\eta) - \mathrm{R}(\rho)| \\ \le \varepsilon^{\alpha}, \sup_{x \in \mathbb{Z} \setminus \{I_{\mathrm{R}} \cup I_{\mathrm{L}}\}} |\mathcal{A}_x^{(\ell)}(\eta) - \mathcal{A}_x^{(\ell)}(\rho)| \le \varepsilon^{\alpha} \Bigg\}, \\ \ell = \text{ integer part of } \varepsilon^{-\beta}, \quad 0 < \alpha < \beta < 1, \tag{5.17}$$

where $\mathrm{L}(\eta)$ and $\mathrm{R}(\eta)$ are defined in (2.1), and I_{L} is the smallest $\mathcal{D}^{(\ell)}$ measurable interval which contains both $\mathrm{L}(\eta)$ and $\varepsilon^{-1}\mathrm{L}(\rho)$; I_{R} is defined analogously with reference to $\mathrm{R}(\eta)$ and $\mathrm{R}(\rho)$.

Proposition 3 *Let $\rho \in \mathcal{R}$ and $\eta_0^{\varepsilon,\delta,\pm} \in \mathcal{G}_{\varepsilon,\alpha,\beta}(\rho)$, then for any $\alpha' \in (0,\alpha)$ such that $\alpha' < \min\{\frac{\beta}{2}, 1-\beta, \frac{1}{4}\}$ the following holds: for any $k \ge 1$ there are coefficients c_k so that*

$$P\left[\eta_{\varepsilon^{-2}\delta}^{\varepsilon,\delta,\pm} \in \mathcal{G}_{\varepsilon,\alpha',\beta}(\rho_\delta^{\delta,\pm})\right] \ge 1 - c_k \varepsilon^k \tag{5.18}$$

Remark By iteration the result extends to any finite macroscopic time interval and we also have:

Corollary 3 *Under the same assumptions of Proposition 3, for any integer $m \ge 1$ and for any $k \ge 1$ there are coefficients c_k so that*

$$P\left[\bigcap_{n=1}^{m} \{\eta_{\varepsilon^{-2}n\delta}^{\varepsilon,\delta,\pm} \in \mathcal{G}_{\varepsilon,\alpha',\beta}(\rho_{n\delta}^{\delta,\pm})\}\right] \ge 1 - c_k \varepsilon^k \tag{5.19}$$

To show Proposition 3 we need to control the position of the quantiles R^a and L^b of the process evolving with the exclusion by the macroscopic time $\delta\varepsilon^{-2}$. Here $a = A_{\varepsilon^{-2}t}$, $b = B_{\varepsilon^{-2}t}$ which are Poisson processes of parameter εj. These bonds only depend on the exclusion dynamics governed by L_0.

Sharp convergence of the exclusion process to the solution of the heat equation Abusing notation denote $(T_t\eta)$ the process in $\mathcal{X}$ with initial configuration η evolving only with the exclusion generator L_0. The SSEP evolution is close to a linear diffusion in the following sense: for any $n \ge 2$ there is c_n so that for any $t > 0$

$$\sup_{(x_1,\dots,x_n) \in \mathbb{Z}^n_{\neq}} |v(x_1,\dots,x_n;t)| \le c_n t^{-n/8} \tag{5.20}$$

where $\mathbb{Z}^n_{\neq}$ is the set of all n-tuple of mutually distinct elements of $\mathbb{Z}$,

$$v(x_1,\dots,x_n;t) = E\left[\prod_{i=1}^{n} \{T_t\eta(x_i) - u_t(x_i)\}\right] \tag{5.21}$$

hereafter called v-functions, and $u_t(x)$ solves the discretized heat equation

$$\frac{du_t(x)}{dt} = \frac{1}{2}\,\Delta u_t(x) = \frac{1}{2}\left(u_t(x+1) + u_t(x-1) - 2u_t(x)\right), \quad u_0 = \eta \tag{5.22}$$

(5.20) is proved in [4]. The solution of (5.22) is

$$u_t(x) = \sum_{y\in\mathbb{Z}} p_t(x,y)\eta(y)$$

$p_t(x,y)$ the transition probability kernel of the symmetric nearest-neighbors random walk. The solution of the heat equation starting from ρ is $G_t\rho$ (recall (5.11)). Thus, since $\eta \in \mathcal{G}_{\varepsilon,\alpha,\beta}(\rho)$

$$\begin{aligned} |u_{\varepsilon^{-2}t}(x) - G_t\rho(\varepsilon x)| &\le c'\left(\varepsilon t^{-1/2} + \varepsilon^{1-\beta}t^{-1/2} + \varepsilon^{\alpha} + \varepsilon t^{-1/2}\varepsilon^{-(1-\alpha)}\right) \\ &\le c\left(\varepsilon^{1-\beta}t^{-1/2} + \varepsilon^{\alpha}\right) \end{aligned} \tag{5.23}$$

The proof of the first inequality is done by changing $u_{\varepsilon^{-2}t}(x)$ into $G_t\rho(\varepsilon x)$ in successive steps:

- Replace $p_{\varepsilon^{-2}t}(x,y)$ by $G_t(\varepsilon x,\varepsilon y)$. By the local central limit theorem the error is bounded by the first term on the right hand side of (5.23).
- Replace $G_t(\varepsilon x,\varepsilon y)$ by its average in the intervals $\varepsilon I_z^{(\ell)}$ of length $\varepsilon^{1-\beta}$, hence the second term on the right hand side of (5.23).
- The contribution of the difference between averages of η and ρ in good intervals (i.e those not in $I_{\mathrm{L}} \cup I_{\mathrm{R}}$) is bounded by ε^{α}, the contribution of the intervals in $I_{\mathrm{L}} \cup I_{\mathrm{R}}$ by $\varepsilon t^{-1/2}\varepsilon^{-(1-\alpha)}$.
- We finally reconstruct in each interval $\varepsilon I_z^{(\ell)}$ the correct term from $G_t\rho(\varepsilon x)$ with an error given again by the second term on the right hand side of (5.23).

Bounds on $|\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(\rho_t)|$. For any $x \in \mathbb{Z}$

$$\begin{aligned} |\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(\rho_t)| &\le |\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(u_{\varepsilon^{-2}t})| + |\mathcal{A}_x^{(\ell)}(u_{\varepsilon^{-2}t}) - \mathcal{A}_x^{(\ell)}(\rho_t)| \\ &\le |\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(u_{\varepsilon^{-2}t})| + c\left(\varepsilon^{1-\beta}t^{-1/2} + \varepsilon^{\alpha}\right), \end{aligned} \tag{5.24}$$

by (5.23). We are going to show that for any integer n,

$$E\left[|\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(u_{\varepsilon^{-2}t})|^{2n}\right] \le c\left(\varepsilon^{\beta n} + [t\varepsilon^{-2}]^{-n/4}\right) \tag{5.25}$$

Proof of (5.25):

- We expand $|\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(u_{\varepsilon^{-2}t})|^{2n}$ getting a sum of products of factors $\eta_{\varepsilon^{-2}t}(z) - u_{\varepsilon^{-2}t}(z)$.

– Each term of the form $(\eta_{\varepsilon^{-2}t}(x) - u_{\varepsilon^{-2}t}(x))^k$ with $k > 1$ can be rewritten as $c + c'(\eta_{\varepsilon^{-2}t}(x) - u_{\varepsilon^{-2}t}(x))$ with constants c and c' not depending on η. c and c' depend on the value of $u_{\varepsilon^{-2}t}(x)$ but that each of them is always smaller (in absolute value) than one.
– Thus $E[|\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(u_{\varepsilon^{-2}t})|^{2n}]$ is a sum of product of constants times v-functions. We then use (5.20) to get (5.25).

Let $\gamma > 0$, then since $\eta \in \mathcal{G}_{\varepsilon,\alpha,\beta}(\rho)$,

$$\sum_{x \geq \varepsilon^{-1-\gamma}} P[T_{\varepsilon^{-2}t}\eta(x) = 1] \leq c'_k \varepsilon^k, \qquad \sum_{x \leq -\varepsilon^{-1-\gamma}} P[T_{\varepsilon^{-2}t}\eta(x) = 0] \leq c'_k \varepsilon^k$$

As a consequence

$$P\left[\mathrm{R}(T_{\varepsilon^{-2}t}\eta) \leq \varepsilon^{-1-\gamma};\ \ \mathrm{L}(T_{\varepsilon^{-2}t}\eta) \geq -\varepsilon^{-1-\gamma}\right] \geq 1 - c''_k \varepsilon^k \tag{5.26}$$

By the hypotheses on ρ,

$$\int_{r \geq \varepsilon^{-\gamma}} G_t\rho(r)\, dr \leq c'_k \varepsilon^k, \qquad \int_{r \leq -\varepsilon^{-\gamma}} dr[1 - G_t\rho(r)] \leq c'_k \varepsilon^k. \tag{5.27}$$

which proves that

$$P\left[\sup_{|x| \geq \varepsilon^{-1-\gamma}} |\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(G_t\rho)| \leq c'_k \varepsilon^k\right] \geq 1 - c'''_k \varepsilon^k. \tag{5.28}$$

We shall use (5.28) to prove that for any α' as in Proposition 3

$$P\left[\sup_{x \in \mathbb{Z}} |\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(G_t\rho)| \leq \varepsilon^{\alpha'}\right] \geq 1 - c_k \varepsilon^k. \tag{5.29}$$

By (5.28) and (5.24) it suffices to prove that for any α' as above

$$P\left[\sup_{|x| \leq \varepsilon^{-1-\gamma}} |\mathcal{A}_x^{(\ell)}(T_{\varepsilon^{-2}t}\eta) - \mathcal{A}_x^{(\ell)}(u_{\varepsilon^{-2}t})| \leq \varepsilon^{\alpha'}\right] \geq 1 - c_k \varepsilon^k. \tag{5.30}$$

which follows using the Chebishev's inequality with power $2n$ for n sufficiently large and (5.25), because $\alpha' < \min\{\frac{\beta}{2}, \frac{1}{4}\}$.
Quantile bounds To complete the proof of (5.18) we fix α' as in Proposition 3 and take $\alpha'' < \min\{\frac{\beta}{2}, 1-\beta, \alpha, \frac{1}{4}\}$ such that $\alpha'' > \alpha'$. Then there is a positive γ such that $\alpha'' - 2\gamma > \alpha'$.

We now fix $t = \delta$ and use (5.29) (with α'') to obtain bounds for the quantiles defined in (5.10) and (5.7). Recalling (5.10) let

$$\mathrm{R}' := \varepsilon^{-1}\mathrm{R}^{j\delta}(G_\delta\rho) + \varepsilon^{-1+\alpha''-2\gamma}.$$

Then

$$P\left(\left|\sum_{\mathrm{R}'\le x\le \varepsilon^{-1-\gamma}} T_{\varepsilon^{-2}\delta}\eta(x)-\varepsilon^{-1}\int_{\varepsilon\mathrm{R}'}^{\varepsilon^{-\gamma}} G_\delta\rho(r)\,dr\right|\le c\varepsilon^{\alpha''}\varepsilon^{-1-\gamma}\right)\ge 1-c_k\varepsilon^k,$$

On the other hand by the definition of the quantile $\mathrm{R}^{j\delta}(G_\delta\rho)$ and by (5.27)

$$\begin{aligned}\int_{\varepsilon\mathrm{R}'}^{\varepsilon^{-\gamma}} G_\delta\rho(r)\,dr &= \int_{\mathrm{R}^{j\delta}(G_\delta\rho)}^{\infty} G_\delta\rho(r)\,dr-\int_{\mathrm{R}^{j\delta}(G_\delta\rho)}^{\varepsilon\mathrm{R}'} G_\delta\rho(r)\,dr-\int_{\varepsilon^{-\gamma}}^{\infty} G_\delta\rho(r)\,dr\\ &\le j\delta-c'\varepsilon^{\alpha''-2\gamma}\end{aligned}$$

with $c'=\min\{G_\delta\rho(r):|r-\mathrm{R}^{j\delta}(G_\delta\rho)|\le 1\}>0$. Hence with probability $\ge 1-c_k\varepsilon^k$,

$$\varepsilon\sum_{\mathrm{R}'\le x\le\varepsilon^{-1-\gamma}} T_{\varepsilon^{-2}\delta}\eta(x)\le j\delta-c'\varepsilon^{\alpha''-2\gamma}+c\varepsilon^{\alpha''}\varepsilon^{-\gamma}<j\delta-\frac{c'}{2}\varepsilon^{\alpha''-2\gamma}$$

Let $\mathrm{R}^b(\eta)$ be the quantile defined in (5.7) with $b=B_{\varepsilon^{-2}\delta}$, B a Poisson process of rate $j\varepsilon$. Observing that for any $\kappa>0$

$$P\left(\left|\varepsilon B_{\varepsilon^{-2}\delta}-j\delta\right|\le\varepsilon^{\frac{1}{2}-\kappa}\right)\ge 1-c_k\varepsilon^k$$

we get that for κ small enough and with probability $\ge 1-c_k\varepsilon^k$,

$$\begin{aligned}\varepsilon\sum_{x\ge\mathrm{R}^b(T_{\varepsilon^{-2}\delta}\eta)} T_{\varepsilon^{-2}\delta}\eta(x)=\varepsilon B_{\varepsilon^{-2}\delta}\ge j\delta-\varepsilon^{\frac{1}{2}-\kappa}\ge j\delta-\frac{c'}{2}\varepsilon^{\alpha''-2\gamma}\\ \ge\varepsilon\sum_{\mathrm{R}'\le x\le\varepsilon^{-1-\gamma}} T_{\varepsilon^{-2}\delta}\eta(x)\end{aligned}$$

that implies $\mathrm{R}^b(T_{\varepsilon^{-2}\delta}\eta)\ge R'=\varepsilon^{-1}\mathrm{R}^{j\delta}(G_\delta\rho)+\varepsilon^{-1+\alpha''-2\gamma}$. Using an analogous argument for the lower bound we get

$$\begin{aligned}&P(|\varepsilon\mathrm{R}^b(T_{\varepsilon^{-2}\delta}\eta)-\mathrm{R}^{j\delta}(G_\delta\rho)|\le\varepsilon^{\alpha''-2\gamma})>1-c_k\varepsilon^k,\\ &P(|\varepsilon\mathrm{L}^a(T_{\varepsilon^{-2}\delta}\eta)-\mathrm{L}^{j\delta}(G_\delta\rho)|\le\varepsilon^{\alpha''-2\gamma})>1-c_k\varepsilon^k;\end{aligned}\tag{5.31}$$

the second inequality is proved by using the same arguments for $a=A_{\varepsilon^{-2}\delta}$, A being a Poisson process of rate εj.

Proof of Proposition 3 By the definitions (5.9) and (5.13),

$$\eta^{\varepsilon,\delta,-}_{\varepsilon^{-2}\delta}=\Gamma^{A_{\varepsilon^{-2}\delta},B_{\varepsilon^{-2}\delta}}T_{\varepsilon^{-2}\delta}\eta,\qquad \rho^{\delta,-}_\delta=\Gamma^{j\delta}G_\delta\rho.$$

Since the left and right boundaries after applying Γ are the quantiles before applying it, inequality (5.18) for the delta$-$ processes follows from (5.29) and (5.31). The same argument applies for the delta$+$ processes. □

5.5 Hydrodynamic limit of interfaces

Proof of Theorem 4 We call $\tau > 0$ the time t fixed in Theorem 4. For each $n \in \mathbb{N}$ we let $\delta \in \{\tau 2^{-n}\}$ and consider the evolutions $(\xi_t^{\varepsilon,\delta,\pm})$ in a bounded time interval, $t \le T = 2^{N+n}\delta = 2^N\tau$, N an arbitrary, fixed non negative integer. We have by (4.14),

$$\begin{aligned}
&\varepsilon\xi^{\varepsilon,\delta,\pm}_{\varepsilon^{-2}(k+1)\delta}(0) - \varepsilon\xi^{\varepsilon,\delta,\pm}_{\varepsilon^{-2}k\delta}(0)\\
&\quad = 2\varepsilon B_{\varepsilon^{-2}(k+1)\delta} - 2\varepsilon B_{\varepsilon^{-2}k\delta} + 2\varepsilon\sum_{x\ge 0}\eta^{\varepsilon,\delta,\pm}_{\varepsilon^{-2}(k+1)\delta}(x) - 2\varepsilon\sum_{x\ge 0}\eta^{\varepsilon,\delta,\pm}_{\varepsilon^{-2}k\delta}(x)
\end{aligned} \tag{5.32}$$

By (4.7) for $k = 1, \dots, 2^{n+N}$ and $x > y$,

$$\varepsilon\xi^{\varepsilon,\delta,\pm}_{\varepsilon^{-2}k\delta}(x) - \varepsilon\xi^{\varepsilon,\delta,\pm}_{\varepsilon^{-2}k\delta}(y) = 2\varepsilon(x-y) - 2\varepsilon\sum_{z=y}^{x-1}\eta^{\varepsilon,\delta,\pm}_{\varepsilon^{-2}k\delta}(z) \tag{5.33}$$

By (5.19) and (5.32)–(5.33)–(5.15) we then get that for any $\gamma > 0$ and any $t \in \{k\delta : k \le 2^{N+n}\}$

$$\lim_{\varepsilon\to 0} P\left[\sup_{x\in\mathbb{Z}} |\varepsilon\xi^{\varepsilon,\delta,\pm}_{\varepsilon^{-2}t}(x) - \phi^{\delta,\pm}_t(\varepsilon x)| \ge \gamma\right] = 0 \tag{5.34}$$

In the next section we shall prove that for any t:

$$\lim_{n\to\infty}\sup_{r\in\mathbb{R}}\left|\phi_t^{\tau 2^{-n},+}(r) - \phi_t^{\tau 2^{-n},-}(r)\right| = 0 \tag{5.35}$$

and that there is a function $\phi_t^{(\tau)}(r)$, $r \in \mathbb{R}$, $t \ge 0$, so that

$$\lim_{n\to\infty}\sup_{r\in\mathbb{R}}\left|\phi_t^{(\tau)}(r) - \phi_t^{\tau 2^{-n},-}(r)\right| = 0 \tag{5.36}$$

Then by (5.34), (5.35), (5.36) and (5.6), for all $\gamma > 0$,

$$\lim_{\varepsilon\to 0} P\left[\sup_{x\in\mathbb{Z}} |\varepsilon\xi^{(\varepsilon)}_{\varepsilon^{-2}t}(x) - \phi^{(\tau)}_t(\varepsilon x)| \ge \gamma\right] = 0, \quad t \in \mathcal{T}(\tau) \tag{5.37}$$

where

$$\mathcal{T}(\tau) = \{k2^{-n}\tau, k \in \mathbb{N}, n \in \mathbb{N}\} \tag{5.38}$$

Since $\tau \in \mathcal{T}(\tau)$ we have proved (5.4) for $t = \tau$ and since τ was arbitrary, Theorem 4 is proved. $\square$

6 The macroscopic evolution

In Sect. 6.1 we prove that as $\delta \to 0$ the macroscopic delta processes converge—that is, we prove (5.35) and (5.36)—and that ϕ_t is well defined by

$$\phi_t = \lim_{\delta \to 0} \phi_t^{\delta,\pm}, \qquad t \geq 0$$

We also collect some properties of the macroscopic evolutions ϕ_t and ρ_t, in particular monotonicity properties of ϕ_t and existence of boundary points for both motions. In Sect. 6.2 we construct macroscopic stationary profiles.

6.1 Existence and regularity of the macroscopic profiles

Proof of (5.35) *and* (5.36) Let $\tau > 0$ and $\delta \in \{\tau 2^{-n}, n \in \mathbb{N}\}$. We shall first prove by induction on k that for any such δ,

$$\sup_{r \in \mathbb{R}} \left| \phi_{k\delta}^{\delta,+}(r) - \phi_{k\delta}^{\delta,-}(r) \right| \leq j\delta. \tag{6.1}$$

(6.1) holds for $k = 0$ because

$$\phi_0^{\delta,+}(r) = \max\{\phi_0(r), j\delta + |r|\}, \quad \phi_0^{\delta,-}(r) = \phi_0(r).$$

Suppose next that (6.1) holds for $k - 1$, then by the maximum principle (for the linear heat equation),

$$\left| \phi_{(k\delta)-}^{\delta,+}(r) - \phi_{(k\delta)-}^{\delta,+}(r) \right| \leq j\delta$$

hence (6.1) holds for k because

$$\phi_{k\delta}^{\delta,+}(r) = \max\left\{\phi_{(k\delta)-}^{\delta,+}(r), j(k+1)\delta + |r|\right\}, \quad \phi_{k\delta}^{\delta,-}(r) = \max\left\{\phi_{(k\delta)-}^{\delta,-}(r), jk\delta + |r|\right\}$$

(6.1) and (5.35) are thus proved. It is not difficult to see that

$$\phi_t^{\delta,-}(r) \leq \phi_t^{\delta',-}(r), \quad \phi_t^{\delta,+}(r) \geq \phi_t^{\delta',-}(r), \qquad \delta = k\delta' \text{ for some integer } k > 0$$

Thus for any $n \in \mathbb{N}$ and $t \geq 0$

$$\phi_t^{\tau 2^{-n},-}(r) \leq \phi_t^{\tau 2^{-(n+1)},-}(r) \leq \phi_t^{\tau 2^{-n},+}(r) \tag{6.2}$$

Hence for any fixed t, $\phi_t^{\tau 2^{-n},-}(r)$ converges pointwise to a function that we call $\phi_t^{(\tau)}(r)$ (it may depend on τ). Since by definition $|\phi_t^{\delta,\pm}(r) - \phi_t^{\delta,\pm}(r')| \le |r - r'|$, the convergence is in sup norm and (5.36) is then proved with $\phi_t^{(\tau)}(r)$ a Lipschitz function with Lipschitz constant 1. □

In the next Theorem we prove that $\phi_t^{(\tau)}$ is independent of τ and also regularity properties of this function.

Theorem 5 *The function $\phi_t^{(\tau)}$ is independent of τ and will be denoted by ϕ_t (the same as in Theorem 4). ϕ_t is continuous in r and t, more precisely there is $c > 0$ so that for all t, t' such that $|t - t'| \le 1$ and all r and r',*

$$|\phi_t(r) - \phi_{t'}(r)| \le c\sqrt{|t - t'|}, \quad |\phi_t(r) - \phi_t(r')| \le |r - r'| \tag{6.3}$$

Denoting by δ_n any sequence of positive numbers such that $\delta_{n+1} = \delta_n/2$ then

$$\phi_t = \lim_{n\to\infty} \phi_t^{\delta_n,\pm}, \qquad \forall t \ge 0, \tag{6.4}$$

with $\phi_t^{\delta_n,-}$ monotonically increasing and $\phi_t^{\delta_n,+}$ monotonically decreasing.

Proof Let $t \ge 0$ and $s > 0$, recalling (5.11) we have

$$G_s\phi_t^{\delta,-}(r) \le \phi_{t+s}^{\delta,-}(r) \le G_s\phi_t^{\delta,-}(r) + j(s+\delta). \tag{6.5}$$

The first inequality is obvious. We have

$$\begin{aligned}\phi_{(k+1)\delta}^{\delta,-} &= \max\{G_\delta\phi_{k\delta}^{\delta,-}, j(k+1)\delta + |r|\} \le \max\{G_\delta\phi_{k\delta}^{\delta,-} + j\delta, j(k+1)\delta + |r|\}\\ &= j\delta + G_\delta\phi_{k\delta}^{\delta,-},\end{aligned}$$

because $\phi_{k\delta}^{\delta,-} \ge jk\delta + |r|$. We then get the last inequality (without the term $j\delta$) for $t = h\delta$, $h \in \mathbb{N}$. If instead $t \in (h\delta, (h+1)\delta)$, then

$$\phi_{(h+1)\delta}^{\delta,-} \le G_{(h+1)\delta - t}\phi_t^{\delta,-} + j\delta$$

hence (6.5).

Since $\phi_t^{\delta,-}(r)$ is Lipschitz, it follows from (6.5) that $|\phi_{t+s}^{\delta,-}(r) - \phi_t^{\delta,-}(r)| \le c\sqrt{s} + j(s + j\delta)$ and, by taking $\delta \to 0$,

$$|\phi_t^{(\tau)}(r) - \phi_{t'}^{(\tau)}(r)| \le c\sqrt{|t - t'|}, \quad \text{for all } r \text{ and } |t - t'| \le 1 \tag{6.6}$$

We shall next prove that $\phi_t^{(\tau)}(r)$ is independent of τ. Obviously $\phi_t^{(\tau)}(r) = \phi_t^{(\tau')}(r)$ if, recalling (5.38), $\tau' \in \mathcal{T}(\tau)$ (or viceversa). We next suppose that τ and τ' are not related in such a way. We fix $T > 0$ and want to prove that $\phi_T^{(\tau)}(r) = \phi_T^{(\tau')}(r)$. Let

$\delta \le \delta'$ and k such that $k\delta < \delta' < (k+1)\delta$. Then

$$\begin{aligned}\phi_{\delta'}^{\delta',-}(r) &= \max\{G_{\delta'-k\delta}\phi_{k\delta}^{\delta',-}(r), j\delta' + |r|\}\\ &\le \max\{G_{\delta'-k\delta}\phi_{k\delta}^{\delta,-}(r) + j(\delta'-k\delta), j\delta' + |r|\} \le \phi_{\delta'}^{\delta,-}(r) + j(\delta'-k\delta)\end{aligned}$$

because $\phi_{k\delta}^{\delta,-}(r) \ge jk\delta + |r|$.
By iteration $\phi_T^{\delta',-} \le \phi_T^{\delta,-} + jN\delta$ if N is the cardinality of $\{k : k\delta' \le T\}$. Thus

$$\phi_T^{\delta',-} \le \phi_T^{\delta,-} + cT\frac{\delta}{\delta'}$$

Take $\delta' = \tau' 2^{-n'}$ and $\delta = \tau 2^{-n}$. Take first $n \to \infty$ and then $n' \to \infty$ to get $\phi_T^{(\tau')} \le \phi_T^{(\tau)}$. The opposite inequality holds as well by interchanging δ and δ' in the previous argument. □

Proof of Theorem 1 From (5.4) we get for all $\gamma > 0$

$$\lim_{\varepsilon\to 0} P\left(\sup_{a<b}\left|\varepsilon \sum_{\varepsilon x\in[a,b]} \eta^{(\varepsilon)}_{t\varepsilon^{-2}}(x) - \frac{1}{2}\{\varepsilon(b-a) - [\phi_t(\varepsilon b) - \phi_t(\varepsilon a)]\}\right| > \gamma\right) = 0$$

Since ϕ_t is Lipschitz there is $\rho_t \in L^1$ such that, given any $r_0 \in \mathbb{R}$,

$$\phi_t(r) = \phi_t(r_0) + \int_{r_0}^{r} \left(1 - 2\rho_t(r')\right) dr' \tag{6.7}$$

and since by (6.3) the Lipschitz constant is 1, ρ_t has (almost surely) values in [0, 1] and this proves (1.2). We prove in Theorem 7 later that $\rho_t \in \mathcal{R}$, i.e. $-\infty < \mathrm{L}(\rho_t) \le \mathrm{R}(\rho_t) < \infty$. □

Proof of Theorem 2 For any t and δ,

$$\int_a^b \rho_t^{\delta,-}(r)\,dr - \int_a^b \rho_t(r)\,dr = \frac{1}{2}\left(\phi_t(b) - \phi_t^{\delta,-}(b) + \phi_t^{\delta,-}(a) - \phi_t(a)\right) \tag{6.8}$$

Let $\delta_n := 2^{-n}\delta$, then from (6.4), for all r

$$\phi_t^{\delta_n,-}(r) \le \phi_t(r) \le \phi_t^{\delta_n,+}(r) \tag{6.9}$$

so that from (6.1) and (6.8)

$$\int_a^b \rho_t^{\delta_n,-}(r)\,dr - \int_a^b \rho_t(r)\,dr \le \frac{1}{2}\left(\phi_t^{\delta_n,+}(b) - \phi^{\delta},\!-_t(b) + \phi_t^{\delta,-}(a) - \phi_t^{\delta_n,-}(a)\right) \le j\delta \tag{6.10}$$

□

6.2 Stationary solutions

We say that a macroscopic interface $\phi \in V_0$ is *stationary* if, $\phi_0 = \phi$ implies $\phi_t = \phi + 2jt$. A macroscopic density $\rho \in \mathcal{R}$ is *stationary* if $\rho_0 = \rho$ implies $\rho_t = \rho$. Here ϕ_t and ρ_t are the dynamics given by Theorems 4 and 1, respectively.

If ϕ is stationary, then the density ρ associated to ϕ via (6.7) is stationary because by (6.7)

$$\int_{r_0}^{r} \left(1 - 2\rho_t(r')\right)\,dr' = \int_{r_0}^{r} \left(1 - 2\rho_0(r')\right)\,dr', \quad \text{for all } r_0,\ r \text{ and } t \ge 0$$

Let

$$\bar\rho(r) := \begin{cases} 0, & \text{for } r \ge \frac{1}{4j} \\ \frac{1}{2} - 2jr, & \text{for } |r| \le \frac{1}{4j} \\ 1, & \text{for } r \le -\frac{1}{4j} \end{cases} \qquad \bar\phi(r) := \begin{cases} 2jr^2 + \frac{1}{8j}, & \text{for } |r| \le \frac{1}{4j} \\ |r|, & \text{for } r \ge \frac{1}{4j}. \end{cases} \tag{6.11}$$

Theorem 6 *The macroscopic interface $\bar\phi$ and the associated macroscopic density $\bar\rho$ are stationary.*

If $\xi_0^{(\varepsilon)}$ approximates $\bar\phi$ in the sense of (5.2), then the theorem says that the rescaled interface process as seen from its vertex converges in the sense of Theorem 4 at any macroscopic time t to the initial value $\bar\phi$ shifted by $2jt$. An analogous statement holds for the particle process (but the stationary density profile does not move).

Theorem 6 is proven in the next section by introducing a deterministic *harness process* on $\mathbb{R}^{\mathbb{Z}}$, a discrete time process that approaches ϕ_t and whose stationary solution is directly computable.

6.3 Monotonicity

We collect some monotonicity properties of the macroscopic interface inherited from the microscopic dynamics. We tacitly suppose hereafter that the initial data $\phi \in V_o$, namely that $\phi(r) = V_o(r) \equiv |r|$ for all $|r|$ large enough.

The following lemma is a direct consequence of the definition of ϕ_t.

Lemma 9 *For any $t > 0$, $\phi_t(r) \geq (|r| + jt)$, $r \in \mathbb{R}$, and*

$$\lim_{|r|\to\infty} |\phi_t(r) - (|r| + jt)| = 0$$

Proof Let $t = \tau k 2^{-n}$. It follows from the inequality $\phi_t^{\tau 2^{-n},-} \leq \phi_t \leq \phi_t^{\tau 2^{-n},+}$ that there is $R = R_{n,\tau}$ so that

$$|r| + jt \leq \phi_t(r) \leq |r| + j(t + 2^{-n}), \quad |r| \geq R.$$

□

We shall next establish inequalities relating evolutions with different values of j, we thus add a superscript j writing $\phi_t^{(j)}$ and $\phi_t^{(j,\delta,\pm)}$.

Lemma 10 *Let $j \leq j'$ then for all $t \geq 0$*

$$\phi_t^{(j)} - jt \geq \phi_t^{(j')} - j't. \tag{6.12}$$

Proof It is enough to prove that

$$\phi_\delta^{(j,\delta,-)}(r) - j\delta \geq \psi_\delta^{(j',\delta,-)}(r) - j'\delta, \quad \text{if } \phi \geq \psi$$

Let L', R' be such that

$$\psi_\delta^{(j',\delta,-)}(r) = G_\delta \psi(r), \quad \mathrm{L}' \leq r \leq \mathrm{R}', \quad \text{and } \psi_\delta^{(j',\delta,-)}(r) = |r| + j'\delta \text{ elsewhere}$$

By the maximum principle $G_\delta \phi \geq G_\delta \psi$, then

$$\phi_\delta^{(j,\delta,-)}(r) \geq G_\delta \phi(r) \geq \psi_\delta^{(j',\delta,-)}(r), \quad \mathrm{L}' \leq r \leq \mathrm{R}'$$

and a fortiori:

$$\phi_\delta^{(j,\delta,-)}(r) \geq \psi_\delta^{(j',\delta,-)}(r) - (j' - j)\delta, \quad \mathrm{L}' \leq r \leq \mathrm{R}'$$

By definition $\phi_\delta^{(j,\delta,-)}(r) \geq |r| + j\delta, \quad r \in \mathbb{R}$ so that

$$\phi_\delta^{(j,\delta,-)}(r) \geq |r| + j\delta = \psi_\delta^{(j',\delta,-)}(r) - (j' - j)\delta, \quad r \notin (\mathrm{L}', \mathrm{R}')$$

which concludes the proof. □

6.4 Existence of boundaries

Recall the definition in Sect. 5.3 of the boundaries $\mathrm{L}(\rho), \mathrm{R}(\rho)$ of a density $\rho \in \mathcal{R}$. They are also the boundaries of the interface ϕ_t which corresponds to ρ_t.

Theorem 7 *The boundaries* $\mathrm{L}(\rho_t)$, $\mathrm{R}(\rho_t)$ *of a density* ρ_t *as defined in Theorem* 1 *starting from* $\rho \in \mathcal{R}$ *are finite. In other words,* $\rho \in \mathcal{R}$ *implies* $\rho_t \in \mathcal{R}$.

Proof We shall prove the theorem in the framework of interfaces. We thus want to prove that the boundary points of the interface are finite, that is, $-\infty < \mathrm{L}(\phi_t^{(j)}), \mathrm{R}(\phi_t^{(j)}) < \infty$ for initial $\rho \in \mathcal{R}$ and all $t \geq 0$. Let $\bar{\phi}^{(j')}$ be the stationary interface for the j'-evolution. If $j' < j$ is small enough, $\phi \leq \bar{\phi}^{(j')}$ so that, by Lemma 10,

$$\phi_t^{(j)} - jt \leq \bar{\phi}_t^{(j')} - j't = \bar{\phi}^{(j')}.$$

This implies that the boundary points of $\phi_t^{(j)}$ are bounded by those of $\bar{\phi}^{(j')}$ and the theorem is proved. □

7 The harness process

We consider now the (deterministic) Harness Process proposed by Hammersley [8] with "moving cone" boundary conditions and prove that with the diffusive scaling this process also converges to the macroscopic evolution ϕ.

Let $H : \mathbb{Z} \to \mathbb{R}$ and define the operator Θ by

$$(\Theta H)(x) = \frac{H(x-1) + H(x+1)}{2} \tag{7.1}$$

but to keep notation light we drop the parentheses and write $\Theta H(x)$. Let $(H_n(x),\ x \in \mathbb{Z},\ n \in \mathbb{Z}^+)$, $H_n(x) \in \mathbb{R}$ be the deterministic process satisfying the discrete heat equation:

$$H_{n+1}(x) := \Theta H_n(x) = \Theta^{n+1} H_0(x) \tag{7.2}$$

Here n is time and x is space.

Duality Let X_n^x be a symmetric nearest neighbors random walk on $\mathbb{Z}$ with $X_0^x = x$ and $p_n(x, y) := P(X_n^x = y)$ the probability that the walk goes from x to y in n steps. Then, a simple recurrence shows that

$$H_n(x) = \sum_{y \in \mathbb{Z}} p_n(x, y) H_0(y) = E(H_0(X_n^x)) \tag{7.3}$$

Traveling wave solutions A family of traveling wave solutions of this process are

$$\bar{H}(x) := ax^2 + b,$$

where a and b are arbitrary constants. Indeed,

$$\bar{H}_n := \Theta^n \bar{H} = \bar{H} + 2an. \tag{7.4}$$

Moving boundaries For $v = (v_1, v_2) \in \mathbb{Z} \times \mathbb{R}$, let $V_v : \mathbb{Z} \to \mathbb{R}$ be the cone defined by $V_v(x) = |x - v_1| + v_2$; call v the vertex of V_v. Let

$$\mathcal{H}_v := \{K : \mathbb{Z} \to \mathbb{R} \,:\, K(x) = V_v(x) \text{ for all but a finite number of } x \in \mathbb{Z}\}$$
$$\mathcal{H} := \cup_{v \in \mathbb{Z}\times\mathbb{R}} \mathcal{H}_v$$

For $K \in \mathcal{H}_v$ define

$$\text{L}(K) := \min\{\ell \in \mathbb{Z} \,:\, K(\ell+1) \neq V_v(\ell+1)\}$$
$$\text{R}(K) := \max\{\ell \in \mathbb{Z} \,:\, K(\ell-1) \neq V_v(\ell-1)\}$$

The harness process with moving cone boundary conditions and initial interface $K_0^{\text{J}} \in \mathcal{H}_{(0,0)}$ is defined for $n \geq 1$ by

$$K_n^{\text{J}}(x) := \max\{\Theta K_{n-1}^{\text{J}}(x), V_{(0,2\text{J}n)}(x)\}. \tag{7.5}$$

So that $K_n^{\text{J}} \in \mathcal{H}_{(0,2\text{J}n)}$.

A travelling wave solution $\bar{K}$ of this process is associated to $\bar{H}$: if the initial interface is given by

$$\bar{K}_0^{\text{J}}(x) := \begin{cases} \frac{1}{8\text{J}} + 2\text{J}x^2, & |x| \leq \frac{1}{4\text{J}} \\ |x|, & |x| \geq \frac{1}{4\text{J}} \end{cases} \tag{7.6}$$

then the evolution (7.5) at time n gives a translation of the initial interface:

$$\bar{K}_n^{\text{J}}(x) = \bar{K}_0^{\text{J}}(x) + 2\text{J}n. \tag{7.7}$$

The sides of the cone $y = |x|$ are tangent to the parabola $y = \frac{1}{8\text{J}} + 2\text{J}x^2$ at the points $(\frac{-1}{4\text{J}}, \frac{1}{4\text{J}})$ and $(\frac{1}{4\text{J}}, \frac{1}{4\text{J}})$.

Hydrodynamic limit Let K be a Lipschitz function on $\mathcal{H}$ and define

$$\Phi_t^{(\varepsilon)}(r) := \varepsilon\left(K_{[\varepsilon^{-2}t]}^{\varepsilon j}([\varepsilon^{-1}r])\right) \tag{7.8}$$

Proposition 4 (Hydrodynamic Limit) *Let ϕ_t be the evolution of Theorem 4 with initial condition ϕ and let $K_n^{\varepsilon j}$ the evolution (7.5) with initial interface $K_0^{(\varepsilon)}(x) = \phi(\varepsilon x)$. Then,*

$$\lim_{\varepsilon\to 0}\sup_{r\in\mathbb{R}} |\Phi_t^{(\varepsilon)}(r) - \phi_t(r)| = 0. \tag{7.9}$$

where the rescaled process $\Phi_t^{(\varepsilon)}(r)$ is defined in (7.8).

The proof of Theorem 6 follows from the above proposition:

Proof of Theorem 6 Taking $\bar{K}^{\varepsilon j}$ as the explicit stationary solution in (7.6) with $\mathrm{J} = \varepsilon j$ and $\bar{\Phi}^{(\varepsilon)}$ the corresponding renormalization as in (7.8),

$$\phi_t(r) = \lim_{\varepsilon\to 0} \bar{\Phi}_t^{(\varepsilon)}(r) = \Phi_0^{(\varepsilon)}(r) + 2jt = \bar{\phi}(r) + 2jt \tag{7.10}$$

where the first identity is consequence of (7.9), the second one comes from the stationarity of $\bar{K}$ given by (7.7) and the last one is a computation based on the explicit expressions of $\bar{K}^{(\varepsilon)}$ and $\bar{\phi}$. $\square$

To prove Proposition 4 we introduce the delta processes associated to K.

The delta harness processes We define the delta harness processes $K_\ell^{\mathrm{J},\delta,-}$, $K_\ell^{\mathrm{J},\delta,+}$ as follows. Take $\delta \geq 1$, fix an initial condition $K_0^{\mathrm{J},\delta,-} \in \mathcal{H}_{(0,0)}$, $K_0^{\mathrm{J},\delta,+} = \max\{K_0^{\mathrm{J},\delta,-}, V_{(0,\delta\mathrm{J})}\}$ and define iteratively

$$\begin{aligned}
K_\ell^{\mathrm{J},\delta,\pm} &:= \Theta^{\ell-[n\delta]} K_{[n\delta]}^{\mathrm{J},\delta,\pm}, && \text{if } \ell \in [[n\delta], [(n+1)\delta]-1],\ n \geq 0 \\
K_{[n\delta]}^{\mathrm{J},\delta,-} &:= \max\{K_{[n\delta]-1}^{\mathrm{J},\delta,-}, V_{(0,n\delta 2\mathrm{J})}\}, && \text{if } n \geq 1 \\
K_{[n\delta]}^{\mathrm{J},\delta,+} &:= \max\{K_{[n\delta]-1}^{\mathrm{J},\delta,+}, V_{(0,(n+1)\delta 2\mathrm{J})}\}, && \text{if } n \geq 1.
\end{aligned} \tag{7.11}$$

Both processes evolve with (7.1) in the time intervals $[[n\delta], [(n+1)\delta]-1] \cap \mathbb{Z}$ and update the interface at times $[n\delta]$: the process delta$-$ takes the max with the cone with vertex $(0, n\delta 2\mathrm{J})$ while the process delta$+$ takes the max with the cone with vertex $(0, (n+1)\delta 2\mathrm{J})$. The following dominating Lemma follows immediately.

Lemma 11 *If $K_0^{\mathrm{J},\delta,-} \leq K_0^{\mathrm{J}} \leq K_0^{\mathrm{J},\delta,+}$, then $K_\ell^{\mathrm{J},\delta,-} \leq K_\ell^{\mathrm{J}} \leq K_\ell^{\mathrm{J},\delta,+}$, for all $\ell \geq 1$; for all $\delta \geq 1$, $\mathrm{J} \geq 0$. Furthermore,*

$$\sup_{x\in\mathbb{Z},\ell\geq 0} \left\{ K_\ell^{\mathrm{J},\delta,+}(x) - K_\ell^{\mathrm{J},\delta,-}(x) \right\} \leq \delta 2\mathrm{J}. \tag{7.12}$$

Hydrodynamic limit of the delta processes Take a macroscopic initial condition ϕ as in Theorem 4. Fix $j > 0$ and take $\mathrm{J} = \varepsilon j$, fix $\delta > 0$ and ε small such that $\delta\varepsilon^{-2} > 1$, take $\delta\varepsilon^{-2}$ in the place of δ and define

$$\Phi_t^{\varepsilon,\delta,\pm}(r) := \varepsilon K_{[\varepsilon^{-2}t]}^{\varepsilon j,\varepsilon^{-2}\delta,\pm}([\varepsilon^{-1}r])$$

with initial $K_0^{(\varepsilon)}(x) = \phi(\varepsilon x)$, which implies

$$\Phi_0^{\varepsilon,\delta,\pm}(r,0) = \Phi_0^{\varepsilon}(r) = \phi(\varepsilon[r\varepsilon^{-1}]).$$

From Lemma 11 we have

$$\Phi_t^{\varepsilon,\delta,-}(r) \le \Phi_t^{\varepsilon}(r) \le \Phi_t^{\varepsilon,\delta,+}(r) \tag{7.13}$$

$$\sup_{r,t,\varepsilon}\left(\Phi_t^{\varepsilon,\delta,+}(r) - \Phi_t^{\varepsilon,\delta,-}(r)\right) \le 2\delta j. \tag{7.14}$$

Proposition 5 (hydrodynamics) *Let ϕ be Lipschitz. Then, there exists a constant $C > 0$ such that,*

$$\sup_r \left|\Phi_t^{\varepsilon,\delta,\pm}(r) - \phi_t^{\delta,\pm}(r)\right| \le Ct\delta^{\beta-1}\varepsilon^{1-2\beta}, \tag{7.15}$$

for any $\beta > 0$.

Proof Take $t < \delta$. In this case ϕ_t obeys the heat equation. Let W_t^r be Brownian motion with starting point r; then the solution ϕ_t is given by $\phi_t(r) = G_t\phi(r) = E\phi(W_t^r)$. By duality (7.3) and assuming X_n and W_t are defined in the same probability space,

$$\begin{aligned}\left|\Phi_t^{\varepsilon,\delta,\pm}(r) - \phi_t^{\delta,\pm}(r)\right| &= \left|\mathbb{E}\left(\phi(\varepsilon X_{[\varepsilon^{-2}t]}^{[\varepsilon^{-1}r]}) - \phi(W_t^r)\right)\right| \\ &\le \mathbb{E}\left|\phi(\varepsilon X_{[\varepsilon^{-2}t]}^{[\varepsilon^{-1}r]}) - \phi(W_t^r)\right| \le \mathbb{E}\left|\varepsilon X_{[\varepsilon^{-2}t]}^{[\varepsilon^{-1}r]} - W_t^r\right| &(7.16)\\ &\le C\delta^{\beta}\varepsilon^{1-2\beta}, \qquad \text{for any } \beta > 0,\quad t < \delta, &(7.17)\end{aligned}$$

where in (7.16) we used that ϕ is Lipschitz and in (7.17) the dyadic KMT coupling between the Brownian motion and the random walk [11], Theorem 7.1.

At $t = \delta$ we have

$$\left|\max\left\{\Phi_{\delta-}^{\varepsilon,\delta,\pm}, V_{(0,[\varepsilon^{-2}\delta]2j\varepsilon^2)}\right\} - \max\left\{\phi_{\delta-}^{\delta,\pm}, V_{(0,\delta 2j)}\right\}\right| \le C\delta^{\beta}\varepsilon^{1-2\beta} + \varepsilon, \tag{7.18}$$

because the two cones differ at most by ε. Changing the constant C, (7.18) is bounded by $C\delta^{\beta}\varepsilon^{1-2\beta}$, so that iterating (7.18) $[(t+1)/\delta]$ times we get (7.15). □

Proof of Proposition 4 As a consequence of (7.13) and (7.14),

$$\begin{aligned}\left|\phi_t - \Phi_t^{(\varepsilon)}(r)\right| &\le \left|\phi_t - \phi_t^{\delta,\pm}\right| + \left|\phi_t^{\delta,\pm} - \Phi_t^{\varepsilon,\delta,\pm}\right| + \left|\Phi_t^{\varepsilon,\delta,\pm} - \Phi_t^{(\varepsilon)}\right| \\ &\le 2\delta + Ct\delta^{1-\beta}\varepsilon^{1-2\beta} + 2\delta.\end{aligned}$$

Taking first $\varepsilon \to 0$ and then $\delta \to 0$, we get (7.9). □

8 Conclusions

In this section we summarize the results we have obtained so far. In Theorems 4 and 1 we have proved convergence in the hydrodynamic limit to a deterministic evolution for the interface and, respectively, the density. The limit interface ϕ_t is Lipschitz continuous in space and continuous in time, see Theorem 5. At each time $t > 0$ it "belongs" to a cone, in the sense that there are real numbers $\mathrm{L}(\phi_t)$ and $\mathrm{R}(\phi_t)$ so that $\phi_t(r) = |r| + jt$ for $r \notin (\mathrm{L}(\phi_t), \mathrm{R}(\phi_t))$, Theorem 7. The limit particle densities inherit analogous properties from the interface.

The interface evolution ϕ_t is characterized in terms of a sequence of upper and lower bounds $\phi_t^{\delta,\pm}$ which in the limit $\delta \to 0$ become identical. $\phi_t^{\delta,\pm}$ are solutions of time discretized free boundary problems obtained by alternating linear heat diffusion and motion of the boundaries.

We miss however a proof that the limit evolution satisfies the free boundary problem described in the introduction for the particle density. We do know however that the stationary solution of (1.7)–(1.8) is indeed stationary for the limit evolution, Theorem 6. The formula for the velocity of the boundaries is quite natural once we observe that the levels of the solution of the heat equation have velocity $-\rho_t''/(2\rho_t')$ (ρ' and ρ'' the space derivatives of ρ). To get (1.8) we need to add the information that at the endpoints $\rho' = -2j$ which is consistent with the analysis of the stationary solution.

We have proved in Theorem 3 that there is a stationary measure for the particle process at fixed $\mathrm{J} = j\varepsilon > 0$; we miss however a proof that in the limit $\varepsilon \to 0$ it becomes supported by the stationary solution of (1.7)–(1.8), even though this is stationary for the limit evolution.

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