

## ANALYTICAL SOLUTIONS FOR RADIATION-DRIVEN WINDS IN MASSIVE STARS. I. THE FAST REGIME

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### ABSTRACT

Accurate mass-loss rate estimates are crucial keys in the study of wind properties of massive stars and for testing different evolutionary scenarios. From a theoretical point of view, this implies solving a complex set of differential equations in which the radiation field and the hydrodynamics are strongly coupled. The use of an analytical expression to represent the radiation force and the solution of the equation of motion has many advantages over numerical integrations. Therefore, in this work, we present an analytical expression as a solution of the equation of motion for radiation-driven winds in terms of the force multiplier parameters. This analytical expression is obtained by employing the line acceleration expression given by Villata and the methodology proposed by Müller & Vink. On the other hand, we find useful relationships to determine the parameters for the line acceleration given by Müller & Vink in terms of the force multiplier parameters.

*Key words:* hydrodynamics – methods: analytical – stars: early-type – stars: mass-loss – stars: winds, outflows

### 1. INTRODUCTION

The study of the winds of massive stars is very important in many aspects. They affect not only the nearby environment of the stars through the input of energy and momentum, but also the evolution of the mass-losing star itself. The interaction of the expelled outflow with the interstellar medium (ISM) contributes to the formation of majestic nebulae, stellar filaments, bow shocks, blown bubbles, bipolar jets, and circumstellar disks/rings. All these fascinating structures are far from being spherical and homogeneous.

Winds of massive stars are mainly driven by the line-radiation force (CAK and m-CAK theories, see Castor et al. 1975; Pauldrach et al. 1986, respectively). Moreover, nowadays, there is more evidence that radiation-driven winds are unstable and highly variable (see, e.g., Puls et al. 2008). Numerical simulations of time-dependent models show that the non-linear evolution of wind instabilities leads to the formation of shocks and spatial structures in both density and velocity (named clumps, Dessart & Owocki 2005). The development of clumps due to the line-deshadowing instability is expected when the wind velocity is large enough to produce shocked structures. However, new observations indicate that clumping is already present close to the stellar photosphere. Such inhomogeneities could then be related to waves produced by the sub-surface convection zone (Cantiello et al. 2009). In addition, any notable inhomogeneity will necessarily result in a mass-loss rate overestimate. Therefore, a clumped-wind theory is essential to resolve the mass-loss discrepancy (Šurlan et al. 2013); however, avoiding time expensive computations of analytical expressions is required.

During stellar evolution, evolved massive stars pass through transient short-lived phases (i.e., blue supergiants, luminous blue variables, B[e] supergiants and red giant branch stars) and expel huge amounts of mass to the ISM. The rate at which massive stars lose mass is quite uncertain and still a subject of debate. Therefore, accurate mass-loss rate estimates are crucial keys in

the study of the wind properties of O-B stars and for testing different evolutionary scenarios. The amount of mass lost via stellar winds can be evaluated by comparing theoretical results with observation and studying their effects on the emergent line spectrum. From a theoretical point of view, this implies solving a complex set of differential equations in which the radiation field and the hydrodynamics are strongly coupled by the radiation force and its interaction with the medium. To treat this system of equations efficiently, in most of the cases, the hydrodynamics solution is approximated by a simple analytical expression, such as a  $\beta$  velocity law (see, e.g., Castor & Lamers 1979) with typical  $\beta$  values determined empirically by modeling the observed line profiles. Furthermore, analytical solutions of the hydrodynamics equations are indispensable to deal with the multidimensional radiative transfer problems of moving media, using Monte Carlo techniques with reasonable accuracy on timescales of a few hours. Therefore, the velocity profile is frequently described with a  $\beta$ -law or a double  $\beta$ -law (Šurlan et al. 2013; Hillier & Miller 1999).

The aim of this work is to deliver an accurate and convenient analytical expression for the solution of the equation of motion for radiation-driven winds by employing a line acceleration expression in terms of the force multiplier parameters ( $\alpha$ ,  $k$ , and  $\delta$ ). The success of these parameters is that they provide scaling laws for the mass-loss rate and terminal speed, as well as an empirically observed Wind–Momentum–Luminosity (WML) relationship that depends on metallicity (Owocki 2009). On the other hand, the CAK line force multipliers  $\alpha$  and  $k$  describe the statistical dependence of the number of lines on frequency position and line strength (e.g., Puls et al. 2000) while the parameter  $\delta$  is related to the change in the ionization of the wind. CAK parameters have been improved using NLTE calculations and they are adequate to describe the optically thin winds of O and B stars. The use of analytical expressions for the radiation force and velocity profile as a function of the force multiplier parameters provides a clear view into how the line-driven mechanism affects the hydrodynamics and clumping. It also might help treat problems with an abrupt discontinuity in the stellar parameters of B supergiants, such as the bi-stability jump

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as a consequence of changes in the wind ionization (Lamers & Pauldrach 1991; Vink et al. 1999; Curé et al. 2005).

An alternative approach to CAK involves the Monte Carlo method developed by Abbott & Lucy (1985) which leads to a parameterized description of the line acceleration that only depends on radius (rather than explicitly on the velocity gradient  $dv/dr$  as in the standard CAK theory). The new parameters that fit the Monte Carlo line acceleration describe very well the dense winds of O stars (Müller & Vink 2014) but they are not that directly related to the main mechanism driving the wind, such as line strength and ionization. Therefore, we also provide useful relationships between the CAK line force parameters and Müller & Vink's (2008) parameterization, making them useful for thin and dense winds.

In a forthcoming paper, we plan to develop other analytical expressions that are adequate to describe radiation-driven winds for stars rotating near the critical rotation velocity (Curé 2004, named  $\Omega$ -slow solutions), or radiation-driven winds with ionization gradients (Curé et al. 2011,  $\delta$ -slow solutions).

In Section 2 we outline the main steps to obtain the dimensionless form of the equation of motion. In addition, we review the analytical expressions of the line accelerations given by Villata (1992) and Müller & Vink (2008). In Section 3, we give our own analytical approximation of the line acceleration and discuss useful expressions that enable us to obtain the parameterization of Müller & Vink's (2008) line acceleration in terms of the force multiplier parameters. A discussion of the results and conclusions are presented in sections Section 4 and Section 5, respectively.

## 2. THE STANDARD HYDRODYNAMICAL WIND MODEL

The *CAK theory* for a radiation-driven wind was developed by Castor et al. (1975), who describe a stationary, one-dimensional, non-rotating, isothermal, outflowing wind with spherical symmetry. Using these hypotheses, and neglecting the effects of viscosity, heat conduction, and magnetic fields, the equations of mass conservation and radial momentum state are

$$4\pi r^2 \rho v = \dot{M} \quad (1)$$

and

$$v \frac{dv}{dr} = -\frac{1}{\rho} \frac{dp}{dr} - \frac{GM_*(1-\Gamma)}{r^2} + g^{\text{line}}(\rho, dv/dr, n_E). \quad (2)$$

Here  $v$  is the fluid radial velocity,  $dv/dr$  is the velocity gradient, and  $g^{\text{line}}$  is the radiative acceleration. All other variables have their standard meaning (for a detailed derivation and definitions of variables, constants, and functions, see Curé 2004).

The CAK theory assumes that the radiation emerges directly from the star (as a source point) and multiple scatterings are not taken into account. A later improvement of this theory (m-CAK) considers the radiation coming from a stellar disk, and therefore, we adopt the m-CAK standard parameterization for the line force term, following the descriptions of Abbott (1982), Friend & Abbott (1986), and Pauldrach et al. (1986), namely,

$$g^{\text{line}} = \frac{C}{r^2} f_D(r, v, dv/dr) \left( r^2 v \frac{dv}{dr} \right)^\alpha \left( \frac{n_E}{W(r)} \right)^\delta, \quad (3)$$

where the coefficient (eigenvalue)  $C$  depends on the mass-loss rate  $\dot{M}$ ,  $W(r)$  is the dilution factor,  $n_E$  is the electron

number density in units of  $10^{-11} \text{ cm}^{-3}$ , and  $f_D$  is the finite disk correction factor.

The momentum equation (Equation (2)) can be expressed in dimensionless form (see, e.g., Müller & Vink 2008) as

$$\hat{v} \frac{d\hat{v}}{d\hat{r}} = -\frac{\hat{v}_{\text{crit}}^2}{\hat{r}^2} + \hat{g}^{\text{line}} - \frac{1}{\rho} \frac{d\rho}{d\hat{r}}, \quad (4)$$

where  $\hat{r}$  is a dimensionless radial coordinate  $\hat{r} = r/R_*$ , and the dimensionless velocities (in units of the isothermal sound speed  $a$ ) are

$$\hat{v} = \frac{v}{a} \quad \text{and} \quad \hat{v}_{\text{crit}} = \frac{v_{\text{esc}}}{a\sqrt{2}}. \quad (5)$$

$v_{\text{crit}}$  is the rotational break-up velocity in the case of a corresponding rotating star, but often it is simply defined by the effective escape velocity  $v_{\text{esc}}$  divided by a factor of  $\sqrt{2}$ . In the same way, we can write the dimensionless line acceleration as

$$\hat{g}^{\text{line}} = \frac{R_*}{a^2} g^{\text{line}}. \quad (6)$$

With the equation of state of an ideal gas ( $p = a^2 \rho$ ) and using Equation (1), the dimensionless equation of motion is the following:

$$\left( \hat{v} - \frac{1}{\hat{v}} \right) \frac{d\hat{v}}{d\hat{r}} = -\frac{\hat{v}_{\text{crit}}^2}{\hat{r}^2} + \frac{2}{\hat{r}} + \hat{g}^{\text{line}}. \quad (7)$$

### 2.1. Line Acceleration

In this section, we review the basic concepts developed by Villata (1992) and Müller & Vink (2008) to derive, later on, a general analytical expression for the velocity profile in the frame of the standard radiation-driven wind solution for massive stars. These basic concepts will then be used in a forthcoming work to obtain analytical expressions for the  $\Omega$ - and  $\delta$ -slow solutions.

For that purpose we analyze two expressions for the line acceleration which are functions only of the radial distance and do not depend either on the velocity or the velocity gradient as the standard m-CAK description does. We also demonstrate that both expressions are related to each other and allow a noticeable simplification to the integration of the equation of motion (Equation (7)), leading to an analytical expression for the fast wind's velocity profile.

#### 2.1.1. Villata's Approximation

Based on the analytic study of radiation-driven stellar winds by Kudritzki et al. (1989), Villata (1992) derived an approximate expression for the line acceleration term. This line acceleration depends only on the radial coordinate:

$$g_{\text{V92}}^{\text{line}}(\hat{r}) = \frac{GM_*(1-\Gamma)}{R_*^2 \hat{r}^2} A(\alpha, \beta, \delta) \left( 1 - \frac{1}{\hat{r}} \right)^{\alpha(2.2\beta-1)}, \quad (8)$$

with

$$A(\alpha, \beta, \delta) = \frac{(1.76\beta)^\alpha}{1-\alpha} [10^{-\delta}(1+\alpha)]^{1/(1-\alpha)} \left[ 1 + \left( \frac{2}{\alpha} \{ 1 - [10^{-\delta}(1+\alpha)]^{1/(\alpha-1)} \} \right)^{1/2} \right]^\alpha, \quad (9)$$

where  $\alpha$  and  $\delta$  are force multiplier parameters (Abbott 1982) and  $\beta$  is the exponent in the  $\beta$  velocity law. This exponent can

be evaluated with a formula given by Kudritzki et al. (1987) in terms of the force multiplier parameters and the escape velocity,  $v_{\text{esc}}$ :

$$\beta = 0.95 \alpha + \frac{0.008}{\delta} + \frac{0.032 v_{\text{esc}}}{500}, \quad (10)$$

with  $v_{\text{esc}}$  in  $\text{km s}^{-1}$ .

Then, using Equation (8) in its dimensionless form (Equation (6)) and inserting it into the dimensionless equation of motion (Equation (7)), it yields

$$\left(\hat{v} - \frac{1}{\hat{v}}\right) \frac{d\hat{v}}{d\hat{r}} = -\frac{\hat{v}_{\text{crit}}^2}{\hat{r}^2} + \frac{2}{\hat{r}} + \frac{1}{a^2} \frac{GM_*(1-\Gamma)}{R_* \hat{r}^2} A(\alpha, \beta, \delta) \left(1 - \frac{1}{\hat{r}}\right)^{\gamma_v}, \quad (11)$$

with  $\gamma_v = \alpha(2.2\beta - 1)$ .

This differential equation, based on the approximation of the line acceleration given by Villata,  $\hat{g}_{\text{V92}}^{\text{line}}(\hat{r})$ , is a solar-like differential equation of motion. In particular, the expression  $\hat{g}_{\text{V92}}^{\text{line}}(\hat{r})$  does not depend on the product  $\hat{v} d\hat{v}/d\hat{r}$ , as in the case of the standard m-CAK theory, and hence the singular point is the sonic point, e.g., when the velocity equals the sound speed. Another important difference between Villata's equation of motion and its equivalent equation in the standard m-CAK theory is that Equation (11) has no eigenvalues. This means that the differential equation does not depend explicitly on the mass-loss rate ( $\dot{M}$ ) of the star. Therefore, Villata derived the following expression for  $\dot{M}$ :

$$\dot{M} = 1.2 \left( \frac{D^\delta \dot{M}_{\text{CAK}}^\alpha}{1 + \alpha} \right)^{1/(\alpha - \delta)}, \quad (12)$$

where  $D$  and  $\dot{M}_{\text{CAK}}$  are given by

$$D = \left( \frac{1 + Z_{\text{He}} Y_{\text{He}}}{1 + 4 Y_{\text{He}}} \right) \left( \frac{9.5 \times 10^{-11}}{\pi m_{\text{H}} R_* v_\infty} \right) \quad (13)$$

and

$$\dot{M}_{\text{CAK}} = \frac{4\pi G M_* \alpha}{\sigma_{\text{E}} v_{\text{th}}} \left[ k \Gamma \left( \frac{1 - \alpha}{1 - \Gamma} \right)^{1 - \alpha} \right]^{1/\alpha}, \quad (14)$$

where  $Z_{\text{He}}$  is the amount of free electrons provided by helium,  $Y_{\text{He}}$  is the helium abundance relative to hydrogen,  $m_{\text{H}}$  is the proton mass,  $\sigma_{\text{E}}$  is the Thomson scattering absorption coefficient per mass density,  $v_\infty$  is the wind terminal velocity,  $v_{\text{th}}$  is the thermal velocity of protons, and  $k$  is a force multiplier parameter; all these parameters are given in cgs units.

Villata solved the equation of motion only by means of a standard numerical integration, obtaining terminal velocities that agree, within 3%–4%, with those computed by Pauldrach et al. (1986) and Kudritzki et al. (1987). In addition, the mass-loss rate he obtained agrees very well with the numerical results computed by Pauldrach et al. (1986; see details in Villata 1992, Table 1).

### 2.1.2. Müller and Vink's Approximation

In the context of stellar wind theory of massive stars, Müller & Vink (2008, hereafter MV08) present an analytical expression for the velocity field using a parameterized description for the line acceleration that (as in Villata's) also depends on the radial coordinate. Müller & Vink (2008) computed the line acceleration using Monte Carlo multi-line radiative transfer

calculations (de Koter et al. 1997; Vink et al. 1999) and a  $\beta$  velocity law. Then, the numerical line acceleration was fitted by the following function:

$$\hat{g}_{\text{MV08}}^{\text{line}}(\hat{r}) = \frac{\hat{g}_0}{\hat{r}^{1+\delta_1}} \left( 1 - \frac{\hat{r}_0}{\hat{r}^{\delta_1}} \right)^\gamma, \quad (15)$$

where  $\hat{g}_0$ ,  $\delta_1$ ,  $\hat{r}_0$ , and  $\gamma$  are the line acceleration parameters.

Replacing Equation (15) in Equation (7), MV08 derived the following dimensionless equation of motion:

$$\left(\hat{v} - \frac{1}{\hat{v}}\right) \frac{d\hat{v}}{d\hat{r}} = -\frac{\hat{v}_{\text{crit}}^2}{\hat{r}^2} + \frac{2}{\hat{r}} + \frac{\hat{g}_0}{\hat{r}^{1+\delta_1}} \left( 1 - \frac{\hat{r}_0}{\hat{r}^{\delta_1}} \right)^\gamma, \quad (16)$$

and a fully *analytical* 1D velocity profile (see Müller & Vink 2008, for details about the methodology used to obtain this solution) as a function of the so-called Lambert W-function (Corless et al. 1993, 1996; Cranmer 2004). The analytical solution reads

$$\hat{v}(\hat{r}) = \sqrt{-W_j(x(\hat{r}))}, \quad (17)$$

with

$$x(\hat{r}) = -\left(\frac{\hat{r}_c}{\hat{r}}\right)^4 \exp \left[ -2 \hat{v}_{\text{crit}}^2 \left( \frac{1}{\hat{r}} - \frac{1}{\hat{r}_c} \right) - \frac{2}{\hat{r}_0 \delta_1 (1 + \gamma)} \right] \times \left[ \left( 1 - \frac{\hat{r}_0}{\hat{r}^{\delta_1}} \right)^{1+\gamma} - \left( 1 - \frac{\hat{r}_0}{\hat{r}_c^{\delta_1}} \right)^{1+\gamma} \right] - 1. \quad (18)$$

The new parameter  $\hat{r}_c$ , which represents the position of the sonic (or critical) point, appears in the last equation. Furthermore, it is important to note that the Lambert W-function ( $W_j$ ) has only two real branches, which are denoted by a sub-index  $j$ , with  $j = -1$  or  $0$ . The sonic point ( $\hat{r}_c$ ) is the boundary between these two branches, i.e.,

$$j = \begin{cases} 0 & \text{for } 1 \leq \hat{r} \leq \hat{r}_c \\ -1 & \text{for } \hat{r} > \hat{r}_c \end{cases}. \quad (19)$$

On the other hand, we can observe that the left-hand side of the equation of motion (Equation (16)) vanishes when  $\hat{v} = 1$  (singularity condition in the CAK formalism); therefore, as in the m-CAK case, a regularity condition must be imposed, which is equivalent to the right-hand side (RHS) of Equation (16) also vanishing at  $\hat{r} = \hat{r}_c$ . That is,

$$-\frac{\hat{v}_{\text{crit}}^2}{\hat{r}_c^2} + \frac{2}{\hat{r}_c} + \frac{\hat{g}_0}{\hat{r}_c^{1+\delta_1}} \left( 1 - \frac{\hat{r}_0}{\hat{r}_c^{\delta_1}} \right)^\gamma = 0, \quad (20)$$

and  $\hat{r}_c$  is obtained by numerically solving this last equation for a given set of the parameters,  $\hat{g}_0$ ,  $\delta_1$ ,  $\hat{r}_0$ , and  $\gamma$ . Then, the function  $x(\hat{r})$  (Equation (18)) is calculated and the velocity profile is derived by replacing it in Equation (17).

## 3. RESULTS

The approximations described before have the advantage of providing analytical expressions to represent the radiation force. This fact considerably simplifies the solution of the equation of motion. However, each one of the mentioned approximations has its own advantages and disadvantages. Even though Villata's approximation of the radiation force is general and can be directly applied to describe the wind of any massive star, one still needs to deal with a numerical integration to

**Table 1**  
Comparison of the Wind Parameters Obtained Via the New Analytical Solutions with Numerical Calculations

Star	$T_{\text{eff}}$ ( $10^3$ K)	$\log g$	$R_*$ ( $R_{\odot}$ )	$k$	$\alpha$	$\delta$	$\hat{r}_c$	$v_{\infty}$ ( $\text{km s}^{-1}$ )	$v_{\infty}^{\text{Analytic}}$ ( $\text{km s}^{-1}$ )	$\dot{M}$ ( $10^{-6} M_{\odot} \text{ yr}^{-1}$ )	$\dot{M}^{\text{Analytic}}$ ( $10^{-6} M_{\odot} \text{ yr}^{-1}$ )
$\epsilon$ Ori	28.5	3.25	37	0.170	0.590	0.090	1.0014	1905	1914	3.71	3.29
$\zeta$ OriA	30.0	3.45	29	0.170	0.590	0.090	1.0017	2226	2205	2.07	1.87
9-Sgr	50.0	4.08	12	0.124	0.640	0.070	1.0035	3422	3484	4.37	3.95
HD 48099	39.0	4.00	11	0.124	0.640	0.070	1.0035	3419	3442	0.70	0.64
HD 42088	40.0	4.05	5.8	0.124	0.640	0.070	1.0027	2534	2656	0.23	0.20
$\lambda$ Cep	42.0	3.70	17	0.124	0.640	0.070	1.0026	2430	2548	5.64	4.96
(O5 V) <sub>Gal</sub>	45.0	4.00	12	0.124	0.640	0.070	1.0034	3284	3343	2.34	2.12
(O5 V) <sub>LMC</sub>	45.0	4.00	12	0.089	0.627	0.100	1.0029	2815	2812	1.25	1.12
(O5 V) <sub>SMC</sub>	45.0	4.00	12	0.097	0.580	0.104	1.0019	2352	2314	0.80	0.72

**Note.** The sample of stars was taken from Villata (1992).

solve the momentum equation.<sup>4</sup> Thanks to Müller & Vink's approximation, an analytical solution of the equation of motion is attained (through the Lambert W-function), in terms of the  $\hat{g}_0$ ,  $\delta_1$ ,  $\hat{r}_0$ , and  $\gamma$  parameters of the star. However, to obtain these parameters the line acceleration still needs to be derived using Monte-Carlo multi-line radiative transfer calculations.

Therefore, here, we propose to obtain a complete analytical description of the 1D hydrodynamical solution for radiation-driven winds, utilizing the advantages of both previous approximations (the use of known parameters and the Lambert W-function).

### 3.1. Analytical Solution in Terms of the Force Multiplier and Stellar Parameters

Our goal is to derive a new analytical expression combining Villata's expression of the equation of motion (Equation (11)), based on Villata's definition of the radiation force,  $\hat{g}_{V92}^{\text{line}}(\hat{r})$ , with the methodology developed by MV08 to solve the equation of motion. This way, we solve Equation (11) through the Lambert W-function (Corless et al. 1993, 1996; Cranmer 2004),

$$\hat{v}(\hat{r}) = \sqrt{-W_j(x(\hat{r}))}, \quad (21)$$

with

$$x(\hat{r}) = -\left(\frac{\hat{r}_c}{\hat{r}}\right)^4 \exp\left[-2\hat{v}_{\text{crit}}^2\left(\frac{1}{\hat{r}} - \frac{1}{\hat{r}_c}\right) - 2(I_{\hat{g}_{V92}^{\text{line}}}(\hat{r}) - I_{\hat{g}_{V92}^{\text{line}}}(\hat{r}_c)) - 1\right], \quad (22)$$

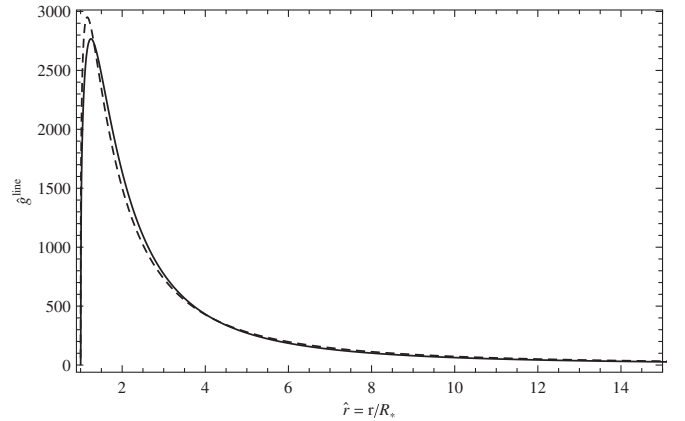
where

$$I_{\hat{g}_{V92}^{\text{line}}}(\hat{r}) = (10^{-\delta}(1+\alpha))^{\frac{1}{1-\alpha}} \left(1 + \sqrt{2} \sqrt{-\frac{(10^{-\delta}(1+\alpha)-1)^{\frac{1}{\alpha-1}}}{\alpha}}\right)^{\alpha} (1.76\beta)^{\alpha} G M_* \left(\frac{\hat{r}-1}{\hat{r}}\right)^{1+\gamma_{\nu}} \frac{\Gamma-1}{(a^2[\alpha-1](1+\gamma_{\nu})R_*)}. \quad (23)$$

From the singular and regular conditions at the critical or some point,  $\hat{r}_c$  can be obtained numerically by making the RHS of Equation (11) equal zero, that is,

$$-\frac{\hat{v}_{\text{crit}}^2}{\hat{r}_c^2} + \frac{2}{\hat{r}_c} + \frac{1}{a^2} \frac{G M_* (1-\Gamma)}{R_* \hat{r}_c^2} A(\alpha, \beta, \delta) \left(1 - \frac{1}{\hat{r}_c}\right)^{\gamma_{\nu}} = 0. \quad (24)$$

<sup>4</sup> The definition of Lambert W-function came one year after Villata's work.



**Figure 1.** Line acceleration as a function of the radial coordinate  $\hat{r} = r/R_*$ . The hydrodynamic line acceleration is shown by the solid line and  $\hat{g}_{V92}^{\text{line}}$  is shown by the dashed line. The calculation is based on a set of values for  $\epsilon$  Ori (see Table 1).

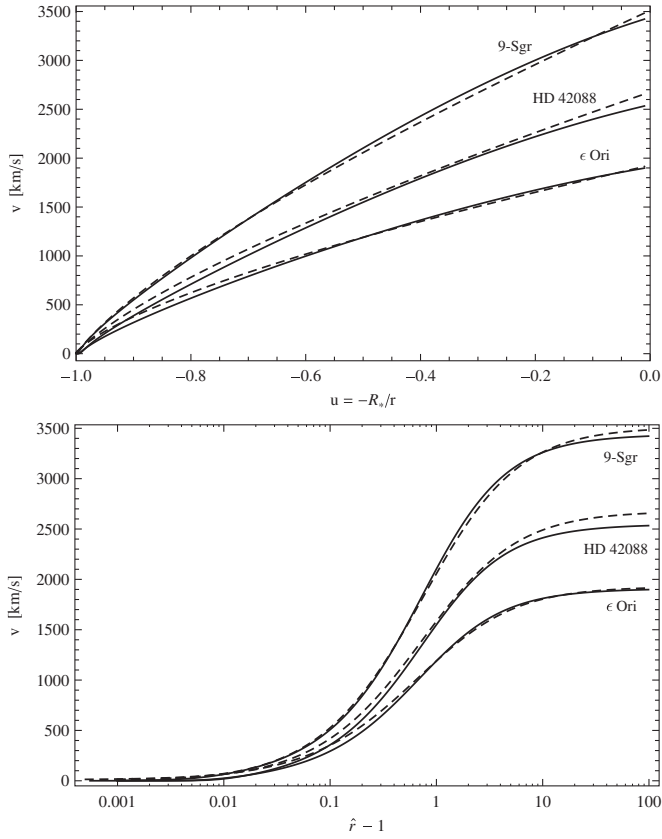
The advantage of using Equation (21) is that not only is it based on the Lambert W-function but it also depends on the force multiplier parameters and the stellar parameters. Both stellar and force multiplier parameters are given for a wide range of spectral types (see, for example, Abbott 1982; Pauldrach et al. 1986; Lamers & Cassinelli 1999).

### 3.2. Comparison of the New General Analytical Expression with the Solution for Radiation-driven Winds

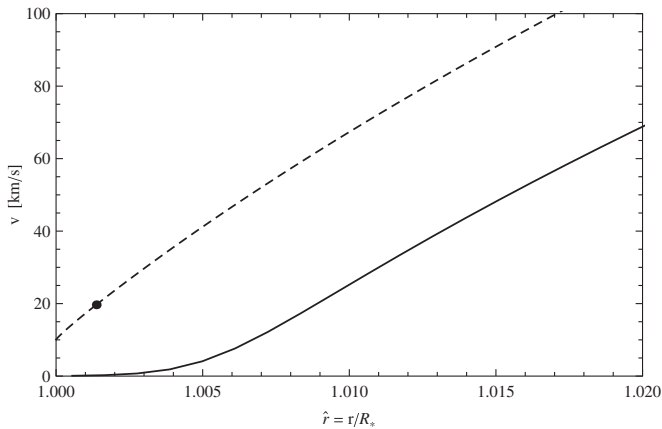
The accuracy of the *new* analytical solution has to be tested by comparing it with rigorous numerical 1D hydrodynamical solutions for radiation-driven winds, as described by Curé (2004). Table 1 gives the values of  $v_{\infty}$  obtained from Equation (21) together with those values calculated from numerical results using the same sample of stars listed by Villata (1992).  $v_{\infty}$  was calculated  $100 R_*$  from the star, and  $\dot{M}$  with the Equation (12) derived by Villata but employing our  $v_{\infty}$  estimates.

As an example, Figure 1 shows the good agreement found between the line acceleration obtained for  $\epsilon$  Ori, using a numerical hydrodynamic model and the expression  $\hat{g}_{V92}^{\text{line}}$ . The comparison of the resulting velocity profiles obtained via the hydrodynamics and the analytical solution of this work is shown in Figure 2 for three stars of the sample ( $\epsilon$  Ori, 9 Sgr, and HD 42088). Both analytical and numerical solutions seem to behave in a similar way. However, if we have a close view to the region near the stellar surface, which is the base of the wind (Figure 3), we can note that the velocity law derived





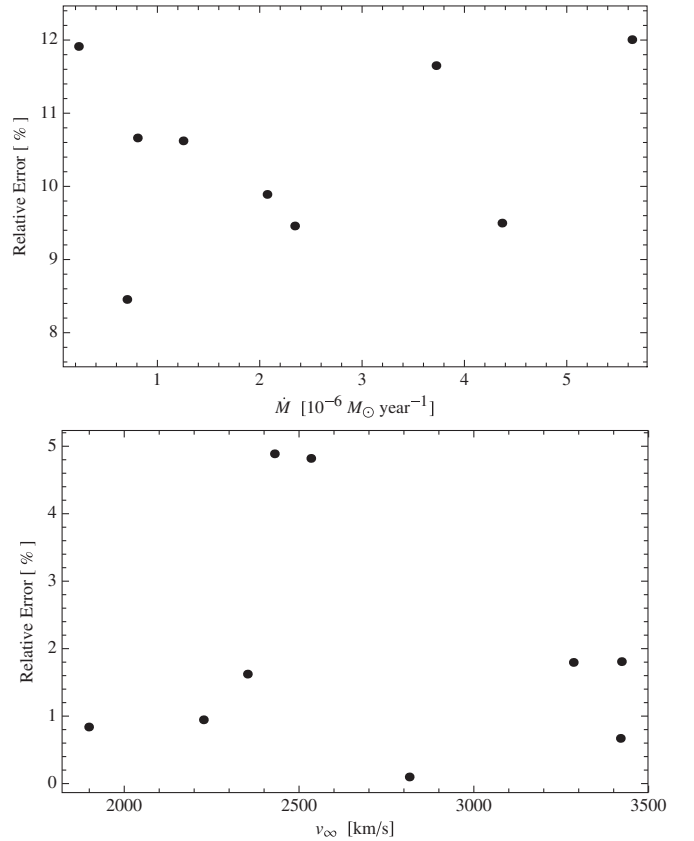
**Figure 2.** Velocity profiles as a function of the inverse radial coordinate  $u = -R_*/r = -1/\hat{r}$  (upper panel) and  $\hat{r} - 1$  in logarithmic scale (lower panel) for three stars of the sample. The hydrodynamic results are shown by the solid line, and the analytical solutions are shown by the dashed line.



**Figure 3.** Velocity profiles as a function of the radial coordinate  $\hat{r} = r/R_*$  in a region very close to the surface of the star ( $\epsilon$  Ori). The numerical hydrodynamic result is shown by the solid line and the analytical solution is shown by the dashed line. The dot symbol indicates the position of the sonic (or critical) radius.

from the analytical solution is steeper than the numerical one. The accuracy of our solution is reflected in Figure 4 which shows the relative error for the mass-loss rate and terminal velocity obtained from our analytical solution and the hydrodynamical code.

In addition, from Table 1, we determined the average value of  $\hat{r}_c$  ( $\hat{r}_c = 1.0026$ ) and re-calculated  $v_\infty$  and  $\dot{M}$  for the sample of selected stars. The results obtained are almost the same as those shown in Table 1. This simplifies by far the numerical



**Figure 4.** Relative error for the mass-loss rate (upper panel) and terminal velocity (lower panel) with respect to the values obtained from the hydrodynamics. The relative error for the terminal velocity has a median of 1.6% and a mean of 1.9%. The median of the mass-loss rate is 10.6% and the mean is 10.5%.

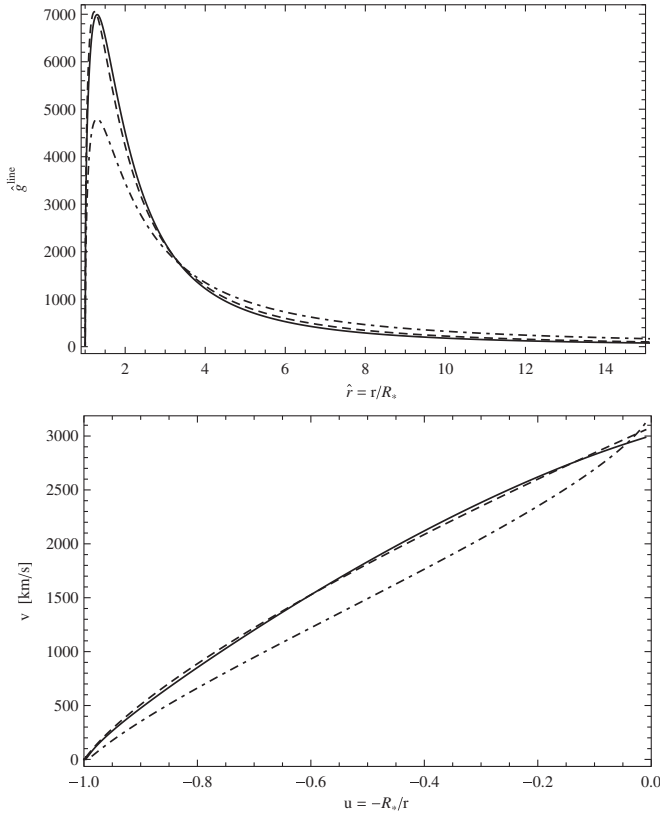
calculation of  $\hat{r}_c$ , thus obtaining a totally analytical solution. It is important to remark that  $\hat{r}_c$  can only be used in the supersonic region of the velocity profile, primarily to obtain the terminal velocity, because of its close connection with both branches of the Lambert W-function, which are discontinuous when we use  $\hat{r}_c$ .

## 4. DISCUSSION

### 4.1. On the Previously Known Analytical Expressions

As mentioned previously, the analytical approach proposed by MV08 was calculated *only* for one set of parameters  $\hat{g}_0$ ,  $\delta$ ,  $\hat{r}_0$ , and  $\gamma$ , which corresponds to an O5 main sequence star with the following stellar parameters:  $T_{\text{eff}} = 40,000$  K,  $R_* = 11.757 R_\odot$ ,  $M = 40 M_\odot$ , and a solar chemical composition. Therefore, the use of  $\hat{g}_{\text{MV08}}^{\text{line}}$  is very restrictive. Instead, Villata (1992) developed a general expression for the line acceleration, based on the standard force multiplier parameters  $k$ ,  $\alpha$ , and  $\delta$ , whose values are tabulated in several works (see, for example, Abbott 1982; Pauldrach et al. 1986; Lamers & Cassinelli 1999).

One could wonder if both analytical expressions that describe the behavior of the radiative acceleration as a function of the radius, as well as their corresponding velocity profiles, are similar. Figure 5 displays the comparison of the solutions given by Villata (dashed line) and MV08 (dot-dashed line, for the set of parameters given in iteration A of MV08) with the hydrodynamical solution (solid line) calculated for an O5 V star using the force multiplier parameters listed by Pauldrach et al. (1986), namely,  $k = 0.124$ ,  $\alpha = 0.640$ , and  $\delta = 0.070$ . The results



**Figure 5.** Comparison of line accelerations as a function of the radial coordinate  $\hat{r} = r/R_*$  (upper panel) and velocity profile as a function of  $u = -R_*/r$  (lower panel) with the numerical hydrodynamic solution for an O5 V star. The line acceleration and the velocity profile calculated using the expressions given by MV08 are shown by the dot-dashed line (for the set of parameters  $\hat{g}_0 = 17\,661$ ,  $\delta_1 = 0.6878$ ,  $\hat{r}_0 = 1.0016$ , and  $\gamma = 0.4758$ ); those corresponding to Villata (1992) are shown by the dashed line and the hydrodynamic calculations are shown by the solid line. The analytical expressions derived by MV08 show disagreements with both Villata (1992) and the numerical calculations in the whole profiles, although the agreement in the terminal velocities of the three models can be considered acceptable.

computed with the expressions and parameters given by MV08 present large discrepancies with the numerical hydrodynamical model, both in the behavior of the line acceleration as a function of radius (upper panel) and the resulting velocity profile as a function of the inverse coordinate  $u = -R_*/r = -1/\hat{r}$  (lower panel). Instead, Villata's approximation and the corresponding numerical velocity solution agree very well with the hydrodynamical results, as shown in Section 3.2.

In principle, we can attribute the discrepancy between Villata and MV08's line accelerations to the lack of knowledge of a relationship between the force multiplier parameters and the  $\hat{g}_0$ ,  $\delta_1$ ,  $\hat{r}_0$ , and  $\gamma$  parameters. This relationship can be obtained by equating  $\hat{g}_{\text{MV08}}^{\text{line}}$  with  $\hat{g}_{\text{V92}}^{\text{line}}$ . Then, we obtain,

$$\hat{g}_0 = \frac{1}{a^2} \frac{GM(1-\Gamma)}{R_*} A(\alpha, \beta, \delta), \quad (25)$$

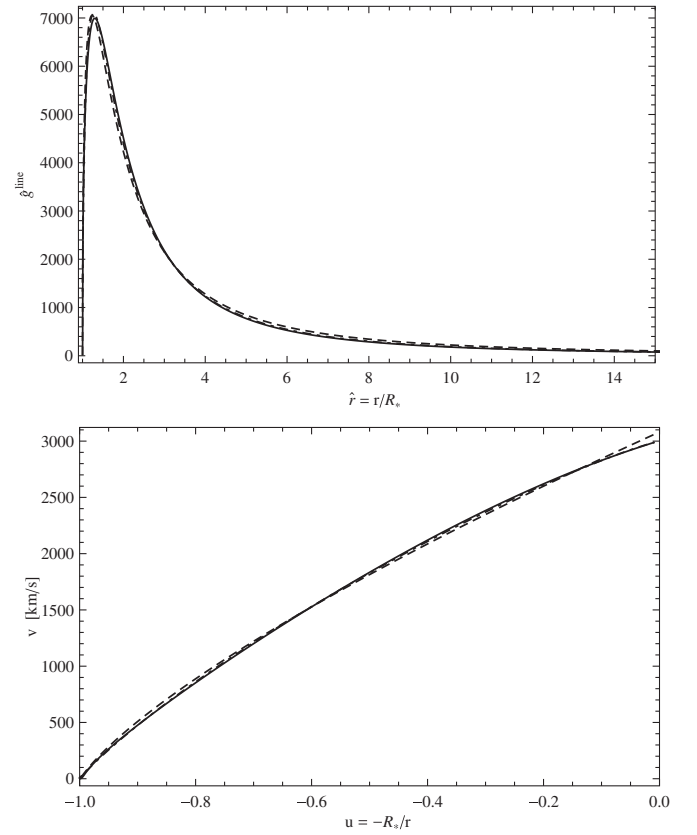
$$\delta_1 = 1, \quad (26)$$

$$\hat{r}_0 = 1 \quad (27)$$

and

$$\gamma = \gamma_V = \alpha(2.2\beta - 1). \quad (28)$$

Based on this, we find a dependence on  $\hat{g}_0$ ,  $\delta_1$ ,  $\hat{r}_0$ , and  $\gamma$  as a function of force multiplier parameters, which agrees very well



**Figure 6.** Comparison of the analytical line accelerations as a function of the radial coordinate  $\hat{r} = r/R_*$  (upper panel) and the velocity profiles as a function of the inverse radial coordinate  $u = -R_*/r$  (lower panel) for an O5 V star, using  $\hat{g}_{\text{MV08}}^{\text{line}}$  (dot-dashed line, with  $\hat{g}_0 = 28\,289.8$ ,  $\delta_1 = 1.1585$ ,  $\hat{r}_0 = 0.9926$ , and  $\gamma = 0.6212$ ) and  $\hat{g}_{\text{V92}}^{\text{line}}$  (dashed lines). The agreement among all the results is almost perfect. Solid line corresponds to the numerical hydrodynamic calculations.

**Table 2**  
Comparison of the  $\hat{g}_0$ ,  $\delta_1$ ,  $\hat{r}_0$ , and  $\gamma$  Parameters Obtained by Fitting a Hydrodynamic Calculation to (28) (Villata's Relationship) with those Derived using Equation (25)

Method	$\hat{g}_0$	$\delta_1$	$\hat{r}_0$	$\gamma$
Numerical solution	28 289.8	1.1585	0.9926	0.6212
Villata's relation	23 396.6	1.0000	1.0000	0.4683

with Villata's approximation and the numerical hydrodynamical solution. Thus, we show that both expressions of the line acceleration are essentially the same approximation and are both based on the m-CAK theory.

Table 2 shows the fitted parameters with the values obtained from Equations (25) to (28) and those derived by fitting Müller & Vink's approximation to the  $g^{\text{line}}$  expressed by the m-CAK theory. Using the values given in Table 2, Villata and MV08's line accelerations and velocity profiles are almost identical (see Figure 6).

#### 4.2. On the New Analytical Solution

From the results of Table 1 we that demonstrated our analytical solution based on Villata and MV08 are very good analytical approximations to the radiation-driven wind theory. Discrepancies in the terminal velocity and mass-loss rate are below 5% and 12%, respectively, when comparing results from

the analytical expression and our numerical hydrodynamical code. In almost all the terminal velocities derived from the analytic solution, we find higher values than those obtained from the numerical hydrodynamics. This could be due to the fact that the velocity rises without limit as the distance increases because of the isothermal temperature assumption. To overcome this problem, MV08 derived an analytical expression for the wind solution in the supersonic approximation by neglecting the corresponding pressure term in the equation of motion, but always depending on the  $\hat{g}_0$ ,  $\delta_1$ ,  $\hat{r}_0$ , and  $\gamma$  parameters.

However, as the MV08 line force parameters can also be expressed in terms of the force multiplier and stellar parameters (via Equation (25)–(28)), both formulations will allow us to discuss the scaling laws and test the WML relationship for a large variety of cases. In particular, the CAK force multiplier parameters required for our analytical solution can be found, e.g., in Pauldrach et al. (1986) for stars with temperature over 20,000 K. Then, both the velocity profile and the mass-loss rate can easily be obtained without extra computing time. For our case, the mass-loss rate results from the approximation derived by Villata.

At the time Kudritzki et al. (1989) developed their analytic formalism, only one solution of the m-CAK equation of motion was known. However, some years later, Curé (2004) and Curé et al. (2011) reported the existence of two new hydrodynamical solutions, namely, the  $\Omega$ -slow and  $\delta$ -slow solutions. The  $\Omega$ -slow solution corresponds to stars rotating with higher than  $\sim 75\%$  of the star critical rotation velocity. On the other hand, it can be shown that the  $\delta$ -slow solution for early-type stars exists when the  $\delta$  line force parameter is greater than 0.25. However, the analytical expressions derived previously (Equation (9)) cannot be applied when the line force parameter is  $\delta \gtrsim 0.3$ , because  $\hat{g}_{V92}^{\text{line}}$  turns complex. Therefore, we leave for a forthcoming paper the search for an analytical solution for the equation of motion that can also be used to describe slow radiation-driven winds.

## 5. CONCLUSIONS

We obtained an analytical expression for the velocity profile, the solution of the equation of motion for radiation-driven winds, in terms of the force multiplier parameters. This analytical expression was obtained by employing the expression of the line acceleration given by Villata (1992) and the methodology proposed by Müller & Vink (2008). We evaluated the accuracy of this new analytical expression for the velocity profile by comparing the mass-loss rate and terminal velocity with numerical 1D hydrodynamical solutions for radiation-driven winds using the code described in Curé (2004), which we will hereafter call HYDWIND. In all cases, the analytical expression provided a very good fit with the numerical solution.

In addition, we demonstrated that the terms  $\hat{g}_{V92}^{\text{line}}$  and  $\hat{g}_{MV08}^{\text{line}}$  are in essence the same, and both are a very good approximation of the m-CAK theory and provide useful relationships to determine the parameters for the line acceleration computed by Müller & Vink (2008) in terms of the force multiplier parameters.

The advantages of deriving analytical expressions for the velocity profile is that the resolution of the radiative transfer problem becomes easier. Unfortunately, the new analytical solution can only be applied to describe the standard m-CAK theory (a fast wind regime), since for values of  $\delta \gtrsim 0.3$ ,  $\hat{g}_{V92}^{\text{line}}$  turns complex. A new and more general study for fast rotating

stars and winds with ionization gradients will be carried out in a future work.

The existence of various kinds of wind solutions opens the possibility of exploring latitudinal-dependent outflowing winds, in which the wind regime might switch from a fast to a slow outflow (Curé et al. 2005; Madura et al. 2007). Therefore, we remark the importance of having analytical expressions to represent the hydrodynamics, mainly if we want to compute the line spectrum in a medium where rotation and the development of inhomogeneities (microclumpings and porosity) also play a fundamental role.

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