$\mathcal{PT}$-symmetric strings

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HIGHLIGHTS

- $\mathcal{PT}$-symmetric Hamiltonians exhibit real eigenvalues when PT symmetry is unbroken.
- We study $\mathcal{PT}$-symmetric strings with complex density.
- They exhibit regions of unbroken $\mathcal{PT}$ symmetry.
- We calculate the critical parameters at the boundaries of those regions.
- There are exact real sum rules for some particular complex densities.

ARTICLE INFO

ABSTRACT

We study both analytically and numerically the spectrum of inhomogeneous strings with $\mathcal{PT}$-symmetric density. We discuss an exactly solvable model of $\mathcal{PT}$-symmetric string which is isospectral to the uniform string; for more general strings, we calculate exactly the sum rules $Z(p) \equiv \sum_{n=1}^{\infty} 1/E_n^p$, with $p = 1, 2, \ldots$ and find explicit expressions which can be used to obtain bounds on the lowest eigenvalue. A detailed numerical calculation is carried out for two non-solvable models depending on a parameter, obtaining precise estimates of the critical values where pair of real eigenvalues become complex.

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1. Introduction

In the last years there has been great interest in the mathematical properties of a class of non-hermitian operators with $\mathcal{PT}$ symmetry (for a review see [1] and references therein). A good deal

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of this research is based on a wide variety of simple models. In particular it is of great interest to determine the conditions of unbroken symmetry under which the eigenvalues are real. This unbroken symmetry takes place for a range of values of a Hamiltonian parameter that in general increases with the quantum number.

The purpose of this paper is the investigation of a new class of $\mathscr{PT}$-symmetric models: inhomogeneous vibrating strings. In a series of papers Amore studied the spectral problems of inhomogeneous strings and drums [2–7]. In this paper we enlarge the class of such problems to include vibrating strings with complex densities $\Sigma(x)$ that satisfy $\Sigma(-x)^* = \Sigma(x)$.

The paper is organized as follows: in Section 2 we introduce the problem, in Section 3 we discuss $\mathscr{PT}$ symmetry, in Section 4 we discuss the application of the Rayleigh–Ritz method to the study of $\mathscr{PT}$ symmetric strings, in Section 5 we introduce a family of $\mathscr{PT}$ symmetric strings, which includes a string isospectral to the uniform string; in Sections 6 and 7 we discuss two examples of $\mathscr{PT}$ symmetric strings which display a mixed spectrum; finally in Section 8 we draw conclusions.

2. $\mathscr{PT}$-symmetric strings

In this paper we consider the problem of an inhomogeneous vibrating string with density $\Sigma(x)$

$$-\frac{d^2}{dx^2}\psi_n(x) = E_n\Sigma(x)\psi_n(x), \quad n = 1, 2, \ldots \tag{1}$$

and Dirichlet boundary conditions at the string ends $\psi_n(\pm 1/2) = 0$. This equation can be straightforwardly converted into [2–7]

$$\frac{1}{\sqrt{\Sigma(x)}}\left[-\frac{d^2}{dx^2}\right]\frac{1}{\sqrt{\Sigma(x)}}\phi_n(x) = E_n\phi_n(x), \tag{2}$$

where $\phi_n(x) \equiv \sqrt{\Sigma(x)}\psi_n(x)$. If $\Sigma(x)$ is a real positive function on $|x| \leq 1/2$, it follows that the operator $\hat{O} = \frac{1}{\sqrt{\Sigma(x)}}\left(-\frac{d^2}{dx^2}\right)\frac{1}{\sqrt{\Sigma(x)}}$ is hermitian. Another advantage of this form of the eigenvalue equation is that the inverse operator $\hat{O}^{-1} = \sqrt{\Sigma(x)}\left(-\frac{d^2}{dx^2}\right)^{-1}\sqrt{\Sigma(x)}$ can be directly expressed in terms of the Green functions of the homogeneous problem [6,7]. In what follows we assume that $\Sigma(x)$ is complex and $\mathscr{PT}$ symmetric.

In particular it is straightforward to generalize the results of [6,7], where exact expressions for the sum rules of inhomogeneous strings and drums have been derived, to the present case. For instance, being E_n the eigenvalues of a $\mathscr{PT}$ symmetric string obeying Dirichlet boundary conditions at its ends, we are interested in obtaining explicit expressions for the sum rules

$$Z_{DD}(s) = \sum_{n=1}^{\infty} E_n^{-s}, \quad s > 0 \tag{3}$$

with $s = 1, 2, \ldots$. Analogous expressions should also be considered for the case of different boundary conditions, as done in [6].

The case corresponding to $s = 1$ can be directly obtained from equation (11) of Ref. [6] and reads[1]

$$Z_{DD}(1) = \int_{-1/2}^{+1/2}\left(\frac{1}{4} - x^2\right)\Re\Sigma(x)dx. \tag{4}$$

Therefore the spectral sum rule $Z_{DD}(1)$ only depends upon the real part of the density.

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[1] We decompose an arbitrary $\Sigma(x)$ into even and odd parts, $\Sigma(x) = (\Sigma(x) + \Sigma(-x))/2 + (\Sigma(x) - \Sigma(-x))/2$ and then use the $\mathscr{PT}$-symmetry to establish that the even part of $\Sigma(x)$ is real, whereas the odd part is imaginary.

3. $\mathscr{PT}$ symmetry

$\mathscr{PT}$ symmetry is related to the antiunitary operator $\hat{P}\hat{T}$, where $\hat{P}$ and $\hat{T}$ are the parity and inversion operators, respectively [1]. In general an antiunitary operator $\hat{A}$ satisfies [8]

$$\hat{A}(f+g) = \hat{A}f + \hat{A}g$$
$$\hat{A}cf = c^*\hat{A}f,$$

(5)

for any pair of vectors f and g and an arbitrary complex number c, where the asterisk denotes complex conjugation. In particular, $\hat{A} = \hat{P}\hat{T}$ satisfies the additional condition $\hat{A}^2 = \hat{1}$.

In order to discuss the $\mathscr{PT}$ symmetry of inhomogeneous strings we rewrite Eq. (1) as

$$\hat{L}\psi = -\left[\frac{d^2}{dx^2} + E\,\Sigma(x)\right]\psi = 0.$$

(6)

It is clear that

$$\hat{A}\hat{L}\hat{A}^{-1}\hat{A}\psi = -\left[\frac{d^2}{dx^2} + E^*\,\Sigma(x)\right]\hat{A}\psi = 0,$$

(7)

provided that $\Sigma(-x)^* = \Sigma(x)$ as already assumed above. We appreciate that the eigenvalues are either real or pair of complex conjugate numbers. In the former case we have

$$\hat{L}\hat{A}\psi = 0.$$

(8)

One-dimensional eigenvalue equations with Dirichlet boundary conditions $\psi(\pm 1/2) = 0$ do not exhibit degeneracy and (8) holds only if $\hat{A}\psi = \lambda\psi$, from which it follows that $\hat{A}^2\psi = |\lambda|^2\psi = \psi$. In particular, when $\lambda = \pm 1$ it follows from $\hat{A}\psi(x) = \psi(-x)^* = \pm\psi(x)$ that the real and imaginary parts of $\psi(x)$ have definite parity: $\Re\psi(-x) = \pm\Re\psi(-x)$, $\Im\psi(-x) = \mp\Im\psi(-x)$. On the other hand, when symmetry is broken the eigenfunctions for the pair of complex conjugate eigenvalues E and E^* are ψ and $\hat{A}\psi$, respectively.

4. Rayleigh–Ritz method

In order to solve Eq. (1) we expand the solution as

$$\psi(x) = \sum_{m=1}^{\infty} c_m u_m(x),$$

(9)

where

$$u_m(x) = \sqrt{2}\sin\left[m\pi(x+1/2)\right].$$

(10)

Thus, the differential equation becomes the infinite matrix equation

$$\mathbf{LC} = \mathbf{0}$$

(11)

where $\mathbf{C}$ is a column vector of the coefficients c_n and $\mathbf{L}$ is a square matrix with elements

$$L_{mn} = n^2\pi^2\delta_{mn} - E\,\Sigma_{mn}, \quad \Sigma_{mn} = \int_{-1/2}^{1/2} u_m(x)\,\Sigma(x)u_n(x)\,dx.$$

(12)

The eigenvalues E_n are given by the roots of

$$F = \det\mathbf{L} = 0.$$

(13)

In practice we truncate the matrices at dimension N and calculate the roots of Eq. (13) for increasing values of N till we get the desired accuracy.

In all the cases discussed in this paper we have $F(E, \alpha) = 0$, where α is a parameter in the string density. The critical values of α are given by $d\alpha/dE = 0$ and we can obtain them from the set of equations

$$\{F(E, \alpha) = 0,\ \partial F(E, \alpha)/\partial E = 0\}.$$

(14)

This strategy proved suitable for the treatment of parameter-dependent $\mathscr{PT}$-symmetric Hamiltonian operators [9].

5. A class of solvable $\mathscr{PT}$-symmetric strings

In the case of a string with Dirichlet boundary conditions at $\pm L$ Amore [4] showed that if the density satisfies the differential equation

$$4\Sigma''(x)\Sigma(x) - 5\Sigma'(x)^2 - 16\kappa\Sigma(x)^3 = 0, \tag{15}$$

where κ is an arbitrary constant, the solution is of the form

$$\phi_n(x) = \sqrt{\frac{2}{\sigma(L)}} \Sigma(x)^{1/4} \sin\frac{n\pi\sigma(x)}{\sigma(L)}, \tag{16}$$

and

$$\sigma(x) \equiv \int_{-L}^{x} \sqrt{\Sigma(y)}dy.$$

The general solution to Eq. (15) for $L = 1/2$ is

$$\Sigma(x) = \frac{256c_1^2}{\left(c_1^2\left(c_2 + x\right)^2 - 256\kappa\right)^2},$$

where $c_{1,2}$ are constants of integration. This solution contains the *Borg string* [10], an inhomogeneous string isospectral to the homogeneous string, as a special case [2]:

$$c_1 = \frac{2+\alpha}{8\alpha}, \qquad c_2 = \frac{(1+\alpha)^2}{\alpha^4},$$

where $\alpha > -1$ is an arbitrary parameter. In this case the density is

$$\Sigma(x) = \frac{16(\alpha+1)^2}{(2\alpha x + \alpha + 2)^4}.$$

Remarkably, the spectrum of the Borg string is independent of α and coincides with the spectrum of a homogeneous string ($\alpha = 0$) of unit length:

$$E_n = n^2\pi^2.$$

The eigenfunctions are

$$\phi_n(x) = \frac{2\sqrt{2}\sqrt{\alpha+1}}{2\alpha x + \alpha + 2} \sin\left(\frac{\pi(\alpha+1)n(2x+1)}{2\alpha x + \alpha + 2}\right).$$

By means of a different choice of the constants of integration, for instance $c_1 = 1$ and $c_2 = i$, we obtain a complex density

$$\Sigma^{(\mathscr{PT})}(x) = \frac{256}{\left(256\kappa - x^2 - 2ix + 1\right)^2}, \tag{17}$$

that is invariant under the $\mathscr{PT}$ transformation.

In particular, the special case

$$\Sigma^{(\mathscr{PT})}(x) = \frac{\left(\alpha^2 + 64\right)^2}{16(\alpha x + 4i)^4} \tag{18}$$

is the $\mathscr{PT}$-symmetric analogous of the Borg string. Fig. 1 shows the first four eigenfunctions of the $\mathscr{PT}$-symmetric Borg string wth $a = 1$.
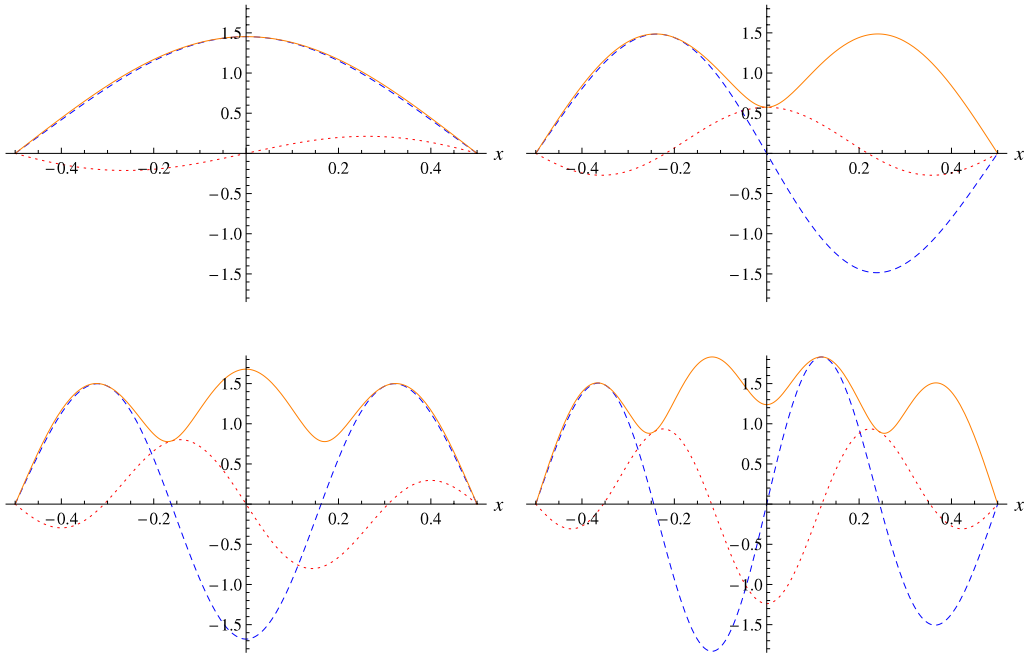
Fig. 1. First four eigenfunctions of the $\mathcal{PT}$-symmetric Borg string for $\alpha = 1$. The dashed and dotted lines are the real and imaginary parts respectively; the solid line is the modulus of the eigenfunction.

Using Eq. (16) we obtain the eigenfunctions of the $\mathcal{PT}$-symmetric Borg string as

$$\phi_n(x) = \sqrt{\frac{1}{2\sigma(L)}}\sqrt{\frac{\alpha^2 + 64}{\alpha^2 x^2 + 16}}\sqrt[4]{e^{-4i\arg(\alpha x + 4i)}}\,\sin\frac{n\pi\sigma(x)}{\sigma(L)} \tag{19}$$

where

$$\sigma(x) = -\frac{\left(\alpha^2 + 64\right)\sqrt{e^{-4i\arg(\alpha x + 4i)}}}{4\alpha(\alpha x - 4i)} - \frac{(\alpha - 8i)e^{\frac{1}{2}i\arg\left(\frac{(\alpha + 8i)^2}{(\alpha - 8i)^2}\right)}}{2\alpha} \tag{20}$$

and $n = 1, 2, \ldots$.

Direct substitution of Eq. (19) inside Eq. (2) shows that these are indeed the exact eigenfunctions of a string with density given in Eq. (18). The eigenvalues are easily obtained

$$E_n = \frac{1}{\phi_n(x)}\hat{O}\phi_n(x) = n^2\pi^2. \tag{21}$$

Thus we see that this string has a real spectrum and that it is isospectral to a homogeneous string with unit density; on the basis of this result we may conclude that one cannot "hear" the density of a $\mathcal{PT}$-symmetric string, if only Dirichlet boundary conditions are imposed, as for the case of a real string.

Having the exact eigenfunctions at our disposal we may easily check that these are orthogonal with respect to the operation

$$\int_{-L}^{+L}\phi_n(x)\phi_m(x)dx = \lambda_n^{-1}\int_{-L}^{+L}\phi_n^{\star}(-x)\phi_m(x)dx = \delta_{nm}. \tag{22}$$

Moreover

$$\overline{\delta}(x, y) \equiv \sum_{n=1}^{\infty}\phi_n(x)\phi_n(y) = \sum_{n=1}^{\infty}\phi_n^{\star}(-x)\phi_n^{\star}(-y) \tag{23}$$

has the Dirac-delta like properties

$$\int_{-L}^{L} \overline{\delta}(x, y) \phi_m(x) dx = \phi_m(y) \tag{24}$$

$$\int_{-L}^{L} \overline{\delta}(x, y) \phi_m^\star(-x) dx = \phi_m^\star(-y). \tag{25}$$

In Fig. 2 we plot the approximation to $\overline{\delta}(x, 0)$ obtained restricting the sum to the first 50 terms, $\overline{\delta}_{50}(x, 0)$, for $\alpha = 1/10$ and $\alpha = 1$ (left and right plots respectively). Notice that for $\alpha = 0$, $\overline{\delta}(x, y)$ reduces to the Dirac delta function and the imaginary part vanishes identically.

All the sum rules (3), $s = 1, 2, \ldots, 9$ calculated analytically by means of the formulas given in Ref. [6] agree with the straightforward sums coming from the spectrum $E_n = n^2 \pi^2$.

This $\mathscr{PT}$-symmetric model is another example like the Hamiltonian $\hat{H} = \hat{p}^2 + \hat{x}^2 + i\alpha\hat{x}$ with real spectrum $E_n(\alpha) = (2n + 1) + \alpha^2/4$ for all real α. We can also add $\hat{H} = \hat{p}^2 + i\alpha\hat{p}$ with the boundary conditions $\psi(\pm L/2) = 0$ with spectrum $E_n(\alpha) = n^2 \pi^2/L^2 + \alpha^2/4$.

6. First example

We consider a string with unit length ($L = 1/2$) with density

$$\Sigma(x) = 1 + i\alpha x. \tag{26}$$

Here we assume that α is a real arbitrary parameter. To begin with, note that if the parameter-dependent string density $\Sigma(\alpha, x)$ satisfies $\Sigma(-\alpha, -x) = \Sigma(\alpha, x)$ then the eigenvalues $E_n(\alpha)$ are symmetric about $\alpha = 0$: $E_n(-\alpha) = E_n(\alpha)$. This is exactly the case of the $\mathscr{PT}$-symmetric density (26).

In this case we use the exact formulas of Ref. [6] and obtain:

$$Z_{DD}(1) = \frac{1}{6} \tag{27}$$

$$Z_{DD}(2) = \frac{1}{90} - \frac{\alpha^2}{5040} \tag{28}$$

$$Z_{DD}(3) = \frac{1}{945} - \frac{\alpha^2}{30240} \tag{29}$$

$$Z_{DD}(4) = \frac{197\alpha^4}{10897286400} - \frac{17\alpha^2}{3742200} + \frac{1}{9450} \tag{30}$$

$$Z_{DD}(5) = \frac{17\alpha^4}{3269185920} - \frac{59\alpha^2}{102162060} + \frac{1}{93555} \tag{31}$$

$$Z_{DD}(6) = -\frac{2341\alpha^6}{1364608498176000} + \frac{16771\alpha^4}{16672848192000} - \frac{359\alpha^2}{5108103000} + \frac{691}{638512875} \tag{32}$$

$$Z_{DD}(7) = -\frac{15773\alpha^6}{22808456326656000} + \frac{46667\alpha^4}{285105704083200} - \frac{1237\alpha^2}{148864716000} + \frac{2}{18243225} \tag{33}$$

$$Z_{DD}(8) = \frac{8458133\alpha^8}{51894121836144107520000} - \frac{689371\alpha^6}{3986227909984320000} + \frac{736579\alpha^4}{30490471131120000} - \frac{68197\alpha^2}{70871446327500} + \frac{3617}{325641566250} \tag{34}$$

$$Z_{DD}(9) = \frac{111789019\alpha^8}{1323300106821674741760000} - \frac{114268283\alpha^6}{330825026705418685440 0} + \frac{5281763\alpha^4}{1577881881035460000} - \frac{627073\alpha^2}{5716963337085000} + \frac{43867}{38979295480125}. \tag{35}$$
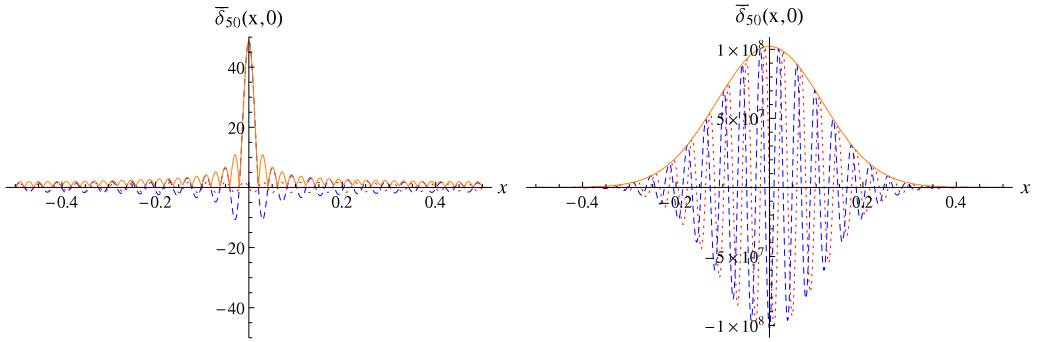
Fig. 2. Approximate $\mathcal{PT}$-delta function $\bar{\delta}_{50}(x, 0)$ for $\alpha = 1/10$ (left plot) and $\alpha = 1$ (right plot). The dashed and dotted lines are the real and imaginary parts respectively; the solid line is the modulus.
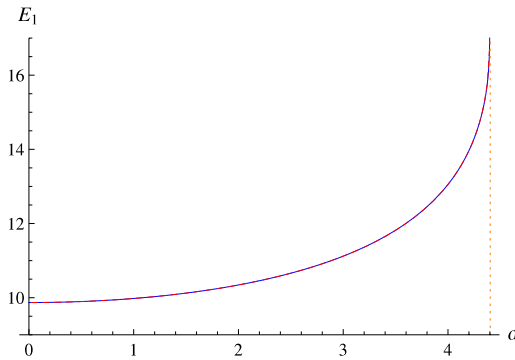

Fig. 3. Lowest eigenvalue as a function of α estimated using the sum rules up to order 9, and performing 3 repeated Shanks transformations. The circles correspond to the numerical values obtained with collocation.

We may estimate the lowest eigenvalue of the string using the inequalities [11]

$$Z_{DD}(s)^{-1/s} \le E_1^{(DD)} \le \frac{Z(s)}{Z(s+1)}. \tag{36}$$

Since the $Z_{DD}(n)$ are polynomials in α, the occurrence of real roots signals that $Z_{DD}(s)^{-1/s}$ can now take complex values, and therefore that the spectrum cannot be completely real.

In Ref. [6] it has been proved that one can use the sequence of approximations to the lowest eigenvalue $E_1 \approx Z_{DD}(n)^{-1/n}$, to obtain very accurate analytical approximations to E_1: using the same strategy we have performed four repeated Shanks transformations obtaining a precise analytical formula. This formula exhibits a singularity at $\alpha^\star \approx 4.40272$ that is quite close to the accurate Rayleigh–Ritz result $\alpha_1 = 4.397159356361900$. Fig. 3 shows the estimate obtained with the Shanks transformations and the actual value of α_1 (vertical line). We have also calculated the eigenvalues of the $\mathcal{PT}$ string by means of a collocation method developed some time ago [12].

At $\alpha = 0$ the eigenvalues of this string are those of the homogeneous string. As $|\alpha|$ increases pairs of eigenvalues start to approach each other and coalesce at a particular critical value, α_n, beyond which they become pairs of complex conjugate numbers. More precisely, pairs of eigenvalues (E_{2n-1}, E_{2n}), $n = 1, 2, \ldots$, coalesce at the critical point α_n where $E_{2n-1}(\alpha_n) = E_{2n}(\alpha_n) = e_n$. It is most interesting that in this case $\alpha_1 > \alpha_2 > \cdots$ so that for each value of $\alpha_{n+1} < \alpha < \alpha_n$ there is a finite number of real eigenvalues! This behaviour is completely different from the one that takes place in a class of $\mathcal{PT}$-symmetric Hamiltonian operators, where $\alpha_1 < \alpha_2 < \cdots$ [1].

By means of the Rayleigh–Ritz method outlined in Section 4 we calculated several pairs of critical parameters $\{e_n, \alpha_n\}$ and carried out nonlinear regressions of the form

$$\alpha_n = b + c|e_n|^{-s}. \tag{37}$$

For this particular string we obtained

$$b = 3.4685067 \pm 0.00090795610$$
$$c = 4.2281164 \pm 0.027739157$$
$$s = 0.53669526 \pm 0.0023416105, \tag{38}$$

which suggests that there is an infinite number of real eigenvalues when $0 < \alpha < b$.

7. A $\mathscr{PT}$-string with real negative eigenvalues

Another most interesting $\mathscr{PT}$ string is given by the density

$$\Sigma(x) = (1 + i\alpha x)^2, \tag{39}$$

where α is a real parameter and $|x| \leq 1/2$ as before.

Once again we use the exact formulas of Ref. [6] and obtain the first 7 sum rules:

$$Z_{DD}(1) = \frac{1}{6} - \frac{\alpha^2}{120} \tag{40}$$

$$Z_{DD}(2) = \frac{\alpha^4}{50400} - \frac{\alpha^2}{630} + \frac{1}{90} \tag{41}$$

$$Z_{DD}(3) = -\frac{29\alpha^6}{432432000} + \frac{\alpha^4}{92400} - \frac{\alpha^2}{4200} + \frac{1}{945} \tag{42}$$

$$Z_{DD}(4) = \frac{251\alpha^8}{1029188160000} - \frac{23\alpha^6}{378378000} + \frac{1499\alpha^4}{567567000} - \frac{\alpha^2}{31185} + \frac{1}{9450} \tag{43}$$

$$Z_{DD}(5) = -\frac{3221\alpha^{10}}{35198235072000 00} + \frac{773\alpha^8}{2514159648000} - \frac{3313\alpha^6}{154378224000}$$
$$+ \frac{83\alpha^4}{170270100} - \frac{691\alpha^2}{170270100} + \frac{1}{93555} \tag{44}$$

$$Z_{DD}(6) = \frac{16965349\alpha^{12}}{4862213796375936000000} - \frac{759931\alpha^{10}}{519467285937600000}$$
$$+ \frac{1646627\alpha^8}{11292767085600000} - \frac{1204631\alpha^6}{230988417660000} + \frac{15047\alpha^4}{192972780000}$$
$$- \frac{\alpha^2}{2027025} + \frac{691}{638512875} \tag{45}$$

$$Z_{DD}(7) = -\frac{5405503\alpha^{14}}{402869143128291840000000} + \frac{211469\alpha^{12}}{315728168595840000 00}$$
$$- \frac{460458127\alpha^{10}}{52095147818313600 0000} + \frac{3219703\alpha^8}{715592689812000 00} - \frac{7523137\alpha^6}{7259635983600000}$$
$$+ \frac{565843\alpha^4}{49497518070000} - \frac{3617\alpha^2}{62026965000} + \frac{2}{18243225}. \tag{46}$$

The fact that $Z_{DD}(n)$ can take negative values signals that part of the spectrum must be complex.

A useful strategy to obtain approximate solutions to the string with density (39) is to apply the Rayleigh–Ritz method as indicated in Section 4 or the collocation approach to the operator $\hat{O}$. In Fig. 4 we show the numerical results for the real and imaginary parts of the first ten eigenvalues of the string with density (39) for $-10 \leq \alpha \leq 10$: these results are obtained using a collocation approach
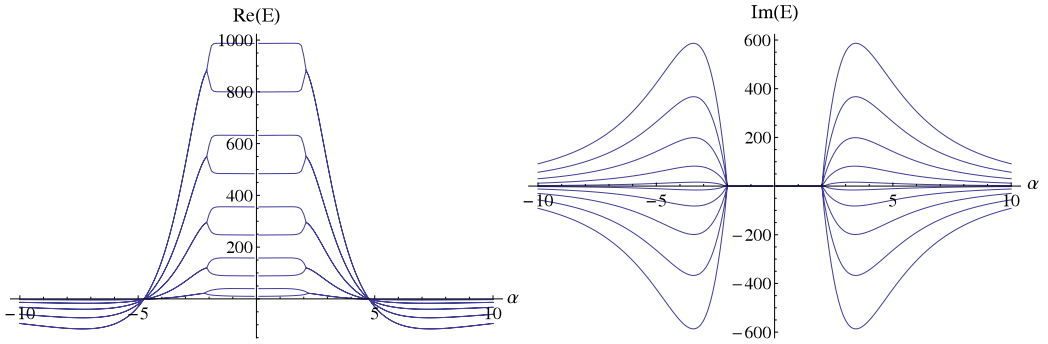
Fig. 4. Real and imaginary parts of the first ten eigenvalues of the string with density (39) for $-10 \le \alpha \le 10$.
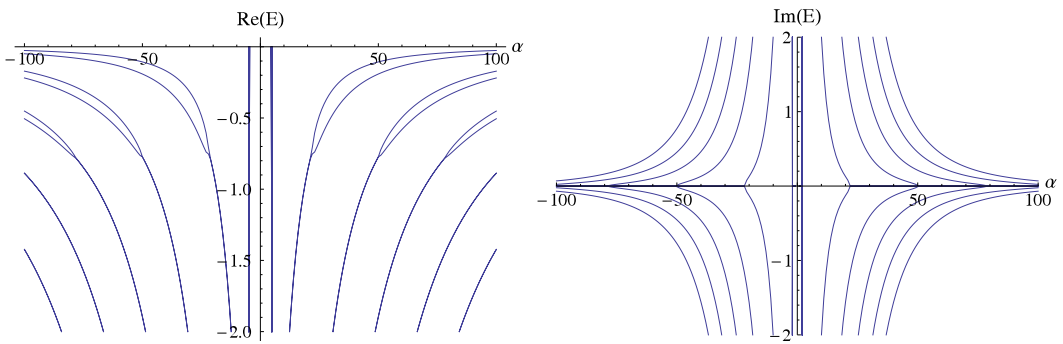

Fig. 5. Real and imaginary parts of the first ten eigenvalues of the string with density (39) for $-100 \le \alpha \le 100$.

with a grid with 2000 points [12]. Looking at the right plot we see that the eigenvalues are real when $-2 \lesssim \alpha \lesssim 2$. In Fig. 5 we show the same results for $-100 \le \alpha \le 100$: in this case pairs of real *negative* eigenvalues appear when α reaches the critical values. The first pair coalesce at $\pm\alpha_1$, where $\alpha_1 = 21.90376732248$.

It is interesting to focus on the second region, where the spectrum contains pairs of *real negative eigenvalues*. In particular, we choose $\alpha = 30$, where a single pair of such eigenvalues appears. In Fig. 6 we plot the real and imaginary parts of the eigenfunctions of the first two modes, whose energies are real and negative. In Fig. 7 we plot the real and imaginary parts of the eigenfunctions of the third and fourth modes, which exhibit complex conjugate eigenvalues. These solutions are numerical approximations to the eigenfunctions of Eq. (1). We may get an idea of the precision of our collocation calculation from the results of Table 1, which compares the numerical sum rules for this string at $\alpha = 30$ with the exact ones shown above. It follows from those figures that $\psi_1(-x)^* = \psi_1(x)$, $\psi_2(-x)^* = -\psi_2(x)$ and that $\psi_3(-x)^* = \psi_4(x)$ in complete agreement with the discussion at the end of Section 3.

In this case the nonlinear fitting yields two sets of critical parameters

$$b = -0.77692697 \pm 2.7920949 \times 10^{-5}$$
$$c = 13.397511 \pm 0.29472502$$
$$s = 1.8798088 \pm 0.0072380433, \tag{47}$$

for $e_n < 0$ and

$$b = 2.0000002 \pm 1.1418782 \times 10^{-5}$$
$$c = 0.70814609 \pm 0.00032819029$$
$$s = 0.50227919 \pm 0.00015369406 \tag{48}$$

Table 1

Comparison between the sum rules for the string (39) at $\alpha = 30$ obtained using the numerical values obtained with collocation on a grid with $N = 2000$ and the exact sum rules.

| q	$Z_{num}^{(DD)}(q)$	$Z_{exact}^{(DD)}(q)$	$\left| Z_{num}^{(DD)}(q)/Z_{exact}^{(DD)}(q) - 1 \right|$
1	$-7.32958160 + 4.13 \times 10^{-10}\, i$	$-\frac{22}{3}$	0.00051
2	$+14.65396825 - 2.58 \times 10^{-9}\, i$	$\frac{4616}{315}$	2.1×10^{-10}
3	$-40.33560515 + 1.24 \times 10^{-8}\, i$	$-\frac{5450752}{135135}$	4.74×10^{-10}
4	$+117.80838771 - 5.31 \times 10^{-8}\, i$	$\frac{9472421696}{80405325}$	7.34×10^{-10}
5	$-353.88875146 + 2.12 \times 10^{-7}\, i$	$-\frac{973145269792}{2749862115}$	1.01×10^{-10}
6	$+1082.41430676 - 8.11 \times 10^{-7}\, i$	$\frac{5139579853771120064}{4748255660523375}$	1.29×10^{-9}
7	$-3351.2084737 + 3.01 \times 10^{-6}\, i$	$-\frac{2636911102632544448}{786853795172445}$	1.56×10^{-9}
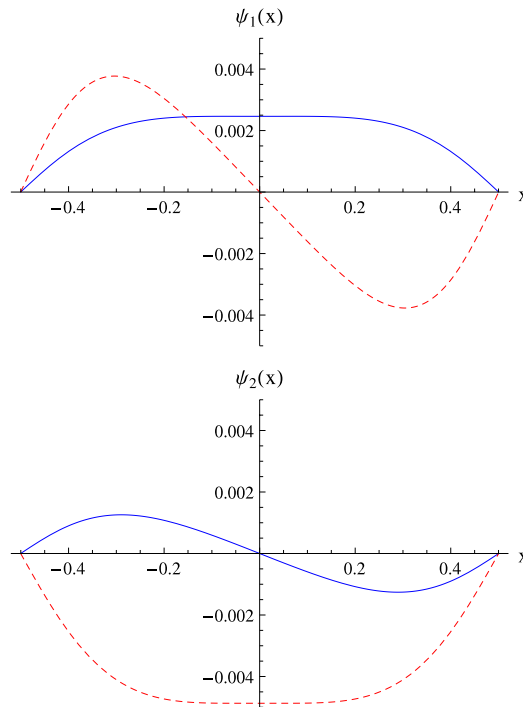
Fig. 6. Real (solid) and imaginary (dashed) parts of the eigenfunctions of the first two modes of the string (39) for $\alpha = 30$.

for $e_n > 0$. In the latter case we conjecture that the exact asymptotic relation may be

$$\alpha_n = 2 + \frac{1}{\sqrt{2e_n}}. \tag{49}$$

8. Conclusions

The purpose of this paper is to enlarge the class of $\mathscr{PT}$-symmetric models with the addition of parameter-dependent inhomogeneous strings with complex densities that satisfy $\Sigma^{\star}(\alpha, -x) = \Sigma(\alpha, x)$. We discussed an exactly solvable example with real spectrum for all values of α. This trivial inhomogeneous string is isospectral with the homogeneous one (a $\mathscr{PT}$-symmetric analogue of the string found by Borg [10] some time ago).
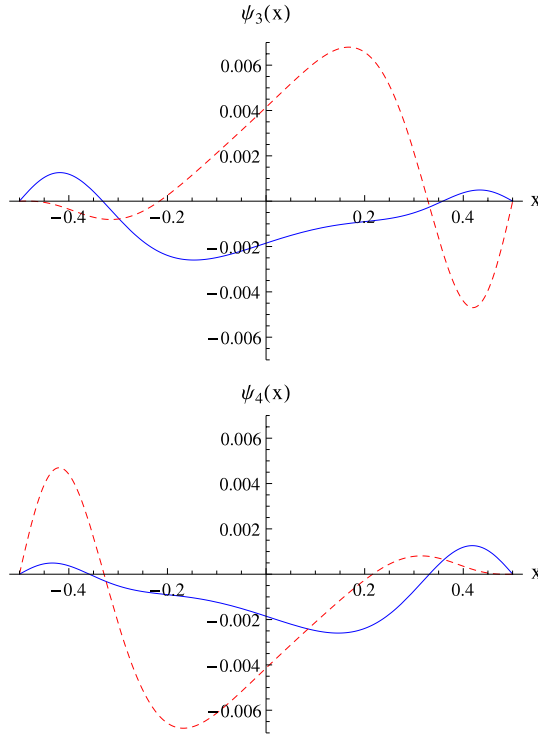
Fig. 7. Real (solid) and imaginary (dashed) parts of the eigenfunctions of the third and fourth modes of the string (39) for $\alpha = 30$.

We also discussed two nontrivial strings for which one can obtain exact sum rules thus extending Amore's result [6] to the $\mathcal{PT}$-symmetric realm. The accurate calculation of the critical parameters revealed that one of the strings exhibits real positive spectrum and the other one both positive and negative eigenvalues. Obviously, such negative eigenvalues cannot take place when the operator $\hat{O}$ is Hermitian.

Another interesting feature of the $\mathcal{PT}$-symmetric strings is that the behaviour of the critical parameters is different from that one observed in $\mathcal{PT}$-symmetric Hamiltonians like $\hat{H} = \hat{p}^2 + \hat{x}^4 + i\alpha\hat{x}$ or $\hat{H} = \hat{p}^2 + i\hat{x}^3 + i\alpha\hat{x}$ (see [1] and references therein).

Acknowledgement

References

[1] C.M. Bender, Rep. Progr. Phys. 70 (2007) 947–1018.
[2] P. Amore, Ann. Phys. 325 (2010) 2679–2696.
[3] P. Amore, J. Math. Phys. 51 (2010) 052105.
[4] P. Amore, Ann. Phys. 326 (2011) 2315–2355.
[5] P. Amore, J. Math. Phys. 53 (2012) 123519.
[6] P. Amore, Ann. Phys. 338 (2013) 341–360.
[7] P. Amore, Ann. Phys. 336 (2013) 223–244.
[8] E. Wigner, J. Math. Phys. 1 (1960) 409–413.
[9] F.M. Fernández, J. Garcia, arXiv:1305.5164 [math-ph].
[10] G. Borg, Acta Math. 78 (1946) 1–96.
[11] M.V. Berry, J. Phys. A 19 (1986) 2281–2296.
[12] P. Amore, M. Cervantes, F.M. Fernández, J. Phys. A 40 (2007) 13047–13062.