RESONANTLY FORCED ECCENTRIC RINGLETS: RELATIONSHIPS BETWEEN SURFACE DENSITY, RESONANCE LOCATION, ECCENTRICITY AND ECCENTRICITY-GRADIENT

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Abstract. We use a simple model of the dynamics of a narrow-eccentric ring, to put some constraints on some of the observable properties of the real systems. In this work we concentrate on the case of the 'Titan ringlet of Saturn'.

Our approach is fluid-like, since our description is based on normal modes of oscillation rather than in individual particle orbits. Thus, the rigid precession of the ring is described as a global $m = 1$ mode, which originates from a standing wave superposed on an axisymmetric background. An integral balance condition for the maintenance of the $m = 1$ standing-wave can be set up, in which the differential precession induced by the oblateness of the central planet must cancel the contributions of self-gravity, the resonant satellite forcing and collisional effects. We expect that in nearly circular narrow rings dominated by self-gravity, the eccentricity varies linearly across the ring. Thus, we take a first order expansion and we derive two integral relationships from the rigid-precession condition. These relate the surface density of the ring with the eccentricity at the centre, the eccentricity gradient and the location of the secular resonance.

These relationships are applied to the Titan ringlet of Saturn, which has a secular resonance with the satellite Titan in which the ring precession period is close to Titan's orbital period. In this case, we estimate the mean surface density and the location of the secular resonance.

Key words: eccentric ringlets, planetary rings

1. Introduction

The dynamical mechanism that maintains the apse alignment of the observed narrow-eccentric planetary rings is basically governed by self-gravity (Goldreich and Tremaine, 1979), which would provide the appropriate contribution to counter-act the differential precession induced by the oblateness of the central planet. However, predictions of the total mass of the ring produced by this model are, in general, not in good agreement with the inferred mass of observed eccentric rings (Tyler et al., 1986; Goldreich and Porco, 1987; Graps et al., 1995). This led to the consideration of other factors that might play an important role in the dynamics. In particular, at their narrowest point, the ring particles are 'close-packed'. In such a situation particle interaction or pressure effects may affect the precession of particle orbits. A simple model where the *pinch* locks the differential precession, was introduced by Dermott and Murray (1980). A more global picture, including the effect of stresses due to particle interactions and neighbouring satellite perturbations, which offered a better agreement with the observations, has also been produced by Borderies et al. (1983). Their dynamical model is described in terms of mutually interacting *streamlines* and the satellite interactions (see Goldreich and Tremaine, 1981) are computed using a resonance-continuum approximation. The standard self-gravity model was later revisited by Chiang and Goldreich (2000), who considered the effects of collisions near the edges, proposing that a sharp increase of an order of magnitude in the surface density should be observed within the last few hundred metres of the ring edges. More recently, employing a pressure term that describes close-packing, Mosqueira and Estrada (2002) obtained surface-density solutions that agree well with the currently available mass estimates.

The eccentric precessing-pattern of the ring can be described as being generated by a normal mode of oscillation of wave-number $m = 1$, which can be viewed as a standing wave. The conditions for the maintenance of steady global $m = 1$ modes have been considered by Papaloizou and Melita (2005). To describe the ring perturbations and the $m = 1$ mode we used the Lagrangian-displacement of the particle orbits from their unperturbed circular ones (see for example Shu et al., 1985). This model includes the dissipation due to interparticle collisions, which would lead to damping of the mode. However, this global $m = 1$ mode can also be perturbed by neighbouring-shepherd satellites, which can inject energy and angular momentum through resonances. In this way, losses due to particle collisions can be balanced. Two conditions for the maintenance of the rings can be derived. The first one is a condition for the steady maintenance of the amplitude or eccentricity associated with the $m = 1$ mode, which requires the external satellite torque to balance the dissipative effects due to collisions (see Papaloizou and Melita, 2005). The second one is the condition of uniform precession of the ring, which, in the absence of satellite resonances with the $m = 1$ mode, only involves self-gravity and the effect of collisions. These conditions can be regarded as continuum forms of the discrete relationships that can be obtained from the 'many streamlines' model (Goldreich and Tremaine, 1979; Longaretti and Rapapport, 1995).

In this work we produce an extension of Papaloizou and Melita (2005) to the case where the pattern frequency of the narrow-eccentric ringlet is in a secular resonance with an external satellite. In this case, there is a contribution to the condition of uniform precession, arising from the resonant secular perturbation. Resonantly forced rings are of particular interest, because there is a real system that the model can be applied to. It is known that the precession frequency of the eccentric Saturnian ringlet at 1.29 R_S (popularly known as the 'Titan ringlet') is in a 1:0 resonance with the orbital frequency of the Saturnian satellite Titan (Porco et al., 1984).

From the rigid precession condition two useful relations are derived. If the eccentricity gradient is approximately constant across the ring, which is expected to be the case for a narrow ringlet dominated by self-gravity, then, these relationships can constrain the mean surface density, the central eccentricity, the eccentricity gradient and the location of the secular resonance. We estimate the mean surface density as a function of the location of the resonance and the form of the ring in the case where physical collisions are neglected.

This article is organized as follows. In Section 2 we set up the equations for the Lagrangian variations starting from the equations of motion in a 2D flat disk approximation. In Section 3 we give adequate approximations for the evolution of the $m = 1$ mode when the precession time is much longer than the orbital period. In Section 4 we derive the condition of rigid precession which incorporates secular satellite resonances and we also compute the contribution from the self gravity of the ring. Two integral relationships are obtained from the rigid-precession condition in Section 5. The relation between the eccentricity gradient, the value of the central eccentricity, the location of the resonance and the surface density is given in Section 6, where we also discuss how to obtain the eccentricity gradient from the other parameters in the linear and the non-linear regimes. In Section 7 we apply these results to the Titan ringlet of Saturn to produce estimates of its mean surface-density. Finally, a discussion of the results is given in Section 8.

2. Equations of Motion and Lagrangian Displacement

We start from the basic equations of motion for a particle in Lagrangian form in 2D:

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = F_r - \frac{\partial \psi}{\partial r}, \tag{1}$$

$$r\frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} + 2\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = F_\theta - \frac{1}{r}\frac{\partial \psi}{\partial \theta}. \tag{2}$$

Here (r, θ) define the cylindrical coordinates of the particle referred to an origin at the centre of mass of the planet. Here $\psi(r)$ denotes the gravitational potential due to both the central planet, the neighbouring satellites and the ring. In addition (F_r, F_θ) denote the radial and azimuthal components of any additional force $\mathbf{F}$ per unit mass respectively. This may arise through internal interactions between particles that might lead to an effective pressure and/or viscosity.

We introduce a Lagrangian description in which the system is supposed to be perturbed from an axisymmetric state in which particles are in circular motion with coordinates such that $r = r_0, \theta = \theta_0 = \Omega(r_0)t + \beta_0$ Here r_0 is the fixed radius of the particle concerned, $\Omega(r_0)$ is the angular velocity and β_0 is a phase factor labeling each particle. In keeping with a Lagrangian description (r_0, β_0) are conserved quantities for a particular particle and so may be used to label it.

In order to describe the system when it is perturbed from the axisymmetric state we introduce the components of the Lagrangian displacement $\boldsymbol{\xi} = (\xi_r, \xi_\theta)$. These are such that the coordinates of each particle satisfy:

$$r = r_0 + \xi_r, \tag{3}$$

and

$$r_0(\theta - \theta_0) = \xi_\theta. \tag{4}$$

To obtain equations for ξ_r and ξ_θ we take variations of Equations (1) and (2). We do this by applying the Lagrangian difference operator, Δ, as defined by Lebovitz (1967) to both sides of Equations (1) and (2). For a given quantity Q, the variation $\Delta(Q)$ is defined by:

$$\Delta(Q) = Q(\mathbf{r}_0 + \boldsymbol{\xi}) - Q_0(\mathbf{r}_0), \tag{5}$$

where Q and Q_0 are the values of the given physical quantity in the perturbed and unperturbed flow, respectively. In contrast, the Eulerian difference operator is defined as:

$$\delta(Q) = Q(\mathbf{r}_0) - Q_0(\mathbf{r}_0). \tag{6}$$

Thus, to first order, they are related through:

$$\Delta = \delta + \boldsymbol{\xi} \cdot \nabla, \tag{7}$$

which gives the linear form of the Lagrangian difference operator.

2.1. EQUATIONS FOR THE LAGRANGIAN DISPLACEMENT

Following Shu et al. (1985) we assume that the components of the displacement are small enough that they can be treated as linear in the sense that $|\xi/r_0| \ll 1$. On the other hand the radial gradient of the radial displacement may be large so that $|\partial\xi_r/\partial r_0|$ may be of order unity. The significance of these assumptions is that although the ring eccentricity is assumed to be everywhere small, the ring surface density perturbation induced by it may be of order unity. Adopting them enables us to perform the variation in the accelerations using the linear form of the difference operator as described above, wherever radial gradients are not involved. These then satisfy:

$$\frac{\mathrm{d}^2\xi_r}{\mathrm{d}t^2} - 2\Omega\frac{\mathrm{d}\xi_\theta}{\mathrm{d}t} + 2\xi_r r_0\Omega\frac{\mathrm{d}\Omega}{\mathrm{d}r_0} = f_r - \frac{\partial\psi'}{\partial r}, \tag{8}$$

$$\frac{\mathrm{d}^2\xi_\theta}{\mathrm{d}t^2} + 2\Omega\frac{\mathrm{d}\xi_r}{\mathrm{d}t} = f_\theta - \frac{1}{r}\frac{\partial\psi'}{\partial\theta}. \tag{9}$$

Here the potential due to the satellite, ψ_S, and that due to the self-gravity of the ring, ψ_{SG} are included in ψ'. Thus $\psi' = \psi_{SG} + \psi_S$. The quantities $f_r = \Delta(F_r)$, $f_\theta = \Delta(F_\theta)$ denote the variational components of the force per unit mass due to particle interactions. The full non-linear Lagrangian variation is retained for ψ' and $\mathbf{F}$ as these may involve the density variation. Contributions coming from the variation of the central planet potential are included on the left hand sides of Equations (8) and (9).

2.2. SURFACE DENSITY PERTURBATION AND LAGRANGIAN TIME DERIVATIVES

We suppose the ring particles to be in eccentric orbits and combine to form a globally eccentric ring. This is described using a surface density distribution $\Sigma(r,\theta)$ and eccentricity distribution $e(r)$ We also consider there to be an axisymmetric reference state for which $e(r) = \xi_r/r_0$ and from which we can regard the eccentric ring as being the result of a perturbation. The perturbation of the surface density is of the form:

$$\Sigma(r,\theta,t) \rightarrow \Sigma(r,\theta) + \Sigma'(r,\theta,t). \tag{10}$$

For linear perturbations $\sum' \propto \cos/\sin(m\theta)$, where the azimuthal mode number, $m = 1$. The eccentric ring can be thought of as being predominantly in a mode with azimuthal mode number $m = 1$. In practice we may assume $|e| \ll 1$.

We further remark that the convective derivative $\mathrm{d}/\mathrm{d}t$ is taken following the fluid motion. In the approximation scheme used here in which the displacements and hence Lagrangian velocity perturbations are small, we may replace the fluid motion by its unperturbed value. Then for any quantity Q

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\partial Q}{\partial t} + \Omega\frac{\partial Q}{\partial\theta_0}. \tag{11}$$

Similarly

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} = \frac{\partial^2 Q}{\partial t^2} + \Omega^2\frac{\partial^2 Q}{\partial\theta_0^2} + 2\Omega\frac{\partial^2 Q}{\partial\theta_0\partial t}. \tag{12}$$

3. Forcing of the m = 1 (Eccentric) Mode

We here consider the forcing of the $m = 1$ mode which causes the ring to become eccentric. The forcing is assumed to be due to an external satellite with mass M_S . With an aim of application to the Titan ringlet around Saturn we consider that the perturbing potential acting on the ring is stationary in a frame rotating with the mean orbital rotation rate when viewed from an inertial frame. In terms of the ring dynamics this pattern rotates at a low frequency, Ω_p, such that $\Omega_p \ll \Omega$. A free $m = 1$ mode that is most easily excited is one that has a global structure in the ring and has a pattern that precesses at a rate comparable to Ω_p. When this precession rate is equal to Ω_p, there is the possibility of resonance and a large response to forcing. The quantity Ω_p^{-1} sets the natural time scale for variations associated with the $m = 1$ mode for the problem on hand.
Accordingly:

$$\frac{\partial}{\partial t} \ll \Omega\left(\frac{\partial}{\partial \theta_0}\right). \tag{13}$$

Recalling that the left hand side of Equation (9) approximated by the linearized form, gives for the azimuthal component of the displacement:

$$\frac{d\xi_\theta}{dt} + 2\Omega\xi_r = Q_{\theta_0}, \tag{14}$$

where the quantity Q_{θ_0} is defined by:

$$\frac{\partial Q_{\theta_0}}{\partial t} + \Omega\frac{\partial Q_{\theta_0}}{\partial \theta_0} = f_\theta - \frac{1}{r}\frac{\partial \psi'}{\partial \theta_0}. \tag{15}$$

Using (13) gives the adequate approximation:

$$\Omega\frac{\partial Q_{\theta_0}}{\partial \theta_0} = f_\theta - \frac{1}{r}\frac{\partial \psi'}{\partial \theta_0}. \tag{16}$$

We comment that the motion is dominated by the central mass and to the lowest order Keplerian, This means that the $m = 1$ component of the displacement satisfies (Shu et al., 1985):

$$\frac{\partial^2 \xi_r}{\partial \theta_0^2} = -\xi_r. \tag{17}$$

Furthermore (14) tells us to lowest order in which Q_{θ_0} and $\frac{\partial}{\partial t}$ may be neglected that:

$$\frac{\partial \xi_\theta}{\partial \theta_0} = -2\xi_r, \tag{18}$$

which applies to Keplerian orbits with small eccentricity.

Also using (14) and Equation (8) one finds that the $m = 1$ component of the displacement satisfies:

$$\frac{\partial^2 \xi_r}{\partial t^2} + 2\Omega \frac{\partial^2 \xi_r}{\partial t \partial \theta_0} - \xi_r(\Omega^2 - \kappa^2) = f_r - \frac{\partial \psi'}{\partial r} + 2\Omega Q_{\theta_0}, \tag{19}$$

Here the square of the epicyclic frequency is given by:

$$\kappa^2 = \frac{2\Omega}{r_0} \frac{\mathrm{d}(r_0^2 \Omega)}{\mathrm{d}r_0}. \tag{20}$$

4. The Condition for a Steady State Response in the Rotating Frame

The $m = 1$ mode responsible for the ring eccentricity has a constant and very small pattern speed as viewed in the inertial frame. This means that individual ring particles appear to be in elliptic orbits that precess at the same rate. In order to achieve this the internal and external forces acting in the mode have to satisfy a constraint that can be view as a non-linear dispersion relation. Our treatment again follows that of Shu et al. (1985) who provided such a relationship for density waves in Saturn's rings. Except here we consider a density wave comprising a global normal mode rather than a forced propagating wave.

Equation (19) can also be written in the form:

$$\frac{\mathrm{d}^2 \xi_r}{\mathrm{d}t^2} + \xi_r \kappa^2 = f_r - \frac{\partial \psi'}{\partial r} + 2\Omega Q_{\theta_0}. \tag{21}$$

We now use an angle that is fixed with respect to a coordinate system rotating at the pattern angular frequency Ω_P namely $\phi_0 = \theta_0 - \Omega_P t$. The radial displacement is taken to be of the form $\xi_r = A(r_0) \cos(\phi_0)$. Following Shu et al. (1985) we note that as the time dependence is contained within ϕ_0, ξ_r only depends on r_0 and ϕ_0.

Multiplying Equation (21) by $\cos(\phi_0)$ and integrating over ϕ_0 , we obtain:

$$\begin{aligned} \frac{1}{2}\left(\frac{\kappa^2}{(\Omega - \Omega_P)^2} - 1\right) A(r_0) = {} & \frac{1}{(\Omega - \Omega_P)^2} (F_{cr} + g_D(r_0) + g_R(r_0) \\ & + \frac{1}{2\pi} \int_0^{2\pi} 2\Omega Q_{\theta_0} \cos(\phi_0) \mathrm{d}\phi_0), \end{aligned} \tag{22}$$

where

$$F_{cr} = \frac{1}{2\pi} \int_0^{2\pi} f_r \cos(\phi_0) \mathrm{d}\phi_0, \tag{23}$$

$$g_D(r_0) = -\frac{1}{2\pi}\int_0^{2\pi}\cos(\phi_0)\frac{\partial\psi_{\mathrm{SG}}}{\partial r}\mathrm{d}\phi_0, \tag{24}$$

and

$$g_R(r_0) = -\frac{1}{2\pi}\int_0^{2\pi}\cos(\phi_0)\frac{\partial\psi_{\mathrm{S}}}{\partial r}\mathrm{d}\phi_0, \tag{25}$$

gives the forcing due to the satellite potential which is here considered to be responsible for the excitation of the $m = 1$ mode.

With the use of Equation (16), the last term in Equation (22) can be re-written and after an integration it reads as:

$$\frac{1}{2\pi}\int_0^{2\pi} 2\Omega Q_\theta\cos(\phi_0)\mathrm{d}\phi_0 = -2\,(F_{c\theta} + g_T), \tag{26}$$

where

$$F_{c\theta} = \frac{1}{2\pi}\int_0^{2\pi} f_\theta\sin(\phi_0)\mathrm{d}\phi_0, \tag{27}$$

and

$$g_T = -\frac{1}{2\pi}\int_0^{2\pi}\sin\phi_0\frac{1}{r}\frac{\partial\psi_{\mathrm{S}}}{\partial\phi_0}\mathrm{d}\phi_0. \tag{28}$$

Equation (22) then becomes:

$$\frac{1}{2}\left(\frac{\kappa^2}{(\Omega-\Omega_{\mathrm{P}})^2} - 1\right)A(r_0) = \frac{g_{\mathrm{int}} + g_{\mathrm{S}}}{(\Omega-\Omega_{\mathrm{P}})^2}, \tag{29}$$

where

$$g_{\mathrm{int}} = (F_{cr} - 2F_{c\theta}) + g_D, \tag{30}$$

and

$$g_{\mathrm{S}} = g_R - 2g_T. \tag{31}$$

Given that $\kappa = \Omega - \omega_{\mathrm{prec}}$ where $\omega_{\mathrm{prec}}\,(r_0)$ is the local radius dependent precession frequency and assuming that $\Omega_{\mathrm{P}} \ll \Omega$ and $\omega_{\mathrm{prec}} \ll \Omega$, Equation (29) can be approximated to first order in Ω_{P} and ω_{prec} as:

$$(\Omega_{\mathrm{P}} - \omega_{\mathrm{prec}})\,A(r_0) = \frac{g_{\mathrm{int}} + g_{\mathrm{S}}}{\Omega}. \tag{32}$$

Equation (32) provides a condition to be satisfied by the excited $m = 1$ mode amplitude. It balances the ring self gravity, internal collisional terms and satellite forcing. For simplicity we shall neglect collisional effects below and thus replace g_{int} by g_D. Note further that for a thin ring of the type considered here, Ω may be taken as constant in (32) and evaluated at the ring centre from now on.

4.1. THE SELF-GRAVITY TERM

In order to calculate g_D we follow Shu et al. (1985). As radial variations are much more rapid than azimuthal ones, the local self-gravity at r_0 is canonically and adequately approximated to be that due to an infinite plane sheet of radial width $\Delta r = r_2 - r_1$, where r_1 and r_2 are the inner and outer bounding radii of the unperturbed ring respectively. Thus:

$$\left(\frac{\partial \psi_{\rm SG}}{\partial r}\right) = 2G \int_{r_1}^{r_2} \frac{\Sigma(r')}{(r - r')} \mathrm{d}r' \tag{33}$$

where G is the gravitational constant, $r = r_0 + \xi_r$ and $r' = r'_0 + \xi'_r$, where

$$\xi_r = \xi_r(r_0) = A(r_0) \cos(\phi_0), \tag{34}$$

and

$$\xi'_r = \xi_r(r'_0) = A(r'_0) \cos(\phi_0).$$

Possible singularities in the integrand are dealt with by evaluating the integral in a principal value sense. In the planar limit, we identify the ring eccentricity as $e(r_0) = 2A(r_0)/(r_1 + r_2)$. Using the tight-winding approximation we have:

$$\Sigma(r')\mathrm{d}r' = \Sigma(r'_0)\mathrm{d}r'_0, \tag{35}$$

which represents conservation of mass. Since $r_0/r \approx 1$, we have:

$$\left(\frac{\partial \psi_{\rm SG}}{\partial r}\right) = 2G \int_{r_1}^{r_2} \frac{\Sigma(r'_0)}{r_0 + \xi_r - r'_0 - \xi'_r} \mathrm{d}r'_0. \tag{36}$$

We can re-write Equation (36) in terms of the eccentricity gradient, q:

$$q = \frac{A(r_0) - A(r'_0)}{r_0 - r'_0}. \tag{37}$$

Then after integrating over ϕ, we obtain (see also Shu et al., 1985):

$$g_D = 2G \int_{r_1}^{r_2} \frac{I(q)}{q} \Sigma(r'_0) \frac{A(r_0) - A(r'_0)}{(r_0 - r'_0)^2} \mathrm{d}r'_0, \tag{38}$$

where

$$I(q) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos(\phi)}{1 - q\cos\phi} \mathrm{d}\phi = \frac{1}{q\sqrt{1 - q^2}} \left(1 - \sqrt{1 - q^2}\right). \tag{39}$$

Notice that the integrand in Equation (38) presents a singularity to be handled in a principal value sense. This can lead to practical complications near ring edges.

5. Two Integral Relations

The practical problem is to solve Equation (32) for the response to the forcing by the external satellite. This is equivalent to calculating the forced eccentricity. To do this requires an accurate specification of the ring surface density profile which may not be available. Instead one may derive two integral relations which contain complete information about the response when it is a linear function of radius. That is equivalent to assuming the constancy of q defined above which is a frequently adopted approximation in planetary ring dynamics (eg. Goldreich and Tremaine, 1979; Borderies et al., 1983; Shu et al., 1985; Chiang and Goldreich, 2000).

A justification for this is that normally (as here) one considers the case when ring self-gravity is strong enough to balance differential precession. In that case the ring precesses at a uniform rate similar to a rigid body. When this process is effective, strong self-gravity precludes short wavelength displacements so that approximating the induced displacement as a linear function of the distance to the ring centre is reasonable as long as the eccentricity response is not too large.

For example, a strict resonance between the pattern speed and the uniform precession frequency of the ring might result in large eccentricities being excited for which the dependence of the precession frequency on eccentricity should not be neglected. We note that if $e^2 > |(\Delta a/\omega_{\rm prec}) \times (d\omega_{\rm prec}/dr)|$, Δa being the ring semi-major axis width, that effect becomes comparable to that due to differential precession and should not be neglected – as we have done – in comparison to that. However, this situation is not encountered for the application considered here.

5.1. THE FIRST RELATION

To obtain this we take Equation (32) multiply by $\Sigma(r_0)$ and integrate over the ring to obtain

$$\int \Sigma(r_0)\Omega(\Omega_{\rm P} - \omega_{\rm prec})A(r_0)dr_0 - \int \Sigma(r_0)g_D dr_0 = \int \Sigma(r_0)g_S dr_0. \quad (40)$$

From Equation (38) the second integral on the left hand side of Equation (40) is zero. The first relation thus simplifies to become

$$\int \Sigma(r_0)\Omega(\Omega_{\rm P} - \omega_{\rm prec})A(r_0)dr_0 = \int \Sigma(r_0)g_S dr_0. \quad (41)$$

5.2. THE SECOND RELATION

To obtain this we take Equation (32) multiply by $\Sigma(r_0)A(r_0)$ and integrate over the ring to obtain

$$\int \Sigma(r_0)\Omega(\Omega_{\rm P} - \omega_{\rm prec})A^2(r_0){\rm d}r_0 - \int \Sigma(r_0)A(r_0)g_D{\rm d}r_0 = \int \Sigma(r_0)A(r_0)g_{\rm S}{\rm d}r_0. \tag{42}$$

In this case we use Equation (38) to evaluate the second term of Equation (42), which, after making use of the symmetry properties of the integral, gives:

$$\int \Sigma(r_0)A(r_0)g_D{\rm d}r_0 = G\int_{r_1}^{r_2}\int_{r_1}^{r_2}\frac{I(q)}{q}\Sigma(r_0')\Sigma(r_0)\frac{(A(r_0)-A(r_0'))^2}{(r_0-r_0')^2}{\rm d}r_0'{\rm d}r_0, \tag{43}$$

which is positive definite. This may also be written entirely in terms of q as

$$\int \Sigma(r_0)A(r_0)g_D{\rm d}r_0 = G\int_{r_1}^{r_2}\int_{r_1}^{r_2} I(q)q\Sigma(r_0')\Sigma(r_0){\rm d}r_0'{\rm d}r_0. \tag{44}$$

If we now specialize to the case when q is constant we accordingly write $A(r_0) = A_c + qx$ as a linear function of radius. Here A_c is constant and $x = r_0 - r_c$ measures the radial coordinate relative to the centre of mass of the unperturbed ring assumed slender. The eccentricity at the ring centre satisfies $e = |A_c|/r_c$, while the sign of A_c determines whether pericentre is in the direction $\phi_0 = 0$, or π if negative or positive respectively. This means that by definition

$$\int_{r_1}^{r_2} x\Sigma{\rm d}x = 0, \tag{45}$$

where the domain of integration for the above and similar integrals below is the extent of the ring. We further adopt a linear form for the precession frequency, thus

$$\omega_{\rm prec} = \omega_{\rm prec,0} + \omega'_{\rm prec,0}x, \tag{46}$$

where $\omega_{\rm prec,0}$ and $\omega'_{\rm prec,0}$ represent the precession frequency and its derivative evaluated at $x = 0$. Consistent with the approximations made here we may also assume the forcing potential term $g_{\rm S}$ as constant throughout the ring.

The first relation (41) then gives a first relation between A_c and q in the form

$$\int \Sigma\Omega(\Omega_{\rm P} - \omega_{\rm prec,0})A_c{\rm d}x - \int \Sigma\Omega\omega'_{\rm prec,0}qx^2{\rm d}x = \int \Sigma g_{\rm S}\cdot{\rm d}x \tag{47}$$

The second relation similarly leads to a second which takes the form

$$\int \Sigma\Omega(\Omega_{\mathrm{P}} - \omega_{\mathrm{prec},0})(A_c^2 + q^2x^2)\mathrm{d}x - 2\int \Sigma\Omega\omega'_{\mathrm{prec},0}A_c q x^2\mathrm{d}x =$$
$$\int \Sigma\Omega\omega'_{\mathrm{prec},0}q^2x^3\mathrm{d}x + G\int_{x_1}^{x_2}\int_{x_1}^{x_2} I(q)q\Sigma(x')\Sigma(x)\mathrm{d}x'\mathrm{d}x + \int \Sigma A_c g_{\mathrm{S}}\mathrm{d}x. \tag{48}$$

We now have two relations which enable the forced response to be calculated under the assumption of constant q. These are A_c, whose magnitude, when divided by the radius at the ring centre gives the eccentricity at the ring center and q itself which is the product of the central radius and the eccentricity gradient. Note that implicit in the thin ring approximation is the requirement that the magnitude of the eccentricity gradient significantly exceeds the ratio of the eccentricity to radius.

6. The Relation Between q and Central Eccentricity

If we multiply the first relation (47) by A_c and subtract it from the second relation (48), the terms involving satellite forcing cancel out and we get a relation between A_c and q which (recalling that q is constant and Ω is evaluated in the centre of the ring) takes the form.

$$q\Omega(\Omega_{\mathrm{P}} - \omega_{\mathrm{prec},0})\int \Sigma x^2\mathrm{d}x - \Omega\omega'_{\mathrm{prec},0}A_c\int \Sigma x^2\mathrm{d}x =$$
$$q\Omega\omega'_{\mathrm{prec},0}\int \Sigma x^3\mathrm{d}x + G\int_{x_1}^{x_2}\int_{x_1}^{x_2} I(q)\Sigma(x')\Sigma(x)\mathrm{d}x'\mathrm{d}x. \tag{49}$$

This may be cast in the very simple form

$$A_c = \frac{q(\Omega_{\mathrm{P}} - \omega_{\mathrm{prec},0})}{\omega'_{\mathrm{prec},0}} - \frac{q\int \Sigma x^3\mathrm{d}x}{\int \Sigma x^2\mathrm{d}x} - \frac{GI(q)(\int \Sigma\mathrm{d}x)^2}{\Omega\omega'_{\mathrm{prec},0}\int \Sigma x^2\mathrm{d}x}. \tag{50}$$

6.1. DETERMINATION OF q

We may now use the relation between q and A_c specified above, to eliminate A_c in the first relation (47) and so obtain q in terms of the satellite forcing. This gives

$$q\left(\frac{(\Omega_{\mathrm{P}} - \omega_{\mathrm{prec},0})^2}{(\omega'_{\mathrm{prec},0})^2} - \frac{(\Omega_{\mathrm{P}} - \omega_{\mathrm{prec},0})}{\omega'_{\mathrm{prec},0}}\frac{\int x^3\Sigma\mathrm{d}x}{\int x^2\Sigma\mathrm{d}x} - \frac{\int \Sigma x^2\mathrm{d}x}{\int \Sigma\mathrm{d}x}\right)$$
$$-\frac{(\Omega_{\mathrm{P}} - \omega_{\mathrm{prec},0})GI(q)(\int \Sigma\mathrm{d}x)^2}{\Omega(\omega'_{\mathrm{prec},0})^2\int \Sigma x^2\mathrm{d}x} = \frac{g_{\mathrm{S}}}{\Omega\omega'_{\mathrm{prec},0}}. \tag{51}$$

Thus Equations (50) and (51) give the response parameters A_c and q directly in terms of the external forcing.

6.2. THE LINEAR REGIME

When the response is in the linear regime q is small and $I(q) = q/2$. In this case the right hand side of (51) is proportional to q and we have

$$a_1 q = \frac{g_S}{\Omega\omega'_{\mathrm{prec},0}}, \tag{52}$$

where

$$a_1 = \frac{(\Omega_P - \omega_{\mathrm{prec},0})^2}{(\omega'_{\mathrm{prec},0})^2} - \frac{\int \Sigma x^2 \mathrm{d}x}{\int \Sigma \mathrm{d}x} - \frac{(\Omega_P - \omega_{\mathrm{prec},0})}{\omega'_{\mathrm{prec},0}} \frac{\int x^3 \Sigma \mathrm{d}x}{\int x^2 \Sigma \mathrm{d}x} - \frac{(\Omega_P - \omega_{\mathrm{prec},0}) G(\int \Sigma \mathrm{d}x)^2}{2\Omega(\omega'_{\mathrm{prec},0})^2 \int \Sigma x^2 \mathrm{d}x}. \tag{53}$$

The response is singular when $a_1 = 0$. Regarding this as an equation for $\Omega_p - \omega_{\mathrm{prec},0}$, we have a quadratic with two real roots indicating a singular response for certain pattern speeds. However, the relation (51) is in fact a non-linear one (through the functional form of $I(q)$) to determine q and the non-linearity present can remove such singularities. This becomes apparent when we expand to the next highest order in q. Then one obtains a cubic equation for q that can always be solved because the coefficients of q^3 and q never vanish simultaneously. This cubic takes the form

$$a_1 q - q^3 \frac{3(\Omega_P - \omega_{\mathrm{prec},0}) G(\int \Sigma \mathrm{d}x)^2}{8\Omega(\omega'_{\mathrm{prec},0})^2 \int \Sigma x^2 \mathrm{d}x} = \frac{g_S}{\Omega\omega'_{\mathrm{prec},0}}. \tag{54}$$

However, as indicated above we must be cautious about using the above determinations of A_c and q when the eccentricity response is large because then the assumption of constant q and the neglect of the dependence of externally induced orbital precession on the eccentricity may no longer be valid.

7. Application to the Titan Ringlet

We here consider the Titan ringlet for which the ring precession rate is close to the orbital frequency of the satellite Titan. This ringlet is therefore a candidate for having its eccentricity forced by Titan. We now consider the forcing potential.

7.1. THE SATELLITE POTENTIAL

For the low frequency $m = 1$ forcing considered here the perturbing effect of the satellite arises through

$$g_S = -\frac{1}{2\pi}\int_0^{2\pi}\left(\cos(\phi_0)\frac{\partial\psi_S}{\partial r} - 2\sin(\phi_0)\frac{1}{r}\frac{\partial\psi_S}{\partial\phi_0}\right)\mathrm{d}\phi_0. \tag{55}$$

For the satellite we neglect the orbital eccentricity and expand ψ_s to leading order in r/a_S, a_S being the semi-major axis of the satellite orbit. For the forcing considered here, recalling that in the frame rotating with the orbital frequency Ω_p, only secular terms $\propto \cos(\phi_0)$ are significant, we obtain, including the indirect potential, to leading order

$$\psi_S = -\frac{3GM_S r^3}{8a_S^4}\cos(\phi_0). \tag{56}$$

Then evaluating at the ring centre $r = r_c$, $r_c \equiv a$, we obtain

$$g_S = \frac{15GM_S r_c^2}{16a_S^4}. \tag{57}$$

We are now ready to apply Equations (50) and (51) to the Titan ringlet.

7.2. PARAMETERIZING THE DYNAMICAL MODEL

We define the dimensionless parameter η through

$$\Omega_P - \omega_{\mathrm{prec},0} = \eta\, \omega'_{\mathrm{prec},0}\, \Delta a. \tag{58}$$

This defines the resonance where the precession rate induced by the planet and Titan match in the approximation that the former can be represented by a first order Taylor expansion about the ring centre. It gives the resonance location at a distance $\eta\Delta a$ from the ring centre.

We also find it convenient to define the dimensionless quantities γ_n related to the ring surface density profile through

$$\gamma_n = \frac{\int x^n \Sigma \mathrm{d}x}{(\Delta a)^n \int \Sigma \mathrm{d}x}. \tag{59}$$

The location of the centre of mass of the ring, r_c, is the origin of the coordinate x (Equation (45)). Then, inside the ring we have $|x| < \Delta a$ – the extreme case is when the mass is concentrated at a ring edge. Thus, it is verified that $|\gamma_n| < 1$. We also notice that $\gamma_n > 0$ when n is even and, when n is odd, $\gamma_n = 0$ if the surface density is symmetric.

Further we introduce the parameter Φ which measures the importance of self-gravity with respect to differential precession:

$$\Phi = \frac{G\overline{\Sigma}}{2\Omega\omega'_{\text{prec},0}\,(\Delta a)^2}, \tag{60}$$

where the mean surface density is defined as:

$$\overline{\Sigma} = \frac{\int \Sigma \mathrm{d}x}{\Delta a}, \tag{61}$$

notice that since the precession frequency decreases with distance (i.e. $\omega'_{\text{prec},0} < 0$), then Φ is defined as a negative quantity.

In terms of these dimensionless parameterizations, Equations (50) and (51) can be reduced to

$$A_c = q\Delta a\, a_2, \tag{62}$$

where

$$a_2 = \eta - \frac{\gamma_3}{\gamma_2} - \frac{\Phi}{\gamma_2}, \tag{63}$$

and

$$q\Delta a = \frac{g_{\text{S}}}{\Omega\omega'_{\text{prec},0}\Delta a}(\eta a_2 - \gamma_2)^{-1}. \tag{64}$$

7.3. ESTIMATION OF THE SURFACE DENSITY AND THE LOCATION OF THE SECULAR RESONANCE

The presently available observations of the Titan ringlet do not enable one to determine all its dynamical and physical parameters. Optical depth profiles give the most accurate description of the ring, thus the surface density can only be inferred by making assumptions on the physical properties of the ring particles or through dynamical models which, like the present one, rely on various assumptions (see for example Goldreich and Tremaine, 1979). Moreover, there are considerable uncertainties in the values of the multiple moments of Saturn (for up-to-date values see the JPL-Solar System Dynamics website: http://ssd.jpl.nasa.gov/sat_gravity.html). In fact, the uncertainties in the precession frequency of the ring due to the uncertainties in multiple moments of Saturn, imply an error in the location of the secular resonance that is of the order of the width of the ring.

Using our model we shall attempt to put some constraints on the value of Φ and so, on the mean surface density, $\bar{\Sigma}$, as well as on the resonance location parameter, η, for the Titan ringlet.

We shall adopt the following values: $a = 77871$ km, $\Delta a = 25$ km, $e = 2.6 \times 10^{-4}$, $\delta \mathrm{e} = (1.4 \pm 0.4) \times 10^{-4}$, (Porco et al., 1984). Thus $q = a\delta e/\Delta a \approx 0.44 \pm 0.18$.

The closest approach between the satellite and the ring occurs at apoapse (Porco et al., 1984). Then, according to the definitions of the satellite potential (Equation (56)) and of the radial displacement (Equation (34)) adopted here, when $\phi = 0$, $\xi_r > 0$. Thus, for the Titan ringlet we have $A_r(r_0) > 0$ as well as $A_c > 0$.

Now we shall rewrite Equations (62) and (64) as:

$$a_3 = \eta\, a_2 - \gamma_2, \tag{65}$$

$$-\Phi = (a_2 - \eta)\gamma_2 + \gamma_3, \tag{66}$$

where

$$a_3 = \frac{g_S}{\Omega\omega'_{\mathrm{prec},0}q(\Delta a)^2}. \tag{67}$$

Notice that $a_3 < 0$ since $\omega'_{\mathrm{prec},0} < 0$ and $g_S > 0$.

We can eliminate η from Equations (65) and (66), to obtain a quadratic expression of Φ as a function of the form factor γ_2:

$$-\Phi = -\frac{1}{a_2}\gamma_2^2 + \left(a_2 - \frac{a_3}{a_2}\right)\gamma_2 + \gamma_3. \tag{68}$$

Similarly, one can eliminate γ_2 and express Φ as a function of η:

$$-\Phi = -a_2\,\eta^2 + (a_2^2 + a_3)\eta - a_3 a_2 + \gamma_3. \tag{69}$$

One must recall that η and γ_2 are related by Equation (65), which acts as a constraint for η since it must be verified that $\gamma_2 < 1$.

Note also that a_2 can be expressed as a *relative eccentricity*,

$$a_2 \approx e_c/\delta e,$$

through the use of Equation (62). The quadratic terms in Equations (68) and (69) depend on a_2. Then, if the relative eccentricity is large, is: $-\Phi \approx a_2\gamma_2 + \gamma_3$. Whereas if a_2 is small, $-\Phi \approx a_3\eta_2 - a_3 a_2 + \gamma_3$.

One may expect to find observed values of γ_2 between those of two critical cases. One extreme case for the shape of the surface-density distribution is when it is constant throughout the ring,

$$\Sigma^{(1)} \neq \Sigma^{(1)}(x),$$

the other when the whole mass is concentrated at both edges, as:

$$\Sigma^{(2)} \propto \delta(x - \Delta a/2) + \delta(x + \Delta a/2),$$

where δ is the Dirac-δ function. In these cases the values of γ_2 are given by $\gamma_2^{(1)} = 1/12$ and $\gamma_2^{(2)} = 1/4$, respectively.

We can estimate the values of a_2 and a_3 for the Titan ringlet as:

$$a_2 \approx \left(\frac{e}{\delta e}\right)_{\mathrm{Obs}} \approx 1.86, \tag{70}$$

$$a_3 \approx -1.38, \tag{71}$$

where these estimates are obtained from the observed parameters previously quoted and, in Equation (71), the precession frequency gradient, $\omega'_{\mathrm{prec},0}$, is computed using the J_2, J_4 and J_6 coefficients associated with the multipole moments of Saturn given in the JPL-SSD website: $J_2 = (16292 \pm 7) \times 10^{-6}$, $J_4 = (931 \pm 32) \times 10^{-6}$, $J_6 = (91 \pm 32) \times 10^{-6}$ (R. A. Jacobson, submitted). We computed the precession rate of the ring due to the oblateness of Saturn and the secular perturbation by Titan[1] and we obtain a value of $a_{\mathrm{Res}} = (77846. \pm 11.)$ km, for the location of the 1:0 secular resonance, where the errors only consider the uncertainties in the values of Saturn's J-coefficients.

For the symmetric case, we have plotted the relationships set by Equations (65), (68) and (69) for the values of a_2 and a_3 corresponding to the Titan ringlet.

We may recall that, if the surface density is symmetric with respect to the centre of mass, then $\gamma_3 = 0$ and that inside the ring $|x| \leqslant \Delta a/2$ and $|\eta| \leqslant 0.5$.

Figure (1) shows $\bar{\Sigma}$ versus γ_2 – where $\bar{\Sigma}$ is related to Φ by Equation (60). It can be seen that the scale of the mean surface density is tens of grams per cubic centimetre, in particular when $\gamma_2^{(1)} \leqslant \gamma_2 \leqslant \gamma_2^{(2)}$. In the symmetric case, the agreement with the solution quoted in Porco et al. (1984) occurs at a value of $\gamma_2 \approx \gamma_2^{(1)}$, however, in that case the numerical values of various parameters are different.

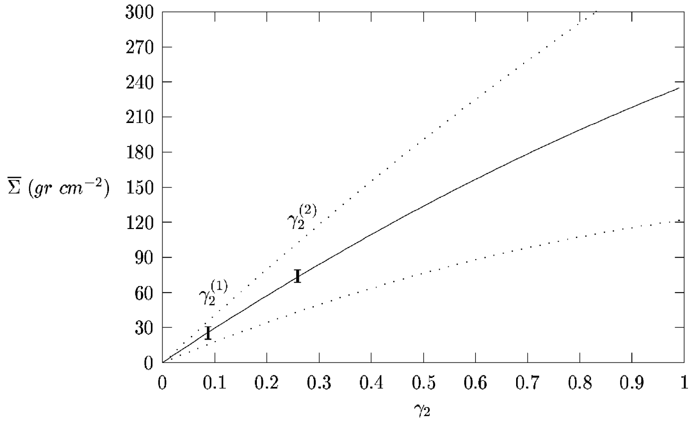

Figure 1. The estimated mean surface density, $\bar{\Sigma}$, as a function of γ_2 for a symmetric profile ($\gamma_3 = 0$). Notice the value of $\bar{\Sigma}$ when γ_2 is between the critical values $\gamma_2^{(1)} = 1/12$ and $\gamma_2^{(2)} = 1/4$ (indicated by the markers). The dotted lines correspond to the non-symmetric case where $|\gamma_3| = \gamma_2$.

[1] We use an expansion up to $(R_{\mathrm{Sat}}/a)^6$ in the oblateness term. For the secular term, we used an approximation up to first order in e for the perturbing function (see for example Murray and Dermott, 1999), where we have used a value of $a_{\mathrm{Titan}} = 1221900$ km for the semimajor axis of the satellite and of $m_{\mathrm{Titan}} = 1345.5 \times 10^{23}$ gr, for its mass.

In Figure (2) we plot η versus γ_2. This figure gives, for a symmetric profile, the location of the secular resonance as a function of the form of the surface density, such that the ring precesses rigidly when the effects of inter-particle collisions are negligible. The allowed values of η, set by Equation (65) and $\gamma_2 < 1$, imply that, given the condition of solid precession, the location of the secular resonance lies outside the ring ($|\eta| > 0.5$) for the symmetric case with $\gamma_2^{(1)} \leqslant \gamma_2 \leqslant \gamma_2^{(2)}$. We may recall that if $\eta \sim 0$, then the assumptions made to obtain these relationships break down, because the eccentricities resonantly excited may become very large. Thus, the rather large values of η obtained assure us that the first order approximation adopted here is consistent with the case. However, as mentioned earlier, the uncertainties involved are large.

Finally, in Figure (3) we plot $\bar{\Sigma}$ as a function of η. This figure presents the required values of the mean surface-density to produce the balance as a function of the location of the secular resonance. Naturally, the result reflects the particular choice of parameters adopted. All the solutions obtained require $\eta < 0$, hence the centre of mass of the ring must be located *interior* (i.e. towards the satellite) with respect to the secular resonance (see Equation (58)).

If the surface density profile were not to be symmetric then these estimates can be somewhat altered. By definition, the magnitude of γ_3 is bounded by that of γ_2 as:

$$|\gamma_3| \leqslant \frac{|x_{\max}|}{\Delta a} \gamma_2 \leqslant \gamma_2.$$

In Figure (1) we also show $\bar{\Sigma}$ as a function of γ_2, for the extreme cases in which $|\gamma_3| = \gamma_2$.

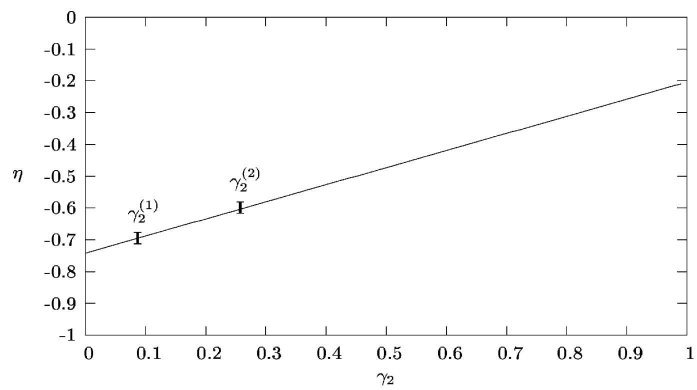

Figure 2. The location of the resonance η, as a function of γ_2 ($\gamma_3 = 0$) that gives the correct balance enabling the rigid precession of the ring, when the collisional terms are neglected. Notice that solutions with $\gamma^{(1)} \leqslant \gamma \leqslant \gamma^{(2)}$, correspond to resonance locations outside the ring ($|\eta| > 0.5$).

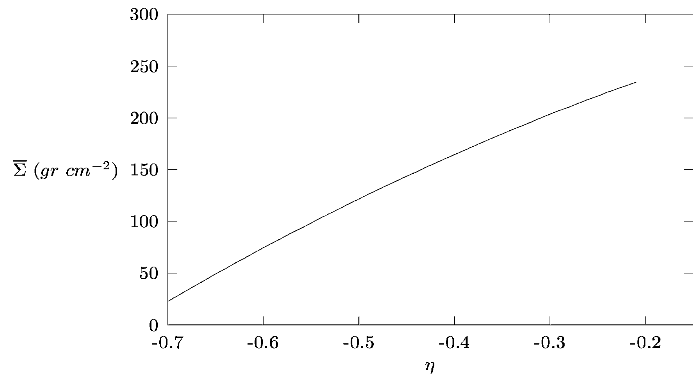

Figure 3. The estimated mean surface density, $\bar{\Sigma}$, as a function of $\eta(\gamma_3 = 0)$. A negative value of η implies that $\Omega_P > \omega_{prec,0}$ (Equation (58)), i.e. the center of mass of the ring is exterior – from the planet – to the location of secular resonance.

On the other hand, we do not expect the effect of the collisional terms to be negligible, as assumed in these calculations. We will explore their exact significance in a forthcoming paper. We will discuss in the next section how physical particle interactions could affect the results obtained here.

8. Discussion

In this paper we have used a model of a thin slender self-gravitating ring in orbit about a dominant central mass to produce relationships between its physical parameters. These relationships can then be applied to real systems to make predictions about their physical and/or dynamical properties.

We view the ring in uniform precession as sustaining a global non-axisymmetric $m = 1$ mode of oscillation. We have considered particularly the case in which the precession frequency is in a secular resonance with the orbital frequency of an external satellite, which is the case of the Titan ringlet of Saturn.

A condition for the ring to be able to maintain a $m = 1$ mode with single slow pattern speed is obtained. It can be expressed as an integral condition for the pattern or normal mode to precess at a uniform rate (Equation (29)) that requires the correct balance between differential precession, self-gravity, secular-resonant forcing and collisional effects. From this condition we have obtained two relationships (Equations (41) and (42)) which combine different observable parameters of the ringlets.

We argued that the scale of the perturbations is of the order of the width of the ring, because we are interested in cases in which the differential precession is compensated mainly by self gravity and the eccentricity is small.

Hence, a first order expansion is a good approximation and so, we take the eccentricity gradient as constant, which, indeed, is the case in real systems (Graps et al., 1995).

In the linear case, the two expressions obtained give simple relationships (Equations (47) and (48)) between the eccentricity at the centre, e_c, the eccentricity gradient, q, the location of the secular resonance, η and the mean value of the surface density $\bar{\Sigma}$, and its form, expressed by the γ_n's. The relationships can be further simplified by assuming that the system is also linear in q (Equations (50) and (52)).

Finally we have applied this model to the Titan ringlet. The scale of the surface density obtained is consistent with previous estimations (Porco et al., 1985), set at the order of ~10 g cm^{-2}. Our symmetric solutions imply that the secular resonance is outside the ring (see Figure (3)). However, the distance between the ring and the location of the secular resonance obtained is smaller than the total uncertainty in the location of the later, given the errors in the determination of the multipole moments of Saturn.

Moreover, additional physics may need to be considered. The fact that all the observed systems have shown a positive value of q is an indication that the differential precession is mainly balanced by self-gravity (Borderies et al., 1983; Papaloizou and Melita, 2005). However, it is also very likely that physical collisions play an important role in setting up the rigid precession, particularly at the edges, where the relative motion is larger (see for example Chiang and Goldreich, 2001). It is also believed that the real systems are close-packed at their narrowest point – as it seems to be indicated by the observations of the Uranian rings (see for example French et al., 1984). The narrowest points of the observed eccentric ringlets occur at periapse, since $q > 0$. Thus, the collisional contribution can be estimated to be impulsive, i.e. entirely concentrated at the *pinch*. At periapse, the geometry of the collisions at the edges is such that the collisionally produced orbital phase-shift produced *reinforces* the differential-precession induced by the oblateness of the central planet (see for example Moulton, 1935). Thus, if a considerable collisional contribution arises mainly at the pinch, self-gravity must counteract a greater differential precession. If that is the case, the values of the masses of the real eccentric ringlets will be *larger* than the estimates produced here. In fact, the masses of eccentric ringlets estimated from observations have always turned out to be greater than the theoretical estimates obtained with models that consider only the self-gravity of the ringlet (see for example Mosqueira and Estrada, 2001). It is straightforward to include collisional effects when they are assumed to act impulsively at pericentre. – more terms are to be retained when considering the two conditions for rigid precession given by (Equations (40) and (42)). We can gain some knowledge of these collisional effects if an observed profile is available – from which we can extract density estimates. We will present a more detailed analysis of some known eccentric ringlets in a forthcoming article.

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