



Brief paper

Stability preserving maps for finite-time convergence: Super-twisting sliding-mode algorithm[☆]J. Picó^{a,1}, E. Picó-Marco^a, A. Vignoni^a, H. De Battista^b^a Institut d'Automàtica i Informàtica Industrial, Universitat Politècnica de València, Spain^b CONICET. LEICI, Facultad de Ingeniería, Universidad Nacional de La Plata, La Plata, Argentina

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ABSTRACT

The super-twisting algorithm (STA) has become the prototype of second-order sliding mode algorithm. It achieves finite time convergence by means of a continuous action, without using information about derivatives of the sliding constraint. Thus, chattering associated to traditional sliding-mode observers and controllers is reduced. The stability and finite-time convergence analysis have been jointly addressed from different points of view, most of them based on the use of scaling symmetries (homogeneity), or non-smooth Lyapunov functions. Departing from these approaches, in this contribution we decouple the stability analysis problem from that of finite-time convergence. A nonlinear change of coordinates and a time-scaling are used. In the new coordinates and time-space, the transformed system is stabilized using any appropriate standard design method. Conditions under which the combination of the nonlinear coordinates transformation and the time-scaling is a stability preserving map are given. Provided convergence in the transformed space is faster than $\mathcal{O}(1/\tau)$ —where τ is the transformed time—convergence of the original system takes place in finite-time. The method is illustrated by designing a generalized super-twisting observer able to cope with a broad class of perturbations.

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1. Introduction

Sliding mode is a powerful technique used both in controller and observer design to reject matched disturbances (Utkin, 1977). The idea is to drive the state trajectory to a prescribed constraint surface where specifications are met and, from then on, to slide on it thanks to an intensive switching action. Because of its robustness and other attractive features, sliding mode has been successfully implemented in a wide variety of real processes (Chiu, 2012; Hung, Gao, & Hung, 1993; Šabanovic, 2011). However, its underlying chattering phenomenon may be inadmissible in some control applications and may add severe noise in estimation (Young, Utkin, & Ozgüner, 1999). Additionally, the prescribed constraint must have unit relative degree. Therefore the control action must explicitly appear in the first time derivative of the constraint function

(Sira-Ramírez, 1989). High order sliding mode (HOSM) has been developed to relax the relative degree limitation and, at the same time, to alleviate chattering (Bartolini, Ferrara, & Usai, 1998; Levant, 1993). Among all HOSM algorithms the super-twisting second-order one (SOSM) distinguishes because it achieves finite time convergence by means of a continuous action without using information about derivatives of the sliding function (Dávila, Fridman, & Levant, 2005; Levant, 1998). This algorithm handles a relative degree equal to one, so it can directly replace standard sliding mode algorithms when the disturbance is smooth and with bounded gradient. The main difference with respect to standard sliding mode is that discontinuity appears in the second derivative of the switching function, whereas a non-Lipschitz term appears in the first derivative to achieve finite-time convergence.

For many years, dissemination and acceptance of HOSM have been resisted due to the lack of powerful design tools and readable stability proofs. Originally, stability conditions were obtained geometrically using worst-case trajectory bounds (Levant, 1998). More recently, homogeneity concepts have been exploited to prove stability of some HOSM algorithms (Levant, 2005). Homogeneous systems have a scaling symmetry which allows for stability analysis. Despite its advantages, this approach does not provide the convergence time and is a limitation to design new algorithms to deal with broader classes of disturbances since the homogeneity property may be lost during the design process. Lyapunov-based stability analysis appeared for the first time in

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E-mail addresses: jpico@ai2.upv.es (J. Picó), enpimar@ai2.upv.es (E. Picó-Marco), alvig2@upvnet.upv.es (A. Vignoni), deba@ing.unlp.edu.ar (H. De Battista).

¹ Tel.: +34 96 3879770; fax: +34 96 3879579.

Moreno and Osorio (2008). Since then, very intensive research is being followed in this area (Cruz-Zabala, Moreno, & Fridman, 2011; Polyakov & Poznyak, 2009; Santiesteban, Fridman, & Moreno, 2010; Shtessel, Moreno, Plestan, Fridman, & Poznyak, 2010; Utkin, 2010). Different Lyapunov functions have been proposed for different sliding algorithms. For instance, in Moreno and Osorio (2008), stability of the super-twisting algorithm (STA) is proved by means of a non-smooth Lyapunov function. Utkin (2010) uses also non-smooth functions for the twisting and super-twisting algorithms. A strict Lyapunov function to prove stability of the twisting algorithm is presented in Santiesteban et al. (2010), and a generalization of the method of characteristics is presented in Polyakov and Poznyak (2009). In general, the Lyapunov approach allows to obtain less conservative designs, to compute the convergence time and, most importantly, to generalize the original algorithm to deal with more general system dynamics and disturbance structures (Cruz-Zabala et al., 2011; De Battista, Picó, Garelli, & Vignoni, 2011; Efimov & Fridman, 2011; Moreno, 2010; Pisano, Orlov, & Usai, 2011). The main drawback of current Lyapunov-based approaches is their dependence on complex tools – e.g. non-smooth analysis, solution of partial differential equations, etc. – to cope with the requirement of finite-time convergence together with the stability analysis. The use of time-scale, already used in other contexts – e.g. achieving feedforward form Moya, Ortega, Netto, Praly, and Picó (2002), or observer linearization and design of observers with linearizable error dynamics (Guay, 2002; Respondek, Pogromsky, & Nijmeijer, 2004) – has not been exploited.

The rest of the paper is organized as follows. In Section 2 a nonlinear coordinates transformation, and a time-scale one are used so as to transform the original system into a new one amenable for constructively finding a smooth control Lyapunov function. This allows to modify the super-twisting error injection terms so as to cope with a broader class of perturbations. The time-scale is chosen so that convergence faster than asymptotic in the transformed space corresponds to finite-time converge in the original one. In Section 3 the technique of stability preserving maps (Michel & Wang, 1995) is used to prove the original system is also stable. In Section 4 a bound on the finite-time convergence is obtained, and simulation results are provided in Section 5.

2. Constructive design of a generalized super-twisting algorithm

Consider the system:

$$\begin{aligned} \dot{x}_1 &= \varphi(t)x_2 + u_1(x_1, t) + \rho_1(x, t) \\ \dot{x}_2 &= u_2(x_1, t) + \rho_2(x, t) \end{aligned} \quad (1)$$

where $\varphi(t)$ is a known, possibly discontinuous, bounded positive function of time. The functions $\rho_1(x, t)$ and $\rho_2(x, t)$ are perturbation terms for which we assume the structure:

$$\begin{aligned} \rho_1(x, t) &= \varrho_1(t)p_1(x_1) |x_1|^{\frac{1}{2}}, & \|p_1(x_1)\| &\leq \bar{p}_1(x_1) \\ \rho_2(x, t) &= \varrho_2(t)p_2(x_1, x_2), & \|p_2(x_1, x_2)\| &\leq \bar{p}_2(x_1) \end{aligned} \quad (2)$$

where $\bar{p}_1(x_1), \bar{p}_2(x_1)$ are known bounded functions for any bounded x_1 (e.g. class \mathcal{K}_∞ functions), and $\varrho_j(t)$ bounded noises with $\bar{\varrho}_j = \|\varrho_j(\tau)\|_\infty$ for $j = 1, 2$. Notice that $\rho_2(x, t)$ may be not vanishing and discontinuous at $x_1 = 0$, while $\rho_1(x, t)$ vanishes at the origin, and is continuous w.r.t. x_1 . System (1) may represent a process to be controlled or the error dynamics for some observer design. In this later case, $\rho_1(x, t)$ allows to represent the approximation error of some dynamics on the first process state to be estimated, and $\rho_2(x, t)$ the unknown derivative of the second process state (De Battista et al., 2011).

The goal is to design the input signals u_1, u_2 so as to robustly stabilize the origin in finite-time. Besides $\varphi(t)$, only the state x_1 is

assumed to be measured. To this end, we first apply the coordinates transformation given by the homeomorphism (Moreno, 2010):

$$(z_1, z_2) \longrightarrow (|x_1|^{\frac{1}{2}} \text{sign}(x_1), x_2) \quad (3)$$

transforming system (1) into:

$$\begin{aligned} \dot{z}_1 &= \frac{1}{2} |z_1|^{-1} [\varphi(t)z_2 + u_1(z_1, t) + \rho_1(z, t)] \\ \dot{z}_2 &= u_2(z_1, t) + \rho_2(z, t). \end{aligned} \quad (4)$$

Now, apply the time-scaling

$$t = \int |z_1| d\tau. \quad (5)$$

In the new (z, τ) -coordinates:

$$\begin{aligned} z_1' &= \frac{\varphi(\tau)}{2} z_2 + \frac{1}{2} u_1(z_1, \tau) + \frac{1}{2} \varrho_1(\tau) |z_1| p_1(z_1) \\ z_2' &= |z_1| u_2(z_1, \tau) + |z_1| \varrho_2(\tau) p_2(z_1, z_2) \end{aligned} \quad (6)$$

with $z' \triangleq dz/d\tau$. The goal now is to asymptotically stabilize system (6). Let us apply, for instance, the Lyapunov redesign methodology. To this end, consider the control signal u_1 is decomposed as:

$$u_1(z_1, \tau) = u_{1b}(z_1, \tau) - \eta_1 \bar{p}_1(z_1) z_1 \quad (7)$$

with $\eta_1 \geq \bar{\varrho}_1$ so that $\forall z_1 \in \mathbb{R}$

$$\Psi_1(z_1, \tau) \triangleq \eta_1 \bar{p}_1(z_1) - \varrho_1(\tau) p_1(z_1) \geq 0. \quad (8)$$

Now, consider the Lyapunov function $V_1 = \frac{1}{2} z_1^2$. The control signals u_{1b} and u_2 will be designed later to force $\dot{z}_2 = \eta_2 z_1$ for some constant $\eta_2 > 0$ and achieve $V_1' \leq 0$. Taking τ -time derivative:

$$V_1' = \frac{\varphi(\tau)}{2} \eta_2 z_1^2 + \frac{1}{2} u_{1b}(z_1) z_1 - \frac{1}{2} z_1^2 \Psi_1(z_1, \tau). \quad (9)$$

Thus, choosing

$$u_{1b}(z_1, \tau) = -[\eta_2 \varphi(\tau) + k_1(z_1, \tau)] z_1 \quad (10)$$

the dynamics of the error signal $\bar{z}_2 = z_2 - \eta_2 z_1$ are:

$$\begin{aligned} \bar{z}_2' &= -\frac{\eta_2 \varphi(\tau)}{2} \bar{z}_2 + |z_1| u_2(z_1) \\ &\quad + z_1 \left[\bar{\varrho}_2(\tau) p_2(z_1, z_2) + \frac{\eta_2}{2} [k_1(z_1, \tau) + \Psi_1(z_1, \tau)] \right] \end{aligned} \quad (11)$$

where $\bar{\varrho}_2(\tau) = \varrho_2(\tau) \text{sign}(z_1)$. Consider now the augmented Lyapunov function:

$$V_2 = \frac{1}{2} z_1^2 + \frac{1}{2} \bar{z}_2^2. \quad (12)$$

Taking τ -time derivative, choosing

$$u_2(z_1, \tau) = -\frac{1}{2} [\varphi(\tau) + \eta_2 [k_1(z_1, \tau) + \eta_1 \bar{p}_1(z_1)]] \text{sign}(z_1) \quad (13)$$

and defining $\mathbf{z} = [z_1, \bar{z}_2]^T$, we have:

$$V_2' \leq -\frac{1}{2} \mathbf{z}^T \begin{bmatrix} k_1(z_1, \tau) & -q_{12}(z, \tau) \\ -q_{12}(z, \tau) & \eta_2 \varphi(\tau) \end{bmatrix} \mathbf{z} \triangleq -\frac{1}{2} \mathbf{z}^T \mathbf{Q} \mathbf{z} \quad (14)$$

with $q_{12}(z, \tau) \triangleq \bar{\varrho}_2 \bar{p}_2(z_1) + \frac{\bar{\varrho}_2 \eta_2}{2} \bar{p}_1(z_1)$. At this point, it is interesting to summarize and observe the structure of the injected correction terms in the original x -dynamics:

$$\begin{aligned} u_1(x_1, t) &= -[\eta_2 \varphi(t) + k_1(x_1, t) + \eta_1 \bar{p}_1(x_1)] |x_1|^{\frac{1}{2}} \text{sign}(x_1) \\ u_2(x_1, t) &= -\frac{1}{2} [\varphi(t) + \eta_2 [k_1(x_1, t) + \eta_1 \bar{p}_1(x_1)]] \text{sign}(x_1). \end{aligned} \quad (15)$$

Notice that for \mathbf{Q} in Eq. (14) to be positive definite, the polynomial $k_1(z_1, \tau)$ will have to dominate the square of the terms in the secondary diagonal. Therefore, for $k_1(z_1, \tau)$ to be bounded – recall this polynomial will form part of the injected correction terms u_1 , and u_2 – we asked $\bar{p}_j(z_1), j = 1, 2$ to be bounded for bounded z_1 . Recalling $\bar{q}_1 \leq \eta_1$, and condition (8), a sufficient condition for positive definiteness of \mathbf{Q} in Eq. (14) is $k_1(z_1, \tau) = k^2(z_1)/\varphi(\tau) > 0$, and:

$$k(z_1) > \frac{\bar{q}_2}{\sqrt{\eta_2}} \bar{p}_2(z_1) + \frac{\sqrt{\eta_2} \eta_1}{2} \bar{p}_1(z_1). \quad (16)$$

Notice that continuity of $u_1(z_1, \tau)$ w.r.t. z_1 at $z_1 = 0$, requires that of $k_1(z_1, \tau)z_1$ and $\bar{p}_1(z_1)z_1$ at that point.

In the simplest case where $p_1(z_1) \equiv 0$, $p_2(z_1) \equiv 1$, and $\varphi(t) \equiv 1$, choosing $\eta_1 = 0$ and $k_1(z_1(x_1)) = k_1$, retrieves the original super-twisting algorithm. The stability region is defined by the bounds $\eta_2 + k_1 > 2\bar{q}_2$, and $\eta_2 k_1 > \bar{q}_2^2$.

As a more complex example, to be used in Section 5, assume $\varphi(t) \equiv 1$, $p_2(z_1, z_2) = 1$, and $p_1(z_1)$ polynomial so that

$$\rho_1(z, \tau) = \varrho_1(\tau) \left[p_0 + p_1 z_1 + \dots + p_\beta z_1^\beta \right] z_1 \quad (17)$$

with $\beta \geq 0$. A bounding function $\bar{p}_1(z_1)$ is needed to fulfill condition (8). Assume $p_1(z_1)$ is unknown, but for its order and bounds on the coefficients. Define $\bar{p} = \max(p_0, \dots, p_\beta)$, $n_\beta = \beta + 1$, and:

$$\bar{p}_1(z_1) \triangleq n_\beta \bar{p} |z_1|^\gamma, \quad \gamma = \begin{cases} \beta, & |z_1| > 1 \\ 0, & |z_1| \leq 1. \end{cases} \quad (18)$$

Condition (8) is satisfied, and a sufficient condition for positive definiteness of \mathbf{Q} is:

$$k(z_1) > \frac{\bar{q}_2}{\sqrt{\eta_2}} + \frac{\sqrt{\eta_2} \eta_1}{2} n_\beta \bar{p} |z_1|^\gamma \quad (19)$$

which can be fulfilled choosing $k(z_1) = k_a + k_b |z_1|^\gamma$, with $k_a > \frac{\bar{q}_2}{\sqrt{\eta_2}}$, and $k_b > \frac{\sqrt{\eta_2} \eta_1}{2} n_\beta \bar{p}$. Fig. 1 shows the stability region for the particular case $p_1(z_1) = 1$.

3. Stability analysis

The proof will be split into three parts. First we analyze the homomorphism transforming system (1) in the (x, t) -coordinates into (4) in the (z, t) -coordinates, given by the change of coordinates (3) and the identity transformation for the time parameter. Its is proved this homomorphism is a time-invariant homeomorphism, and consequently preserves asymptotic stability.

Let X_i be the set of current states of system s_i . Any set with subscript i refers to a subset of X_i . The following elementary fact will be used in the proof:

Lemma 1. Given any one-to-one function $f : X_i \rightarrow X_j$, if $A_i \subset B_i \subseteq X_i$, then $f(A_i) \subset f(B_i)$.

Theorem 2. Any homomorphism given by a time-invariant homeomorphic coordinate change and the identity transformation for the time parameter, i.e. with no time-scaling, preserves asymptotic stability.

Proof. Consider a homeomorphism $f : X_1 \rightarrow X_2$. By continuity, around any point $x_{1e} \in X_1$, $\forall \epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x_2 - x_{2e}| < \delta$, $|x_1 - x_{1e}| < \epsilon$, with $x_{2e} = f(x_{1e}) \in X_2$.

Now take $\epsilon_1 = \epsilon$, and $\epsilon_2 = \delta$. If the goal system s_2 is stable, and assuming $x_{2e} = 0$ without loss of generality, then, $\forall \epsilon_2 > 0$, $\exists \delta_2 > 0$ such that if $|x_{20}| < \delta_2$ then $\forall t > t_0$, $|x_2| < \epsilon_2$. Using Lemma 1, $f^{-1}(\delta_2) \subset \epsilon_1$, and $|x_1| < \epsilon_1$ whenever $|x_{10}| < f^{-1}(\delta_2)$. So the first system $s_1(X_1)$ is also stable.

The same reasoning is valid for attractivity, but now orbits are inside an ϵ -ball from a given $t = t_\epsilon$. Notice t_ϵ is the same for both systems since there is no time-scaling. \square

Lemma 3. The change of coordinates given by $(z_1, z_2) \mapsto (|x_1|^{1/2} \text{sign}(x_1), x_2)$ is a homeomorphism.

Corollary 4. The homomorphism transforming system (1) in the (x, t) -coordinates into (4) in the (z, t) -coordinates, given by the change of coordinates (3) and the identity transformation for the time parameter – i.e. with no time-scaling – is a time-invariant homeomorphism, and consequently a stability preserving map.

Secondly, we analyze the homomorphism transforming system (4) in the (z, t) -coordinates into system (6) in the (z, τ) -coordinates, given by the identity coordinate transformation and the time-scaling (5). For time invariant systems, stability of equilibrium points in the sense of Lyapunov is a property of the orbits independent of their parameterization. Consequently, under the conditions for equivalence of regular curves explained below (Kühnel, 2005), any reparameterization (i.e. time-scaling) will preserve stability.

Definition 5. A regular curve is an equivalence class of regular parameterized curves, where the equivalence relationship is given by regular (orientation preserving) parameter transformations $\xi : [\alpha, \beta] \rightarrow [a, b]$, with $\dot{\xi} > 0$, and ξ bijective and continuously differentiable.

In our context, we will define the required time scaling to be a regular parameter transformation ξ , i.e. $\dot{\xi} \neq 0$ must hold everywhere. Since equilibrium points are singularities, we will consider the system orbits to be represented by the curves with time interval I spanning from initial conditions to the equilibrium. To fulfill the definition above, the time scaling ξ must be one-to-one, and onto. The fact that ξ must be onto ensures the whole orbit (a regular curve) is covered by both parameterizations. The condition $\dot{\xi} > 0$, except perhaps in a set of measure zero, ensures both “times” go forward since it makes ξ strictly increasing. Finally, for a function ξ to be one-to-one with positive derivative everywhere, it is sufficient to prove it is strictly increasing. Then we must prove it is also onto and hence a bijection.

Proposition 6. Under the conditions for equivalence of regular curves given in Definition 5, any reparameterization (i.e. time-scaling) will preserve stability.

Proof. The proof is straightforward from the standard definition of stability in the sense of Lyapunov, and the facts that reparameterizing the orbit will not change it and time in both parameterizations moves in the same direction. \square

Now it is proved the above conditions are also sufficient for attractivity and, consequently asymptotic stability. Denote by T_{t_0} the set $\{t \in T : t \geq t_0\}$ for some initial time instant t_0 .

Theorem 7. Any homomorphism given by the identity transformation for the coordinates and a time-scaling defined by a strictly increasing and onto function $\xi : t \rightarrow \tau$ preserves attractivity.

Proof. Since the coordinates transform to themselves, the homomorphism only introduces a reparameterization of the orbits (which are regular curves) by means of the time-scaling ξ . Since ξ is strictly increasing, there is ξ^{-1} with positive derivative everywhere. There are two possible cases:

- (1) ξ^{-1} is unbounded, mapping T_{t_0} to T_{t_0} . Then $\forall T, \exists \tau_T : \forall \tau > \tau_T, t = \xi^{-1}(\tau) > T$. Taking $T = \xi^{-1}(\tau_\epsilon)$, and given that ξ^{-1} is strictly increasing, eventually $t > T$ and, if $|z(\tau)| < \epsilon$, then $|x(t)| < \epsilon$ too.

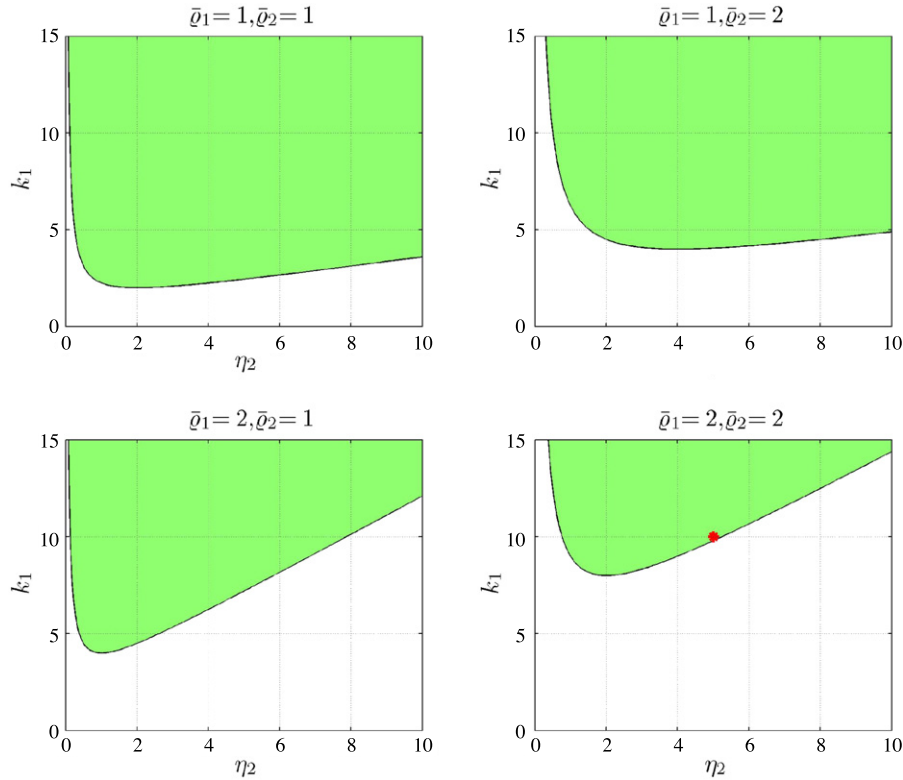


Fig. 1. Stability region in the η_2, k_1 parameters space for the case $p_1(z_1) = 1$ (i.e. $\rho_1(x, t) = 2Q_1(t)|x_1|^{\frac{1}{2}}$, $p_2(z_1, z_2) = 1$, and $\varphi(t) \equiv 1$). Two different values of the perturbation bounds \bar{q}_1 and \bar{q}_2 are shown.

(2) ξ^{-1} is bounded. The previous reasoning can be reproduced willy-nilly, but now we must additionally prove that ξ^{-1} is onto. Since ξ^{-1} is strictly increasing and bounded there is t_f such that $t \in [t_0, t_f]$ and $t \rightarrow t_f$ when $\tau \rightarrow \infty$. Therefore, if $z(\tau) \rightarrow z_{eq}$ then $x(t_f) = z_{eq} = x_{eq}$ by the identity of coordinates, proving finite time convergence to the equilibrium point of the original system. \square

Corollary 8. Any reparameterization as defined in 5 preserves asymptotic stability.

Proof. From Proposition 6, and Theorem 7 any reparameterization as defined in 5 preserves both stability and attractivity. Therefore, it preserves asymptotic stability. \square

Proposition 9. Choosing the time-scaling ξ implicitly, by giving its inverse ξ^{-1} as defined in Eq. (5) the corresponding homomorphism fulfills Corollary 8.

Proof. Due to the modulus function, no matter what $|z_1(\tau)|$ does, except being identically zero, the integral defines a strictly increasing function. So it is an injection and has an inverse, defining the regular parameter transformation ξ we need. \square

Finally, because of transitivity, the composition of stability preserving maps is also a stability preserving map. The combination of the nonlinear change of coordinates (3) and the time-scale transformation (5) is equivalent to the composition of the homomorphisms defined above. Therefore, this combination is a stability preserving map. Consequently, the system (1) in the (x, t) -coordinates is asymptotically stable if and only if the system (6) in the (z, τ) -coordinates is.

In particular, the convergence rate can be obtained from:

$$t_f \triangleq \lim_{\tau \rightarrow \infty} t(\tau) = \lim_{\tau \rightarrow \infty} \int_0^\tau |z_1(\xi)| d\xi. \tag{20}$$

If the integral is divergent as $\tau \rightarrow \infty$ (e.g. $z_1 = \mathcal{O}(1/\tau)$) we are in the first case in Theorem 7. Otherwise, if it is convergent (e.g. $z_1 = \mathcal{O}(1/\tau^2)$), we are in the second case, and finite-time convergence is achieved in the original (x, t) -coordinates.

4. Bound on finite-time convergence

Let us consider again Eq. (14) rewritten as:

$$V' = -\frac{1}{2} \bar{z}^T Q \bar{z} \leq -\frac{1}{2} \bar{z}^T \bar{\lambda}_{\mathbf{Q}}(\tau) \bar{z} = -\underline{\lambda}_{\mathbf{Q}}(\tau) V(\tau) \tag{21}$$

where $\underline{\lambda}_{\mathbf{Q}}(\tau)$ is the minimum eigenvalue of $\mathbf{Q}(\tau)$. Notice, for every initial condition and every τ there is either a minimum eigenvalue of Q or a lower bound since by positive definiteness the eigenvalues are always positive. Application of the comparison lemma leads to

$$|z_1| \leq \|\bar{z}\| \leq \|\bar{z}(\tau_0)\| e^{-\frac{1}{2} \int \underline{\lambda}_{\mathbf{Q}}(\tau) d\tau}. \tag{22}$$

It then follows, using (20) and (22), that

$$t_f \leq \|\bar{z}(\tau_0)\| \int_0^\infty e^{-\frac{1}{2} \int \underline{\lambda}_{\mathbf{Q}}(\tau) d\tau} d\xi \leq \frac{2}{\underline{\lambda}_{\mathbf{Q}, \min}} \|\bar{z}(\tau_0)\| \tag{23}$$

where $\underline{\lambda}_{\mathbf{Q}, \min} \triangleq \min_{\tau} \underline{\lambda}_{\mathbf{Q}}(\tau)$. Since $\frac{1}{\underline{\lambda}_{\mathbf{Q}}} = \frac{\bar{\lambda}_{\mathbf{Q}}}{\underline{\lambda}_{\mathbf{Q}} \bar{\lambda}_{\mathbf{Q}}} = \frac{\bar{\lambda}_{\mathbf{Q}}}{\det \mathbf{Q}}$, then t_f in Eq. (23) can be bounded by

$$t_f < 2 \|\bar{z}(\tau_0)\| \frac{\max_{\tau, z_1} \text{tr} \mathbf{Q}}{\min_{\tau, z_1} \det \mathbf{Q}}. \tag{24}$$

For instance, if $\rho_1(z, \tau) \equiv 0$, and $k_1 \eta_2 > \bar{q}_2^2$ one easily gets $t_f < 2 \|\bar{z}(\tau_0)\| \frac{k_1 + \eta_2}{k_1 \eta_2 - \bar{q}_2^2}$.

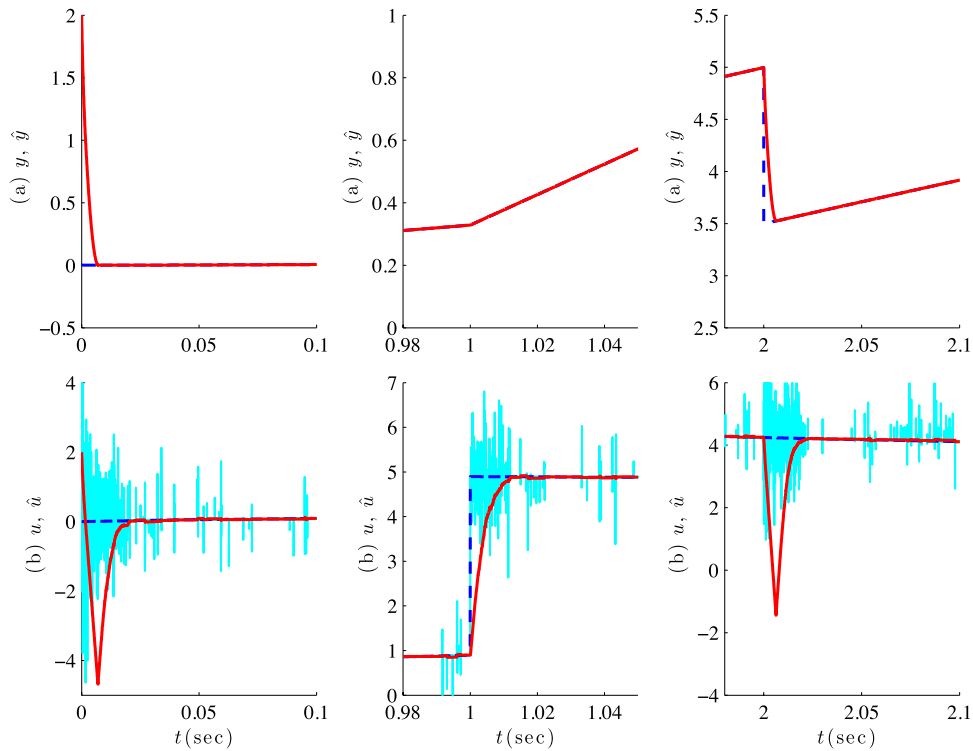


Fig. 2. Super-twisting observer response. Zooms around the time instants $t = 0$ s, $t = 1$ s, and $t = 2$ s.

5. Example

A super-twisting observer has been tuned using the proposed approach. The perturbation term $\rho_1(x, t)$ takes the form of (17), with $p_1(x_1) = 1 + 1.5|x_1|^{1/2}$, and $\varrho_1(t)$ being a bounded noise. The order of $p_1(x_1)$ and an upper bound on its coefficients are assumed to be known so as to build the bounding function $\bar{p}_1(x_1)$ according to (18). The perturbation term $\rho_2(x, t)$ has been obtained as the derivative of an unknown signal $u(t)$ with bounded time derivative $\|\rho_2(x, t)\|_\infty \leq 2$, but at $t = 1$ s, and $t = 2$ s when a step and impulse respectively were injected as disturbances in $u(t)$. The bounds of the unknown signal derivative and the integrator input noise have been set to $\bar{\varrho}_2(x, t) = 2$ and $\bar{\varrho}_1(x, t) = 2$ respectively. Choosing $\eta_1 = 2.5$, $\eta_2 = 5$, and $\bar{p} = 1.5$ the observer parameters were obtained using (19).

The simulation results are shown in Fig. 2. Three zooms around the time instants $t = 0$ s, $t = 1$ s, and $t = 2$ s are shown. At other time instants in between, the real and estimated signals are indistinguishable. The top plot depicts the input signal to the observer $y(t)$ (in blue dashed line), and the estimated signal $\hat{y}(t)$ (in red solid line). The real integrator input $u(t)$ – with noise, in cyan, and without noise in dashed blue – and its estimated value $\hat{u}(t)$ (in red solid line) are displayed in the bottom plot. It is seen that the observer output converges in less than 0.02 s and perfectly tracks the evolution of $u(t)$ when the appropriate conditions hold. At $t = 1$ s and $t = 2$ s the derivative of $u(t)$ is larger than the assumed bound $\bar{\varrho}_2(x, t)$, and the observer output diverges and then converges rapidly, putting in evidence the occurrence of an abrupt fault, as the derivative of $u(t)$ overly differed from the expected one.

6. Conclusions

In this contribution the problem of designing algorithms with finite-time convergence has been addressed by decoupling the stability analysis problem from that of finite-time convergence. This allows simple design methods and stability proofs to be derived in a wide set of cases. In order to show the proposed approach, it

has been applied to give an alternative proof of the super-twisting second-order sliding mode algorithm. This alternative approach allowed a simple design of a generalized SOSM coping with a broad class of perturbations. An estimate of the convergence time can be easily obtained in the transformed time-space. The approach can be extended to systems that can be robustly controlled in the coordinates-time transformed space for any coordinates dependent time-scaling fulfilling Corollary 8. Finite-time convergence will be achieved under the conditions of case 2 in Theorem 7.

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Enric Picó-Marco received the M.S.Eng. degree in Control Engineering and Industrial Informatics, and the Ph.D. in Automatic Control in 2000 and 2004 respectively. Currently he is Senior Lecturer at the Department of Systems Engineering and Control of the Technical University of Valencia (Spain).

His main research interests are in synthetic and systems biology, and algebraic methods for nonlinear control.



Alejandro Vignoni received the B.Eng. in Electronics and Control Systems at the Universidad Nacional de La Plata (Argentina), and the M.S.Eng. degree in Control Engineering, Robotics and Industrial Informatics at the Technical University of Valencia, Spain, in 2008 and 2011, respectively. Currently he is Ph.D. student at the Technical University of Valencia (Spain).

His main research interests are in Synthetic and systems biology, nonlinear and complex systems, coordination, consensus and synchronization, dynamical systems networks, and sliding modes.



Jesús Picó received the M.S.Eng. degree in Engineering Science and the Ph.D. degree in Control Engineering in 1989 and 1996, respectively. Currently he is Full Professor of Automatic Control at the Department of Systems Engineering and Control of the Technical University of Valencia (Spain).

His main research interests are in nonlinear systems, sliding mode observers and control, modeling and control of biotechnological processes, and systems and synthetic biology.



Hernán De Battista received the M.S.Eng. degree in Electronic Engineering, and the Ph.D. in Electronic Engineering in 1994 and 2000 respectively. Currently he is Associate Professor at the Faculty of Engineering, Universidad Nacional de La Plata (Argentina), and a research member of the Argentinean National Research Council (CONICET).

His main research interests are in nonlinear and switched control systems, biotechnological processes, and renewable energy conversion systems.