Diagonal extendible multilinear operators between ℓ_p -spaces
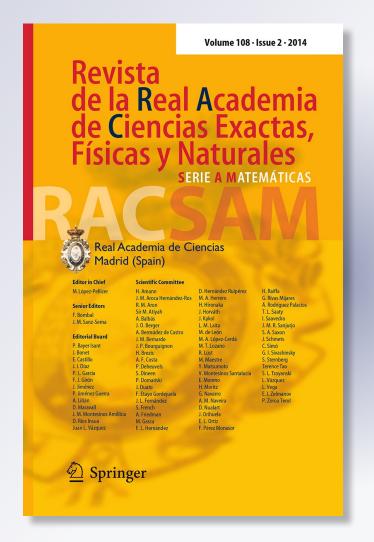
Daniel Carando, Verónica Dimant, Pablo Sevilla-Peris & Román Villafañe

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Diagonal extendible multilinear operators between ℓ_p-spaces

**Daniel Carando · Verónica Dimant ·
Pablo Sevilla-Peris · Román Villafañe**

Abstract We study extendibility of diagonal multilinear operators from ℓ_p to ℓ_q spaces. We determine the values of p and q for which every diagonal n-linear operator is extendible, and those for which the only extendible ones are integral. We address the same question for multilinear forms on ℓ_p.

D. Carando · R. Villafañe
Departamento de Matemática, Pab I, Facultad de Cs. Exactas y Naturales,
Universidad de Buenos Aires, IMAS, CONICET, C1428EGA Buenos Aires, Argentina
e-mail: dcarando@dm.uba.ar

R. Villafañe
e-mail: rvillafa@dm.uba.ar

V. Dimant
Departamento de Matemática, Universidad de San Andrés, CONICET,
Vito Dumas 284, Victoria, B1644BID Buenos Aires, Argentina
e-mail: vero@udesa.edu.ar

P. Sevilla-Peris (✉)
Instituto Universitario de Matemática Pura y Aplicada and DMA, ETSIAMN,
Universitat Politècnica de València, Cmno Vera s/n, 46022 Valencia, Spain
e-mail: psevilla@mat.upv.es

1 Introduction

There is no Hahn–Banach Theorem for linear/multilinear operators, nor for multilinear forms. This makes *extendibility* of such mappings a subject of interest, which has connections with different branches of functional analysis and Banach space theory (local theory, tensor norms, geometry of Banach spaces, etc). A multilinear operator $T \in \mathcal{L}(^n E, F)$ is *extendible* (see [10,16,21,22]) if for every Banach space X containing E there exists $\tilde{T} \in \mathcal{L}(^n X, F)$ extending T. For fixed Banach spaces E and F, one may wonder which multilinear operators from E to F are extendible. There are two extreme cases which are particularly important: sometimes extendible n-linear operators are "as few as possible" and sometimes they are "as many as possible". More precisely, since integral multilinear operators are always extendible, the "as few as possible"-case occurs when only the integral mappings are extendible. On the other hand, the "as many as possible"-situation is when all the multilinear operators are extendible. This problem, for bilinear forms on Banach sequence spaces, has been recently addressed in [15].

In this article we focus on particular sequence spaces: ℓ_p-spaces. The study of diagonal operators on ℓ_p-spaces started in the early 1970's with the work of (among others) Carl [17], König [23] and Pietsch [30] and by now it is a well established part of the theory. On the other hand, in [10,12] we have studied diagonal multilinear forms on these spaces (see also [13] where multilinear forms on Lorentz sequence spaces were also considered). In this paper we carry on with the study of diagonal multilinear forms, and also extend it to diagonal multilinear operators. In this context, we want to characterize the extendible elements, and the extreme cases mentioned above must be rephrased for diagonal mappings. Thus, we want to determine which values of p, q and n make the set of diagonal extendible n-linear operators from ℓ_p to ℓ_q equal to the set of integral mappings, which make it equal to the whole space of continuous diagonal n-linear operators from ℓ_p to ℓ_q, and which make it something in between. For this, we relate properties of a multilinear mapping to summability properties of its coefficients. The conclusions of the main result of the article (Theorem 4.1) are illustrated in Tables 1 and 2 in Sect. 2.

Section 3 is devoted to diagonal multilinear forms. Several steps in the scalar valued settings were given in [10,12]. It is known that in ℓ_1 every continuous (and, in particular, every extendible) diagonal form is integral. Also, every diagonal form on c_0 or ℓ_∞ is nuclear. For the non-trivial range of p, Proposition 3.1 in [12] shows that for $p \geq 2$ a diagonal n-linear form on ℓ_p is extendible if and only if it is nuclear. Corollary 3.1 in [12] shows that there exists a diagonal n-linear form that is extendible but not nuclear in every ℓ_p with $\frac{2(n-1)}{2n-3} < p < 2$, leaving unsolved the question for the remaining values of p. We complete here these results. For example, a consequence of Theorem 3.1 is that there are extendible diagonal n-linear forms that are not integral on ℓ_p for every $1 < p < 2$ and $n \geq 3$. Another one is that there are non-extendible diagonal n-linear forms in every ℓ_p for $1 < p < \infty$.

Our main theorem, regarding multilinear operators, and its proof is the content of Sect. 4. This result gives a rather complete account of the summability conditions of the coefficients of a multilinear operator T from ℓ_p to ℓ_q which are necessary and sufficient for T being integral or extendible. As a consequence, the existence (or lack) of diagonal multilinear operators which are not extendible, or which are extendible but not integral, is established for every p and q (see Table 2).

As a byproduct of some of our results we show that spaces of diagonal multilinear forms/operators behave very differently to spaces of all multilinear forms/operators. For example, there are always non-extendible diagonal bilinear forms on $\ell_p \times \ell_1$ if $p > 1$ but, surprisingly, every diagonal trilinear form on $\ell_p \times \ell_1 \times \ell_1$ is extendible (see the comments

after Lemma 3.2). See Proposition 4.4 and the subsequent comments for another unexpected behaviour.

For the theory of polynomials and multilinear mappings on Banach spaces we refer the reader to the books of Dineen [20] and Mujica [27].

2 Preliminaries

We consider Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For simplicity, we write some of our proofs for complex spaces, but all the results hold for both the real and complex cases. For Banach spaces E and F, we denote by $\mathcal{L}(^n E, F)$ the space of continuous n-linear mappings from $E \times \cdots \times E$ into F. This is a Banach space if we consider the norm

$$\|T\| = \sup\{\|T(x_1, \ldots, x_n)\| : x_i \in B_E, \ i = 1, \ldots, n\}.$$

Here B_E denotes the closed unit ball of E.

We are going to work mainly with Minkowski ℓ_p-spaces and nuclear, integral and extendible multilinear mappings between them. We recall their definitions. Given $1 < p < \infty$, we denote its conjugate by p', that is the number satisfying $1 = \frac{1}{p} + \frac{1}{p'}$. As usual, 1 and ∞ are conjugate to each other.

An n-linear mapping $T \in \mathcal{L}(^n \ell_p, \ell_q)$ is said to be **nuclear** (see [1]) if it can be written as

$$T(x_1, \ldots, x_n) = \sum_{j=1}^{\infty} \gamma_1^{(j)}(x_1) \cdots \gamma_n^{(j)}(x_n) y_j \text{ for all } x_1, \ldots, x_n \in \ell_p, \tag{1}$$

where $\gamma_i^{(j)} \in \ell_{p'}$, $y_j \in \ell_q$, for every i, j, and $\sum_{j=1}^{\infty} \|\gamma_1^{(j)}\| \cdots \|\gamma_n^{(j)}\| \cdot \|y_j\| < \infty$. We denote by $\mathcal{N}(^n \ell_p, \ell_q)$ the space of nuclear n-linear mappings from $\ell_p \times \cdots \times \ell_p$ to ℓ_q. This is a Banach space if we endow it with the norm

$$\|T\|_{\mathcal{N}} = \inf \left\{ \sum_{j=1}^{\infty} \|\gamma_1^{(j)}\| \cdots \|\gamma_n^{(j)}\| \cdot \|y_j\| \right\},$$

where the infimum is taken over all the nuclear representations of T as in (1). An n-linear mapping $T \in \mathcal{L}(^n \ell_p, \ell_q)$ is said to be **integral** if there exists a regular ℓ_q-valued measure G of bounded variation on the product $B_{\ell_{p'}} \times \cdots \times B_{\ell_{p'}}$ such that

$$T(x_1, \ldots, x_n) = \int_{B_{\ell_{p'}} \times \cdots \times B_{\ell_{p'}}} \gamma_1(x_1) \cdots \gamma_n(x_n) \, dG(\gamma_1, \ldots, \gamma_n),$$

for all $(x_1, \ldots, x_n) \in \ell_p \times \cdots \times \ell_p$. The integral norm of T is defined as the infimum of the total variation of G over all measures G representing T. With this norm $\mathcal{I}(^n \ell_p, \ell_q)$, the space of all integral n-linear mappings from $\ell_p \times \cdots \times \ell_p$ to ℓ_q is a Banach space.

There are two general definitions of integral n-linear operators in the literature (Grothendieck and Pietsch integral) but it is known [33] that in our context both coincide.

We denote by $\mathcal{E}(^n \ell_p, \ell_q)$ the space consisting of all extendible mappings in $\mathcal{L}(^n \ell_p, \ell_q)$; it is a Banach space if we endow it with the norm

$$\|T\|_{\varepsilon} = \inf\{C > 0 :$$
$$\forall E \supset \ell_p \text{ there exists an extension } \widetilde{T} \in \mathcal{L}(^n E, \ell_q) \text{ with } \|\widetilde{T}\| \leq C\}.$$

Every nuclear n-linear mapping is clearly integral. The reverse inclusion sometimes holds: for $1 < p < \infty$, the space ℓ_p is Asplund and then integral multilinear operators on ℓ_p are nuclear [1]. On the other hand, by [14], integral n-linear operators are extendible. Thus, we have the following chain of embeddings:

$$\mathcal{N}(^n\ell_p, \ell_q) \subseteq \mathcal{I}(^n\ell_p, \ell_q) \subseteq \mathcal{E}(^n\ell_p, \ell_q) \subseteq \mathcal{L}(^n\ell_p, \ell_q). \tag{2}$$

Regarding the extendible operators we have two extreme cases: when only the integral operators are extendible or when all operators are so. Our main aim in this paper is to try to characterize these two cases for a class of distinguished multilinear operators, the so called 'diagonal mappings'. We will also determine when nuclear and integral diagonal mappings coincide (this last question has interest only for $p = 1$ and $p = \infty$, by the above mentioned result in [1]).

An n-linear operator $T \in \mathcal{L}(^n\ell_p, \ell_q)$ is said to be **diagonal** if there exists a sequence $\alpha = (\alpha(k))_k$ such that for all $x_1, \ldots, x_n \in \ell_p$ we can write

$$T(x_1, \ldots, x_n) = \sum_k \alpha(k) x_1(k) \cdots x_n(k) \, e_k,$$

where e_k denotes the kth canonical unit vector: $e_k(j) = \delta_{k,j}$ for $k, j \in \mathbb{N}$. We denote by T_α the diagonal multilinear mapping given by the sequence α.

Recall that a Banach sequence space is a Banach space $E \subseteq \mathbb{K}^{\mathbb{N}}$ of sequences in $\mathbb{K}$ such that $\ell_1 \subseteq E \subseteq \ell_\infty$ satisfying that if $x \in \mathbb{K}^{\mathbb{N}}$ and $y \in E$ are such that $|x(k)| \leq |y(k)|$ for all $k \in \mathbb{N}$ then $x \in E$ and $\|x\| \leq \|y\|$.

Definition 2.1 For a Banach ideal of n-linear mappings $\mathfrak{A} = \mathcal{N}, \mathcal{I}, \mathcal{E}, \mathcal{L}$ we define the set

$$\ell_n(\mathfrak{A}, p, q) = \{\alpha \in \ell_\infty : T_\alpha \in \mathfrak{A}(^n\ell_p, \ell_q)\},$$

which is a Banach sequence space with the norm $\|\alpha\|_{\ell_n(\mathfrak{A}, p, q)} = \|T_\alpha\|_{\mathfrak{A}(^n\ell_p, \ell_q)}$.

Then, $\ell_n(\mathfrak{A}, p, q)$ describes the space of diagonal n-linear mappings from ℓ_p to ℓ_q that belong to the ideal $\mathfrak{A}$. For example, for $\mathcal{L}$ the ideal of continuous multilinear operators, it is easy to check using Hölder's inequality that

$$\ell_n(\mathcal{L}, p, q) \overset{1}{=} \begin{cases} \ell_\infty & \text{if } p \leq nq \\ \ell_r & \text{if } p > nq \end{cases}$$

where r is defined by $\frac{1}{r} = \frac{1}{q} - \frac{n}{p}$. The notation $E \overset{1}{=} F$ means that E and F are isometrically isomorphic. Then the chain of embeddings (2) implies that

$$\ell_n(\mathcal{N}, p, q) \subseteq \ell_n(\mathcal{I}, p, q) \subseteq \ell_n(\mathcal{E}, p, q) \subseteq \ell_n(\mathcal{L}, p, q)$$

Our aim is to study when these inclusions are strict or not. A description of these sequence spaces is given in our main result (Theorem 4.1). As an immediate consequence of it we obtain the following tables.

Table 1 Nuclear versus integral multilinear operators

$\ell_n(\mathcal{N}, p, q) \neq \ell_n(\mathcal{I}, p, q)$	$p = 1$ and $q = \infty$
$\ell_n(\mathcal{N}, p, q) = \ell_n(\mathcal{I}, p, q)$	otherwise

Table 2 Extendible versus integral and bounded multilinear operators.

$\ell_n(\mathcal{I}, p, q) = \ell_n(\mathcal{E}, p, q) = \ell_n(\mathcal{L}, p, q)$	$p = 1$ and $q = \infty$ $p = \infty$ and $q = 1$
$\ell_n(\mathcal{I}, p, q) = \ell_n(\mathcal{E}, p, q) \neq \ell_n(\mathcal{L}, p, q)$	$2 \leq p < \infty$ and $q = 1$
$\ell_n(\mathcal{I}, p, q) \neq \ell_n(\mathcal{E}, p, q) = \ell_n(\mathcal{L}, p, q)$	$p = 1$ and $1 \leq q < \infty$ $1 < p < \infty$ and $q = \infty$ $p = \infty$ and $1 < q \leq \infty$
$\ell_n(\mathcal{I}, p, q) \neq \ell_n(\mathcal{E}, p, q) \neq \ell_n(\mathcal{L}, p, q)$	$1 < p < 2$ and $q = 1$ $1 < p < \infty$ and $1 < q < \infty$

3 Diagonal multilinear forms in ℓ_p spaces

There is a natural isometric identification between $\mathcal{L}(^n\ell_p, \ell_q)$ and the space of continuous $(n + 1)$-linear applications from $\ell_p \times \cdots \times \ell_p \times \ell_{q'}$ to $\mathbb{K}$ (for $q = 1$, we put c_0 instead of ℓ_∞ for the identification to be onto). It can be easily seen that this identification is also an isometric isomorphism for the classes of nuclear and integral mappings (with the obvious modifications in the definitions). However, this is not the case for the extendible operators. An extendible $(n + 1)$-linear form defined on $\ell_p \times \cdots \times \ell_p \times \ell_{q'}$ produces an extendible n-linear operator from $\ell_p \times \cdots \times \ell_p$ to ℓ_q (since ℓ_1 is complemented in its bidual, for $q = 1$ we can use either ℓ_∞ or c_0 in the place of $\ell_{q'}$). However, the converse is not true, as we will see in Remark 4.6. Anyway, it will be helpful for our general goal to look first at multilinear forms.

One of the aims in [10,12] was precisely to determine when the diagonal extendible multilinear forms are few (that is, they coincide with the integral multilinear forms) and when they are as many as they can (all continuous multilinear forms are extendible). The case $1 < p < 2$ was left open and we fill this gap here.

For a sequence α, the diagonal n-linear form $\phi_\alpha \in \mathcal{L}(^n\ell_p)$ associated to α is given by

$$\phi_\alpha(x_1, \ldots, x_n) = \sum_k \alpha(k)x_1(k) \cdots x_n(k)$$

for $x_1, \ldots, x_n \in \ell_p$ (whenever this mappings is well defined). Also, like in Definition 2.1, for $\mathfrak{A} = \mathcal{N}, \mathcal{I}, \mathcal{E}, \mathcal{L}$ we define $\ell_n(\mathfrak{A}, p) = \{\alpha \in \ell_\infty : \phi_\alpha \in \mathfrak{A}(^n\ell_p)\}$, which again is a Banach sequence space with the norm $\|\alpha\|_{\ell_n(\mathfrak{A}, p)} = \|\phi_\alpha\|_{\mathfrak{A}(^n\ell_p)}$.

The following theorem describes the sequence spaces $\ell_n(\mathcal{N}, p)$, $\ell_n(\mathcal{I}, p)$, $\ell_n(\mathcal{E}, p)$ and $\ell_n(\mathcal{L}, p)$. One consequence is that the inclusion $\ell_n(\mathcal{I}, p) \subset \ell_n(\mathcal{E}, p)$ is strict for all $1 < p < 2$ and $n \geq 3$, while the inclusion $\ell_n(\mathcal{E}, p) \subset \ell_n(\mathcal{L}, p)$ is strict for $1 < p < \infty$ and $n \geq 2$.

Theorem 3.1 (1) $\ell_n(\mathcal{N}, 1) = c_0 \subsetneq \ell_\infty = \ell_n(\mathcal{I}, 1) = \ell_n(\mathcal{E}, 1) = \ell_n(\mathcal{L}, 1)$.

(2) *If* $1 < p < 2$ *then*

$$\ell_2(\mathcal{N}, p) = \ell_2(\mathcal{I}, p) = \ell_2(\mathcal{E}, p) = \ell_{\frac{p'}{2}} \subsetneq \ell_\infty = \ell_2(\mathcal{L}, p),$$

$$For\ n \geq 3, \quad \ell_n(\mathcal{N}, p) = \ell_n(\mathcal{I}, p) = \ell_{\max(\frac{p'}{n}, 1)} \subsetneq \ell_{\frac{p'}{2}} = \ell_n(\mathcal{E}, p) \subsetneq \ell_\infty = \ell_n(\mathcal{L}, p).$$

(3) *If* $2 \leq p < \infty$ *then*

$$\ell_n(\mathcal{N}, p) = \ell_n(\mathcal{I}, p) = \ell_n(\mathcal{E}, p) = \ell_1 \subsetneq \ell_n(\mathcal{L}, p) = \begin{cases} \ell_{\frac{p}{p-n}} & \text{for } n < p. \\ \ell_\infty & \text{for } n \geq p. \end{cases}$$

(4) $\ell_n(\mathcal{N}, \infty) = \ell_n(\mathcal{I}, \infty) = \ell_n(\mathcal{E}, \infty) = \ell_n(\mathcal{L}, \infty) = \ell_1.$

The description of $\ell_n(\mathcal{L}, p)$ follows easily from Hölder's inequality. Also, the characterizations of $\ell_n(\mathcal{N}, p)$ and $\ell_n(\mathcal{I}, p)$ and most of the relations with $\ell_n(\mathcal{E}, p)$ where already proved in [10,12], but we have decided to bring them all here to have a more complete picture. The only thing we need to prove is, in item (2), the equality $\ell_n(\mathcal{E}, p) = \ell_{\frac{p'}{2}}$. This will be showed in Proposition 3.3 below. We remark that the case $1 < p < 2$ shows an important difference between bilinear forms and n-linear forms with $n \geq 3$. In fact, a consequence of a deep result of Pisier [31] is that every extendible bilinear form defined on a Banach space with cotype 2 is integral (see [10,16], where Pisier's result is "read" in a fashion more akin to our framework). In particular, this holds for bilinear forms on ℓ_p with $1 \leq p \leq 2$. However, by the previous theorem, for n-linear forms with $n \geq 3$ and every $1 < p < 2$ we have diagonal extendible n-linear forms which are not integral.

For the following result, we need to define some mappings that will be used again later. We consider a Walsh matrix; i.e., a matrix $(a_{kr})_{kr=1}^N$ such that $|a_{kr}| = 1$ and

$$\sum_{r=1}^N a_{kr}\bar{a}_{rl} = N\,\delta_{k,l}.$$

Assume also that the matrix is symmetric. Examples of such matrices in the complex case are the Fourier matrices, given by $a_{kr} = e^{\frac{2\pi i}{N}rk}$ [18, Sect. 8.5]. For the real case one can consider Hadamard matrices (see [26, Sect. 3]) defined for N that are powers of 2 and that are generated by blocks as follows

$$A_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad A_{2^{n+1}} = \begin{pmatrix} A_{2^n} & A_{2^n} \\ A_{2^n} & -A_{2^n} \end{pmatrix}.$$

We define a Toeplitz-like operator $\xi_N : \ell_p^N \to \ell_\infty^N$ by

$$\xi_N(x) = \sum_{k=1}^N \left(\sum_{r=1}^N \bar{a}_{kr} x(r) \right) e_k. \tag{3}$$

This operator satisfies $\|\xi_N : \ell_p^N \to \ell_\infty^N\| \leq \|id : \ell_p^N \to \ell_1^N\| \|\xi_N : \ell_1^N \to \ell_\infty^N\| \leq N^{\frac{1}{p'}}$. Finally, for $n \geq 3$ we introduce the following modification of the n-linear form on ℓ_∞^N studied by Bohnenblust and Hille in [4, Sect. 2]:

$$L_N(x_1, \ldots, x_n) = \sum_{j,k,l=1}^N a_{jl}a_{lk}x_1(j)x_2(k)x_3(l) \cdots x_n(l). \tag{4}$$

Actually, this n-linear form is built taking the trilinear mapping defined by Bohnenblust and Hille and giving an n-linear extension of this definition maintaining the norm of the original trilinear form. Since on ℓ_∞^N the extendible and the usual norms coincide, we have by [4, Sect. 2]

$$\|L_N\|_{\mathcal{E}(^n \ell_\infty^N)} = \|L_N\|_{\mathcal{L}(^n \ell_\infty^N)} = N^2. \tag{5}$$

Thanks to the properties of the coefficients $(a_{kr})_{kr}$ we have:

$$
\begin{aligned}
L_N(\xi_N(x_1), \xi_N(x_2), x_3, \ldots, x_n) &= \sum_{j,k,l=1}^{N} a_{jl} a_{lk} \cdot \xi_N(x_1)(j) \cdot \xi_N(x_2)(k) \cdot x_3(l) \cdots x_n(l) \\
&= \sum_{r,s,l=1}^{N} x_1(r) x_2(s) x_3(l) \cdots x_n(l) \sum_{j=1}^{N} a_{jl} \bar{a}_{jr} \sum_{k=1}^{N} a_{lk} \bar{a}_{ks} \\
&= \sum_{r,s,l=1}^{N} x_1(r) x_2(s) x_3(l) \cdots x_n(l) N \delta_{l,r} N \delta_{l,s} \\
&= N^2 \sum_{r=1}^{N} x_1(r) x_2(r) x_3(r) \cdots x_n(r). \tag{6}
\end{aligned}
$$

Lemma 3.2 *Every diagonal n-linear form on $\ell_1 \times \ell_1 \times \ell_{p_1} \times \cdots \times \ell_{p_{n-2}}$ with $1 \leq p_i \leq \infty$ and $n \geq 2$, is extendible.*

Proof The case $n = 2$ is immediate: diagonal bilinear forms on $\ell_1 \times \ell_1$ are integral, and therefore extendible. For $n \geq 3$, any diagonal n-linear form $\phi_\alpha : \ell_1 \times \ell_1 \times \ell_{p_1} \times \cdots \times \ell_{p_{n-2}} \to \mathbb{K}$ has a factorization as

where $D_\alpha : \ell_1 \to \ell_1$ is the diagonal operator given by $D_\alpha((x(k))_k) = (\alpha(k)x(k))_k$ and

$$\Phi(x_1, \ldots, x_n) = \sum_{k=1}^{\infty} x_1(k) \cdots x_n(k). \tag{7}$$

Then by the ideal property of extendible multilinear forms, it is enough to show that the n-linear form Φ is extendible on $\ell_1 \times \ell_1 \times \ell_{p_1} \times \cdots \times \ell_{p_{n-2}}$. We denote by Φ_N the form defined by summing the first N terms in (7).

We consider the operator ξ_N defined in (3) with domain ℓ_1 (thus $\|\xi_N\| \leq 1$) and the n-linear form L_N as in (4). As a consequence of (6) and (5) we have

$$
\begin{aligned}
\|\Phi_N\|_{\mathcal{E}(^n \ell_1 \times \ell_1 \times \ell_{p_1} \times \cdots \times \ell_{p_{n-2}})} &\leq \frac{1}{N^2} \|L_N\|_{\mathcal{E}(^n \ell_\infty^N)} \|\xi_N\|^2 \prod_{i=1}^{n-2} \|id : \ell_{p_i}^N \to \ell_\infty^N\| \\
&\leq \frac{1}{N^2} N^2 = 1.
\end{aligned}
$$

Therefore, the extendible norms of the n-linear forms Φ_N ($N \in \mathbb{N}$) are uniformly bounded. A multilinear version of the density lemma [18, Sect. 13.4], stated in [13, Lemma 5.4], implies that Φ is extendible (and its extendible norm is one). $\qquad\square$

The previous lemma is meant to be a tool which will be used in the next proposition and also in the following section. However, as a byproduct of this lemma, a rather unexpected behaviour of diagonal multilinear forms is illustrated. It is easy to see that, if *every n-linear* form on $E_1 \times \cdots \times E_n$ (E_j Banach spaces) is extendible, then every $(n-1)$-linear form on any $(n-1)$-tuple of the previous spaces is extendible. This is a property shared by most classes of multilinear forms, and is related to the notions of "Property B", "coherence" and ideals "closed under differentiation" developed in [5,7,11]. However, this is no longer the case when we restrict ourselves to diagonal multilinear forms. Indeed, Lemma 3.2 shows that every diagonal trilinear form on $\ell_1 \times \ell_1 \times \ell_2$ is extendible, while we know from [15] that there are non-extendible diagonal bilinear forms on $\ell_1 \times \ell_2$.

In order to prove the next proposition, recall that, for $s \leq rn$, an n-linear form $\phi \in \mathcal{L}(^n\ell_p)$ is said to be **absolutely (r;s)-summing** [2,25] if there exists a constant $K > 0$ such that for every $x_j^{(k)} \in \ell_p$, $j = 1, \ldots, n$, $k = 1, \ldots, N$,

$$\left(\sum_{k=1}^{N} |\phi(x_1^{(k)}, \ldots, x_n^{(k)})|^r \right)^{\frac{1}{r}} \leq K \cdot w_s \left((x_1^{(k)})_{k=1}^{N} \right) \cdots w_s \left((x_n^{(k)})_{k=1}^{N} \right),$$

where the weakly s-summing norm w_s of the sequence $(x_j^{(k)})_{k=1}^{N}$ is given by

$$w_s \left((x_j^{(k)})_{k=1}^{N} \right) = \sup_{\gamma \in B_{\ell_{p'}}} \left(\sum_{k=1}^{N} |\gamma(x_j^{(k)})|^s \right)^{\frac{1}{s}}.$$

There are several possible extensions of the notion of absolutely summing operators to the multilinear setting [9,28]. Although the one given above is not considered, in some sense, a good generalization of the linear concept, it will prove useful for our purposes. Extendible multilinear mappings are those that can be factored through an $\mathcal{L}_\infty$ space (due to the injectivity of these spaces). Thus, Grothendieck's multilinear inequality allows us to derive (see [3] or [29, Corollary 2.5]) that any extendible n-linear form is absolutely $(1; 2)$-summing (for Grothendieck inequality and the notions of cotype and absolutely summing linear operators we refer the reader to the classical book [19]). From this result and an interpolation technique, a stronger statement is obtained in [6, Theorem 3.15]: all extendible n-linear forms are absolutely $(r; 2r)$-summing, for any $r \geq 1$.

Proposition 3.3 *Let* $1 < p < 2$ *and* $n \geq 2$. *Then,* $\ell_n(\mathcal{E}, p) = \ell_{\frac{p'}{2}}$.

Proof Let $\phi_\alpha \in \mathcal{E}(^n\ell_p)$ be a diagonal extendible n-linear form. Since ϕ_α is absolutely $(\frac{p'}{2}; p')$-summing, there is a constant K such that, for all N,

$$\left(\sum_{k=1}^{N} |\alpha_k|^{\frac{p'}{2}} \right)^{\frac{2}{p'}} = \left(\sum_{k=1}^{N} |\phi_\alpha(e_k, \ldots, e_k)|^{\frac{p'}{2}} \right)^{\frac{2}{p'}} \leq K \cdot w_{p'} \left((e_k)_{k=1}^{N} \right)^n = K,$$

which means that α belongs to $\ell_{\frac{p'}{2}}$.

For the reverse inclusion, let $\alpha \in \ell_{\frac{p'}{2}}$ and define $(\sigma(k))_k = (\alpha(k)^{\frac{1}{2}})_k$ (with the suitable modification in the real case). We consider the diagonal operator $D_\sigma : \ell_p \to \ell_1$, given by $D_\sigma((x(k))_k) = (\sigma(k)x(k))_k$ and we have the following commutative diagram

$$
\begin{array}{ccccccccc}
\ell_p & \times & \ell_p & \times & \ell_p & \times & \cdots & \times & \ell_p \xrightarrow{\phi_\alpha} \mathbb{K} \;. \\
D_\sigma \downarrow & & D_\sigma \downarrow & & i \downarrow & & & & i \downarrow \;\; \nearrow \Phi \\
\ell_1 & \times & \ell_1 & \times & \ell_p & \times & \cdots & \times & \ell_p
\end{array}
$$

Since Φ is extendible and has extendible norm one (by Lemma 3.2 and its proof) we obtain that ϕ_α is extendible and

$$
\|\phi_\alpha\|_{\mathcal{E}(^n \ell_p)} \leq \|\Phi\|_{\mathcal{E}(^n \ell_1 \times \ell_1 \times \ell_p \times \cdots \times \ell_p)} \cdot \|D_\sigma\|^2 = \|\alpha\|_{\ell_{\frac{p'}{2}}}.
$$

$\square$

4 Diagonal multilinear operators

In this section, we state and prove our main result. For p, q, n fixed we define the following numbers, that will be used along the rest of the paper:

$$
r = \left(\frac{1}{q} - \frac{n}{p}\right)^{-1}
$$

$$
t = \max\left\{ \left(\frac{n}{p'} + \frac{1}{q}\right)^{-1}, 1\right\}. \tag{8}
$$

Theorem 4.1 *For $n \geq 1$, the following assertions hold.*

(1) *Let $p = 1$,*

    (a) *if $1 \leq q < \infty$, then*

$$
\ell_n(\mathcal{N}, 1, q) = \ell_n(\mathcal{I}, 1, q) = \ell_q \subsetneqq \ell_\infty = \ell_n(\mathcal{E}, 1, q) = \ell_n(\mathcal{L}, 1, q);
$$

    (b) *if $q = \infty$, then*

$$
\ell_n(\mathcal{N}, 1, \infty) = c_0 \subsetneqq \ell_\infty = \ell_n(\mathcal{I}, 1, \infty) = \ell_n(\mathcal{E}, 1, \infty) = \ell_n(\mathcal{L}, 1, \infty).
$$

(2) *Let $1 < p < 2$,*

    (a) *if $q = 1$, then*

$$
\ell_n(\mathcal{N}, p, 1) = \ell_n(\mathcal{I}, p, 1) = \ell_1 \subsetneqq \ell_{\frac{p'}{2}} = \ell_n(\mathcal{E}, p, 1) \subsetneqq \ell_\infty = \ell_n(\mathcal{L}, p, 1);
$$

    (b) *if $p' < q < \infty$, then*

$$
\ell_n(\mathcal{N}, p, q) = \ell_n(\mathcal{I}, p, q) = \ell_t \subsetneqq \ell_q = \ell_n(\mathcal{E}, p, q) \subsetneqq \ell_\infty = \ell_n(\mathcal{L}, p, q);
$$

    (c) *if $1 < q \leq p'$, then, for all $\varepsilon > 0$,*

$$
\ell_n(\mathcal{N}, p, q) = \ell_n(\mathcal{I}, p, q) = \ell_t \subsetneqq \ell_q \subseteq \ell_n(\mathcal{E}, p, q) \subseteq \ell_{p'+\varepsilon} \subsetneqq \ell_\infty = \ell_n(\mathcal{L}, p, q);
$$

(d) *if $q = \infty$, then*

$$\ell_n(\mathcal{N}, p, \infty) = \ell_n(\mathcal{I}, p, \infty) = \ell_t \subsetneq \ell_\infty = \ell_n(\mathcal{E}, p, \infty) = \ell_n(\mathcal{L}, p, \infty).$$

(3) *Let $2 \leq p < \infty$,*

(a) *if $q = 1$, then*

$$\ell_n(\mathcal{N}, p, 1) = \ell_n(\mathcal{I}, p, 1) = \ell_n(\mathcal{E}, p, 1) = \ell_1 \subsetneq \ell_n(\mathcal{L}, p, 1) = \ell_\infty \text{ or } \ell_r;$$

(b) *if $1 < q < \infty$, then*

$$\ell_n(\mathcal{N}, p, q) = \ell_n(\mathcal{I}, p, q) = \ell_1 \subsetneq \ell_n(\mathcal{E}, p, q) = \ell_q \subsetneq \ell_n(\mathcal{L}, p, q) = \ell_\infty \text{ or } \ell_r;$$

(c) *if $q = \infty$, then*

$$\ell_n(\mathcal{N}, p, \infty) = \ell_n(\mathcal{I}, p, \infty) = \ell_1 \subsetneq \ell_\infty = \ell_n(\mathcal{E}, p, \infty) = \ell_n(\mathcal{L}, p, \infty).$$

(4) *Let $p = \infty$*

(a) *if $q = 1$, then*

$$\ell_n(\mathcal{N}, \infty, 1) = \ell_n(\mathcal{I}, \infty, 1) = \ell_n(\mathcal{E}, \infty, 1) = \ell_n(\mathcal{L}, \infty, 1) = \ell_1;$$

(b) *if $1 < q \leq \infty$, then*

$$\ell_n(\mathcal{N}, \infty, q) = \ell_n(\mathcal{I}, \infty, q) = \ell_1 \subsetneq \ell_q = \ell_n(\mathcal{E}, \infty, q) = \ell_n(\mathcal{L}, \infty, q).$$

The proof of Theorem 4.1 will be splitted in several propositions. The first ones deal with nuclear and integral diagonal mappings.

Proposition 4.2 *Let $T_\alpha \in \mathcal{L}(^n \ell_1, \ell_\infty)$. Then:*

(i) *T_α is integral and $\|T_\alpha\|_{\mathcal{I}} = \|\alpha\|_{\ell_\infty}$.*
(ii) *T_α is nuclear if and only if $\alpha \in c_0$. In this case, $\|T_\alpha\|_{\mathcal{N}} = \|\alpha\|_{\ell_\infty}$.*

Proof The result follows from the isometric identifications

$$\mathcal{L}(^n \ell_1, \ell_\infty) \cong \mathcal{L}(^{n+1} \ell_1),$$
$$\mathcal{I}(^n \ell_1, \ell_\infty) \cong \mathcal{I}(^{n+1} \ell_1),$$
$$\mathcal{N}(^n \ell_1, \ell_\infty) \cong \mathcal{N}(^{n+1} \ell_1),$$

which map diagonal n-linear operators into diagonal $(n + 1)$-linear forms, along with the corresponding scalar-valued result [12, Proposition 1.2]. □

We need to recall the definition of the injective tensor norm of order n. In the n-fold tensor product of Banach spaces $E_1 \otimes \cdots \otimes E_n$ the ε-tensor norm is given by

$$\left\| \sum_{j=1}^N x_1^{(j)} \otimes \cdots \otimes x_n^{(j)} \right\|_\varepsilon = \sup_{\gamma_i \in B_{E_i'}} \left| \sum_{j=1}^N \gamma_1(x_1^{(j)}) \cdots \gamma_n(x_n^{(j)}) \right|.$$

The space of integral n-linear forms on $\ell_{p_1} \times \cdots \times \ell_{p_n}$ is the dual of the (complete) injective tensor product of the spaces (see the monographs [18] and [32] where the 2-fold/bilinear case is treated in detail).

Proposition 4.3 *Let $p > 1$. Then $T_\alpha \in \mathcal{L}(^n \ell_p, \ell_q)$ is nuclear if and only if $\alpha \in \ell_t$, where t is defined in (8). In this case, $\|T_\alpha\|_{\mathcal{N}} = \|\alpha\|_{\ell_t}$.*

Proof If T_α is nuclear, then it is integral and the associated $(n + 1)$-linear diagonal form $\phi_{T_\alpha} : \ell_p \times \cdots \times \ell_p \times \ell_{q'} \to \mathbb{K}$ is also integral with $\|\phi_{T_\alpha}\|_{\mathcal{I}} = \|T_\alpha\|_{\mathcal{I}}$ (for the case $q = 1$ we take $\ell_{q'}$ as c_0 instead of ℓ_∞). Equivalently, ϕ_{T_α} is ε-continuous. Then, we have

$$\left| \phi_{T_\alpha} \left(\sum_{k=1}^{N} \beta_k e_k \otimes \cdots \otimes e_k \otimes e_k' \right) \right| \le \|\phi_{T_\alpha}\|_{\mathcal{I}} \left\| \sum_{k=1}^{N} \beta_k e_k \otimes \cdots \otimes e_k \otimes e_k' \right\|_{\varepsilon},$$

which implies that

$$\left| \sum_{k=1}^{N} \alpha_k \beta_k \right| \le \|T_\alpha\|_{\mathcal{I}} \cdot \left(\sup_{\varphi_1,\ldots,\varphi_n \in B_{\ell_{p'}}, \psi \in B_{\ell_q}} \left| \sum_{k=1}^{N} \beta_k \varphi_1(e_k) \cdots \varphi_n(e_k) \cdot \psi(e_k') \right| \right)$$

$$= \|T_\alpha\|_{\mathcal{I}} \cdot \left(\sup_{\varphi_1,\ldots,\varphi_n \in B_{\ell_{p'}}} \left\| \sum_{k=1}^{N} \beta_k \varphi_1(k) \cdots \varphi_n(k) \cdot e_k' \right\|_{\ell_{q'}} \right)$$

$$= \|T_\alpha\|_{\mathcal{I}} \cdot \left\| (T_\beta)_N \right\|_{\mathcal{L}(^n\ell_{p'}, \ell_{q'})}$$

$$= \|T_\alpha\|_{\mathcal{I}} \cdot \left\| (\beta_k)_{k=1}^{N} \right\|_{\ell_{t'}}.$$

Hence $\alpha \in \ell_t$ and $\|\alpha\|_{\ell_t} \le \|T_\alpha\|_{\mathcal{I}}$.

We see now that, if $\alpha \in \ell_t$, then T_α is nuclear and $\|T_\alpha\|_{\mathcal{N}} \le \|\alpha\|_{\ell_t}$. For $t = 1$ the conclusion is immediate, so we assume $t > 1$. Let us consider the following factorization:

$$
\begin{array}{ccccc}
\ell_p \times \cdots \times & \ell_p & \xrightarrow{T_\alpha} & \ell_q & \\
D_\eta \downarrow & D_\eta \downarrow & & \uparrow D_\nu & \\
\ell_1 \times \cdots \times & \ell_1 & \xrightarrow{\Psi} & \ell_\infty &
\end{array}
$$

where $\Psi = T_{(1,1,\ldots)}$ and D_η and D_ν are the diagonal linear operators associated with the sequences $\eta(k) = \alpha(k)^{\frac{t}{p'}}$ and $\nu(k) = \alpha(k)^{\frac{t}{q}}$.

By Proposition 4.2, the n-linear operator Ψ is integral with $\|\Psi\|_{\mathcal{I}} = 1$. Thus, it follows that T_α is also integral. Since $t > 1$, we have $1 < p < \infty$ and ℓ_p is Asplund. So, T_α is actually nuclear, and its nuclear norm coincides with its integral norm [1]. Therefore,

$$\|T_\alpha\|_{\mathcal{N}} = \|T_\alpha\|_{\mathcal{I}} \le \|D_\nu\| \|\Psi\|_{\mathcal{I}} \|D_\eta\|^n = \|\alpha\|_{\ell_t}.$$

$\square$

We finally study the remaining case $p = 1$ and $q < \infty$.

Proposition 4.4 *Let $q < \infty$ and $T_\alpha \in \mathcal{L}(^n\ell_1, \ell_q)$. Then the following are equivalent:*

(i) T_α *is integral.*
(ii) T_α *is nuclear.*
(iii) $\alpha \in \ell_q$.

When these equivalences hold, $\|T_\alpha\|_{\mathcal{I}} = \|T_\alpha\|_{\mathcal{N}} = \|\alpha\|_{\ell_q}$.

Proof The equivalence between (i) and (iii) follows as in Proposition 4.3. So we only have to prove that (iii) implies (ii). Given $\alpha \in \ell_q$, we estimate the nuclear norm of $T_\alpha^{(s,l)} :=$
$\sum_{k=s}^{s+l} \alpha(k) \cdot e_k' \otimes \cdots \otimes e_k' \cdot e_k$. For this, we factor $T_\alpha^{(s,l)}$ as

$$
\begin{array}{ccccc}
\ell_1 \times \cdots \times & \ell_1 & \xrightarrow{T_\alpha^{(s,l)}} & \ell_q \\
\Pi^{(s,l)} \downarrow \quad\quad & \Pi^{(s,l)} \downarrow & & \uparrow D_\alpha^{(s,l)} \\
\ell_1^{l+1} \times \cdots \times & \ell_1^{l+1} & \xrightarrow[\Psi_{l+1}]{} & \ell_\infty
\end{array}
$$

where $\Pi^{(s,l)} = \sum_{k=s}^{s+l} e_k' \cdot e_{k-s+1}$ is the (norm-one) projection on the coordinates $(s, \ldots, s+l)$ and $D_\alpha^{(s,l)} := \sum_{k=1}^{l+1} \alpha(k+s-1) \cdot e_k' \cdot e_{k+s-1}$. From the equalities $\|\Psi_{l+1}\|_{\mathcal{N}} = \|\Psi_{l+1}\|_{\mathcal{I}} = 1$, and $\|D_\alpha^{(s,l)}\| = \|(\alpha_j)_{j=s}^{s+l}\|_{\ell_q}$, we obtain

$$
\|T_\alpha^{(s,l)}\|_{\mathcal{N}} \leq \|D_\alpha^{(s,l)}\| \, \|\Psi_{l+1}\|_{\mathcal{N}} \, \|\Pi^{(s,l)}\|^n \leq \|(\alpha_j)_{j=s}^{s+l}\|_{\ell_q}.
$$

Since α belongs to ℓ_q, this inequality shows that the series $\sum_{k=1}^{\infty} \alpha(k) \cdot e_k' \otimes \cdots \otimes e_k' \cdot e_k$ defining T_α is Cauchy in nuclear norm and thus T_α is nuclear.   □

As an immediate consequence of the previous propositions we have

$$
\ell_n(\mathcal{N}, p, q) \overset{1}{=} \ell_n(\mathcal{I}, p, q) \overset{1}{=} \ell_t \quad\quad \text{for } (p, q) \neq (1, \infty); \tag{9}
$$
$$
\ell_n(\mathcal{N}, 1, \infty) \overset{1}{=} c_0 \quad \text{and} \quad \ell_n(\mathcal{I}, 1, \infty) \overset{1}{=} \ell_\infty.
$$

Before turning our attention to extendibility, we comment on another behaviour of diagonal multilinear forms/operators. In general, if two classes of multilinear forms do not coincide on some Banach space E, the corresponding classes of vector valued multilinear operators will not coincide (for any range space F). More precisely, given ideals of multilinear mappings $\mathfrak{A}$ and $\mathfrak{B}$ and a Banach space E, if $\mathfrak{A}(^n E) \neq \mathfrak{B}(^n E)$, then $\mathfrak{A}(^n E, F) \neq \mathfrak{B}(^n E, F)$ for every Banach space F. Indeed, if we take a multilinear form ϕ, say, in $\mathfrak{A}(^n E) \setminus \mathfrak{B}(^n E)$, then for any nonzero $y \in F$ the multilinear operator $\phi \cdot y$ will belong to $\mathfrak{A}(^n E, F)$ but not to $\mathfrak{B}(^n E, F)$. It should be noted that, if ϕ is a diagonal n-linear form (on some sequence space), the operator $\phi \cdot y$ will fail to be diagonal. Let us see that, when restricted to diagonal multilinear forms or operators, things are different. For example, Proposition 4.4 or (9) show that diagonal integral n-linear mappings from ℓ_1 to ℓ_q are nuclear for every $1 \leq q < \infty$. Note, however, that there are (scalar-valued) diagonal n-linear forms on ℓ_1 which are integral but not nuclear, as Theorem 3.1 shows.

We focus now on the problem of describing the space of diagonal extendible mappings. We take for a moment a more general point of view, considering Banach sequence spaces. We recall that the Köthe dual of a Banach sequence space E is defined as

$$
E^{\times} := \left\{ z \in \mathbb{K}^{\mathbb{N}} : \sum_{j \in \mathbb{N}} |z(j)x(j)| < \infty \text{ for all } x \in E \right\}.
$$

This is a Banach sequence space with the norm given by

$$
\|z\|_{E^{\times}} := \sup_{\|x\|_E \leq 1} \sum_{j \in \mathbb{N}} |z(j)x(j)|
$$

(see [24, p. 29] where the analogous notion in the more general context of Banach lattices is developed). We say that a Banach sequence space E is **Köthe reflexive** if the canonical inclusion of E into $E^{\times\times}$ is surjective.

Lemma 4.5 *Let E and F be Banach sequence spaces. Then the diagonal operator T_α : $E \times \cdots \times E \to F$ is extendible for every $\alpha \in F$, and its extendible norm is at most $\|\alpha\|_F$. In other words,*

$$F \subseteq \ell_n(\mathcal{E}, E, F)$$

with norm one inclusion.

Proof We take $\alpha \in F$ and factor T_α as

$$
\begin{array}{ccccccc}
E & \times & \cdots & \times & E & \xrightarrow{\ T_\alpha\ } & F \ , \\
\downarrow{\scriptstyle i} & & & & \downarrow{\scriptstyle i} & \nearrow & \\
& & & & & {\scriptstyle S_\alpha} & \\
\ell_\infty & \times & \cdots & \times & \ell_\infty & &
\end{array}
$$

where i is the natural inclusion and S_α is just T_α acting on $\ell_\infty \times \cdots \times \ell_\infty$. Since ℓ_∞ has the metric extension property, S_α is extendible and $\|S_\alpha\|_{\mathcal{E}(^n\ell_\infty, F)} = \|S_\alpha\| = \|\alpha\|_F$. Therefore $T_\alpha \in \mathcal{E}(^n E, F)$ and also $\|T_\alpha\|_{\mathcal{E}(^n E, F)} \leq \|S_\alpha\| \cdot \|i\|^n = \|\alpha\|_F$. □

Remark 4.6 A consequence of this lemma was anticipated at the beginning of Sect. 3: the canonical identification between diagonal n-linear operators and $(n + 1)$-linear forms does not preserve extendibility. If we take α in $\ell_2 \setminus \ell_1$, then the n-linear operator $T_\alpha \in \mathcal{L}(^n\ell_2, \ell_2)$ is extendible by the previous lemma. But Theorem 3.1 tells us that its associated $(n + 1)$-linear form is not extendible.

Proposition 4.7 *Let F be a Köthe reflexive sequence space and $2 \leq p \leq \infty$. Then T_α : $\ell_p \times \cdots \times \ell_p \to F$ is extendible if and only if $\alpha \in F$. In other words,*

$$\ell_n(\mathcal{E}, p, F) = F.$$

Proof One direction follows directly from the previous lemma. For the converse, suppose $T_\alpha \in \mathcal{E}(^n\ell_p, F)$. For any $\beta \in F^\times$ we denote by $\gamma_\beta : F \to \mathbb{K}$ the linear functional on F defined by β:

$$\gamma_\beta(x) = \sum_{k \in \mathbb{N}} \beta(k) \cdot x(k).$$

Then, the scalar valued multilinear form $\phi_{\alpha\beta} = \gamma_\beta \circ T_\alpha$ belongs to $\mathcal{E}(^n\ell_p)$, and has extendible norm not greater than $\|\beta\|_{F^\times} \cdot \|\alpha\|_{\ell_n(\mathcal{E}; p, F)}$. Since $p \geq 2$, by [12, Proposition 3.1] we know that $\alpha\beta \in \ell_1$ and

$$\|\alpha\beta\|_{\ell_1} \leq K_G^{n-1} \cdot \|\phi_{\alpha\beta}\|_{\mathcal{E}(^n\ell_p)},$$

where K_G is the constant in Grothendieck's inequality. This shows that $\alpha \in F$ and $\|\alpha\|_F \leq K_G^{n-1} \cdot \|\alpha\|_{\ell_n(\mathcal{E}, p, F)}$. □

Note that if we consider an arbitrary Banach sequence space F (not necessarily Köthe reflexive), the above argument gives the following inclusions:

$$F \hookrightarrow \ell_n(\mathcal{E}, p, F) \hookrightarrow F^{\times\times}.$$

Moreover, the result in Proposition 3.1 of [12] cited in the proof remains true if we change the space ℓ_p (with $p \geq 2$) to any 2-convex Banach sequence space E. Hence, Proposition 4.7 is also valid for diagonal multilinear maps from $E \times \cdots \times E$ to a Köthe reflexive sequence space F. For instance, this applies when E is a Lorentz sequence space $d(w, p)$ with $p \geq 2$.

Taking $F = \ell_q$ in the previous proposition gives

$$\ell_n(\mathcal{E}, p, q) = \ell_q,$$

for $2 \leq p \leq \infty$ and $1 \leq q \leq \infty$. If $p = 1$, we can also precisely describe the set $\ell_n(\mathcal{E}, p, q)$. This follows from Lemma 3.2.

Corollary 4.8 *Every diagonal multilinear mapping from $\ell_1 \times \cdots \times \ell_1$ to ℓ_q $(1 \leq q \leq \infty)$ is extendible:*

$$\ell_n(\mathcal{E}, 1, q) \overset{1}{=} \ell_\infty.$$

Proof Given $T_\alpha \in \mathcal{L}(^n\ell_1, \ell_q)$ we consider its canonical associated diagonal $(n+1)$-linear mapping $\phi_\alpha : \ell_1 \times \cdots \times \ell_1 \times \ell_{q'} \to \mathbb{K}$. By Lemma 3.2, ϕ_α is extendible and its extendible norm equals its usual norm. Then, it is clear that T_α is extendible with

$$\|T_\alpha\|_{\mathcal{E}(^n\ell_1, \ell_q)} \leq \|\phi_\alpha\|_{\mathcal{E}(^{n+1}\ell_1 \times \cdots \times \ell_1 \times \ell_{q'})} = \|\phi_\alpha\|_{\mathcal{L}(^{n+1}\ell_1 \times \cdots \times \ell_1 \times \ell_{q'})} = \|T_\alpha\|_{\mathcal{L}(^n\ell_1, \ell_q)}.$$

$\square$

We have already described $\ell_n(\mathcal{E}, p, q)$ for $p = 1$ and $p \geq 2$. For $1 < p < 2$, we present a characterization only for the cases $q = 1$ and $q > p'$. For the remaining situation ($1 < p < 2$ and $1 < q \leq p'$) we just obtain an *estimate* of $\ell_n(\mathcal{E}, p, q)$.

Proposition 4.9 *Let $1 < p < 2$.*

(1) *For $q = 1$, $\ell_n(\mathcal{E}, p, 1) = \ell_{\frac{p'}{2}}$.*
(2) *For $q > p'$, $\ell_n(\mathcal{E}, p, q) = \ell_q$.*
(3) *For $1 < q \leq p'$, $\ell_n(\mathcal{E}, p, q) \subseteq \ell_{p'+\varepsilon}$, for every $\varepsilon > 0$.*

Proof (1) If $\alpha \in \ell_{\frac{p'}{2}}$, the same argument of the proof of Proposition 3.3 shows that $\phi_\alpha : \ell_p \times \cdots \times \ell_p \times \ell_\infty \to \mathbb{K}$ is extendible with extendible norm not bigger than $\|\alpha\|_{\ell_{\frac{p'}{2}}}$. Then, $T_\alpha : \ell_p \times \cdots \times \ell_p \to \ell_1$ is extendible and $\|T_\alpha\|_{\mathcal{E}(^n\ell_p, \ell_1)} \leq \|\alpha\|_{\ell_{\frac{p'}{2}}}$.

Suppose now that $T_\alpha \in \mathcal{E}(^n\ell_p, \ell_1)$. Then, for every $\beta \in \ell_\infty$, we have $\phi_\beta \circ T_\alpha = \phi_{\alpha\beta} \in \mathcal{E}(^n\ell_p)$. By Proposition 3.3 we derive that $\alpha\beta = (\alpha(k)\beta(k))_k$ belongs to $\ell_{\frac{p'}{2}}$. Since this happens for every $\beta \in \ell_\infty$, we conclude that $\alpha \in \ell_{\frac{p'}{2}}$.

(2) Since $q > p'$, ℓ_q has cotype $q > 2$. Then, we can apply [8, Proposition 3.4] to derive that if $T_\alpha \in \mathcal{E}(^n\ell_p, \ell_q)$ then T_α is absolutely $(q; p')$-summing. So, there exists $K > 0$, such that, for every N we have

$$\left(\sum_{k=1}^N |\alpha_k|^q\right)^{\frac{1}{q}} = \left(\sum_{k=1}^N \|\phi_\alpha(e_k, \ldots, e_k)\|^q\right)^{\frac{1}{q}} \leq K \cdot w_{p'}\left((e_k)_{k=1}^N\right)^n = K.$$

Hence, $\alpha \in \ell_q$. The other inclusion was already shown in Lemma 4.5.

(3) Let $T_\alpha \in \mathcal{E}(^n\ell_p, \ell_q)$ then, by [6, Proposition 5.3], T_α is absolutely $(p'+\varepsilon; p')$-summing, for every $\varepsilon > 0$. Reasoning as in the previous item, we see that $\alpha \in \ell_{p'+\varepsilon}$.

$\square$

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