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Uniform Convergence Series to Solve Nonlinear Partial Differential Equations: Application to Beam Dynamics

C. P. FILIPICH[1,2] and M. B. ROSALES[1,3]
[1]*Department of Engineering, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina*
[2]*Mechanical Systems Analysis Group, FRBB, Universidad Tecnológica Nacional, 8000 Bahía Blanca, Argentina*
[3]*CONICET, 8000 Bahía Blanca, Argentina*

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Abstract. Extended trigonometric series of uniform convergence are proposed as a method to solve the nonlinear dynamic problems governed by partial differential equations. In particular, the method is applied to the solution of a uniform beam supported at its ends with nonlinear rotational springs and subjected to dynamic loads. The beam is assumed to be both material and geometrically linear and the end springs are of the Duffing type. The action may be a continuous load $q = q(x, t)$ within a certain range and/or concentrated dynamic moments at the boundaries. The adopted solution satisfies the differential equation, the initial conditions, and the nonlinear boundary conditions. It has been previously demonstrated that, due to the uniform convergence of the series, the method yields arbitrary precision results. An illustration example shows the efficiency of the method.

Keywords: Uniform convergence series, partial differential equations, nonlinear boundary conditions.

1. Introduction

Classically, the dynamic behavior of problems governed by partial differential equations are addressed by means of a Galerkin-type technique with a separation of variables, using a trial function in the space domain. Other alternatives include the harmonic balance or multiple scale methods. Also, purely numerical methods such as Runge–Kutta and other time-integrative techniques are in common use.

In any case, the solution obtained is approximated. Since the spatial mode shape is not evident when the boundary conditions are nonlinear, the measure of the magnitude is not available with some of these methods. Nor is it possible to obtain using symbolic algebra packages or standard finite element algorithms.

In this paper, the nonlinear differential problem is solved using an original methodology which yields arbitrary precision results. The method is based on the use of an extended trigonometric series of uniform convergence that has been applied previously for finding the solution of a wide variety of boundary-value problems [6–9]. The tool is called WEM1 or simply WEM and leads to a generalized solution. Also, initial conditions (IC) problems were tackled with this technique [10]. When dealing with partial differential equations, with boundary conditions (BC) + IC, this type of methodology is usually applied to the space domain and integration techniques (central difference, Newmark methods, and so on) are used for the time domain [11]. In particular, the title problem was solved in [12] using WEM1 for the space variable. A linear problem governed by a partial differential equation (BC + IC) was analyzed using WEM1 and reported in [13].

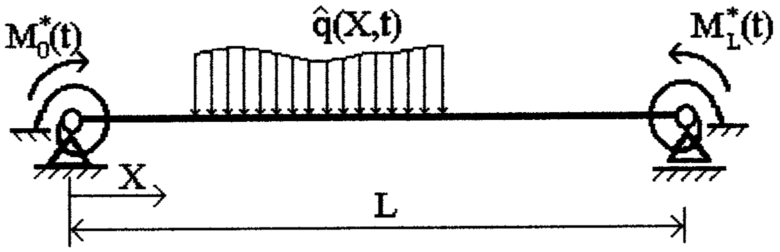

Figure 1. Linear beam supported on nonlinear springs and subjected to a dynamic load.

Alternatively, two-dimensional series of *a-priori* uniform convergence have shown great efficiency in finding a nonvariational solution that satisfies, in the domain, the differential equation and all its boundary conditions. The first author has described this methodology in [14] where alternative variants are shown. For the sake of order, let us call the method herein employed WEM2.

So as to illustrate the type of extended series to be used, described briefly in Appendix A, let us introduce $\phi = \phi(x, y)$, a continuous function in $\{D : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ such that

$$|\phi_{MN} - \phi| \to 0, \quad M, N \to \infty \quad \text{in} \quad D,$$

i.e. uniform convergence will hold. $\phi_{MN}(x, y)$ is a two-dimensional extended uniform convergence series. Among infinite possibilities and with an appropriate selection of the constants $\gamma_{ij}, \gamma_{0j}, \gamma_{i0}, \gamma_{00}, \varepsilon_j, \varepsilon_i, a_0, b_0$ and k, we may write

$$\begin{aligned}\phi_{MN}(x, y) \; = \; & \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{\gamma_{ij}}{\alpha_i \alpha_j} s_i(x) s_j(y) + x \left[a_0 + \sum_{j=1}^{N} \frac{\gamma_{0j} s_j(y)}{\alpha_j} \right] \\ & + y \left[b_0 + \sum_{i=1}^{M} \frac{\gamma_{i0} s_i(x)}{\alpha_i} \right] + \gamma_{00} x y + \sum_{j=1}^{N} \frac{\varepsilon_j s_j(y)}{\alpha_j} + \sum_{i=1}^{M} \frac{\varepsilon_i s_i(x)}{\alpha_i} + k.\end{aligned}$$

The following notation was introduced

$$\begin{aligned}&\alpha_m \equiv m\pi, \\ &s_m(x) \equiv \sin \alpha_m x, \quad s_m(y) \equiv \sin \alpha_m y, \quad c_m(x) \equiv \cos \alpha_m x, \quad c_m(y) \equiv \cos \alpha_m y, \\ &(m - \text{integer}, m > 0, m = i, j).\end{aligned}$$

In this paper, the dynamic response of a uniform straight beam, supported at its ends with simple supports and nonlinear rotational springs of the Duffing type, is studied. Several authors have addressed the study of systems with nonlinear boundary conditions (see, for instance, [1–5]). The beam is subjected to a continuous load and/or dynamic concentrated moments acting on the springs. Methodology WEM2, using the above introduced sequence, yields arbitrary precision results. The validity and efficiency of the method proposed in this paper are illustrated through a numerical example.

2. Problem Statement

Let $u = \hat{u}(X, t)$ be the dynamic response of a homogeneous, uniform, and straight beam supported at its ends by simple supports and nonlinear rotational springs and subjected to a load $q^* = \hat{q}(X, t)$ continuous in $0 \leq X \leq L$ and $T^* \geq t \geq 0$, as shown in Figure 1. The beam is elastic with modulus E, mass density ρ, length L, cross section area F, and baricentric moment of inertia J. Its motion is governed by the partial differential equation

$$\hat{u}_{XXXX} + a^{*2}\ddot{\hat{u}} = q^* \tag{1}$$

and, $\forall t$, subjected to the following nonlinear boundary conditions (BC)

$$\text{BC}\begin{cases} EJ\hat{u}_{XX}(0,t) = \alpha_0^*\hat{u}_X(0,t) + \gamma_0^*\hat{u}_X^3(0,t) - M_0^*(t), & \text{(a)} \\ EJ\hat{u}_{XX}(L,t) = -(\alpha_L^*\hat{u}_X(L,t) + \gamma_L^*\hat{u}_X^3(L,t) + M_L^*(t)), & \text{(b)} \\ \hat{u}(0,t) = 0, & \text{(c)} \\ \hat{u}(L,t) = 0, & \text{(d)} \end{cases} \tag{2}$$

and also verify, $\forall X$, the following initial conditions (IC):

$$\text{IC}\begin{cases} \hat{u}(X,0) = U_0^*(X), & \text{(a)} \\ \dot{\hat{u}}(X,0) = V_0^*(X), & \text{(b)} \end{cases} \tag{3}$$

where $a^{*2} \equiv \rho F/EJ$, α^* and γ^* are the end spring constants, and M_0^* and M_L^* are the bending moments function of t applied at $X = 0$ and $X = L$, respectively ($M_0^*(t) = \mu_0^*\hat{f}(t)$, $M_L^*(t) = \mu_L^*\hat{g}(t)$). We denote $(\bullet)_X \equiv \partial(\bullet)/\partial X$, $(\dot{\bullet}) \equiv \partial(\bullet)/\partial t$ and so on and $U_0^*(X)$, $V_0^*(X)$ are known functions of X that fulfill BC (2) at $t = 0$.

In conclusion, this work deals with the dynamic study of a (both material and geometrically) linear beam that should fit end supports with springs of nonlinear response. In the present analysis, the reacting moment is assumed as a cubic function of the end rotations. This nonlinearity is chosen in order to fix ideas. Other types of nonlinearities may be introduced in a similar way.

The solution of the problem will be tackled by means of an extended series (of the type introduced in Section 1). Then a change of variable becomes necessary so that a 'unitary' domain $\{D : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ is considered. We define

$$\begin{cases} x \equiv \dfrac{X}{L}, & \text{(a)} \\ y \equiv \dfrac{t}{T}, & \text{(b)} \end{cases} \tag{4}$$

and $[0, T]$ is the arbitrary interval of time of interest $0 < T \leq T^*$ for which the response is found. Since we denote $(\bullet)' \equiv \partial(\bullet)/\partial x$, $(\bar{\bullet}) \equiv \partial(\bullet)/\partial y$, it may be observed that $(\bullet)_X = (\bullet)'/L$, $(\dot{\bullet}) = (\bar{\bullet})/T$, etc.

We also introduce the following notation

$$a^2 \equiv a^{*2}L^4/T^2; \quad \alpha_0 \equiv \alpha_0^*L/EJ; \quad \gamma_0 \equiv \gamma_0^*/LEJ; \quad \alpha_1 \equiv \alpha_L^*L/EJ; \quad \gamma_1 \equiv \gamma_L^*/LEJ$$

and

$$\begin{cases} v = v(x,y) \equiv \hat{u}(Lx, Ty) & \text{(a)} \\ q = q(x,y) \equiv \hat{q}(Lx, Ty)\dfrac{L^4}{EJ} & \text{(b)} \\ U_0 = U_0(x) = U_0^*(Lx) & \text{(c)} \\ V_0 = V_0(x) = TV_0^*(Lx) & \text{(d)} \\ M_0(y) \equiv M_0^*(Ty)\dfrac{L^2}{EJ};\ M_1(y) \equiv M_L^*(Ty)\dfrac{L^2}{EJ}. & \text{(e)} \end{cases} \tag{5}$$

Finally, the nondimensionalized differential problem is

$$\nu'''' + a^2\bar{\bar{\nu}} = q(x, y), \tag{6}$$

$$\text{BC}\begin{cases} \nu''(0, y) = \alpha_0\nu'(0, y) + \gamma_0\nu'^3(0, y) - M_0(y), & \text{(a)} \\ \nu''(1, y) = -[\alpha_1\nu'(1, y) + \gamma_1\nu'^3(1, y) + M_1(y)], & \text{(b)} \\ \nu(0, y) = 0, & \text{(c)} \\ \nu(1, y) = 0, & \text{(d)} \end{cases} \tag{7}$$

$$\text{IC}\begin{cases} \nu(x, 0) = U_0(x), & \text{(a)} \\ \bar{\nu}(x, 0) = V_0(x). & \text{(b)} \end{cases} \tag{8}$$

For the sake of simplicity and without loss of generality, we impose

$$\begin{cases} q(x, y) \equiv 0, & \text{(a)} \\ \hat{f}(t) = \hat{g}(t) = \hat{f}(Ty) = \hat{g}(Ty) = f(y), & \text{(b)} \\ M_0(y) = \mu_0 f(y), & \text{(c)} \\ M_1(y) = \mu_1 f(y). & \text{(d)} \end{cases} \tag{9}$$

That is, we accept, that the motion is produced by $M_0(y)$ and $M_1(y)$ at the boundaries, which vary with different magnitude in a similar fashion according to $f(y)$ (known). It is assumed (always possible) that $f(0) = 1$.

3. Proposition

This proposition will lead to the solution of the nonlinear problem and consists in expressing the largest order derivatives in each variable by a series of uniform convergence in two dimensions. The methodology imposes the following type of expanded series in sines (recall the Introduction):

$$\begin{aligned} \varphi_{MN}(x, y) &= \sum_{i=1}^{M}\sum_{j=1}^{N}\frac{\gamma_{ij}}{\alpha_i\alpha_j}s_i s_j + x\left(a_0 + \sum_{j=1}^{N}\frac{\gamma_{0j}s_j}{\alpha_j}\right) + y\left(b_0 + \sum_{i=1}^{M}\frac{\gamma_{i0}s_i}{\alpha_i}\right) \\ &\quad + \gamma_{00}xy + \sum_{i=1}^{M}\frac{\varepsilon_i s_i}{\alpha_i} + \sum_{j=1}^{N}\frac{\varepsilon_j s_j}{\alpha_j} + k, \end{aligned} \tag{10}$$

which is selected to represent both ν'''' and $\bar{\bar{\nu}}$, naming them ν_1'''' and $\bar{\bar{\nu}}_2$, respectively (see Equations (11) and (12) below). In this equation and in what follows, $s_i \equiv s_i(x)$, $s_j \equiv s_i(y)$, etc. From P_7 of Appendix A, since $\nu(0, y) = \nu(1, y) = 0$, (Equations (7c) and (d)), it is true that $\bar{\nu}(0, y) = \bar{\nu}(1, y) = \bar{\bar{\nu}}(0, y) = \bar{\bar{\nu}}(1, y) = 0$. If the satisfaction of the differential equation (6) with $q = 0$ at $x = 0, 1$ is required, the additional result yields $\nu''''(0, y) = \nu''''(1, y) = 0$. From these conditions applied in Equation (10) and taking into account expansions of the type (A13–A15) in the variable y, significant reductions are obtained, as is shown below.

Consequently, by imposing ν_1'''' and $\bar{\bar{\nu}}_2$, and integrating successively, we have

$$\left\{\begin{array}{ll}
\nu_{1_{MN}}'''' = \sum_{i=1}^{M}\sum_{j=1}^{N} A_{ij} s_i s_j + y \sum_{i=1}^{M} A_{i0} s_i + \sum_{i=1}^{M} a_{i0} s_i, & \text{(a)} \\
\nu_{1_{MN}}''' = -\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{A_{ij} c_i s_j}{\alpha_i} - y \sum_{i=1}^{M} \frac{A_{i0} c_i}{\alpha_i} - \sum_{i=1}^{M} \frac{a_{i0} c_i}{\alpha_i} + F_1(y), & \text{(b)} \\
\nu_{1_{MN}}'' = -\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{A_{ij} s_i s_j}{\alpha_i^2} - y \sum_{i=1}^{M} \frac{A_{i0} s_i}{\alpha_i^2} - \sum_{i=1}^{M} \frac{a_{i0} s_i}{\alpha_i^2} + x F_1(y) + F_2(y), & \text{(c)} \\
\nu_{1_{MN}}' = \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{A_{ij} c_i s_j}{\alpha_i^3} + y \sum_{i=1}^{M} \frac{A_{i0} c_i}{\alpha_i^3} + \sum_{i=1}^{M} \frac{a_{i0} c_i}{\alpha_i^3} + \frac{x^2}{2} F_1(y) & \\
\qquad + x F_2(y) + F_3(y), & \text{(d)} \\
\nu_{1_{MN}} = \sum_{i=1}^{M}\sum_{j=1}^{N} \frac{A_{ij} s_i s_j}{\alpha_i^4} + y \sum_{i=1}^{M} \frac{A_{i0} s_i}{\alpha_i^4} + \sum_{i=1}^{M} \frac{a_{i0} s_i}{\alpha_i^4} + \frac{x^3}{6} F_1(y) & \\
\qquad + \frac{x^2}{2} F_2(y) + x F_3(y) + F_4(y), & \text{(e)}
\end{array}\right. \tag{11}$$

where A_{ij}, A_{i0}, a_{i0} are unknown constants and $F_m(y)$ $(m = 1, \ldots, 4)$ are integration functions. All the series are uniformly convergent. We use the same reasoning to expand the derivatives with respect to y.

$$\left\{\begin{array}{ll}
\bar{\bar{\nu}}_{2_{MN}} = \sum_{i=1}^{M}\sum_{j=1}^{N} B_{ij} s_i s_j + y \sum_{i=1}^{M} B_{i0} s_i + \sum_{i=1}^{M} b_{i0} s_i, & \text{(a)} \\
\bar{\nu}_{2_{MN}} = -\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{B_{ij} s_i c_j}{\alpha_j} + \frac{y^2}{2} \sum_{i=1}^{M} B_{i0} s_i + y \sum_{i=1}^{M} b_{i0} s_i + G_1(x), & \text{(b)} \\
\nu_{2_{MN}} = -\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{B_{ij} s_i s_j}{\alpha_j^2} + \frac{y^3}{6} \sum_{i=1}^{M} B_{i0} s_i + \frac{y^2}{2} \sum_{i=1}^{M} b_{i0} s_i + y G_1(x) + G_2(x), & \text{(c)}
\end{array}\right. \tag{12}$$

where B_{ij}, B_{i0}, b_{i0} are unknown constants and $G_m(x)$ $(m = 1, 2)$ are integration functions. Again, the uniform convergence is verified. With this methodology, not only should the differential problem (6–8) be verified but also the obvious consistence condition

$$\nu_{1_{MN}} = \nu_{2_{MN}} \quad \forall (x, y). \tag{13}$$

Without loosing generality, we assume the case of $q = 0$. As is shown in Appendix B, in order for the IC to be satisfied, M_0 and M_1 should not be simultaneously null. Before going on with the algebra, let us write the expressions of $U_0(x)$ and $V_0(x)$ (see, Appendix B for more details).

$$\left\{\begin{array}{ll}
U_0(x) = x(1-x)[\theta_0 - (\theta_0 + \theta_1)x], & \text{(a)} \\
V_0(x) = x(1-x)[\Omega_0 - (\Omega_0 + \Omega_1)x]. & \text{(b)}
\end{array}\right. \tag{14}$$

Recall that we have introduced the notation

$$\left\{\begin{array}{lll}
\theta_0 \equiv \nu'(0,0); & \theta_1 \equiv \nu'(1,0), & \text{(a)} \\
\Omega_0 \equiv \bar{\nu}'(0,0); & \Omega_1 \equiv \bar{\nu}'(1,0). & \text{(b)}
\end{array}\right. \tag{15}$$

The relationship among the Ω's and the θ's is stated in Appendix B as well as the μ_0 and μ_1 expressions as function of θ_0 and θ_1.

From the boundary conditions (7c) and (7d), it is found that

$$\begin{cases} F_4(y) = 0, & \text{(a)} \\ \dfrac{F_1(y)}{6} + \dfrac{F_2(y)}{2} + F_3(y) = 0, \forall y, & \text{(b)} \\ G_1(0) = G_2(0) = 0, & \text{(c)} \\ G_1(1) = G_2(1) = 0. & \text{(d)} \end{cases} \tag{16}$$

The fulfillment of the initial conditions (8) gives place to (taking into consideration also (16b))

$$\begin{cases} \displaystyle\sum_{i=1}^{M} \frac{a_{i0} s_i}{\alpha_i^4} + \frac{F_{10}}{6} x(x^2 - 1) + \frac{F_{20}}{2} x(x - 1) = U_0(x), & \text{(a)} \\ G_2(x) = U_0(x), & \text{(b)} \\ \displaystyle -\sum_{i=1}^{M} \sum_{j=1}^{N} \frac{B_{ij} s_i}{\alpha_j} + G_1(x) = V_0(x), & \text{(c)} \end{cases} \tag{17}$$

where we introduced

$$\begin{cases} F_j(0) \equiv F_{j0} \quad (j = 1, 2), & \text{(a)} \\ F_j(1) \equiv F_{j1} \quad (j = 1, 2). & \text{(b)} \end{cases} \tag{18}$$

Before stating generally the remaining boundary conditions (7a) and (7b), let us write them in $y = 0$, making use of (11c), (15a), (16b) and Appendix B. Thus, we obtain

$$F_{10} = 6(\theta_0 + \theta_1); \qquad F_{20} = -2(2\theta_0 + \theta_1). \tag{19}$$

Now expressions (15a) with (11d) and (16b) for $y = 0$ may be written as

$$\begin{cases} \displaystyle\sum_{i=1}^{M} \frac{a_{i0}}{\alpha_i^3} - \frac{F_{10}}{6} - \frac{F_{20}}{2} = \theta_0, & \text{(a)} \\ \displaystyle\sum_{i=1}^{M} \frac{(-1)^i a_{i0}}{\alpha_i^3} + \frac{F_{10}}{3} + \frac{F_{20}}{2} = \theta_1. & \text{(b)} \end{cases} \tag{20}$$

It is interesting to find from (17a), (19) and (14a) that

$$a_{i0} = 0 \quad (i = 1, 2, \ldots, M). \tag{21}$$

Finally, the statement of boundary conditions (7a) and (7b) for any y using Equations (11), (16b) and (21), yields

$$\begin{cases} F_2(y) = -\alpha_0[\Lambda_0(y) + \varphi_0(y)] - \gamma_0[\Lambda_0(y) + \varphi_0(y)]^3 - \mu_0 f(y), & \text{(a)} \\ F_1(y) + F_2(y) = \alpha_1[\Lambda_1(y) + \varphi_1(y)] + \gamma_1[\Lambda_1(y) + \varphi_1(y)]^3 - \mu_1 f(y), & \text{(b)} \end{cases} \tag{22}$$

where the following notation has been introduced:

$$\begin{cases} \Lambda_0(y) \equiv a^2[S_0(y) + y Z_0], & \text{(a)} \\ \Lambda_1(y) \equiv a^2[S_1(y) + y Z_1], & \text{(b)} \end{cases} \tag{23}$$

$$\begin{cases} \varphi_0(y) \equiv \frac{1}{6}[F_1(y) + 3F_2(y)], & \text{(a)} \\ \varphi_1(y) \equiv -\frac{1}{6}[2F_1(y) + 3F_2(y)], & \text{(b)} \end{cases} \quad \text{or}$$

$$\begin{cases} F_1(y) = -6[\varphi_0(y) + \varphi_1(y)], & \text{(a)} \\ F_2(y) = 2[2\varphi_0(y) + \varphi_1(y)]. & \text{(b)} \end{cases} \tag{24}$$

and, at the same time,

$$\begin{cases} a^2 S_0(y) \equiv -\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{A_{ij}s_j}{\alpha_i^3}; & a^2 Z_0 \equiv -\sum_{i=1}^{M} \frac{A_{i0}}{\alpha_i^3}, \quad \text{(a)} \\ a^2 S_1(y) \equiv -\sum_{i=1}^{M}\sum_{j=1}^{N} \frac{(-1)^i A_{ij}s_j}{\alpha_i^3}; & a^2 Z_1 \equiv -\sum_{i=1}^{M} \frac{(-1)^i A_{i0}}{\alpha_i^3}. \quad \text{(b)} \end{cases} \tag{25}$$

From Equations (24), we may write $F_1(y)$ and $F_2(y)$ as functions of $\varphi_1(y)$ and $\varphi_2(y)$. Then Equations (22) may be written as a system of cubic equations in $\varphi_0(y)$ and $\varphi_1(y)$

$$\begin{cases} \gamma_0\varphi_0^3 + 3\Lambda_0\gamma_0\varphi_2^2 + (3\gamma_0\Lambda_0^2 + 4 + \alpha_0)\varphi_0 \\ \qquad + \gamma_0\Lambda_0^3 + \alpha_0\Lambda_0 + \mu_0 f(y) + 2\varphi_1 = 0, \quad \text{(a)} \\ \gamma_1\varphi_1^3 + 3\Lambda_1\gamma_1\varphi_1^2 + (3\gamma_1\Lambda_1^2 + 4 + \alpha_1)\varphi_1 \\ \qquad + \gamma_1\Lambda_1^3 + \alpha_1\Lambda_1 - \mu_1 f(y) + 2\varphi_0 = 0. \quad \text{(b)} \end{cases} \tag{26}$$

NOTE: In the linear problem, $\gamma_0 = \gamma_1 = 0$, the system (26) is 2×2 linear in $\varphi_0(y)$ and $\varphi_1(y)$.

In order to finish this cumbersome algebra, one should impose the fulfillment of the differential Equation (6) with $q = 0$ taking into account (11a), (12a) and (21), and one obtains

$$\sum_{i=1}^{M}\sum_{j=1}^{N}(A_{ij} + a^2 B_{ij})s_i s_j + y\sum_{i=1}^{M}(A_{i0} + a^2 B_{i0})s_i + a^2\sum_{i=1}^{M} b_{i0}s_i = 0. \tag{27}$$

From the theory of extended series and factoring $\sum_{i=1}^{M} s_i$, it holds that

$$\begin{cases} A_{ij} = -a^2 B_{ij}, & \text{(a)} \\ A_{i0} - a^2 B_{i0}, & \text{(b)} \\ b_{i0} = 0, & \text{(c)} \\ (i = 1, 2, \ldots, M; \quad j = 1, 2, \ldots, N). \end{cases} \tag{28}$$

Now, the consistence condition (13) can be also written as the following extended sine series of uniform convergence:

$$\sum_{j=1}^{N} P_j(x)s_j + yP_0(x) + \varepsilon(x) = \phi(x, y), \tag{29}$$

where

$$\begin{cases} P_j(x) \equiv \sum_{i=1}^{M} \left(\frac{a^2}{\alpha_i^4} - \frac{1}{\alpha_j^2} \right) B_{ij} s_i \quad (j = 1, 2, \ldots, N) & \text{(a)} \\ P_0(x) \equiv G_1(x) + a^2 \sum_{i=1}^{M} \frac{B_{i0} s_i}{\alpha_i^4}, & \text{(b)} \\ \varepsilon(x) \equiv G_2(x), & \text{(c)} \\ \phi(x, y) \equiv \frac{x(x^2-1)}{6} F_1(y) + \frac{x(x-1)}{2} F_2(y) - \frac{y^3}{6} \sum_{i=1}^{M} B_{i0} s_i, & \text{(d)} \end{cases} \tag{30}$$

where Equations (16b) and (28) have been used. In order for (29) to be true, the following conditions (see Appendix A) should hold:

$$\begin{cases} \varepsilon(x) = \phi(x, 0), & \text{(a)} \\ P_0(x) = \phi(x, 1) - \phi(x, 0), & \text{(b)} \\ P_j(x) = 2\{[\phi(x, y) - y P_0(x) - \varepsilon(x)], s_j\}. & \text{(c)} \end{cases} \tag{31}$$

Furthermore, since

$$\phi(x, 0) = \frac{x(x^2-1)}{6} F_{10} + \frac{x(x-1)}{2} F_{20} \tag{32}$$

and considering expressions (19), (17b), and (14a), the condition (31a) is identically verified. That is

$$\varepsilon(x) = G_2(x) = \phi(x, 0) = U_0(x). \tag{33}$$

Also, considering (30b), (30d) and (31b), we find that

$$\begin{aligned} P_0(x) &= G_1(x) + a^2 \sum_{i=1}^{M} \frac{B_{i0} s_i}{\alpha_i^4} \\ &= \frac{x(x^2-1)}{6} (F_{11} - F_{10}) + \frac{x(x-1)}{2} (F_{21} - F_{20}) - \frac{1}{6} \sum_{i=1}^{M} B_{i0} s_i. \end{aligned} \tag{34}$$

From this equation and (17c),

$$\begin{aligned} &\sum_{i=1}^{M} B_{i0} \left(\frac{1}{6} + \frac{a^2}{\alpha_i^4} \right) s_i + x \left[\frac{(F_{11} - F_{10})}{6} + \frac{(F_{21} - F_{20})}{2} \right] \\ &\quad = \frac{x^3 (F_{11} - F_{10})}{6} + \frac{x^2 (F_{21} - F_{20})}{2} - V_0(x) - \sum_{i=1}^{M} \sum_{j=1}^{N} \frac{B_{ij} s_i}{\alpha_i^4}. \end{aligned} \tag{35}$$

Again, from the theory of extended series it is possible to write

$$\left(\frac{a^2}{\alpha_i^4} + \frac{1}{6} \right) B_{i0} = \frac{\Delta_i^{31}}{3} (F_{11} - F_{10}) + \Delta_i^{21} (F_{21} - F_{20}) - 2[V_0(x), s_i] - \sum_{j=1}^{N} \frac{B_{ij}}{\alpha_j}, \tag{36}$$

with

$$\left.\begin{array}{ll} \Delta_k^{mn} = L_k^m - L_k^n, & \text{(a)} \\ L_k^m \equiv (x^m, s_k), & \text{(b)} \end{array}\right\} \quad k = i \quad \text{or} \quad j. \tag{37}$$

Finally, equating Equations (30a) and (31c) and introducing

$$\left\{\begin{array}{ll} \beta_{1j} \equiv [F_1(y), s_j], & \text{(a)} \\ \beta_{2j} \equiv [F_2(y), s_j], & \text{(b)} \end{array}\right. \tag{38}$$

we may write

$$\begin{aligned} P_j(x) &= \sum_{i=1}^{M} \left(\frac{a^2}{\alpha_i^4} - \frac{1}{\alpha_j^2} \right) B_{ij} s_i \\ &= \frac{x(x^2-1)}{3}\beta_{1j} + x(x-1)\beta_{2j} - \frac{L_j^3}{3} \sum_{i=1}^{M} B_{i0} s_i - 2L_j^1 P_0(x) - 2L_j^0 \varepsilon(x). \end{aligned} \tag{39}$$

The solution of expression (39), now with respect to x, and the use of the definitions and expressions introduced above, permits us to obtain

$$\begin{aligned} \left(\frac{a^2}{\alpha_i^4} - \frac{1}{\alpha_j^2} \right) B_{ij} &= -\frac{B_{i0}}{3}\Delta_j^{31} + \frac{2}{3}\Delta_i^{31}(\beta_{1j} - L_j^1 F_{11} + \Delta_j^{10} F_{10}) \\ &\quad + 2\Delta_i^{21}(\beta_{2j} - L_j^1 F_{21} + \Delta_j^{10} F_{20}). \end{aligned} \tag{40}$$

At this stage the problem is fully stated and theoretically solved. However, this would be true if $F_1(y)$ and $F_2(y)$ were known. In effect, we could find the B_{ij}'s from Equation (40) and with them the B_{i0}'s from (36) with which the procedure would be completed. Unfortunately, as may be observed from Equation (22), $F_1(y)$ and $F_2(y)$ depend on B_{ij} and B_{i0}, even if the problem is linear.

The problem is solved by means of an iterative algorithm. This way permits us to address with arbitrary precision any type of nonlinearity and it is not restricted to a cubic one as tackled herein. In short, and for given M, N (number of terms in the series) and *NS* (number of steps in the numerical integration algorithm) the iteration is performed over the following steps:

1: The initial sets $\{B_{i0}\}^0$ and $\{B_{ij}\}^0$ are given.
2: After using equivalences (28), summations (25) are calculated.
3: System (26) is solved for φ_0 and φ_1.
4: $F_1(y)$ and $F_2(y)$ are solved from system (24).
5: β_{1j} and β_{2j} are found from expressions (38).
6: $\{B_{ij}\}^1$ are found from Equation (40) and $\{B_{i0}\}^1$ using expression (36).
7: If the fixed precision is not attained, the iteration continues at step 2.
8: If finished, $\nu_{1_{MN}}$ or $\nu_{2_{MN}}$ are completely determined.

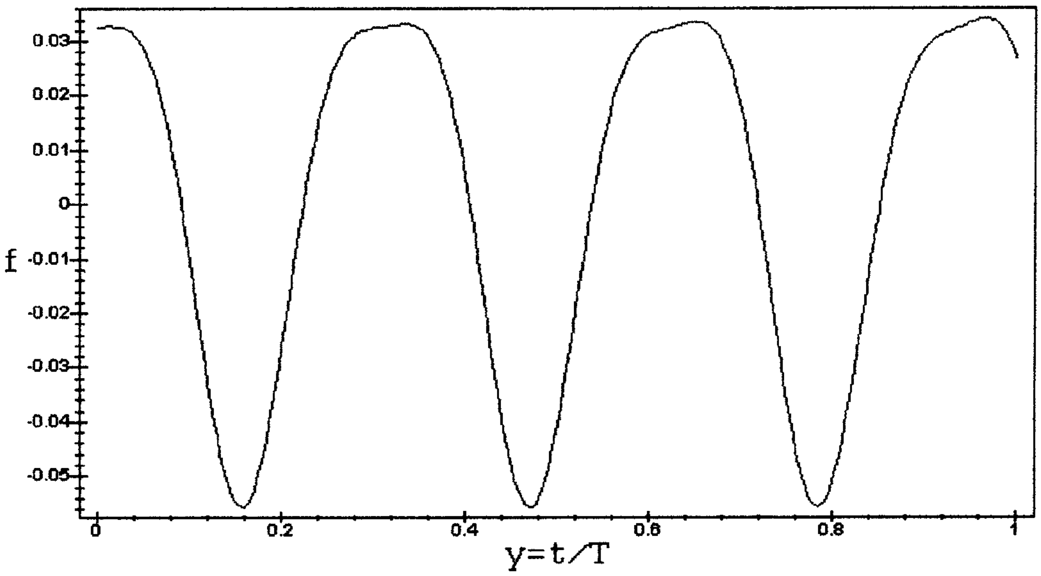

Figure 2a. Beam on nonlinear supports subjected to dynamic end moments. $\alpha_0 = \alpha_1 = 0$; $\gamma_0 = \gamma_1 = 2$; $\theta_0 = 5°$; $\theta_1 = -10°$; $M = N = 150$; $NS = 300$. Time-displacement (at the center of the beam) curve. Forcing frequency $\omega = 5$ rad/sec.

4. Numerical Example

The data are given by the values of the stiffness constant of the springs α_0, α_1, γ_0 and γ_1 and the extreme rotations θ_0 and θ_1 for $t = y = 0$. Also we adopt

$$\hat{f}(t) = \cos(\omega t), \tag{41}$$

where ω is the chosen forcing frequency. Then

$$f(y) = \cos \omega T y. \tag{42}$$

Consequently, $f(0) = 1$ and $\bar{f}(0) = 0$, and with this, $V_0(x) = 0$. A numerical example is now reported. The dynamic behavior of a beam with symmetric nonlinear supports was analyzed with the following parameters: $\alpha_0 = \alpha_1 = 0$; $\gamma_0 = \gamma_1 = 2$. At $t = 0$, the rotations at the ends of the beam were assumed to be $\theta_0 = 5°$, $\theta_1 = -10°$. For all the cases, the number of terms in the series was adopted as $M = N = 150$ and also in a numerical integration algorithm (Simpson) $NS = 300$. The interval of interest was taken as $T = 4$ sec.

Figures 2 show the response of the beam subjected to the end moments of frequency $\omega = 5$ rad/sec (less than π^2). From the *time-displacement* (at the center of the beam) curve (Figure 2a) and the phase plane (Figure 2b) a period-1 oscillation may be observed. When the forcing frequency is increased to $\omega = 15$ rad/sec, a period-2 motion is now evident (Figures 3). Finally, the frequency $\omega = 25$ rad/sec yields different curves (Figures 4).

5. Final Comments

An alternative methodology was proposed to address problems governed by partial differential equations, even nonlinear ones. In particular, the behavior of a beam on nonlinear supports

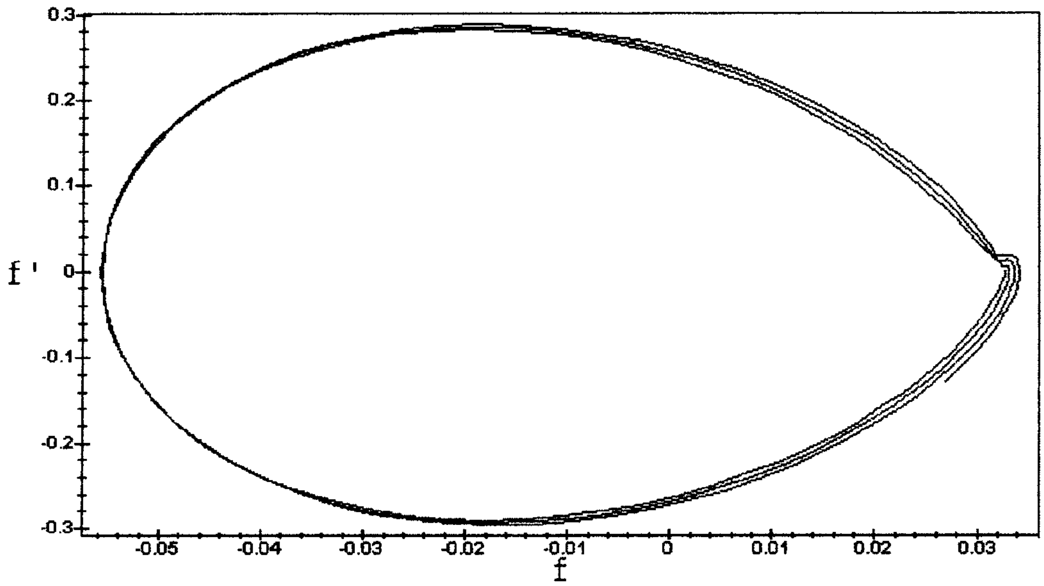

Figure 2b. Beam on nonlinear supports subjected to dynamic end moments. $\alpha_0 = \alpha_1 = 0$; $\gamma_0 = \gamma_1 = 2$; $\theta_0 = 5°$; $\theta_1 = -10°$; $M = N = 150$; $NS = 300$. Phase diagram. f: displacement at the center of the beam; f': nondimensional velocity. Forcing frequency $\omega = 5$ rad/sec.

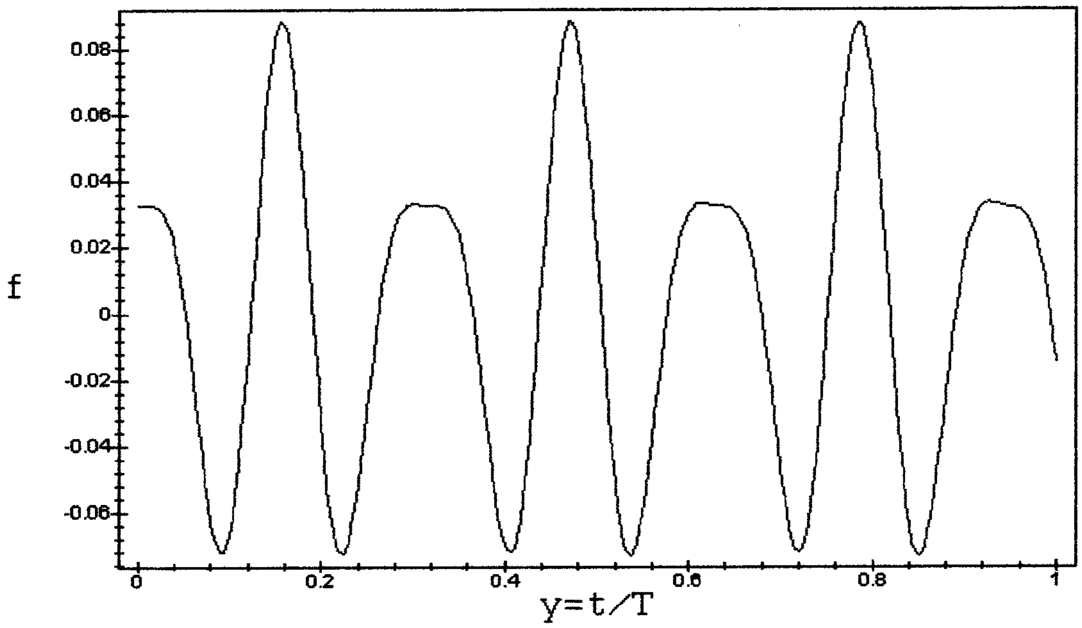

Figure 3a. Beam on nonlinear supports subjected to dynamic end moments. $\alpha_0 = \alpha_1 = 0$; $\gamma_0 = \gamma_1 = 2$; $\theta_0 = 5°$; $\theta_1 = -10°$; $M = N = 150$; $NS = 300$. Forcing frequency $\omega = 15$ rad/sec.

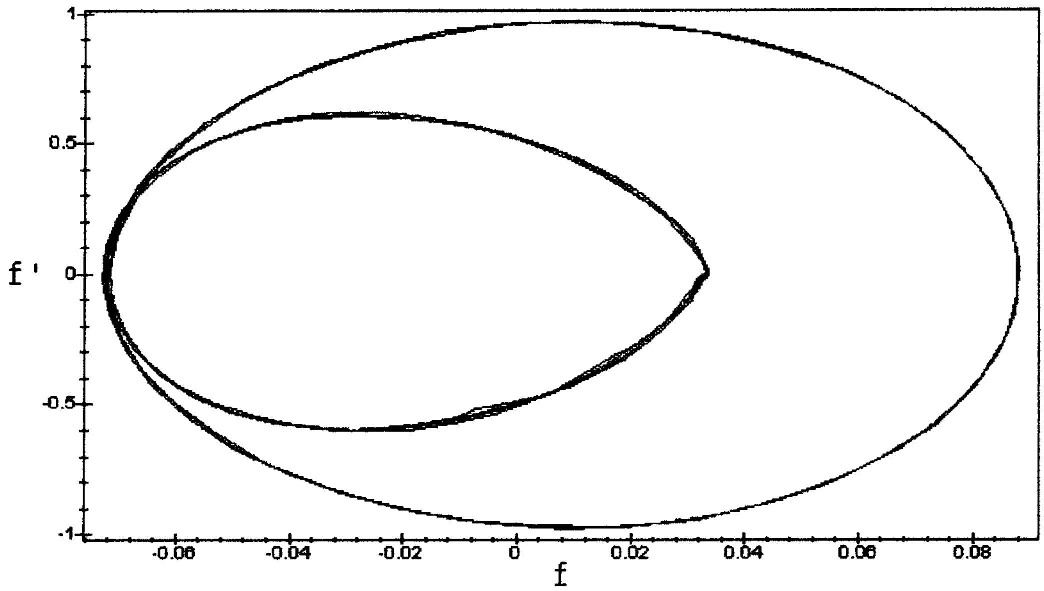

Figure 3b. Beam on nonlinear supports subjected to dynamic end moments. $\alpha_0 = \alpha_1 = 0$; $\gamma_0 = \gamma_1 = 2$; $\theta_0 = 5°$; $\theta_1 = -10°$; $M = N = 150$; $NS = 300$. Phase diagram. f: displacement at the center of the beam; f': nondimensional velocity. Forcing frequency $\omega = 15$ rad/sec.

subjected to dynamic forces was tackled by this means. A superposition type of method appears to be inapplicable in this example.

The use of two-dimensional trigonometric series of *a-priori* uniform convergence permits us to satisfy the differential equation, initial conditions, and nonlinear boundary conditions. Unlike Galerkin techniques, each coordinate function satisfies neither the essential nor the natural boundary conditions and this requirement is fulfilled by the complete sequence. This feature makes the statement systematic. On the other hand, here the variables space and time are dealt with in the same way. The usual semi-discrete method of solving the resulting time equations by some numerical integration scheme is not used here.

The response is analyzed in the time domain and the phase diagrams are also given. In the particular example, a period doubling is present. A further study would be necessary to investigate the possibility of chaos. In effect we are dealing with Duffing-type nonlinearities that, as is known in SDOF systems, may lead to chaotic responses.

Appendix A. About the Extremizing Sequences

A.1. Definitions in One Dimension

Let $x \in D$ ($D = [0, 1]$) and $f = f(x)$ and $g = g(x)$ be two arbitrary square integrable functions.

Internal product in $L_2(x)$

$$(f, g) \equiv \int_0^1 f(\eta) g(\eta)\, d\eta. \tag{A1}$$

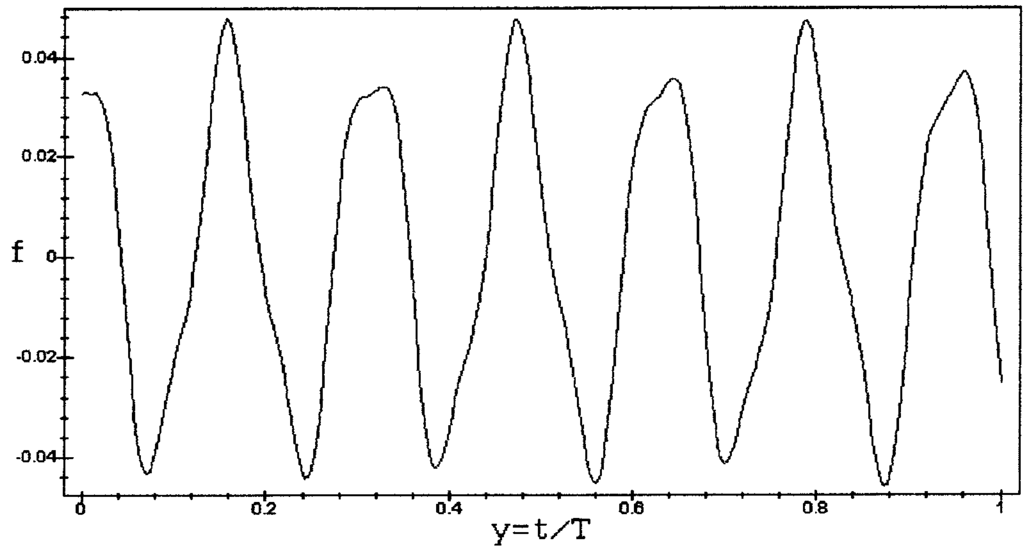

Figure 4a. Beam on nonlinear supports subjected to dynamic end moments. $\alpha_0 = \alpha_1 = 0$; $\gamma_0 = \gamma_1 = 2$; $\theta_0 = 5°$; $\theta_1 = -10°$; $M = N = 150$; $NS = 300$. Time-displacement (at the center of the beam) curve. Forcing frequency $\omega = 25$ rad/sec.

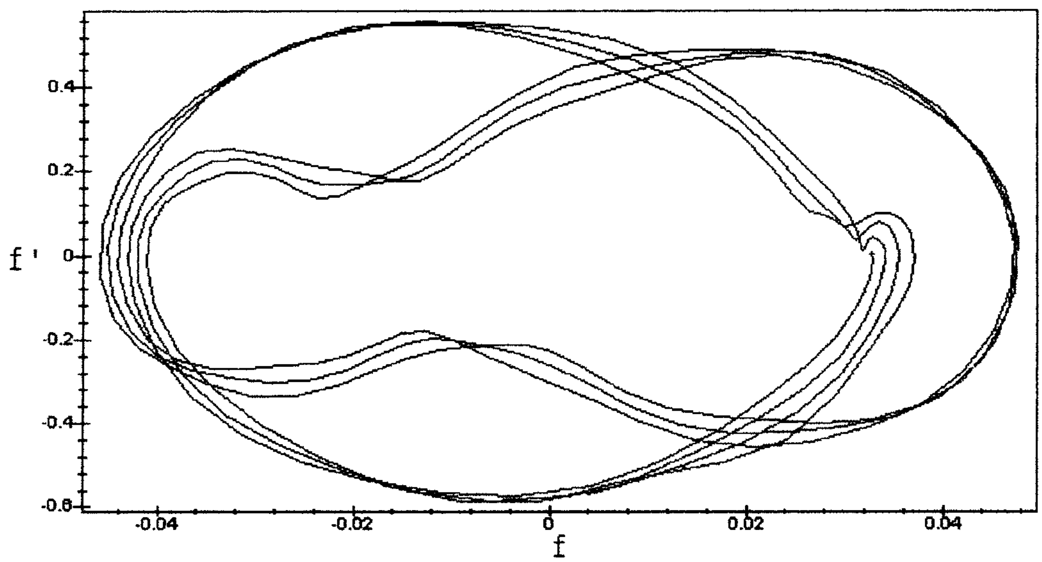

Figure 4b. Beam on nonlinear supports subjected to dynamic end moments. $\alpha_0 = \alpha_1 = 0$; $\gamma_0 = \gamma_1 = 2$; $\theta_0 = 5°$; $\theta_1 = -10°$; $M = N = 150$; $NS = 300$. Phase diagram. f: displacement at the center of the beam; f': nondimensional velocity. Forcing frequency $\omega = 25$ rad/sec.

Norm in $L_2(x)$

$$\begin{cases} 0 \leq \|f^2\|^2 \equiv (f, f) \equiv \int_0^1 f^2(\eta)\,d\eta, & \text{(a)} \\ 0 \leq \|g^2\|^2 \equiv (g, g) \equiv \int_0^1 g^2(\eta)\,d\eta. & \text{(b)} \end{cases} \tag{A2}$$

We make use of Reimann integrals, but within the Hilbert spaces, the integral definitions are generalized in the Lebesgue sense. Then *convergence in* L_2 or *convergence in the mean* may be defined. Let $F_M(x)$ be a series that linearly combines infinite continuous functions $\varphi_i(x)$ $(i = 1, 2, 3, \ldots)$, as follows

$$F_M(x) = \sum_{i=0}^{M} a_i \varphi_i(x), \tag{A3}$$

where a_i are arbitrary constants. We choose $\{\varphi_i(x)\}$ such that they constitute a *complete set* in L_2. It is possible then that, with an adequate selection of a_i's, the following is verified:

$$\|F(x) - F_M(x)\| < \varepsilon \tag{A4}$$

with $\varepsilon \to 0$ as $M \to \infty$. $F(x)$ is a square integrable function.

A.2. Fourier Trigonometric Series

In the series (A3), we assume that

$$\varphi_i(x) \begin{cases} \sin i\pi x, & \text{(a)} \\ \text{or} & \qquad (i = 0, 1, 2, \ldots). \\ \cos i\pi x, & \text{(b)} \end{cases} \tag{A5}$$

From Fourier theory, both sets are complete in L_2 in $x \in [0, 1]$. To reduce the notation, $\alpha_i = i\pi$; $s_i = \sin i\pi x = \sin \alpha_i x$; $c_i = \cos i\pi x = \cos \alpha_i x$. Then

$$\begin{cases} S_M^*(x) = \sum_{i=1}^{M} A_i s_i, & \text{(a)} \\ C_M^*(x) = \sum_{i=1}^{M} B_i c_i + B_0, & \text{(b)} \end{cases} \tag{A6}$$

are the two types of Fourier series in the unitary domain [0, 1] and A_i, B_i and B_0 are unknown coefficients. Using the definitions introduced above we find

$$(c_i, c_j) = (s_i, s_j) = \begin{cases} 0, & i \neq j, \\ \frac{1}{2}, & i = j, \end{cases} \qquad (i, j = 1, 2, \ldots) \tag{A7}$$

or

$$\|s_i\|^2 = \|c_i\|^2 = \frac{1}{2} \quad (i = 1, 2, \ldots). \tag{A8}$$

The following convergence properties hold:

$$\begin{aligned} &\|F(x) - S_M^*(x)\| \to 0, \quad M \to \infty, \\ &\|F(x) - C_M^*(x)\| \to 0, \quad M \to \infty, \end{aligned} \tag{A9}$$

selecting

$$\begin{cases} A_i = 2(F, s_i), & \text{(a)} \\ \qquad (i = 1, 2, \ldots) & \\ \begin{cases} B_i = 2(F, c_i), \\ B_0 = (F, 1), \end{cases} & \text{(b)} \end{cases} \tag{A10}$$

A.3. SPECIAL PROPERTY OF C_M^*

The series (A6b) with (A10b) verifies a property relevant for our goal (as will be shown below). In case $F(x)$ is continuous and not only square integrable, besides (A9b), also the following convergence property will be true:

$$|F(x) - C_M^*| \to 0, \quad M \to 0, \quad \forall x \in [0, 1]. \tag{A11}$$

That is, the expansion $C_M^*(x)$ yields *uniform convergence* (UC) when $F(x)$ is continuous in [0, 1]. Evidently, with the sine series, if $F(x)$ is continuous and $F(0) = F(1) = 0$, we would also attain UC but, in general, the sine expansions only yield convergence in L_2. This feature will lead us to introduce an extension of S_M^* below.

A.4. SOME PROPERTIES OF THE FOURIER SERIES

P_1^*: the derivative series $\mathrm{d}S_M^*(x)/\mathrm{d}x$ loses the convergence in L_2 property .
P_2^*: the derivative series $\mathrm{d}C_M^*(x)/\mathrm{d}x$ is convergent in L_2.
P_3^*: the successive integrations of $S_M^*(x)$ and $C_M^*(x)$ keep their convergence properties in L_2, as the uniform one (if they originally had them).
P_4^*: the addition of arbitrary functions in x to $S_M^*(x)$ and $C_M^*(x)$ at least maintains the L_2 convergence. Eventually it could change to UC.
P_5^*: the multiplication of $S_M^*(x)$ and $C_M^*(x)$ by bounded, continuous arbitrary functions in x does not modify their convergence.

These are some of the properties that will be eventually used.

A.5. EXTENDED TRIGONOMETRIC SERIES

The special property of $C_M^*(x)$ described in Section A.3 leads to a search of the sine series, which would yield UC. This simple challenge that will be developed here, is the basis of the methodology named WEM and it is essential for arriving at a solution of any differential equation.

Actually the reference of the square integrable function $F(x)$ (in general $F(0) \neq 0$, $F(1) \neq 0$) is shifted. That is, the following series

$$S_M(x) = S_M^*(x) + \underline{xA_0 + a_0} \tag{A12}$$

is introduced. The underlined terms constitute the *support function*, i.e.,

$$S_M(x) = \sum_{i=1}^{M} A_i s_i + xA_0 + a_0, \tag{A13}$$

where

$$\begin{aligned} a_0 &= F(0), \\ A_0 &= F(1) - F(0), \end{aligned} \tag{A14}$$

but now

$$A_i = 2\{[F(x) - xA_0 - a_0], s_i\}. \tag{A15}$$

As may be observed, Equation (A15) differs from (A10a). In this way, from property P_4^* it is verified that

$$\|F(x) - S_M(x)\| \to 0, \quad M \to \infty, \tag{A16}$$

but, if also $F(x)$ is continuous in $x \in [0, 1]$, UC will be attained too.

$$|F(x) - S_M(x)| \to 0, \quad M \to \infty, \forall x.. \tag{A17}$$

In what follows, series $S_M(x)$ will be used instead of $S_M^*(x)$ and $C_M(x) \equiv C_M^*(x)$. Additionally, property P_1^* should be changed into

P_1: the series $\mathrm{d}S_M(x)/\mathrm{d}x$ is convergent in L_2

and we add

P_2: if $F(x)$ is continuous, from Section A.3 and P_1, $S_M(x)$ and $\mathrm{d}S_M(x)/\mathrm{d}x$ have UC.

Let us give two other properties that verify integrals where $S_M^*(x)$ and $C_M^*(x)$, $S_M(x)$ and $C_M(x)$ are involved. They are not evident but, making use of the Cauchy–Schwarz inequality ($f(x)$ and $g(x)$ square integrable),

$$|(f, g)| \leq \|f(x)\| \, \|g(x)\|, \tag{A18}$$

it is easily demonstrated that if $h = h(x)$ is a square integrable function, it is true as $M \to \infty$, that

$$P_3: \quad (S_M^*, h) = (S_M, h), \tag{A19}$$

$$P_4: \quad (C_M^*, h)_1 = (C_M, h). \tag{A20}$$

P_5: the successive integration of $S_M(x)$ or $C_M(x)$ are UC to the corresponding successive integrals of $F(x)$ if $F(x)$ is continuous. Instead, if $F(x)$ is square integrable, $S_M(x)$ and $C_M(x)$ yield convergence in the mean. But if the first integral, $\int F(x)\,\mathrm{d}x$, is continuous, then the first integrals of $S_M(x)$ and $C_M(x)$ will be UC.

A.6. Extended Series in 2D

A natural extension of the above-mentioned for dimension one allows us to state the extended series in domains of larger dimensions.

Let $F(x, y)$ be a square integrable function in $\{D : 0 \leq x \leq 1; 0 \leq y \leq 1\}$. We introduce the following extended series:

$$F_M(x, y) = \sum_{i=1}^{M} A_i(y)s_i + xA_0(y) + a_0(y) \tag{A21}$$

or also

$$F_M(x, y) = \sum_{i=1}^{M} B_i(y)c_i + B_0(y), \tag{A22}$$

in both cases

$$\|F_M(x, y) - F(x, y)\| < \varepsilon, \tag{A23}$$

and $\varepsilon \to 0$ if $M \to \infty$ with analogous definition of norm in two-dimensional domains.

After expanding $A_i(y)$ ($i = 1, 2, \ldots$), $A_0(y)$ and $a_0(y)$ in turn, according to the alternatives seen in the one dimension domain, infinite possibilities of extended series in the two-dimensional domain are obtained. In a similar fashion, this technique applies to larger dimensions. The functions $A_i(y)$, $A_0(y)$ and $a_0(y)$ are found with expressions similar to (A10b), (A14) and (A15) (for instance, $a_0(y) = F(0, y)$). If $F(x, y)$ in D is square integrable in the sense of x and continuous in the sense of y, a new property holds:

P_6: $\partial F_M(x, y)/\partial y$ is a L_2 convergence series in x and UC or L_2 convergent in y, depending on the feature of $\partial F(x, y)/\partial y$.

One of the infinite UC series in two-dimensional domains is expression (10) derived from Equation (A21). Another series of usefulness is (using (A22))

$$\varphi_{MN}(x, y) = \sum_{i=1}^{M}\sum_{j=1}^{N} p_{ij} c_i c_j + \sum_{i=1}^{M} p_{i0} c_i + \sum_{j=1}^{N} p_{0j} c_j + p_{00},$$

which with constants chosen in a certain way approximates with uniform convergence the function $\phi = \phi(x, y)$ in the domain D.

P_7: If $F(0, y) = F(1, y) = 0 \Rightarrow \partial^n F(x, y)/\partial y^n\big|_{x=0,1} = 0$. In effect, if Equation (A21) is selected with $F(0, y) = F(1, y) = 0$, then $F_M(x, y) = \sum_{i=1}^{M} A_i(y) s_i$ of UC.

Appendix B. About the Initial Conditions

Let us analyze the initial conditions starting with

$$\nu(x, 0) = ax^3 + bx^2 + cx + d = U_0(x). \tag{B1}$$

We denote

$$\nu'(0, 0) \equiv \theta_0 \quad \text{and} \quad \nu'(1, 0) \equiv \theta_1. \tag{B2}$$

Since, we are dealing with a supported beam,

$$\nu(0, 0) = 0 \quad \text{and} \quad \nu(1, 0) = 0. \tag{B3}$$

If θ_0 and θ_1 are considered to be known, through (B2) and (B3) we find that

$$a = \theta_0 + \theta_1; \quad b = -(2\theta_0 + \theta_1); \quad c = \theta_0; \quad d = 0 \tag{B4}$$

and the displacement at $t = 0$ is written as

$$\nu(x, 0) = U_0(x) = x(1 - x)[\theta_0 - (\theta_0 + \theta_1)x]. \tag{B5}$$

Let us now obtain the values of μ_0 and μ_1 (recall that $\mu_0 f(0) = \mu_0$ and $\mu_1 f(0) = \mu_1$ since $f(0)$ was assumed to be equal to unity). μ_0 and μ_1 represent the moments acting at the ends of the beam for $t = 0$ and their values should satisfy the boundary conditions imposed by the nonlinear springs with rotations θ_0 and θ_1.

$$\begin{aligned} v''(0,0) &= \alpha_0\theta_0 + \gamma_0\theta_0^3 - \mu_0, \\ v''(1,0) &= -(\alpha_1\theta_1 + \gamma_1\theta_1^3 + \mu_1), \end{aligned} \tag{B6}$$

from which μ_0 and μ_1 result

$$\mu_0 = (4+\alpha_0)\theta_0 + 2\theta_1 + \gamma_0\theta_0^3, \tag{B7}$$

$$\mu_1 = -[(4+\alpha_1)\theta_1 + 2\theta_0 + \gamma_1\theta_1^3]. \tag{B8}$$

Now let us analyze the other initial condition (velocity at $t = 0$)

$$\bar{v}(x,0) = V_0(x) = Px^3 + Qx^2 + Rx + S. \tag{B9}$$

Since

$$\bar{v}(0,0) = 0 \quad \text{and} \quad \bar{v}(1,0) = 0 \tag{B10}$$

after denoting $\bar{v}'(0,0) \equiv \Omega_0$ and $\bar{v}'(1,0) \equiv \Omega_1$, it is possible to write

$$V_0(x) = x(x-1)[\Omega_0 - (\Omega_0 + \Omega_1)x]. \tag{B11}$$

Now we will find out whether or not Ω_0 and Ω_1 are independent of θ_0 and θ_1. From conditions (7a) and (7b)

$$\bar{v}''(0,y) = \alpha_0\bar{v}'(0,y) + 3\gamma_0 v'^2(0,y)\bar{v}'(0,y) - \bar{M}_0(y), \tag{B12}$$

$$\bar{v}''(1,y) = -[\alpha_1\bar{v}'(1,y) + 3\gamma_1 v'^2(1,y)\bar{v}'(1,y) + \bar{M}_1(y)], \tag{B13}$$

where

$$\begin{cases} \bar{M}_0(y) = \mu_0\bar{f}(y), \\ \bar{M}_1(y) = \mu_1\bar{f}(y). \end{cases} \tag{B14}$$

Now expressions (B12) and (B13), written at $y = 0$ ($t = 0$) yield

$$\bar{v}''(0,0) = (\alpha_0 + 3\gamma_0\theta_0^2)\Omega_0 - \mu_0\bar{f}(0), \tag{B15}$$

$$\bar{v}''(1,0) = -[(\alpha_1 + 3\gamma_1\theta_1^2)\Omega_1 + \mu_1\bar{f}(0)]. \tag{B16}$$

But, on the other hand,

$$\bar{v}''(0,0) = V_0''(0) = -2(2\Omega_0 + \Omega_1), \tag{B17}$$

$$\bar{v}''(1,0) = V_0''(1) = 2(\Omega_0 + 2\Omega_1). \tag{B18}$$

Consequently, we end up with the following linear system in Ω_0 and Ω_1

$$\mu_0\bar{f}(0) = [(4+\alpha_0) + 3\gamma_0\theta_0^2]\Omega_0 + 2\Omega_1, \tag{B19}$$

$$\mu_1 \bar{f}(0) = -[(4+\alpha_1) + 3\gamma_1\theta_1^2]\Omega_1 + 2\Omega_0, \tag{B20}$$

where μ_0, μ_1, θ_0 and θ_1 are assumed to be known. The solution is written as

$$\Omega_0 = \frac{\nabla_0}{\nabla}\bar{f}(0) \quad \text{and} \quad \Omega_1 = \frac{\nabla_1}{\nabla}\bar{f}(0), \tag{B21}$$

in which

$$\nabla \equiv 12 + 4(\alpha_0+\alpha_1) + \alpha_0\alpha_1 + 3\gamma_0\theta_0^2(4+\alpha_1) + 3\gamma_1\theta_1^2(4+\alpha_0) + 9\gamma_0\theta_0^2\gamma_1\theta_1^2, \tag{B22}$$

$$\nabla_0 \equiv (4+\alpha_1+3\gamma_1\theta_1^2)\mu_0 + 2\mu_1, \tag{B23}$$

$$\nabla_1 \equiv -(4+\alpha_0+3\gamma_0\theta_0^2)\mu_1 + 2\mu_0. \tag{B24}$$

In effect, Ω_0 and Ω_1 are not independent of θ_0 and θ_1 and, hence, once θ_0 and θ_1 are imposed, it is possible to know $U_0(x)$ and $V_0(x)$.

NB if $\bar{f}(0) = 0 \Rightarrow V_0(x) \equiv 0 \quad (\forall\theta_0$ and $\theta_1)$,
NB if $\theta_0 = \theta_1 = 0 \Rightarrow U_0(x) = V_0(x) = 0$.

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