Collapse times for attractive Bose–Einstein condensates

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Abstract
We argue that the main mechanism for condensate collapse in an attractive Bose–Einstein condensate is the loss of coherence between atoms a finite distance apart, rather than the growth of the occupation number in non-condensate modes. Since the former mechanism is faster than the latter by a factor of approximately 3/2, this helps to dispel the apparent failure of field theoretical models in predicting the collapse time of the condensate.

1. Introduction

The so-called *Bose Nova* experiment on the collapse of a Bose–Einstein condensate with attractive interactions [1–3] has opened up a fascinating window in the far out-of-equilibrium behaviour of these systems. The experiment has been analysed from a number of perspectives [4–10] and is fair to say that we have a good qualitative understanding of the phenomenon. However, at the quantitative level certain anomalies persist.

In this paper, we shall deal with the apparent failure of existing models in predicting the collapse time scale t_c for the condensate, in the regime where the scattering length a is just below the critical value $-a_c$. In [7] the scaling law

$$t_c \propto \left[\frac{|a|}{a_c} - 1 \right]^{-1/2} \qquad (1)$$

is proposed, which fits well the experimental results. However, the proportionality constant is not derived. The authors of [7] claimed that the proper proportionality constant could be derived from a complete field theoretic calculation, but when the calculation was actually done [11–13], it failed to produce a satisfactory prediction.

In this paper, we shall present a qualitative analysis of the collapse time for a condensate trapped in a flat box [5] with periodic boundary conditions. Unlike previous analysis, we shall assume that the total number of particles in the condensate remains fixed [14–18]. Under these constraints, the condensate occupation number is properly defined as the greatest eigenvalue of the one-particle density matrix (to be defined below) [19]. Given the assumed geometry, the corresponding eigenmode is necessarily homogeneous, so the eigenvalue is just the integral of the one-particle density

matrix with one argument fixed, and the other ranging over the confining box.

Since the overall normalization of the one-particle density matrix is determined by the overall density of the gas (see below), the most important factor in the evolution of the condensate occupation number is how fast the density matrix falls off, measuring the degree of coherence among atoms at finite distances. We shall argue below that the one-particle density matrix is approximately Gaussian with a variance which decays in time as $\exp\{-\gamma t\}$, with

$$\gamma = \tau^{-1} \sqrt{\frac{|a|}{a_c} - 1}, \qquad (2)$$

where

$$\tau^{-1} = \frac{2\pi^2 \hbar}{ML^2} \qquad (3)$$

is the frequency of the first excited states for a particle in the box; here M is the mass of an atom and L is the size of the box. Therefore, after integrating over the three-dimensional box we obtain that the condensate occupation number decays as $\exp\{-3\gamma t\}$.

The expectation number in the first excited state, as computed from the Gross–Pitaevskii equation, the Hartree–Fock–Bogoliubov or the Popov approximation would grow only as $\exp\{2\gamma t\}$. Therefore, condensate collapse from the loss of coherence between atoms is faster than the estimate from the loss of particles to excited modes by a factor of 3/2. For comparison, note that a detailed calculation of the collapse time for $a = -10a_c$ yields a predicted value of 10 ms against an experimental value of 6 ± 1 ms [13]. Therefore, a factor of three halves goes a long way to solve the existing puzzle.

J. Phys. B: At. Mol. Opt. Phys. **41** (2008) 025302

E Calzetta

This paper is organized as follows. In the following section, we present the model of a cold-trapped Bose gas, introduce a suitable set of density and phase variables and solve the Heisenberg equations in the linearized approximation. In section 3, we apply these results to derive the evolution of the condensate particle number and thereby our main result. In section 4, we compare this result to the particle number conserving, Hartree–Fock–Bogoliubov and Popov approaches. We close the paper with some brief final remarks, and give supplementary technical details in the appendix.

2. The model

The idea is to analyse the Bose Nova experiment with the tools we have developed to handle the Mott transition in [20]. The starting point is a second-quantized field operator $\hat{\Psi}(\mathbf{x}, t)$ which removes an atom at the location $\mathbf{x}$ at times t. It obeys the canonical commutation relations

$$[\hat{\Psi}(\mathbf{x}, t), \hat{\Psi}(\mathbf{y}, t)] = 0, \tag{4}$$

$$[\hat{\Psi}(\mathbf{x}, t), \hat{\Psi}^{\dagger}(\mathbf{y}, t)] = \delta(\mathbf{x} - \mathbf{y}). \tag{5}$$

The dynamics of this field is given by the Heisenberg equations of motion

$$-\mathrm{i}\hbar \frac{\partial}{\partial t}\hat{\Psi} = [\hat{\mathbf{H}}, \hat{\Psi}], \tag{6}$$

where $\hat{\mathbf{H}}$ is the Hamiltonian. The theory is invariant under a global phase change of the field operator

$$\hat{\Psi} \rightarrow \mathrm{e}^{\mathrm{i}\theta}\hat{\Psi}, \qquad \hat{\Psi}^{\dagger} \rightarrow \mathrm{e}^{-\mathrm{i}\theta}\hat{\Psi}^{\dagger}. \tag{7}$$

The constant of motion associated with this invariance through the Noether theorem is the total particle number.

To progress further, we need a specific model for the atom–atom interactions. In principle, we should specify the atom–atom interaction potential. However, in many applications it is enough to know the cross-section σ for low-energy spherically symmetric scattering of two identical bosons. We introduce the scattering length a through $\sigma \equiv 8\pi a^2$, where the factor 8π involves both the integration over all scattering angles and the Bose enhancement factors. We shall adopt as model atom–atom interaction a contact potential $U\delta(\mathbf{x})$. This is expected to be a good approximation as long as the distance between atoms is much greater than both the scattering length and the distance out to which the fundamental atom–atom interaction is important [21]. To reproduce the right scattering length we need $U = 4\pi\hbar^2 a/M$, where M is the mass of the atoms. A positive value of a means a repulsive interaction; we adopt the convention that an attractive interaction is described by a negative value of a.

Assuming a contact atom–atom potential we get the Hamiltonian

$$\hat{\mathbf{H}} = \int \mathrm{d}\mathbf{x} \left\{ \hat{\Psi}^{\dagger}\hat{H}\hat{\Psi} + \frac{U}{2}\hat{\Psi}^{\dagger 2}\hat{\Psi}^2 \right\}. \tag{8}$$

The single-particle Hamiltonian $\hat{H}$ is given by

$$\hat{H}\hat{\Psi} = -\frac{\hbar^2}{2M}\nabla^2\hat{\Psi} + V_{\mathrm{trap}}(\mathbf{x})\hat{\Psi}, \tag{9}$$

where V_{trap} denotes a confining trap potential. Then, the Heisenberg equation of motion

$$\mathrm{i}\hbar \frac{\partial}{\partial t}\hat{\Psi} = \hat{H}\hat{\Psi} + U\hat{\Psi}^{\dagger}\hat{\Psi}^2 \tag{10}$$

is also the classical equation of motion derived from the action

$$S = \int \mathrm{d}t\, \mathrm{d}\mathbf{x}\, \mathrm{i}\hbar \Psi^* \frac{\partial}{\partial t}\Psi - \int \mathrm{d}t\, \mathbf{H} \tag{11}$$

placing hats everywhere. For simplicity we shall replace the trap potential by a flat bounding box of volume $V = L^3$ with periodic boundary conditions. Yurovsky has demonstrated that this is enough for a qualitative treatment of the *Bose Nova* [5]. We also assume that we have a finite total number of particles N, which remains fixed through the evolution (that is, there is no particle loss to the environment).

2.1. Density and phase variables in the CTP formulation

To analyse further this model we shall adopt density-phase variables [22, 23]. These variables have been extensively used to study dynamical problems, including the Mott transition [24]. This will set the stage for a further canonical transformation to a more convenient set of degrees of freedom.

In the path integral representation, quantum amplitudes are given in terms of functional integrals over complex fields Ψ and $\Psi^{\dagger}$ associated with the destruction and creation operators. Our starting point is the Madelung representation [22, 23]

$$\Psi(\mathbf{x}, t) = [\exp(-\mathrm{i}\varphi(\mathbf{x}, t))]\sqrt{\rho(\mathbf{x}, t)}, \tag{12}$$

$$\Psi^{\dagger}(\mathbf{x}, t) = \sqrt{\rho(\mathbf{x}, t)}[\exp(\mathrm{i}\varphi(\mathbf{x}, t))]. \tag{13}$$

In the canonical formalism, the fields ρ and φ become operators with commutation relations

$$[\hat{\rho}(\mathbf{x}, t), \hat{\varphi}(\mathbf{y}, t)] = -\mathrm{i}\delta(\mathbf{x} - \mathbf{y}). \tag{14}$$

Within the path integral we allow the phases φ to take all real values, and therefore so do the conjugated density operators ρ [20, 25]. This makes the square roots in (12) and (13) problematic. It is best to adopt a new set of variables where square roots do not appear, as we shall do presently. For further discussion of density-phase variables in continuum theories see [26].

We adopt the formalism developed in [20] to describe the transition from the superfluid to the Mott insulator state in an optical lattice. To compute expectation values, we shall use the closed time-path formalism, where we choose the independent variables as follows. In the first branch, we define a new (complex) variable $\chi^1(\mathbf{x}, t)$ from

$$\Psi^1(\mathbf{x}, t) = \exp[-\mathrm{i}\chi^1(\mathbf{x}, t)], \tag{15}$$

$$\Psi^{1\dagger}(\mathbf{x}, t) = \rho^1(\mathbf{x}, t)\exp[\mathrm{i}\chi^1(\mathbf{x}, t)]. \tag{16}$$

In the second branch, we write instead

$$\Psi^{2\dagger}(\mathbf{x}, t) = \exp[\mathrm{i}\chi^{2\dagger}(\mathbf{x}, t)], \tag{17}$$

$$\Psi^2(\mathbf{x}, t) = \exp[-\mathrm{i}\chi^{2\dagger}(\mathbf{x}, t)]\rho^2(\mathbf{x}, t). \tag{18}$$

In the canonical formulation, the fields χ and ρ become operators with commutation relations [20]

$$[\hat{\rho}(\mathbf{x}, t), \hat{\chi}(\mathbf{y}, t)] = -i\delta(\mathbf{x} - \mathbf{y}). \tag{19}$$

The dynamics of these operators is given by the Hamiltonian

$$\hat{H}(\hat{\rho}, \hat{\chi}) = \int d\mathbf{x} \left\{ \frac{\hbar^2}{2M} (\hat{\rho}\nabla\hat{\chi} - i\nabla\hat{\rho})\nabla\hat{\chi} + \frac{U}{2}\hat{\rho}(\hat{\rho} - 1) \right\} \tag{20}$$

plus the necessary terms to enforce a fixed total particle number [20]. Observe that in the new variables, the action is explicitly analytical.

We now split all variables into a homogeneous and an inhomogeneous part:

$$\hat{\rho}(\mathbf{x}, t) = n + \hat{r}(\mathbf{x}, t), \tag{21}$$

$$\hat{r}(\mathbf{x}, t) = \sum_{\mathbf{p} \neq 0} \hat{r}_{\mathbf{p}}(t) f_{\mathbf{p}}(\mathbf{x}), \tag{22}$$

where the $f_{\mathbf{p}}$ are plane waves,

$$f_{\mathbf{p}}(\mathbf{x}) = \frac{1}{V^{1/2}} \exp\{i\mathbf{p}\mathbf{x}/\hbar\}, \tag{23}$$

and the allowed values of the components p_μ, $\mu = 1$–3, of the momentum $\mathbf{p}$ are the integer multiples of $2\pi\hbar/L$, and similarly

$$\hat{\chi}(\mathbf{x}, t) = \frac{\hat{X}_0}{V^{1/2}} + \sum_{\mathbf{p} \neq 0} \hat{X}_{\mathbf{p}}(t) f_{\mathbf{p}}(\mathbf{x}). \tag{24}$$

Observe that the homogeneous part of the density operator is constrained to be the c-number $n = N/V$, and the homogeneous part of the phase is a collective coordinate [27] which couples only to the homogeneous density. It does not affect the dynamics of the inhomogeneous modes.

Consider the lowest order theory which is obtained by keeping only the 'free' quadratic part of the Hamiltonian

$$\hat{H}_{\text{free}}(\hat{r}_{\mathbf{p}}, \hat{X}_{\mathbf{p}}) = \sum_{\mathbf{p} \neq 0} \left\{ \frac{v_p}{2}(n\hat{X}_{-\mathbf{p}}\hat{X}_{\mathbf{p}} - i\hat{r}_{-\mathbf{p}}\hat{X}_{\mathbf{p}}) + \frac{U}{2}\hat{r}_{-\mathbf{p}}\hat{r}_{\mathbf{p}} \right\}, \tag{25}$$

where $v_p = p^2/M$, $p = |\mathbf{p}|$. The Heisenberg equations of motion are

$$\hbar\frac{d}{dt}\hat{X}_{\mathbf{p}} = \frac{-iv_p}{2}\hat{X}_{\mathbf{p}} + U\hat{r}_{\mathbf{p}}, \tag{26}$$

$$-\hbar\frac{d}{dt}\hat{r}_{\mathbf{p}} = nv_p\hat{X}_{\mathbf{p}} + \frac{-iv_p}{2}\hat{r}_{\mathbf{p}}, \tag{27}$$

where

$$Un = \frac{4\pi\hbar^2 Na}{ML^3}. \tag{28}$$

In the Bose Nova scenario, we have $U = 0$ if $t \leqslant 0$. Therefore, the frequencies are $\hbar\omega_p^< = v_p/2$. If we call $A_{\mathbf{p}}$ the destruction operator which kills the initial state, then

$$r_{\mathbf{p}}(0^-) = (-i)\sqrt{n}[A_{\mathbf{p}} - A_{-\mathbf{p}}^\dagger], \tag{29}$$

$$X_{\mathbf{p}}(0^-) = \frac{1}{\sqrt{n}}A_{\mathbf{p}}. \tag{30}$$

For $t > 0$, we have $U < 0$ instead, and

$$X_{\mathbf{p}}(t) = \frac{1}{\sqrt{n}} \left\{ \left[\cos[\omega_p t] - i\left(\frac{v_p}{2} + Un\right)\frac{\sin[\omega_p t]}{\hbar\omega_p} \right] A_{\mathbf{p}} + iUn\frac{\sin[\omega_p t]}{\hbar\omega_p}A_{-\mathbf{p}}^\dagger \right\} \tag{31}$$

with the dispersion relation

$$\omega_p = \frac{1}{\hbar}\sqrt{v_p\left(Un + \frac{v_p}{4}\right)}. \tag{32}$$

3. The one-particle density matrix

We may now turn to computing the one-particle density matrix

$$\sigma(\mathbf{x}, \mathbf{y}, t) = \langle\hat{\Psi}^\dagger(\mathbf{x}, t)\hat{\Psi}(\mathbf{y}, t)\rangle$$
$$\equiv \langle\exp i[\chi^{2*}(\mathbf{x}, t) - \chi^1(\mathbf{y}, t)]\rangle. \tag{33}$$

In the last term, the 1, 2 superindex indicates closed time-path ordering: operators with a 2 superindex always go to the left of operators with a 1 superindex. Observe that in our variables, the observable to be computed is a pure exponential: there are no square roots to be developed. This is the whole point of introducing the new variables.

As in the previous section, we separate the variables χ^{2*} and χ^1 into their homogeneous and inhomogeneous parts. Observe that the homogeneous terms may affect the overall normalization of the one-particle density functional but not its shape. The overall normalization, on the other hand, is determined by the requirement that $\sigma(\mathbf{x}, \mathbf{x}, t) = n$. So we may simply continue to disregard the homogeneous terms.

Since we have restricted ourselves to a Hamiltonian which is quadratic in the inhomogeneous modes, we may use the Wick theorem result

$$\langle e^{iA} \rangle = \langle 1 \rangle \text{Exp}\left\{ \frac{-1}{2}\langle A^2 \rangle \right\}, \tag{34}$$

with

$$A = A(\mathbf{x}, \mathbf{y}) = \chi^{2*}(\mathbf{x}, t) - \chi^1(\mathbf{y}, t). \tag{35}$$

Decomposing the field operators in modes, with due attention to the closed time-path ordering, we obtain

$$\langle A^2 \rangle = \text{const} + 2\sum_{\mathbf{p} \neq 0}\left[\frac{1}{V} - f_{-\mathbf{p}}(\mathbf{x})f_{\mathbf{p}}(\mathbf{y})\right]\langle X_{\mathbf{p}}^\dagger X_{\mathbf{p}}\rangle. \tag{36}$$

Using the decomposition (31),

$$\langle A^2 \rangle = \text{const} + \frac{4(Un)^2}{N}\sum_{\mathbf{p} \neq 0}$$
$$\times \left\{ \sin\left[\frac{\mathbf{p}(\mathbf{x} - \mathbf{y})}{2\hbar}\right] \right\}^2 \left[\frac{\sin[\omega_p t]}{\hbar\omega_p} \right]^2. \tag{37}$$

To continue, we consider only the contribution from the unstable modes. The condition for instability is $U < 0$ with $|Un| > v_p/4$. Since the lowest possible nontrivial value of p is h/L, we get the critical scattering length as $a_c = \pi L/4N$. For a close enough to the critical value, the six modes with $L^2 p^2 = h^2$ are the only unstable ones. Their frequency is

$\omega = -i\gamma$, where γ is given in (2). Setting $\mathbf{y} = 0$ and $x = |\mathbf{x}|$, we get

$$\sum_{p=h/L} \left\{ \sin\left[\frac{\mathbf{px}}{2\hbar}\right] \right\}^2 \left[\frac{\sin[\omega_p t]}{\hbar\omega_p}\right]^2$$

$$\sim 2\left[\frac{\pi \sinh[\gamma t]}{\hbar\gamma}\right]^2 \left(\frac{x}{L}\right)^2, \qquad (38)$$

therefore

$$\sigma(\mathbf{x}, t) = n \, \mathrm{Exp}\left\{ -\left[\frac{2\pi U n \sinh[\gamma t]}{N^{1/2}\hbar\gamma L}\right]^2 x^2 \right\}. \qquad (39)$$

The condensate occupation number N_c is obtained by integrating over $\mathbf{x}$, so, once the Gaussian approximation becomes valid

$$N_c \propto e^{-3\gamma t}. \qquad (40)$$

We therefore obtain the same scaling law as in [7], but the coefficient is 3/2 times larger. As noted in the introduction, this correction is enough to account for the anomaly observed in [13].

4. Comparison with other approaches

In this section, we will compare the result above for the one-particle density matrix with other approaches in the literature, namely the particle-number conserving (PNC) formalism and the Hartree–Fock–Bogoliubov (HFB) and Popov approximations. We shall not discuss the so-called truncated Wigner approximation, but refer the reader to the detailed treatment in [13]. See [28] for further details on these approaches.

4.1. The equations of motion in the PNC approach

The PNC formalism [14–18] is usually presented as an expansion in the inverse powers of the total particle number N. In the preparation for this, it is convenient to scale the interaction term, writing $U = u/N$.

The basic insight of the PNC approach is that if the total particle number remains constant, then each particle above the condensate corresponds to a hole in the condensate, so we may speak of *particle–hole* (PH) pairs.

Let us consider the expansion of the field operator in plane waves

$$\Psi(\mathbf{x}, t) = \sum_{\mathbf{p}} a_{\mathbf{p}}(t) f_{\mathbf{p}}(\mathbf{x}), \qquad (41)$$

a_0 reduces the number of particles in the condensate by one. Following Arnowitt and Girardeau, let us introduce the operator

$$\beta = \frac{1}{\sqrt{\hat{N}_0 + 1}} a_0 = a_0 \frac{1}{\sqrt{\hat{N}_0}}, \qquad (42)$$

where

$$\hat{N}_0 = N - \sum_{\mathbf{p}\neq 0} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \qquad (43)$$

is the condensate number Heisenberg operator. Observe that for a number eigenstate $\beta|N_0\rangle = |N_0 - 1\rangle$ unless $N_0 = 0$, in which case $\beta|0\rangle = 0$. Therefore, β preserves the norm for all states orthogonal to the state with no particles in the zeroth mode (which is much stronger than not having a condensate). If there is a condensate, any physically meaningful state will satisfy this requirement, and β may be considered a unitary operator, with inverse

$$\beta^\dagger = \frac{1}{\sqrt{\hat{N}_0}} a_0^\dagger = a_0^\dagger \frac{1}{\sqrt{\hat{N}_0 + 1}}. \qquad (44)$$

We now introduce the destruction operator of a PH with the particle in mode $\mathbf{p}$

$$\lambda_{\mathbf{p}} = \beta^\dagger a_{\mathbf{p}}. \qquad (45)$$

If we consider the β's as unitary, then the λ's satisfy bosonic canonical commutation relations. This relationship may be inverted

$$a_{\mathbf{p}} = \beta \lambda_{\mathbf{p}}. \qquad (46)$$

The number of particles in a given mode is equal to the number of PH

$$a_{\mathbf{p}}^\dagger a_{\mathbf{p}} = \lambda_{\mathbf{p}}^\dagger \lambda_{\mathbf{p}}. \qquad (47)$$

We write the field operator restricted to the subspace with a well-defined total number of particles N as $\Psi = \sqrt{N}\beta\phi$,

$$\phi = \phi_0 + \frac{1}{\sqrt{N}}\lambda(\mathbf{x}, t) - \frac{1}{2N}F[\delta n(t)]\phi_0, \qquad (48)$$

where for a homogeneous condensate we must have $\phi_0 = V^{-1/2}$,

$$\lambda(\mathbf{x}, t) = \sum_{\mathbf{p}\neq 0} \lambda_{\mathbf{p}}(t) f_{\mathbf{p}}(\mathbf{x}), \qquad (49)$$

$$\delta n(t) = \int d^3\mathbf{x} \, \lambda^\dagger \lambda, \qquad (50)$$

$$F(x) = 2N\left[1 - \sqrt{1 - \frac{x}{N}}\right] \sim x + O(N^{-1}). \qquad (51)$$

Within our approximations β commutes with ϕ. To lowest order in N^{-1}, λ evolves according to

$$0 = -i\hbar\lambda_{,t} + H\lambda + Un(\lambda + \lambda^\dagger) + O(N^{-1/2}) \qquad (52)$$

(see the appendix).

4.2. The HFB and Popov approximations

Before proceeding to compute the one-particle density matrix in the PNC approach, let us show that the HFB and Popov approximations give essentially equivalent results.

The HFB and Popov approximations are implementations of the symmetry breaking approach to condensation, where the formation of a BEC is associated with the spontaneous breaking of the $U(1)$ symmetry (7) [29]. The field operator develops a c-number expectation value, which by translation symmetry may depend only on time,

$$\langle\Psi\rangle = e^{-i\Theta(t)}\Phi(t). \qquad (53)$$

More generally

$$\Psi = e^{-i\Theta(t)}[\Phi(t) + \psi]. \qquad (54)$$

In the HFB approach, we use this decomposition in the Heisenberg equations of motion, where we also replace the products of two fluctuation operators by their expectation value, and use the so-called Hartree approximation:

$$\psi^\dagger \psi^2 \sim 2\tilde{n}\psi + \tilde{m}\psi^\dagger, \tag{55}$$

where

$$\tilde{n} = \langle \psi^\dagger \psi \rangle, \tag{56}$$

$$\tilde{m} = \langle \psi^2 \rangle. \tag{57}$$

The Heisenberg equations decompose into equations for the mean fields and equations for the fluctuations

$$i\hbar \frac{d}{dt}\Phi + \eta\Phi = U\Phi^3 + 2U\tilde{n}\Phi + U\tilde{m}\Phi, \tag{58}$$

$$i\hbar \frac{\partial}{\partial t}\psi + \eta\psi = H\psi + 2U(\Phi^2 + \tilde{n})\psi + U(\Phi^2 + \tilde{m})\psi^\dagger, \tag{59}$$

where

$$\eta = \hbar \frac{d\Theta}{dt}. \tag{60}$$

The HFB approximation has the serious drawback that it is not gapless, and therefore is hardly reliable in a problem such as the Bose Nova, which depends critically on the behaviour of long wavelength modes. The Popov approximation overcomes this problem by further neglecting $\tilde{m}$. Then we obtain

$$\eta = U\Phi^2 + 2U\tilde{n}, \tag{61}$$

and the fluctuation equation becomes

$$i\hbar \frac{\partial}{\partial t}\psi = H\psi + U\Phi^2(\psi + \psi^\dagger). \tag{62}$$

Under this approximation Φ^2 remains constant. This may be avoided by including explicitly the effect of particle loss through three-body recombination. However, the final results are robust against these terms [8, 10, 13], and we shall not consider them in detail. On the other hand, the total number of particles is not conserved.

If we assume that the temperature is effectively absolute zero, then $\Phi^2 = n$ initially and remains close to it until much later in the collapse; the effect of finite temperature is discussed in [13] and is seen to be minor. If we just replace $\Phi^2 = n$, the Popov equation for the fluctuations reduces to the PNC equation for the inhomogeneous modes (52). This approximation gives a reasonable description of early jet and burst formation [7], so it may be considered reliable.

4.3. The one-particle density matrix in the PNC approach

We now return to the calculation of the one-particle reduced density matrix:

$$\sigma(\mathbf{x}, \mathbf{y}, t) = \langle \hat{\Psi}^\dagger(\mathbf{x}, t)\hat{\Psi}(\mathbf{y}, t) \rangle$$

$$\equiv n\left\{ 1 - \frac{1}{N}[\langle \delta n \rangle - V\langle \lambda^\dagger(\mathbf{x}, t)\lambda(\mathbf{y}, t)\rangle] \right\}. \tag{63}$$

Decomposing into modes, we get

$$\sigma(\mathbf{x}, \mathbf{y}, t) = n\left\{ 1 - \frac{1}{n}\sum_{\mathbf{p}\neq 0}\left[\frac{1}{V} - f_{-\mathbf{p}}(\mathbf{x})f_{\mathbf{p}}(\mathbf{y})\right]\langle \lambda_{\mathbf{p}}^\dagger \lambda_{\mathbf{p}}\rangle \right\}. \tag{64}$$

Each mode evolves according to

$$i\hbar \frac{d\lambda_{\mathbf{p}}}{dt} = \frac{\nu_p}{2}\lambda_{\mathbf{p}} + Un(\lambda_{\mathbf{p}} + \lambda_{-\mathbf{p}}^\dagger). \tag{65}$$

The dispersion relation is given by (32). At $t = 0$, λ must destroy the physical state, so $\lambda_{\mathbf{p}}(0) = e^{i\varphi_{\mathbf{p}}}A_{\mathbf{p}}$ for some phase $\varphi_{\mathbf{p}}$. From the equation of motion we derive the initial velocity

$$i\hbar \frac{d\lambda_{\mathbf{p}}}{dt}(0) = \frac{\nu_p}{2}e^{i\varphi_{\mathbf{p}}}A_{\mathbf{p}} + Un(e^{i\varphi_{\mathbf{p}}}A_{\mathbf{p}} + e^{-i\varphi_{\mathbf{p}}}A_{-\mathbf{p}}^\dagger). \tag{66}$$

Therefore,

$$\lambda_{\mathbf{p}}(t) = \left\{ \left[\cos[\omega_p t] - i\left(\frac{\nu_p}{2} + Un\right)\frac{\sin[\omega_p t]}{\hbar\omega_p}\right]e^{i\varphi_{\mathbf{p}}}A_{\mathbf{p}} \right.$$

$$\left. - iUn\frac{\sin[\omega_p t]}{\hbar\omega_p}e^{-i\varphi_{\mathbf{p}}}A_{-\mathbf{p}}^\dagger \right\}. \tag{67}$$

This equation and (31) show that

$$\langle \lambda_{\mathbf{p}}^\dagger \lambda_{\mathbf{p}}\rangle = n\langle X_{\mathbf{p}}^\dagger X_{\mathbf{p}}\rangle \tag{68}$$

and therefore the PNC result (64) is just the first term in the expansion of our earlier result (36) in inverse powers of $N^{1/2}$.

Indeed, the representations of the field operators (15) and (48) are equivalent, to next to leading order in $N^{-1/2}$, provided we identify $e^{i\varphi_{\mathbf{p}}} = -i$ and $\exp\{-i\hat{X}_0/V^{1/2}\} = n^{1/2}\beta$.

5. Final remarks

After this point, it only remains to comment on the reasons why this proposal works.

From the formal point of view, our expression for the one-particle reduced density matrix is seen to agree with the perturbative implementation of the particle number conserving approach to next to leading order. This agreement suggests that, more generally, our approach implements a resummation of the PNC expansion. A key feature is that we use variables that keep the exponential structure of the one-particle density matrix. Therefore, the method suggested amounts to a perturbative evaluation of the exponent, but is non-perturbative with respect to the final result.

This formal advantage of the proposed method correlates with a shift in the physical emphasis, from particle creation in the excited modes to the loss of coherence among atoms. Comparing this to other formal studies of decoherence, it comes as no surprise that the latter process is faster than the former [30, 31].

Observe that in principle the main result of this paper can be obtained from standard approaches, such as PNC, HFB or Popov, provided the output of those approaches is interpreted as the short time behaviour of the one-particle density matrix (e.g., 'resumming' (64) into (36)). This would signal that, taking into account the short time limitation in dealing perturbatively with unstable modes, the essential physics involved in producing the main result, when exponential parametrizations are used, is already present in the standard, linearized time-dependent mean-field approximations.

We submit this minor contribution with the expectation that it will help clear the way to a full quantitative understanding of this fascinating experiment.

Appendix. Derivation of (52)

The idea is to seek a solution of the Heisenberg equations of motion for Ψ where the β and λ have developments in inverse powers of N. Define a q-number chemical potential $\hat{\mu}$ from

$$\beta^{\dagger}\frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{-\mathrm{i}\hat{\mu}}{\hbar}, \tag{A.1}$$

we have

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\phi = (H - \hat{\mu})\phi + u\phi^{\dagger}\phi^2. \tag{A.2}$$

We then find

$$0 = -\hat{\mu}\phi_0 + u\phi_0^3 + \frac{1}{\sqrt{N}}\big[-\mathrm{i}\hbar\lambda_{,t} + (H - \hat{\mu})\lambda$$
$$+ u\phi_0^2(2\lambda + \lambda^{\dagger})\big] + O(N^{-1}). \tag{A.3}$$

Taking the expectation value, we find

$$0 = -\langle\hat{\mu}\rangle\phi_0 + u\phi_0^3 - \frac{1}{\sqrt{N}}\langle\hat{\mu}\lambda\rangle + O(N^{-1}). \tag{A.4}$$

Recall that $\hat{\mu}$ is Hermitian. So we may decompose this equation into

$$0 = -\langle\hat{\mu}\rangle\phi_0 + u\phi_0^3 - \frac{1}{2\sqrt{N}}\langle\hat{\mu}\lambda + \lambda^{\dagger}\hat{\mu}\rangle + O(N^{-1}) \tag{A.5}$$

and

$$0 = \frac{1}{2\sqrt{N}}\langle\hat{\mu}\lambda - \lambda^{\dagger}\hat{\mu}\rangle + O(N^{-1}). \tag{A.6}$$

Subtracting the expectation value from the Heisenberg equation, we get

$$0 = (\langle\hat{\mu}\rangle - \hat{\mu})\phi_0 + \frac{1}{\sqrt{N}}\big[-\mathrm{i}\hbar\lambda_{,t} + (H - \hat{\mu})\lambda$$
$$+ u\phi_0^2(2\lambda + \lambda^{\dagger})\big] + \frac{1}{\sqrt{N}}\langle\hat{\mu}\lambda\rangle + O(N^{-1}), \tag{A.7}$$

and from (A.7), (A.6) and (A.5), we get

$$\hat{\mu} = \langle\hat{\mu}\rangle + O(N^{-1}) \sim \frac{u}{V} = Un. \tag{A.8}$$

Observe that this implies

$$\langle\hat{\mu}\lambda\rangle = O(N^{-1/2}), \tag{A.9}$$

the equation for λ simplifies to (52).

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