EPJ B
Condensed Matter and Complex Systems

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Eur. Phys. J. B (2015) 88: 265 DOI: 10.1140/epjb/e2015-60522-6

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Società Italiana di Fisica
Springer

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THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Generalized Nagumo model: interesting effects of parameters' fluctuations

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Received 30 June 2015 / Received in final form 28 August 2015
Published online 19 October 2015 –

Abstract. In this paper, we used an analytical method to calculate the effects that produce the parameter's fluctuations characterizing a generalization of Nagumo model. (The extinction option is replaced by one of low density homogeneous population.) Moreover, we also check the results by means of numerical simulations of the corresponding stochastic process. We find that these fluctuations have a strong impact on the solutions producing interesting changes.

1 Introduction

Results about constructive effects of noise have been reported for many years now [1–19]. Particularly, an intuitive vision has been used in some articles. This vision considers that a multiplicative noise tends to expel to the system out of field region where its intensity is higher [10,11]. In order to explore this idea, we have presented and studied multiplicative noises that were able to push the homogeneous solutions (HS), that correspond to a given dynamic, toward predetermined field values by the multiplicative noise factor. Here, the objective is to stimulate the pattern formation [20–23] and dissipative solitons [24]. We have recently reported results about a comprehensive study that shows how by means of a suitably designed multiplicative noise, the HS can be not only pushed but also confined into a field region predetermined by its multiplicative factor [25]. We have named this phenomenon as "Pusher noise" and we have observed that its forcefulness grows proportionally to the negative multiplicative noise factor's derivative with respect to the field.

Here, we have addressed a case that considers Nagumo model's parameters fluctuations [26]. This is a simple model appropriate to show how a multiplicative noise generated by fluctuations of its parameters may cause significant changes in the behaviour of the system. In the future we will consider more complex models such as the FitzHugh-Nagumo-model, which describes a prototype of an excitable system (e.g., a neuron) [27–30]. It is well known, the Nagumo-model describes the evolution of a species that is restricted by adverse factors, with two options (extinction or survival), with a cubic nonlinear equation

$$F(u) = u(u - \alpha b)(b - u);$$

it is characterized by two parameters: the HS ($u = b$), representing the survival, and the so-called adversity (α), indicating the limit between the attraction basins of each HS by means of the unstable solution $u = b\alpha$. We established this model by replacing the extinction option with a low density homogeneous population (this generalization has also been applied before [31]), which adds another parameter: β associated to the new solution as $u = \beta b$ (instead of $u = 0$).

As aforementioned, our aim is to study the fluctuations' effects of each of these parameters over the Nagumo model solutions, including the unstable one. With this in mind, we obtain a zero-dimensional stochastic dynamic driven by the generalized Nagumo model's nonlinearity as well as a multiplicative noise with a factor that depends on the positive parabola shaped field (the corresponding Langevin equations are to be interpreted in the Stratonovich sense). Such dynamic can be turned into relaxation one, being the corresponding coefficient the square of multiplicative factor of the noise [21–23,25]. This way, not only the noise fulfills the dissipation-fluctuation theorem [10,32,33], but the corresponding average dynamics can also be easily described. Thus, we can observe how the multiplicative noise pushes to the affected solution toward the values that minimize its factor. In addition to the corresponding analytic calculation, we obtain the stationary probability distribution function, using a recently reported numeric method [34]. This way, we check the interesting predictions of the analytic calculation.

2 Proposal development and results

By applying the Nagumo model we consider that each individual requires a minimum vital space (a space that cannot be invaded by another individual). Thus we define u as the covering of the space available (with u a dimensionless

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variable normalized to 1). As aforementioned, we also consider a low density homogeneous population: $u = \beta b$ as an alternative proposal to extinction. Under these conditions, the modified Nagumo nonlinearity can be expressed as: $F(u) = (u - \beta b)(u - \alpha b)(b - u)$. This also shows two uniform attractors ($u = \beta b$ and $u = b$) and an ejector (αb), where α is known as the *adversity factor*. The ejector $u = \alpha b$ marks a limit between domains (low/high population density). When u is lower than this value, the system evolves toward the low population density $u = \beta b$ and when u is higher, the system evolves toward a population with density $u = b$.

Furthermore, we consider the effects of fluctuations of each parameter separately by using a Gaussian white noise with zero mean and correlation

$$\langle \eta(t)\eta(t')\rangle = 2\lambda^2\delta(t - t'). \tag{1}$$

Thus, by describing β's fluctuations as: $\beta = \beta_0 + \eta(t)$, the modified zero-dimensional dynamic that the Nagumo model imposes can be expressed as:

$$\dot{u} = F_{\beta_o}(u) + \Gamma^{1/2}(u)\,\eta(x,t), \tag{2}$$

where $F_{\beta_o}(u) = (u - \beta_0 b)(u - \alpha b)(b - u)$ and $\Gamma^{1/2}(u) = -b(u - \alpha b)(b - u)$. We note that $\Gamma^{1/2}(u)$ is a parabola with positive curvature and has a minimum value located right at the midpoint between the two roots which are not affected by fluctuations ($u_{min} = b\frac{1+\alpha}{2}$).

In absence of fluctuations, we can rewrite equation (2) by multiplying and dividing $F_{\beta_o}(u)$ by $-\Gamma(u)$. This indicates that the dynamic is relaxational in some free-energy function $\mathcal{F}(u)$ with a field-dependent kinetic coefficient $\Gamma(u)$. The fictitious relaxation function so defined is written as:

$$\mathcal{F}(u) = -\int du \frac{F_{\beta_o}(u)}{\Gamma(u)} = \ln\left\{\frac{|u - b|^{\frac{1-\beta_0}{(1-\alpha)b^2}}}{|u - \alpha b|^{\frac{\alpha-\beta_0}{(1-\alpha)b^2}}}\right\}. \tag{3}$$

It is clear that the fictitious relaxation function does not include the solution that is affected by fluctuations. However, the real dynamic does not change because of the action of the relaxation coefficient which reintroduces the aforementioned solution.

Therefore, equation (2) written in terms of the fictitious relaxation function is:

$$\partial_t u = -\Gamma(u)\frac{d\mathcal{F}(u)}{du} + \Gamma^{1/2}(u)\,\eta(t). \tag{4}$$

When raising this issue, we observe that the fluctuations fulfill the fluctuation − dissipation theorem [10,32,33].

Under these conditions, the stationary probability distribution function (SPDF) for the field $P_{st}(u)$ is of the Boltzmann's type and can be described by effective relaxation function [10,11]:

$$P_{th}(u) \propto \exp\left\{-\frac{\mathcal{F}_{\text{eff}}(u)}{\lambda^2}\right\} = \frac{|u - \alpha b|^{e^{\alpha b}}}{b^2|u - b|^{e^b}}, \tag{5}$$

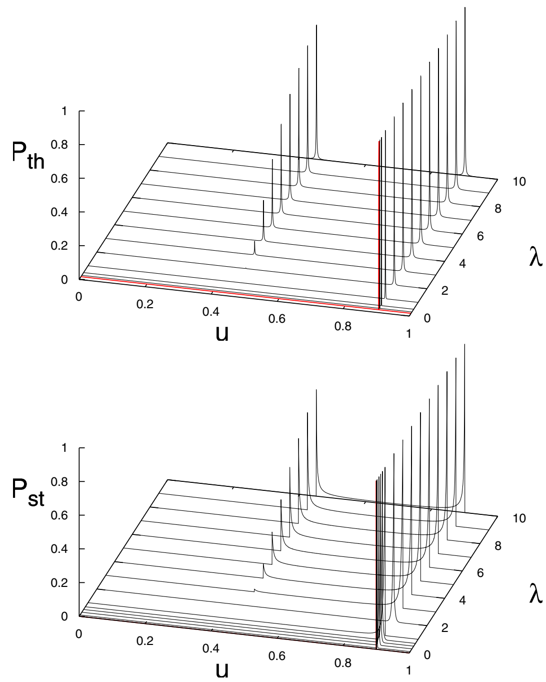

Fig. 1. SPDF vs. u and λ for β parameter fluctuations. (Top) calculated analytically (P_{th}). (Bottom) calculated by simulation of stochastic process (P_{st}). $\alpha = 0.5$, $\beta_0 b = 0.001$ and $b = 0.9$ (curves are referenced to their maximum value).

where $\mathcal{F}_{\text{eff}}(u) = \mathcal{F}(u) + \lambda^2 \ln[\Gamma(u)]$, $e^{\alpha b} = \frac{\alpha-\beta_0-2(1-\alpha)b^2\lambda^2}{(1-\alpha)b^2\lambda^2}$ and $e^b = \frac{1-\beta_0+2(1-\alpha)b^2\lambda^2}{(1-\alpha)b^2\lambda^2}$.

On the other hand, we also simulate the stochastic process described by equation (4) and calculate the stationary probability density $P_{st}(u)$, using an efficient algorithm that was recently reported[1] [34]. Figure 1 shows the curves obtained both analytically and numerically. It becomes evident that the fluctuations-affected solution is absent even in very low values of λ. We also observe that, starting a (λ_c) critical value of noise intensity onward, the unstable solution in noise absence is stabilized by the noise effect. The latter is predicted by the theoretical calculation when the exponent $e^{\alpha b}$ in equation (5) changes its sign. Therefore we obtain $\lambda_c = \sqrt{\frac{\alpha-\beta_0}{2(1-\alpha)b^2}}$ requiring a zero value for said exponent. For the parameters of Figure 1, we obtain $\lambda_c \sim 0.79$ by theoretical calculation, while we obtain $\lambda_c \sim 1.5$ by numerical calculation[2]. Although the numbers do not match, it is important to be able to predict such phenomenon. Figure 2 shows the exponent corresponding to both solutions; ($-e^{\alpha b}$ and e^b) $\mapsto 2$ when $\lambda \mapsto \infty$.

Another property that is worth taking into account is the ratio amongs the peaks of the SPDF. Figure 1 shows

[1] We test distinct symmetrical initial conditions (including a Gaussian and other uniform distribution) getting the same results regardless the symmetrical initial condition used.

[2] While for $\lambda \sim 1$, we observe a balanced competition between the two solutions, for $\lambda \sim 2$, $u = \alpha b$ is evidently the stable solution.

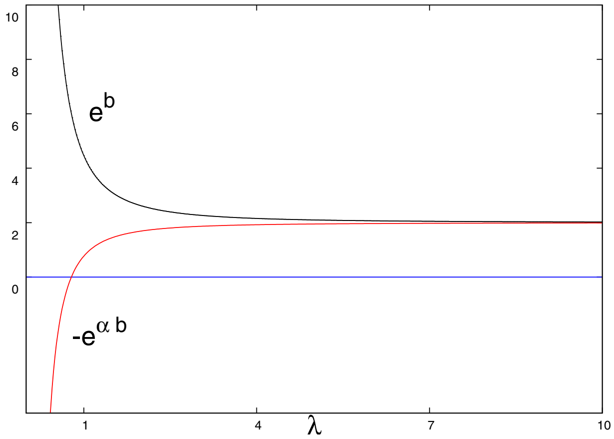

Fig. 2. Exponent vs. noise intensity. Upper curve corresponds to e^b and bottom curve corresponds to $-e^{\alpha b}$. $\alpha = 0.5$, $\beta_0 b = 0.001$ and $b = 0.9$.

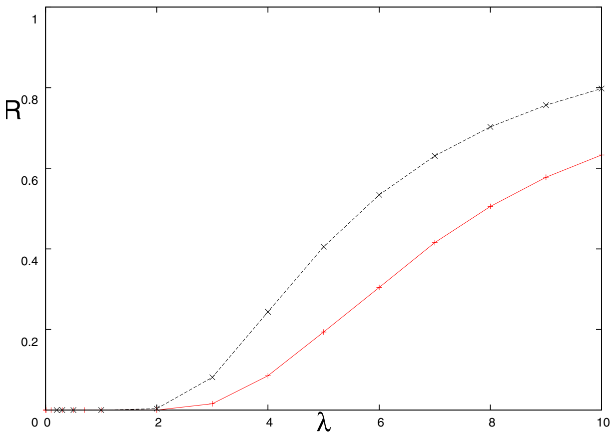

Fig. 3. Ratio vs. noise intensity. Upper curve corresponds to theoretical calculation and bottom curve corresponds to numerical calculation. $\alpha = 0.5$, $\beta_0 b = 0.001$ and $b = 0.9$.

that for higher intensities of noise the two stable solutions become more equiprobable. Although equation (5) clearly shows that the height of those peaks is infinite, they are comparable when measured from a given distance to the solutions. Figure 3 shows the ratio R vs. λ. The curves differ by an additive constant, whose origin we assume is related to the difference between the λ_c values.

Since the lack of the fluctuations-affected solution (even for very low noise intensities) is intriguing to us, we try to find evidence of its existence. The effective relaxation function $\mathcal{F}_{\text{eff}}(u)$ enables a description of the zero-dimensional average dynamics through one effective nonlinearity [10,11,21–23]:

$$F_{\text{eff}}^{\beta_0} = F_{\beta_0} - \lambda^2 \frac{d\Gamma(u)}{du} = \left(1+4b^2\lambda^2\right)(u-u_{st})(u-\alpha b)(b-u), \tag{6}$$

where $u_{st} = b\frac{\beta_0+2b^2\lambda^2(1+\alpha)}{1+4b^2\lambda^2}$ is the "missing" solution. For low λ values

$$u_{st} \approx b[\beta_0 + 2b^2\lambda^2(1+\alpha-2\beta_0)].$$

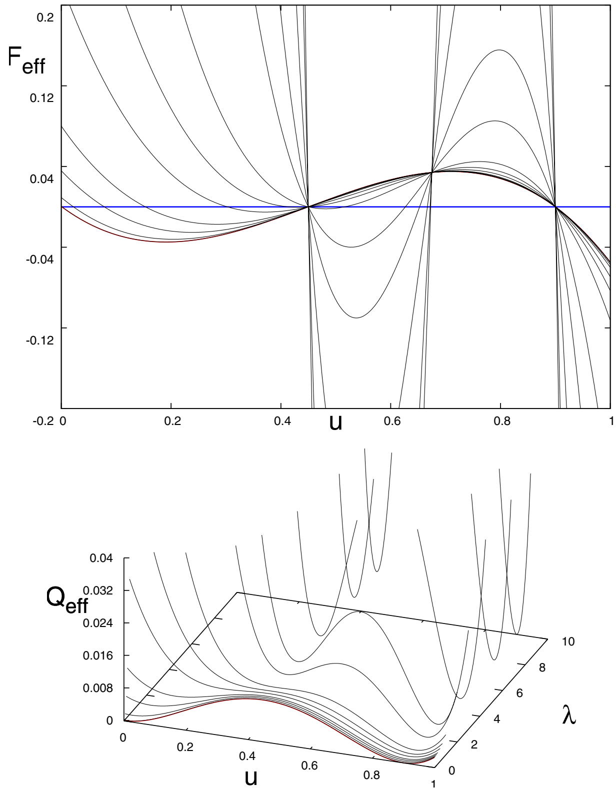

Fig. 4. (Top) $F_{\text{eff}}^{\beta_0}$ vs. u for different values of λ. (Bottom) Q_{eff} vs. u for different values of λ. $\alpha = 0.5$, $\beta_0 b = 0.001$ and $b = 0.9$.

So, it can be seen that u_{st} is displaced toward unstable solution $u = \alpha b$, by effect of the fluctuations. In fact, for $\lambda \mapsto \infty$, u_{st} is located right in the middle point between $u = \alpha b$ and $u = b$ ($u_{st}^{\infty} = b\frac{1+\alpha}{2} = u_{min}$). This means that u_{st} become an unstable solution creating an exchange of roles between both solutions. This result explains why the solution $u = \alpha b$ appears in the curves shown in Figure 1. The solution that is unstable in absence of noise becomes stable for higher noise intensities (in fact, λ_c can also be obtained as: $u_{st} = b\alpha$). Furthermore; due to the fact that for very high noise intensities, u_{st} (being now the unstable solution) is located right in the middle between the other two solutions, both stable solutions ($u = \alpha b$ and $u = b$) become almost equiprobable (as shown in Fig. 1). In order to better illustrate this result, Figure 4 shows curves of effective nonlinearity ($F_{\text{eff}}^{\beta_0}$) and effective potential Q_{eff} (integral over u of $F_{\text{eff}}^{\beta_0}$) for different noise intensities. Curves clearly show the displacement of the solution that is affected by fluctuations until the roles of unstable ($u = \alpha b$) and stable (u_{st}) solutions are exchanged. Then, for $\lambda > \lambda_c$, regardless of the noise intensity's value, the now stable solution remains in $u = \alpha b$ while the affected one by fluctuations is now unstable. Therefore, the now unstable solution (u_{st}) changes its value while the noise intensity increases, and so does the relationship between the weights of both stable solutions.

Based on the above results, we believe that the "missing" solution should be noticeable through the simulation of stochastic processes. We test this by calculating

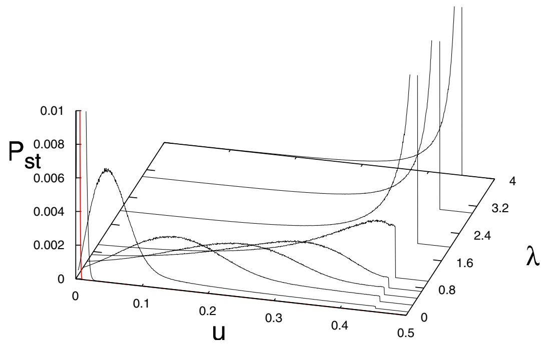

Fig. 5. SPDF vs. u and λ (for β parameter fluctuations) calculated by simulation of stochastic process (P_{st}) by introducing a form which is non-zero uniform over the corresponding basin around the affected solution and in the rest zero as the initial condition. $\alpha = 0.5$, $\beta_0 b = 0.001$ and $b = 0.9$.

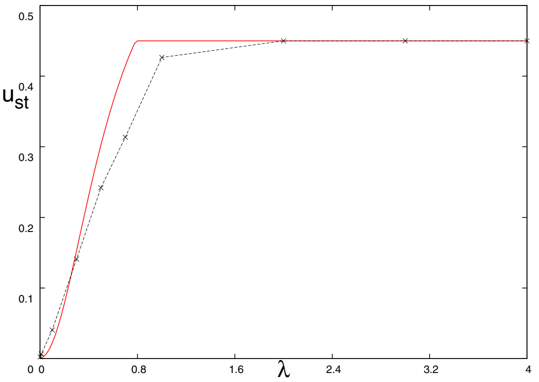

Fig. 6. u_{st} vs. λ. Solid line: analytic values and crosses line: numerical values. $\alpha = 0.5$, $\beta_0 b = 0.001$ and $b = 0.9$.

the SPDF with a very skewed initial condition over the fluctuations-affected solution. We particularly suggest a non-zero uniform form over the corresponding basin around the said solution and in the rest zero. This way, we expect the solution we assume as meta-stable to remain on such basin. Figure 5 highlights the existence of the missing solution. Now, we can not only detect the fluctuations-affected solution but also observe its displacement as λ increases, until it overlaps with the unstable solution. Figure 6 completes this idea by comparing u_{st} values that are obtained analytically with those obtained numerically, for different intensities of noise.

2.1 Fluctuations of b parameter

When instead of β (without any of the other two parameters doing it) b fluctuates, we expect a similar behavior to the previous case. Therefore, the corresponding zero-dimensional dynamic is described by

$$\dot{u} = F_{b_o}(u) + \Gamma^{1/2}(u)\,\eta(x,t), \tag{7}$$

where $F_{b_o}(u) = (u - \beta b_0)(u - \alpha b_0)(b_0 - u)$ and $\Gamma^{1/2}(u) = b_0(u - \alpha b_0)(u - \beta b_0)$. Here we also observe that $\Gamma^{1/2}(u)$ is a parabola with positive curvature and a minimum value located right at the midpoint between the two roots which are not affected by fluctuations ($u_{\min} = b_0 \frac{\beta+\alpha}{2}$).

In the same way as before we calculate the fictitious relaxation function $\mathcal{F}(u)$

$$\mathcal{F}(u) = -\int du \frac{F_{b_0}(u)}{\Gamma(u)} = \ln\left\{\frac{|u - b_0\beta|^{\frac{1-\beta}{(\alpha-\beta)b_0^2}}}{|u - \alpha b_0|^{\frac{1-\alpha}{(\alpha-\beta)b_0^2}}}\right\}. \tag{8}$$

Then, by calculating $\mathcal{F}_{\text{eff}}(u) = \mathcal{F}(u) + \lambda^2 \ln[\Gamma(u)]$, we obtain the corresponding SPDF:

$$P_{th}(u) \propto \exp\left\{-\frac{\mathcal{F}_{\text{eff}}(u)}{\lambda^2}\right\} = \frac{|u - \alpha b_0|^{e^{\alpha b_0}}}{b_0^2|u - \beta b_0|^{e^{\beta b_0}}}, \tag{9}$$

where $e^{\alpha b_0} = \frac{1-\alpha-2(\alpha-\beta)b_0^2\lambda^2}{(\alpha-\beta)b_0^2\lambda^2}$ and $e^{\beta b_0} = \frac{1-\beta_0+2(\alpha-\beta)b_0^2\lambda^2}{(\alpha-\beta)b_0^2\lambda^2}$. Subsequently, just as we have done before, we also numerically calculate the corresponding SPDF for same parameters values. Figure 7 shows the results for both calculations. Indeed, a critical value of λ is observed from which the fluctuations always stabilize the solution $u = \alpha b_0$. Unlike the previous case, here we can detect the fluctuations-affected solution when the noise intensities are very low. This result is not surprising, since the current problem is not symmetric with the previous one. In fact, when we consider fluctuations of β for $\beta_0 = 0.1$, $\alpha = 0.5$ and $b = 0.9$, we also observe low pecks for very low λ value. Moreover, the behavior of the ratio between the peaks of

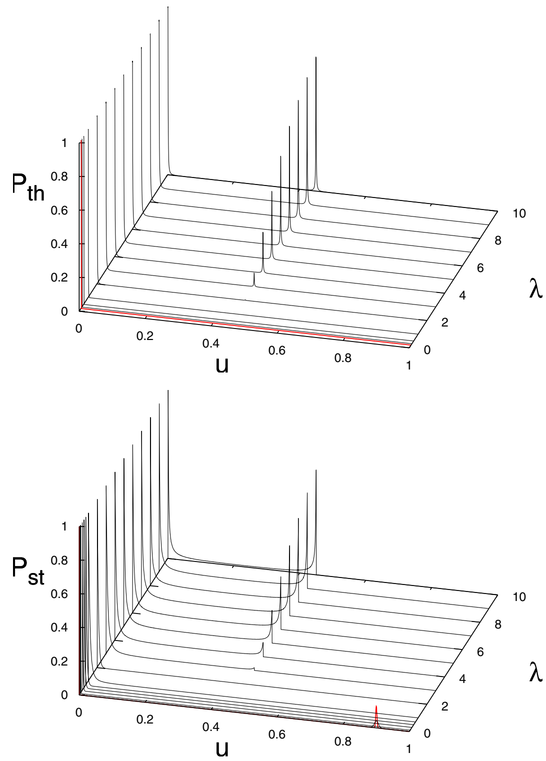

Fig. 7. SPDF vs. u and λ for b parameter fluctuations. (Top) calculated analytically (P_{th}). (Bottom) calculated by simulation of stochastic process (P_{st}). $\alpha = 0.5$, $\beta_0 b = 0.001$ and $b_0 = 0.9$ (curves are referenced to its maximum value).

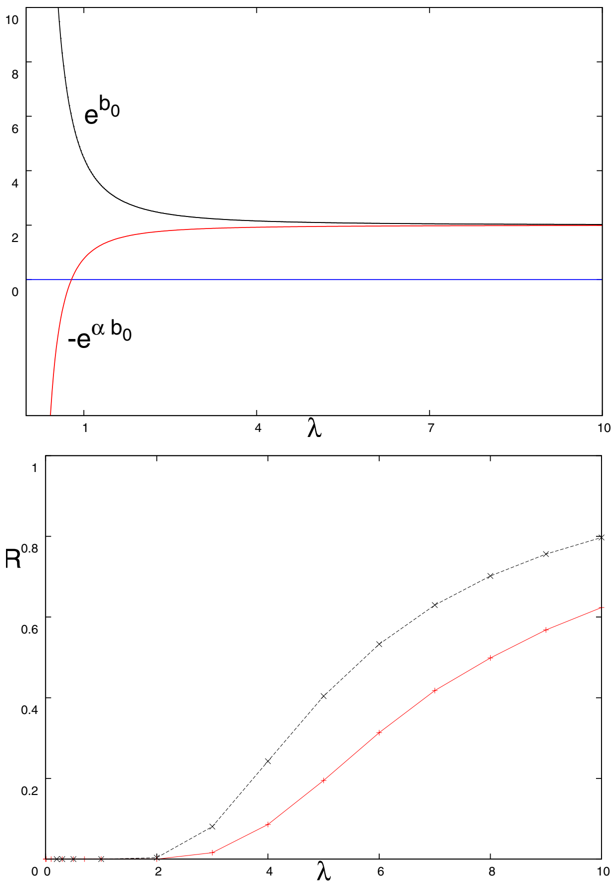

Fig. 8. (Top) exponent vs. noise intensity; upper curve corresponds to $e^{\beta b_0}$ and bottom curve corresponds to $-e^{\alpha b_0}$. (Bottom) ratio vs. noise intensity; upper curve corresponds to theoretical calculate and bottom curve corresponds to numerical calculate. $\alpha = 0.5$, $\beta b = 0.001$ and $b_0 = 0.9$.

the SPDF and also of the exponents ($e^{\alpha b_0}$ and $e^{\beta b_0}$), in relation to the noise intensity, is similar to the previous case. This can be observed in Figure 8.

For this case we also reveal the "missing" solution by means of a calculation that is similar to the previous case. Therefore, we obtain the effective non-linear equation as:

$$F^{b_0}_{\mathrm{eff}} = F_{b_0} - \lambda^2 \frac{d\Gamma(u)}{du} = (1+4b_0^2\lambda^2)(u-u_{st})(u-\alpha b)(u-\beta b_0), \tag{10}$$

where $u_{st} = b_0 \frac{1+2b_0^2\lambda^2(\beta+\alpha)}{1+4b_0^2\lambda^2}$ is the "missing" solution. Here u_{st} is also displaced by the noise toward the unstable solution until the last is reached. Figure 9 shows $F^{b_0}_{\mathrm{eff}}(u)$ and $Q_{\mathrm{eff}}(u)$ with λ increasing from $\lambda = 0.002$. The displacement of fluctuations-affected solution and strong decrease in the size of its basin can be observed clearly in the shown figure. As noted before, the critical noise intensity can be calculated as both $u_{st} = \alpha b_0$ and $e^{\alpha b_0} = 0$ resulting in $\lambda_c = \sqrt{\frac{1-\alpha}{2(\alpha-\beta)b_0^2}}$.

Finally, we also calculate the SPDF with a very skewed initial condition over the fluctuations-affected solution. Figure 10 shows the corresponding outcomes. The displacement of affected solution induced by fluctuations can

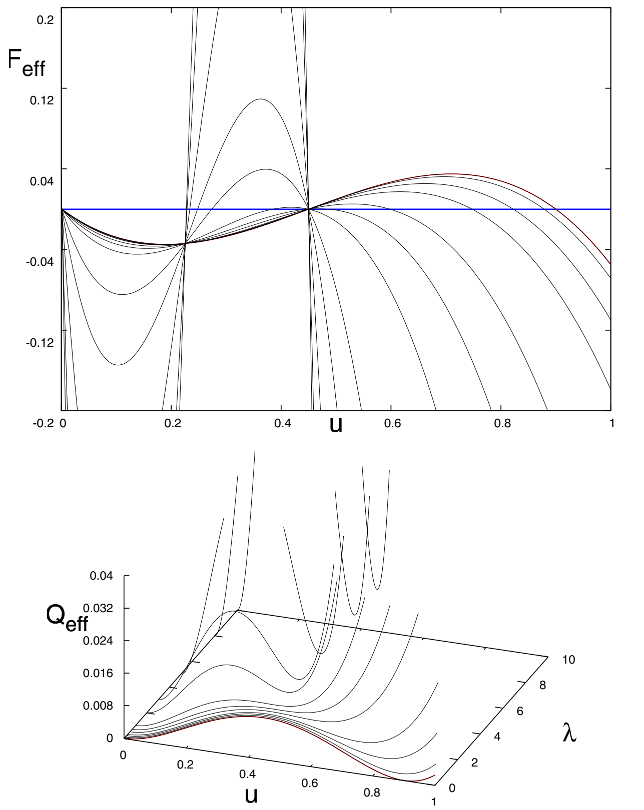

Fig. 9. (Top) $F^{b_0}_{\mathrm{eff}}$ vs. u for different values of λ. (Bottom) Q_{eff} vs. u for different values of λ. $\alpha = 0.5$, $\beta b = 0.001$ and $b_0 = 0.9$.

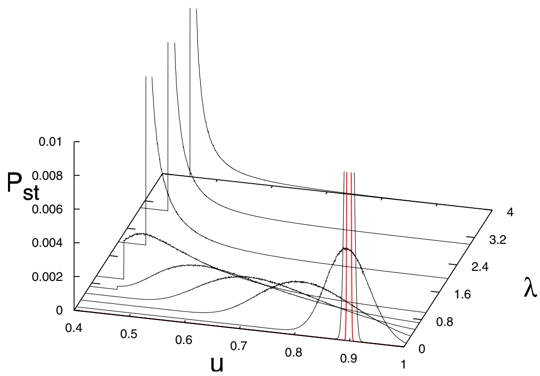

Fig. 10. SPDF vs. u and λ (for b parameter fluctuations) calculated by simulation of stochastic process (P_{st}) by introducing a form which is non-zero uniform over the corresponding basin around the affected solution and the in rest zero, as initial condition. $\alpha = 0.5$, $\beta b = 0.001$ and $b_0 = 0.9$.

be clearly observed. This solution approaches to the unstable solution as the noise intensity increases, until both solutions overlap. Then, for higher noise intensities, the roles between both solutions are exchanged.

One result that is worth emphasizing is that, for both cases studied, as $\lambda \mapsto \infty$ fluctuations-affected solution is set right at the midpoint between the two roots which are not affected by fluctuations, corresponding to the minimum value of multiplicative factor of noise. This way, the solution not only becomes unstable but also tends to

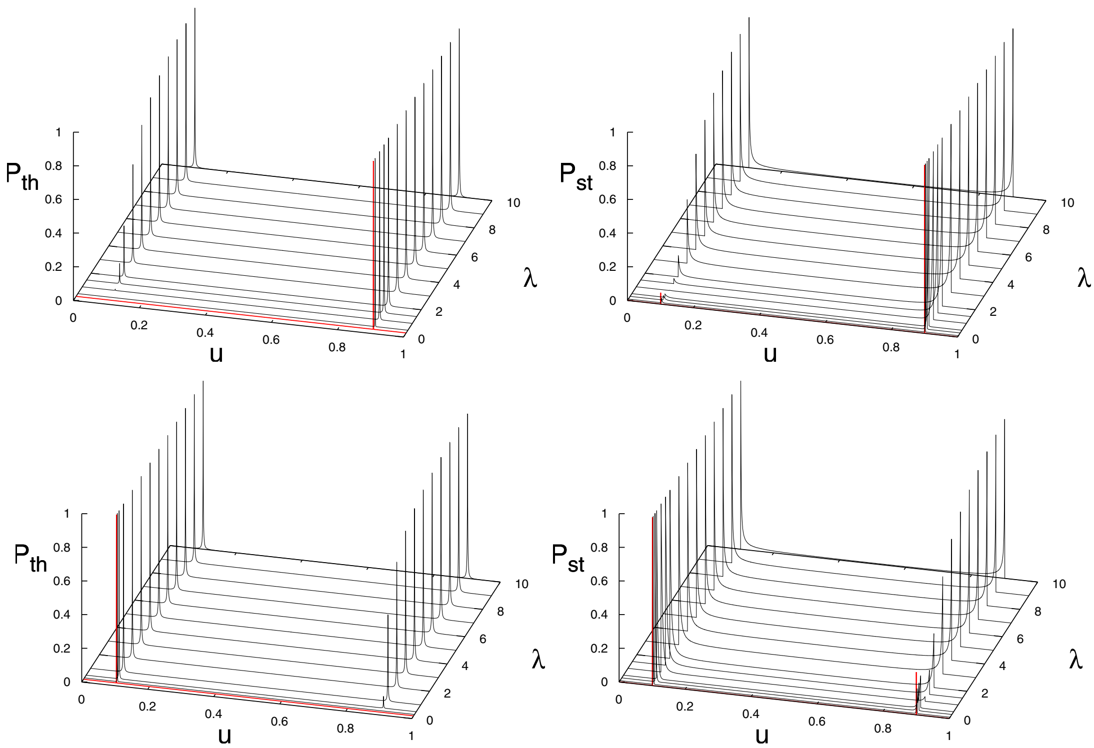

Fig. 11. SPDF vs. u and λ for α parameter fluctuations. (Upper left) $\alpha_0 = 0.3$, calculated analytically (P_{th}). (Upper right) $\alpha_0 = 0.3$, calculated by simulation of stochastic process (P_{st}). (Lower left) $\alpha_0 = 0.7$, calculated analytically (P_{th}). (Lower right) $\alpha_0 = 0.7$, calculated numerically (P_{st}). $\beta b = 0.1$ and $b = 0.9$ (curves are referenced to their maximum value).

balance the weight between the attraction basins corresponding to each stable solution.

2.2 Fluctuations of α parameter

This case is different from the previous one, since the stable solutions are not affected by the fluctuations and therefore remain on its original values. Following the same line of work as before, we described the corresponding zero-dimensional dynamic as:

$$\dot{u} = F_{\alpha_o}(u) + \Gamma^{1/2}(u)\,\eta(x,t), \tag{11}$$

where $F_{\alpha_o}(u) = (u-\beta b)(u-\alpha_0 b)(b-u)$ and $\Gamma^{1/2}(u) = b(u-\beta b)(b-u)$. Here we also observe that $\Gamma^{1/2}(u)$ is a parabola with positive curvature and a minimum value located right at the midpoint between the two roots which are not affected by fluctuations ($u_{\min} = b\frac{1+\beta}{2}$). Then, we calculate the corresponding fictitious relaxation function:

$$\begin{aligned}\mathcal{F}(u) &= -\int du \frac{F_{\alpha_0}(u)}{\Gamma(u)} \\ &= \ln\left\{ |u-b\beta|^{\frac{\alpha_0-\beta}{(1-\beta)b^2}} |u-b|^{\frac{1-\alpha_0}{(1-\beta)b^2}} \right\}\end{aligned} \tag{12}$$

and the SPDF:

$$P_{th}(u) \propto \exp\left\{-\frac{\mathcal{F}_{\text{eff}}(u)}{\lambda^2}\right\} = \frac{|u-b\beta|^{-e^{\beta b}}|u-b|^{-e^{b}}}{b^2}, \tag{13}$$

where $\mathcal{F}_{\text{eff}}(u)$ is defined as before, $e^{\beta b} = \frac{\alpha_0-\beta+2(1-\beta)b^2\lambda^2}{(1-\beta)b^2\lambda^2}$ and $e^{b} = \frac{1-\alpha_0+2(1-\beta)b^2\lambda^2}{(1-\beta)b^2\lambda^2}$. Of course, for this case we also calculated numerically the corresponding SPDF.

Here, we report studies on two particular cases which, in absence of fluctuations, differ by the weight ratio between the respective attraction basins. The affected parameter by fluctuations (adversity) is the relevant one to determine such ratio. We consider $\alpha_0 = 0.3$ (dominant solution: $u = b$) and $\alpha_0 = 0.7$ (dominant solution: $u = \beta b$). Figure 11 showing the SPDFs calculated analytically and numerically for both cases. It can be observed that for noise intensities $\lambda > 1$ the analytical result corresponds to the numeric result. Under this condition, we can see that weight of the non-dominant solution increases with λ until it is equalized with the weight of the other solution for very high noise intensities. However, we observe differences between results of both calculations for values of $\lambda < 1$.

In order to interpret this result we calculate the corresponding effective nonlinearity:

$$F_{\text{eff}}^{\alpha_0} = F_{\alpha_0} - \lambda^2\frac{d\Gamma(u)}{du} = (1+4b^2\lambda^2)(u-u_{st})(b-u)(u-\beta b), \tag{14}$$

where $u_{st} = b\frac{\alpha_0+2b^2\lambda^2(1+\beta)}{1+4b^2\lambda^2}$ is the fluctuations-affected solution and, of course, is the unstable one (effective adversity) that separates the basins corresponding to two stable solutions. The effective adversity grows with λ tending asymptotically toward the midpoint between the two

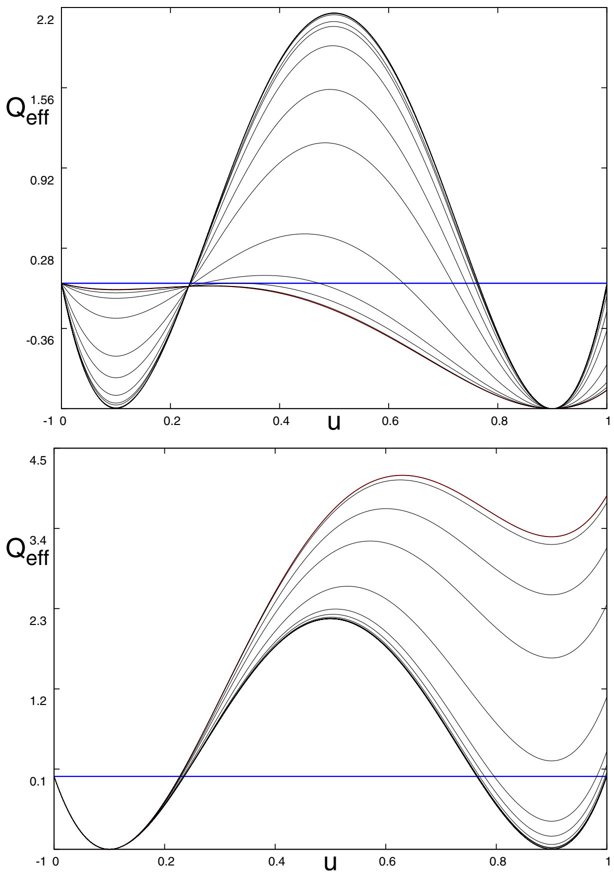

Fig. 12. $Q_{\rm eff}$ vs. u for different values of λ. The more asymmetric curve corresponds to $\lambda = 0$, then the curves are increasingly asymmetric with λ increasing. (Top) $\alpha_0 = 0.3$. (Bottom) $\alpha_0 = 0.7$. Values of others parameters: $\beta b = 0.1$ and $b_0 = 0.9$.

solutions not affected by the fluctuations: $u_{st} = \frac{(1+\beta)b}{2}$ when $\lambda \mapsto \infty$ (also coincident with the value that minimizes the multiplicative noise factor). This means that the size of the attraction basins tends to balance out as the noise intensity increases. In order to illustrate this result we show the effective potential $Q_{\rm eff}$ on a graph. Figure 12 shows the corresponding curves for different noise intensities. We can see that the "imbalance" of effective potential in absence of fluctuations becomes "balanced" for very high noise intensities, with a gradual connection between these two situations as λ varies from zero to infinite. Therefore, regardless of the fact that the adversity promotes one or another solution in absence of fluctuations, whenever the noise intensity is high enough, the weight of both solutions is being balanced by effect to fluctuations.

On the other hand, we know that a deterministic dynamic prevails for very low noise intensity and therefore, when numerically calculating the SPDF starting from uniform distribution, each attraction basin captures an amount of individuals proportional to the basin's size. This means that under these conditions the two peaks must emerge. Then, as noise intensity increases, a probability flow is generated towards the larger basin which tends to empty the other one. Hence, the probability peaks height of the smaller basin decreases with λ. Nevertheless, for even higher noise intensities, the weight ratio between the two basins tends to reach an equilibrium when λ increases, and so do the peaks' weight. The behavior for low λ cannot be detected by the analytic calculation because the effective relaxation function loses the fluctuations-affected solution, which arise when the deterministic dynamic prevails. In fact, the first does not describe it properly when $\lambda \mapsto 0$.

3 Analysis and conclusions

The aim of this paper is to research the effect of parameters fluctuations corresponding to a Nagumo's Model generalization, which consists in replacing the extinction option with the low density homogeneous population ($u = \beta b$). First we obtain analytic results to later confirm them by numerical simulation of stochastic process. This leads us to obtain the stationary probability distribution function [34]. So, we separately calculate the fluctuations' effects of the three parameters (β, α and b) that characterize this model.

In relation to the parameters that define the stable solutions in absence of fluctuations (β and b), we observe that the fluctuations always "displace" the affected solution toward the middle point between the two non-affected solutions. Moreover, for enough high noise intensities, the affected solution exchanges its role with the adversity (α), therefore, the previously unstable solution becomes stable and the affected by the fluctuations plays the role of the unstable one. In a normal situation, a larger (smaller) adversity value means to promote the low (high) density population solution, but with the contribution of fluctuations, once λ_c is overcome, the affected solution (now in the adversity role) gives the previously unstable (and now stable) solution more weight (by being displaced toward the aforementioned middle point), while the noise intensity increases, until both stable solutions reach an equilibrium for higher intensities. Moreover, the aforementioned middle point coincides with the value that minimizes the multiplicative factor of the noise, which suggests the possible existence of an underlying average dynamic that is able to push the system toward such minimum, until balanced by the deterministic forces. This vision is enhanced when we consider that the forcefulness of this effect grows proportionally to the negative multiplicative noise factor's derivative with respect to the field [25]. Then, when the noise intensity is high enough, the stochastic average dynamic prevails over the deterministic dynamic and, therefore, the system is localized in a solution that minimizes the multiplicative factor of the noise.

We also observe that the fluctuations-affected solution is displaced toward one of the multiplicative factor's zeros, provided said solution remains stable ($\lambda < \lambda_c$). Then, when this solution turns into unstable ($\lambda > \lambda_c$), it is displaced toward field's values that minimize the multiplicative factor. Therefore, for the latter situation, the stable solutions are always those that are non-affected by the fluctuations. By visualizing the population as a set of individuals, our results confirm the intuitive knowledge, which states that these individuals move away from those circumstances that change randomly. To this extend, the individuals choose to exist with such density values in order to minimize the effect of the fluctuations.

Since the adversity determines the size ratio between the attraction basins, its fluctuations affect said ratio. Here, our results show that when the adversity is affected by fluctuations, an effective adversity arises (α_{eff}) which moves itself toward the middle point ($\frac{1+\beta}{2}$) between the two (stable) non-affected solutions as noise intensity increases. This tends to balance the size ratio between the attraction basins and the probabilities of the two stable solutions. Then, when the noise intensity is very high $\alpha_{\mathrm{eff}} = \frac{1+\beta}{2}$, meaning that both stable solutions are equally probable. Again, the said middle point also coincides with the value that minimizes the multiplicative factor of noise.

Finally, we emphasize two issues. On the one hand, we have raised and systematized an analytic method (which has been checked numerically) for studying the effects of parameters fluctuations that characterize a given system, inspired by Ibañes et al. [10,11]. We have also observed that this method is reliable when the noise intensities are high enough. On the other hand, we have observed strong and interesting effects of fluctuations over the population's model studied.

Financial support from CONICET, Argentina; and UNMdP, Argentina, EXA603/12.

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