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Multilinear Cesàro maximal operators ☆

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ABSTRACT

The study of the almost everywhere convergence of the product of m Cesàro-α averages leads to the characterization of the boundedness of the associated multi(sub)linear maximal operator. We characterize weighted weak type and strong type inequalities for this operator, extending results by Lerner et al. [A. Lerner, S. Ombrosi, C. Pérez, R. Torres, R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, Adv. Math. 220 (2009) 1222–1264.]. We also study the restricted weak type inequalities which are of particular interest in our case (they were not considered by Lerner et al.).

1. Introduction

Let us consider the dilated functions $\varphi_R(x) = \frac{1}{R^n}\varphi(\frac{x}{R})$, $R > 0$, of a nonnegative integrable function φ defined on $\mathbb{R}^n$ such that $\int \varphi = 1$. It is well known that the study of the a.e. convergence and the convergence in the L^p norm of the averages $P_R f = f * \varphi_R$ to f as $R \to 0$ is related to the behavior of the maximal operator $M_\varphi f(x) = \sup_{R>0} |f| * \varphi_R(x)$.

When $\varphi(x) = \varphi^\alpha(x) = C_{n,\alpha}(1-|x|_\infty)^\alpha \chi_{Q(0,1)}(x)$, where $x = (x_1, \ldots, x_n)$, $|x|_\infty = \max_{1\le i\le n} |x_i|$, $-1 < \alpha \le 0$, and $C_{n,\alpha}$ is such that $\int \varphi^\alpha = 1$, we have

$$P_R f(x) = P_R^\alpha f(x) = \frac{2^{n+\alpha} C_{n,\alpha}}{|Q(x,R)|^{1+\alpha/n}} \int_{Q(x,R)} f(y) d(y, \partial Q(x,R))^\alpha \, dy,$$

where $Q(x,R) = [x_1 - R, x_1 + R] \times \cdots \times [x_n - R, x_n + R]$, $d(y, \partial Q(x,R))$ is the distance in the infinity norm from y to the boundary of $Q(x,R)$ and $|E|$ is the Lebesgue measure of the set E. These averages are called Cesàro-α averages. The maximal operator associated to these averages is (essentially)

$$M_\alpha^c f(x) = \sup_{R>0} \frac{1}{|Q(x,R)|^{1+\alpha/n}} \int_{Q(x,R)} |f(y)| d(y, \partial Q(x,R))^\alpha \, dy.$$

If $\alpha = 0$, M_α^c is the centered Hardy–Littlewood maximal operator which is simply denoted by M^c. The non-centered Hardy–Littlewood maximal operator is $Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy$, where the supremum is taken over all the cubes Q

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such that $x \in Q$ (throughout the paper, a cube will be a cube with sides parallel to the axis). The non-centered version of the Cesàro-α maximal operator is given by

$$M_\alpha f(x) = \sup_{x\in Q} \frac{1}{|Q|^{1+\alpha/n}} \int_Q |f(y)| d(y, \partial Q)^\alpha \, dy.$$

The operators M and M^c are pointwise equivalent. Classical results assert that M and M^c are of weak type $(1, 1)$ and of strong type (p, p), $1 < p \le +\infty$, with respect to the Lebesgue measure. The characterizations of the boundedness of M^c (M) in weighted spaces are well-known (see [1] and [2]).

It follows from the results in [3] that M^c_α, $-1 < \alpha \le 0$, is of restricted weak type $(\frac{1}{1+\alpha}, \frac{1}{1+\alpha})$ and, consequently, it is of strong type (p, p) for $p > \frac{1}{1+\alpha}$ with respect to the Lebesgue measure. It is not clear if M_α and M^c_α are pointwise equivalent for $\alpha \neq 0$. Therefore, the boundedness of M_α cannot be obtained from the corresponding result for M^c_α. However, the weighted inequalities for M_α are equivalent to the corresponding one for M^c_α (see [4]). We shall state below the results obtained in [4]. In order to state them, we introduce definitions and notations.

A non-negative measurable function (a weight) w satisfies $A_{p,\alpha}$, $-1 < \alpha \le 0$, $1 < p < +\infty$, and we write $w \in A_{p,\alpha}$, if there exists $C > 0$ such that

$$A_{p,\alpha}: \quad \left(\int_Q w(y)\, dy\right)^{1/p} \left(\int_Q w^{1-p'}(y) d(y, \partial Q)^{\alpha p'} dy\right)^{1/p'} \le C|Q|^{1+\frac{\alpha}{n}},$$

for every cube Q, where p' is the conjugate exponent of p. Observe that $A_{p,0} = A_p$ is the Muckenhoupt A_p class. We recall that w satisfies A_1 if there exists $C > 0$ such that $Mw(x) \le Cw(x)$ for a.e. x. By definition, $A_{1,0}$ is A_1. It is clear that $A_{p,\alpha} \subset A_p$. Consequently, if $w \in A_{p,\alpha}$ and w is not the function zero a.e. then w and $w^{1-p'}$ are locally integrable and $0 < w, w^{1-p'} < \infty$ a.e. Other property is in the following theorem (Corollary 3.4 in [4]).

Theorem 1 *([4]). Let $-1 < \alpha \le 0$, $1 < p < +\infty$, and let w be a weight on $\mathbb{R}^n$. If w satisfies $A_{p,\alpha}$, then there exists $s \in (1, p)$ such that w satisfies $A_{\frac{p}{s},\alpha}$.*

The weighted weak L^p-norm of a measurable function f is defined by $\|f\|_{L^{p,\infty}(w)} = \sup_{\lambda>0} \lambda[w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\})]^{\frac{1}{p}}$ and $L^{p,\infty}(w) = \{f : \|f\|_{L^{p,\infty}(w)} < \infty\}$, where $w(E) = \int_E w$. The weighted L^p-norm of f is $\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f|^p w\right)^{\frac{1}{p}}$ and $L^p(w) = \{f : \|f\|_{L^p(w)} < \infty\}$. The following result (see [4]) characterizes the weighted weak and strong type inequalities for M^c_α and M_α.

Theorem 2 *([4]). Let w be a weight on $\mathbb{R}^n$, $1 < p < \infty$ and $-1 < \alpha \le 0$. Let S_α be either M_α or M^c_α. The following conditions are equivalent.*

(a) $w \in A_{p,\alpha}$.
(b) *There is $C > 0$ such that $\|S_\alpha f\|_{L^{p,\infty}(w)} \le C\|f\|_{L^p(w)}$ for all $f \in L^p(w)$.*
(c) *There is $C > 0$ such that $\|S_\alpha f\|_{L^p(w)} \le C\|f\|_{L^p(w)}$ for all $f \in L^p(w)$.*

From this result, it is easy to prove that for all $f \in L^p(w)$ with $w \in A_{p,\alpha}$, the averages $P^\alpha_R f$ converge to f a.e. and in the $L^p(w)$-norm as $R \to 0$.

The restricted weak type inequalities require other classes of weights. A weight w satisfies $RA_{p,\alpha}$, $1 \le p < +\infty$, if there exists $C > 0$ such that

$$RA_{p,\alpha}: \left(\int_Q w\right)^{\frac{1}{p}} \left(\int_Q \chi_E(y) d(y, \partial Q)^\alpha dy\right) \le C|Q|^{1+\frac{\alpha}{n}} \left(\int_Q \chi_E w\right)^{\frac{1}{p}} \tag{3}$$

for all cubes Q and all measurable sets E. It was proved (see [5]) that $RA_{p,0} = RA_p$ characterizes the restricted weak type for M.

Theorem 4 *([4]). Let w be a weight on $\mathbb{R}^n$, $1 \le p < \infty$ and $-1 < \alpha \le 0$. Let S_α be either M_α or M^c_α. The following conditions are equivalent.*

(a) $w \in RA_{p,\alpha}$.
(b) *There is $C > 0$ such that $\|S_\alpha \chi_E\|_{L^{p,\infty}(w)} \le C\|\chi_E\|_{L^p(w)} = (w(E))^{\frac{1}{p}}$ for every measurable set E.*

As before, it is easy to prove that if $w \in RA_{p,\alpha}$ then the averages $P^\alpha_R f$ converge to f almost everywhere for all $f \in L^{p,1}(w)$ as $R \to 0$, where $L^{p,1}(w) = \{f : \|f\|_{L^{p,1}(w)} = \frac{1}{p}\int_0^\infty \lambda^{1/p} w(\{x \in \mathbb{R}^n : |f(x)| > \lambda\}) \frac{1}{\lambda} d\lambda < \infty\}$.

Remark 5. We point out that we can only have non-trivial restricted weak-type inequalities if $p \ge 1/(1+\alpha)$ (the proof is as the proof in Theorem 8, the multilinear case, in Section 3). It follows from Theorem 1 that if there is a non-trivial function in $A_{p,\alpha}$, $\alpha \neq 0$, then $p > 1/(1+\alpha)$.

Let us fix a natural number $m > 1$. Let us take real numbers p_i and α_i, $i = 1, \dots, m$, $p_i > 1$ and $-1 < \alpha_i \le 0$. Let $f_i \in L^{p_i}(w_i)$ with $w_i \in A_{p_i,\alpha_i}$. Then, $\lim_{R\to 0} \prod_{i=1}^m P_R^{\alpha_i} f_i = \prod_{i=1}^m f_i$ a.e. and in the $L^p(\nu)$-norm, where $\nu = \prod_{i=1}^m w_i^{\frac{p}{p_i}}$ and $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$. The conditions $w_i \in A_{p_i,\alpha_i}$ are sufficient to obtain that $\prod_{i=1}^m M_{\alpha_i}$ satisfies the following (multilinear) weighted strong type inequality: $\left\| \prod_{i=1}^m M_{\alpha_i}(f_i) \right\|_{L^p(\nu)} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}$.

The maximal operator associated to $\prod_{i=1}^m P_R^{\alpha_i} f_i$ is, up to a constant,

$$\mathcal{M}_{\vec{\alpha}}^c(\vec{f})(x) = \sup_{R>0} \prod_{i=1}^m \frac{1}{|Q(x,R)|^{1+\frac{\alpha_i}{n}}} \int_{Q(x,R)} |f_i(y)|(dy, \partial Q(x,R))^{\alpha_i}\, dy,$$

where $\vec{f} = (f_1, \dots, f_m)$ and $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$. Since $\mathcal{M}_{\vec{\alpha}}^c(\vec{f}) \le \prod_{i=1}^m M_{\alpha_i}(f_i)$ it is reasonable to think that weaker conditions on the weights would imply the boundedness of $\mathcal{M}_{\vec{\alpha}}^c$ in the weighted space. In this way we reach to the aim of this article: to characterize the weights for which we have restricted weak type, weak type and strong type inequalities for $\mathcal{M}_{\vec{\alpha}}^c$. We follow the ideas in [6] where the authors studied the case $\vec{\alpha} = \vec{0} = (0, \dots, 0)$, that is, the multi(sub)linear version of the Hardy–Littlewood maximal operator $\mathcal{M}^c := \mathcal{M}_{\vec{0}}^c$. This operator is used in [6] "to obtain a precise control on multilinear singular integral operators of Calderón–Zygmund type". The results in [6] are used in [7] to study the convergence of ergodic multilinear averages. The study of $\mathcal{M}_{\vec{\alpha}}^c$ in this paper will be useful to study the Cesàro-α ergodic multilinear averages, extending the study in [7].

2. Statement of the main results

We collect in the next theorem some of the main results in [6]. The space $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ will be denoted by $\prod_{i=1}^m L^{p_i}(w_i)$.

Theorem 6 (*see [6]*)**.** *For $i = 1, \dots, m$, let $1 \le p_i < \infty$ and let p be such that $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}$. Let w_i be weights and $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}$. The following statements are equivalent.*

(i) *There is C such that $\|\mathcal{M}^c(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}$ $(\vec{f} \in \prod_{i=1}^m L^{p_i}(w_i))$.*

(ii) *$\vec{w} \in \mathcal{A}_{\vec{p}}$, that is, there exists $C > 0$ such that for all cubes Q*

$$\left(\frac{1}{|Q|}\int_Q \nu_{\vec{w}}\right)^{1/p} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\right)^{1/p_i'} \le C,$$

where $\left(\frac{1}{|Q|}\int_Q w_i^{1-p_i'}\right)^{1/p_i'}$ is understood in the case $p_i = 1$ as $(\operatorname{ess\,inf}_{x\in Q} w_i(x))^{-1}$.

If $p_i > 1$ for every i, then (i) *and* (ii) *are equivalent to the following.*

(iii) *There is C such that $\|\mathcal{M}^c(\vec{f})\|_{L^p(\nu_{\vec{w}})} \le C \prod_{i=1}^m \|f_i\|_{L^{p_i}(w_i)}$ $(\vec{f} \in \prod_{i=1}^m L^{p_i}(w_i))$.*

$\mathcal{A}_{\vec{p}}$ is related to Muckenhoupt A_p conditions in the following way (see [6]):

$$\vec{w} \in \mathcal{A}_{\vec{p}} \Leftrightarrow \nu_{\vec{w}} \in A_{mp} \quad \text{and} \quad w_i^{1-p_i'} \in A_{mp_i'}, \quad i = 1, \dots, m, \tag{7}$$

where $w_i^{1-p_i'} \in A_{mp_i'}$ in the case $p_i = 1$ is understood as $w_i^{1/m} \in A_1$.

Our aim is to extend this theorem to the Cesàro maximal operator $\mathcal{M}_{\vec{\alpha}}^c$. This extension is not straightforward. We explain in the next few lines one of the difficulties that appear in the study of $\mathcal{M}_{\vec{\alpha}}^c$.

The results in Theorem 6 are stated for the noncentered multilinear Hardy–Littlewood maximal operator $\mathcal{M}(\vec{f})(x) = \sup_{x\in Q} \prod_{i=1}^m \frac{1}{|Q|}\int_Q |f_i(y)|\,dy$. Since $\mathcal{M}$ and $\mathcal{M}^c$ are pointwise equivalent, it is clear that the boundedness in (i) and (iii) is equivalent to the corresponding ones for $\mathcal{M}$. The pointwise equivalence of $\mathcal{M}$ and $\mathcal{M}^c$ makes easier some parts of the proof. However that is not the situation when we work with the Cesàro maximal operators. Therefore, we have to work harder to obtain the condition on the weights from the boundedness of $\mathcal{M}_{\vec{\alpha}}^c$. For future reference, we define the non-centered multilinear version of the Cesàro maximal operator by

$$\mathcal{M}_{\vec{\alpha}}(\vec{f})(x) = \sup_{x\in Q} \prod_{i=1}^m \frac{1}{|Q|^{1+\frac{\alpha_i}{n}}} \int_Q |f_i(y)|(d(y, \partial Q))^{\alpha_i}\, dy.$$

The restricted weak-type inequalities are relevant for the case $\vec{\alpha} \ne \vec{0}$. These inequalities were not studied in [6] for $\vec{\alpha} = \vec{0}$ but the proof in this case is easy. Since the proof for arbitrary $\vec{\alpha}$ exemplifies the difference indicated above, we start our study of $\mathcal{M}_{\vec{\alpha}}$ and $\mathcal{M}_{\vec{\alpha}}^c$ characterizing the restricted weak-type weighted inequalities. To state our first main result we introduce some notation: given m measurable sets E_i, we denote $(\chi_{E_1}, \dots, \chi_{E_m})$ by $\vec{\chi}_{\vec{E}}$.

Theorem 8. *For each $i = 1, \ldots, m$, let $-1 < \alpha_i \leq 0$, $\bar{\alpha} = \alpha_1 + \cdots + \alpha_m$, $1 \leq p_i < \infty$ and let p be such that $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}$. Let w_i be weights and $\nu_{\vec{w}} = \prod_{i=1}^{m} w_i^{\frac{p}{p_i}}$. The following statements are equivalent.*

(i) *There is $C > 0$ such that $\|\mathcal{M}_{\bar{\alpha}}(\vec{\chi}_{\vec{E}})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{i=1}^{m} \|\chi_{E_i}\|_{L^{p_i}(w_i)}$ for all measurable sets E_i, $i = 1, \ldots, m$.*
(ii) *There is $C > 0$ such that $\|\mathcal{M}^c_{\bar{\alpha}}(\vec{\chi}_{\vec{E}})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{i=1}^{m} \|\chi_{E_i}\|_{L^{p_i}(w_i)}$ for all measurable sets E_i, $i = 1, \ldots, m$.*
(iii) *$\vec{w} \in RA_{\vec{p},\vec{\alpha}}$, that is, there is $C > 0$ such that*

$$\left(\int_Q \nu_{\vec{w}}\right)^{\frac{1}{p}} \prod_{i=1}^{m} \left(\frac{1}{|Q|^{1+\frac{\alpha_i}{n}}} \int_Q \chi_{E_i}(y) d(y, \partial Q)^{\alpha_i} dy \right) \leq C \prod_{i=1}^{m} \left(\int_Q \chi_{E_i} w_i \right)^{\frac{1}{p_i}}$$

for all cubes Q and all measurable sets E_i, $i = 1, \ldots, m$.

If $\nu_{\vec{w}}$ is not 0 a.e. and statement (iii) holds then $p_i \geq \frac{1}{1+\alpha_i}$ for all i.

As we have already said, this theorem is new even in the case $\vec{\alpha} = \vec{0}$. However, we remark that it is not very difficult to see that (i) $\Leftrightarrow$ (iii). The proof of (i) $\Rightarrow$ (iii) is straightforward. To prove (iii) $\Rightarrow$ (ii) we follow the ideas in [6] but we have to use that (iii) implies that $\nu_{\vec{w}}$ satisfies the doubling condition (this is not necessary when $\vec{\alpha} = \vec{0}$ because in this case it suffices to estimate $\mathcal{M}^c_{\vec{0}} = \mathcal{M}^c$). The stronger difficulty appears when $\vec{\alpha} \neq \vec{0}$ in the proof of (ii) $\Rightarrow$ (iii) since we cannot use that $\mathcal{M}_{\vec{\alpha}}$ and $\mathcal{M}^c_{\vec{\alpha}}$ are pointwise equivalent. In particular, as in [4], we establish the implication proving that (ii) implies certain conditions of one-sided nature.

Our next step is the characterization of the weak type inequalities.

Theorem 9. *For $i = 1, \ldots, m$, let $-1 < \alpha_i \leq 0$, $\bar{\alpha} = \alpha_1 + \cdots + \alpha_m$, $\widetilde{\alpha_i} = \bar{\alpha} - \alpha_i$, $\frac{1}{1+\alpha_i} \leq p_i < \infty$. Let p be such that $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}$. Let w_i be weights and $\nu_{\vec{w}} = \prod_{i=1}^{m} w_i^{p/p_i}$. The following statements are equivalent.*

(i) *There is C such that $\|\mathcal{M}_{\vec{\alpha}}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}$ ($f_i \in L^{p_i}(w_i)$).*
(ii) *There is C such that $\|\mathcal{M}^c_{\vec{\alpha}}(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}$ ($f_i \in L^{p_i}(w_i)$).*
(iii) *$\vec{w} \in \mathcal{A}_{\vec{p},\vec{\alpha}}$, that is, there exists $C > 0$ such that for all cubes Q*

$$\left(\frac{1}{|Q|} \int_Q \nu_{\vec{w}}\right)^{1/p} \prod_{i=1}^{m} \left(\frac{1}{|Q|^{1+\frac{\alpha_i p_i'}{n}}} \int_Q w_i(y)^{1-p_i'} d(y, \partial Q)^{\alpha_i p_i'} dy \right)^{1/p_i'} \leq C,$$

where $(|Q|^{-1-\frac{\alpha_i p_i'}{n}} \int_Q w_i(y)^{1-p_i'} d(y, \partial Q)^{\alpha_i p_i'} dy)^{1/p_i'}$ in the case $p_i = 1$ (consequently, $\alpha_i = 0$) is understood as $(\operatorname{essinf}_{x \in Q} w_i(x))^{-1}$.
(iv) *The following conditions hold.*
(a) *$\nu_{\vec{w}} \in A_{mp, \frac{\bar{\alpha}}{m}}$,*
(b) *$w_i^{1-p_i'} \in A_{mp_i', \frac{\widetilde{\alpha_i}}{m}}$, for all $i = 1, \ldots, m$ and*
(c) *$w_i^{\frac{1}{r_i}} \in A_{\frac{mp_i}{r_i}, \frac{\alpha_i}{m}}$, for all $i = 1, \ldots, m$, where $r_i = (m-1)p_i + 1$.*

If $\nu_{\vec{w}}$ is not 0 a.e. and (iii) holds then $p_i > \frac{1}{1+\alpha_i}$ for all i such that $\alpha_i \neq 0$.

To prove the theorem we follow the ideas in [6], except when we work with the centered multilinear maximal operator. To prove (ii) $\Rightarrow$ (iii) we have to show (ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii). As in the first theorem, for the implication (ii) $\Rightarrow$ (iv) we prove that (ii) implies conditions of one-sided nature.

When $\vec{\alpha} = \vec{0}$ then conditions (b) and (c) in statement (iv) are equivalent because, for $1 < q < \infty$, $u \in A_q \Leftrightarrow u^{1-q'} \in A_{q'}$. Therefore, when $\vec{\alpha} = \vec{0}$ the equivalence between (iii) and (iv) is nothing but the equivalence (7).

If $m = 1$ ($\vec{p} = p$, $\vec{w} = w$ and $\vec{\alpha} = \alpha$), the conditions in (iv) yield $\nu_{\vec{w}} = w \in A_{p,\alpha}$, $w^{1-p'} \in A_{p',0} = A_{p'}$ and $w \in A_{p,\alpha}$. Since $A_{p,\alpha} \subset A_p$, they are equivalent to $\nu_{\vec{w}} = w \in A_{p,\alpha}$ (statement (iv) is statement (a) in Theorem 2).

The following theorem shows that $\mathcal{A}_{\vec{p},\vec{\alpha}}$ also characterizes the weighted strong type inequalities for the multilinear maximal Cesàro operators.

Theorem 10. *For $i = 1, \ldots, m$, let $-1 < \alpha_i \leq 0$ and $\frac{1}{1+\alpha_i} < p_i < \infty$. Let p, w_i and $\nu_{\vec{w}}$ be as in Theorem 9. The following statements are equivalent.*

(i) *There is C such that $\|\mathcal{M}_{\vec{\alpha}}(\vec{f})\|_{L^{p}(\nu_{\vec{w}})} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}$ ($f_i \in L^{p_i}(w_i)$).*
(ii) *There is C such that $\|\mathcal{M}^c_{\vec{\alpha}}(\vec{f})\|_{L^{p}(\nu_{\vec{w}})} \leq C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(w_i)}$ ($f_i \in L^{p_i}(w_i)$).*
(iii) *$\vec{w} \in \mathcal{A}_{\vec{p},\vec{\alpha}}$.*

The main tool to prove this theorem is the characterization of the weak type and the following openness property of the condition $\mathcal{A}_{\vec{p},\vec{\alpha}}$.

Theorem 11. *For $i = 1, \dots, m$, let α_i, p_i, p, w_i and $\nu_{\vec{w}}$ be as in Theorem 10. If $\vec{w} \in \mathcal{A}_{\vec{p},\vec{\alpha}}$ then there exists $r > 1$ such that $\frac{p_i}{r} > 1$, $\beta_i = \alpha_i \frac{1+r(m-1)}{m} \in (-1, 0]$ for all $i = 1, \dots, m$, and $\vec{w} \in \mathcal{A}_{\frac{\vec{p}}{r},\vec{\beta}}$.*

Notice that if $\vec{q} = \frac{\vec{p}}{r}$ and q_i are the coordinates of $\vec{q}$ then $q_i < p_i$ for all i. If $m > 1$ then $\beta_i < \alpha_i$ and we have that $\vec{w} \in \mathcal{A}_{\vec{p},\vec{\alpha}} \Rightarrow \vec{w} \in \mathcal{A}_{\vec{q},\vec{\beta}}$ for some $\vec{q}$ and some $\vec{\beta}$ such that $q_i < p_i$ and $\beta_i < \alpha_i$; this is an improvement of the result in [6] in the case $\vec{\alpha} = \vec{0}$, where this implication was proved with $\vec{\beta} = \vec{0}$. We point out that our argument in the proof of Theorem 11 is not the same as the corresponding result in [6]. When $m = 1$, the above theorem gives the result proved in [4]: if $w \in A_{p,\alpha}$ then there exists $r > 1$ such that $w \in A_{\frac{p}{r},\alpha}$.

Throughout the paper, if $1 < p < \infty$ then p' is its conjugate exponent and the letter C is used to denote positive constants whose values may change from line to line.

3. Proof of Theorem 8

First of all we give a result about the weights w_i and $\nu_{\vec{w}}$.

Proposition 12. *For each $i = 1, \dots, m$, let p_i, p, w_i and $\nu_{\vec{w}}$ be as in Theorem 8. Then there exists $C > 0$ (we can take $C = 2^m$) such that for each measurable set E of finite measure we can choose measurable subsets $G_1, \dots, G_m \subset E$ such that $C|G_i| \geq |E|$, $i = 1, \dots, m$, and $\prod_{i=1}^m \left(\int_{G_i} w_i\right)^{\frac{1}{p_i}} \leq C \left(\int_E \nu_{\vec{w}}\right)^{\frac{1}{p}}$.*

Proof. For any measurable set F of finite measure and any $\lambda > 0$, let $A_{i,F}^{\lambda} := \{x \in F : w_i(x) \leq \lambda \nu_{\vec{w}}(x)\}$. Notice that there exists λ such that $|A_{i,F}^{\lambda}| > \frac{1}{2}|F|$.

Let us fix any measurable set E of finite measure. Let $F_1 = E$ and $\lambda_1 = \inf\{\lambda : |A_{1,F_1}^{\lambda}| > \frac{1}{2}|F_1|\}$. It is clear that $\lambda_1 > 0$. Let μ_1 be such that $\frac{\lambda_1}{2} < \mu_1 < \lambda_1$. Then $|A_{1,F_1}^{\lambda_1}| \geq \frac{1}{2}|F_1|$ and $|A_{1,F_1}^{\mu_1}| \leq \frac{1}{2}|F_1|$. Let $F_2 = F_1 \setminus A_{1,F_1}^{\mu_1}$. Clearly $|F_2| \geq \frac{1}{2}|F_1| = \frac{1}{2}|E|$. Let $\lambda_2 = \inf\{\lambda : |A_{2,F_2}^{\lambda}| > \frac{1}{2}|F_2|\}$ and let μ_2 be any number such that $\frac{\lambda_2}{2} < \mu_2 < \lambda_2$; then $|A_{2,F_2}^{\lambda_2}| \geq \frac{1}{2}|F_2| \geq \frac{1}{4}|E|$ and $|A_{2,F_2}^{\mu_2}| \leq \frac{1}{2}|F_2|$. We continue in this way choosing sets F_i, numbers λ_i and μ_i, $i = 1, \dots, m-1$, such that $|F_i| \geq \frac{1}{2^{i-1}}|E|$, $|A_{i,F_i}^{\lambda_i}| \geq \frac{1}{2^i}|E|$, $|A_{i,F_i}^{\mu_i}| \leq \frac{1}{2}|F_i|$ and $\frac{\lambda_i}{2} < \mu_i < \lambda_i$. Let $F_m = F_{m-1} \setminus A_{m-1,F_{m-1}}^{\mu_{m-1}}$. Notice that $|F_m| \geq \frac{1}{2}|F_{m-1}| \geq \frac{1}{2^m}|E|$. Let us take $G_i = A_{i,F_i}^{\lambda_i}$ for $i = 1, \dots, m-1$ and $G_m = F_m$. Clearly $|G_i| \geq \frac{1}{2^m}|E|$ for all $i = 1, \dots, m$. By the definition of the sets G_i we have

$$w_i(x) \leq \lambda_i \nu_{\vec{w}}(x), \quad x \in G_i,\ i = 1, \dots, m-1. \tag{13}$$

For G_m we have the following inequality:

$$w_m(x) < \prod_{i=1}^{m-1} (\mu_i)^{-\frac{p_m}{p_i}} \nu_{\vec{w}}(x), \quad x \in G_m. \tag{14}$$

In fact, since $F_m = \bigcap_{i=1}^{m-1} \left(E \setminus A_{i,F_i}^{\mu_i}\right)$ we have that $w_i(x) > \mu_i \nu_{\vec{w}}(x)$, for $x \in F_m$ and $i \neq m$. Consequently, $\nu_{\vec{w}} = \prod_{i=1}^m w_i^{\frac{p}{p_i}} > \prod_{i=1}^{m-1} (\mu_i)^{\frac{p}{p_i}} \nu_{\vec{w}}^{\frac{p}{p_1}+\dots+\frac{p}{p_{m-1}}} w_m^{\frac{p}{p_m}}$ and (14) follows. Putting together (13) and (14) we obtain

$$\begin{aligned}\prod_{i=1}^m \left(\int_{G_i} w_i\right)^{\frac{1}{p_i}} &\leq C \prod_{i=1}^{m-1} \left(\frac{\lambda_i}{\mu_i}\right)^{\frac{1}{p_i}} \prod_{i=1}^m \left(\int_{G_i} \nu_{\vec{w}}\right)^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p}-\frac{1}{p_m}} \left(\int_E \nu_{\vec{w}}\right)^{\frac{1}{p}} \leq 2^{m-1} \left(\int_E \nu_{\vec{w}}\right)^{\frac{1}{p}}. \quad \square\end{aligned}$$

To prove (ii) $\Rightarrow$ (iii) in Theorem 8, we introduce some classes of weights.

Definition 15. For each $i = 1, \dots, m$, let α_i, $\bar{\alpha}$, p_i, p, w_i and $\nu_{\vec{w}}$ be as in Theorem 8. It is said that $\vec{w}$ satisfies $RA_{\vec{p},\vec{\alpha},k}^{-}$, $k = 1, \dots, n$, if there exists $C > 0$ such that

$$RA_{\vec{p},\vec{\alpha},k}^{-} : \left(\int_{U_k} \nu_{\vec{w}}\right)^{\frac{1}{p}} \prod_{i=1}^m \left(\int_{V_k \cap E_i} d(y, \partial Q)^{\alpha_i}\, dy\right) \leq C|Q|^{m+\frac{\bar{\alpha}}{n}} \prod_{i=1}^m \left(\int_{V_k \cap E_i} w_i\right)^{\frac{1}{p_i}} \tag{16}$$

for all cubes $Q = Q(z, R)$, $z = (z_1, \dots, z_m)$, and all measurable sets E_i, $i = 1, \dots, m$, where

$$U_k = \mathcal{K}_k(z, R) \cap \{y : y_k \geq z_k\}, \qquad V_k = \mathcal{K}_k(z, R) \cap \{y : y_k \leq z_k\} \tag{17}$$

and $\mathcal{K}_k(z, R) = \{y \in Q(z, R) : |y_j - z_j| \leq |y_k - z_k|, j = 1, \dots, n\}$.

The class $RA^{+}_{\vec{p},\vec{\alpha},k}$ is defined analogously changing the roles of U_k and V_k.

Notice that if $m = 1$ then $RA_{\vec{p},\vec{0}}$ coincides with the class $RA_p := RA_{p,0}$ introduced in (3). If $m = 1$ then the classes $RA^{\pm}_{\vec{p},\vec{0},k}$ will be simply denoted by $RA^{\pm}_{p,k}$. We notice that it follows from the results in [4] that

$$RA_p = \bigcap_{k=1}^{n} (RA^{+}_{p,k} \cap RA^{-}_{p,k}), \quad p \geq 1. \tag{18}$$

Taking into account these notations, we have the following result.

Proposition 19. *Let p_i, p, w_i, and $\nu_{\vec{w}}$ be as in Definition 15. Let $\widetilde{RA}_{\vec{p},\vec{\alpha}} := \bigcap_{k=1}^{n}(RA^{-}_{\vec{p},\vec{\alpha},k} \cap RA^{+}_{\vec{p},\vec{\alpha},k})$. If $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{\alpha}}$ then $\nu_{\vec{w}} \in RA_{pm}$.*

Proof. Let Q be any cube and let E be any measurable set. Assume that $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{\alpha}}$. Clearly $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{0}}$. Therefore, for $k = 1, \dots, n$, and for each family of measurable sets E_i, $i = 1, \dots, m$, we have that

$$\left(\int_{U_k} \nu_{\vec{w}}\right)^{\frac{1}{p}} \prod_{i=1}^{m} |E_i \cap V_k| \leq C|Q|^m \prod_{i=1}^{m} \left(\int_{V_k \cap E_i} w_i\right)^{\frac{1}{p_i}}, \tag{20}$$

$$\left(\int_{V_k} \nu_{\vec{w}}\right)^{\frac{1}{p}} \prod_{i=1}^{m} |E_i \cap U_k| \leq C|Q|^m \prod_{i=1}^{m} \left(\int_{U_k \cap E_i} w_i\right)^{\frac{1}{p_i}}. \tag{21}$$

By Proposition 12 with $E = E \cap V_k$ and taking $E_i = G_i$ in (20) we have that

$$\left(\int_{U_k} \nu_{\vec{w}}\right)^{\frac{1}{mp}} |E \cap V_k| \leq C|Q| \left(\int_{V_k \cap E} \nu_{\vec{w}}\right)^{\frac{1}{mp}},$$

for $k = 1, \dots, m$. In a similar way, Proposition 12 with $E = E \cap U_k$ and (21) give

$$\left(\int_{V_k} \nu_{\vec{w}}\right)^{\frac{1}{mp}} |E \cap U_k| \leq C|Q| \left(\int_{U_k \cap E} \nu_{\vec{w}}\right)^{\frac{1}{mp}}.$$

Therefore, $\nu_{\vec{w}} \in \bigcap_{k=1}^{n}(RA^{+}_{mp,k} \cap RA^{-}_{mp,k})$ and then, by (18), $\nu_{\vec{w}} \in RA_{mp}$. $\square$

Remark 22. It is clear that if $\nu_{\vec{w}} \in RA_{mp}$ then $\nu_{\vec{w}}$ is an A_∞ weight, so it is doubling. As a consequence, or taking $E = U_k$ and $E = V_k$ in the definition of RA_{mp}, we get that $\nu_{\vec{w}}(Q) \leq C\nu_{\vec{w}}(U_k)$ and $\nu_{\vec{w}}(Q) \leq C\nu_{\vec{w}}(V_k)$.

The following lemma shows the relationship between the one-sided conditions $RA^{+}_{\vec{p},\vec{\alpha},k}$ and $RA^{-}_{\vec{p},\vec{\alpha},k}$ with the general condition $RA_{\vec{p},\vec{\alpha}}$ (see (iii) in Theorem 8). This lemma is a key result in the proof of Theorem 8.

Lemma 23. *Let p_i, p, w_i, and $\nu_{\vec{w}}$ be as in Definition 15. Then $\widetilde{RA}_{\vec{p},\vec{\alpha}} = RA_{\vec{p},\vec{\alpha}}$.*

Proof. It is obvious that $RA_{\vec{p},\vec{\alpha}} \subset \widetilde{RA}_{\vec{p},\vec{\alpha}}$ because the sets $U_k, V_k \subset Q$. Let $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{\alpha}}$ and let us see that $\vec{w} \in RA_{\vec{p},\vec{\alpha}}$. Assume first that $\vec{\alpha}$ is equal to $(0, \dots, \alpha_j, \dots, 0)$, that is, all the numbers α_i are zero, except possibly the one in the place j. It follows from Proposition 19 and Remark 22 that there exists $C > 0$ such that for all cubes Q and every $k = 1, \dots, n$, we have $\nu_{\vec{w}}(Q) \leq C\nu_{\vec{w}}(U_k)$ and $\nu_{\vec{w}}(Q) \leq C\nu_{\vec{w}}(V_k)$, where U_k and V_k are the sets associated to the cube Q in Definition 15.

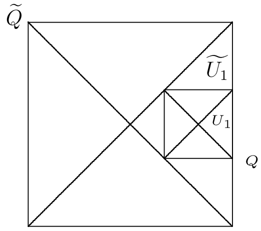
We have to prove that $\vec{w} \in RA_{\vec{p},\vec{\alpha}}$, that is, there exists a constant $C > 0$ such that for all cubes Q and all measurable sets E_i, $i = 1, \dots, m$,

$$\left(\int_Q \nu_{\vec{w}}\right)^{\frac{1}{p}} \left(\int_Q \chi_{E_j}(y) d(y, \partial Q)^{\alpha_j}\, dy\right) \prod_{i=1, i\neq j}^{m} |E_i \cap Q| \leq C|Q|^{m+\frac{\alpha_j}{n}} \prod_{i=1}^{m} \left(\int_{E_i} w_i\right)^{\frac{1}{p_i}}.$$

We estimate the left hand side in the following way:

$$\begin{aligned}\left(\int_Q \nu_{\vec{w}}\right)^{\frac{1}{p}} \left(\int_Q \chi_{E_j}(y) d(y, \partial Q)^{\alpha_j}\, dy\right) \prod_{i=1, i\neq j}^{m} |E_i \cap Q| &= \sum_{k=1}^{n} \left(\int_Q \nu_{\vec{w}}\right)^{\frac{1}{p}} \left(\int_{U_k} \chi_{E_j}(y) d(y, \partial Q)^{\alpha_j}\, dy\right) \prod_{i=1, i\neq j}^{m} |E_i \cap Q| \\ &+ \sum_{k=1}^{n} \left(\int_Q \nu_{\vec{w}}\right)^{\frac{1}{p}} \left(\int_{V_k} \chi_{E_j}(y) d(y, \partial Q)^{\alpha_j}\, dy\right) \prod_{i=1, i\neq j}^{m} |E_i \cap Q| = I + II.\end{aligned}$$

We shall only estimate I since II is estimated in a similar way. It will suffice to obtain the estimate for each term in the sum of I. We shall do it for $k = 1$ and without loss of generality we assume $j = 1$. Let $\widetilde{Q}$ be a cube such that for all $y \in U_1$, $d(y, \partial Q) = d(y, \partial\widetilde{Q})$, $|Q|$ is comparable to $|\widetilde{Q}|$ and $Q \subset \widetilde{U}_1$, where $\widetilde{U}_1$ is the set U_1 associated to $\widetilde{Q}$ (see the figure below).



Then, since $d(y, \partial Q) = d(y, \partial\widetilde{Q})$, for all $y \in U_1$,

$$\left(\int_Q \nu_{\vec{w}}\right)^{\frac{1}{p}} \left(\int_{U_1} \chi_{E_1}(y) d(y, \partial Q)^{\alpha_1} dy\right) \prod_{i=2}^m |E_i \cap Q| \le \left(\int_{\widetilde{Q}} \nu_{\vec{w}}\right)^{\frac{1}{p}} \left(\int_{\widetilde{U_1}} \chi_{E_1\cap Q}(y) d(y, \partial\widetilde{Q})^{\alpha_1} dy\right) \prod_{i=2}^m |E_i \cap Q|.$$

If $\widetilde{V_1}$ is the set V_1 associated to $\widetilde{Q}$ we have $\nu_{\vec{w}}(\widetilde{Q}) \le C\nu_{\vec{w}}(\widetilde{V_1})$ by the doubling property of $\nu_{\vec{w}}$. This inequality and the assumption $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{\alpha}}$ give

$$\begin{aligned}\left(\int_Q \nu_{\vec{w}}\right)^{\frac{1}{p}} \left(\int_{U_1} \chi_{E_1}(y) d(y, \partial Q)^{\alpha_1} dy\right) \prod_{i=2}^m |E_i \cap Q| &\le C \left(\int_{\widetilde{V_1}} \nu_{\vec{w}}\right)^{\frac{1}{p}} \left(\int_{\widetilde{U_1}} \chi_{E_1\cap Q}(y) d(y, \partial\widetilde{Q})^{\alpha_1}\, dy\right) \prod_{i=2}^m |E_i \cap Q \cap \widetilde{U_1}| \\ &\le C|\widetilde{Q}|^{m+\frac{\alpha_1}{n}} \prod_{i=1}^m \left(\int_{\widetilde{U_1}} \chi_{E_i\cap Q} w_i\right)^{\frac{1}{p_i}} \le C|Q|^{m+\frac{\alpha_1}{n}} \prod_{i=1}^m \left(\int_Q \chi_{E_i} w_i\right)^{\frac{1}{p_i}}.\end{aligned}$$

This proves the lemma in the particular case $\vec{\alpha} = (0, \dots, \alpha_j, \dots, 0)$.

We prove now the general case. Let us fix a cube Q and sets $E_1, \dots, E_m$. By Proposition 12 applied to the set $E = Q$ there exist subsets $G_1, \dots, G_m$ such that

$$C|G_i| \ge |Q|, \quad i = 1, \dots, m, \quad \text{and} \quad \prod_{i=1}^m \left(\int_{G_i} w_i\right)^{\frac{1}{p_i}} \le C \left(\int_Q \nu_{\vec{w}}\right)^{\frac{1}{p}}. \tag{24}$$

Since $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{\alpha}}$ then $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{\alpha}_j}$, where $\vec{\alpha}_j = (0, \dots, \alpha_j, \dots, 0)$ for all j. By what we have already proved we have that $\vec{w} \in RA_{\vec{p},\vec{\alpha}_j}$. For fixed j, we apply that $\vec{w} \in RA_{\vec{p},\vec{\alpha}_j}$ in the cube Q and with the sets $E_i^j = G_i$ if $i \ne j$ and $E_j^j = E_j$. Then we obtain

$$\nu_{\vec{w}}(Q)^{\frac{1}{p}} \left(\int_Q \chi_{E_j}(y) d(y, \partial Q)^{\alpha_j}\, dy\right) \prod_{i=1, i\ne j}^m |G_i \cap Q| \le C|Q|^{m+\frac{\alpha_j}{n}} \left(\int_{E_j} w_j\right)^{\frac{1}{p_j}} \prod_{i=1,i\ne j}^m \left(\int_{G_i} w_i\right)^{\frac{1}{p_i}},$$

for some constant independent of j and the sets. Multiplying these inequalities on $j = 1, \dots, m$ we have

$$\nu_{\vec{w}}(Q)^{\frac{m}{p}} \left(\prod_{j=1}^m \int_Q \chi_{E_j}(y) d(y, \partial Q)^{\alpha_j} dy\right) \left(\prod_{i=1}^m |G_i|\right)^{m-1} \le C|Q|^{m^2+\frac{\bar{\alpha}}{n}} \prod_{j=1}^m \left(\int_{E_j} w_j\right)^{\frac{1}{p_j}} \prod_{i=1}^m \left(\int_{G_i} w_i\right)^{\frac{m-1}{p_i}}.$$

Then, the lemma follows from (24) and the last inequality. $\square$

Now we are ready to prove the characterization of the restricted weak type weighted inequalities for $\mathcal{M}_{\vec{\alpha}}^c$.

Proof of Theorem 8. The implication (i) $\Rightarrow$ (ii) is obvious. To prove (ii) $\Rightarrow$ (iii) it will suffice to see that $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{\alpha}} = \bigcap_{k=1}^n \left(RA^-_{\vec{p},\vec{\alpha},k} \cap RA^+_{\vec{p},\vec{\alpha},k}\right)$.

Assume that (ii) holds and consider for each $k = 1, \dots, n$, the following non-centered multi(sub)linear maximal operators

$$\mathcal{N}^-_{\vec{\alpha},k}\vec{f}(x) = \sup_{x\in U_k(z,R)} \prod_{i=1}^m \frac{1}{|Q(z,R)|^{1+\frac{\alpha_i}{n}}} \int_{V_k(z,R)} |f_i(y)| d(y, \partial Q)^{\alpha_i}\, dy \tag{25}$$

and

$$\mathcal{N}^+_{\vec{\alpha},k}\vec{f}(x) = \sup_{x\in V_k(z,R)} \prod_{i=1}^m \frac{1}{|Q(z,R)|^{1+\frac{\alpha_i}{n}}} \int_{U_k(z,R)} |f_i(y)| d(y, \partial Q)^{\alpha_i}\, dy, \tag{26}$$

where $U_k = U_k(z, R)$ and $V_k = V_k(z, R)$ are the sets considered in Definition 15 associated to the cube $Q = Q(z, R)$. It is easy to see that $\mathcal{N}^-_{\vec{\alpha},k}\vec{f}(x) \le C\mathcal{M}^c_{\vec{\alpha}}\vec{f}(x)$ and $\mathcal{N}^+_{\vec{\alpha},k}\vec{f}(x) \le C\mathcal{M}^c_{\vec{\alpha}}\vec{f}(x)$, for all $k = 1, \dots, n$ and all measurable function f (see the proof of Proposition 2.3 in [4]). Then we get the inequality in (ii) for the operators $\mathcal{N}^-_{\vec{\alpha},k}$ and $\mathcal{N}^+_{\vec{\alpha},k}$. In order to see that $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{\alpha}}$, let $Q = Q(z, R)$ be a fixed cube, and let $U_k = U_k(z, R)$ and $V_k = V_k(z, R)$, $k = 1, \dots, n$, its associated sets considered in Definition 15. Let $E_1, \dots, E_m$ be measurable sets. For each k we consider the multilinear function $\vec{\chi}_{\vec{E}^k} = (\chi_{E_1^k}, \dots, \chi_{E_m^k})$ with $E_i^k = E_i \cap V_k$. For $x \in U_k$ we have $\mathcal{N}^-_{\vec{\alpha},k}\left(\vec{\chi}_{\vec{E}^k}\right)(x) \ge \prod_{i=1}^m \frac{1}{|Q|^{1+\frac{\alpha_i}{n}}}\int_{V_k}\chi_{E_i^k}(y)d(y,\partial Q)^{\alpha_i}dy = \lambda$. Then, using that $\mathcal{N}^-_{\vec{\alpha},k}$ satisfies (ii),

$$\nu_{\vec{w}}(U_k) \le \nu_{\vec{w}}\left(\{x : \mathcal{N}^-_{\vec{\alpha},k}\left(\vec{\chi}_{\vec{E}^k}\right)(x) \ge \lambda\}\right) \le \frac{C}{\lambda^p}\prod_{i=1}^m \|\chi_{E_i^k}\|^p_{p_i,w_i}.$$

Therefore

$$\left(\int_{U_k}\nu_{\vec{w}}\right)^{\frac{1}{p}}\prod_{i=1}^m\left(\int_{V_k}\chi_{E_i^k}(y)d(y,\partial Q)^{\alpha_i}dy\right) \le C|Q|^{m+\frac{\bar{\alpha}}{n}}\prod_{i=1}^m\left(\int_{V_k}\chi_{E_i}w_i\right)^{\frac{1}{p_i}}.$$

Then we get that $\vec{w} \in RA^-_{\vec{p},\vec{\alpha},k}$. In a similar way, using that $\mathcal{N}^+_{\vec{\alpha},k}$ satisfies (ii) we obtain that $\vec{w} \in RA^+_{\vec{p},\vec{\alpha},k}$. Thus $\vec{w} \in \widetilde{RA}_{\vec{p},\vec{\alpha}}$ as we wished to prove.

Finally, we prove (iii) $\Rightarrow$ (i). Using the assumption (iii) we get, for every cube Q, that

$$\begin{aligned}\prod_{i=1}^m \frac{1}{|Q|^{1+\frac{\alpha_i}{n}}}\int_Q \chi_{E_i}(y)d(y,\partial Q)^{\alpha_i}dy &\le C\prod_{i=1}^m\left(\int_Q\chi_{E_i}w_i\right)^{1/p_i}\frac{1}{\left(\int_Q\nu_{\vec{w}}\right)^{1/p}}\\ &= \prod_{i=1}^m\left(\frac{1}{\nu_{\vec{w}}(Q)}\int_Q\chi_{E_i}\frac{w_i}{\nu_{\vec{w}}}\nu_{\vec{w}}\right)^{1/p_i} \le \prod_{i=1}^m\left(M_{\nu_{\vec{w}}}(\chi_{E_i}w_i\nu_{\vec{w}}^{-1})(x)\right)^{1/p_i},\end{aligned}$$

where $M_{\nu_{\vec{w}}}$ is the Hardy–Littlewood maximal operator associated to the measure $\nu_{\vec{w}}(x)dx$ defined as $M_{\nu_{\vec{w}}}f(x) = \sup_{x\in Q}\frac{1}{\nu_{\vec{w}}(Q)}\int_Q|f(y)|\nu_{\vec{w}}(y)dy$.

By Proposition 19, Lemma 23 and Remark 22 we have that $\nu_{\vec{w}}$ is a doubling weight. Then $M_{\nu_{\vec{w}}}$ is of weak type $(1,1)$ with respect to $\nu_{\vec{w}}$. It follows from the last inequality, Hölder's inequality for weak-spaces [8, p.15] and the weak-type $(1,1)$ inequality for $M_{\nu_{\vec{w}}}$ that

$$\begin{aligned}\|\mathcal{M}_{\vec{\alpha}}(\vec{\chi}_{\vec{E}})\|_{L^{p,\infty}(\nu_{\vec{w}})} &\le C\prod_{i=1}^m\left\|\left[M_{\nu_{\vec{w}}}(\chi_{E_i}w_i\nu_{\vec{w}}^{-1})\right]^{1/p_i}\right\|_{L^{p_i,\infty}(\nu_{\vec{w}})}\\ &= C\prod_{i=1}^m\|M_{\nu_{\vec{w}}}(\chi_{E_i}w_i\nu_{\vec{w}}^{-1})\|^{1/p_i}_{L^{1,\infty}(\nu_{\vec{w}})} \le C\prod_{i=1}^m\|\chi_{E_i}\|_{L^{p_i}(w_i)},\end{aligned}$$

and we are done.

Finally, we are going to prove that if $\vec{w}$ is not the function zero a.e. and $\vec{w} \in RA_{\vec{p},\vec{\alpha}}$ then $p_i \ge 1/(1+\alpha_i)$. We shall prove it for $i = 1$. Since $\vec{w} \in RA_{\vec{p},\vec{\alpha}}$ implies that $\vec{w} \in RA_{\vec{p},\vec{\beta}}$ where $\beta = (\alpha_1, 0, \dots, 0)$, we may assume that $\alpha_i = 0$ for all $i \ge 2$. There exists N such that the sets $F_{i,N} = \{x : w_i(x) < N\}$ have positive measure for all i. Let z be a Lebesgue point of the locally integrable functions $\nu_{\vec{w}}$, $w_i\chi_{F_{i,N}}$ and $\chi_{F_{i,N}}$, $i = 1, \dots, m$, such that $\nu_{\vec{w}}(z) \ne 0$. Without loss of generality, we may assume that $z = 0$. Let $Q = Q(0, R)$ and $E_\lambda = Q(0, (1-\lambda)R) \setminus Q(0, (1-2\lambda)R)$, $0 < \lambda < 1/4$. Notice that if $x \in E_\lambda$ then $d(x,\partial Q)^{\alpha_1} \ge (2\lambda R)^{\alpha_1} = \lambda^{\alpha_1}|Q|^{\alpha_1/n}$ and $C_1\lambda \le \frac{|E_\lambda|}{|Q|} \le C_2\lambda$ (C_1 and C_2 depend on n but they are independent of λ). Now, keeping in mind these inequalities, we apply that $\vec{w} \in RA_{\vec{p},\vec{\alpha}}$ with the cube Q and the sets $E_1 = E_\lambda \cap F_{1,N}$ and $E_i = F_{i,N}$, $i \ge 2$. Then we obtain

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q\nu_{\vec{w}}\right)^{\frac{1}{p}}\lambda^{1+\alpha_1}|Q|^{\frac{1}{p}}\frac{|E_\lambda\cap F_{1,N}|}{|E_\lambda|}\prod_{i=2}^m\frac{|F_{i,N}\cap Q|}{|Q|}\\ &\le C(\lambda|Q|)^{\frac{1}{p_1}}\left(\frac{1}{|E_\lambda|}\int_{E_\lambda}\chi_{F_{1,N}}w_1\right)^{\frac{1}{p_1}}\prod_{i=2}^m\left(\frac{1}{|Q|}\int_Q\chi_{F_{i,N}}w_i\right)^{\frac{1}{p_i}}|Q|^{\frac{1}{p}-\frac{1}{p_1}}.\end{aligned}$$

Notice that, for fixed λ, the family $E_\lambda = E_\lambda(R)$ is a regular family which shrinks nicely to 0. If we let R tend to 0 we obtain $\lambda^{1+\alpha_1-\frac{1}{p_1}} \le C$. Since λ can be chosen so small as we wish, we have $1+\alpha_1-\frac{1}{p_1} \ge 0$. $\square$

4. Proof of Theorem 9

If $\alpha_i = 0$ for all i then our theorem is Theorem 6 (proved in [6]). From now on, we assume $\alpha_i \neq 0$ for some i which implies $p_i > 1$ and $pm > 1$. We study only the case $m > 1$ since the case $m = 1$ is contained in Theorem 2.

To prove (ii) $\Rightarrow$ (iii), we work with the one-sided multi(sub)linear maximal operators $\mathcal{N}^-_{\vec{\alpha},k}$ and $\mathcal{N}^+_{\vec{\alpha},k}$ defined in (25) and (26). We also need to introduce the following multilinear classes of weights: for $k = 1, \ldots, n$, it is said that $\vec{w} = (w_1, \ldots, w_m)$ satisfies $\mathcal{A}^-_{\vec{p},\vec{\alpha},k}$ if there is $C > 0$ such that

$$\mathcal{A}^-_{\vec{p},\vec{\alpha},k} : \left(\frac{1}{|Q|}\int_{U_k} \nu_{\vec{w}}\right)^{1/p} \prod_{i=1}^{m} \left(\frac{1}{|Q|^{1+\frac{\alpha_i p_i'}{n}}}\int_{V_k} w_i(y)^{1-p_i'} d(y,\partial Q)^{\alpha_i p_i'} dy\right)^{1/p_i'} \leq C \tag{27}$$

for all cubes $Q = Q(z, R)$, where U_k and V_k are the sets defined in (17) (if $p_i = 1$ then $(|Q|^{-1-\frac{\alpha_i p_i'}{n}} \int_{V_k} w_i(y)^{1-p_i'} d(y,\partial Q)^{\alpha_i p_i'} dy)^{1/p_i'}$ is understood as $(\operatorname{essinf}_{V_k} w_i)^{-1}$). The class $\mathcal{A}^+_{\vec{p},\vec{\alpha},k}$ is defined changing the roles of U_k and V_k.

The following theorem contains the proof of (iii) $\Leftrightarrow$ (iv) and shows the relationship between the multilinear classes of weights with a family of suitable linear conditions.

Theorem 28. *For $i = 1, \ldots, m$, let α_i, $\bar{\alpha}$, $\widetilde{\alpha}_i$, p_i, w_i and $\nu_{\vec{w}}$ be as in Theorem 9 and assume that $\alpha_i \neq 0$ for some i. Let $\vec{w} = (w_1, \ldots, w_m)$. Let $\widetilde{\mathcal{A}}_{\vec{p},\vec{\alpha}} = \bigcap_{k=1}^{n}(\mathcal{A}^-_{\vec{p},\vec{\alpha},k} \cap \mathcal{A}^+_{\vec{p},\vec{\alpha},k})$. Then the following statements are equivalent.*

(i) $\vec{w} \in \widetilde{\mathcal{A}}_{\vec{p},\vec{\alpha}}$.

(ii) *The following conditions hold.*
   (a) $\nu_{\vec{w}} \in A_{mp,\frac{\bar{\alpha}}{m}}$,
   (b) $w_i^{1-p_i'} \in A_{mp_i',\frac{\widetilde{\alpha}_i}{m}}$, *for all $i = 1, \ldots, m$ where in the case $p_i = 1$ is understood as $w_i^{1/m} \in A_1$ and*
   (c) $w_i^{\frac{1}{r_i}} \in A_{\frac{mp_i}{r_i},\frac{\alpha_i}{m}}$, *for all $i = 1, \ldots, m$, with $r_i = (m-1)p_i + 1$.*

(iii) $\vec{w} \in \mathcal{A}_{\vec{p},\vec{\alpha}}$.

Proof of Theorem 28. In this proof we shall use the following lemma (see [4]). From now on, when $m = 1$ the classes $\mathcal{A}^-_{\vec{p},\vec{\alpha},k}$ and $\mathcal{A}^+_{\vec{p},\vec{\alpha},k}$ are denoted by $A^-_{p,\alpha,k}$ and $A^+_{p,\alpha,k}$, respectively.

Lemma 29 ([4]). *Let w be a weight on $\mathbb{R}^n$ and let $-1 < \alpha < 0$. If $1 < p < \infty$ then w satisfies $A_{p,\alpha}$ if and only if $w \in \bigcap_{k=1}^{n} \left(A^-_{p,\alpha,k} \bigcap A^+_{p,\alpha,k}\right)$.*

In the proof we consider the sets $J = \{i : p_i = 1\}$ and $H = \{i : p_i > 1\}$.

(i) $\Rightarrow$ (ii). Let us prove (a). Let $k \in \{1, \ldots, n\}$. It follows from the assumptions that $pm > 1$ since $p_i > 1$ for some i. By Hölder's inequality with exponents $s_i = \frac{p_i(mp-1)}{p(p_i-1)}$ if $i \in H$ and $s_i = +\infty$ if $i \in J$, we have that

$$\begin{aligned}\int_{V_k} \nu_{\vec{w}}(y)^{1-(mp)'} d(y,\partial Q)^{\frac{\bar{\alpha}}{m}(mp)'} dy &= \int_{V_k} \prod_{i=1}^{m} w_i(y)^{-\frac{p}{p_i(mp-1)}} d(y,\partial Q)^{\alpha_i \frac{p}{mp-1}} dy \\ &\leq C \prod_{i\in H}\left(\int_{V_k} w_i(y)^{1-p_i'} d(y,\partial Q)^{\alpha_i p_i'} dy\right)^{\frac{p(p_i-1)}{p_i(mp-1)}} \prod_{i\in J}\left(\operatorname{ess\,inf}_{y\in V_k} w_i(y)\right)^{-\frac{p}{mp-1}}.\end{aligned}$$

Since $\vec{w} \in \widetilde{\mathcal{A}}_{\vec{p},\vec{\alpha}} \subset \mathcal{A}^-_{\vec{p},\vec{\alpha},k}$, we get that $\nu_{\vec{w}} \in A^-_{mp,\frac{\bar{\alpha}}{m},k}$. Changing the role of the sets U_k and V_k, we also obtain that $\nu_{\vec{w}} \in A^+_{mp,\frac{\bar{\alpha}}{m},k}$ for all k. By Lemma 29 we get (a). Since $A_{p,\alpha} \subset A_p$ for any p and α, then (a) implies that $\nu_{\vec{w}}$ is doubling.

Assume $p_i > 1$. To prove (b), it suffices to show that for all k

$$\left(\int_{E_k} w_i^{1-p_i'}\right)^{\frac{1}{mp_i'}} \left(\int_{F_k}\left(w_i(y)^{1-p_i'}\right)^{-\frac{1}{mp_i'-1}} d(y,\partial Q)^{\frac{\widetilde{\alpha}_i}{m}(mp_i')'} dy\right)^{1-\frac{1}{mp_i'}} \leq C|Q|^{1+\frac{\widetilde{\alpha}_i}{mn}}, \tag{30}$$

with $E_k = U_k, V_k$ and $F_k = U_k, V_k$. These inequalities imply that $w_i^{1-p_i'} \in A^-_{mp_i',\frac{\widetilde{\alpha}_i}{m},k}$ and $w_i^{1-p_i'} \in A^+_{mp_i',\frac{\widetilde{\alpha}_i}{m},k}$ for all k; consequently, by Lemma 29 we have $w_i^{1-p_i'} \in A_{mp_i',\frac{\widetilde{\alpha}_i}{m}}$. We shall prove (30) with $(E_k, F_k) = (U_k, U_k)$, being similar the proof of the case $(E_k, F_k) = (V_k, V_k)$. Once these inequalities have been proved, we observe that (30) with $(E_k, F_k) = (U_k, U_k)$ implies the same inequality for $\widetilde{\alpha}_i = 0$ which is equivalent to $w_i^{1-p_i'} \in A_{mp_i'}$. Therefore, $w_i^{1-p_i'}$ is doubling. In particular, $\int_{U_k} w_i^{1-p_i'} \leq C\int_{V_k} w_i^{1-p_i'}$

and $\int_{V_k} w_i^{1-p_i'} \le C \int_{U_k} w_i^{1-p_i'}$. These inequalities together with (30) in the cases $(E_k, F_k) = (U_k, U_k)$ and $(E_k, F_k) = (V_k, V_k)$ imply (30) with $(E_k, F_k) = (U_k, V_k)$ and $(E_k, F_k) = (V_k, U_k)$.

Now we prove (30) in the case $(E_k, F_k) = (U_k, U_k)$. Observe that

$$\left(\int_{U_k} w_i^{1-p_i'}\right)^{\frac{1}{mp_i'}} \le C|Q|^{-\frac{\alpha_i}{nm}} \left(\int_{U_k} w_i(y)^{1-p_i'} d(y, \partial Q)^{\alpha_i p_i'} dy\right)^{\frac{1}{mp_i'}}. \tag{31}$$

On the other hand, with $r_i = (m-1)p_i + 1$, we write

$$\begin{aligned} I_i &= \left(\int_{U_k} \left(w_i(y)^{1-p_i'}\right)^{-\frac{1}{mp_i'-1}} d(y, \partial Q)^{\frac{\widetilde{\alpha}_i}{m}(mp_i')'} dy\right)^{1-\frac{1}{mp_i'}} \\ &= \left(\int_{U_k} w_i(y)^{\frac{1}{r_i}} \prod_{j=1, j\neq i}^{m} w_j(y)^{\frac{p_i}{p_j r_i}} w_j(y)^{-\frac{p_i}{p_j r_i}} \prod_{j=1, j\neq i}^{m} d(y, \partial Q)^{\frac{\alpha_j p_i}{r_i}} dy\right)^{\frac{r_i}{mp_i}}. \end{aligned} \tag{32}$$

By Hölder's inequality with $s_i = \frac{pr_i}{p_i}$ and $s_j = \frac{p_j r_i}{(p_j-1)p_i}$ for $j \in H, j \neq i$, and $s_j = +\infty$ for $j \in J$, we get that

$$\begin{aligned} I_i &\le \left[\int_{U_k} \left(w_i(y)^{\frac{1}{r_i}} \prod_{j=1, j\neq i}^{m} w_j(y)^{\frac{p_i}{p_j r_i}}\right)^{s_i} dy\right]^{\frac{r_i}{s_i m p_i}} \\ &\quad \times \prod_{j\in H, j\neq i} \left[\int_{U_k} \left(w_j(y)^{-\frac{p_i}{p_j r_i}} d(y, \partial Q)^{\alpha_j \frac{p_i}{r_i}}\right)^{s_j} dy\right]^{\frac{r_i}{s_j m p_i}} \prod_{j\in J} \left[\operatorname{ess\,inf}_{y\in U_k} w_j(y)\right]^{-\frac{1}{m}}. \end{aligned}$$

Then, since $\nu_{\vec{w}}$ is a doubling weight, we have that

$$I_i \le C \left(\int_{V_k} \nu_{\vec{w}}\right)^{\frac{1}{mp}} \prod_{j\in H, j\neq i}^{m} \left(\int_{U_k} w_j^{1-p_j'}(y) d(y, \partial Q)^{\alpha_j p_j'} dy\right)^{\frac{1}{mp_j'}} \prod_{j\in J} \left[\operatorname{ess\,inf}_{y\in U_k} w_j(y)\right]^{-\frac{1}{m}}. \tag{33}$$

Putting together (31), (33) and using (i) we get (30) with $(E_k, F_k) = (U_k, U_k)$.

To prove (c), we argue as before and, consequently, we only have to prove

$$\left(\int_{U_k} w_i^{\frac{1}{r_i}}\right)^{\frac{r_i}{mp_i}} \left(\int_{U_k} \left(w_i(y)^{\frac{1}{r_i}}\right)^{-\frac{1}{\frac{mp_i}{r_i}-1}} d(y, \partial Q)^{\frac{\alpha_i}{m}(\frac{mp_i}{r_i})'} dy\right)^{\frac{1}{(\frac{mp_i}{r_i})'}} \le C|Q|^{1+\frac{\alpha_i}{nm}}. \tag{34}$$

To prove it we use (33) (see the definition of I_i in (32)) and we obtain

$$\begin{aligned} \left(\int_{U_k} w_i^{\frac{1}{r_i}}\right)^{\frac{r_i}{mp_i}} &\le |Q|^{-\frac{\widetilde{\alpha}_i}{nm}} \left(\int_{U_k} w_i(y)^{\frac{1}{r_i}} d(y, \partial Q)^{\frac{\widetilde{\alpha}_i p_i}{r_i}} dy\right)^{\frac{r_i}{mp_i}} = |Q|^{-\frac{\widetilde{\alpha}_i}{nm}} I_i \\ &\le C|Q|^{-\frac{\widetilde{\alpha}_i}{nm}} \left(\int_{V_k} \nu_{\vec{w}}\right)^{\frac{1}{mp}} \prod_{j\in H, j\neq i}^{m} \left(\int_{U_k} w_j(y)^{1-p_j'} d(y, \partial Q)^{\alpha_j p_j'} dy\right)^{\frac{1}{mp_j'}} \\ &\quad \times \prod_{j\in J} \left[\operatorname{ess\,inf}_{y\in U_k} w_j(y)\right]^{-\frac{1}{m}}. \end{aligned}$$

Then, using that $\vec{w} \in \widetilde{\mathcal{A}}_{\vec{p},\vec{\alpha}} \subset \mathcal{A}^{+}_{\vec{p},\vec{\alpha},k}$, we obtain (34) as follows:

$$\begin{aligned} &\left(\int_{U_k} w_i^{\frac{1}{r_i}}\right)^{\frac{r_i}{mp_i}} \left(\int_{U_k} \left(w_i(y)^{\frac{1}{r_i}}\right)^{-\frac{1}{\frac{mp_i}{r_i}-1}} d(y, \partial Q)^{\frac{\alpha_i}{m}(\frac{mp_i}{r_i})'} dy\right)^{\frac{1}{(\frac{mp_i}{r_i})'}} \\ &\le C|Q|^{-\frac{\widetilde{\alpha}_i}{nm}} \left(\int_{U_k} \nu_{\vec{w}}\right)^{\frac{1}{mp}} \prod_{j\in H} \left(\int_{U_k} w_j(y)^{1-p_j'} d(y, \partial Q)^{\alpha_j p_j'} dy\right)^{\frac{1}{mp_j'}} \\ &\quad \times \prod_{j\in J} \left[\operatorname{ess\,inf}_{y\in U_k} w_j(y)\right]^{-\frac{1}{m}} \le C|Q|^{-\frac{\widetilde{\alpha}_i}{nm}} |Q|^{1+\frac{\bar{\alpha}}{nm}} = C|Q|^{1+\frac{\alpha_i}{nm}}. \end{aligned}$$

Now we prove (b) and (c) for $i \in J$, that is $w_i^{1/m} \in A_1$. It suffices to prove that for all cubes Q and some k

$$\frac{1}{|Q|}\int_{U_k} w_i^{1/m} \le C \operatorname{ess\,inf}_{U_k} w_i^{1/m}. \tag{35}$$

Inequality (35) implies that $w_i^{1/m} \in A_1$ as follows: as in the proof of Lemma 23, given a cube Q there exists a cube $\widetilde{Q}$ such that $|Q|$ is comparable to $|\widetilde{Q}|$ and $Q \subset \widetilde{U_k}$, where $\widetilde{U_k}$ is the set U_k associated to $\widetilde{Q}$; by (35)

$$\frac{1}{|Q|}\int_{Q} w_i^{1/m} \le \tfrac{C}{|\widetilde{Q}|}\int_{\widetilde{U_k}} w_i^{1/m} \le C \operatorname{ess\,inf}_{\widetilde{U_k}} w_i^{1/m} \le C \operatorname{ess\,inf}_{Q} w_i^{1/m}.$$

Let us prove (35). By Hölder's inequality with exponent pm, we have

$$\int_{U_k} w_i^{1/m} \le \left(\int_{U_k} w_i^p \prod_{j\in H} w_j^{\frac{p}{p_j}}\right)^{\frac{1}{mp}} \left(\int_{U_k} \prod_{j\in H} w_j^{-\frac{p}{p_j(pm-1)}}\right)^{\frac{pm-1}{mp}}. \tag{36}$$

Let $s_j = (m-1/p)p_j'$, $j \in H$. Assume first that $s_j > 1$ for all $j \in H$. Applying Hölder's inequality with exponents s_j and the inequality $d(y,\partial Q)^{-\alpha_j p_j'} \le |Q|^{-\frac{\alpha_j p_j'}{n}}$, we obtain

$$\int_{U_k} w_i^{1/m} \le \left(\int_{U_k} w_i^p \prod_{j\in H} w_j^{\frac{p}{p_j}}\right)^{\frac{1}{mp}} \prod_{j\in H}\left(\int_{U_k} w_j^{1-p_j'}(y) d(y,\partial Q)^{\alpha_j p_j'}\, dy\right)^{\frac{1}{mp_j'}} |Q|^{-\frac{\alpha_j}{mn}}. \tag{37}$$

If $s_{j_0} = 1$ for some j_0 then $p_i = 1$ for all $i \ne j_0$. Consequently, $H = \{j_0\}$ and inequality (37) follows from (36) since $d(y,\partial Q)^{-\alpha_j p_j'} \le |Q|^{-\frac{\alpha_j p_j'}{n}}$ (in this case we do not need to use Hölder's inequality). By (i) and using that $\nu_{\vec{w}}$ is doubling, we obtain (35) as follows:

$$\begin{aligned}
\int_{U_k} w_i^{1/m} &\le C\left(\int_{U_k} w_i^p \prod_{j\in H} w_j^{\frac{p}{p_j}}\right)^{\frac{1}{mp}} \left(\int_{U_k} \nu_{\vec{w}}\right)^{-\frac{1}{mp}} \prod_{j\in I} (\operatorname{ess\,inf}_{U_k} w_j)^{1/m}|Q| \\
&\le C\left(\int_{U_k} w_i^p \prod_{j\in I, j\ne i} w_j^p \prod_{j\in H} w_j^{\frac{p}{p_j}}\right)^{\frac{1}{mp}} \left(\int_{U_k} \nu_{\vec{w}}\right)^{-\frac{1}{mp}} (\operatorname{ess\,inf}_{U_k} w_i)^{1/m}|Q| \\
&= C \operatorname{ess\,inf}_{U_k} w_i^{1/m}|Q|.
\end{aligned}$$

(ii) $\Rightarrow$ (iii). We consider again the sets J and H. Let $\#(H)$ be the number of elements of H. Notice that

$$\begin{aligned}
|Q|^{\frac{\bar{\alpha}\#(H)}{nm^2}+1} &= C\int_Q d(y,\partial Q)^{\frac{\bar{\alpha}\#(H)}{m^2}} \nu_{\vec{w}}(y)^{\frac{1}{m^2 p}} \nu_{\vec{w}}(y)^{-\frac{1}{m^2 p}}\, dy \\
&= C\int_Q d(y,\partial Q)^{\frac{\bar{\alpha}}{m^2}} d(y,\partial Q)^{\frac{\bar{\alpha}(\#(H)-1)}{m^2}} \nu_{\vec{w}}(y)^{-\frac{1}{m^2 p}} \prod_{i=1}^m w_i^{\frac{1}{m^2 p_i}}(y)\, dy \\
&= C\int_Q \nu_{\vec{w}}(y)^{-\frac{1}{m^2 p}} d(y,\partial Q)^{\frac{\bar{\alpha}}{m^2}} \prod_{i\in H} w_i^{\frac{1}{m^2 p_i}}(y) d(y,\partial Q)^{\frac{\widetilde{\alpha}_i}{m^2}} \prod_{i\in J} w_i^{\frac{1}{m^2}}(y)\, dy.
\end{aligned}$$

By Hölder's inequality with $s_0 = \frac{m^2 p}{mp-1}$ and $s_i = \frac{m^2 p_i}{r_i}$, $i = 1, \dots, m$, we get

$$\begin{aligned}
|Q|^{\frac{\bar{\alpha}\#(H)}{nm^2}+1} &\le C\left(\int_Q \left(\nu_{\vec{w}}(y)^{-\frac{1}{m^2 p}}\right)^{s_0} d(y,\partial Q)^{\frac{\bar{\alpha}s_0}{m^2}}\, dy\right)^{\frac{1}{s_0}} \\
&\quad \times \prod_{i\in H}\left[\int_Q w_i(y)^{\frac{s_i}{p_i m^2}} d(y,\partial Q)^{\frac{s_i\widetilde{\alpha}_i}{m^2}}\, dy\right]^{\frac{1}{s_i}} \prod_{i\in J}\left[\int_Q w_i(y)^{\frac{s_i}{m^2}}\, dy\right]^{\frac{1}{s_i}} \\
&= C\left(\int_Q \nu_{\vec{w}}(y)^{-\frac{1}{mp-1}} d(y,\partial Q)^{\frac{\bar{\alpha}p}{mp-1}}\, dy\right)^{\frac{mp-1}{m^2 p}} \\
&\quad \times \prod_{i\in H}\left(\int_Q w_i(y)^{\frac{1}{r_i}} d(y,\partial Q)^{\frac{\widetilde{\alpha}_i p_i}{r_i}}\, dy\right)^{\frac{r_i}{m^2 p_i}} \prod_{i\in J}\left[\int_Q w_i(y)^{\frac{1}{m}}\, dy\right]^{\frac{1}{m}}.
\end{aligned}$$

Raising to the mth power,

$$1 \le C\left(\frac{1}{|Q|^{1+\frac{\bar\alpha p}{n(mp-1)}}}\int_Q \nu_{\vec w}(y)^{-\frac{1}{mp-1}} d(y,\partial Q)^{\frac{\bar\alpha p}{mp-1}}\,dy\right)^{\frac{mp-1}{mp}}$$
$$\times \prod_{i\in H}\left(\frac{1}{|Q|^{1+\frac{\tilde\alpha_i p_i}{nr_i}}}\int_Q w_i(y)^{\frac{1}{r_i}} d(y,\partial Q)^{\frac{\tilde\alpha_i p_i}{r_i}}\,dy\right)^{\frac{r_i}{mp_i}} \prod_{i\in J}\frac{1}{|Q|}\int_Q w_i^{\frac{1}{m}}. \tag{38}$$

By Hölder's inequality with $s_i = \frac{mp_i}{r_i}$ we get

$$1 \le \prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q w_i^{\frac{1}{r_i}}\right)^{\frac{r_i}{mp_i}}\left(\frac{1}{|Q|}\int_Q w_i^{-\frac{1}{p_i-1}}\right)^{\frac{1}{mp_i'}}, \tag{39}$$

where $\left(\frac{1}{|Q|}\int_Q w_i^{-\frac{1}{p_i-1}}\right)^{\frac{1}{mp_i'}}$ is understood as $\left(\operatorname{ess\,inf}_Q w_i\right)^{-1/m}$ if $p_i = 1$. Multiplying inequalities (38) and (39) and the result by

$$\left(\frac{1}{|Q|}\int_Q \nu_{\vec w}\right)^{\frac{1}{mp}}\prod_{i=1}^m\left(\frac{1}{|Q|^{1+\frac{\alpha_i p_i'}{n}}}\int_Q w_i(y)^{1-p_i'} d(y,\partial Q)^{\alpha_i p_i'}\,dy\right)^{\frac{1}{mp_i'}}, \tag{40}$$

and using (ii) we obtain that (40) is smaller than a constant C, as we wished to prove.

Finally, the implication (iii) $\Rightarrow$ (i) is obvious since $U_k, V_k \subset Q$. $\square$

Proof of Theorem 9. (i) $\Rightarrow$ (ii) is obvious. To prove (ii) $\Rightarrow$ (iii) we consider the operators $\mathcal{N}^-_{\vec\alpha,k}$ and $\mathcal{N}^+_{\vec\alpha,k}$. Since $\mathcal{N}^-_{\vec\alpha,k}\vec f(x) \le C\mathcal{M}^c_{\vec\alpha}\vec f(x)$ and $\mathcal{N}^+_{\vec\alpha,k}\vec f(x) \le C\mathcal{M}^c_{\vec\alpha}\vec f(x)$, we get that the inequality in (ii) holds for $\mathcal{N}^-_{\vec\alpha,k}$ and $\mathcal{N}^+_{\vec\alpha,k}$. Proceeding as in Lemma 2.4 in [4] we get that $\vec w \in \widetilde{\mathcal{A}}_{\vec p,\vec\alpha}$ which is equivalent to $\vec w \in \mathcal{A}_{\vec p,\vec\alpha}$ by Theorem 28.

To prove (iii) implies (i), notice that by Hölder's inequality we get if $p_i > 1$

$$\int_Q f_i(y) d(y,\partial Q)^{\alpha_i} dy \le \left(\int_Q f_i^{p_i} w_i\right)^{1/p_i}\left(\int_Q w_i(y)^{-\frac{1}{p_i-1}} d(y,\partial Q)^{\alpha_i p_i'} dy\right)^{1/p_i'},$$

and if $p_i = 1$ ($\alpha_i = 0$) then $\int_Q f_i(y) d(y,\partial Q)^{\alpha_i} dy \le \left(\int_Q f_i w_i\right)\left(\operatorname{ess\,inf}_Q w_i\right)^{-1}$. Now, by using the hypothesis we get

$$\prod_{i=1}^m \frac{1}{|Q|^{1+\frac{\alpha_i}{n}}}\int_Q f_i(y) d(y,\partial Q)^{\alpha_i} dy \le C\prod_{i=1}^m\left(\int_Q f_i^{p_i} w_i\right)^{1/p_i}\left(\int_Q \nu_{\vec w}\right)^{-1/p}.$$

It follows that $\mathcal{M}_{\vec\alpha}(\vec f)(x) \le C\prod_{i=1}^m\left[M_{\nu_{\vec w}}(f_i^{p_i} w_i \nu_{\vec w}{}^{-1}(x))\right]^{1/p_i}$, where $M_{\nu_{\vec w}}$ is the Hardy–Littlewood maximal operator associated to the doubling measure $\nu_{\vec w}(x)dx$ (by Theorem 28, $\nu_{\vec w} \in A_{mp,\frac{\bar\alpha}{m}} \subset A_{mp}$). Using Hölder's inequality for weak-spaces [8, p.15] and the weak-type $(1,1)$ inequality for $M_{\nu_{\vec w}}$ we get

$$\begin{aligned}\|\mathcal{M}_{\vec\alpha}(\vec f)\|_{L^{p,\infty}(\nu_{\vec w})} &\le C\prod_{i=1}^m\left\|\left[M_{\nu_{\vec w}}(f_i^{p_i} w_i \nu_{\vec w}{}^{-1})\right]^{1/p_i}\right\|_{L^{p_i,\infty}(\nu_{\vec w})}\\ &\le C\prod_{i=1}^m\left\|M_{\nu_{\vec w}}(f_i^{p_i} w_i \nu_{\vec w}{}^{-1})\right\|^{1/p_i}_{L^{1,\infty}(\nu_{\vec w})} \le C\prod_{i=1}^m\|f_i\|_{L^{p_i}(w_i)}.\end{aligned}$$

It follows from Remark 5 and $w_i^{\frac{1}{r_i}} \in A_{\frac{mp_i}{r_i},\frac{\alpha_i}{m}}$ that $\frac{mp_i}{r_i} > \frac{1}{1+\frac{\alpha_i}{m}}$, $\alpha_i \ne 0$, which is equivalent to $p_i > \frac{1}{1+\alpha_i}$. $\square$

5. Proof of Theorems 11 and 10

Proof of Theorem 11. Let $\vec w \in \mathcal{A}_{\vec p,\vec\alpha}$. Using Theorem 28 and Theorem 1 we get that there exists $r_0 > 1$ such that $w_i^{\frac{1}{r_i}} \in A_{\frac{mp_i}{r_0 r_i},\frac{\alpha_i}{m}}$. We also have $\vec w \in \mathcal{A}_{\vec p}$. By Hölder's inequality with $s_i = \frac{mp_i}{r_i}$ and $s_i' = \frac{mp_i}{p_i-1}$,

$$|Q|^{r_0 m} = \prod_{i=1}^m\left(\int_Q w_i^{\frac{1}{mp_i}} w_i^{-\frac{1}{mp_i}}\right)^{r_0} \le \prod_{i=1}^m\left(\int_Q w_i^{\frac{1}{r_i}}\right)^{\frac{r_0 r_i}{mp_i}}\left(\int_Q w_i^{-\frac{1}{p_i-1}}\right)^{\frac{r_0}{mp_i'}}. \tag{41}$$

Since $w_i^{\frac{1}{r_i}} \in A_{\frac{mp_i}{r_0 r_i}, \frac{\alpha_i}{m}}$, we have

$$\prod_{i=1}^{m}\left(\int_Q w_i^{\frac{1}{r_i}}\right)^{\frac{r_0 r_i}{mp_i}}\left(\int_Q w_i(y)^{-\frac{r_0}{mp_i - r_0 r_i}} d(y,\partial Q)^{\frac{\alpha_i p_i}{mp_i - r_0 r_i}}\, dy\right)^{1-\frac{r_0 r_i}{mp_i}} \le C\prod_{i=1}^{m}|Q|^{1+\frac{\alpha_i}{nm}} = C|Q|^{m+\frac{\bar{\alpha}}{nm}}. \tag{42}$$

Using (41), (42) and $\vec{w} \in \mathcal{A}_{\vec{p}}$ we obtain

$$|Q|^{r_0 m}\left(\int_Q \nu_{\vec{w}}\right)^{\frac{r_0}{mp}}\prod_{i=1}^{m}\left(\int_Q w_i(y)^{-\frac{r_0}{mp_i - r_0 r_i}} d(y,\partial Q)^{\frac{\alpha_i p_i}{mp_i - r_0 r_i}}\, dy\right)^{1-\frac{r_0 r_i}{mp_i}} \le C|Q|^{r_0+m+\frac{\bar{\alpha}}{nm}}.$$

Taking $r = \frac{r_0}{m-(m-1)r_0}$ and $\beta_i = \alpha_i \frac{1+r(m-1)}{m}$ we get that $\frac{-r_0}{p_i m - r_0 r_i} = 1 - (\frac{p_i}{r})'$ and $\frac{\alpha_i p_i}{mp_i - r_i r_0} = \beta_i (\frac{p_i}{r})'$. Then the above inequality can be written as

$$\left(\int_Q \nu_{\vec{w}}\right)^{r/p}\prod_{i=1}^{m}\left(\int_Q w_i(y)^{1-(\frac{p_i}{r})'} d(y,\partial Q)^{\beta_i(\frac{p_i}{r})'}dy\right)^{\frac{mr}{r_0}-\frac{rr_i}{p_i}} \le C|Q|^{m+\frac{\bar{\beta}}{n}},$$

where we have used $\frac{mr}{r_0}\left(m - r_0(m-1) + \frac{\bar{\alpha}}{mn}\right) = m + \frac{\bar{\beta}}{n}$. Since $\frac{mr}{r_0} - \frac{rr_i}{p_i} = [(\frac{p_i}{r})']^{-1}$, we are done. □

Proof of Theorem 10. Since (i) ⇒ (ii) is trivial and by Theorem 9 we get that (ii) ⇒ (iii), we shall only prove (iii) ⇒ (i). As in the proof of Theorem 3.7 in [6] it is enough to prove that there exists $\delta < 1$ such that

$$\mathcal{M}_{\vec{\alpha}}(\vec{f})(x) \le C\prod_{i=1}^{m}\left[M_{\nu_{\vec{w}}}\left(|f_i|^{\delta p_i} w_i^{\delta}\nu_{\vec{w}}^{-\delta}\right)(x)\right]^{\frac{1}{\delta p_i}}. \tag{43}$$

In fact, by Hölder's inequality with exponents $\frac{p_i}{p}$ and the strong type $(\frac{1}{\delta}, \frac{1}{\delta})$ inequality of $M_{\nu_{\vec{w}}}$ with respect to the measure $\nu_{\vec{w}}$ we get

$$\begin{aligned}\left(\int |\mathcal{M}_{\vec{\alpha}}(\vec{f})|^p \nu_{\vec{w}}\right)^{\frac{1}{p}} &\le C\left[\int \prod_{i=1}^{m}\left[M_{\nu_{\vec{w}}}\left(|f_i|^{p_i\delta} w_i^{\delta}\nu_{\vec{w}}^{-\delta}\right)\right]^{\frac{p}{\delta p_i}}\nu_{\vec{w}}\right]^{\frac{1}{p}}\\ &\le C\prod_{i=1}^{m}\left[\int \left[M_{\nu_{\vec{w}}}(|f_i|^{\delta p_i} w_i^{\delta}\nu_{\vec{w}}^{-\delta})\right]^{\frac{1}{\delta}}\nu_{\vec{w}}\right]^{\frac{1}{p_i}} \le C\prod_{i=1}^{m}\left(\int |f_i|^{p_i} w_i\right)^{\frac{1}{p_i}}.\end{aligned}$$

Now, we shall prove (43). Let δ be such that $\max_{i=1,\dots,m}\left\{\frac{1}{p_i(1-\frac{1}{r})+1}\right\} < \delta < 1$, where r is the number in Theorem 11. By Hölder's inequality with exponent $p_i\delta$,

$$\begin{aligned}\int_Q |f_i(y)| d(y,\partial Q)^{\alpha_i} dy &\le \left(\int_Q |f_i|^{p_i\delta} w_i^{\delta}\nu_{\vec{w}}^{1-\delta}\right)^{\frac{1}{\delta p_i}}\left(\int_Q w_i(y)^{-\frac{\delta}{p_i\delta-1}}\nu_{\vec{w}}(y)^{\frac{\delta-1}{p_i\delta-1}} d(y,\partial Q)^{\frac{\alpha_i p_i\delta}{p_i\delta-1}}\, dy\right)^{\frac{p_i\delta-1}{p_i\delta}}\\ &= \left(\int_Q |f_i|^{p_i\delta} w_i^{\delta}\nu_{\vec{w}}^{1-\delta}\right)^{\frac{1}{\delta p_i}} I_i.\end{aligned}$$

Let $\lambda = \frac{1+r(m-1)}{mr}$. It is clear that $\lambda \in (0,1)$ and

$$I_i = \left(\int_Q w_i(y)^{-\frac{\delta}{p_i\delta-1}}\nu_{\vec{w}}(y)^{\frac{\delta-1}{p_i\delta-1}} d(y,\partial Q)^{\frac{\alpha_i\lambda p_i\delta}{p_i\delta-1}} d(y,\partial Q)^{\frac{\alpha_i(1-\lambda)\delta p_i}{\delta p_i-1}}\, dy\right)^{\frac{\delta p_i-1}{\delta p_i}}.$$

Let $s_i = \frac{r(\delta p_i-1)}{\delta(p_i-r)}$. We have that $s_i > 1$ and, for later computations, $\frac{1}{s_i'} = \frac{\delta p_i(r-1)+r(\delta-1)}{r(p_i\delta-1)}$. By Hölder inequality with exponent s_i we get that

$$I_i \le \left(\int_Q w_i(y)^{-\frac{r}{p_i-r}} d(y,\partial Q)^{\frac{\beta_i p_i}{p_i-r}} dy\right)^{\frac{p_i-r}{p_i r}} II_i,$$

where $II_i = \left(\int_Q \nu_{\vec{w}}(y)^{\frac{(\delta-1)s_i'}{\delta p_i-1}} d(y,\partial Q)^{\frac{\alpha_i(1-\lambda)p_i\delta s_i'}{p_i\delta-1}}\, dy\right)^{\frac{p_i\delta-1}{p_i\delta s_i'}}$ and $\beta_i = \alpha_i\frac{1+r(m-1)}{m}$. Now, choosing δ close to 1 such that

$$t_i = \frac{\delta p_i - 1}{(1-\delta)s_i'(mp-1)} > 1 \quad \text{and} \quad \frac{\alpha_i(1-\lambda)p_i\delta s_i' t_i'}{p_i\delta - 1} > -1,$$

and applying Hölder's inequality with t_i we get

$$II_i \le \left(\int_Q \nu_{\vec{w}}^{-\frac{1}{mp-1}}\right)^{\frac{(1-\delta)(mp-1)}{\delta p_i}} |Q|^{\frac{\alpha_i(1-\lambda)}{n}+\frac{p_i\delta-1}{p_i\delta s_i' t_i'}}.$$

(For subsequent computations, notice that $\frac{1}{s_i't_i'} = \frac{1}{s_i'} - \frac{(1-\delta)(mp-1)}{p_i\delta-1}$.) Putting together the above estimates and using that $\nu_{\vec{w}} \in A_{mp}$ we get that

$$\prod_{i=1}^m \int_Q |f_i(y)| d(y,\partial Q)^{\alpha_i} dy \le C \prod_{i=1}^m \left(\frac{1}{\nu_{\vec{w}}(Q)} \int_Q |f_i(y)|^{\delta p_i} w_i(y)^\delta \nu_{\vec{w}}(y)^{1-\delta} dy\right)^{\frac{1}{\delta p_i}}$$
$$\times \prod_{i=1}^m \left(\left(\int_Q w_i(y)^{-\frac{r}{p_i-r}} d(y,\partial Q)^{\frac{\beta_i p_i}{p_i-r}}\right)^{\frac{p_i-r}{p_i r}} \left(\int_Q \nu_{\vec{w}}\right)^{\frac{1}{p}} \prod_{i=1}^m |Q|^{\frac{(1-\delta)mp}{\delta p_i}+\frac{\alpha_i(1-\lambda)}{n}+\frac{p_i\delta-1}{p_i\delta s_i' t_i'}}.$$

Then, using Theorem 11 we get

$$\frac{1}{|Q|^{m+\frac{\bar{\alpha}}{n}}} \prod_{i=1}^m \int_Q |f_i(y)| d(y,\partial Q)^{\alpha_i} dy \le C \prod_{i=1}^m \left(\frac{1}{\nu_{\vec{w}}(Q)} \int_Q |f_i|^{\delta p_i} w_i^\delta \nu_{\vec{w}}^{1-\delta}\right)^{\frac{1}{\delta p_i}}$$
$$\times \left(|Q|^{-m-\frac{\bar{\alpha}}{n}} |Q|^{\frac{m}{r}+\frac{\bar{\beta}}{nr}} |Q|^{\frac{(1-\delta)m}{\delta}} |Q|^{\frac{\bar{\alpha}(1-\lambda)}{n}} \prod_{i=1}^m |Q|^{\frac{p_i\delta-1}{p_i\delta s_i' t_i'}}\right).$$

Since the last factor is 1 we are done. □

6. Remarks and examples

We start establishing some properties of the linear classes $A_{p,\alpha}$. Assume that $p(1+\alpha) > 1$. Applying Hölder's inequality, it is easy to see that $A_q \subset A_{p,\alpha}$ for $1 < q < p(1+\alpha)$. Using that $w \in A_p \Rightarrow A_{p-\varepsilon}$ we have $A_{p(1+\alpha)} \subset A_{p,\alpha}$. Other interesting relation is that $w \in A_1 \Rightarrow w \in RA_{\frac{1}{1+\alpha},\alpha}$. We can also see that a power weight

$$w(x) = |x|^\beta \in A_{p,\alpha} \Leftrightarrow -n < \beta < p(n+\alpha) - n \Leftrightarrow |x|^\beta \in A_{p(1+\frac{\alpha}{n})}. \tag{44}$$

These properties in the one-sided case can be found in [9].

Example 45. We work in $\mathbb{R}$ $(n = 1)$ with $m = 2$. We assume that $\vec{p}, \vec{\alpha}$ satisfy the assumptions on Theorem 9. Let $w_1(x) = |x|^{\gamma_1}$ and $w_2(x) = |x|^{\gamma_2}$. We are going to determine the conditions on γ_1 and γ_2 to have $\vec{w} \in \mathcal{A}_{\vec{p},\vec{\alpha}}$ or, equivalently, (a) $\nu_{\vec{w}} \in A_{2p,\frac{\bar{\alpha}}{2}}$, (b) $w_i^{1-p_i'} \in A_{2p_i',\frac{\alpha_i}{2}}$, for $i = 1, 2$ and (c) $w_i^{\frac{1}{r_i}} \in A_{\frac{2p_i}{r_i},\frac{\alpha_i}{2}}$, for $i = 1, 2$, where $r_i = p_i + 1$. Using (44), $\vec{w} \in \mathcal{A}_{\vec{p},\vec{\alpha}}$ if and only if the following relations hold: $-1 < \frac{p}{p_1}\gamma_1 + \frac{p}{p_2}\gamma_2 < 2p + p(\alpha_1+\alpha_2) - 1$, $-p_1 - 1 - p_1\alpha_2 < \gamma_1 < p_1 - 1$, $-p_2 - 1 - p_2\alpha_1 < \gamma_2 < p_2 - 1$, $-p_1 - 1 < \gamma_1 < p_1 + p_1\alpha_1 - 1$ and $-p_2 - 1 < \gamma_2 < p_2 + p_2\alpha_2 - 1$.

Let $p_1 = 2$, $p_2 = 4$, $p = 4/3$, $\alpha_1 = -1/3$, $\alpha_2 = -1/2$.

(1) If $\gamma_1 = -1$, $\gamma_2 = 0$ then $\vec{w} \in \mathcal{A}_{\vec{p},\vec{\alpha}}$ but $w_1 \notin A_{p_1,\alpha_1}$. Therefore, $\mathcal{M}_{\vec{\alpha}}$ applies $L^{p_1}(w_1) \times L^{p_2}(w_2)$ into $L^p(\nu_{\vec{w}})$ but M_{α_1} is not bounded in $L^{p_1}(w_1)$.
(2) If $\gamma_1 = -1 = \gamma_2$ then (b) and (c) are satisfied but (a) is not satisfied.
(3) If $\gamma_1 = 1/3$, $\gamma_2 = -11/3 + \varepsilon$, ε small, then (a) and (b) are satisfied but (c) is not satisfied (the second condition in (c) is satisfied).

Notice that in this example (a) and (c) imply (b).

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