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EXTENDIBILITY OF BILINEAR FORMS ON BANACH SEQUENCE SPACES

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ABSTRACT

We study Hahn–Banach extensions of multilinear forms defined on Banach sequence spaces. We characterize c_0 in terms of extension of bilinear forms, and describe the Banach sequence spaces in which every bilinear form admits extensions to any superspace.

1. Introduction

One of the fundamental results in Functional Analysis is the Hahn–Banach theorem. It was proved independently by Hahn in 1927 [18] and by Banach in 1929 [5] (see also [6, Chapitre IV, §2]). In one of its forms, it states that if X is a subspace of a normed space Z, then every continuous, linear functional

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$f : X \to \mathbb{K}$ can be extended to Z preserving the norm. It soon became clear that a multilinear version of this result was not possible in general, and this started the search of situations in which such multilinear extension theorems are possible. A particular positive result was given by Arens in 1951, where he showed how to extend the product on a Banach algebra to its bidual and, also, how to extend bilinear operators defined on a couple of Banach spaces to their corresponding biduals [2, 3]. This is one of the lines to find extension theorems: given a space, find a superspace to which every multilinear mapping can be extended. Aron and Berner went further on this line and showed in 1978 that every holomorphic function on a Banach space can be extended to an open subset of the bidual [4].

Another line is to fix a Banach space X and consider the problem of extending bilinear forms defined on subspaces of X. Maurey's extension theorem [15, Corollary 12.23] is a classical example of this natural point of view, in which relevant advances have been obtained in the last years [10, 26].

A third way to face the extension problem is to find the bilinear mappings (on a fixed Banach space) that can be extended to every superspace. This was the point of view taken by Grothendieck: in 1956 he showed in his théorème fondamental [17, page 60] that these are precisely those bilinear mappings factoring through Hilbert spaces via 2-summing operators. We say that a bilinear form $T : X \times Y \to \mathbb{K}$ is **extendible** (see, e.g., [7, 11, 20, 22]) if for all Banach spaces $E \supset X$, $F \supset Y$, there exists a bilinear form defined on $E \times F$ that extends T. Our aim, which can be framed in this last approach, is to describe those spaces which enjoy a bilinear (or multilinear) Hahn–Banach theorem, in the sense that every bilinear form is extendible. Examples of such spaces are $A(\mathbb{D})$, $H^\infty(\mathbb{D})$, $\mathscr{L}^\infty$-spaces and Pisier spaces, but a complete characterization is still unknown. In this line, our main result is the following theroem, which solves the problem among Banach spaces with unconditional basis.

Theorem 1.1: *The only Banach space with an unconditional basis on which every bilinear form is extendible is c_0.*

This theorem will follow as a consequence of Theorem 2.2 below. We also characterize the Banach sequence spaces satisfying a bilinear Hahn–Banach theorem as those "between" c_0 and ℓ_∞ (see Corollary 2.4). As a byproduct, we obtain a partial answer to the following open problem: if a sequence X_n of n-dimensional Banach spaces is uniformly complemented in some $\mathscr{L}_\infty$, must these spaces be uniformly isomorphic to ℓ_∞^n? Corollary 2.3 gives a positive answer for sections of a Banach sequence space (see Proposition 2.5).

1.1. Preliminaries. We briefly collect here some basic definitions that will be used throughout the paper. We will consider real or complex Banach spaces, that will be denoted X, $Y, \ldots$. Unless otherwise stated they will be assumed to be infinite dimensional. The duals will be denoted by X^*, Y^*, ... Given two Banach spaces X and Y, we write $X \approx Y$ if they are isomorphic and $X \overset{1}{\approx} Y$ if they are isometrically isomorphic. We refer to [1, 25] for basic concepts and notations on Banach spaces.

We denote by $\mathscr{L}^2(X, Y)$ the Banach space of all scalar valued, continuous, bilinear mappings (in short, bilinear forms) on $X \times Y$. We write $\mathscr{L}^2(X)$ whenever $Y = X$.

The space of extendible bilinear forms is denoted by $\mathscr{E}^2(X, Y)$. The **extendible norm**

$$\|T\|_{\mathscr{E}} = \|T\|_{\mathscr{E}^2(X,Y)} := \inf\{c > 0 : \text{ for all } W \supseteq X, Z \supseteq Y \text{ there is } \text{an extension of } T \text{ to } W \times Z \text{ with norm} \leq c\}$$

makes $\mathscr{E}^2(X, Y)$ a Banach space. Since every $\ell_\infty(I)$ space is injective (in fact, has the metric extension property), every bilinear form on such spaces is extendible and the extendible and uniform norm coincide. Moreover, a bilinear form T on $X \times Y$ is extendible if and only if it extends to $\ell_\infty(I) \times \ell_\infty(J)$, for some $\ell_\infty(I) \supset X$ and $\ell_\infty(J) \supset Y$. The supremum defining the extendible norm can be taken only over the extensions to $\ell_\infty(I) \times \ell_\infty(J)$.

We write $\mathscr{L}(X; Y)$ for the space of all (continuous, linear) operators $u : X \to Y$. We denote by $\Pi_1(X; Y)$ the space of absolutely summing operators, $\Gamma_\infty(X; Y)$ for the ∞-factorable and $\Delta_2(X; Y)$ for the 2-dominated. Their corresponding norms are, respectively, π_1, γ_∞ and δ_2 (see [13, 15] for definitions and basic properties).

We are going to use the theory of tensor products and operator ideals as presented in [13]. We recall some notation and definitions for completeness. The projective tensor norm π is defined, for a z in the tensor product $X \otimes Y$, by

$$\pi(z) = \inf\left\{\sum_{j=1}^{r} \|x_j\| \, \|y_j\|\right\},$$

where the infimum is taken over all the representations of z of the form $z = \sum_{j=1}^r x_j \otimes y_j$. The right-injective associate of π is denoted by $\pi\backslash$. This tensor norm is the greatest right-injective tensor norm and makes the following inclusion an isometry:

$$X \otimes_{\pi\backslash} Y \overset{1}{\hookrightarrow} X \otimes_\pi \ell_\infty(B_{Y^*}),$$

where B_{Y^*} is the unit ball of Y^* (see [13, Theorem 20.7.] for details). Likewise, the injective associate $/\pi\backslash$ is the largest injective tensor norm and is induced by the isometric inclusion

$$X \otimes_{/\pi\backslash} Y \overset{1}{\hookrightarrow} \ell_\infty(B_{X^*}) \otimes_\pi \ell_\infty(B_{Y^*}).$$

The metric extension property of $\ell_\infty(I)$ spaces implies that extendible bilinear forms are precisely the $/\pi\backslash$-continuous ones:

$$\mathscr{E}^2(X,Y) \overset{1}{=} \big(X \otimes_{/\pi\backslash} Y\big)^*.$$

We refer to [13, 15] for all the basic (and not so basic) facts and any undefined notation on tensor norms and operator ideals.

Given a family $\{X_n\}_n$ of Banach spaces where $\dim X_n = n$, we say that X_n are K-uniformly complemented in X if for each n we have a mapping $i_n : X_n \to X$ and $q_n : X \to X_n$ such that $q_n \circ i_n$ is the identity on X_n and $\|i_n\|\,\|q_n\| \leq K$. In this case we also say that X contains X_n uniformly complemented. We note that if X contains uniform copies of ℓ_∞^n (i.e., $\mathbb{K}^n$ with the sup norm), then the ℓ_∞^n are uniformly complemented since they are injective spaces.

1.2. Banach sequence spaces. By a Banach sequence space (also known as Köthe sequence space) we will mean a Banach space $X \subseteq \mathbb{K}^{\mathbb{N}}$ of sequences in $\mathbb{K}$ such that $\ell_1 \subseteq X \subseteq \ell_\infty$ with norm one inclusions satisfying that if $x \in \mathbb{K}^{\mathbb{N}}$ and $y \in X$ are such that $|x_n| \leq |y_n|$ for all $n \in \mathbb{N}$, then x belongs to X and $\|x\| \leq \|y\|$.

If X is a Banach sequence space, we denote by $\{e_n\}_n$ the sequence of canonical vectors, which is always a 1-unconditional basic sequence. We define $X_N = \operatorname{span}\{e_1, \dots, e_N\}$ and $X_0 = \overline{\operatorname{span}}\{e_n\}_n$. This last space is usually referred to as the minimal kernel of X. Given $x \in X$ we write $x^N = (x_1, \dots, x_N)$. There are inclusions $i_N^X : X_N \hookrightarrow X$ and projections $\pi_N^X : X \to X_N$ given by $i_N^X(x_1, \dots, x_N) = (x_1, \dots, x_N, 0, 0, \dots)$ and $\pi_N^X(x) = x^N$. The inclusions are isometric and the projections have norm 1. For the case $X = \ell_p$ $(1 \leq p \leq \infty)$, we write ℓ_p^N for X_N.

Given a Banach sequence space X, its Köthe dual is defined as

$$X^\times = \{(z_n)_n \in \mathbb{K}^{\mathbb{N}} \colon \textstyle\sum_n |z_n x_n| < \infty \text{ for all } x \in X\}\,.$$

With the norm $\|z\|_{X^\times} = \sup_{\|x\|_X \leq 1} \sum_n |z_n x_n|$ it is again a Banach sequence space.

Following [24, 1.d], a Banach sequence space X is said to be r-convex (with $1 \leq r < \infty$) if there exists a constant $\kappa > 0$ such that for any choice $x_1, \ldots, x_N \in X$ we have

$$\left\| \left(\left(\sum_{j=1}^N |x_j(k)|^r \right)^{1/r} \right)_{k=1}^\infty \right\|_X \leq \kappa \left(\sum_{j=1}^N \|x_j\|_X^r \right)^{1/r}.$$

On the other hand, X is s-concave (with $1 \leq s < \infty$) if there is a constant $\kappa > 0$ such that

$$\left(\sum_{j=1}^N \|x_j\|_X^s \right)^{1/s} \leq \kappa \left\| \left(\left(\sum_{j=1}^N |x_j(k)|^s \right)^{1/s} \right)_{k=1}^\infty \right\|_X$$

for all $x_1, \ldots, x_N \in X$.

It is well known that ℓ_p is r convex for $1 \leq r \leq p$ and s-concave for $p \leq s < \infty$.

The following result is probably known. However, we were not able to find a proper reference of this fact and we include here a short proof. It is modelled along the same lines as the proof of the fact that if the canonical vectors form a basis of X then both duals coincide.

Proposition 1.2: *If X is a Banach sequence space, its Köthe dual $X^\times$ is a 1-complemented subspace of the usual dual X^*.*

Proof. Let us see first that the mapping $i : X^\times \to X^*$ defined by $i(z) = \varphi_z : X \to \mathbb{K}$, with $\varphi_z(x) = \sum_n z_n x_n$, is an isometry. It is clearly well defined; moreover

$$\|\varphi_z\| = \sup_{x \in B_X} \left| \sum_n z_n x_n \right| \leq \sup_{x \in B_X} \sum_n |z_n x_n| = \|z\|_{X^\times}\,.$$

To see the reverse inequality, for any $t, s \in \mathbb{K}$ we take $\varepsilon(t,s) \in \mathbb{K}$ with $|\varepsilon(t,s)| = 1$ such that $|st| = \varepsilon(t,s)st$; then for every $x \in B_X$ and every $z \in X^\times$ we have

$$\sum_n |z_n x_n| = \sum_n \varepsilon(z_n, x_n) z_n x_n = \Big| \sum_n \varepsilon(z_n, x_n) z_n x_n \Big| \leq \sup_{a \in B_X} \Big| \sum_n z_n a_n \Big| = \|\varphi_z\|,$$

which gives $\|z\|_{X^\times} \leq \|\varphi_z\|$.

On the other hand, the mapping $q : X^* \to X^\times$ given by $q(\varphi) = \big(\varphi(e_n)\big)_n$ defines a norm-one projection. Indeed, given $x \in X$ and fixed N we have

$$\sum_{n=1}^N |x_n \varphi(e_n)| = \sum_{n=1}^N \varepsilon(x_n, \varphi(e_n)) x_n \varphi(e_n) = \varphi\Big(\sum_{n=1}^N \varepsilon(x_n, \varphi(e_n)) x_n e_n \Big) \leq \|\varphi\| \, \Big\| \sum_{n=1}^N \varepsilon(x_n, \varphi(e_n)) x_n e_n \Big\|_X \leq \|\varphi\| \; \|x\| \, .$$

This shows that $\sum_{n=1}^\infty |x_n \varphi(e_n)| \leq \|\varphi\| \; \|x\|$, which gives that q is well defined and $\|q(\varphi)\| \leq \|\varphi\|$. Furthermore, q is a projection, since clearly $q \circ i(z) = q(\varphi_z) = (z_n)_n = z$. ∎

2. Extension of bilinear forms on Banach sequence spaces

In what follows K_G denotes Grothendieck's constant. We begin by proving the following known fact, which was stated as Theorem 3.4 in [20] without the estimates for the norms (see [11, Lemma 2.4] for a result in the same spirit).

Proposition 2.1: *If every bilinear form $B : X \times Y \to \mathbb{K}$ is extendible with $\|B\|_{\mathscr{E}} \leq K\|B\|$, then every operator $u : X^* \to \ell_2$ is absolutely 1-summing and $\pi_1(u) \leq K_G K \|u\|$.*

Proof. We first note that, by definition of the tensor norm $/\pi\backslash$ (see [13, Section 20.7]), $\mathscr{E}^2(X,Y)$ is isometrically the dual of $X \otimes_{/\pi\backslash} Y$. Then, our hypothesis is equivalent to the inequality $\pi \leq K /\pi\backslash$ on $X \otimes Y$ and, as a consequence, we also have $\pi\backslash \leq K /\pi\backslash$ on $X \otimes Y$. Since both $\pi\backslash$ and $/\pi\backslash$ are right-injective, an application of Dvoretzky's theorem [15, 19.1] and the previous inequality gives an isomorphism

$$X \otimes_{/\pi\backslash} \ell_2^N \longrightarrow X \otimes_{\pi\backslash} \ell_2^N \tag{1}$$

with norm at most K. Since ℓ_2^N is finite dimensional, $\mathscr{L}(X;\ell_2^N)$ and $X^*\otimes\ell_2^N$ coincide as sets. Then, the embedding in [13, Section 17.6] is actually surjective and [13, Sections 21.5 and 27.2] give $X^*\otimes_{w_\infty}\ell_2^N \overset{1}{\approx} \Gamma_\infty(X;\ell_2^N)$ (see [15, Chapter 9] or [13, Section 18] for the definition of $\Gamma_\infty(X;Y)$). Therefore,

$$(2)\qquad \left(X\otimes_{\pi\backslash}\ell_2^N\right)^{**} \overset{1}{\approx} \left(\Gamma_\infty(X;\ell_2^N)\right)^{*} \overset{1}{\approx} \left(X^*\otimes_{w_\infty}\ell_2^N\right)^{*} \overset{1}{\approx} \Pi_1(X^*;\ell_2^N)\,.$$

Now, by [13, Sections 17.12 and 27.2], the operator ideal Δ_2 is associated to w_2 and Π_1 is associated to $\pi\backslash$. On the other hand, Grothendieck's inequality [13, Section 20.17] states that $w_2 \geq K_G/\pi\backslash$ and clearly we have $\mathscr{L}(X^*;\ell_2^N) \overset{1}{\approx} \Delta_2(X^*;\ell_2^N)$. Using (2) to take biduals in (1) we have an isomorphism

$$(3)\qquad \mathscr{L}(X^*;\ell_2^N) \longrightarrow \left(X\otimes_{/\pi\backslash}\ell_2^N\right)^{**} \longrightarrow \left(X^{**}\otimes_{\pi\backslash}\ell_2^N\right)^{**} \overset{1}{\approx} \Pi_1(X^*;\ell_2^N),$$

where the first mapping has norm bounded by K_G and the second one by K. Since both $\mathscr{L}$ and Π_1 are maximal operator ideals, the same holds if we put ℓ_2 instead of ℓ_2^N. ∎

With this result we can now prove the following one, from which Theorem 1.1 follows as an immediate consequence.

THEOREM 2.2: *Let X be a Banach space with an unconditional basis and Y be any infinite dimensional Banach space such that every bilinear form on $X\times Y$ is extendible. Then $X\approx c_0$.*

Proof. Let us see first that, under our assumptions, the basis of X must be shrinking. Suppose it is not. Since it is unconditional, by James theorem [19, Corollary 2] (see also [1, Theorem 3.3.1]) X must contain a complemented copy of ℓ_1. Since the property of all bilinear forms being extendible is inherited by complemented subspaces, it follows that every bilinear form on $\ell_1\times Y$ is extendible. This implies [22, Lemma 6] that every continuous linear operator from Y to ℓ_∞ is absolutely 2-summing. By the so-called $\mathfrak{L}_p$-Local Technique Lemma for Operator Ideals [13, Section 23.1], the same holds for every operator from Y to $\ell_\infty(I)$, for any index set I. But this is not possible, since there exists an isometric embedding from Y into some $\ell_\infty(I)$, and this cannot be absolutely 2-summing (otherwise, the identity on Y would be so, but Y is infinite dimensional).

This means that the canonical basis of X must be shrinking. We can assume that the basis is 1-unconditional, so that the coordinate basis is an 1-unconditional basis of X^*. We also know from Proposition 2.1 that all operators from X^* to ℓ_2 are absolutely 1-summing. By [23] (see also [25, Theorem 8.21]) this implies that the basis of X^* is $(K_G K)^2$-equivalent to the basis of ℓ_1. Although ℓ_1 has many non-isomorphic preduals, if the coordinate basis is equivalent to that of ℓ_1, a standard computation shows that the canonical basis on X must be $(K_G K)^2$-equivalent to the basis of c_0. ∎

Note that the proof not only shows that X must be isomorphic to c_0, but also gives an estimation of the Banach–Mazur distance between X and c_0 whenever X has a 1-unconditional basis. As a consequence, we can also characterize the pairs of Banach sequence spaces on which every bilinear form is extendible.

Corollary 2.3: *If X and Y are Banach sequence spaces, then the following are equivalent.*

(i) $\mathscr{L}^2(X,Y) = \mathscr{E}^2(X,Y)$ *and* $\|B\|_{\mathscr{E}} \leq K_1 \|B\|$ *for all bilinear form* B *on* $X \times Y$.

(ii) *The canonical basic sequence of X and Y is K_2-equivalent to the canonical basis of c_0.*

(iii) $K_3 := \sup\{d(X_N, \ell_\infty^N), d(Y_N, \ell_\infty^N) : N \in \mathbb{N}\}$ *is finite (where d denotes the Banach–Mazur distance).*

(iv) *The spaces X_N and Y_N ($N \in \mathbb{N}$) are K_4-uniformly complemented in some $L_\infty(\mu)$.*

Moreover, we have $K_4 \leq K_3 \leq K_2 \leq (K_G\, K_1)^2$ *and* $K_1 \leq K_2^2 \leq K_G^4 K_4^8$.

Proof. If (i) holds on $X \times Y$, then the same holds for $X_N \times Y_N$ for any N and, by the density lemma [13, Section 13.4], for $X_0 \times Y_0$. By Theorem 2.2, both bases are $(K_G K)^2$-equivalent to the basis of c_0.

The implications (ii) ⇒ (iii) ⇒ (iv) are immediate, as well as the inequalities $K_4 \leq K_3 \leq K_2$.

If (iv) holds, bilinear forms on $X_N \times Y_N$ are extendible with $\|\cdot\|_{\mathscr{E}} \leq K_4^2 \|\cdot\|$ and, as before, the same holds for $X_0 \times Y_0$. By Theorem 2.2 their canonical bases are $K_G^2 K_4^4$-equivalent to the canonical basis of c_0, which is (ii).

Now suppose (ii) holds and take a bilinear form $B : X \times Y \to \mathbb{K}$. We know from Proposition 1.2 that $X^\times$ is 1-complemented in X^*. Then $(X^\times)^*$ is isometrically a (complemented) subspace of X^{**}. Since (ii) implies $X^\times = \ell_1$, we also have $(X^\times)^* = X^{\times\times} = \ell_\infty$. The same holds for Y, so we obtain the following diagrams:

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X^{\times\times} & \xrightarrow{i_2} & X^{**} \\ & & \Big\uparrow{\scriptstyle u} & & \\ & & \ell_\infty & & \end{array} \qquad \begin{array}{ccccc} Y & \xrightarrow{j_1} & Y^{\times\times} & \xrightarrow{j_2} & Y^{**} \\ & & \Big\uparrow{\scriptstyle v} & & \\ & & \ell_\infty & & \end{array},$$

where i_1, i_2, j_1 and j_2 are isometric injections and u and v are isomorphisms with

$$\|u\| \, \|u^{-1}\| \leq K_2 \quad \text{and} \quad \|v\| \, \|v^{-1}\| \leq K_2. \tag{4}$$

We can extend B (in the canonical way) to a bilinear form $\tilde{B} : X^{**} \times Y^{**} \to \mathbb{K}$ with the same norm as B, and then define a bilinear form $\widehat{B}$ on $\ell_\infty \times \ell_\infty$ by $\widehat{B} = \tilde{B} \circ (i_2 \circ u, j_2 \circ v)$. We have obtained the factorization $B = \widehat{B} \circ (u^{-1} \circ i_1, v^{-1} \circ j_1)$. Since on $\ell_\infty \times \ell_\infty$ every bilinear form is extendible (with the extendible norm equal to the usual norm), from the ideal property of extendible bilinear forms and inequalities (4) we conclude that B is extendible and $\|B\|_{\mathscr{E}} \leq K_2^2 \|B\|$. ∎

It follows from the previous corollary (and its proof) that a Banach sequence space on which every bilinear form is extendible must satisfy the sublattice inclusions

$$c_0 \subset X \subset \ell_\infty. \tag{5}$$

Conversely, let X be a Banach sequence space satisfying (5). By a closed graph argument, both inclusions are continuous and it is easy to check that X satisfies the equivalent conditions of Corollary 2.3. Note also that a Banach sequence space satisfies (5) if and only if its Köthe dual is ℓ_1. As a consequence, we have the following version of Theorem 1.1 for Banach sequence spaces.

Corollary 2.4: *The Banach sequence spaces X on which every bilinear form is extendible are those satisfying* (5). *Also, this happens if and only if $X^\times = \ell_1$.*

Examples of such spaces are $c_0 \oplus \ell_\infty$, $c_0(\ell_\infty)$ and $\ell_\infty(c_0)$. It is not hard to see that these spaces are mutually non-isomorphic Banach sequence spaces (see also [12], where the authors show that $c_0(\ell_\infty)$ and $\ell_\infty(c_0)$ are not isomorphic even as Banach spaces).

If a sequence X_n of n-dimensional Banach spaces is uniformly complemented in some $\mathscr{L}_\infty$, it is an open problem if these spaces have to be uniformly isomorphic to ℓ_∞^n. Taking $X = Y$ in Corollary 2.3, the implication (iv)$\Rightarrow$ (iii) gives the following partial answer.

PROPOSITION 2.5: *If the N-dimensional sections X_N of a Banach sequence space X are uniformly complemented in some $L_\infty(\mu)$, then they must be uniformly isomorphic to ℓ_∞^N.*

Note also that, since ℓ_∞ and c_0 are the only symmetric Banach sequence spaces satisfying (ii) of Corollary 2.3, these two are the only symmetric Banach sequence spaces on which every bilinear form is extendible.

If $X_1, \dots, X_n$ are Banach spaces such that every n-linear form on $X_1 \times \cdots \times X_n$ is extendible, then it is known (and easy to see) that so is every bilinear form on $X_i \times X_j$ for each pair $i \neq j$. Indeed, given $B \in \mathscr{L}^2(X_i \times X_j)$, we can multiply it by linear functionals to obtain a n-linear form on $X_1 \times \cdots \times X_n$. This is extendible by our hypothesis. From this, it is rather immediate to conclude that B is extendible. As a consequence, multilinear versions of our results follow directly from the bilinear ones.

If X and Y are Banach sequence spaces such that every bilinear form on $X \times Y$ is extendible, then we know from Theorem 2.3 (iii) that both X and Y contain the ℓ_∞^N uniformly. We can extend this statement to subspaces of Banach lattices.

PROPOSITION 2.6: *Let X_1, X_2 be subspaces of Banach lattices such that every n-linear form on X_1, X_2 is extendible. Then every infinite dimensional complemented subspace of each X_j contains the ℓ_∞^N uniformly.*

Proof. Suppose that there exists a complemented subspace E of X_1 that does not contain the ℓ_∞^N uniformly. By [21, Corollary 1], E must contain uniformly complemented N-dimensional subspaces E_N such that $\sup_N d(E_N, \ell_p^N) < \infty$ for $p = 1$ or 2. Since E is complemented in X_1, the E_N are also uniformly complemented in X_1. On the other hand, again by [21, Corollary 1], X_2 must contain uniformly complemented N-dimensional subspaces F_N such that $\sup_N d(F_N, \ell_q^N) < \infty$ for $q = 1, 2$ or ∞. Our hypotheses ensure that bilinear forms on $X_1 \times X_2$ are extendible. Since E_N and F_N are uniformly complemented in X_1 and X_2 and they are (uniformly) isomorphic to ℓ_p^N and ℓ_q^N, there must exist $K > 0$ such that $\|B\|_{\mathscr{E}^2(\ell_p^N,\ell_q^N)} \leq K\|B\|_{\mathscr{L}^2(\ell_p^N,\ell_q^N)}$ for all N. Now, the density lemma [13, Section 13.4] implies that every bilinear form on $\ell_p \times \ell_q$ (or $\ell_p \times c_0$) must be extendible, which contradicts Theorem 2.2. ∎

The converse of Proposition 2.6 does not hold. For example, the Schreier space is c_0-saturated and there are non-extendible bilinear forms on it (since there are bilinear forms which are not weakly sequentially continuous). Another conterexample of the converse is $d_*(w,1)$, the predual of the Lorentz sequence space $d(w,1)$ (see [27] or [16] for a description of the predual). Since these examples are Banach sequence spaces, they also show that assertion (iii) in Theorem 2.3 is strictly stronger than containing ℓ_∞^N uniformly.

In Banach sequence spaces, diagonal bilinear forms are the simplest ones. These are the bilinear forms $T_\alpha : X_1 \times X_2 \to \mathbb{C}$ given by

$$T_\alpha(x^1, x^2) = \sum_{k=1}^{\infty} \alpha_k x_k^1 x_k^2 ,$$

for some sequence $(\alpha_k)_k$ of scalars. We end this note showing, under some assumptions, which are the spaces on which all diagonal bilinear forms are extendible.

Following standard notation, given a symmetric Banach sequence space X we consider the fundamental function of X, given by $\lambda_X(N) := \big\| \sum_{k=1}^N e_k \big\|_X$ for $N \in \mathbb{N}$.

Given two sequences of real numbers $(a_n)_n$ and $(b_n)_n$ we write $a_n \preceq b_n$ whenever there is a universal constant $C > 0$ such that $a_n \leq C b_n$ for every n. If $a_n \preceq b_n$ and $b_n \preceq a_n$, we write $a_n \asymp b_n$.

THEOREM 2.7: *Let X and Y be symmetric Banach sequence spaces, each being 2-convex or 2-concave. Then all diagonal bilinear forms on $X \times Y$ are extendible if and only if either $X = Y = \ell_1$ or $X, Y \in \{c_0, \ell_\infty\}$.*

Proof. The *if* part is clear: by [8, Proposition 2.3] (see also [9, Proposition 1.2]) on ℓ_1 diagonal bilinear forms are integral (and, therefore, extendible), and in the other cases all bilinear forms are extendible.

For the converse, we consider the diagonal bilinear form given by $\phi_N(x,y) = \sum_{i=1}^N x_i y_i$. It is easily computed that $\|\phi_N\|_{\mathscr{L}^2(\ell_2^N)} = 1$; on the other hand, by [8, Proposition 1.1] or [11, Proposition 2.5] we have $\|\phi_N\|_{\mathscr{E}^2(\ell_2^N)} = \|\phi_N\|_{\mathscr{N}^2(\ell_2^N)} = N$.

Let now $\mathrm{id}_X^N : \ell_2^N \to X_N$ and $\mathrm{id}_Y^N : \ell_2^N \to Y_N$ be the identity mappings. Comparing the usual and extendible norms of the bilinear forms ϕ_N and $\phi_N \circ ((\mathrm{id}_X^N)^{-1}, (\mathrm{id}_Y^N)^{-1})$, we get

$$N \preceq \| \mathrm{id}_X^N \| \| \| \mathrm{id}_Y^N \| \| \| (\mathrm{id}_X^N)^{-1} \| (\mathrm{id}_Y^N)^{-1} \|.$$

Is there \| missing?

By [28, 16.4] (see also [14, page 138]), since X is a symmetric Banach sequence space we have $d(\ell_2^N, X_N) = \| \mathrm{id}_X^N \| \, \| (\mathrm{id}_X^N)^{-1} \|$ (and the same for Y_N). Therefore,

$$N \preceq \| \mathrm{id}_X^N \| \| \| \mathrm{id}_Y^N \| \| \| (\mathrm{id}_X^N)^{-1} \| (\mathrm{id}_Y^N)^{-1} \| = d(\ell_2^N, X_N) d(\ell_2^N, Y_N).$$

Is there \| missing?

Since we always have $d(\ell_2^N, X_N) \leq \sqrt{N}$ and $d(\ell_2^N, Y_N) \leq \sqrt{N}$, we can conclude that $\sqrt{N} \asymp d(\ell_2^N, X_N) = \| \mathrm{id}_X^N \| \, \| (\mathrm{id}_X^N)^{-1} \|$ (and the same for Y_N). We now apply [14, Lemma 1 (i)] and get

$$\max \Big(\frac{1}{\lambda_X(N)}, \frac{\lambda_X(N)}{N} \Big) \asymp 1.$$

From this we can conclude that X must be ℓ_1, c_0 or ℓ_∞. Indeed, suppose we split the natural numbers $\mathbb{N} = I \cup J$, so that $\big(\frac{1}{\lambda_X(N)}\big)_{N \in I} \asymp 1$ and $\big(\frac{\lambda_X(N)}{N}\big)_{N \in J} \asymp 1$. We have then that $(\lambda_X(N))_{N \in I}$ is bounded and $(N)_{N \in J} \preceq (\lambda_X(N))_{N \in J}$. Since $(\lambda_X(N))_{N \in \mathbb{N}}$ is non-decreasing, either I or J must be finite. If J is finite, then $(\lambda_X(N))_{N \in \mathbb{N}}$ is bounded and then the norm in X is equivalent to the sup norm and X is c_0 or ℓ_∞. If I is finite, then $N \preceq (\lambda_X(N))_{N \in \mathbb{N}}$. Although the fundamental sequence of a symmetric Banach sequence space does not characterize the norm, for this extreme case it is possible to prove that the norm on X must be isomorphic to ℓ_1: from the estimate $N \preceq (\lambda_X(N))_{N \in \mathbb{N}}$ we easily obtain $\lambda_{X^\times}(N) \asymp 1$ and, by the previous case, $X^\times$ must be ℓ_∞. Then we have $X = \ell_1$. Proceeding in the same way, Y has to be either ℓ_1, c_0 or ℓ_∞.

It remains to show that on $c_0 \times \ell_1$ and on $\ell_1 \times \ell_\infty$ there are non-extendible diagonal bilinear forms. The mapping $c_0 \times \ell_1$ given by $(x, x') \mapsto x'(x)$ is the diagonal bilinear form induced by the formal identity. An extension of this mapping to $c_0 \times \ell_\infty$ would give a projection from ℓ_∞ to ℓ_1 (see [13, 1.5]), which does not exist. For $\ell_1 \times \ell_\infty$ we can reason in a similar way. ∎

Both assumptions on symmetry and concavity/convexity in the “only if” part of the previous theorem cannot be omitted. Indeed, if we take $c_0 \oplus \ell_1$ (that seen as a sequence space is neither symmetric nor 2-concave or 2-convex), then every diagonal bilinear form on $(c_0 \oplus \ell_1) \times (c_0 \oplus \ell_1)$ is the sum of a diagonal bilinear form on $c_0 \times c_0$ and a diagonal bilinear form on $\ell_1 \times \ell_1$, and is therefore extendible.

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