Physics Letters A 374 (2010) 1481–1487

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Beyond the rotating wave approximation. An intensity dependent nonlinear coupling model in two-level systems

Horacio Grinberg *,1

Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Pabellón 1, Ciudad Universitaria, (1428) Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 16 November 2009
Received in revised form 12 January 2010
Accepted 20 January 2010
Available online 26 January 2010
Communicated by A.R. Bishop

Keywords:
Rotating wave approximation
Photon
Bimodal cavity
Normal squeezing
Variance squeezing
Entropy squeezing
Shannon information theory
Entanglement

ABSTRACT

An intensity dependent nonlinear coupling model of a two-level system interacting with a bimodal cavity field via two-photon transitions is investigated in a scenario where the rotating wave approximation is lifted. The model is numerically tested against simulations of normal squeezing variance and entropy squeezing factors based on the Heisenberg uncertainty principle and Shannon information theory derived from entangled states.

1. Introduction

The matter–radiation interaction is a central problem in quantum optics. The simplest model to deal with this is the Rabi model [1], which describes the interaction of a two-level atom with a single mode of the quantized electromagnetic field. Although widely studied over the past few decades, up to now an exact analytical solution is lacking and only numerical [2,3] and approximate analytical solutions are available [4,5], despite the conjecture by Reik and Doucha [6,7] that an exact solution of the Rabi Hamiltonian in terms of known functions is possible. The commonest analytical approach to solving the Rabi model is to make use of the rotating wave approximation (RWA), where the counter-rotating terms are neglected. In this limit, the Rabi Hamiltonian is known as Jaynes–Cummings (JC) Hamiltonian and can be integrated exactly [8,9]. In spite of the simplicity of the JC model, the dynamics have turned out to be very rich and complex. In fact, this model has revealed interesting phenomena related to the quantum nature of the light, encompassing the granular nature of the electromagnetic field, revealed through the existence of nonclassical effects such as revival of the atomic inversion, Rabi oscillations, squeezing [10,11], and atom–atom or atom–field entanglement [12].

The manipulation of atom–field interaction has been employed in cavity quantum electrodynamics, as well as in the atomic teleportation process, which have contributed to a fast development of quantum information theory [13]. With the experimental progress of some systems it was found that the coupling between the systems may be made very large, and the RWA breaks down so that only the Rabi model describes the dynamics correctly [14]. In addition, recent papers have questioned the validity of the RWA [14,15] and proposed alternative analytical approximate methods [4,5]. Moreover, it has been shown that the counter-rotating terms are responsible for several novel quantum-mechanical phenomena [16].

Strict analysis of the validity of the RWA is not usually considered in concrete applications, and the range of system parameters where the results are meaningful remains uncertain. Therefore, it is of great interest to explore approximate solutions of more complex Hamiltonians that contain the counter-rotating terms for a wide range of the system parameters, and compare them with the RWA results. Such studies are useful for determining the limits of validity of the JC models. In the present work the counter-rotating terms are explicitly taken into consideration along with a simultaneous inclusion of an intensity dependent coupling in the bosonic part of the interacting Hamiltonian. Further incorporation of two

* Tel.: +54 11 4576 3353; fax: +54 11 4576 3357.
E-mail address: grinberg@df.uba.ar.

[1] Member of the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) – República Argentina.

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2010.01.047

laser fields with different amplitudes and frequencies in the quantum regime involving two-photon transitions generates in essence a highly nonlinear Hamiltonian. The previously uninvestigated resulting two-level model will be tackled through time-dependent perturbation expansion of the evolution operator. Thus, density operator matrix elements which are necessary to compute different nonclassical properties arising from an excited state and its decay to the ground state in the two-level system are to be computed through first- and second-order of the chronologically ordered time-dependent perturbation expansion of the interaction picture evolution operator for the ground and excited states, respectively. Such a generalization is of considerable interest because of its relevance to the study of the coupling between a single atom and the radiation field with the atom making k-photon transitions. To illustrate the applicability of the model numerical simulations of the normal squeezing variance and entropy squeezing factors based on the Heisenberg uncertainty principle and Shannon information theory derived from entangled states are presented.

2. Atom–cavity interaction generalized model

Let us consider a bosonic system S, with Hilbert space $\mathcal{S}^{(S)}$ which is coupled with a two-level atom, with Hilbert space $\mathcal{S}^{(B)}$. Let us assume that the complete system is in thermal equilibrium with a reservoir at temperature β^{-1}. It is important to keep in mind that the presence of the reservoir only takes the atom (or in fact, any other two-level system) and the bosonic modes in thermal equilibrium. Let us denote by $\mathcal{H}_S$, $\mathcal{H}_B$, and $\mathcal{H}_I$ the Hamiltonians of the bosonic field, the two-level atom, and the interaction between both systems, respectively. The Hamiltonian for the total system can be written as

$$\mathcal{H} = \mathcal{H}_S \otimes I_B + I_S \otimes \mathcal{H}_B + \mathcal{H}_I \equiv \mathcal{H}_0 + \mathcal{H}_I, \tag{2.1}$$

where I_S and I_B denote the identities in the Hilbert spaces of the bosonic field and the atomic system.

The aim of this section is to introduce the necessary formalism to develop generalized models of two-level systems in which the counter-rotating terms are not ignored and an intensity dependent nonlinear coupling is explicitly incorporated in the Hamiltonian. It is further assumed that the electromagnetic field is associated with two modes of monocromatic radiation and induces two-photon transitions. This makes the present model to be strongly nonlinear and therefore only approximate solutions can be developed. The system dynamics will be explored through the density operator formalism emerging from the chronologically ordered time-dependent perturbation Dyson expansion. The complete Hamiltonian for such model reads

$$\mathcal{H} = \hbar \sum_{j=1}^{2} \nu_j a_j^\dagger a_j \otimes I_B + I_S \otimes \hbar/2\omega\sigma_z + \hbar \sum_{j=1}^{2} g_j\left(R_j^k + R_j^{\dagger k}\right) \otimes \sigma_x \equiv \mathcal{H}_0 + \mathcal{H}_I, \tag{2.2}$$

where ν_j and g_j are the photon frequency and atom–field coupling constant (vacuum Rabi frequency) for the mode j respectively; $a_j^\dagger$ (a_j) is the creation (annihilation) bosonic operator for the mode j, the zero-point energy of the bosonic field was omitted, and a constant term $1/2(\omega_a + \omega_b)$, where ω_a and ω_b are the energies of the ground ($|a\rangle$) and excited ($|b\rangle$) states of the two-level system, was ignored. As usual, we are using the pseudo-spin operators $\sigma^+ = |b\rangle\langle a|$, $\sigma^- = |a\rangle\langle b|$, $\sigma_x = |b\rangle\langle a| + |a\rangle\langle b|$, and $\sigma_z = |b\rangle\langle b| - |a\rangle\langle a|$ for the two-level atom which satisfy the standard angular momentum commutation relations corresponding to spin 1/2 Pauli operators, and therefore, they constitute a basis of the $SU(2)$ algebra. Thus, these atom-flip operators characterize the effective two-level system with transition frequency $\omega = \omega_b - \omega_a$. In Eq. (2.2) R_j^k and $R_j^{\dagger k}$ are intensity dependent *shifting* operators involving k photons, i.e.,

$$R_j^k = a_j^k\left(a_j^\dagger a_j\right)^{1/2}, \tag{2.3}$$

and its Hermitian conjugate

$$R_j^{\dagger k} = \left(a_j^\dagger a_j\right)^{1/2} a_j^{\dagger k}. \tag{2.4}$$

In the present generalized model the counter-rotating terms $R_j^k\sigma^- + R_j^{\dagger k}\sigma^+$, ignored under WRA, are retained in the Hamiltonian (2.2). It is convenient to work in the interaction picture with the Hamiltonian given by

$$\mathcal{V}(t) = \mathcal{U}_0^\dagger(t)\mathcal{H}_I\mathcal{U}_0(t), \tag{2.5}$$

where $\mathcal{U}_0$ is the unitary time evolution operator for the unperturbed Hamiltonian and which merely contributes a phase factor in each atomic subspace. Using the expansion

$$\exp(\alpha A)B\exp(-\alpha A) = B + \alpha[A,B] + \alpha^2/2!\big[A,[A,B]\big] + \cdots, \tag{2.6}$$

along with the commutation relations

$$\left[a_j^\dagger a_j, R_i^k\right] = -kR_i^k\delta_{ij}, \tag{2.7}$$

$$\left[a_j^\dagger a_j, R_i^{\dagger k}\right] = kR_i^{\dagger k}\delta_{ij}, \tag{2.8}$$

and consequently noting that

$$\exp\left(i\nu_j a_j^\dagger a_j t\right)R_j^k\exp\left(-i\nu_j a_j^\dagger a_j t\right) = R_j^k\exp(-i\nu_j kt), \tag{2.9}$$

$$\exp\left(i\nu_j a_j^\dagger a_j t\right)R_j^{\dagger k}\exp\left(-i\nu_j a_j^\dagger a_j t\right) = R_j^{\dagger k}\exp(i\nu_j kt), \tag{2.10}$$

$$\exp(i\omega t\sigma_z/2)\sigma_x\exp(-i\omega t\sigma_z/2) = \begin{pmatrix} 0 & \exp(i\omega t) \\ \exp(-i\omega t) & 0 \end{pmatrix}, \tag{2.11}$$

the interaction picture Hamiltonian (2.5) can be written as

$$\mathcal{V}(t) = \hbar\begin{pmatrix} 0 & \exp(i\omega t) \\ \exp(-i\omega t) & 0 \end{pmatrix}\sum_{j=1}^{2} g_j\left(R_j^k\exp(i\Delta_j t) + \text{h.c.}\right), \tag{2.12}$$

with the detuning parameter Δ_j for the mode j given by

$$\Delta_j = \omega - k\nu_j. \tag{2.13}$$

These detunings between the cavity mode and the atomic transition can have an important influence on the nonclassical effects, as recently reported in the case of a two-level atom coupled to a single mode of cavity fields [17].

The dynamics of the present model is not stationary and depends on the initial conditions of the system and the cavity field. Thus, it is assumed that, initially, the field modes are in coherent states and the atomic system is in the excited state $|b\rangle$, that is, the atomic system and the field are initially in a disentangled state. It is further assumed that at $t = 0$, the two modes have the same photon distribution with density operator

$$\rho(0) = \big|\psi(0)\big\rangle\big\langle\psi(0)\big| = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} c_{n_1n_2}(0)c^*_{m_1m_2}(0)|b;m_1m_2\rangle\langle b;n_1n_2| \tag{2.14}$$

where

$$c_{m_1m_2}(0)=\big[\rho_{m_1m_1}(0)\rho_{m_2m_2}(0)\big]^{1/2}. \tag{2.15}$$

Since in this model system $\mathcal{H}_0$ does not commute with $\mathcal{H}_I$, the set of interaction picture Hamiltonians $\mathcal{V}(t_1),\mathcal{V}(t_2),\dots$, taken at different times $t_1,t_2,\dots$, fail to commute. In fact, after some rather lengthy algebra, the commutator $[\mathcal{H}(t_1),\mathcal{H}(t_2)]$ is found to be

$$\begin{aligned}
&\big[\mathcal{V}(t_1),\mathcal{V}(t_2)\big]\\
&\quad=2i\sum_{j=1}^{2}g_j^2\bigg[\sin\big(\omega(t_1-t_2)\big)\sigma_z\big(R_j^{2k}\exp\big(i\Delta_j(t_1+t_2)\big)+\text{h.c.}\big)\\
&\qquad+\sin\big(\Delta_j(t_1-t_2)\big)\begin{pmatrix}0 & \exp(i\omega(t_1-t_2))\\ \exp(-i\omega(t_1-t_2)) & 0\end{pmatrix}\\
&\qquad\times\big[R_j^k,R_j^{\dagger k}\big]\bigg]\\
&\qquad+2ig_1g_2\sin\big(\omega(t_1-t_2)\big)\sigma_z\big[\big(R_1^kR_2^k\alpha(t_1,t_2)+\text{h.c.}\big)\\
&\qquad+\big(R_1^{\dagger k}R_2^k\beta(t_1,t_2)+\text{h.c.}\big)\big],
\end{aligned}\tag{2.16}$$

with the two-time dependent scalar functions $\alpha(t_1,t_2)$ and $\beta(t_1,t_2)$ given by

$$\begin{aligned}
\alpha(t_1,t_2)&=\exp\big[i(\Delta_2t_1+\Delta_1t_2)\big]\\
&\quad+\exp\big[i(\Delta_1t_1+\Delta_2t_2)\big],
\end{aligned}\tag{2.17}$$

$$\begin{aligned}
\beta(t_1,t_2)&=\exp\big[i(\Delta_2t_1-\Delta_1t_2)\big]\\
&\quad+\exp\big[i(-\Delta_1t_1+\Delta_2t_2)\big].
\end{aligned}\tag{2.18}$$

We deal with off-resonant states (this corresponds obviously to the nondegenerate case of the two modes of the field), i.e., $\Delta_j\neq 0$ ($j=1,2$) with $\Delta_1\neq\Delta_2$ and with exact resonance of one mode (i.e., $\Delta_1\neq 0$, $\Delta_2=0$). Therefore, perturbation expansion of the time evolution operator matrix elements truncated to a finite-order has to be used. The time evolution operator in the interaction picture reads (Dyson expansion)

$$\mathcal{U}_I(t)=\mathcal{F}\exp\left[-\frac{i}{\hbar}\int_0^t\mathcal{V}(t)\,dt\right], \tag{2.19}$$

where $\mathcal{F}$ is the time-ordering operator, which is a shorthand notation for the expansion

$$\begin{aligned}
&\mathcal{F}\exp\left[-\frac{i}{\hbar}\int_0^t\mathcal{V}(t)\,dt\right]\\
&\quad=1-\frac{i}{\hbar}\int_0^t\mathcal{V}(t_1)\,dt_1+\left(\frac{-i}{\hbar}\right)^2\int_0^t dt_1\int_0^{t_1}dt_2\,\mathcal{V}(t_1)\mathcal{V}(t_2)\\
&\qquad+\left(\frac{-i}{\hbar}\right)^3\int_0^t dt_1\int_0^{t_1}dt_2\int_0^{t_2}dt_3\,\mathcal{V}(t_1)\mathcal{V}(t_2)\mathcal{V}(t_3)+\cdots\\
&\quad\equiv\sum_{n=0}^{\infty}\mathcal{U}_{In}(t).
\end{aligned}\tag{2.20}$$

The different contributions to the interaction picture time evolution operator up to second-order are given by ($\mathcal{U}_{I0}=1$)

$$\mathcal{U}_{I1}(t)=\sum_{j=1}^{2}g_j\big(A_j(\Delta_j;t)R_j^k+A_j(-\Delta_j;t)R_j^{\dagger k}\big), \tag{2.21}$$

where the *two-signature* matrix $A_j(\Delta_j;t)$ is given by

$$A_j(\Delta_j;t)=\begin{pmatrix}0 & \phi_j(\omega;\Delta_j;t)\\ \phi_j(-\omega;\Delta_j;t) & 0\end{pmatrix}, \tag{2.22}$$

with the time-dependent scalar functions $\phi_j(\omega;\Delta_j;t)$

$$\phi_j(\omega;\Delta_j;t)=\frac{1-\exp(i(\omega+\Delta_j)t)}{\omega+\Delta_j}. \tag{2.23}$$

The second-order contribution is

$$\begin{aligned}
\mathcal{U}_{I2}(t)=\sum_{i,j=1}^{2}g_ig_j\big(&R_i^kR_j^kA_{ij}(\Delta_i;\Delta_j;t)\\
&+R_i^kR_j^{\dagger k}A_{ij}(\Delta_i;-\Delta_j;t)+R_i^{\dagger k}R_j^kA_{ij}(-\Delta_i;\Delta_j;t)\\
&+R_i^{\dagger k}R_j^{\dagger k}A_{ij}(-\Delta_i;-\Delta_j;t)\big),
\end{aligned}\tag{2.24}$$

where the *four-signature* matrix $A_{ij}(\Delta_i;\Delta_j;t)$ reads

$$\begin{aligned}
&A_{ij}(\Delta_i;\Delta_j;t)\\
&\quad=\begin{pmatrix}\phi_{ij}(\omega;\Delta_i;\Delta_j;t) & 0\\ 0 & \phi_{ij}(-\omega;\Delta_i;\Delta_j;t)\end{pmatrix},
\end{aligned}\tag{2.25}$$

with the time-dependent scalar functions $\phi_{ij}(\omega;\Delta_i;\Delta_j;t)$ given by

$$\begin{aligned}
\phi_{ij}(\omega;\Delta_i;\Delta_j;t)=\frac{\exp(i\Delta_jt)}{\omega-\Delta_i}&\left(\frac{\exp(i\omega t)}{\omega+\Delta_j}-\frac{\exp(i\Delta_it)}{\Delta_i+\Delta_j}\right)\\
&+\frac{1}{(\omega+\Delta_j)(\Delta_i+\Delta_j)},
\end{aligned}\tag{2.26}$$

along with the diagonal contributions

$$\phi_{ii}(\omega;\pm\Delta_i;\mp\Delta_i;t)=\frac{\exp(i(\omega\mp\Delta_i)t)}{(\omega\mp\Delta_i)^2}-\frac{it}{\omega\mp\Delta_i}. \tag{2.27}$$

The entangled interaction picture state vector at any time t emerges from the coherent state $|\psi(0)\rangle$ implicit in Eq. (2.14) via the unitary time-evolution operator $\mathcal{U}_I(t)$

$$\big|\psi_I(t)\big\rangle=\sum_{n=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}c_{m_1m_2}(0)\mathcal{U}_{In}(t)|b;m_1m_2\rangle. \tag{2.28}$$

Up to this point the developed formalism is completely general, allowing to investigate k photons transitions within the framework of time-dependent perturbation theory. The present study will be restricted to two-photon transitions, i.e., $k=2$ in Eqs. (2.21) and (2.24). Different contributions to the time-evolution operator matrix elements can be evaluated through

$$a_i^p|n_i\rangle=\sqrt{\frac{n_i!}{(n_i-p)!}}|n_i-p\rangle, \tag{2.29}$$

$$a_j^{\dagger q}|n_j\rangle=\sqrt{\frac{(n_j+q)!}{n_j!}}|n_j+q\rangle, \tag{2.30}$$

with $i,\ j=1,2$ and $p\leqslant n_i$. Thus, projection of $\psi_I(t)$ onto $\langle a;n_1n_2|$ allows to write the first-order contribution to the time-evolution operator matrix element as

$$\langle a;n_1n_2|\mathcal{U}_{I1}(t)|b;m_1m_2\rangle=\sum_{i\neq j}g_i\big(\mathcal{U}_{I1;ij}^{+}(t)+\mathcal{U}_{I1;ij}^{-}(t)\big), \tag{2.31}$$

with the signatures

$$\mathcal{U}_{I1;ij}^{+}(t)=Q_{m_i}\phi_i^{+}(-\omega;t)\delta_{n_im_i-2}\delta_{n_jm_j}, \tag{2.32}$$

and

$$\mathcal{U}^{-}_{I1;ij}(t)=Q_{m_i+2}\phi^{-}_i(-\omega;t)\delta_{n_1m_i+2}\delta_{n_j,m_j},\tag{2.33}$$

where the notations $\phi_i^+(-\omega;t)\equiv\phi_i(-\omega;\Delta_i;t)$ and $\phi_i^-(-\omega;t)\equiv\phi_i(-\omega;-\Delta_i;t)$ have been used and where $Q_{m_i}=m_i(m_i-1)^{1/2}$. With these ingredients the density operator diagonal matrix element for the ground state $|a\rangle$ and for specific quantum numbers n_1, n_2 of the bosonic field is obtained as

$$\rho^{aa}_{n_1n_2}(t)=\left|\sum_{i\neq j}g_i\left(\sqrt{\rho^{+}_{ij}(t)}+\sqrt{\rho^{-}_{ij}(t)}\right)\right|^2,\tag{2.34}$$

where $\sqrt{\rho^{+}_{ij}(t)}$ and $\sqrt{\rho^{-}_{ij}(t)}$ are obtained from Eq. (2.28) using (2.15) for the initial condition at $t=0$, and contracting the resulting ket $|\psi_I(t)\rangle$ by means of the orthonormality of the photon states $|n_1n_2\rangle=|n_1\rangle\otimes|n_2\rangle$. This procedure yields

$$\sqrt{\rho^{+}_{ij}(t)}=Q_{n_i+2}\phi^{+}_i(-\omega;t)\sqrt{\rho_{n_i+2n_i+2}(0)\rho_{n_jn_j}(0)},\tag{2.35}$$

and

$$\sqrt{\rho^{-}_{ij}(t)}=Q_{n_i}\phi^{-}_i(-\omega;t)\sqrt{\rho_{n_i-2n_i-2}(0)\rho_{n_jn_j}(0)}.\tag{2.36}$$

Similar arguments show that the nonvanishing zeroth- and second-order contributions to the time dependent evolution operator matrix element for the excited state $|b\rangle$ can be written as

$$\langle b;n_1n_2|U_{I0}|b;m_1m_2\rangle=\delta_{n_1m_1}\delta_{n_2m_2},\tag{2.37}$$

and

$$\langle b;n_1n_2|\mathcal{U}_{I2}(t)|b;m_1m_2\rangle=\sum_{i\neq j}\left(\mathcal{U}^{++}_{I2;ij}(t)+\mathcal{U}^{+-}_{I2;ij}(t)+\mathcal{U}^{-+}_{I2;ij}(t)+\mathcal{U}^{--}_{I2;ij}(t)\right),\tag{2.38}$$

respectively. Here the different signatures are given by

$$\mathcal{U}^{++}_{I2;ij}(t)=g_i^2Q_{n_i}Q_{n_i-2}\phi^{++}_{ii}(\omega;t)\delta_{n_im_i-4}\delta_{n_jm_j}+g_ig_jQ_{n_i}Q_{n_j}\phi^{++}_{ij}(\omega;t)\delta_{n_im_i-2}\delta_{n_jm_j-2},\tag{2.39}$$

with $\phi^{++}_{ij}(\omega;t)\equiv\phi_{ij}(\omega;\Delta_i;\Delta_j;t)$;

$$\mathcal{U}^{+-}_{I2;ij}(t)=g_i^2Q^2_{n_i+2}\phi^{+-}_{ii}(\omega;t)\delta_{n_im_i}\delta_{n_jm_j}+g_ig_jQ_{n_i}Q_{n_j+2}\phi^{+-}_{ij}(\omega;t)\delta_{n_1m_i-2}\delta_{n_jm_j+2},\tag{2.40}$$

with $\phi^{+-}_{ij}(\omega;t)\equiv\phi_{ij}(\omega;\Delta_i;-\Delta_j;t)$;

$$\mathcal{U}^{-+}_{I2;ij}(t)=g_i^2Q^2_{n_i}\phi^{-+}_{ii}(\omega;t)\delta_{n_im_i}\delta_{n_jm_j}+g_ig_jQ_{n_i+2}Q_{n_j}\phi^{-+}_{ij}(\omega;t)\delta_{n_im_i+2}\delta_{n_jm_j-2},\tag{2.41}$$

with $\phi^{-+}_{ij}(\omega;t)\equiv\phi_{ij}(\omega;-\Delta_i;\Delta_j;t)$;

$$\mathcal{U}^{--}_{I2;ij}(t)=g_i^2Q_{n_i+2}Q_{n_i+4}\phi^{--}_{ii}(\omega;t)\delta_{n_im_i+4}\delta_{n_jm_j}+g_ig_jQ_{n_i+2}Q_{n_j+2}\phi^{--}_{ij}(\omega;t)\delta_{n_im_i+2}\delta_{n_jm_j+2},\tag{2.42}$$

with $\phi^{--}_{ij}(\omega;t)\equiv\phi_{ij}(\omega;-\Delta_i;-\Delta_j;t)$. Therefore, the density operator diagonal matrix element for the excited state $|b\rangle$ becomes

$$\rho^{bb}_{n_1n_2}(t)=\left|\sqrt{\rho_{n_1n_1}(0)\rho_{n_2n_2}(0)}+\sum_{i\neq j}\left(\sqrt{\rho^{++}_{ij}(t)}+\sqrt{\rho^{+-}_{ij}(t)}+\sqrt{\rho^{-+}_{ij}(t)}+\sqrt{\rho^{--}_{ij}(t)}\right)\right|^2,\tag{2.43}$$

with the different signatures of the time dependent density operators given by

$$\sqrt{\rho^{++}_{ij}(t)}=g_i^2Q_{n_i+4}Q_{n_i+2}\phi^{++}_{ii}(\omega;t)\sqrt{\rho_{n_i+4n_i+4}(0)\rho_{n_jn_j}(0)}+g_ig_jQ_{n_i+2}Q_{n_j+2}\phi^{++}_{ij}(\omega;t)\sqrt{\rho_{n_i+2n_i+2}(0)\rho_{n_j+2n_j+2}(0)},\tag{2.44}$$

$$\sqrt{\rho^{+-}_{ij}(t)}=g_i^2Q^2_{n_i+2}\phi^{+-}_{ii}(\omega;t)\sqrt{\rho_{n_in_i}(0)\rho_{n_jn_j}(0)}+g_ig_jQ_{n_i+2}Q_{n_j}\phi^{+-}_{ij}(\omega;t)\sqrt{\rho_{n_i+2n_i+2}(0)\rho_{n_j-2n_j-2}(0)},\tag{2.45}$$

$$\sqrt{\rho^{-+}_{ij}(t)}=g_i^2Q^2_{n_i}\phi^{-+}_{ii}(\omega;t)\sqrt{\rho_{n_in_i}(0)\rho_{n_jn_j}(0)}+g_ig_jQ_{n_i}Q_{n_j+2}\phi^{-+}_{ij}(\omega;t)\sqrt{\rho_{n_i-2n_i-2}(0)\rho_{n_j+2n_j+2}(0)},\tag{2.46}$$

$$\sqrt{\rho^{--}_{ij}(t)}=g_i^2Q_{n_i-2}Q_{n_i}\phi^{--}_{ii}(\omega;t)\sqrt{\rho_{n_i-4n_i-4}(0)\rho_{n_jn_j}(0)}+g_ig_jQ_{n_i}Q_{n_j}\phi^{--}_{ij}(\omega;t)\sqrt{\rho_{n_i-2n_i-2}(0)\rho_{n_j-2n_j-2}(0)},\tag{2.47}$$

while the off-diagonal matrix elements $\rho^{ab}_{n_1n_2}(t)$ satisfy

$$\left|\rho^{ab}_{n_1n_2}(t)\right|^2\leqslant\rho^{aa}_{n_1n_2}(t)\rho^{bb}_{n_1n_2}(t),\tag{2.48}$$

with $\rho^{ab}_{n_1n_2}(t)=\rho^{ba*}_{n_1n_2}(t)$ and the equality holding only for pure states. Thus, the density operator matrix elements $\rho^{aa}_{n_1n_2}(t)$, $\rho^{bb}_{n_1n_2}(t)$, and $\rho^{ab}_{n_1n_2}(t)$ were generated from the initial condition at $t=0$ via $\rho_{n_1n_2}(0)$ and from certain time dependent scalar functions arising from the time dependent Dyson expansion of the evolution operator. Various nonclassical effects in the present model can be generated by choosing different initial states of the field and the system. As the number of excitations, field plus atom, is a constant of motion in the JC model, there are only two physically relevant parameters: detuning Δ_j between the atomic transition frequency ω and the field mode frequency ν_j and atom–field coupling g_j. In the present generalized model, however, the number of excitations is not conserved and all three parameters are of importance. In the next section the normal and entropy squeezing, based on the Heisenberg (variance) relation and on the information quantum entropy will be discussed through numerical simulations for specific values of these parameters.

3. Variance and Shannon information entropy squeezing

Squeezing phenomenon is one of the most interesting phenomena in the field of quantum optics [18]. It reflects the nonclassical behavior for the quantum systems. Squeezed light has less noise in one of the field quadratures than the vacuum level and an excess of noise in the other quadrature such that the Heisenberg uncertainty principle is satisfied. To discuss the normal squeezing we use the squeezing factor

$$Q_{ij}=1-4\left\langle\left(\Delta X^{(i)}_j\right)^2\right\rangle,\tag{3.1}$$

where $\langle(\Delta X_j^{(i)})^2\rangle$ is the variance for the mode i and field quadrature operator j, defined in terms of the creation $a_i^\dagger(t)$ and annihilation $a_i(t)$ photon operators. These variances satisfy the Heisenberg uncertainty principle and therefore, by definition, squeezing is said to exist whenever the variance $\langle(\Delta X_j^{(i)})^2\rangle$ is below the standard quantum limit. Thus $0 < Q_{ij} \leqslant 1$ for squeezing. The required expectation values of the normally-ordered operators involved in Eq. (3.1) can be computed via density operator traces standard techniques [11] with the initial state of the field given in terms of the Poisson distribution

$$\rho_{n_i n_i}(0) = \frac{\langle n_i\rangle^{n_i} e^{-\langle n_i\rangle}}{n_i!}, \tag{3.2}$$

where $\langle n_i\rangle$ is the initial mean photon number for the mode i.

Fig. 1 shows the time evolution of the second-order quadrature variance $\langle(\Delta X_1^{(2)})^2\rangle$ computed for two different sets of numerical parameters, assuming that the initial state of the atomic system is the excited state and the field modes are initially in coherent states. In Fig. 1(a) the following parameters were used: $\omega = 83\ \text{cm}^{-1}$, $\Delta_1 = 53\ \text{cm}^{-1}$, $\Delta_2 = 37\ \text{cm}^{-1}$, $g_1 = 400\ \text{cm}^{-1}$, $g_2 = 833\ \text{cm}^{-1}$, $\langle n_1\rangle = 25$, $\langle n_2\rangle = 12$. In the short time regime it is observed that the time evolution does bring about noise reduction in this quadrature when the counter-rotating terms are retained (solid line). However, incorporation of the rotating wave approximation (dashed line) does not produce any evidence of such reduction over the time scale considered. This is to be expected since the ratio g_i/Δ_i corresponds in this case to the limit of strong coupling for both modes, where the RWA is not applicable. The persistent behavior of noise reduction in the limit of no rotating wave approximation is also evident in the limit of exact resonance of one of the field modes with the transition frequency of the system. This is evident in Fig. 1(b), where the system parameters are $\omega = 1267\ \text{cm}^{-1}$, $\Delta_1 = 833\ \text{cm}^{-1}$, $\Delta_2 = 0$, $g_1 = 3.3\ \text{cm}^{-1}$, $g_2 = 600\ \text{cm}^{-1}$, $\langle n_1\rangle = 28$, $\langle n_2\rangle = 10$. This figure shows a reasonable amount of squeezing (more than 25%) over the whole range of the time scale considered. A comparison of Figs. 1(a) and 1(b) shows that the curves monotonically approach a plateau as it should for an irreversible process. In both of these cases the system remains unsqueezed and squeezed, respectively, for all t. Thus, in off-resonant and nondegenerate states ($0 \neq \Delta_1 \neq \Delta_2 \neq 0$) as well as in states of exact resonance of one mode ($\Delta_1 \neq 0$, $\Delta_2 = 0$) it is then apparent that the value of the coupling constants as well as the atomic transition and field mode frequencies all play an important role in determining the behavior of the normal second-order squeezing. Squeezed states, even those appearing in transient times, could be of interest for quantum information processing.

For a two-level system, characterized by the Pauli operators $(\sigma_x, \sigma_y, \sigma_z)$, the Heisenberg uncertainty relation is given by

$$\Delta\sigma_i \Delta\sigma_j \geqslant \left|\langle\sigma_k\rangle\right|\epsilon_{ijk} \tag{3.3}$$

where $\Delta\sigma = [\langle\sigma^2\rangle - \langle\sigma\rangle^2]^{1/2}$, with the commutation relations $\sigma \wedge \sigma = 2i\sigma$. Fluctuations in the component σ_i of the atomic system are said to be squeezed if σ_i satisfies the condition

$$V(\sigma_i) = \Delta\sigma_i - \left|\langle\sigma_j\rangle\right|^{1/2} < 0, \quad i \neq j. \tag{3.4}$$

However, as claimed by Mao-Fa Fang et al. [19] the value of $\langle\sigma_i\rangle$ is highly dependent on the atomic states used to perform the average, and may be zero for some states, in which case the uncertainty relation Eq. (3.3) is trivially satisfied (as $\Delta\sigma_i \geqslant 0$) and fails to provide any useful information. For example, for some states one can have $\langle\sigma_i\rangle = 0$, and therefore is not possible to obtain any information on squeezing from the inequality (3.3). Actually, these states may be considered to be maximally squeezed states

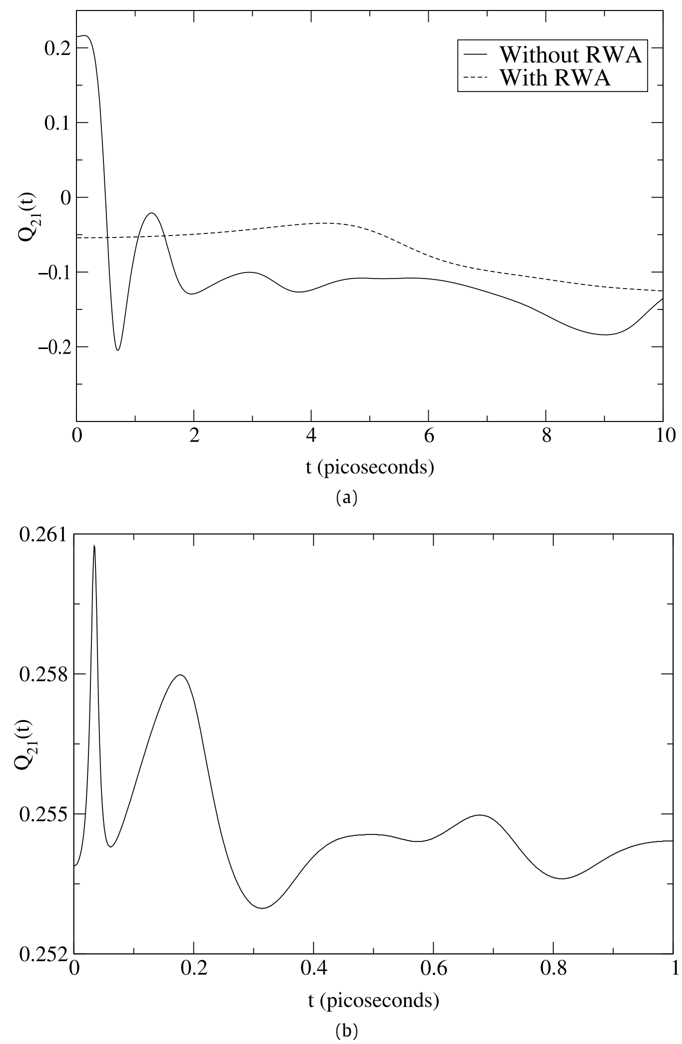

Fig. 1. Time evolution of the normal squeezing factor Q_{ij} of a two-level atom interacting with a bimodal cavity. The atom is initially in the excited state and the field in the coherent state. The value of the different parameters is given in the text. (a) Without RWA and with RWA; (b) without RWA.

of the atomic system from the entropy point of view. To overcome the limitations of the Heisenberg uncertainty relation quantum entropy theory based on Shannon information theory must be used [19,20].

In an even N-dimensional Hilbert space, the investigation of the optimal entropic uncertainty relation for sets of $N+1$ complementary observables with nondegenerate eigenvalues can be described by the inequality [21,22]

$$\sum_{i=1}^{N+1} H(\sigma_i) \geqslant \frac{N}{2}\ln\left(\frac{N}{2}\right) + \left(1+\frac{N}{2}\right)\ln\left(1+\frac{N}{2}\right), \tag{3.5}$$

where $H(\sigma_i)$ represents the Shannon information entropy associated to the observable σ_i

$$H(\sigma_i) = -\operatorname{Tr}\rho(\sigma_i)\ln\rho(\sigma_i), \tag{3.6}$$

and $\rho(\sigma_i)$ is the reduced density operator associated to the atomic distribution, obtained by taking the trace of the density matrix over the bosonic field. For a pure disentangled state $H(\sigma_i) = 0$.

Since the uncertainty relation of the entropy can be used as a general criterion for the squeezing in the entropy of an atom, for a two-level atom where $N = 2$, we have $0 \leqslant H(\sigma_i) \leqslant \ln 2$, and hence

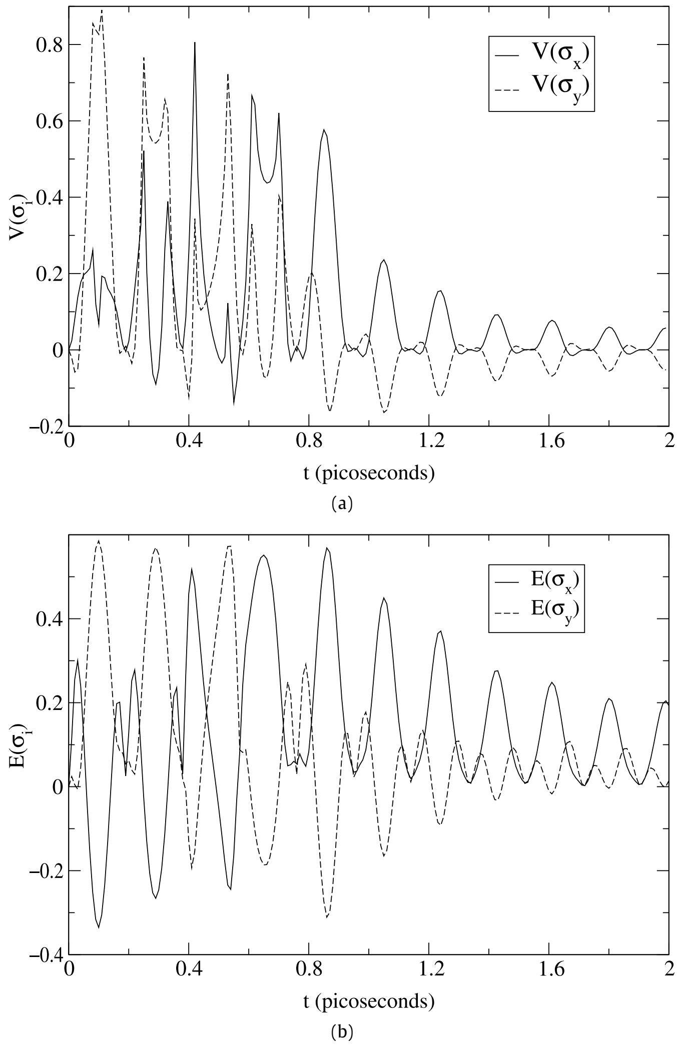

Fig. 2. Time evolution of the variance entropy squeezing factors of a two-level atom interacting with a bimodal cavity without RWA. The atom is initially in the excited state and the field in a coherent state. The value of the different parameters is given in the text. (a) Variance squeezing factor based on the Heisenberg uncertainty relation $V(\sigma_i)$; (b) entropy squeezing factor based on the information entropy $E(\sigma_i)$.

it follows that the information entropies of the operators σ_x, σ_y, and σ_z will satisfy the inequality

$$\delta H(\sigma_x)\delta H(\sigma_y)\delta H(\sigma_z) \geqslant 4, \tag{3.7}$$

where it was defined $\delta H(\sigma_i) = \exp[H(\sigma_i)]$. $\delta H(\sigma_i) = 1$ corresponds to the atom being in a pure state and $\delta H(\sigma_i) = 2$ corresponds to the atom being in a completely mixed state. In fact, the Hamiltonian given by Eq. (2.2) leads to the entanglement between the two-mode entangled coherent field and the atom, so that the states of the two-mode entangled coherent field may evolve into a mixed state. In this case, the degree of entanglement between the two-mode field and the atomic system can be measured by the quantum relative entropy of the entanglement. Thus, fluctuations in the component σ_i of the atomic dipole are said to be squeezed in entropy if the information entropy $H(\sigma_i)$ satisfies the condition

$$E(\sigma_i) = \delta H(\sigma_i) - \frac{2}{\sqrt{\delta H(\sigma_j)}} < 0, \quad i \neq j. \tag{3.8}$$

The time evolution of the squeezing factors, $V(\sigma_x)$, $V(\sigma_y)$, $E(\sigma_x)$, and $E(\sigma_y)$ are shown in Fig. 2. In this figure the values of the different parameters for the first and second quadratures of the variance and entropy squeezing are $\omega = 600\ \text{cm}^{-1}$, $\Delta_1 = 500\ \text{cm}^{-1}$, $\Delta_2 = 0$, $g_1 = 1.3\ \text{cm}^{-1}$, $g_2 = 0.8\ \text{cm}^{-1}$, with the atom initially in the excited state and the field in a coherent state with average photon numbers $\langle n_1\rangle = 15$, $\langle n_2\rangle = 9$. It is seen that at $t \geqslant 0.4$ ps squeezing occurs several times in the second quadrature of the entropy squeezing $E(\sigma_y)$, but it is absent from the first quadrature $E(\sigma_x)$. For the variance squeezing the situation is similar where the squeezing occurs twice in $V(\sigma_x)$ at times less than 0.6 ps and several times from the quadrature $V(\sigma_y)$ over the whole range of the time scale considered. It is also noted that the period of the oscillations in both quadratures of the entropy and variance squeezing is very similar, with a regular pattern above 0.8 ps, where squeezing is only observed for the second quadrature of the entropy and variance squeezing. In addition, rapid fluctuations occurring at $t < 0.6$ ps are observed in the first quadrature of variance squeezing, with some interferences between the patterns pronounced in $E(\sigma_x)$ and $V(\sigma_x)$. However, we realize that for $t > 0.8$ ps there is a decreasing in the variance and entropy squeezing amount in the second quadrature, with regular fluctuations in both quadratures. On the other hand, a reduction in the amplitudes in both quadratures of squeezing factors $V(\sigma_x)$ and $E(\sigma_x)$ is clearly observed. In other words, the time evolution of the degree of entanglement between the two-mode field and the atomic system carries out a damping oscillation, which means that the atom and two-mode field combined system cannot recover its initial maximal entangled state periodically. In the long time regime this clearly corresponds to an irreversible phenomenon due to the decoherence resulting from the energy exchange between the atom and the field modes during the interaction process. In the vicinity of $t \simeq 0.8$ ps a ghost is observed in the variance and entropy squeezing factors in the form of a sudden small variation in the entropy. This dip indicates a competition between the irreversible effects of chaos and reversible effects.

Having discussed the general behavior of the variance and entropy squeezing it should be addressed some particular points. For example, $E(\sigma_x)$ in Fig. 2(b) exhibits an almost optimal entropy squeezing $E(\sigma_x) \simeq -0.34 < 0$ at the time 0.1 ps, just 0.07 units above the optimal entropy squeezing factor,[2] i.e., at this time the atom has achieved an almost pure state with a small entanglement ($\mathrm{Tr}\,\rho^2(t) \simeq 0.97$ at $t = 0.1$ ps). This state is just an eigenstate of the atomic operator σ_x. By contrast, at $t = 0.1$ ps the second quadrature σ_y has an entropy squeezing factor $E(\sigma_y) \simeq 0.586 > 0$ footnote 2 and therefore this state evolves towards a completely mixed entanglement state. This shows that the operator σ_x exhibits an almost optimal entropy squeezing, while no entropy squeezing occurs in the operator σ_y. In fact, if the atom was exactly in an eigenstate of σ_x, we would have $\Delta\sigma_x = 0$, its smallest possible value. However, $V(\sigma_x)$, as it can be seen in Fig. 2(a), does not exhibit any variance squeezing, since the atomic inversion satisfies $\langle\sigma_z\rangle = 0$ at that time, and the value of $V(\sigma_x)$ has no particular significance. In this case, the information provided by the Heisenberg uncertainty relation (3.3) is not helpful. A similar analysis can be made for the other two maxima (peaked at $E(\sigma_y) \simeq 0.586$) and the concomitant two minima observed for the second and first quadrature entropy squeezing factors respectively.

4. Final remarks

In this Letter a previously uninvestigated highly nonlinear model consisting of a two-level atom interacting with a bimodal cavity field via two-photon transitions was implemented. The pro-

[2] Using Eqs. (3.6) and (3.7), the information entropies of the atomic operators σ_x, σ_y, σ_z are obtained as $H(\sigma_x) = 0$, $H(\sigma_y) = H(\sigma_z) = \ln 2$. Correspondingly, we find $\delta H(\sigma_x) = 1$ and $\delta H(\sigma_y) = \delta H(\sigma_z) = 2$. Using these results we calculate the entropy squeezing factors $E(\sigma_x) \simeq -0.414 < 0$ and $E(\sigma_y) \simeq 0.586 > 0$ for the disentangled and entangled states respectively.

cedure, based on the Dyson expansion of the time evolution operator matrix elements for the initial (excited) and ground states, retains the counter-rotating terms in the Hamiltonian along with an intensity dependent nonlinear coupling, which renders the present method to be a novel procedure allowing to further investigate nonclassical effects in two-level systems. This is especially relevant in the strong coupling regime, where other approaches such as the RWA fail to give reasonable results or tend to be cumbersome. The method developed is purely analytical, but once the time evolved density matrix elements are obtained, all various physical quantities are easily numerically obtainable. Thus, the normal and entropy squeezing, based on the Heisenberg (variance) and on the entropic uncertainty relations were discussed. The results obtained show that the Shannon information entropy satisfactorily explains the entanglement phenomenon in terms of a measure of the quantum uncertainty of atomic operators and seem to confirm the idea that the concept of entropic squeezing is preferable to that of variance squeezing based on the Heisenberg uncertainty principle containing only second-order statistical moments, in agreement with previous findings [19,23]. The model may be meaningful to explore the dynamics of nonclassical effects in more general models, such as two-mode Raman coupled model [24] and resonance fluorescence, e.g., squeezing of a two-level atom resonantly driven by a laser field [19]. Work along these lines is in progress and will be reported at a later time.

Acknowledgements

This work has been made possible by research grants in aid from the University of Buenos Aires (Project No. X-017/08), and the Consejo Nacional de Investigaciones Científicas y Técnicas (PIP No. 05098/05). The author is grateful to the Department of Physics, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, for facilities provided during the course of this work.

References

[1] I.I. Rabi, Phys. Rev. 49 (1937) 324;
I.I. Rabi, Phys. Rev. 51 (1937) 652.
[2] C. Emary, Int. J. Mod. Phys. B 17 (2003) 5477.
[3] V. Fessatidis, J.D. Mancini, S.P. Bowen, Phys. Lett. A 297 (2002) 100.
[4] A. Pereverzev, E.R. Bitnet, Phys. Chem. Chem. Phys. 8 (2006) 1378.
[5] E.K. Irish, Phys. Rev. Lett. 99 (2007) 173601.
[6] H.G. Reik, M. Doucha, Phys. Rev. Lett. 57 (1986) 787.
[7] H.G. Reik, P. Lais, M.E. Stutzle, M. Doucha, J. Phys. A 20 (1987) 6327.
[8] E.T. Jaynes, F.W. Cummings, Proc. IEEE 51 (1963) 89.
[9] B.W. Shore, P.L. Knight, J. Mod. Opt. 40 (1993) 1195.
[10] H. Grinberg, Phys. Lett. A 344 (2005) 170;
H. Grinberg, Phys. Lett. A 350 (2006) 428.
[11] H. Grinberg, J. Phys. Chem. B 112 (2008) 16140, and references therein.
[12] S.J.D. Phoenix, P.L. Knight, Phys. Rev. A 44 (1991) 6023.
[13] M.A. Nielsen, I.L. Chuang, Quantum Computation and Quantum Information, Cambridge Univ. Press, Cambridge, 2000.
[14] J. Larson, Phys. Scr. 76 (2007) 146, and references therein.
[15] A.B. Klimov, J.L. Romero, C. Saavedra, Phys. Rev. A 64 (2001) 063802.
[16] T. Werlang, A.V. Dodonov, E.I. Duzzioni, C.J. Villas-Bôas, Phys. Rev. A 78 (2008) 053805, and references therein.
[17] Fu-li Li, S. Gao, Phys. Rev. A 62 (2000) 043809.
[18] Zou Yan, Li Yong-Ping, Chinese Physics B 18 (2009) 2794.
[19] Mao-Fa Fang, Peng Zhou, S. Swain, J. Mod. Opt. 47 (2000) 1043.
[20] H. Grinberg, Int. J. Mod. Phys. B, in press.
[21] J. Sánchez-Ruiz, Phys. Lett. A 201 (1995) 125.
[22] J. Sánchez-Ruiz, Phys. Lett. A 244 (1998) 189.
[23] M. Sebawe Abdalla, E.M. Khalil, A.S.-F. Obada, Ann. Phys. 322 (2007) 2554.
[24] Zhang Jian, Shao Bin, Zon Jian, Chinese Physics B 18 (2009) 1517.