# Between coloring and list-coloring: $\mu$-coloring

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**Abstract**

A new variation of the coloring problem, $\mu$-coloring, is defined in this paper. A *coloring* of a graph $G = (V, E)$ is a function $f : V \to \mathbb{N}$ such that $f(v) \neq f(w)$ if $v$ is adjacent to $w$. Given a graph $G = (V, E)$ and a function $\mu : V \to \mathbb{N}$, $G$ is $\mu$-*colorable* if it admits a coloring $f$ with $f(v) \leq \mu(v)$ for each $v \in V$. It is proved that $\mu$-coloring lies between coloring and list-coloring, in the sense of generalization of problems and computational complexity. Furthermore, the notion of perfection is extended to $\mu$-coloring, giving rise to a new characterization of cographs. Finally, a polynomial time algorithm to solve $\mu$-coloring for cographs is shown.

*Keywords*: cographs, coloring, list-coloring, $\mu$-coloring, M-perfect graphs, perfect graphs.

## 1 Introduction

Let $G = (V, E)$ be a graph, with vertex set $V(G)$ and edge set $E(G)$. Denote by $n$ and $m$ the cardinalities of $V(G)$ and $E(G)$, respectively.

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Denote by $N_G(v)$ the set of neighbors of $v \in V(G)$, and by $d(v)$ the cardinality of $N_G(v)$, that is, the *degree* of $v$. Denote by $\overline{G}$ the complement of $G$.

Denote by $K_n$ the complete graph of $n$ vertices. A *clique* of a graph is a subset of its vertices inducing a complete subgraph maximal under inclusion. Let $X$ and $Y$ be two sets of vertices of $G$. We say that $X$ is *complete to* $Y$ if every vertex in $X$ is adjacent to every vertex in $Y$, and that $X$ is *anticomplete to* $Y$ if no vertex of $X$ is adjacent to a vertex of $Y$.

A *coloring* of a graph $G = (V, E)$ is a function $f : V \to \mathbb{N}$ such that $f(v) \neq f(w)$ if $v$ is adjacent to $w$. A *$k$-coloring* is a coloring $f$ for which $f(v) \leq k$ for every $v \in V$. A graph $G$ is *$k$-colorable* if there exists a $k$-coloring of $G$.

Several variations of the coloring problem have been studied in the literature (see reviews in [10, 14], and recent works in [11, 16]). One of them is *list-coloring* [15]. Given a graph $G = (V, E)$ and a finite list $L(v) \subseteq \mathbb{N}$ of colors for each vertex $v \in V$, $G$ is *list-colorable* if there is a coloring $f$ such that $f(v) \in L(v)$ for each $v \in V$.

We define here *$\mu$-coloring* as follows. Given a graph $G = (V, E)$ and a function $\mu : V \to \mathbb{N}$, $G$ is *$\mu$-colorable* if it admits a coloring $f$ with $f(v) \leq \mu(v)$ for each $v \in V$. In terms of problems, list-coloring generalizes $\mu$-coloring, which in turn generalizes $k$-coloring. Moreover, the same chain arises when considering the relation between these problems in the sense of computational complexity.

Given a graph $G$ and an integer $k$, a polynomial time reduction from $k$-coloring to $\mu$-coloring can be achieved maintaining the graph $G$ and defining $\mu(v) = k$ for every vertex $v$ of $G$. Similarly, given a graph $G$ and a function $\mu : V(G) \to \mathbb{N}$, a polynomial time reduction from $\mu$-coloring to list-coloring can be achieved maintaining the graph $G$ and defining $L(v) = \{1, \ldots, \min\{\mu(v), |V(G)|\}\}$ for each vertex $v$ of $G$.

It is important to remark that these reductions do not involve changes in the graph. Therefore, for every graph class $\mathcal{C}$ list-coloring is harder than $\mu$-coloring, which is in turn harder than $k$-coloring, in terms of computational complexity. That is, if $k$-coloring is NP-complete over $\mathcal{C}$, so is $\mu$-coloring and if $\mu$-coloring is NP-complete over $\mathcal{C}$, so is list-coloring. In this sense, it can be said that $\mu$-coloring lies *between* $k$-coloring and list-coloring. We show in this work that the betweenness is strict, that is, there is a class of graphs (bipartite graphs) for which $\mu$-coloring is

NP-complete while $k$-coloring is in P, and there is another class of graphs (cographs) for which list-coloring is NP-complete while $\mu$-coloring is in P.

The $\mu$-coloring problem arises in a natural way from resources assignment with some incompatibilities between users. This problem could be modelled as a coloring problem, but sometimes not every resource can be used by every user. In some special cases, modelling it as a list-coloring problem seems to be too general. For example, if the resources can be ordered by quality or size and each user has a minimum size/quality requirement (each user can receive a resource being "good enough" for him/her), it is natural to see it as a $\mu$-coloring problem.

We say that a coloring $f$ is *minimal* when for every vertex $v$, and every $i < f(v)$, $v$ has a neighbor $w_i$ with $f(w_i) = i$. Note that every $k$-coloring or $\mu$-coloring can be transformed into a minimal one.

The three coloring problems considered here are decision problems. The $k$-coloring problem has an optimization problem naturally associated to it: to find the chromatic number of a graph. The *chromatic number* of a graph $G$ is the minimum $k$ such that $G$ is $k$-colorable, and it is denoted by $\chi(G)$. An obvious lower bound is the maximum cardinality of the cliques of $G$, the *clique number* of $G$, denoted by $\omega(G)$. A graph $G$ is *perfect* [1] when $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$. Perfect graphs have very interesting properties: they are a self-complementary class of graphs [13], the $k$-coloring problem is solvable in polynomial time for perfect graphs [8], they have been characterized by minimal forbidden subgraphs [4] and they may be recognized in polynomial time [3].

The concept of perfection was studied for some variations of the coloring problem, see for example [11, 16]. In this work we project the notion of perfection to define the M-perfect graphs, and show that they are exactly the cographs. A *cograph* is a $P_4$-free graph, i.e., a graph with no induced path of four vertices. It follows from this equivalence that M-perfect graphs are a self-complementary class of graphs and can be recognized in linear time [6]. Moreover, we show that the $\mu$-coloring problem is solvable in polynomial time for this class of graphs.

Preliminary results of this work appear published in [2].

## 2 Cographs and M-perfect graphs

A graph $G$ is perfect when $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$. If we restrict our attention to the $k$-coloring problem, we can reformulate the definition of perfect graphs in the following way: "$G$ is perfect when for every induced subgraph $H$ of $G$ and for every $k$, $H$ is $k$-colorable if and only if every clique of $H$ is $k$-colorable".

We define M-perfect graphs by analogy with this second definition of perfect graphs. A graph $G$ is *M-perfect* when for every induced subgraph $H$ of $G$ and for every function $\mu : V \to \mathbb{N}$, $H$ is $\mu$-colorable if and only if every clique of $H$ is $\mu$-colorable.

**Proposition 1** *If $G$ is a graph, the following statements are equivalent:*

- $(i)$ *$G$ is M-perfect*
- $(ii)$ *for every function $\mu : V \to \mathbb{N}$, $G$ is $\mu$-colorable if and only if every clique of $G$ is $\mu$-colorable.*

*Proof.* $(i) \Rightarrow (ii)$) It follows from the definition.

$(ii) \Rightarrow (i)$) Let $G$ be a graph verifying $(ii)$, $H$ an induced subgraph of $G$, and $\mu : V(H) \to \mathbb{N}$ a function for which all the cliques of $H$ are $\mu$-colorable. Let us extend that function $\mu$ to a function $\mu'$ defined for the vertex set of $G$, $\mu' : V(G) \to \mathbb{N}$, such that: $\mu'(v) = \mu(v)$ for every $v$ in $H$ and $\mu'(w) = |V(G)|$ for every $w$ in $V(G) \setminus V(H)$.
Let $K$ be a clique of $G$ and $K_H = K \cap V(H)$. The set $K_H$ induces a complete subgraph of $H$, and therefore it is included in a clique of $H$, and since it can be $\mu$-colored, it can be $\mu'$-colored, as well. We can extend this $\mu'$-coloring to $K$, since the vertices of $K \setminus K_H$ have $\mu'$ equal to $|V(G)| \geq |K|$. Then all the cliques of $G$ can be $\mu'$-colored, and since $G$ verifies $(ii)$, $G$ can be $\mu'$-colored. But this $\mu'$-coloring restricted to $H$ is also a $\mu$-coloring, because $\mu'|_H = \mu$. □

A complete graph is $k$-colorable if and only if it has size at most $k$. It is easy to know when a complete graph is $\mu$-colorable, as well. Let $\mu : V(K_n) \to \mathbb{N}$, and a vertex ordering $v_1, \ldots, v_n$ such that $\mu(v_i) \leq \mu(v_j)$ if $i \leq j$, then $K_n$ is $\mu$-colorable if and only if $i \leq \mu(v_i)$ for $1 \leq i \leq n$.

M-perfect graphs are also perfect because perfection is equivalent to M-perfection with $\mu$ restricted to constant functions. The converse is not true. We will show that the graph $P_4$ is not M-perfect, although it is perfect. In fact, M-perfect graphs are exactly the cographs. In order to prove it we need the next general result about minimal colorings on cographs.

**Lemma 1** *Let $G$ be a cograph and $x \in V(G)$. Let $f$ be a minimal coloring of $G - x$, and $T \in \mathbb{N}$. If $f$ cannot be extended to $G$ in such a way that $f(x) \leq T$, then there is a subset of vertices $H \subseteq N_G(x)$ of size $T$, inducing a complete subgraph in $G$, and such that $f(H) = \{1, \ldots, T\}$.*

*Proof.* Let $G$ be a cograph and $x \in V(G)$. Let $f$ be a minimal coloring of $G - x$, and $T \in \mathbb{N}$. Let us prove the result by induction on $T$. First, suppose that $T = 1$. If $f$ cannot be extended to $G$ coloring $x$ with color 1, then there exists $v \in N_G(x)$ such that $f(v) = 1$. In this case, $H = \{v\}$. Now, suppose that the result holds for $T = s - 1$ and let us see that it holds for $T = s$, $s \geq 2$. If $f$ cannot be extended to $G$ coloring $x$ with a color less or equal to $s$, in particular the same holds for $s - 1$, and so, by induction hypothesis, there is a subset of vertices $H \subseteq N_G(x)$ of size $s - 1$, inducing a complete subgraph in $G$ and using all the colors from 1 to $s - 1$. On the other hand, since $x$ cannot use color $s$, it must be a vertex $v \in N_G(x)$ such that $f(v) = s$. Let us consider the subgraph $\widetilde{G}$ of $G - x$ induced by $\{w \in G - x : f(w) \leq s - 1\} \cup \{v\}$ and let $\widetilde{f}$ be the coloring $f$ restricted to $\widetilde{G} - v$. By the minimality of $f$ it follows that $\widetilde{f}$ is minimal and it cannot be extended to $\widetilde{G}$ coloring $v$ with a color smaller or equal to $s - 1$, so, by induction hypothesis, there is a subset of vertices $F \subseteq N_{\widetilde{G}}(v)$ of size $s - 1$, inducing a complete subgraph in $\widetilde{G}$ and using all the colors from 1 to $s - 1$.
If $H = F$ then $H \cup \{v\}$ is a subset of vertices of size $s$ in the neighborhood of $x$, inducing a complete subgraph in $G$ and using all the colors from 1 to $s$. Suppose that they are not equal. Then $F \setminus H$ and $H \setminus F$ have the same cardinality and use the same colors. Let $v_H$ in $H \setminus F$, and let $v_F$ in $F \setminus H$ such that $f(v_F) = f(v_H)$. Since $f$ is a coloring of $G - x$, $v_F$ and $v_H$ are not adjacent. Since $G$ is $P_4$-free, $v_H, x, v, v_F$ do not induce a $P_4$, so $x$ is adjacent to $v_F$ or $v$ is adjacent to $v_H$. If all the vertices of $H \setminus F$ are adjacent to $v$, then $H \cup \{v\}$ is a subset of vertices of size $s$ in the neighborhood of $x$, inducing a complete subgraph in $G$ and using all the colors from 1 to $s$.
Thus, we may assume that the set $H_v = \{w \in H : (w, v) \notin E(G)\}$ is non empty, and define $F_v = \{z \in F : \exists z_H \in H_v \text{ with } f(z) = f(z_H)\}$. Please note that $H_v \subseteq H \setminus F$ thus $F_v \subseteq F \setminus H$. Moreover, $F_v$ and $H_v$ have the same cardinality and use the same colors. Since $H_v$ is anticomplete to $v$, it follows that $F_v$ must be complete to $x$. If $H \setminus H_v$ is empty, then $F = F_v$ is complete to $x$ and $F \cup \{v\}$ is a subset of vertices of size $s$ in

the neighborhood of $x$, inducing a complete subgraph in $G$ and using all the colors from 1 to $s$.
Suppose now that $H \setminus H_v$ is non-empty, and let us see that $F_v$ is complete to $H \setminus H_v$. Let $z_F \in F_v$ and $w \in H \setminus H_v$. Let $z_H \in H_v$ such that $f(z_H) = f(z_F)$. Then $z_H$ is neither adjacent to $z_F$ nor to $v$ and since $H$ induces a complete subgraph, $z_H$ and $w$ are adjacent. Besides, $w$ is adjacent to $v$ because $w$ lies in $H \setminus H_v$. Since the vertices $z_H, w, v, z_F$ do not induce a $P_4$, the vertex $w$ must be adjacent to $z_F$ (a scheme of this situation is depicted in Figure 1). Therefore $F_v$ is complete to $H \setminus H_v$. Hence $\widetilde{H} = (H \setminus H_v) \cup F_v \cup \{v\}$ is a subset of vertices in $N_G(x)$ of size $s$, inducing a complete subgraph in $G$ and such that $f(\widetilde{H}) = \{1, \ldots, s\}$. □

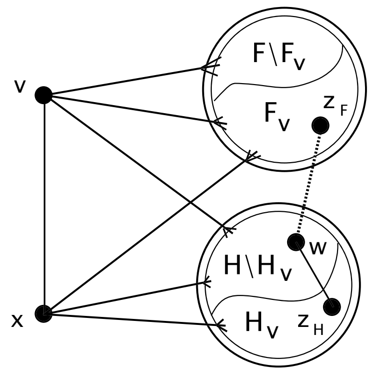


Figure 1: Scheme for the notation used in the proof of Lemma 1.

**Theorem 1** *If $G$ is a graph, the following statements are equivalent:*

$(i)$ *$G$ is M-perfect*

$(ii)$ *$G$ is a cograph.*

*Proof.* $(i) \Rightarrow (ii)$) Suppose that $v_1v_2v_3v_4$ induce a $P_4$, and let $\mu$ be defined as follows: $\mu(v_1) = \mu(v_4) = 1$, $\mu(v_2) = \mu(v_3) = 2$. Clearly, every clique is $\mu$-colorable, but the whole graph is not.

$(ii) \Rightarrow (i)$) Suppose that there is a $P_4$-free graph that is not M-perfect. Let $G$ be a minimal one, that is, $G$ is $P_4$-free and it is not M-perfect, but for every vertex $x$ of $G$, $G - x$ is M-perfect.
Let $\mu : V(G) \to \mathbb{N}$ be a function such that the cliques of $G$ are $\mu$-colorable but $G$ is not. Let $x$ be a vertex of $G$ with $\mu(x)$ maximum. The graph

$G - x$ is M-perfect, and since the cliques of $G$ are $\mu$-colorable, also those of $G - x$ are, so $G - x$ is $\mu$-colorable. Let $f$ be a minimal $\mu$-coloring of $G - x$.
Since $G$ is not $\mu$-colorable, $f$ cannot be extended to a $\mu$-coloring of $G$. Hence by Lemma 1, $N_G(x)$ contains a complete subgraph of size $\mu(x)$. But then $G$ contains a complete subgraph of size $\mu(x) + 1$ for which the upper bounds of all of its vertices are at most $\mu(x)$ (we have chosen $x$ with maximum value of $\mu$). This is a contradiction, because all the cliques of $G$ are $\mu$-colorable.
Therefore there is not a minimal M-imperfect $P_4$-free graph, and that concludes the proof. □

## 3 Algorithm for $\mu$-coloring cographs

The greedy coloring algorithm consists on coloring iteratively the vertices of the graph in a given order with the minimum possible color.

This sequential algorithm does not solve the general problem of $\mu$-coloring, even with $\mu$ restricted to be a constant function. Nevertheless, for any given graph, there is an ordering of the vertices in which the sequential greedy algorithm $\mu$-colors the graph, when it is $\mu$-colorable. Then a strategy to find polynomial time algorithms to solve $\mu$-coloring for some classes of graphs consists on finding such an order in polynomial time.

Using Lemma 1 we can prove the following result.

**Theorem 2** *Given $G$ a cograph, the greedy coloring algorithm applied to its vertices in non-decreasing order of $\mu$ gives a $\mu$-coloring of $G$ if and only if $G$ is $\mu$-colorable. Moreover, this coloring is minimal and uses only the first $\chi(G)$ colors.*

*Proof.* By induction on the number of vertices of $G$. The theorem is valid for $|V(G)| = 1$. Suppose it is also valid for any $G$ such that $|V(G)| < n$. Let $G$ be a cograph with $n$ vertices, $\mu$ be a function such that $G$ is $\mu$-colorable, and $S = \{v_1, \ldots, v_n\}$ a sequence of non-decreasingly ordered vertices ($\mu(v_i) \leq \mu(v_j)$ for $i \leq j$).
The graph $G - v_n$ is $\mu$-colorable because $G$ is $\mu$-colorable. By induction hypothesis we can obtain $f$, a minimal $\mu$-coloring for $G - v_n$, using the sequential greedy algorithm.
Suppose that we cannot extend the coloring $f$ to $v_n$ using a color smaller

or equal to $\mu(v_n)$. Then, by Lemma 1, there exists a complete subgraph $K$ of size $\mu(v_n)$ in the neighborhood of $v_n$. But since we order the vertices so that $\mu(v_i) \leq \mu(v_j)$ for $i \leq j$, $V(K) \cup \{v_n\}$ induces a complete subgraph of size $\mu(v_n) + 1$ such that no vertex of it can use a color greater than $\mu(v_n)$. This subgraph is clearly not $\mu$-colorable, a contradiction.
Because of the way of choosing the color for each vertex, the obtained coloring is minimal. Let us see that it uses only the colors from 1 to $\chi(G)$ inclusive. As we just saw, if the algorithm uses the color $k$ to color a vertex, then the graph contains a complete subgraph of size $k$, so it is not $(k-1)$-colorable. $\square$

A nice corollary of this theorem is the following result, proved previously by Chvátal [5].

**Corollary 1** *The greedy coloring algorithm gives an optimal coloring for cographs, independently of the order of the vertices.*

**Theorem 3** *The $\mu$-coloring algorithm described in Theorem 2 has time complexity $O(n \log n + m)$.*

*Proof.* Sorting the vertices by $\mu$ has time complexity $O(n \log n)$. Then we color each vertex $v$ with the smallest color available. It can be found in $O(d(v))$ if the graph is given by adjacency lists. For instance, we can remark in an array the colors used by the neighbors of $v$, and then search for the smallest free (always smaller or equal to $d(v) + 1$). Since $\sum_{v \in V(G)} d(v) = 2m$ the overall time complexity of the algorithm is $O(n \log n + m)$. $\square$

A little improvement in the greedy algorithm allows us to find a non $\mu$-colorable clique when the graph is not $\mu$-colorable, that is, make it a robust algorithm. The second part of the algorithm, once a vertex $v_j$ that cannot be $\mu$-colored is found, it will be:

> $K := \{v_j\}$;
> $L := \{v_i \in N(v_j) : i < j\}$;
> `for` $c$ `from` $\mu(v_j)$ `down to` 1 `do`
>     find $v_k$ in $L$ so that $f(v_k) = c$;
>     $K := K \cup \{v_k\}$;
>     $L := L \cap N(v_k)$;
> `end for`
> `return` $K$;

**Theorem 4** *The robust algorithm is correct, and it has time complexity $O(n \log n + m)$.*

*Proof.* By Lemma 1, we know that there is a complete subgraph of size $\mu(v_j)$ in the neighborhood of $v_j$, which has been already colored by $f$ with colors 1 to $\mu(v_j)$. Therefore, the candidates to form part of this complete subgraph are the vertices of the set $\{v_1, \dots, v_{j-1}\}$ that are in the neighborhood of $v_j$, and these vertices will be the initial elements of the set $L$. Then, for each color $c$ from $\mu(v_j)$ down to 1, the algorithm will look for a vertex $w_c$ (equal to $v_k$ for some $k$, $1 \leq k < j$) such that $f(w_c) = c$ and $w_c$ is adjacent to $w_{c+1}, \dots, w_{\mu(v_j)}, v_j$. It is clear that if in each step such a vertex exists, the algorithm will finish with the complete subgraph we were looking for.

We shall prove by induction on the number of steps $i$, $0 \leq i \leq \mu(v_j) - 1$, that if $w_{\mu(v_j)}, w_{\mu(v_j)-1}, \dots, w_{\mu(v_j)-i+1}$ induce a complete subgraph, then for every $r$, $1 \leq r \leq \mu(v_j) - i$, there exists a vertex $v_k$ with $k < j$, adjacent to all of them and to $v_j$ and such that $f(v_k) = r$. For $i = 0$, since $v_j$ cannot be colored with a color smaller or equal to $\mu(v_j)$, then clearly for every $r$, $1 \leq r \leq \mu(v_j)$, there is a neighbor $v_k$ of $v_j$, with $k < j$, such that $f(v_k) = r$. Now, suppose that the hypothesis is valid for $i = s$, and let us see that it also holds for $i = s + 1$, with $s + 1 \leq \mu(v_j) - 1$. Suppose that $w_{\mu(v_j)}$, $w_{\mu(v_j)-1}$, $\dots$, $w_{\mu(v_j)-s}$ induce a complete subgraph and let $r$ be a natural number, $1 \leq r \leq \mu(v_j) - s - 1$. By induction hypothesis there is a vertex $v_k$ with $k < j$, adjacent to $v_j$, $w_{\mu(v_j)}$, $w_{\mu(v_j)-1}, \dots, w_{\mu(v_j)-s+1}$, and such that $f(v_k) = r$. If $v_k$ is adjacent to $w_{\mu(v_j)-s}$, we have finished. Otherwise, and because $w_{\mu(v_j)-s}$ was colored with the smallest possible color $\mu(v_j) - s$, it turns out that there must be a vertex $v_{k'}$ with $k' < j$, adjacent to $w_{\mu(v_j)-s}$, such that $f(v_{k'}) = r$. Since $G$ is a cograph, $v_{k'} w_{\mu(v_j)-s} w v_k$ cannot induce a $P_4$ for $w \in \{v_j, w_{\mu(v_j)}, w_{\mu(v_j)-1}, \dots, w_{\mu(v_j)-s+1}\}$. And since $v_k$ and $v_{k'}$ cannot be adjacent because they are colored with the same color, and $v_k$ is not adjacent to $w_{\mu(v_j)-s}$, $v_{k'}$ must be adjacent to $w$. Thus $v_{k'}$ is the vertex we were searching for, concluding the proof of correctness of the algorithm.

If the graph is given by ordered adjacency lists, the representation of the set $L$ can be implemented with an ordered list, and then each step of the search will have time complexity $O(|L|)$, the intersection with the neighborhood of $v$ will have time complexity $O(|L| + |N(v)|)$, and the list $L$ for the next step will have size $O(|N(v)|)$. If the vertices of the obtained complete subgraph are $w_1, \dots, w_s$, then the time complexity of this part of the robust algorithm is $O(\sum_{i=1}^{s} d(w_i))$, that is $O(m)$. Therefore, the overall time complexity of the complete robust algorithm is $O(n \log n + m)$. □

Jansen and Scheffler [10] proved that list-coloring is NP-complete for cographs, hence $\mu$-coloring is strictly easier than list-coloring in terms of computational complexity, unless P=NP.

**Note:** The fact that $k$-coloring is polynomial for perfect graphs and $\mu$-coloring is polynomial for M-perfect graphs makes us wonder of a possible intrinsic relation between polynomial time coloring and perfectness. As a third example, let us consider the definition of perfectness applied to list-coloring: let a graph $G$ be $\mathbb{L}$-*perfect* if for any family of lists $L$, $G$ is $L$-colorable if and only if all of its cliques are $L$-colorable. It is easy to see that if we have a $P_3 = v_1v_2v_3$ and we assign to $v_1$ the list $\{1\}$, to $v_2$ the list $\{1, 2\}$, and to $v_3$ the list $\{2\}$, then both cliques are $L$-colorable but the whole graph is not. Therefore, the only $\mathbb{L}$-perfect graphs are $P_3$-free graphs, that is, the graphs whose connected components are complete, also known as *cluster* graphs. For these graphs $L$-coloring is polynomial, since it can be easily reduced to maximal matching in a bipartite graph.

# 4 Bipartite graphs

It follows from Theorem 2 that a cograph $G$ that is $\mu$-colorable can be $\mu$-colored using the first $\chi(G)$ colors. This does not happen for bipartite graphs, not even for trees.

Define the family $\{T_n\}_{n\in\mathbb{N}}$ of rooted trees and the corresponding family $\{\mu_n\}_{n\in\mathbb{N}}$ of functions as follows: $T_1 = \{v\}$ is a trivial tree, and $\mu_1(v) = 1$. The tree $T_{n+1}$ is obtained from $T_1, \ldots, T_n$ by adding a new root $w$ adjacent to the roots of $T_1, \ldots, T_n$. Function $\mu_{n+1}$ extends $\mu_1, \ldots, \mu_n$ and is defined at $w$ as $\mu_{n+1}(w) = n+1$. Some trees in this family can be seen in Figure 2.

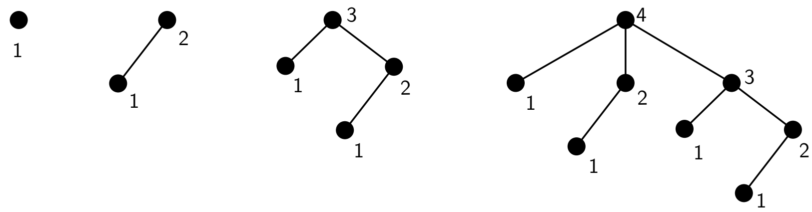

Figure 2: Family of binomial trees $\{T_n\}_{n\in\mathbb{N}}$ and their corresponding functions $\{\mu_n\}_{n\in\mathbb{N}}$.

These trees are known as *binomial trees*, and the following two results can be also obtained from some results on greedy colorings [7].

**Proposition 2** *The tree $T_n$ has $2^{n-1}$ vertices and requires $n$ colors to be $\mu_n$-colored. Moreover, every vertex $v$ must be colored with $\mu_n(v)$.*

*Proof.* By induction on $n$. The tree $T_1$ has $2^0 = 1$ vertex and needs one color to be $\mu_1$-colored. Its only vertex $v$ has $\mu_1(v) = 1$, and therefore the property holds.
Suppose it holds for $n \leq k$, and let us see what happens with $T_{k+1}$. Since we obtain $T_{k+1}$ by joining the roots of the trees $T_1, \ldots, T_k$ to a new root $w$, and $\mu_{k+1}$ extends $\mu_1, \ldots, \mu_k$, by induction hypothesis each vertex $v$ in $T_1 \cup \cdots \cup T_k$ must use the color $\mu_{k+1}(v)$. In particular, the root of $T_i$ uses the color $i$, for each $1 \leq i \leq k$, and therefore $w$ must use the color $k+1 = \mu_{k+1}(w)$. It remains to see that $|V(T_{k+1})| = 2^k$. By induction hypothesis, $|V(T_i)| = 2^{i-1}$, for $1 \leq i \leq k$. Thus, we can compute $|V(T_{k+1})| = \sum_{1 \leq i \leq k} 2^{i-1} + 1 = 2^k - 1 + 1 = 2^k$. □

This family is optimal on the number of vertices, in fact the following property holds.

**Theorem 5** *Let $T$ be a tree, and let $\mu$ be a function such that $T$ is $\mu$-colorable. Then $T$ can be $\mu$-colored using at most the first $log_2(|V(T)|)+1$ colors.*

*Proof.* Given a tree $T$, and a function $\mu$, we will see that if a minimal $\mu$-coloring $f$ of $T$ uses $r$ colors, then $T$ has at least $2^{r-1}$ vertices.
By induction on $r$. It is trivially true for $r = 1$. Suppose it holds for $r \leq k$, and let $f$ be a minimal $\mu$-coloring of $T$. If there is a vertex $v$ with $f(v) = k + 1$, then among its neighbors there are $k$ vertices using the colors $1, \ldots, k$. Therefore, in $T - v$ there are $k$ disjoint trees $T_1, \ldots, T_k$ such that each $T_i$ is minimally $\mu$-colored by $f$, and has a vertex that uses the color $i$. By induction hypothesis, $|V(T_i)| \geq 2^{i-1}$ and therefore $|V(T)| \geq \sum_{1 \leq i \leq k} |V(T_i)| + 1 \geq \sum_{1 \leq i \leq k} 2^{i-1} + 1 = 2^k$. Now, let us take a $\mu$-coloring of $T$ that minimizes $\max_{v \in V(T)} f(v)$. It is clear that we can take a minimal one. Thus, if it uses $r$ colors, $|V(T)| \geq 2^{r-1}$ and then $r \leq log_2(|V(T)|) + 1$. □

A similar result can be obtained for bipartite graphs. Define the family $\{B_n\}_{n \in \mathbb{N}}$ of bipartite graphs and the corresponding family $\{\mu_n\}_{n \in \mathbb{N}}$ of functions as follows: $B_1 = \{v\}$ is a trivial graph, and $\mu_1(v) = 1$. The bipartite graph $B_{n+1} = (V, W, E)$ has $V = \{v_1, \ldots, v_n\}$, $W = \{w_1, \ldots, w_n\}$; $v_i$ is adjacent to $w_j$ for $i \neq j$; $v_n$ is adjacent to $w_n$, and $v_i$ is not adjacent to $w_i$ for $i < n$; $\mu_{n+1}(v_i) = \mu_{n+1}(w_i) = i$ for $i < n$; $\mu_{n+1}(v_n) = n$ and $\mu_{n+1}(w_n) = n + 1$. Some graphs in this family can be seen in Figure 3.

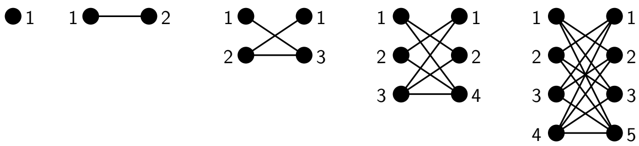


Figure 3: Family of bipartite graphs $\{B_n\}_{n\in\mathbb{N}}$ and their corresponding functions $\{\mu_n\}_{n\in\mathbb{N}}$.

**Proposition 3** *The bipartite graph $B_n$ has $2n - 2$ vertices (if $n \geq 2$) and requires $n$ colors to be $\mu_n$-colored. Moreover, every vertex $v$ must be colored with $\mu_n(v)$.*

*Proof.* It is clear that the proposition holds for $n = 1$. So we shall show that, for $n \geq 1$, there is only one $\mu_{n+1}$-color $f$ for $B_{n+1}$, and that $f$ uses $n + 1$ colors, since $f(v) = \mu_{n+1}(v)$ for every vertex $v$ in $B_{n+1}$. First, let us see that $f(v_i) = f(w_i) = i$ for $i < n$, by induction on $i$. It is valid for $i = 1$, and suppose that it holds for any $i \leq k$. The vertex $v_{k+1}$ is adjacent to every $w_j$ with $j \leq k$, which by induction hypothesis use all the colors from 1 to $k$. But $\mu(v_{k+1}) = k + 1$, so $v_{k+1}$ can be colored only with color $k + 1$. Likewise, $f(w_{k+1}) = k + 1$. Finally, $v_n$ is adjacent to every $w_j$ with $j < n$, which as we just proved, use the colors from 1 to $n - 1$. Since $\mu(v_n) = n$, it must be $f(v_{n+1}) = n + 1$. Now, $w_n$ is adjacent to every $v_j$ with $j \leq n$, which use the colors from 1 to $n$. Since $\mu(w_n) = n + 1$, it must be $f(w_n) = n + 1$. □

Analogously, the following property holds.

**Theorem 6** *Let $B$ be a bipartite graph, and let $\mu$ be a function such that $B$ is $\mu$-colorable. Then $B$ can be $\mu$-colored using at most the first $\frac{(|V(B)|+2)}{2}$ colors.*

*Proof.* Given $B = (V, W, E)$, let us take a minimal $\mu$-coloring $f$ of $B$ minimizing $\max_{v\in V(B)} f(v)$. Suppose there is a vertex $v$ in $V$ using the color $k$. Since $f$ is minimal, $v$ has at least $k-1$ neighbors $w_1, \ldots, w_{k-1}$ in $W$ for which $f(w_i) = i$. Therefore, $|W| \geq k-1$. On the other hand, $f(w_{k-1}) = k-1$, and because $f$ is minimal, $w_{k-1}$ has at least $k-2$ neighbors in $V$ using the colors 1 to $k-2$. But as $f(v) = k$, all of them are different from $v$, thus $|W| \geq k-2+1 = k-1$, then $|V(B)| \geq 2(k-1)$ and finally $k \leq \frac{(|V(B)|+2)}{2}$. □

Hujter and Tuza [9] proved that list-coloring is NP-complete for bipartite graphs, and the same holds for $\mu$-coloring.

**Theorem 7** *The $\mu$-coloring problem is NP-complete for bipartite graphs.*

*Proof.* We will show a reduction from list-coloring of bipartite graphs, which is NP-complete [9], to $\mu$-coloring of bipartite graphs. Consider an instance of bipartite graphs list-coloring, i.e., suppose that a bipartite graph $G = (X, Y, E)$ and a finite list $L(v) \subseteq \mathbb{N}$ of colors for each $v \in V(G)$ are given. Let $k = |\bigcup_{v \in V(G)} L(v)|$. Without loss of generality, we can assume that $L(v) \subseteq \{1, \ldots, k\}$. In order to build an instance $(G', \mu)$ of bipartite $\mu$-coloring, add two $k$-element sets of vertices $X' = \{x'_1, \ldots, x'_k\}$ and $Y' = \{y'_1, \ldots, y'_k\}$ to $G$ such that $X, Y, X', Y'$ are pairwise disjoint. Furthermore, take a bipartition $(X \cup X', Y \cup Y')$ of the new graph $G'$, and for any $x \in X$, $y \in Y$, and $i, j \in \{1, \ldots, k\}$, define the following new adjacency relations: $x'_i$ is adjacent to $y'_j$ if and only if $i \neq j$; $x'_i$ is adjacent to $y$ if and only if $i \notin L(y)$; $y'_i$ is adjacent to $x$ if and only if $i \notin L(x)$. Finally, define $\mu(x'_i) = \mu(y'_i) = i$ and $\mu(x) = \mu(y) = k$. Then $G$ is list-colorable if and only if $G'$ is $\mu$-colorable. The transformation can be made in polynomial time, and this completes the proof. □

Coloring is trivially in P for bipartite graphs, hence $\mu$-coloring is strictly harder than $k$-coloring in terms of computational complexity, unless P=NP.

# 5 Further characterizations of M-perfect graphs

The way of defining M-perfect graphs by analogy with one of the possible definitions of perfect graphs may seem somehow arbitrary. However, we propose in this section some alternative definitions and show that they lead to the same class of graphs.

Given a graph $G$ and a function $\mu : V(G) \to \mathbb{N}$ such that $G$ is $\mu$-colorable, define $\chi_\mu(G)$ as the minimum number of colors needed in a $\mu$-coloring of $G$. Clearly, $\chi(G) \leq \chi_\mu(G) \leq |V(G)|$. Let $\chi_M(G) = \max\{\chi_\mu(G) : \mu : V(G) \to \mathbb{N}$ and $G$ is $\mu$-colorable$\}$. We may define that a graph $G$ is perfect with respect to $\mu$-coloring when $\chi_M(H) = \chi(H)$ for every induced subgraph $H$ of $G$. But, in fact, the following equivalence holds.

**Proposition 4** *If $G$ is a graph, the following are equivalent:*

*(i) $\chi_M(H) = \chi(H)$ for every induced subgraph $H$ of $G$*
*(ii) $G$ is a cograph.*

*Proof.* It follows from Theorem 2 that $(ii) \Rightarrow (i)$. On the other hand, the graph $T_3$ and its corresponding function $\mu_3$ in Section 4 show that $\chi_M(P_4) > \chi(P_4)$, hence $(i) \Rightarrow (ii)$. □

A different, more algorithmic, approach to the definition of perfection relative to $\mu$-coloring could be the following. Let $G$ be a graph and $\mu : V(G) \to \mathbb{N}$. Let $\mu_{\min} = \min\{|V(G)|, \min_{v \in V(G)} \mu(v)\}$ and $\mu_{\max} = \min\{|V(G)|, \max_{v \in V(G)} \mu(v)\}$. Let $M(G, \mu)$ be the graph obtained from $G$ by adding a complete graph with vertices $w_{\mu_{\min}+1}, \ldots, w_{\mu_{\max}}$, and by joining each vertex $v \in V(G)$ to $w_j$, for every $j > \mu(v)$ [12]. The transformation $G \to M(G, \mu)$ allows us to reduce the $\mu$-coloring problem to the $k$-coloring problem, namely $G$ admits a $\mu$-coloring if and only if $M(G, \mu)$ admits a $\mu_{\max}$-coloring. So, if $M(G, \mu)$ is perfect the $\mu$-colorability of $G$ can be decided in polynomial time, and we say that $G$ is *$\mu$-perfect*. We may define that a graph $G$ is perfect with respect to $\mu$-coloring when it is $\mu$-perfect independently of the choice of $\mu$. But, again, the following equivalence holds.

**Theorem 8** *If $G$ is a graph, the following are equivalent:*

*(i) $G$ is $\mu$-perfect for every function $\mu : V(G) \to \mathbb{N}$*
*(ii) $G$ is a cograph.*

*Proof.* $(ii) \Rightarrow (i)$) Let $G$ be a cograph and $\mu : V(G) \to \mathbb{N}$ a function, and let us suppose that $G$ is not $\mu$-perfect. By the Strong Perfect Graph Theorem [4], a graph $H$ is perfect if and only if neither $H$ nor $\overline{H}$ contains a chordless cycle of odd length at least five. Then either $M(G, \mu)$ or $\overline{M(G, \mu)}$ contains such a cycle $C$. Since $N_{M(G,\mu)}(w_i) \subseteq N_{M(G,\mu)}(w_j)$ and $N_{\overline{M(G,\mu)}}(w_j) \subseteq N_{\overline{M(G,\mu)}}(w_i)$ when $i < j$, $C$ contains at most one vertex of the set $\{w_{\mu_{\min}+1}, \ldots, w_{\mu_{\max}}\}$. Therefore $G$ contains a chordless path of length four, a contradiction.

$(i) \Rightarrow (ii)$) Let us suppose that $v_1v_2v_3v_4$ induce a chordless path of length four in $G$, and let $\mu : V(G) \to \mathbb{N}$ such that $\mu(v_1) = \mu(v_4) = 1$, $\mu(v_2) = 2$ and $\mu(v_3) = 3$. Then $v_1v_2v_3v_4w_2$ induce a chordless cycle of length five in $M(G, \mu)$, hence $G$ is not $\mu$-perfect. $\square$

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