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Bipolar varieties and real solving of a singular polynomial equation†

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Abstract

We introduce the concept of a *bipolar* variety of a real algebraic hypersurface. This notion is then used for the design and complexity estimations of a novel type of algorithms that finds algebraic sample points for the connected components of a singular real hypersurface. The complexity of these algorithms is polynomial in the maximal geometric degree of the bipolar varieties of the given hypersurface and in this sense intrinsic.

Keywords: Real polynomial equation solving; singular hypersurface; polar varieties.

MSC: Primary 14P05, 14Q10, 14B05; Secondary 14Q15, 68W30.

§1. Introduction

This paper is based on the concept of polar varieties which classically goes back to F. Severi and J. A. Todd in the 1930's and beyond that to the work of J.-V. Poncelet in the period of 1813–1829. The modern theory started in 1975 with essential contributions due to R. Piene [19] (global theory), B. Teissier and D. T. Lê [29] and [17], J. P. Henry and M. Merle [13] (local theory), J. P. Brasselet and others (see [30], [19] and [7] for a historical account and references). The aim was a deeper understanding of the structure of singular varieties by means of the considerations of *generic* classic polar varieties. On the other hand, classic (and dual) polar varieties became about ten years ago

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a fundamental tool for the design of efficient computer procedures with *intrinsic* complexity which find real algebraic sample points for the connected components of complete intersection varieties with *smooth* real trace ([1], [2], [3], [4],[26], [27]). Our method for finding smooth sample points in *singular* real hypersurfaces relies on a deformation which leads us to the consideration of *non–generic* polar varieties. The same problem is treated by a different (the so–called "critical point") method in [24] and [28].

§2. Geometric foundation and computational complexity

Let V be a complex affine algebraic variety given by a polynomial $f(x_1, \ldots, x_n)$ of degree $d \geq 2$ over the rational numbers. We assume that the real trace $V_{\mathbb{R}}$ of V is non–empty and that the gradient $J(f) := (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$ of f does not vanish identically on any connected component of $V_{\mathbb{R}}$. Thus $V_{\mathbb{R}}$ is a real hypersurface. Suppose that f is given by a division–free arithmetic circuit (straight–line program) of size L (see [8] for an exact definition and the underlying complexity model).

For any $1 \leq i \leq n-1$ let $A := [a_{k,l}]_{\substack{1 \leq k \leq n-i \\ 1 \leq l \leq n}}$ be a (complex or real) $((n-i) \times n)$–matrix of maximal rank rk $A = n - i$. Recall that the i–th open (classic) polar variety $P_i(A)$ of V associated with A consists of the smooth points of V where the tangent plane does not intersect transversally the kernel of A. If A is generic, $P_i(A)$ coincides with the usual notion of a (classic) polar variety ([19] and [29]). In this case we call the polar variety $P_i(A)$ (fully) *generic*.

Suppose that the polar variety $P_i(A)$ is generic and non–empty. Then $P_i(A)$ is of pure codimension i in V, normal and Cohen–Macaulay at any point which is smooth in V. Moreover there exist canonical equations of degree nd that describe $P_i(A)$ locally as transversal (and hence as reduced, complete) intersection outside of a subvariety at least of codimension one. If $V_{\mathbb{R}}$ is smooth and compact and A is real then the real trace $P_i(A) \cap \mathbb{R}^n$ contains an algebraic sample point for every connected component of $V_{\mathbb{R}}$ and therefore $P_i(A)$ is non–empty (see [3, 4] and [5]). This may happen to be wrong in case that $V_{\mathbb{R}}$ is not smooth anymore, even if $V_{\mathbb{R}}$ is compact.

Fortunately, we have at our disposal the canonical desingularization of determinantal varieties – *à la Room–Kempf* [25] – in order to inspire us. Let us introduce the incidence variety defined by the condition:

$$f(x) = 0, \qquad J(f)^T(x)\, \lambda + A^T\, \mu \;=\; 0 \qquad\qquad (*).$$

Here $J(f)^T$ denotes the transposed gradient of f and x has to be interpreted as a point of the n–dimensional complex affine space $\mathbb{A}^n$, A is a complex $((n-i) \times n)$–matrix of maximal rank, λ

belongs to $\mathbb{A}^1$ and μ to $\mathbb{A}^{n-i}$, with $\mu \neq 0$. we denote the set of points $(x, A, \lambda, \mu) \in \mathbb{A}^n \times \mathbb{A}^{(n-i)\times n} \times \mathbb{A}^1 \times \mathbb{A}^{(n-i)}$ which satisfy the condition $(*)$ and the open restrictions above by $\mathcal{H}_i$.

In the special case $i := n-1$, the matrix $A := a_{n-1}$ is a n–tuple, and if $A \in \mathbb{R}^n$, then the real trace of $\mathcal{H}_i$ describes the extremal points and the Lagrange multipliers of the real valued function induced by A on $V_{\mathbb{R}}$.

Moreover, the image of $\mathcal{H}_i$ under its canonical projection into $\mathbb{A}^n$ is exactly the set of non–singular points of V. It is somewhat lengthy, but not difficult to show that $\mathcal{H}_i$ is a locally closed and smooth algebraic subvariety of the affine ambient space $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times n} \times \mathbb{A}^1 \times \mathbb{A}^{n-i}$. Furthermore $\mathcal{H}_i$ is of pure dimension $(n-i)(n+1)$ and the equations $(*)$ intersect transversally at any point of $\mathcal{H}_i$.

We consider now the configuration space

$$E_i := \{(x, A, \lambda, \mu) \in \mathbb{A}^n \times \mathbb{A}^{(n-i)\times n} \times \mathbb{A}^1 \times \mathbb{A}^{n-i} \mid \operatorname{rank} A = n-i,\ \mu \neq 0\}.$$

E_i is an open subset of $\mathbb{A}^n \times \mathbb{A}^{(n-i)\times n} \times \mathbb{A}^1 \times \mathbb{A}^{n-i}$ and hence a smooth algebraic variety.
The algebraic group $G_i := GL(n-i) \times GL(1)$ acts in the following way from the right on E_i:

$$\text{For } g := (b, t) \in G_i \text{ and } e := (x, A, \lambda, \mu) \in E_i \text{ let } e \cdot g := (x, b^T A,\ t\lambda,\ tb^{-1} \cdot \mu).$$

We denote by E_i^* the (topological) orbit space of E_i with respect to G_i. Since the algebraic group G_i is linearly reductive, E_i^* owns a natural structure of an algebraic variety (see [18]). It turns out that E_i^* is smooth and equidimensional. Observe that $\mathcal{H}_i$ is a subvariety of E_i and that the action of G_i on E_i leaves $\mathcal{H}_i$ invariant.

By standard arguments of algebraic geometry one sees that E_i^* is a smooth and equidimensional algebraic variety of dimension $r_i := n + (i+1)(n-i)$ with a canonical atlas of $N_i := \binom{n}{n-i}(n-i)$ open charts $U_k,\ 1 \leq k \leq N_i$ which are all isomorphic to $\mathbb{A}^{r_i}$. Let $\varphi_i : E_i \to E_i^*$ be the morphism of algebraic varieties which maps each point of E_i onto its G_i–orbit. Then φ_i is a smooth morphism of analytic manifolds. Let $S_i := \varphi_i(\mathcal{H}_i)$ and $S_{i,k} := S_i \cap U_k$ for $1 \leq k \leq N_i$. The geometric main issue is the following result.

Lemma 2.1. *Let $1 \leq k \leq N_i$. Then, identifying U_k with $\mathbb{A}^{r_i}$, the constructible set $S_{i,k}$ becomes a smooth closed subvariety of the affine space $\mathbb{A}^{r_i}$. Moreover, $S_{i,k}$ is equidimensional of dimension $D_i := (i+1)(n-i) - 1$ and given as a transversal intersection of $n+1$ equations of degree d which have a circuit representation of size $O(L + n + i(n-i))$. In particular, S_i is a smooth subvariety of E_i^* and the varieties $S_{i,k},\ 1 \leq k \leq N_i$, form an open atlas of S_i.*

Proof. Without loss of generality we may identify U_k with

$$\{(x,A,\lambda,\mu)\in E_i \mid A=[I_{n-i},\tilde{A}],\ \tilde{A}\in\mathbb{A}^{(n-i)\times i},\ \mu=(1,\tilde{\mu}),\ \tilde{\mu}\in\mathbb{A}^{n-i-1}\},$$

where I_{n-i} denotes the $((n-i)\times(n-i))$ identity matrix and $[I_{n-i},\tilde{A}]$ the complex $((n-i)\times n)$–matrix whose first $n-i$ columns form I_{n-i} and whose remaining i columns form the $((n-i)\times i)$–matrix $\tilde{A}$.

Then $S_{i,k}$ may be identified with the points $(x,\tilde{A},\lambda,\tilde{\mu})\in\mathbb{A}^n\times\mathbb{A}^{(n-i)\times i}\times\mathbb{A}^1\times\mathbb{A}^{n-i-1}$ with $\tilde{A}=[\tilde{a}_{k,l}]_{\substack{1\le k\le n-i\\ n-i+1\le l\le n}}$ and $\tilde{\mu}=(\tilde{\mu}_2,\dots,\tilde{\mu}_{n-i})$, satisfying the polynomial conditions

$$(**)\qquad \begin{aligned} f(x)&=0,\\ \frac{\partial f}{\partial x_1}(x)\,\lambda+1&=0,\\ \frac{\partial f}{\partial x_l}(x)\,\lambda+\mu_l&=0, &\quad& 2\le l\le n-i,\\ \frac{\partial f}{\partial x_l}(x)\,\lambda+\tilde{a}_{1,l}+\sum_{k=2}^{n-i}\tilde{a}_{k,l}\,\mu_k&=0, && n-i<l\le n. \end{aligned}$$

Let $z=(x,\tilde{A},\lambda,\tilde{\mu})$ be any such point. For $1\le j\le i$ let $Z_j:=Z_j(z)$ be the complex $(i\times(n-i))$–matrix whose rows are all zero except the row number j which is $(1,\tilde{\mu}_2,\dots,\tilde{\mu}_{n-i})$. The Jacobian of the polynomial equation system $(*)$ at z is

$$M(z):=\begin{bmatrix} J(f)(x) & 0 & O & O & \cdots & O\\ * & \frac{\partial f}{\partial x_1}(x) & O & O & \cdots & O\\ * & * & I_{n-i-1} & O & \cdots & O\\ * & * & * & Z_1 & \cdots & Z_i \end{bmatrix}.$$

From the second equation of $(**)$ we deduce $\frac{\partial f}{\partial x_1}(x)\neq 0$ and therefore $M(z)$ has maximal rank $n+1$. This implies that the equations of $(**)$ intersect transversally at the point z. Since z was an arbitrary point satisfying the polynomial conditions $(**)$, the lemma follows by standard arguments of differential geometry and commutative algebra. ■

Since the manifold $S_{i,k}$ is a smooth, closed algebraic subvariety of $\mathbb{A}^{r_i}$ we may apply the algorithmic procedure designed in [3] and [4] or [28] to the equation system $(**)$ in order to find algebraic sample points for the connected components of the real variety $(S_{i,k})_{\mathbb{R}}$. The complexity of this algorithm is linear in L and polynomial in d, n and $\delta_{i,k}$, where $\delta_{i,k}$ is the maximal degree of the generic dual polar varieties of $S_{i,k}$. We denote for $1 \leq j \leq D_i$, $1 \leq k \leq N_i$ these polar varieties by $B_{i,j,k}$ and call them *bipolar* varieties of V. In the sequel we shall suppose without loss of generality that the bipolar varieties of V are associated with real matrices.

For a given index $1 \leq i \leq n-i$ the maximal geometric degree of the bipolar variety of V, namely

$$\delta_i := \max\{\deg B_{i,j,k} \,|\, 1 \leq j \leq D_i,\, 1 \leq k \leq N_i\} = \max\{\delta_{i,k} \,|\, 1 \leq k \leq N_i\},$$

is an invariant of V. For fixed indices $1 \leq i \leq n-1$ and $1 \leq k \leq N_i$, the bipolar varieties of V are organized with decreasing codimension j in strictly ascending chains as follows:

$$B_{i,D_i,k} \subset \cdots \subset B_{i,j,k} \subset \cdots \subset B_{i,1,k} \subset B_{i,0,k} = S_{i,k}.$$

Finally with i running from $n-1$ to 1, we obtain a three–dimensional lattice of bipolar varieties. A *walk* in this lattice is a path which starts with some n–tuple of zero–dimensional bipolar varieties $(B_{i_1,D_{i_1},k})_{1 \leq k \leq n}$ and ends with some orbit variety S_{i_2}. At each step, the index i or the codimension j decreases and the bipolar varieties visited along the walk, modulo suitable sections and identifications, form ascending chains of bipolar varieties of dimension exactly increasing by one. Their real trace is dense. Running through a given walk in the reverse mode, we obtain an algorithmic strategy, which as soon that it finds smooth real points on the bipolar varieties, projects them on smooth real points of V. Observe that we obtain as a bonus certain $((n-i) \times n)$–matrices A with real algebraic entries and with them real algebraic sample points of the the (non–generic) polar varieties $P_i(A)$.

This argumentation explains fairly well the geometric ideas behind our approach to point finding in singular real hypersurfaces.

For given $1 \leq i \leq n-1$, an algorithm which follows textually our explanations would involve computations with polynomials in $O(n+(n-i)^2)$ variables. This would lead to a worst case complexity estimation of $d^{O(n+(n-i)^2)}$ whereas the expected worst case complexity is $d^{O(n)}$. There are two ways out of this dilemma. One way is to choose i close to $n-1$ or to remodel the deformation of $J(f)$ used in the equation system $(*)$ in the spirit of the concept of *meagerly generic* varieties introduced in [5].

If we choose $i := n-1$ we obtain an intrinsic variant of the so called "critical point" method, which is often used in a geometrically unstructured way with extrinsic complexity bounds.

Summarizing we have the following complexity result.

Theorem 2.2. *Let $f(x_1,\ldots,x_n)$ be a polynomial of degree $d \geq 2$ defining as before complex and real hypersurfaces V and $V_{\mathbb{R}}$. Suppose that f is given by a straight–line program of size L. Then each walk $\mathcal{W}$ yields a procedure $\mathcal{R}_{\mathcal{W}}$ that finds at least one real algebraic sample point for each connected component of $V_{\mathbb{R}}$. The sequential time complexity of the procedure $\mathcal{R}_{\mathcal{W}}$ is linear in L and polynomial in d, n and a suitable geometric quantity $\delta_{\mathcal{W}}$. The quantity $\delta_{\mathcal{W}}$ is the maximal degree of the bipolar varieties of V visited during the walk and is therefore an intrinsic invariant of V and $\mathcal{W}$. It bounds also the number and the algebraic degree of the sample points produced by $\mathcal{R}_{\mathcal{W}}$.*

As said before, for suitably chosen walks $\mathcal{W}$ the quantity $\delta_{\mathcal{W}}$ and the complexity of the procedure $\mathcal{R}_{\mathcal{W}}$ are in worst case of order $d^{O(n)}$, which meets all previously known algorithmic bounds (see [12], [9], [20] and [14] for original contributions and [15], [21, 22, 23] and [6] for forthcoming work). Following [11], [10] and [16] $d^{\Omega(n)}$ is also a lower bound for the worst case complexity of elimination problems like the one under consideration. The only way out of this dilemma is the introduction of intrinsic complexity measures like $\delta_{\mathcal{W}}$.

§3. A computational example

We are going now to illustrate our approach to find sample points in singular real hypersurfaces V by an explicit calculation in the case $n := 2$, i.e., when $V_{\mathbb{R}}$ is a singular algebraic curve in the plane $\mathbb{R}^2$.

Let f be a non–constant square-free polynomial in two variables x and y with real coefficients, V the complex plane curve defined by f and let $V_{\mathbb{R}}$ be the real trace of V. We suppose as before that $V_{\mathbb{R}}$ is a real curve, i.e., $V_{\mathbb{R}}$ is non–empty and does not contain isolated points. This implies that $J(f)$ does not vanish identically on any connected component of $V_{\mathbb{R}}$. For the sake of simplicity, we suppose that $\frac{\partial f}{\partial x}$ satisfies already this condition (this may be achieved by a generic linear coordinate transformation of x and y). The smooth variety S_1 introduced in the previous section has two charts $S_{1,1}$ and $S_{1,2}$. Without loss of generality we may identify $S_{1,1}$ with the subvariety W of $\mathbb{C}^4$ consisting of all complex solutions of the system

$$
\begin{aligned}
f(x,y) &= 0\\
\lambda\frac{\partial f}{\partial x}(x,y)+1 &= 0\\
\lambda\frac{\partial f}{\partial y}(x,y)+a &= 0.
\end{aligned}
\tag{1}
$$

Since, by assumption, $V_{\mathbb{R}}$ contains points where $\frac{\partial f}{\partial x}$ does not vanish, we conclude that the real variety $W_{\mathbb{R}}$ is non–empty. If $V_{\mathbb{R}}$ is singular, then $W_{\mathbb{R}}$ is non–compact.

Let $(\alpha, \beta, \gamma, \delta) \in \mathbb{R}^4$ be an arbitrary point belonging to an open set O of genericity (the genericity condition of O will become clear in the sequel). The bipolar variety $B := B_{1,1,1}$ which depends on the polynomial and the chart $S_{1,1}$, is a dual polar variety of the smooth (and non-compact) variety W whose real points are the critical points of the distance function to the point $(\alpha, \beta, \gamma, \delta)$, restricted to $W_{\mathbb{R}}$. In terms of equations, B is defined by the system (1) and by

$$\det \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & 0 & 0 \\ \lambda \frac{\partial^2 f}{\partial x^2} & \lambda \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial f}{\partial x} & 0 \\ \lambda \frac{\partial^2 f}{\partial x \partial y} & \lambda \frac{\partial^2 f}{\partial y^2} & \frac{\partial f}{\partial y} & 1 \\ \alpha - x & \beta - y & \gamma - \lambda & \delta - a \end{bmatrix} = 0.$$

We are now going to show that B contains real points (at least one in each connected component of $W_{\mathbb{R}}$).

For every point (x, y, a, λ) of $W_{\mathbb{R}}$ the matrix

$$M := \begin{bmatrix} \frac{\partial f}{\partial x} & 0 & 0 \\ \lambda \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x} & 0 \\ \lambda \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial f}{\partial y} & 1 \end{bmatrix}$$

is regular, since $\frac{\partial f}{\partial x}$ does not vanish at any point of $W_{\mathbb{R}}$.

Thus we may infer that y is a parameter of the real variety $W_{\mathbb{R}}$ Therefore, $W_{\mathbb{R}}$ may be interpreted as the graph of a map $(x(y), \lambda(y), a(y))$.

Differentiating the equation system (1) with respect to y, we conclude that for any point $(x, y, a, \lambda) \in W_{\mathbb{R}}$

$$(2) \qquad M \begin{bmatrix} x' \\ \lambda' \\ a' \end{bmatrix} + \begin{bmatrix} \frac{\partial f}{\partial y} \\ \\ \lambda \frac{\partial^2 f}{\partial x \partial y} \\ \\ \lambda \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = 0.$$

holds.

Choosing the set of genericity O outside of the real variety $W_{\mathbb{R}}$, we have $(\alpha, \beta, \gamma, \delta) \notin W_{\mathbb{R}}$. Since $W_{\mathbb{R}}$ is closed in $\mathbb{R}^4$, there exists a point (x, y, λ, a) in $W_{\mathbb{R}}$ such that the distance $(\alpha - x)^2 + (\beta - y)^2 + (\gamma - \lambda)^2 + (\delta - a)^2$ is minimal (in fact, such a point exists in every connected component of $W_{\mathbb{R}}$). It is obvious that such a point satisfies the condition:

$$(3) \qquad (\alpha - x)x' + (\beta - y) + (\gamma - \lambda)\lambda' + (\delta - a)a' = 0.$$

We consider now the augmented matrix:

$$\widetilde{M} := \begin{bmatrix} \frac{\partial f}{\partial x} & 0 & 0 \\ \\ \lambda \frac{\partial^2 f}{\partial x^2} & \frac{\partial f}{\partial x} & 0 \\ \\ \lambda \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial f}{\partial y} & 1 \\ \alpha - x & \gamma - \lambda & \delta - a \end{bmatrix}.$$

From the equations (2) and (3) we deduce

$$\widetilde{M} \begin{bmatrix} x' \\ \lambda' \\ a' \end{bmatrix} + \begin{bmatrix} \frac{\partial f}{\partial y} \\ \\ \lambda \frac{\partial^2 f}{\partial x \partial y} \\ \\ \lambda \frac{\partial^2 f}{\partial y^2} \\ \\ \beta - y \end{bmatrix} = 0.$$

This implies

$$(4) \qquad \det \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & 0 & 0 \\ \lambda\frac{\partial^2 f}{\partial x^2} & \lambda\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial f}{\partial x} & 0 \\ \lambda\frac{\partial^2 f}{\partial x \partial y} & \lambda\frac{\partial^2 f}{\partial y^2} & \frac{\partial f}{\partial y} & 1 \\ \alpha - x & \beta - y & \gamma - \lambda & \delta - a \end{bmatrix} = 0,$$

This means that the minimizing point considered above belongs to the bipolar variety B. From the argumentation in the previous section we conclude now that for a generic choice of $(\alpha,\, \beta,\, \gamma,\, \delta)$ in $\mathbb{R}^4$ the bipolar variety B is zero-dimensional or empty. Our preceding calculation excludes the second alternative. Therefore, B contains a real point (x, y, a, λ). Thus (x, y) is a real algebraic point of $V_{\mathbb{R}}$ which satisfies the condition $\frac{\partial f}{\partial x}(x, y) \neq 0$.

Let us illustrate this argumentation by the simple example of an algebraic curve consisting of two non–collinear lines. The intersection of this two lines is a double point of the curve. Thus let us consider $f(x, y) := x^2 - y^2$ and observe that the Gauss–map of $V := \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$ is degenerated. The system (1) takes now the following form:

$$(1') \qquad \begin{aligned} x^2 - y^2 &= 0 \\ 2\lambda x + 1 &= 0 \\ -2\lambda y + a &= 0. \end{aligned}$$

Consequently the equation (4) becomes

$$(4') \qquad \det \begin{bmatrix} x & -y & 0 & 0 \\ \lambda & 0 & x & 0 \\ 0 & -2\lambda & -2y & 1 \\ \alpha - x & \beta - y & \gamma - \lambda & \delta - a \end{bmatrix} = 0.$$

We consider now the connected component of $V_{\mathbb{R}}$ which is defined by the equation $x - y = 0$ (the case of the component of $V_{\mathbb{R}}$ given by the equation $x + y = 0$ is treated similarly). On this connected component equation $(4')$ becomes:

$$\det \begin{bmatrix} x & -x & 0 & 0 \\ -\frac{1}{2x} & 0 & x & 0 \\ 0 & \frac{1}{x} & -2x & 1 \\ \alpha - x & \beta - x & \gamma + \frac{1}{2x} & \delta + 1 \end{bmatrix} = 0.$$

This latter equation may be rewritten as

$$0 = -(\alpha - x)x^2 + (\beta - x)x^2 - (\gamma + \frac{1}{2x})\frac{1}{2} - 2(\delta + 1)x = (\alpha - \beta)x^2 + \frac{\gamma}{2} + \frac{1}{4x} - 2(\delta + 1)x,$$

i.e., as

$$(5') \qquad (\alpha - \beta)x^3 + 2(\delta + 1)x^2 + \frac{\gamma}{2}x + \frac{1}{4} = 0.$$

Since $(\alpha, \beta, \gamma, \delta)$ was chosen generically in $\mathbb{R}^4$, we have $\alpha \neq \beta$. Therefore the equation $(5')$ has a real solution $u \neq 0$.

This implies that the real algebraic point $(u, u, -\frac{1}{2u}, -1)$ belongs to the bipolar variety B, and hence $(u, u) \in \mathbb{R}^2$, the point found by our procedure, is an algebraic sample point of $V_{\mathbb{R}}$ which belongs to the connected component under consideration.

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