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ENUMERATION OF SURFACES CONTAINING AN ELLIPTIC QUARTIC CURVE

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Dedicated to Steve Kleiman on the occasion of his 70th birthday

ABSTRACT. A very general surface of degree at least four in $\mathbb{P}^3$ contains no curves other than intersections with surfaces. We find a formula for the degree of the locus of surfaces in $\mathbb{P}^3$ of degree at least five which contain some elliptic quartic curve. We also compute the degree of the locus of quartic surfaces containing an elliptic quartic curve, a case not covered by that formula.

1. INTRODUCTION

The Noether-Lefschetz theorem asserts that all curves contained in a very general surface F of degree at least four in $\mathbb{P}^3$ are complete intersections. This is usually rephrased saying that the Picard group is $\mathbb{Z}$. Noether-Lefschetz theory shows that, roughly speaking, each additional generator for Pic$F$ decreases the dimension of the locus of such F in $\mathbb{P}^N = |\mathcal{O}_{\mathbb{P}^3}(d)|$, $d \geq 4$.

Let W be a closed, irreducible subvariety of the Hilbert scheme of curves in $\mathbb{P}^3$ with Hilbert polynomial $p_W(t)$. Let us denote by $NL(W, d)$ the subset of $\mathbb{P}^N$ defined by the requirement that the surface contain some member of W.

The purpose of this note is to address the question of determining the degree of $NL(W, d)$ for the family of elliptic quartic curves in $\mathbb{P}^3$.

When W is the family of lines, or conics, or twisted cubics, formulas for $NL(W, d)$ have been found in [9]. There as here, we follow the strategy of using Bott's formula as explained in [5]. We get a polynomial formula (4.3) valid for $d \geq 5$. We also compute the degree (**38475**) of the locus of quartic surfaces containing an elliptic quartic curve. The case of quartic surfaces is not covered by the formula essentially because the map that forgets the curve shrinks dimensions: generically, it contracts a pair of disjoint pencils, see 4.2.

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2. THERE IS A POLYNOMIAL FORMULA

Let W be a closed, irreducible subvariety of the Hilbert scheme of curves in $\mathbb{P}^3$ with Hilbert polynomial $p_W(t)$. Let

$$\begin{tikzcd} W \times \mathbb{P}^N \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathbb{P}^N \\ W & \end{tikzcd} \tag{2.1}$$

denote the projection maps from $W \times \mathbb{P}^N$. Castelnuovo-Mumford regularity [12] shows that for all $d >> 0$, the subset $\widetilde{NL}(W, d)$ of pairs (C, F) in $W \times \mathbb{P}^N$ such that the curve C is contained in the surface F is a projective bundle over W via p_1. We have

$$\operatorname{codim}_{W\times\mathbb{P}^n} \widetilde{NL}(W, d) = p_W(d).$$

For instance, if W is the Grassmannian of lines in $\mathbb{P}^3$, then $p_W(d) = d+1$ and so $\dim \widetilde{NL}(W, d) = N - (d - 3)$, $d \geq 1$.

Let us denote by $NL(W, d)$ the subset of $\mathbb{P}^N$ defined by the requirement that the surface contain some member of W. In other words, with notation as in (2.1),

$$NL(W, d) = p_2(\widetilde{NL}(W, d)).$$

We assume henceforth that the general member of W is a smooth curve.

2.1. **Proposition.** *For fixed W we have that $\deg NL(W, d)$ is a polynomial in d of degre $\leq 3 \dim W$, for all $d >> 0$.*

Proof. Let $\widetilde{C} \subset W \times \mathbb{P}^3$ be the universal curve. Likewise, let $\widetilde{F} \subset \mathbb{P}^N \times \mathbb{P}^3$ be the universal surface of degree d. Write $\widehat{C}, \widehat{F}$ for their pullbacks to $W \times \mathbb{P}^N \times \mathbb{P}^3$. We have the diagram of sheaves over $Y := W \times \mathbb{P}^N \times \mathbb{P}^3$,

$$\begin{tikzcd} \mathcal{O}_Y \arrow[r] \arrow[dr, "\rho"'] & \mathcal{O}_Y(\widehat{F}) \arrow[r] \arrow[d] & \mathcal{O}_{\widehat{F}}(\widehat{F}) \\ & \mathcal{O}_{\widehat{C}}(\widehat{F}) & \end{tikzcd}$$

By construction, the slant arrow ρ vanishes at a point $(C, F, x) \in W \times \mathbb{P}^N \times \mathbb{P}^3$ if and only if $x \in F \cap C$. We have $C \subset F$ when the previous condition holds for all $x \in C$ (point with values in any $\mathbb{C}$-algebra). Thus $\widetilde{NL}(W, d)$ is equal to the scheme of zeros of ρ along the fibers of the projection $p_{12} : \widehat{C} \to W \times \mathbb{P}^N$. Recalling [1, (2.1),p. 14], this is the same as the zeros of the adjoint section of the direct image vector bundle $p_{12\star}(\mathcal{O}_{\widehat{C}}(\widehat{F}))$. Let

$$\begin{tikzcd} W \times \mathbb{P}^3 \arrow[r, "q_2"] \arrow[d, "q_1"'] & \mathbb{P}^3 \\ W & \end{tikzcd}$$

denote the projection maps from $W \times \mathbb{P}^3$. Since $\mathcal{O}(\widetilde{F}) = \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_{\mathbb{P}^3}(d)$, by projection formula we have to make and do with a section of $\mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{E}_d$, where

$$\mathcal{E}_d = q_{1\star}(\mathcal{O}_{\widetilde{C}}(d)). \tag{2.2}$$

By Castelnuovo-Mumford and base change theory, there is an integer d_0 such that $\mathcal{E}_d$ is a vector bundle of rank $p_W(d)$ for all $d \geq d_0$ (=regularity, see Remark 2.2.). In fact, it fits into the exact sequence of vector bundles over W,

$$\begin{tikzcd} 0 \arrow[r] & q_{1\star}(\mathcal{I}_{\widetilde{C}} \otimes \mathcal{O}_{\mathbb{P}^3}(d)) \arrow[r] \arrow[d, equal] & q_{1\star}(\mathcal{O}_{\mathbb{P}^3}(d)) \arrow[r] \arrow[d, equal] & q_{1\star}(\mathcal{O}_{\widetilde{C}}(d)) \arrow[r] \arrow[d, equal] & 0 \\ & \mathcal{D}_d \arrow[r, tail] & \mathcal{F}_d \arrow[r, two heads] & \mathcal{E}_d & \end{tikzcd} \tag{2.3}$$

where we set for short

$$\mathcal{F}_d = H^0(\mathcal{O}_{\mathbb{P}^3}(d)),$$

(trivial vector bundle with fiber) the space of polynomials of degree d. Taking the projectivization, and pulling back to $\mathbb{P}^N \times W$, we get

$$\begin{tikzcd} & \mathcal{O}_{\mathbb{P}^N}(-1) \arrow[d, tail] \arrow[dr, "\overline{\rho}"] & \\ \mathcal{D}_d \arrow[r, tail] & \mathcal{F}_d \arrow[r, two heads] & \mathcal{E}_d \end{tikzcd}$$

By construction, $\overline{\rho}$ vanishes precisely over $\widetilde{NL}(W,d)$. This shows that we actually get

$$\widetilde{NL}(W,d) = \mathbb{P}(\mathcal{D}_d). \tag{2.4}$$

Since rank of $\mathcal{E}_d$ and codimension of $\widetilde{NL}(W,d)$ agree, it follows that $\widetilde{NL}(W,d)$ represents the top Chern class of $\mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{E}_d$ (cf. [6, 3.2.16, p. 61]). This is the key to the calculation of degrees below. The map

$$\begin{array}{ccc} W \times \mathbb{P}^N \supset \widetilde{NL}(W,d) & \xrightarrow{p_2} & NL(W,d) \subset \mathbb{P}^N \\ (C,F) & \mapsto & F \end{array} \tag{2.5}$$

is generically injective by Noether-Lefschetz theory [8], cf. Corollary 4.2 below. Therefore the degree of $NL(W,d)$ can be computed upstairs. Namely, setting

$$m = \dim \widetilde{NL}(W,d), H = c_1\mathcal{O}_{\mathbb{P}^N}(1),$$

we have

$$\deg NL(W,d) = \int H^m \cap \widetilde{NL}(W,d) = \int H^m c_\tau(\mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{E}_d),$$

where $\tau = \mathrm{rank}\mathcal{E}_d$. Expanding the top Chern class and pushing forward to W, we arrive at

$$\deg NL(W,d) = \int_W c_w(\mathcal{E}_d), \tag{2.6}$$

with $w = \dim W$. Since $\mathcal{E}_d$ is the pushforward of a sheaf on $W \times \mathbb{P}^3$, we may apply Grothendieck-Riemann-Roch [6, p. 286] to express the Chern character of $\mathcal{E}_d$ as

$$\mathrm{ch}(\mathcal{E}_d) = \mathrm{ch}(q_{1!}(\mathcal{O}_{\widetilde{C}}(d)) = q_{1\star}\left(\mathrm{ch}(\mathcal{O}_{\widetilde{C}})\,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^3}(d))\mathrm{todd}\mathbb{P}^3\right).$$

Notice that the right hand side is a polynomial in d of degree ≤ 3. Since c_w is a polynomial of degree w on the coefficients of the Chern character, we deduce that $c_w(\mathcal{E}_d)$ is a polynomial in d of degree $\leq 3w$. $\square$

2.2. **Remarks.** (1) The assertion that $\mathcal{E}_d$, as defined in (2.2), is a vector bundle of rank $p_W(d)$ holds for all d beyond $d_0 =$ the maximal Castelnuovo-Mumford regularity of the members of W. For instance, if W is the family of lines in $\mathbb{P}^3$, then $d_0 = 1$.
(2) For the case of elliptic quartic curves presented below, we note that the regularity of the ideals $\langle x_1^2, x_2^2\rangle$ and $\langle x_1x_2, x_1^2, x_2^3\rangle$ is 3, whereas for $\langle x_0^2, x_0x_1, x_0x_2^2, x_1^4\rangle$ it is 4. The last two ideals are representatives of the closed orbits in W. An argument of semi-continuity shows that $d_0 = 4$ works for all members of W, see [2]. Nevertheless, the map in display (2.5) is generically injective only for $d \geq 5$, cf. (4.2) below. Notice that the full $\mathrm{Hilb}^{4t}\mathbb{P}^3$ has a "ghost" component with regularity index ≥ 5, see [7].

3. ELLIPTIC QUARTICS

We consider now the case of surfaces of degree ≥ 4 containing an elliptic quartic curve in $\mathbb{P}^3$. Thus, a general member C_4 of W is the intersection of two quadric surfaces. The parameter space W is described in [2] and has been used in [5] to enumerate curves in Calabi-Yau 3-folds. For the convenience of the reader, we summarize below its main features.

The Noether–Lefschetz locus of quartic surfaces containing some C_4 is slightly exceptional. This is a case when the map (2.5) fails to be generically injective (cf. Corollary 4.2): it actually shrinks $\dim \widetilde{NL}(W,d) = 34$ to $\dim NL(W,d) = 33$. Indeed, if a quartic surface F contains some general elliptic quartic C_4, then F must contain the two pencils $|C_4|$ and $|C_4'|$, where C_4' is the residual intersection of F with a quadric containing C_4, *i.e.*, $2H = C_4 + C_4'$, with $H =$ plane section. We show in §3.2 that $NL(W,4)$ is a hypersurface of degree **38475** in $\mathbb{P}^{34} = |\mathcal{O}_{\mathbb{P}^3}(4)|$.

3.1. Next we give an outline of the calculation. Put

$$\mathbb{X} = \mathbb{G}(2,\mathcal{F}_2),$$

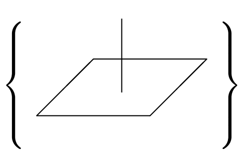
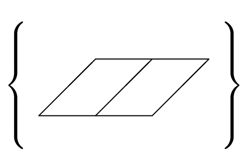
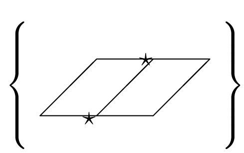
the Grassmannian of pencils of quadrics in $\mathbb{P}^3$.

The diagram below summarizes the construction of W as explained in [2].

$$\begin{tikzcd}
\mathbb{G}(19,\mathcal{F}_4) \supset W \arrow[r, equal] & \widehat{\mathbb{X}} \arrow[d] & & & & \widehat{\mathbb{E}} \arrow[llll, hook'] \arrow[d] \\
\mathbb{G}(8,\mathcal{F}_3)\times\mathbb{X} \supset & \widetilde{\mathbb{X}} \arrow[d] & \supset & \widetilde{\mathbb{E}} \arrow[d] & \supset & \widetilde{Y} \arrow[d] \\
\mathbb{G}(2,\mathcal{F}_2) \arrow[r, equal] & \mathbb{X} & \supset & Z & \supset & Y
\end{tikzcd} \tag{3.1}$$

where

$$\begin{cases} Z \cong \check{\mathbb{P}}^3 \times \mathbb{G}(2,\mathcal{F}_1) \ \text{ consists of pencils with a fixed plane;} \\ Y \cong \{(p,\ell)\,|\, p \supset \ell\} = \ \text{ closed orbit of } \ Z; \\ \widetilde{Y} \ \longrightarrow \ Y = \mathbb{P}^2 - \text{bundle of degree 2 divisors on the varying } \ \ell \subset p; \\ \widehat{X} = \ \text{ the blow-up of } \widetilde{X} \ \text{ along } \widetilde{Y} \text{ and} \\ \widetilde{X} = \ \text{ the blow-up of } X \text{ along } Z. \end{cases}$$

$Z =$  $\supset Y =$  $\longleftarrow \widetilde{Y} =$ 

Let

$$\mathcal{A} \subset \mathcal{F}_2 \times \mathbb{X}$$

be the tautological subbundle of rank 2 over our Grassmannian of pencils of quadrics. There is a natural map of vector bundles over $\mathbb{X}$ induced by multiplication,

$$\mu_3 : \mathcal{A} \otimes \mathcal{F}_1 \longrightarrow \mathcal{F}_3 \times \mathbb{X}$$

with generic rank 8. It drops rank precisely over Z. It induces a rational map $\kappa : \mathbb{X} \dashrightarrow \mathbb{G}(8, \mathcal{F}_3)$. Blowing up $\mathbb{X}$ along Z, we find the closure $\widetilde{\mathbb{X}} \subset \mathbb{G}(8, \mathcal{F}_3) \times \mathbb{X}$ of the graph of κ. Similarly, up on $\widetilde{\mathbb{X}}$ we have a subbundle $\mathcal{B} \subset \mathcal{F}_3 \times \widetilde{\mathbb{X}}$ of rank 8 and a multiplication map

$$\mu_4 : \mathcal{B} \otimes \mathcal{F}_1 \longrightarrow \mathcal{F}_4 \times \widetilde{\mathbb{X}}$$

with generic rank 19. The scheme of zeros of $\bigwedge^{19} \mu_4$ is equal to $\widetilde{Y}$. Indeed, it can be checked that each fiber of $\mathcal{B}$ is a linear system of cubics which

- either has base locus equal to a curve with Hilbert polynomial $p_W(t) = 4t$
- or is of the form $p \cdot \mathcal{F}_2^{\star\star}$, meaning the linear system with fixed componente a plane p, and $\mathcal{F}_2^{\star\star}$ denoting an 8-dimensional space of quadrics which define a subscheme of p of dimension 0 and degree 2.

The exceptional divisor $\widehat{\mathbb{E}}$ is a $\mathbb{P}^8$-bundle over $\widetilde{Y}$. The fiber of $\widehat{\mathbb{E}}$ over $(p, y_1 + y_2) \in \widetilde{Y}$ is the system of quartic curves in the plane p which are singular at the doublet $y_1 + y_2$. Precisely, if $x_0, \ldots, x_3$ denote homogeneous coordinates on $\mathbb{P}^3$, assuming $p := x_0, \ell = \langle x_0, x_1 \rangle$, a typical doublet has homogeneous ideal of the form $\langle x_0, x_1, f(x_2, x_3) \rangle$, with $\deg f = 2$. Our system of plane quartics lies in the ideal $\langle x_1, f \rangle^2 = \langle x_1^2, x_1 f, f^2 \rangle$. Given a non-zero quartic g in this ideal, we may form the ideal $J = \langle x_0^2, x_0 x_1, x_0 f, g \rangle$. It can be checked that the J contains precisely 19 independent quartics and its Hilbert polynomial is correct. In fact, any such ideal is 4-regular (in the sense of Castelnuovo-Mumford). Moreover, up on $\widehat{\mathbb{X}}$ we get a subbundle

$$\mathcal{C} \subset \mathcal{F}_4 \times \widehat{\mathbb{X}}$$

of rank 19. Each of its fibers over $\widehat{\mathbb{X}}$ is a system of quartics which cut out a curve with the correct Hilbert polynomial. The multiplication map

$$\mathcal{C} \otimes \mathcal{F}_{d-4} \longrightarrow \mathcal{F}_d \times \widehat{\mathbb{X}}$$

is of constant rank $\binom{d+3}{3} - 4d$. The image

$$\mathcal{D}_d \subset \mathcal{F}_d \times \widehat{\mathbb{X}}$$

is a subbundle as in (2.3). We have

$$\widetilde{NL}(W, d) = \mathbb{P}(\mathcal{D}_d) \subset \mathbb{P}^N \times \widehat{\mathbb{X}}.$$

Now the map $\widetilde{NL}(W, d) \to NL(W, d)$ is generically injective for $d \geq 5$ in view of Corollary 4.2 (ii) below. Hence the degree of the image $NL(W, d) \subset \mathbb{P}^N$ is given by $\int c_{16} \mathcal{E}_d$, cf. (2.6).

The above description suffices to feed in Bott's localization formula with all required data. Indeed, $\widehat{\mathbb{X}}$ inherits a $\mathbb{C}^\star$-action, with (a lot) of isolated fix points. The vector bundle $\mathcal{D}_d \to \widehat{\mathbb{X}}$ is equivariant; ditto for $\mathcal{E}_d$. Bott's formula reads [4],

$$\int c_{16} \mathcal{E}_d = \sum_{p \in \text{fixpts}} \frac{c_{16}^T (\mathcal{E}_d)_p}{c_{16}^T \tau_p} \,. \tag{3.2}$$

The equivariant classes on the r.h.s are calculated in two steps. We set below $d = 5$ for simplicity. First, find the $\mathbb{C}^\star$–weight decomposition of the fibers of the vector bundles $\mathcal{E}_d$ and τ at each fixed point. Then, since the fix points are isolated, the equivariant Chern classes c_i^T are just the symmetric functions of the weights.

For instance, for the tangent bundle τ, say at the fixed point corresponding to the pencil $p = \langle x_0^2, x_1^2\rangle \in \mathbb{X}$, we have

$$\tau_p = \mathrm{Hom}(p, \mathcal{F}_2/p) = p^\vee \otimes \mathcal{F}_2/p = \tfrac{x_0x_1}{x_0^2} + \tfrac{x_0x_2}{x_0^2} + \tfrac{x_1x_2}{x_0^2} + \cdots + \tfrac{x_3^2}{x_1^2}\cdot$$

Each of the 16 fractions , $\frac{x_ix_j}{x_kx_l}$, on the right hand side symbolizes a 1-dimensional subspace of τ_p where $\mathbb{C}^\star$ acts with character $t^{x_i+x_j-x_k-x_l}$. The denominator in (3.2) is the corresponding equivariant top Chern class, to wit

$$(x_1 - x_0)(x_2 - x_0)(x_1 + x_2 - 2x_0)\cdots(2x_3 - 2x_1).$$

The numerator in (3.2) requires finding the monomials of degree 5 that survive modulo the ideal

$$\langle x_0^2, x_1^2\rangle \cdot \langle x_0, \dots, x_3\rangle^3.$$

We are left with 20 (=rank $\mathcal{E}_5$) terms,

$$x_2^3x_1x_0 + x_3x_2^3x_1 + \cdots + x_3^4x_0.$$

Now the equivariant Chern class $c_{16}^T(\mathcal{E}_d)_p$ is the coefficient of t^{16} in the product

$$(t + x_0 + x_1 + 3x_2)(t + x_1 + 3x_2 + x_3)\cdots(t + x_0 + 4x_3) \quad .$$

(20 factors.) In practice, all these calculations are made substituting x_i for suitable numerical values, cf. the computer algebra scripts in [13].

3.2. **the case $d = 4$.** Presently $p_2 : \widetilde{NL}(W,4) \to NL(W,4)$ is no longer generically injective. It shrinks dimension by one: a general fiber is a disjoint union of $\mathbb{P}^1$'s (cf. Corollary 4.2 (iii) below). Explicitly, say $F_4 = A_1Q_1 + A_2Q_2$, $\deg A_i = \deg Q_i = 2$, everything in sight as general as needed. Then

$$F_4 = (A_1 - tQ_2)Q_1 + (A_2 + tQ_1)Q_2,$$

so F_4 contains the pencil of elliptic quartics $\langle A_1 - tQ_2, A_2 + tQ_1\rangle, t \in \mathbb{P}^1$; setting $t = \infty$, we find $\langle Q_1, Q_2\rangle$. Similarly, get $\langle Q_1 - tA_2, Q_2 + tA_1\rangle$. This is one and the same pencil. But there is also $\langle A_1 - tA_2, Q_2 + tQ_1\rangle$. In general, these 2 pencils are disjoint. Looking at them as curves in $\mathbb{X} = \mathbb{G}(2, \mathcal{F}_2)$, we actually get a Plücker–embedded conic, $(A_1 - tA_2) \wedge (Q_2 + tQ_1) = A_1 \wedge Q_2 + t(A_1 \wedge Q_1 - A_2 \wedge Q_2) - t^2 A_2 \wedge Q_1$, disjoint from Y (see (3.1)). In particular, capping each conic against the Plücker hyperplane class $\Pi = -c_1\mathcal{A}$, we find 2. As before, we may write

$$\deg NL(W,d) = \int H^{33} \cap NL(W,d).$$

The cycle $p_2^\star H^{33} \cap NL(W,d)$ can be represented by a sum of $\deg NL(W,d)$ disjoint unions of pairs of $\mathbb{P}^1$'s. Hence

$$\deg NL(W,d) = \frac{1}{4}\int_{\widehat{\mathbb{X}}} \Pi\cdot H^{33} \cap \widetilde{NL}(W,d) = \frac{1}{4}\int_{\widehat{\mathbb{X}}} \Pi\cdot (p_1)_\star H^{33} \cap \widetilde{NL}(W,d)$$

$$= \frac{1}{4}\int_{\widehat{\mathbb{X}}} \Pi\cdot c_{15}(\mathcal{E}_4).$$

The latter integral can be computed via Bott's formula and we get **38475**, cf. the script in [13]. This has been found independently in [3] with different techniques, using [10].

4. The fibers of p_2

The main result needed to validate the above enumeration is the following.

4.1. **Proposition.** *Let $C \subset \mathbb{P}^3$ be a smooth irreducible curve of degree e and genus g. Let $d \gg 0$ and let $F \subset \mathbb{P}^3$ be a general surface of degree d containing C. Then C is the only effective divisor of degree e and arithmetic genus g on F.*

Proof. By [8, Cor.II.3.8] we have that $\mathrm{Pic}(F)$ is freely generated by its hyperplane section H and C. Let C' be an effective divisor of degree e and arithmetic genus g on F. Then there are two integers a, b such that, on F, we have $C' \sim aH + bC$. Now $e = H \cdot C' = ad + be$, so that

$$a = \frac{e}{d}(1 - b). \tag{4.1}$$

By adjunction formula we have

$$C^2 = 2p_a(C) - 2 - K_F \cdot C = 2g - 2 - e(d-4) = (C')^2 < 0 \tag{4.2}$$

as $d \gg 0$. Now

$$C^2 = (C')^2 = (aH + bC)^2 = a^2 d + b^2 C^2 + 2abe$$

and using (4.1) we get

$$(1 - b^2)(e^2 - dC^2) = 0.$$

Note that $e^2 - dC^2 > 0$ by (4.2), whence $b = \pm 1$. If $b = -1$ we have from (4.1) $a = \frac{2e}{d} \notin \mathbb{Z}$, as $d \gg 0$. Therefore we deduce that $b = 1$ and $a = 0$, that is $C' \sim C$. Since $C^2 < 0$ we must have $C' = C$. □

4.2. **Corollary.** *Let W be an irreducible subvariety of a Hilbert scheme component of curves in $\mathbb{P}^3$ of degree e and arithmetic genus g with general member smooth. Let $\mathbb{P}^N = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^3}(d)))$. Then*

(i) *there is a d_0 such that for all $d \geq d_0$ the projection map $p_2 : \widetilde{NL}(W,d) \to \mathbb{P}^N$ is generically injective.*

(ii) *If W is the family of elliptic quartics then we can take $d_0 = 5$,* i.e., *p_2 is generically one-to-one for $d \geq 5$ and*

(iii) *for $d = 4$, the general fiber of $\widetilde{NL}(W,4) \xrightarrow{p_2} NL(W,4)$ is two disjoint $\mathbb{P}^1$'s.*

Proof. We know from (2.4) that $p_1 : \widetilde{NL}(W,d) \to W$ is a projective bundle. Hence $NL(W,d) = p_2(\widetilde{NL}(W,d))$ is irreducible and a general element $F \in NL(W,d)$ can be identified with a general hypersurface of degree d containing a general $C \in W$. Hence assertion (i) follows from the propostion. Assertion (ii) also follows, except for $d = 8$. In this case, with the notation as in (4.1), if $b = -1$ we would get $a = 1$, whence $C' \sim H - C$ so that C would be contained in a plane, absurd. If $b = 1$, may proceed as at the end of the proof of the above proposition.

If $d = 4$ then we get instead $C' \sim 2H - C$. The exact sequence

$$0 \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_F(C) \longrightarrow \mathcal{O}_C(C) = \mathcal{O}_C \longrightarrow 0$$

shows that $|\mathcal{O}_F(C)| \cong \mathbb{P}^1$ and similarly $|\mathcal{O}_F(2H - C)| \cong |\mathcal{O}_F(C')| \cong \mathbb{P}^1$. Moreover there is no curve D on F such that $D \sim C$ and $D \sim 2H - C$ for then $2C \sim 2H$ giving the contradiction $0 = C^2 = H^2 = 4$. This proves that, in this case, the general fiber of p_2 is two disjoint $\mathbb{P}^1$'s,

$$p_2^{-1}(F) = |\mathcal{O}_F(C)| \cup |\mathcal{O}_F(2H - C)|. \tag{4.3}$$

□

4.3. **The formula.** In view of Prop. 2.1, it suffices to find the degrees of $NL(W,d)$ for $3 \cdot 16+1$ values of $d \geq 5$ and interpolate. This is done in [13]. We obtain

$$\begin{gathered}
\tbinom{d-2}{3}\Big(106984881d^{29} - 3409514775d^{28} + 57226549167d^{27} - \\
643910429259d^{26} + 5267988084411d^{25} - 31628193518727d^{24} + 126939490699539d^{23} \\
-144650681793207d^{22} - 2701978741671631d^{21} + 28913126128882647d^{20} - \\
182919422241175163d^{19} + 858473373993063183d^{18} - 3061191057059772423d^{17} \\
+7448109470245631187d^{16} - 3841505361473930575d^{15} - 80644842327962348733d^{14} + \\
568059231910087276234d^{13} - 2560865812030993315212d^{12} + 9159430737614259196104d^{11} \\
-27608527286339077691280d^{10} + 71605637662357479581024d^{9} \\
-160009170853633152594240d^{8} + 303685692157317249665152d^{7} \\
-473993548940769326728704d^{6} + 571505502502703378479104d^{5} \\
-459462480152611231457280d^{4} + 111908571251948243582976d^{3} \\
+251116612534424272896000d^{2} - 328452832055501940326400d \\
+136886449647246114816000\Big)\Big/(2^{27} \cdot 3^{9} \cdot 5^{2} \cdot 7^{2} \cdot 11 \cdot 13).
\end{gathered}$$

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