# P-means and the solution of a functional equation involving Cauchy differences

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**Abstract**

Solutions to the functional equation

$$f(x+y)-f(x)-f(y)=2f(\Phi(x,y)),\ x,y>0, \tag{1}$$

are sought for the admissible pairs $(f,\Phi)$ constituted by a strictly monotonic function $f$ and a strictly increasing in both variables mean $\Phi$. A related class of means, P-means, is introduced, studied and then employed in solving (1) under additional hypotheses on $\Phi$. For instance, R. Ger has proved that the unique P-mean which is also quasiarithmetic is the geometric mean $G(x,y)=\sqrt{xy}$. An elementary proof to this result is given in this paper. Moreover, as a consequence of a fundamental result on the uniqueness of representation of P-means it is proved that the geometric mean $G$ is the unique homogeneous P-mean.

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## 1 Introduction

The study of the functional equation

$$f(x+y)-f(x)-f(y)=2f(\Phi(x,y)),\ x,y>0, \tag{2}$$

begun in [6] (see also [5]) in connection with the Pythagorean Theorem. The main interest in that paper was the characterization of the pair constituted by the quadratic function and the geometric mean; i.e.,

$$f(x)=cx^2,\quad \Phi(x,y)=\sqrt{xy}, \tag{3}$$

among all the possible pairs $(f,\Phi)$ solving (2), after which the Pythagorean law was obtained in the form of the associative operation

$$x\bigtriangleup y=f^{-1}(f(x)+f(y)),\ x,y>0.$$

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For that purpose, the class of admissible solutions was supposed to be the pairs $(f, \Phi)$ composed by a strictly monotonic and continuous function $f$ and a reflexive function $\Phi$. In this class, the pair (3) turns out to be the unique solution to (2) giving rise to a (positively) homogeneous operation $x \triangle y$.

Displacing the focus of interest from Geometry to Functional Equations, the study of the equation (2) was continued in [12] and [7]. In the first of these papers, the restrictions on the class of admissible solution were strengthened up to the point of imposing on $\Phi$ a rigid functional form, proving that the pair (3) is the unique solution to (2) when $\Phi$ is a quasiarithmetic mean (and $f$ is, as before, a strictly monotonic and continuous function). A similar result was obtained in [7] by assuming that $\Phi$ is a (regular) Lagrangian mean. In this way, means have played a relevant role in the investigations on the equation (2) accomplished up to date and the present one will not be an exception, so that let us digress briefly on them.

Let $I$ be a real interval. A (two variables) *mean* $M$ defined on $I$ is a function $M : I \times I \to I$ which is *internal* (v.g. [9], pgs. xxvi-xxvii); i.e., it satisfies the property

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\},\ x, y \in I. \tag{4}$$

The mean is said to be *strict* when the inequalities in (4) are strict whenever $x \neq y$ (*strict internality*). Since the equality

$$M(x, x) = x,\ x \in I, \tag{5}$$

holds for every mean $M$, means are *reflexive* functions. A mean $M$ is said to be *symmetric* when

$$M(x, y) = M(y, x),\ x, y \in I. \tag{6}$$

If $I = \mathbb{R}^+$; then, a mean $M$ is (positively) *homogeneous* provided that it satisfies

$$M(\lambda x, \lambda y) = \lambda M(x, y),\ x, y > 0. \tag{7}$$

A *continuous mean* is a mean that is continuous on $I \times I$. Clearly, a reflexive and strictly increasing in both variables function $M$ defined on $I \times I$ is a strict mean on $I$. The means considered throughout this paper will generally belong to this class; for instance, the *quasiarithmetic* or *Lagrangian* means $A_f$ and $L_f$, respectively defined by

$$A_f(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right),\ x, y \in I, \tag{8}$$

and

$$L_f(x, y) = \begin{cases} f^{-1}\left(\frac{1}{y-x}\int_x^y f(\xi)\, d\xi\right), & x \neq y \\ x, & x = y \end{cases}. \tag{9}$$

In (8) and (9), $f : I \to \mathbb{R}$ is a strictly monotonic and continuous function named *generator function* of the (corresponding) mean. Setting $f(x) = x$ in (8) or (9) yields the arithmetic mean $A(x, y) = (x + y)/2$. The representations $A_f$ and $L_f$ are essentially unique, in the sense specified by the following:

**Theorem 1** *Two strictly monotonic and continuous functions $f,\ g : I \to \mathbb{R}$ are generator functions of the same quasiarithmetic (or Lagrangian) mean if and only if there exist $\alpha,\ \beta \in \mathbb{R},\ \ \alpha \neq 0$, such that*

$$g(x) = \alpha f(x) + \beta,\ \ x \in I.$$

In Cor. 5, pg. 246, of [3] can be found a proof of the quasiarithmetic case, while the Lagrangian one is covered by [8] or [9], Theor. 29, pg. 404.

In continuing the main exposition let us note that, unlike what occurs with the second members of (8) and (9), the functional form

$$\Phi(x,y) = f^{-1}\left(\frac{f(x+y) - f(x) - f(y)}{2}\right),\ \ x, y > 0, \tag{10}$$

does not necessarily define a mean when $f$ satisfies the sole conditions of monotonicity and continuity. For example, substituting $f$ by the power function $x^p,\ \ x > 0,\ \ (p \neq 0)$, in (10) yields

$$\Phi(x,y) = \left(\frac{(x+y)^p - x^p - y^p}{2}\right)^{\frac{1}{p}},$$

and $\Phi$ turns out to be reflexive only when $p = 2$. If so, (10) gives $\Phi(x,y) = \sqrt{xy} = G(x,y)$, the geometric mean which, as many times remarked along this paper, constitutes a very singular case among the means admitting this functional form.

Throughout this paper, a symmetric and strictly increasing in both variables mean $\Phi$ defined on $\mathbb{R}^+$ will be named a *P-mean* when the representation (10) is admitted by $\Phi$. It will be indistinctly said that $f$ is the *generator function* of $\Phi$ or that $\Phi$ *is generated* by $f$, which will be denoted by writing $\Phi_f$ when necessary. The class of P-means will be denoted by $\mathcal{PM}$.

Now, consider a reflexive and strictly increasing in both variables function $\Phi$ which, as said above, is a strict mean. The problem of deciding whether $\Phi$ is a P-mean can be reformulated as that of finding solutions to the functional equation (2) in the class of strictly monotonic functions $f$ defined on $\mathbb{R}^+$: $f$ solves equation (2) if and only if $\Phi$ can be written in the form (10). In this paper, a systematic exploration of P-means is undertaken or, what amounts to the same thing, of equation (2) in the class of pair $(f, \Phi)$ composed by a strictly monotonic function $f$ and a reflexive and strictly increasing in both variables function $\Phi$. The functions $f$ generating a symmetric and strictly increasing in both variables mean $\Phi$ through the formula (10) are characterized in Section 2, thus obtaining a complete description of the class of P-means. The properties of such generator functions are studied in Section 3, where basic facts on their Cauchy differences and other useful results are established. Subarithmeticity, duplication and other simple properties of the P-means are gather together in Section 4 and then used to offer an elementary proof of the main result in [12]: $G$ is the unique quasiarithmetic P-mean. A characterization of P-means based on the co-cycle equation is also given there. A fundamental result, which is to

P-means as Theorem 1 is to quasiarithmetic or Lagrangian means, is proved in the final Section 5. As a consequence of it, it is shown that a P-mean satisfies an equation of the form $\Phi(\lambda_0 x, \lambda_0 y) = \lambda_0 \Phi(x, y),\ x, y > 0$, with $\ln \lambda_0 / \ln 2 \notin \mathbb{Q}$, if and only if $\Phi = G$. In particular, the pair (3) turns out to be the the unique solving equation (2) in the class of pairs composed by a strictly monotonic function $f$ and a strictly increasing in both variables and homogeneous mean $\Phi$. Unlike what happen when a given functional form is imposed on $\Phi$ (as made in [12] and [7]), this set of hypotheses may be visually contrastable by using simple figures, so that the result can be applied in the geometric context in which equation (2) was originated.

# 2 P-means

Our first result gives necessary and sufficient conditions in order that the replacement of a strictly monotonic function $f$ in (10) yields an increasing in both variables mean $\Phi$.

**Theorem 2** *A strictly monotonic function $f : \mathbb{R}^+ \to \mathbb{R}$ generates a symmetric and increasing in both variables mean $\Phi$ through the expression (10) if and only if $f$ fulfills the following two conditions:*

**i)** *$f$ solves the Schröder equation*

$$f(2x) = 4f(x),\ x > 0; \tag{11}$$

**ii)** *$f$ is strictly convex or strictly concave provided that is increasing or decreasing, respectively.*

The following result of the type "monotonicity $\Rightarrow$ continuity" on solutions to the equation (2) will be a key part of the proof of Theorem 2.

**Theorem 3** *Let $f$ be a strictly monotone solution to equation (2) in which $\Phi$ is assumed to be a function increasing in both variables. Then $f$ and $\Phi$ are continuous functions; moreover, $f$ is convex or concave provided that $f$ is increasing or decreasing, respectively.*

A result of this kind was stated by Z. Páles in [17].

**Proof.** First, let us assume that $f$ is increasing. Then, its increment $f(x + y) - f(x)$ is, as a function of $x$, a strictly monotonic function; namely, $x \mapsto f(x + y) - f(x) = f(y) + 2f(\Phi(x, y))$ is (strictly) increasing for every $y > 0$. This fact together with the monotonicity of $f$ implies its continuity. Indeed, if $x_0$ was a jump of size $\alpha > 0$ of the (increasing) function $f$; then, taking a small enough $\delta > 0$ and choosing a point of continuity $x_1$ of $f$ such that $x_1 > x_0$, it could be written

$$\begin{aligned}\alpha &< f\left(x_0+\frac{\delta}{2}\right)-f\left(x_0-\frac{\delta}{2}\right)\\ &= f\left(\left(x_0-\frac{\delta}{2}\right)+\delta\right)-f\left(x_0-\frac{\delta}{2}\right)\\ &< f(x_1+\delta)-f(x_1),\end{aligned}$$

an inequality which is, in view of the arbitrariness of $\delta$, in contradiction with the continuity of $f$ at $x_1$. Since $f$ is continuous and strictly increasing, so is $f^{-1}$ and therefore, solving equation (2) for $\Phi$ yields the representation (10), so that $\Phi$ is also continuous.

Now, the Jensen convexity of $f$ follows from the monotonicity of $x \mapsto f(x+y)-f(x)$. In fact, the inequality

$$f((x-y)+y)-f(x-y) < f(x+y)-f(x)$$

holds for $x > y > 0$, and setting $X = x+y,\ Y = x-y$, it can be rewritten in the form

$$f\left(\frac{X+Y}{2}\right) < \frac{f(X)+f(Y)}{2},\ X,Y > 0,$$

whence $f$ is Jensen convex. Since f is continuous, it turns out to be convex, as claimed.

When $f$ is decreasing, the previous argument applies to $-f$, so that $-f$ turns out to be convex and therefore, $f$ is concave. ■

Another proof of Theorem 3 can be given along the following lines (see the proof of Lemma 2.2 in [13]). From the strict monotonicity of $x \mapsto f(x+y)-f(x)$, it is deduced that $f$ is either strictly Wright-convex or concave, and therefore, by a result due to C. T. Ng ([16]), $f$ has the form

$$f(x) = g(x)+A(x),\ x > 0,$$

where $g$ is strictly convex or strictly concave and $A$ is additive. Since $f$ is monotonic, $A(x)$ turns out to be a linear function; i.e., $A(x) = \alpha x,\ (\alpha \in \mathbb{R})$, and Theorem 3 follows.

**Proof of Theorem 2.** Observe in the first place that $f$ is a solution to equation (11) if and only if $\Phi$ is reflexive. Now, if a strict symmetric mean $\Phi$ can be represented by (10) in terms of a strictly monotonic function $f$ ; then, $f$ satisfies the condition **ii)** by Theorem 3. The converse follows by realizing that the argument in the proof of this theorem can be reversed. Assuming, for instance, that $f$ is strictly increasing; then, the strict convexity of $f$ implies that $x \mapsto f(x+y)-f(x)$ is a strictly increasing function ([14], Theor. 7.3.4) and therefore, $\Phi$ turns out to be strictly increasing in the variable $x$. ■

In what follows, the class of functions fulfilling the conditions of Theorem 2 will be denoted by $\mathcal{P}(\mathbb{R}^+)$. After Theorem 2, P-means are continuous means and a precise description for the class $\mathcal{PM}$ of P-means can be given: $\Phi \in \mathcal{PM}$ if and only if $\Phi$ is given by (10) for any function $f \in \mathcal{P}(\mathbb{R}^+)$; i.e.,

$$\mathcal{PM} = \{f^{-1}((f(x+y)-f(x)-f(y))/2) : f \in \mathcal{P}(\mathbb{R}^+)\}.$$

## 3 Properties of the functions in $\mathcal{P}(\mathbb{R}^+)$

The class $\mathcal{P}(\mathbb{R}^+)$ naturally splits in two subclasses $\mathcal{P}(\mathbb{R}^+) = \mathcal{P}_+(\mathbb{R}^+) \cup \mathcal{P}_-(\mathbb{R}^+)$, where $\mathcal{P}_+(\mathbb{R}^+)$ is composed by the strictly increasing and strictly convex solutions to the Schröder equation (11) and $\mathcal{P}_-(\mathbb{R}^+) = -\mathcal{P}_+(\mathbb{R}^+)$. Clearly, the subclasses $\mathcal{P}_+(\mathbb{R}^+)$ and $\mathcal{P}_-(\mathbb{R}^+)$ are disjoint; i.e., $\mathcal{P}_+(\mathbb{R}^+) \cap \mathcal{P}_-(\mathbb{R}^+) = \varnothing$. As observed in the Introduction, the quadratic function $f(x) = x^2$ belongs to the class $\mathcal{P}_+(\mathbb{R}^+)$; thus, the geometric mean $G(x, y) = \sqrt{xy}$ is a P-mean.

The general solution of the Schröder equation (11) is given by

$$f(x) = x^2 P_1\left(\frac{\ln x}{\ln 2}\right),\ x > 0, \tag{12}$$

where $P_1 : \mathbb{R} \to \mathbb{R}$ is an arbitrary periodic function of period 1 (cf. [4]), so that $\mathcal{P}_+(\mathbb{R}^+)$ is composed by the strictly increasing and strictly convex functions of the form (12). Assuming that $P_1$ is a $\mathcal{C}^2$ function, from (12) it is obtained

$$\begin{aligned} f'(x) &= x\left(2P_1\left(\tfrac{\ln x}{\ln 2}\right) + \tfrac{1}{\ln 2}P_1'\left(\tfrac{\ln x}{\ln 2}\right)\right), \\ f''(x) &= 2P_1\left(\tfrac{\ln x}{\ln 2}\right) + \tfrac{3}{\ln 2}P_1'\left(\tfrac{\ln x}{\ln 2}\right) + \tfrac{1}{\ln^2 2}P_1''\left(\tfrac{\ln x}{\ln 2}\right); \end{aligned} \tag{13}$$

in this way, a $\mathcal{C}^2$ function $f$ belongs to $\mathcal{P}_+(\mathbb{R}^+)$ if and only if it can be written in the form (12) with a 1-periodic function $P_1$ satisfying the inequalities

$$\left\{\begin{array}{l} P_1(x) > 0 \\ 2P_1(x) + \frac{1}{\ln 2}P_1'(x) > 0 \\ 2P_1(x) + \frac{3}{\ln 2}P_1'(x) + \frac{1}{\ln^2 2}P_1''(x) > 0 \end{array}\right.,\ x > 0. \tag{14}$$

It is easy to see that all these are satisfied by the trigonometric polynomial

$$P_1(x) = \sum_{k=0}^{n}\left(A_k \sin 2k\pi x + B_k \cos 2k\pi x\right)$$

provided that $B_0 > 0$ is great enough; and hence, there exist $B_0 > 0$ such that

$$f(x) = Ax^2 + x^2\sum_{k=1}^{n}\left(A_k \sin\left(\frac{2k\pi}{\ln 2}\ln x\right) + B_k \cos\left(\frac{2k\pi}{\ln 2}\ln x\right)\right)$$

is a member of $\mathcal{P}_+(\mathbb{R}^+)$ for every $A > B_0$. A similar assertion is true when $P_1$ is represented as a Fourier series. Note in passing that the Fourier series corresponding to the periodic function $P_1$ in (12) is always convergent when $f \in \mathcal{P}_+(\mathbb{R}^+)$. In fact, by (12),

$$\ln P_1(x) = \ln f(2^x) - (2\ln 2)\, x,\ x > 0,$$

is the difference of the two increasing functions $x \mapsto \ln f(2^x)$ and $x \mapsto (2\ln 2)\,x$; therefore, $\ln P_1(x)$ is of bounded variation on every compact interval $[a,b]$. Being $x \mapsto \exp x$ a locally Lipschitz-continuous function, it turns out to be that $P_1(x) = \exp(\ln P_1(x))$ is of bounded variation on every compact interval $[a,b]$. Thus, by a classical result on the pointwise convergence of a Fourier series (see, for instance, [18], pg. 175), the Fourier series of the (continuous) function $P_1$ converges to $P_1(x)$ for every $x \in [0,1]$.

For a strictly monotonic function $f$ satisfying the condition **ii)** of Theorem 2, the limit $f(0^+)$ is finite. Moreover, if $f$ satisfies condition **i)**; then, $f(0^+) = 4f(0^+)$ by continuity and therefore, $f(0^+) = 0$. In this way, a function belonging to $\mathcal{P}(\mathbb{R}^+)$ does not change of sign; more precisely, $f(x) > 0,\ x > 0$, when $f \in \mathcal{P}_+(\mathbb{R}^+)$ and $f(x) < 0,\ x > 0$, when $f \in \mathcal{P}_-(\mathbb{R}^+)$. On the other hand, a function $f \in \mathcal{P}(\mathbb{R}^+)$ can not be bounded above when $f \in \mathcal{P}_+(\mathbb{R}^+)$ (or below when $f \in \mathcal{P}_-(\mathbb{R}^+)$). Thus, $f(\mathbb{R}^+) = \pm\mathbb{R}^+$; and therefore, if $f \in \mathcal{P}(\mathbb{R}^+)$, then $f$ or $-f$ is an homeomorphism onto $\mathbb{R}^+$. Finally, let us note that, being $f \in \mathcal{P}_+(\mathbb{R}^+)$ a continuous function, the periodic function $P_1$ in (12) must be also continuous and therefore, a bounded function; i.e., there exist two constants $\alpha,\, \beta > 0$ such that $\alpha \le P_1(x) \le \beta,\ x \in \mathbb{R}$. Thus, inequalities of the type $\alpha x^2 \le f(x) \le \beta x^2,\ x > 0$ hold for functions $f \in \mathcal{P}_+(\mathbb{R}^+)$. Clearly, inequalities of the type $\alpha x^2 \le -f(x) \le \beta x^2,\ x > 0,\ (\alpha,\, \beta > 0)$, are satisfied by functions $f \in \mathcal{P}_-(\mathbb{R}^+)$.

Synthesizing the content of the previous paragraph, the following result is stated.

**Proposition 4** *Let $f$ be a function belonging to the class $\mathcal{P}(\mathbb{R}^+)$; then,*

**i)** $f(0^+) = 0$ *and* $f(x) \neq 0,\ x > 0$*;*

**ii)** $f(x) > 0,\ x > 0$, *when* $f \in \mathcal{P}_+(\mathbb{R}^+)$ *and* $f(x) < 0,\ x > 0$, *when* $f \in \mathcal{P}_-(\mathbb{R}^+)$*;*

**iii)** *there exist* $\alpha,\, \beta > 0$ *such that* $\alpha x^2 \le f(x) \le \beta x^2,\ x > 0$, *when* $f \in \mathcal{P}_+(\mathbb{R}^+)$*; the same inequalities with* $\alpha,\, \beta < 0$ *are satisfied when* $f \in \mathcal{P}_-(\mathbb{R}^+)$ *;*

**iv)** $f$ *is an homeomorphism and* $f(\mathbb{R}^+) = \pm\mathbb{R}^+$*.*

**Proof.** See the discussion above. ∎

The Cauchy difference $\mathcal{C}f$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$\mathcal{C}f(x,y) = f(x+y) - f(x) - f(y),\ x, y > 0.$$

A direct computation shows that the *co-cycle equation*

$$F(x+y,z) + F(x,y) = F(x,y+z) + F(y,z),\ x,y,z > 0, \tag{15}$$

is satisfied by the Cauchy difference of every function $f : \mathbb{R}^+ \to \mathbb{R}$. Conversely, if a symmetric function $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ satisfies (15); then, $F$ is the Cauchy difference of some function $f : \mathbb{R}^+ \to \mathbb{R}$ (see [11]). In the following proposition, the main properties of the Cauchy difference of a function $f \in \mathcal{P}(\mathbb{R}^+)$ are stated.

**Proposition 5** *Let $f$ belong to the class $\mathcal{P}_+(\mathbb{R}^+)$ and, for every $y > 0$, $\psi_y$ be the function defined by*

$$\psi_y(x) = \mathcal{C}f(x, y),\ \ x > 0.$$

*Then $\psi_y$ is a strictly increasing and continuous function satisfying $\psi_y(0^+) = 0$, $\psi_y(+\infty) = +\infty$. The one-parameter family $\{\psi_y : y > 0\}$ is strictly increasing and continuous in the parameter $y$. Moreover, the relationships*

$$\psi_y(x) = \psi_x(y) \text{ and } \psi_{2y}(x) = 4\psi_y(x/2),\ \ x, y > 0,$$

*hold for the members of this family.*

Thus, $\psi_y$ is an increasing homeomorphism from $\mathbb{R}^+$onto $\mathbb{R}^+$ when $f \in \mathcal{P}_+(\mathbb{R}^+)$, a property not generally enjoyed by a strictly convex function $f$, even though the condition $f(0^+) = 0$ be fulfilled by $f$. Mutatis mutandis, the proposition holds also for functions in $\mathcal{P}_-(\mathbb{R}^+)$.

**Proof.** As seen in the proof of Theorem 2, the Cauchy difference $\mathcal{C}f(x, y)$ of a strictly convex function $f$ is a strictly increasing function of $x$ (and $y$!) and therefore, given a strictly convex function satisfying $f(0^+) = 0$ and a fixed $y > 0$, the inequality

$$\psi_y(x) = \mathcal{C}f(x, y) > \lim_{x\downarrow 0}[f(x + y) - f(x) - f(y)] = 0, \tag{16}$$

holds for every $x > 0$. After Proposition 4-**i)**, (16) holds for the functions $f \in \mathcal{P}_+(\mathbb{R}^+)$. The same fact shows that $y \mapsto \psi_y$ is strictly increasing, while its continuity, as well as the continuity of $\psi_y$, follows from the continuity of $f$. That the variable $x$ and the parameter $y$ in $\psi_y(x)$ can be interchanged with each other is a simple consequence of the symmetry of $\mathcal{C}f$. In its turn, the relationship $\psi_{2y}(x) = 4\psi_y(x/2)$ follows easily from (11).

Now let us show that, when $f \in \mathcal{P}_+(\mathbb{R}^+)$, the function $\psi_y$ is not bounded above for any $y > 0$. In fact, if the inequality

$$f(x + y_0) - f(x) - f(y_0) \le A,\ \ x > 0,$$

was true for a constant $A > 0$ and a $y_0 > 0$; then, replacing $x$ by $x+(k-1)y_0$, $2 < k \in \mathbb{N}$, it is obtained

$$f(x + ky_0) - f(x + (k-1)y_0) - f(y_0) \le A,\ \ x > 0,\ k \in \mathbb{N},$$

and then

$$\begin{aligned} & f(x + 2^n y_0) - f(x) - 2^n f(y_0) \\ = \ & \sum_{k=1}^{2^n} [f(x + ky_0) - f(x + (k-1)y_0) - f(y_0)] \le 2^n A,\ \ x > 0,\ n \in \mathbb{N}. \end{aligned}$$

Replacing $x$ by $2^n x$ in this last inequality, and recalling that $f$ satisfies (11), yields

$$\begin{aligned} & 4^n [f(x + y_0) - f(x)] - 2^n f(y_0) \\ = \ & f(2^n x + 2^n y_0) - f(2^n x) - 2^n f(y_0) \le 2^n A,\ \ x > 0, \end{aligned}$$

or

$$f(x+y_0)-f(x)\leq \frac{A+f(y_0)}{2^n},\ x>0,\, n\in\mathbb{N}.$$

By taking limits when $n\uparrow +\infty$ in this last inequality, it is obtained

$$f(x+y_0)-f(x)\leq 0,$$

thus contradicting the strictly increasing character of $f$. Then, $\psi_y$ is not bounded above for any $y>0$ and the proof is finished. ■

The proof given in Section 5 for the Theorem 11 on the uniqueness of the representation (10) depends on particular properties of the transformation $T:(\mathbb{R}^+)^3\to\mathbb{R}^3$ defined by $T(x,y,z)=(u,v,w)$, with

$$\left\{\begin{array}{l} u=\mathcal{C}f(x,y)\\ v=\mathcal{C}f(x+y,z)\\ w=\mathcal{C}f(y,z)\end{array}\right. . \tag{17}$$

In the sequel, a one-parameter family of function $\{\phi_u : u>0\}$ is defined and then used to study the properties of the transformation $T$. Assuming that $f\in\mathcal{P}_+(\mathbb{R}^+)$, let us pay attention to the first equality in (17). By Proposition 5, for every $u>0$ there exists a unique function $\phi_u:\mathbb{R}^+\to\mathbb{R}^+$ implicitly defined by

$$u=\mathcal{C}f(x,\phi_u(x)). \tag{18}$$

$\phi_u$ is continuous by the Implicit Function Theorem and, observing that $\phi_u(x)=\psi_x^{-1}(u),\ x>0,\ (u>0)$, it turns out to be that $\{\phi_u\}$ is a family of strictly decreasing functions which is strictly increasing and continuous in the parameter $u$. Thus, there exist (finite or infinite) the limits $\phi_u(0^+)$ and $\phi_u(+\infty)$. Indeed, if $\phi_u(0^+)=A<+\infty$; then, making $x\downarrow 0$ in (18) it is obtained $u=\mathcal{C}f(0,A)=0$, an absurdity; hence $\phi_u(0^+)=+\infty$. That $\phi_u(+\infty)=0$ it is shown in a similar way. On the other side, the symmetry of $\mathcal{C}f$ implies that $\phi_u=\phi_u^{-1}$ and therefore, $\phi_u^2=\mathrm{id}$, the identity map on $\mathbb{R}^+$. Finally, recalling that $f$ satisfies (11) and multiplying by 4 the two members of the equality (18), it is obtained

$$4u=4\mathcal{C}f(x,\phi_u(x))=\mathcal{C}f(2x,2\phi_u(x)),$$

or replacing $x$ by $x/2$,

$$4u=\mathcal{C}f\left(x,2\phi_u\left(\frac{x}{2}\right)\right),$$

whence $\phi_{4u}(x)=2\phi_u(x/2),\ x>0$.

Summarizing these considerations, let us state the following:

**Proposition 6** *The one-parameter family $\{\phi_u : u>0\}$ defined by (18) turns out to be a one-parameter family of strictly decreasing and continuous functions satisfying $\phi_u(0^+)=+\infty$ and $\phi_u(+\infty)=0$. The family is strictly increasing and continuous in the parameter $u$, and the relationships*

$$\phi_u^2=\mathrm{id}\ \ \textit{and}\ \phi_{4u}(x)=2\phi_u(x/2),\ x>0,$$

*where $\mathrm{id}(x)=x,\ x>0$, hold for every $u>0$.*

Note that $\phi_u$ is a decreasing homeomorphism from $\mathbb{R}^+$onto $\mathbb{R}^+$. A similar result holds, mutatis mutandis, for functions in $\mathcal{P}_-(\mathbb{R}^+)$.

**Proof.** See the discussion above. ■

The remaining of this section is devoted to prove the following:

**Proposition 7** *Let $f$ belong to $\mathcal{P}_+(\mathbb{R}^+)$; then the transformation $T$ defined by (17) is a continuous transformation from $(\mathbb{R}^+)^3$ onto*

$$T\left(\left(\mathbb{R}^+\right)^3\right) = \left\{(u,v,w) \in \left(\mathbb{R}^+\right)^3 : \exists\, \xi > 0 \,/\; \phi_v(\phi_w(\xi)) - \phi_u(\xi) = \xi\right\}. \quad (19)$$

*The open and non empty set*

$$U = \left\{(u,v,w) \in \left(\mathbb{R}^+\right)^3 : \exists\, \xi > 0 \,/\; \phi_v(\phi_w(\xi)) - \phi_u(\xi) > \xi\right\},$$

*is contained in the image set $T\left(\left(\mathbb{R}^+\right)^3\right)$.*

An analogous result holds when $f$ belongs to $\mathcal{P}_-(\mathbb{R}^+)$. Note that, given $f \in \mathcal{P}_+(\mathbb{R}^+)$, $v = \mathcal{C}f(x+y,z) > \mathcal{C}f(y,z) = w$ for every $x,y,z > 0$; and therefore,

$$T\left(\left(\mathbb{R}^+\right)^3\right) \subseteq \left\{(u,v,w) \in \left(\mathbb{R}^+\right)^3 : v > w\right\}.$$

**Proof.** Let $f$ belong to $\mathcal{P}_+(\mathbb{R}^+)$ and consider $T$ defined by (17). As seen in the proof of Proposition 5, given a strictly convex function $f : \mathbb{R}^+ \to \mathbb{R}$ with $f(0^+) = 0$, the inequality $\mathcal{C}f(X,Y) > 0$ holds for every $X, Y > 0$, and hence $u, v, w \in \mathbb{R}^+$.

In terms of the functions $\phi_u$, $\phi_v$ and $\phi_w$, the equalities (17) can be written in the form

$$\begin{cases} x = \phi_u(y) \\ x + y = \phi_v(z) \\ z = \phi_w(y) \end{cases} ;$$

thus

$$y + \phi_u(y) = \phi_v(\phi_w(y)). \quad (20)$$

In this way, $T(\phi_u(\xi), \xi, \phi_w(\xi)) = (u,v,w)$ for every solution $\xi > 0$ of (20), what proves equality (19). To prove the remaining assertion, observe that after Proposition 6, the function $h : \mathbb{R}^+ \times (\mathbb{R}^+)^3 \to \mathbb{R}$ defined by

$$h(x; u,v,w) = \phi_v(\phi_w(x)) - \phi_u(x) \quad (21)$$

is a continuous function satisfying

$$h(0^+; u,v,w) = -\infty, \quad h(+\infty; u,v,w) = +\infty, \quad (22)$$

for every $u,v,w > 0$. As a consequence, the set

$$U = \bigcup_{\xi > 0} \left\{(u,v,w) \in \left(\mathbb{R}^+\right)^3 :\; \phi_v(\phi_w(\xi)) - \phi_u(\xi) > \xi\right\}$$

is open.

Now, if $(u,v,w) \in U$; then, there exists $\xi_1 > 0$ such that $h(\xi_1; u,v,w) > \xi_1$. But the first equality in (22) shows that $h(\xi_0; u,v,w) < 0$ for a certain $\xi_0 > 0$ and therefore, there exists $\xi > 0$ such that $h(\xi; u,v,w) = \xi$ by continuity; i.e., $(u,v,w) \in T\left((\mathbb{R}^+)^3\right)$. This proves that $U \subseteq T\left((\mathbb{R}^+)^3\right)$.

It remains to show that $U \neq \varnothing$. To this end, let us prove that every $(u,v,w) \in (\mathbb{R}^+)^3$ is a member of $U$ provided that $v \geq 4w$. In fact, under this assumption, $\phi_v(x) \geq \phi_{4w}(x)$ by Proposition 6, and then

$$\begin{aligned}
\phi_v(\phi_w(x)) - \phi_u(x) &\geq \phi_{4w}(\phi_w(x)) - \phi_u(x) \\
&= 2\phi_w(\phi_w(x)/2)) - \phi_u(x) \\
&\geq 2\phi_w(\phi_w(x)) - \phi_u(x) \\
&= 2x - \phi_u(x), \; x > 0.
\end{aligned}$$

Proposition 6 was repeatedly employed in writing this inequalities. Now, there exists $\xi > 0$ such that

$$2\xi - \phi_u(\xi) > \xi,$$

since in other case, the inequality

$$2x - \phi_u(x) \leq x,$$

or

$$\phi_u(x) \geq x \tag{23}$$

would hold for every $x > 0$. Then, in view of $\phi_u^2(x) = \mathrm{id}$, the equality $\phi_u(x) = x, \; x > 0$, would be derived from (23), which is clearly a contradiction. Thus, $\phi_v(\phi_w(\xi)) - \phi_u(\xi) > \xi$ and $(u,v,w) \in U$, as affirmed. ■

# 4 Duplication and subarithmeticity

Several properties of the P-means are to be established in this section. First, assume that $f \in \mathcal{P}(\mathbb{R}^+)$ and observe that, since $\Phi_f$ is (strictly) increasing in both variables, the limits $\Phi_f(x, 0^+) = \lim_{y \downarrow 0} \Phi_f(x,y)$ and $\Phi_f(x, +\infty) = \lim_{y \uparrow +\infty} \Phi_f(x,y)$ there exist (finite or infinite) for every $x > 0$. Since $f(0^+) = 0$ and $f(+\infty) = \pm\infty$ by Proposition 4, and $\psi_x(y) = \mathcal{C}f(x,y)$ satisfies $\psi_x(0^+) = 0, \; \psi_x(+\infty) = \pm\infty, \; x > 0$, by Proposition 5, taking limits for $y \downarrow 0$ and $y \uparrow +\infty$ in (2) yields, respectively,

$$0 = 2f(\Phi(x, 0^+)) \text{ and } \pm\infty = 2f(\Phi(x, +\infty)),$$

whence

$$\Phi(x, 0^+) = 0 \text{ and } \; \Phi_f(x, +\infty) = +\infty, \; x > 0. \tag{24}$$

In other words, a P-mean $\Phi$ can be always continuously extended to $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{0, +\infty\}$ paying the price of losing the strict internality, a fact expressed by saying that $\Phi$ *degenerates* both at 0 and $+\infty$. For example, the arithmetic mean $A$ is not a P-mean, since it can be continuously extended by $A(x,0) = A(0,x) = x/2$ as a strict mean to $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$, and neither the harmonic mean $H(x,y) = 2xy/(x+y),\ x,y>0$, is a P-mean, because $H(x,+\infty) = 2x$; i.e., $H$ does not degenerate at $+\infty$. The logarithmic mean $\Lambda(x,y) = (x-y)/(\ln x - \ln y),\ x,y>0,\ x \neq y$, (and $\Lambda(x,x) = x,\ x>0$) degenerates both at 0 and $+\infty$; nevertheless, $\Lambda$ is not a P-mean (this was established in [7] by resorting to the fact that $\Lambda = L_{1/x}$ is a regular Lagrangian mean and it will be shown in a different way in the following section).

Excepting the geometric mean $G$, the P-means are not homogeneous (see Theorem 12 below); nevertheless, every $\Phi \in \mathcal{PM}$ enjoys the *duplication* property; i.e.,

$$\Phi(2x,2y) = 2\Phi(x,y),\ x,y>0. \tag{25}$$

This is an immediate consequence of the fact that the equation (11) is solved by the generator function $f$ of $\Phi$.

Now, suppose that $f \in \mathcal{P}_+(\mathbb{R}^+)$; then, for every $x,y>0$,

$$f(x+y) = f\left(2\frac{x+y}{2}\right) = 4f\left(\frac{x+y}{2}\right) \leq 4\frac{f(x)+f(y)}{2} = 2(f(x)+f(y));$$

whence

$$f(x+y) - f(x) - f(y) \leq \frac{1}{2}f(x+y) = 2f\left(\frac{x+y}{2}\right),$$

and therefore

$$\Phi_f(x,y) \leq \frac{x+y}{2} = A(x,y),\ x,y>0. \tag{26}$$

The *subarithmeticity* of $\Phi_f$ is expressed by this inequality, which holds also when $f \in \mathcal{P}_-(\mathbb{R}^+)$. Really, it is a strict inequality unless $x=y$. In this way, the quadratic mean $Q(x,y) = \sqrt{(x^2+y^2)/2},\ x,y>0$, is not a P-mean.

If $A_f$ is the quasiarithmetic mean generated by $f$; then, it is easy to see that the inequality

$$A(x,y) \leq A_f(x,y),\ x,y>0, \tag{27}$$

holds provided that $f \in \mathcal{P}(\mathbb{R}^+)$ (equality occurs in (27) if and only if $x=y$). Thus, when $f \in \mathcal{P}(\mathbb{R}^+)$,

$$\Phi_f(x,y) \leq A(x,y) \leq A_f(x,y); \tag{28}$$

so that $\Phi_f$ can be considered as a subarithmetic counterpart of the quasiarithmetic mean $A_f$. The idea is reinforced by the identity

$$A_f(\Phi_f(x,y), A_f(x,y)) = A(x,y),\ x,y>0, \tag{29}$$

which is shown to be true after a simple computation.

Let us register the above facts in the following:

**Theorem 8** *A P-mean $\Phi$ is a subarithmetic mean which satisfy the duplication property and degenerates both at* 0 *and* $+\infty$*. Furthermore, the inequalities (28) and the identity (29) hold for* $\Phi$*.*

**Proof.** See the discussion above. ∎

As an example of application of this theorem, let us give another proof of the result by R. Ger mentioned in the Introduction. Using the above introduced terminology, the result can be stated as follows:

**Theorem 9 (R. Ger, [12])** *A quasiarithmetic mean $M$ defined on $\mathbb{R}^+$ is a P-mean if and only if $M = G$, the geometric mean.*

The proof offered by R. Ger depends on a deep characterization due to A. Járai, Gy. Maksa and Zs. Páles ([13]) of the functions whose Cauchy difference has the functional form $p(q(x)+q(y))$ with $p$ and $q$ suitable monotonic functions. A few elements besides the simple Theorem 1 and Theorem 9.5.1 in [15] (on the solution to simultaneous Schröder equations) are required by the proof below.

**Proof.** The "if" part of the theorem is straightforward. Now, assume that a mean $M$ has the twofold representation $M = \Phi_f = A_g$, where $\Phi_f$ is the P-mean generated by $f$ and $A_g$ is the quasiarithmetic mean generated by $g$. Without loss of generality, it can be suppossed that $f \in \mathcal{P}_+(\mathbb{R}^+)$ and that $g$ is strictly increasing and continuous. As noted in [7], to prove that $M = G$ it is sufficient to show that

$$M(x, 4x) = 2x,\ x > 0. \tag{30}$$

Indeed, the substitution $y = 4x$ in the equation (2) yields

$$f(5x) - f(x) - f(4x) = 2f(M(x, 4x)) = 2f(2x)$$

or, in view of (11),

$$f(5x) = f(x) + 16f(x) + 8f(x) = 25f(x),\ x > 0. \tag{31}$$

In this way, the Schröder equations (11) and (31) are simultaneously satisfied by $f$; therefore, Theorem 9.5.1 in [15] applies to show that $f$ has the form $f(x) = cx^2,\ x > 0,\ (c \neq 0)$, and hence $M = G$.

Now, let us see that if a quasiarithmetic mean $A_g$ satisfies the duplication property and degenerates both at 0 and $+\infty$ (like in the present case, due to $A_g = \Phi_f$); then, for a certain $\beta > 0$, the linear functional equation

$$g(2x) = g(x) + \beta,\ x > 0, \tag{32}$$

must be satisfied by its generator function $g$. In fact, since $A_g$ satisfies (25), it can be written

$$A_g(x, y) = \frac{1}{2} A_g(2x, 2y) = A_{g(2\times\cdot)}(x, y),\ x, y > 0,$$

where $g(2\times x) = g(2x),\ x > 0$; thus, by Theorem 1, there exists $\alpha,\ \beta \in \mathbb{R},\ \alpha \neq 0$, such that

$$g(2x) = \alpha g(x) + \beta,\ x > 0. \tag{33}$$

In the sequel, the equalities

$$g(2^n x) = \alpha^n g(x) + \frac{\alpha^n - 1}{\alpha - 1}\beta,\ x > 0,\ (n \in \mathbb{N}), \tag{34}$$

obtained by iterating (33) when $\alpha \neq 1$ and easily proved by induction will be useful.

From the fact that $g$ and $g(2 \times \cdot)$ are (strictly) increasing, it is deduced that $\alpha > 0$ in (33). Now, assuming that $0 < \alpha < 1$ and taking limits when $n \uparrow +\infty$ in (34), it is obtained

$$g(+\infty) = \lim_{n \uparrow +\infty} g(2^n x) = \lim_{n \uparrow +\infty} \left( \alpha^n g(x) + \frac{\alpha^n - 1}{\alpha - 1} \beta \right) = \frac{\beta}{1 - \alpha} < +\infty,$$

whence

$$A_g(x, +\infty) = \lim_{y \uparrow +\infty} g^{-1} \left( \frac{g(x) + g(y)}{2} \right) = g^{-1} \left( \frac{g(x) + \beta/(1 - \alpha)}{2} \right).$$

This shows that $A_g$ does not degenerate at $+\infty$, thus contradicting the above made assumptions.

The equalities

$$g\left(\frac{x}{2^n}\right) = \alpha^{-n} g(x) - \frac{1 - \alpha^{-n}}{\alpha - 1} \beta, \; x > 0, \; (n \in \mathbb{N}),$$

are immediate from (34); so that assuming that $\alpha > 1$ and passing once again to the limit, it is deduced

$$g(0^+) = \lim_{n \uparrow +\infty} g\left(\frac{x}{2^n}\right) = \lim_{n \uparrow +\infty} \left( \alpha^{-n} g(x) - \frac{1 - \alpha^{-n}}{\alpha - 1} \beta \right) = \frac{\beta}{1 - \alpha} > -\infty.$$

Reasoning like in the case $0 < \alpha < 1$, it is shown that $A_g$ would not degenerate at 0, a contradiction again.

In this way, the equation (32) must be satisfied by $g$ and, in view of $g$ is strictly increasing, with a certain $\beta > 0$.

The proof finishes by observing that the equality (30) holds for $M = A_g$ provided that the equation (32) is satisfied by $g$. In fact, from (32) it is obtained

$$g(4x) = g(x) + 2\beta, \; x > 0,$$

and

$$2g^{-1}(x) = g^{-1}(x + \beta), \; x > 0;$$

whence,

$$A_g(x, 4x) = g^{-1} \left( \frac{g(x) + g(4x)}{2} \right) = g^{-1} \left( g(x) + \beta \right) = 2x, \; x > 0.$$

■

By exploiting the characterization of symmetric functions as solutions to the co-cycle equation (15), a characterization of P-means is established in the following result.

**Theorem 10** *A symmetric and strictly increasing in both variables mean $\Phi$ is a P-mean if and only if*

**i)** $\Phi$ *has the duplication property (25);*

**ii)** *there exists a strictly monotonic solution $f$ to the equation (11) such that $f \circ \Phi$ satisfies the co-cycle equation; i.e.,*

$$f(\Phi(x+y,z)) + f(\Phi(x,y)) = f(\Phi(x,y+z)) + f(\Phi(y,z)),\ x,y,z > 0.$$

**Proof.** Let $\Phi$ be a symmetric and strictly increasing in both variables mean. The necessity of conditions **i)** and **ii)** is immediate from Theorem 8 and (2). In regards to the sufficiency, note that the function $f(\Phi(x,y))$, being a symmetric solution of the co-cycle equation for a certain $f$, must be a Cauchy difference; i.e., there exists a function $g$ such that

$$f(\Phi(x,y)) = g(x+y) - g(x) - g(y),\ x,y > 0.$$

Setting $x = y$ in this equality yields

$$f(x) = g(2x) - 2g(x),\ x > 0,$$

so that, in view of $\Phi$ has the duplication property and of $f$ satisfies (11), it can be written

$$\begin{aligned}
4f(\Phi(x,y)) &= f(2\Phi(x,y)) \\
&= f(\Phi(2x,2y)) \\
&= g(2(x+y)) - g(2x) - g(2y) \\
&= f(x+y) + 2g(x+y) - (f(x) + 2g(x)) - (f(y) + 2g(y)) \\
&= f(x+y) - f(x) - f(y) + 2\left(g(x+y) - g(x) - g(y)\right) \\
&= f(x+y) - f(x) - f(y) + 2f(\Phi(x,y)),\ x,y > 0.
\end{aligned}$$

Hence,

$$2f(\Phi(x,y)) = f(x+y) - f(x) - f(y),\ x,y > 0;$$

and therefore, taking into account that $f$ is strictly monotonic, it turns out to be that $\Phi$ is a P-mean. ∎

An argument analogous to the employed in proving this theorem shows that the unique symmetric mean $M$ such that $M(2x,2y) = 2M(x,y),\ x,y > 0$, and

$$[M(x+y,z)]^2 + [M(x,y)]^2 = [M(x,y+z)]^2 + [M(y,z)]^2,\ x,y,z > 0,$$

is the geometric mean $G$.

# 5 Uniqueness of the representation and some consequences

The following theorem answers the question of uniqueness of the representation (10); i.e., given two P-means $\Phi_f$ and $\Phi_g$, the result furnishes necessary and sufficient condition on the generator functions $f$ and $g$ in order that $\Phi_f = \Phi_g$.

**Theorem 11** *Let $\Phi_f$ and $\Phi_g$ two P-means; then, $\Phi_f = \Phi_g$ if and only if there exists a constant $\alpha \neq 0$ such that*

$$g(x) = \alpha f(x), \ x > 0. \tag{35}$$

When the means $\Phi_f$ and $\Phi_g$ are continuously differentiable functions, this result is a consequence of Theorem 2 in [10].

**Proof.** Let $f$, $g$ belong to $\mathcal{P}_+(\mathbb{R}^+)$. The general case in which $f, g \in \mathcal{P}(\mathbb{R}^+)$, is reduced to this by a suitable replacement of $f$ by $-f$ or $g$ by $-g$. The equality $\Phi_f = \Phi_g$ can be written as follows

$$f(x+y) - f(x) - f(y) = H\left(g(x+y) - g(x) - g(y)\right), \ x, y > 0, \tag{36}$$

where

$$H(x) = 2f\left(g^{-1}\left(\frac{x}{2}\right)\right), \ x > 0. \tag{37}$$

Being the second member of (36) a Cauchy difference, it turns out to be that it must satisfy the co-cycle equation so that, for every $x, y, z > 0$,

$$\begin{aligned} & H\left(g(x+y+z) - g(x+y) - g(z)\right) + H\left(g(x+y) - g(x) - g(y)\right) \\ = \ & H\left(g(x+y+z) - g(x) - g(y+z)\right) + H\left(g(y+z) - g(y) - g(z)\right). \end{aligned} \tag{38}$$

In terms of the transformation $T$ defined by (17), the equation (38) takes the form

$$H(u) + H(v) = H(u+v-w) + H(w), \ u, v, w \in T\left(\left(\mathbb{R}^+\right)^3\right). \tag{39}$$

Now consider a component $U_0$ of the open set $U$ defined in Proposition 7. As a consequence of (39) and Proposition 7, the restricted Pexider equation

$$H(u) + H(v) = G(u+v), \tag{40}$$

holds for every $(u, v) \in R$, being $G$ defined by

$$G(u) = H(u-w) + H(w),$$

and $R$ by

$$R = \{(u, v) : \text{there exists } w \text{ such that } (u, v, w) \in U_0\}.$$

Taking into account that $R = \pi_{(u,v)}(U_0)$ for the (open) projection $\pi_{(u,v)}(u, v, w) = (u, v)$ and that $U_0$ is an open and connected set, $R$ turns out to be open and connected, so that Theorem 4 (pg. 80) in [2] applies to the equation (40) to show that

$$H(u) = A(u) + b \tag{41}$$

where $A$ is an additive function and $b \in \mathbb{R}$. Since $H$ is strictly increasing and $H(0^+) = 0$, it turns out to be $A(u) = \alpha u$ with $\alpha > 0$ and $b = 0$ in (41); so that from (37) it is obtained

$$2f(x) = H(2g(x)) = 2\alpha g(x), \ x > 0.$$

In the general case, $\alpha$ can be also a negative real number. This proves the necessity of (35). The sufficiency follows by observing that $\Phi_{\alpha f}$ really coincides with $\Phi_f$ when $\alpha \neq 0$. ■

As an application of Theorem 11, let us prove that the geometric mean $G$ is the unique homogeneous P-mean. In fact, if $\Phi_f$ is homogeneous; then, for every $\lambda > 0$,

$$\Phi_f(\lambda x, \lambda y) = \lambda \Phi_f(x, y), \ x, y > 0, \tag{42}$$

whence the equality $\Phi_f = \Phi_g$ is satisfied provided that $g \in \mathcal{P}(\mathbb{R}^+)$ is given by $g(x) = f(\lambda x), \ x > 0$. After Theorem 11, for every $\lambda > 0$ there exists $\alpha(\lambda) \neq 0$ such that

$$f(\lambda x) = \alpha(\lambda) f(x), \ x, \lambda > 0.$$

The general continuous solution $f$ to this Pexider equation is given ([1], Theor. 4, pg. 144) by

$$f(x) = cx^p, \ x > 0, \tag{43}$$

where $p$, $c \in \mathbb{R}$ and, taking into account that $f \in \mathcal{P}(\mathbb{R}^+)$, $p \neq 0 \neq c$. Furthermore, the equation (11) must be satisfied by $f$; whence $p = 2$ and $\Phi = G$, as claimed.

Since $\Lambda$, the logarithmic mean, is homogeneous and differs from the geometric mean $G$ ($\Lambda(x, y) > G(x, y)$ provided that $x \neq y$), it turns out to be that $\Lambda$ is not a P-mean.

The assumption of homogeneity can be weakened in the above assertion without changing the conclusion. Indeed, it can be proved the following:

**Theorem 12** *Let $\lambda_0$ be a positive number such that $\lambda_0 \neq 2^r$ for every $r \in \mathbb{Q}$. If a P-mean $\Phi_f$ satisfies the equality (42) for $\lambda = \lambda_0$; then, $f(x) = cx^2, \ x > 0, \ (c \neq 0)$, and $\Phi_f = G$.*

Another statement of this theorem is as follows: the pair (3) is the unique solution of the equation (2) in the class of pairs $(f, \Phi)$ composed by a strictly monotonic function $f$ and a increasing in both variables mean $\Phi$ satisfying $\Phi_f(\lambda_0 x, \lambda_0 y) = \lambda_0 \Phi_f(x, y), \ x, y > 0$, for a certain $\lambda_0 > 0$ such that $\ln \lambda_0 / \ln 2 \notin \mathbb{Q}$.

**Proof.** Reasoning like in the previous discussion it is seen that, besides of the equation (11), $f$ must satisfy, for a certain $\alpha \neq 0$, the Schröder equation

$$f(\lambda_0 x) = \alpha f(x), \ x > 0.$$

Since $\ln \lambda_0 / \ln 2 \notin \mathbb{Q}$, Theorem 9.5.1 in [15] applies to show that $f$ has the form (43), with $c \neq 0$. Hence, $p = 2$ and the proof is finished. ■

To end this paper, let us remark the general interest of extending Theorem 11 to other classes of means. Besides of properly completing the definition of the means in the class, results like Theorem 12 hold for every one of these extensions. As an example, denote by $\mathcal{S}(\mathbb{R}^+)$ the class of strictly monotonic functions $f \in \mathcal{C}^1(\mathbb{R}^+)$ such that $f'$ does not reduce to a constant on any non void open subinterval of $\mathbb{R}^+$, and consider the class $\mathcal{SM}$ constituted by the means defined on $\mathbb{R}^+$ admitting the representation (10) with $f \in \mathcal{S}(\mathbb{R}^+)$. As a consequence of Theorem 2 in [10], Theorem 11 also holds when $\Phi_f$ and $\Phi_g$ are members of $\mathcal{SM}$.

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