REPORTS ON MATHEMATICAL LOGIC
37 (2003), 101–112

Manuel ABAD, José Patricio DíAZ VARELA
and Marta ZANDER

BOOLEAN ALGEBRAS WITH A DISTINGUISHED AUTOMORPHISM

A b s t r a c t. In this paper we investigate a subvariety $\mathcal{BA}$ of tense algebras, which we call Boolean algebras with a distinguished automorphism. This variety provides a unifying framework for the algebras studied by Monteiro in [4] and by Moisil in [5, 6]. Among others we prove that $\mathcal{BA}$ is generated by its finite members and we characterize the locally finite subvarieties of $\mathcal{BA}$.

1. Introduction and Preliminaries

The variety of *tense algebras* [7] is defined as the variety of algebras $\langle A; \wedge, \vee, -, T, T', 0, 1\rangle$, where $\langle A; \wedge, \vee, -, 0, 1\rangle$ is a Boolean algebra and T and T' are unary operators satisfying the following identities: $T(1) = 1$, $T'(1) = 1$; $T(x \wedge y) = T(x) \wedge T(y)$, $T'(x \wedge y) = T'(x) \wedge T'(y)$; $-x \vee T(-T'(-x)) = 1$; $-x \vee T'(-T(-x)) = 1$.

Received 30 October 2002
Mathematics Subject Classification (2000): 06E25, 03G25
Keywords: Varieties, Boolean Algebras, Tense Algebras.

In [4], Monteiro defined the k-cyclic Boolean algebras, k a fixed integer ≥ 1, as Boolean algebras with a new unary operation T with the properties of an automorphism such that $T^k = Id$. This class of algebras is a variety, denoted by V_k. Monteiro characterized the subdirectly irreducible members of V_k and described its finitely generated free objects. As a consequence of Monteiro's work V_k turns out to be locally finite.

Some particular cases had been previously studied by Moisil [5, 6] in connection with the theory of switching circuits. Moisil proved that the 2-cyclic Boolean algebra $\langle B_2; T\rangle$ can be identified with the Galois field $GF(2^2)$ and he also proved that a ring structure such that $2x = 0$ and $x^4 = x$ can be defined on any 2-cyclic Boolean algebra, and conversely. These results were generalized by Cendra [3] for arbitrary k, and recently by Abad et al. [1] for cyclic Post algebras and finite fields.

The purpose of this paper is to investigate a subvariety of tense algebras, which we call Boolean algebras with a distinguished automorphism. Such variety contains all the varieties of Monteiro and provides a unifying framework to prove results that generalize his.

By a *Boolean algebra with a distinguished automorphism* we understand a tense algebra $\langle A; \wedge, \vee, -, T, T', 0, 1\rangle$, such that T is an automorphism of A (and so $T' = T^{-1}$).

The class of Boolean algebras with a distinguished automorphism is denoted by $\mathcal{BA}$. $\mathcal{BA}$ can be characterized by the identities $T(x \wedge y) = T(x) \wedge T(y)$, $T(-x) = -T(x)$, and $T(T^{-1}(x)) = T^{-1}(T(x)) = x$, so $\mathcal{BA}$ is a variety.

As it is standard practice, we will frequently denote with the same letter an algebra and its underlying universe. Hence, by a *subalgebra* of an algebra $A \in \mathcal{BA}$ we understand a Boolean subalgebra of A which is closed under T and T^{-1}, and by a *homomorphism* from A to B, $A, B \in \mathcal{BA}$, we understand a Boolean homomorphism $h : A \to B$ such that $h(T(a)) = T(h(a))$ and $h(T^{-1}(a)) = T^{-1}(h(a))$, for all $a \in A$.

Typical examples of Boolean algebras with a distinguished automorphism are the following algebras introduced by Monteiro.

Let B_k denote the Boolean algebra of k-tuples $(x_1, \ldots x_k)$, with $x_i \in \{0, 1\}$. Let a_1, a_2, $\ldots$, a_k be the atoms of this algebra, that is, $(a_i)_j = 1$ if $i = j$ and $(a_i)_j = 0$ otherwise, and let T be the automorphism of B_k such that $T(a_1) = a_2, T(a_2) = a_3, \ldots, T(a_{k-1}) = a_k, T(a_k) = a_1$. It is clear that $T^k(x) = x$ for every $x \in B_k$. It is clear that B_k with the operations T and T^{-1} is a Boolean algebra with an automorphism. Then B_k is a simple k-cyclic Boolean algebra and $V_k = V(B_k)$, the variety generated by B_k.

The lattice of subalgebras of B_k ordered by inclusion is isomorphic to the lattice of divisors of k, ordered by divisibility. If d is a divisor of k, in symbols $d|k$, the corresponding subalgebra is isomorphic to the algebra B_d, where T acts transitively on the atoms of B_d [4]. V_k is a discriminator variety [7]. Indeed, let us write $\Delta_k(a) = a \wedge T(a) \wedge T^2(a) \wedge \ldots \wedge T^{k-1}(a)$. Then, for $a \in B_k$, $\Delta_k(a) = 1$ if $a = 1$ and $\Delta_k(a) = 0$ otherwise. Then $t(x, y, z) = (z \wedge \Delta_k(x \leftrightarrow y)) \vee (x \wedge -\Delta_k(x \leftrightarrow y))$, where $x \leftrightarrow y = (x \vee -y) \wedge (-x \vee y)$, is a discriminator term for B_k. In particular, every finite k-cyclic algebra is a direct product of simple algebras [4] and every finite subdirectly irreducible algebra in V_k is simple [2].

For $A \in \mathcal{BA}$, a subset F of A is a T-*filter* if F is a filter such that $T(x)$, $T^{-1}(x) \in F$, whenever $x \in F$. If $W \subseteq A$, the notion of T-filter generated by W is defined as usual. If $W = \{z\}$, the T-filter $F_T(z)$ generated by $\{z\}$ is the filter generated by $\{T^n(z) : n \in \mathbb{Z}\}$, that is,

$$F_T(z) = \{x \in A : T^{n_1}(z) \wedge \ldots \wedge T^{n_k}(z) \leq x, n_i \in \mathbb{Z}\}$$

Congruences on A are determined by T-filters in the following way: If F is a T-filter of A, then the relation $\equiv$ defined on A by $x \equiv y \pmod F$ if and only if $x \leftrightarrow y \in F$, is a congruence relation. Conversely, if $\equiv$ is a congruence on A, then $F = \{x \in A : x \equiv 1\}$ is a T-filter and $x \equiv y$ if and only if $x \leftrightarrow y \in F$. Therefore, there exists a lattice isomorphism from the set of T-filters of A onto the set of congruences of A.

The notions of T-*ideal* and T-ideal generated by a subset of A are dually defined. Congruences can also be determined by T-ideals: the congruence associated to the T-ideal I is $x \equiv y \pmod I$ if and only if $x \triangle y \in I$, where $x \triangle y = (x \wedge -y) \vee (-x \wedge y)$.

2. The algebra $\mathbf{2}^{\mathbb{Z}}$

The aim of this section is to introduce the crucial example of the algebra $\mathbf{2}^{\mathbb{Z}}$ and prove some of its properties. We will see that this algebra contains an isomorphic copy of every subdirectly irreducible algebra in $\mathcal{BA}$, and consequently it is a generator for $\mathcal{BA}$.

Let $\mathbf{2}^{\mathbb{Z}}$ be the field of subsets of $\mathbb{Z}$ with the set-theoretical operations of union, meet and complementation. Let T be the automorphism of $\mathbf{2}^{\mathbb{Z}}$ induced by the mapping $n \mapsto n+1$, $n \in \mathbb{Z}$. It is clear that $\langle \mathbf{2}^{\mathbb{Z}}, T, T^{-1} \rangle \in \mathcal{BA}$.

The algebra $\mathbf{2}^{\mathbb{Z}}$ is atomic, that is, every element in $\mathbf{2}^{\mathbb{Z}}$ is a (finite or infinite) join of atoms, or equivalently, every element in $\mathbf{2}^{\mathbb{Z}}$ is a meet of coatoms; the atoms of $\mathbf{2}^{\mathbb{Z}}$ are the singletons and the coatoms are their complements.

Lemma 2.1. $\mathbf{2}^{\mathbb{Z}}$ *is non-simple subdirectly irreducible.*

Proof. Let $I_0 = \{x \in \mathbf{2}^{\mathbb{Z}} : x \text{ is a finite subset of } \mathbb{Z}\}$. It is easy to see that I_0 is a non-trivial T-ideal of $\mathbf{2}^{\mathbb{Z}}$, generated by any atom of $\mathbf{2}^{\mathbb{Z}}$. So, $\mathbf{2}^{\mathbb{Z}}$ is not simple.

If $I \neq \{0\}$ is a T-ideal of $\mathbf{2}^{\mathbb{Z}}$, then I contains an element $x \neq 0$. Let a be an atom of $\mathbf{2}^{\mathbb{Z}}$ such that $a \leq x$. Then $a \in I$, that is, I contains an atom of $\mathbf{2}^{\mathbb{Z}}$. Consequently, $I_0 \subseteq I$, that is, I_0 is the only minimal T-ideal of $\mathbf{2}^{\mathbb{Z}}$, and then, $\mathbf{2}^{\mathbb{Z}}$ is subdirectly irreducible. □

Let $A \in \mathcal{BA}$ and let $Ult(A)$ denote the set of ultrafilters of A. Consider the Boolean algebra $\mathbf{2}^{Ult(A)}$. Then the automorphism $T : A \to A$ induces an automorphism, which we denote with the same letter, $T : \mathbf{2}^{Ult(A)} \to \mathbf{2}^{Ult(A)}$ by means of $T(x)(U) = 1$ if and only if $x(T^{-1}(U)) = 1$ for every $x \in \mathbf{2}^{Ult(A)}$, $U \in Ult(A)$.

The following lemma shows that the Boolean Stone embedding is in fact a $\mathcal{BA}$-embedding, that is, it preserves T.

Lemma 2.2. (Stone embedding) *If $A \in \mathcal{BA}$, A is isomorphic to a subalgebra of $\mathbf{2}^{Ult(A)}$.*

Proof. Let $s : A \to \mathbf{2}^{Ult(A)}$ be the Stone embedding

$$s(x)(U) = \begin{cases} 1 \text{ if } x \in U \\ 0 \text{ if } x \notin U \end{cases}$$

It is known that s is a Boolean embedding. In addition we have that $T(s(x))(U) = 1$ if and only if $s(x)(T^{-1}(U)) = 1$, and this is equivalent to $x \in T^{-1}(U)$, that is, $T(x) \in U$, or equivalently, $s(T(x))(U) = 1$. So s is an embedding in $\mathcal{BA}$. □

The *orbit* of an element $a \in A$ is the set $O(a) = \{T^n(a) : n \in \mathbb{Z}\}$.

Now we prove that the algebra $\mathbf{2}^{\mathbb{Z}}$ is a *generator* for the variety $\mathcal{BA}$. That is,

Theorem 2.3. *The variety $\mathcal{BA}$ is generated by the algebra $\mathbf{2}^{\mathbb{Z}}$. Moreover, $\mathcal{BA} = ISP(\mathbf{2}^{\mathbb{Z}})$.*

Proof. By Lemma 2.2, if $A \in \mathcal{BA}$, $A \hookrightarrow \mathbf{2}^{Ult(A)}$. If $\{O(U_j)\}_{j\in\mathfrak{J}}$ is the family of distinct orbits of $Ult(A)$, then $Ult(A) = \bigcup_{j\in\mathfrak{J}} O(U_j)$, and $\mathbf{2}^{Ult(A)} \cong \prod_{j\in\mathfrak{J}} \mathbf{2}^{O(U_j)}$, where the automorphism in $\mathbf{2}^{O(U_j)}$ is the one induced by the action of T in $O(U_j)$. If $O(U_j)$ is finite, then $\mathbf{2}^{O(U_j)} \cong B_n$ for some n, and if $O(U_j)$ is infinite then $\mathbf{2}^{O(U_j)} \cong \mathbf{2}^{\mathbb{Z}}$. Consequently $\mathbf{2}^{Ult(A)}$ is isomorphic to a product $\prod_{j\in\mathfrak{J}} A_j$, where $A_j \cong B_n$ or $A_j \cong \mathbf{2}^{\mathbb{Z}}$. In both cases there exists a monomorphism $A_j \hookrightarrow \mathbf{2}^{\mathbb{Z}}$. Thus $A \hookrightarrow \prod_{j\in\mathfrak{J}} A_j \hookrightarrow (\mathbf{2}^{\mathbb{Z}})^{\mathfrak{J}}$, and so $A \in ISP(\mathbf{2}^{\mathbb{Z}})$. □

Corollary 2.4. *An equation holds in every algebra in the variety $\mathcal{BA}$ if and only if it holds in $\mathbf{2}^{\mathbb{Z}}$.*

For each orbit $O(U_j)$ of $Ult(A)$ consider the $\mathcal{BA}$-homomorphism $f_j : A \to \mathbf{2}^{O(U_j)}$ defined by $f_j(x)(U) = 1$ if and only if $x \in U$, for every $x \in A$ and $U \in O(U_j)$. We say that f_j is the homomorphism associated to $O(U_j)$.

The next theorem tells us where the subdirectly irreducible algebras in $\mathcal{BA}$ are.

Theorem 2.5. *If an algebra $A \in \mathcal{BA}$ is subdirectly irreducible then A is isomorphic to a subalgebra of $\mathbf{2}^{\mathbb{Z}}$.*

Proof. We are going to prove that if A is subdirectly irreducible, there exists an orbit $O(U_j) \in Ult(A)$ such that the associated homomorphism f_j is an embedding.

If A is simple, then we can choose any orbit of $Ult(A)$. Indeed, for any orbit $O(U_j)$, $Ker\ (f_j) \neq A$, and consequently $Ker\ (f_j) = \{1\}$.

Suppose that A is non-simple subdirectly irreducible and let M be the minimal proper T-filter of A. For an orbit $O(U_j)$, if $Ker\ (f_j) \neq \{1\}$, then $M \subseteq Ker\ (f_j)$. Thus if we suppose that for every orbit $O(U_j)$, $Ker\ (f_j) \neq \{1\}$, then $M \subseteq \bigcap Ker\ (f_j) = \bigcap\{U : U \in Ult(A)\} = \{1\}$, which is a contradiction. Hence there exists an orbit $O(U_j)$ such that f_j is an embedding. By Lemma 3.1, A is infinite, so this orbit cannot be finite. Thus $A \hookrightarrow \mathbf{2}^{O(U_j)} \simeq \mathbf{2}^{\mathbb{Z}}$. □

It is worth pointing out that Theorem 2.5 in conjunction with Lemma 2.1 show that $\mathcal{BA}$ is residually small, with residual bound $(2^\omega)^+$.

The next theorem proves that the variety $\mathcal{BA}$ is *generated by its finite members*, and consequently, $\mathcal{BA}$ provides the unifying framework for the varieties V_k of Monteiro. In the proof, any element $x \in \mathbf{2}^{\mathbb{Z}}$ will be written as a sequence $x = (x_i)_{i\in\mathbb{Z}}$, where $x_i = 0$ or $x_i = 1$, for every $i \in \mathbb{Z}$. Similarly, any element in B_m will be represented as a finite sequence of m 0's and 1's, without commas between them.

Theorem 2.6. $\mathbf{2}^{\mathbb{Z}} \in V(\{B_m : m > 0\})$.

Proof. Consider the algebra $\prod_{k>0} B_{2k+1}$, and for $x = (x_i)_{i\in\mathbb{Z}} \in \mathbf{2}^{\mathbb{Z}}$, define

$$f(x) = (x_{-1}\,x_0\,x_1,\; x_{-2}\,x_{-1}\,x_0\,x_1\,x_2, \ldots) \in \prod_{k>0} B_{2k+1}.$$

f is a Boolean homomorphism.

Let I be the T-ideal of $\prod_{k>0} B_{2k+1}$ generated by

$$\{T(f(x)) \triangle f(T(x)), x \in \mathbb{Z}\}$$

and let $\overline{f} : \mathbf{2}^{\mathbb{Z}} \to \prod_{k>0} B_{2k+1}/I$ be the mapping $\overline{f}(x) = \overline{f(x)}$, for $x \in \mathbf{2}^{\mathbb{Z}}$.

$\overline{f}$ is a Boolean homomorphism, and it is clear that $\overline{f}(T(x)) = T(\overline{f}(x))$. Observe that $T^{-1}(f(x)) \triangle f(T^{-1}(x)) \in I$, and so, $\overline{f}(T^{-1}(x)) = T^{-1}(\overline{f}(x))$. In order to prove that $\overline{f}$ is injective it suffices to show that $\overline{f}(z) = 0$ implies $z = 0$.

Observe that from the definition of f and taking into account that $(T(x))_i = (x)_{i-1}$, we have that $T(f(x)) \triangle f(T(x)) \leq (100, 10000, \ldots) \in \prod_{k>0} B_{2k+1}$.

An element $w \in I$ if and only if $w \leq \bigvee_{j\in J} T^j(T(f(x_j)) \triangle f(T(x_j)))$, J finite, and then there exists m such that $w \leq \bigvee_{j=-m}^{j=m} T^j(100, 10000, \ldots)$, that is

$$(w)_k \leq \left(\bigvee_{j=-m}^{j=m} T^j(100, 10000, \ldots)\right)_k \leq \underbrace{11\ldots1}_{m+1}00\ldots0\underbrace{11\ldots1}_{m} \in B_{2k+1}.$$

In particular, if $\overline{f}(z) = \overline{f(z)} = 0$, for $z = (z_i)_{i\in\mathbb{Z}}$, then $f(z) \in I$, and then there exists m such that for every k,

$$(f(z))_k = z_{-k}z_{-(k-1)}\ldots z_{-1}z_0z_1\ldots z_{k-1}z_k \leq \underbrace{11\ldots1}_{m+1}00\ldots0\underbrace{11\ldots1}_{m} \in B_{2k+1}.$$

Then it is clear that $z_i = 0$ for every $i \in \mathbb{Z}$, that is, $z = 0$. □

Corollary 2.7. *$\mathcal{BA}$ is generated by its finite members.*

3. Other generators for $\mathcal{BA}$

In the present section we obtain some more insight into the structure of $\mathcal{BA}$. We first study some subalgebras of $\mathbf{2}^{\mathbb{Z}}$, namely, the subalgebra of m-periodic subsets of $\mathbb{Z}$ for each m, the subalgebra of finite and cofinite subsets of $\mathbb{Z}$, and the subalgebra $B_{\mathbb{N}}$ of the elements of finite order of $\mathbf{2}^{\mathbb{Z}}$. We use these subalgebras to deduce some properties of $\mathcal{BA}$.

Let m be a positive integer. A subset x of $\mathbb{Z}$ is called *m-periodic* if it coincides with the set obtained by adding m to each of its elements.

It is clear that if x is m-periodic, then $T(x)$ and $T^{-1}(x)$ are m-periodic, so the set of m-periodic subsets of $\mathbb{Z}$ is a subalgebra of $\mathbf{2}^{\mathbb{Z}}$. If we consider the congruence modulo m in $\mathbb{Z}$ and we use the notation $[x]_m$ for the equivalence class of x, then $[0]_m, [1]_m, \ldots, [m-1]_m$ are the atoms of this subalgebra. T acts transitively on the atoms, and thus, this subalgebra is isomorphic to the algebra B_m of Monteiro [4]. From now on, the notation B_m will also be used to denote the subalgebra of m-periodic subsets of $\mathbf{2}^{\mathbb{Z}}$. In particular, the algebras B_m are simple, and the lattice of subalgebras of B_m is isomorphic to the lattice of divisors of m.

Lemma 3.1. *For a finite algebra algebra $A \in \mathcal{BA}$, A is subdirectly irreducible if and only if A is isomorphic to B_m, for some m, that is, A is subdirectly irreducible if and only if A is simple.*

Proof. If A is finite, $T^k = Id$, for some k. Then A is a finite k-cyclic algebra, and then $A \cong \prod_{m|k}(B_m)^{\alpha_m}$, where $m|k$ stands for m a divisor of k. As A is subdirectly irreducible, then $A \cong B_m$, for some m. The converse is trivial. □

Another important subalgebra of $\mathbf{2}^{\mathbb{Z}}$ is the subalgebra of finite-cofinite subsets of $\mathbb{Z}$. Recall that a subset of a set X is said to be *cofinite* if its complement in X is finite.

Let $FC = \{x \in \mathbf{2}^{\mathbb{Z}} : x \text{ is either finite or cofinite}\}$. FC is the subalgebra of $\mathbf{2}^{\mathbb{Z}}$ generated by the atoms of $\mathbf{2}^{\mathbb{Z}}$, or equivalently by an atom of $\mathbf{2}^{\mathbb{Z}}$.

The proof of the following lemma is similar to that of Lemma 2.1.

Lemma 3.2. *FC is non-simple subdirectly irreducible.*

Proof. $I_0 = \{x \in FC : x \text{ is a finite subset of } \mathbb{Z}\}$ is the only (non-trivial) minimal T-ideal of FC. □

In addition, if I is a T-ideal of FC such that $I_0 \subset I$ and $I_0 \neq I$, then if $x \notin I_0$, then x is a cofinite subset of $\mathbb{Z}$. So, $-x \in I_0 \subseteq I$, which implies $1 = x \vee -x \in I$, that is, $I = FC$. In particular, the lattice of T-ideals of FC is a three-element chain.

Observe that FC has no proper subalgebras other than the two-element one. Indeed, if $S \neq \{0,1\}$ is a subalgebra of FC, then S contains an element x, $x \neq 0, 1$ which is a finite subset of $\mathbb{Z}$. As before, S contains an atom of $\mathbf{2}^{\mathbb{Z}}$, which is a generator of FC, so $S = FC$.

Now we are going to show some subalgebras of $\mathbf{2}^{\mathbb{Z}}$ with the property that they are simple and non atomic.

A family $\{A_i\}_{i\in\mathfrak{I}}$ of subalgebras of an algebra is said to be *directed* if for $i, j \in \mathfrak{I}$ there exists $k \in \mathfrak{I}$ such that $A_i \subseteq A_k$ and $A_j \subseteq A_k$.

If $\{A_i\}_{i\in\mathfrak{I}}$ is a directed family of subalgebras of $\mathbf{2}^{\mathbb{Z}}$, then $A_{\mathfrak{I}} = \bigcup_{i\in\mathfrak{I}} A_i$ is a subalgebra of $\mathbf{2}^{\mathbb{Z}}$, and thus $A_{\mathfrak{I}} \in \mathcal{BA}$.

In particular, we are interested in the algebras $B_{\mathfrak{I}} = \bigcup_{i\in\mathfrak{I}} B_i$, where $\{B_i\}_{i\in\mathfrak{I}}$, $\mathfrak{I} \subseteq \mathbb{N}$, is a directed family of subalgebras B_i introduced above.

We say that an element $x \in \mathbf{2}^{\mathbb{Z}}$ is *of finite order* if $x \in B_m$, for some m. Otherwise we say that x is *of infinite order*.

An algebra A is *locally finite* if and only if every finitely generated subalgebra of A is finite. A class of algebras K is locally finite if and only if every member of K is locally finite. A variety is *finitely generated* if it is generated by a finite set of finite algebras.

Lemma 3.3. *If $\mathfrak{I}$ is infinite, the algebra $B_{\mathfrak{I}}$ is simple, atomless and locally finite. If $\mathfrak{I} = \mathbb{N}$, $B_{\mathbb{N}}$ consists of the elements of finite order of $\mathbf{2}^{\mathbb{Z}}$.*

Proof. Let F be a proper T-filter of $B_{\mathfrak{I}}$ and $x \neq 1$ an element of F. Since $x \in B_{\mathfrak{I}}$, $x \in B_i$ for some $i \in \mathfrak{I}$, and consequently $0 = \triangle_i x = x \wedge T(x) \wedge \ldots \wedge T^{i-1}(x) \in F$. So $F = B_{\mathfrak{I}}$. Hence $B_{\mathfrak{I}}$ is simple.

Let us see that $B_{\mathfrak{I}}$ has no atoms. For $x \in B_{\mathfrak{I}}$, $x \in B_i$, for some $i \in \mathfrak{I}$. Consider $i' \in \mathfrak{I}$ a multiple of i, $i' \neq i$. B_i is a proper subalgebra of $B_{i'}$, and since T acts transitively on the atoms or $B_{i'}$, it follows that x can be not an atom of $B_{i'}$. In particular, there exists an atom a of $B_{i'}$ such that $0 < a < x$, that is x is not an atom of $B_{\mathfrak{I}}$.

Every element of $B_{\mathfrak{I}}$ is of finite order, and consequently $B_{\mathfrak{I}}$ is locally finite.

Finally, it is clear that for $x \in \mathbf{2}^{\mathbb{Z}}$, $x \in B_{\mathbb{N}}$ if and only if $T^m(x) = x$, for some m. □

From Theorem 2.6, as B_n is a subalgebra of $B_{\mathbb{N}}$ for every n, we get that $B_{\mathbb{N}}$ is another generator for $\mathcal{BA}$, that is, $\mathcal{BA} = V(B_{\mathbb{N}})$.

It is known that every finitely generated subvariety is locally finite. Let us prove that in $\mathcal{BA}$ the converse also holds.

Lemma 3.4. *A subvariety V of $\mathcal{BA}$ is locally finite if and only if it is finitely generated.*

Proof. Let V be a locally finite subvariety of $\mathcal{BA}$ and suppose that V is not finitely generated. Let $X = \{A_i, i \in I\}$ be a set of non-isomorphic subdirectly irreducible algebras that generate V. Then X is not a finite set of finite algebras.

Suppose that X contains an infinite algebra $A_i = A$. If A has an element x of infinite order, then the subalgebra $\langle x \rangle$ generated by x is not finite. So V is not locally finite. Suppose that every element $x \in A$ is of finite order. Since A is subdirectly irreducible, we may assume that A is isomorphic to a subalgebra of $\mathbf{2}^{\mathbb{Z}}$. Then for every $x \in A$, $\langle x \rangle$ is isomorphic to B_k, for some k. Then $A \simeq \bigcup_{k \in K} B_k$. If K is finite, then infinitely many different subalgebras if A are isomorphic to B_k for some k. But A is a subalgebra of $\mathbf{2}^{\mathbb{Z}}$ and therefore contains only one subalgebra isomorphic to B_k for every k. So, K must be infinite, $K \subseteq \mathbb{N}$. Thus we have that $\prod_{k \in K} B_k \in V$ and $\prod_{k \in K} B_k$ is not locally finite, contrary to the assumption.

So X does not contain an infinite algebra, and consequently X contains infinitely many non-isomorphic finite algebras A_i. In this case $\prod_{i \in I} A_i \in V$ and $\prod_{i \in I} A_i$ is not locally finite, since it contains an element of infinite order, again a contradiction. □

It is also known that every locally finite subvariety is generated by its finite members. The converse does not hold in $\mathcal{BA}$ as it is the case for the subvariety generated by all the finite algebras.

Remarks.

1. An algebra $A \in \mathcal{BA}$ can be locally finite and it still may not be in a locally finite variety (the variety generated by A is not locally finite). For instance, $B_{\mathbb{N}}$ is locally finite and we will see that the variety generated by $B_{\mathbb{N}}$ is the whole variety $\mathcal{BA}$.

2. The locally finite simple algebras in $\mathcal{BA}$ are subalgebras of $B_{\mathbb{N}}$. It is a consequence of the proof of Lemma 3.4.

3. The locally finite subdirectly irreducible algebras in $\mathcal{BA}$ are simple, and consequently, subalgebras of $B_{\mathbb{N}}$. (Observe that every subalgebra of $B_{\mathbb{N}}$ is simple: the filter generated by $x \neq 1$ is the whole algebra).

4. As a consequence, every locally finite algebra $A \in \mathcal{BA}$ is semisimple, and consequently, A can be embedded into $(B_{\mathbb{N}})^k$.

The locally finite varieties are the varieties generated by finitely many algebras B_n. An equation that characterizes the subvariety $V(B_{m_1}) \vee \ldots \vee V(B_{m_n})$ is $x \to (T^{m_1}(x) \vee \ldots \vee T^{m_n}(x)) = 1$. The proof follows by simple inspection.

In the rest of this section we exhibit other generators for the variety $\mathcal{BA}$. In fact, we prove that for every infinite directed family $\{B_i\}_{i \in \mathfrak{I}}$, the algebra $B_{\mathfrak{I}}$ generates $\mathcal{BA}$, and that the algebra FC of finite-cofinite subsets of $\mathbb{Z}$ generates $\mathcal{BA}$ as well.

Lemma 3.5. *If $\{B_i, i \in \mathfrak{I}\}$ is an infinite family of non-isomorphic simple finite algebras of $\mathcal{BA}$, then $B_n \in V(\{B_i, i \in \mathfrak{I}\})$, for every n.*

Proof. Let $\mathfrak{I}' = \{i \in \mathfrak{I}, i \geq n\}$. Let $b_1, \ldots, b_s$ be the atoms of B_s and consider the element $a = (a_i)_{i \in \mathfrak{I}'} \in A = \prod_{i \in \mathfrak{I}'} B_i$ defined by

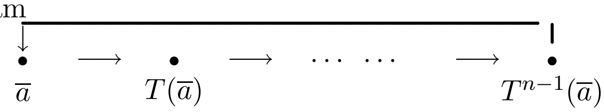
$$a_i = b_1 \vee T^n(b_1) \vee T^{2n}(b_1) \vee \ldots \vee T^{n \cdot m_i}(b_1),$$

with m_i the greatest integer such that $n(m_i + 1) + 1 \leq i$.

Consider the T-ideal I generated by $a \triangle T^n(a)$. An element $x \in A$ belongs to I if and only if $x \leq \bigvee_{j \in J} T^j(a \triangle T^n(a))$, J finite, $J \subseteq \mathbb{Z}$. But the coordinate of $a \triangle T^n(a)$ corresponding to B_i is $(a \triangle T^n(a))_i = b_1 \vee T^{n(m_i+1)+1}(b_1)$, that is, $(a \triangle T^n(a))_i$ is a join of two atoms. Hence each coordenate of $\bigvee_{j \in J} T^j(a \triangle T^n(a))$ is a join of at most $2|J|$ atoms.

Let us see that in the quotient A/I, the elements $\overline{a}$, $\overline{T(a)}$, $\ldots \overline{T^{n-1}(a)}$ form a partition of 1. Indeed, it is clear that the meet of any two of them is 0. In addition, the coordinate of $(a \vee T(a) \vee \ldots \vee T^{n-1}(a)) \triangle 1$ that corresponds to B_i has at most the last n atoms of B_i, that is $(a \vee T(a) \vee \ldots \vee T^{n-1}(a)) \triangle 1 \leq \bigvee_{s=1}^{n} T^{-s}(a \triangle T^n(a)) \in I$. Then $\overline{a} \vee \overline{T(a)} \vee \ldots \vee \overline{T^{n-1}(a)} = 1$.

Consequently, the atoms of the subalgebra $\langle \overline{a} \rangle$ generated by $\overline{a}$ look like the following diagram



and thus $\langle \overline{a} \rangle$ is isomorphic to B_n. □

As easy consequences of this lemma we have that every infinite family $\{B_i, i \in \mathfrak{I}\}$ also generates $\mathcal{BA}$, and $B_{\mathfrak{I}}$ is a generator for $\mathcal{BA}$, for every infinite directed family $\{B_i\}_{i\in\mathfrak{I}}$.

As a corollary of the following lemma, it follows that FC is another generator for the variety $\mathcal{BA}$.

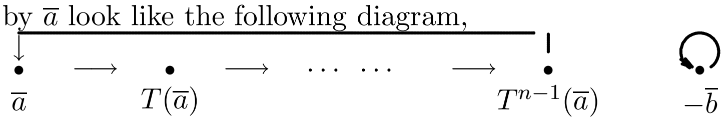
Lemma 3.6. *$B_n \in V(FC)$, for every n.*

Proof. Consider the product $A = \prod_{i\in\mathbb{N}} A_i$, where every $A_i = FC$, and consider the element

$$a = (a_1, a_1 \vee T^n(a_1), \ldots, a_1 \vee T^n(a_1) \vee \ldots \vee T^{nk}(a_1), \ldots),$$

where a_1 is an atom of FC. Arguing as before, we can prove that $\overline{a}$ is of order n in the quotient of A by the T-ideal generated by $a \triangle T^n(a)$ and that $T^i(\overline{a}) \wedge T^j(\overline{a}) = \overline{0}$, for $0 \le i, j \le n-1$. Observe that the coordinates of any element $x \in I$ are finite, and the element $b = a \vee T(a) \vee \ldots \vee T^{n-1}(a)$ is such that the coordinates of $b \triangle 1$ are cofinite. Consequently, $b \triangle 1 \notin I$, that is, $\overline{b} \neq 1$.

So we have that $\overline{a}$, $\overline{T(a)}$, …, $\overline{T^{n-1}(a)}$ and $-\overline{b}$ form a partition of 1, $\overline{a}$ is of order n and $-\overline{b}$ is invariant by T. Hence the atoms of the subalgebra $\langle\overline{a}\rangle$ generated by $\overline{a}$ look like the following diagram,



that is, $\langle\overline{a}\rangle$ is isomorphic to $B_n \times B_1$. Then $B_n \times B_1$ belongs to the variety generated by FC and, accordingly, so does B_n. □

Corollary 3.7. $\mathcal{BA} = V(FC)$

We conclude the paper by conjecturing that any proper subvariety of $\mathcal{BA}$ is locally finite, that is to say, any proper subvariety of $\mathcal{BA}$ is a finite join of $V(B_n)$'s. And related to this, we also conjecture that the converse of Theorem 2.5 holds, that is, every subalgebra of $\mathbf{2}^{\mathbb{Z}}$ is subdirectly irreducible.

References

[1] M. Abad, J. P. Díaz Varela, F. López Martinolich, M. del C. Vannicola, M. Zander, *Varieties Generated by Finite Fields and Cyclic Post Algebras*, submitted for publication.

[2] S. Burris and H. Sankappanavar, A Course in Universal Algebra, Graduate Texts in Mathematics, Vol 78, Springer, Berlin, 1981.

[3] H. Cendra, *Cyclic Boolean Algebras and Galois Fields* $F(2^k)$, Portugaliae Mathematica, **39** (1980), pp.1–4 .

[4] A. Monteiro, *Algèbres de Boole Cycliques*, Revue Roumaine de Mathématiques Pures et Appliquées, XXIII, **1** (1978), pp.71–76.

[5] G. Moisil, *Algebra Schemelor cu Elemente Ventil*, Revista Universitatii C.I. Parhon, Bucharest, Seria St. Nat. **4-5** (1954), pp.9–15.

[6] G. Moisil, *Algèbres Universelles et Automates*, in: Essais sur les logiques non chrysippiennes, Editions de L'Academie de la Republique Socialiste de Roumanie, Bucharest, 1972.

[7] T. Kowalski, *Varieties of Tense Algebras*, Reports on Mathematical Logic, **32** (1998), pp.53–95.

Departamento de MatemáticaUniversidad Nacional del Sur8000 Bahía BlancaArgentina

{imabad, usdiavar, mzander}@criba.edu.ar