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Rotational Motion Control of a Spacecraft

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Abstract

The paper adopts the energy shaping method to control of rotational motion. A global representation of the rigid body motion is given in the canonical form by a quaternion and its conjugate momenta. A general method for motion control on a cotangent bundle to the 3-sphere is suggested. The design algorithm is validated for three-axis spacecraft attitude control.

Keywords

Hamiltonian methods, nonlinear control, stability theory, attitude control.

I. INTRODUCTION

Over the last 50 years since the first spacecraft was launched the subject of attitude control has become mature. A new demand on the aerospace/control engineering has come up. The design phase has to be reduced in time and thereby in cost. A way for achieving this goal is to establish a general design method for an on-board attitude control. Here energy shaping seems to be a good candidate. The objective of this work is to adopt energy shaping to rotational motion control of a spacecraft.

Stabilization by the energy shaping of a Hamiltonian system was first proposed by [1]. The control action was the sum of the gradient of potential energy and the dissipation force. Such a control law made the system uniformly asymptotically stable to the desired reference point - the point of minimal potential energy, [2] ch. 12. This elegant concept is straightforward in the Euclidean space, nevertheless motion control on an arbitrary differential manifold can only be solved locally in the coordinate neighborhood. Later, the concept was generalized to a coordinate-free setting on a Riemannian manifold in [3]. The paper translated the method to the language of differential mechanics. It showed that the energy shaping applies to rigid body control on $SO_3(\mathbb{R})$. A side effect of the generality of this new approach is the difficulty of designing a potential function on a manifold. In this as well as in more recent publications e.g. [4], [5] the Lyapunov stability methods accessible in the standard literature of control engineering were replaced by the concepts of stability originating from differential mechanics: the classical Lagrange-Dirichlet and Arnold's energy-Casimir method.

In this paper the focus is on motion control on the 3-sphere S^3. The quaternion and its conjugate momentum are used for global representation of motion. This representation corresponds to an inclusion of the cotangent bundle T^*S^3 of dimension 6 in $T^*\mathbb{R}^4$ of dimension 8. The canonical transformations from $2n$ to $2m$ dimensional phase space, where $m > n$ was addressed earlier in the literature of celestial mechanics by [6] and [7]. In this representation of the rigid body motion the spacecraft with three independent torque generators resembles an underactuated system. A straightforward solution is to compute a desired control input on $T^*_\rho\mathbb{R}^4$ first. Its projection on the cotangent space of the 3-sphere corresponds to the control torque. The final part of this paper is devoted to application of the findings for three-axis attitude control problem.

II. CANONICAL FORM FOR A RIGID BODY

To apply the energy shaping as in [1] the rigid body motion is expressed in the canonical form. The standard approach is to use locally a coordinate neighborhood, e.g. Euler angles and their conjugate momenta. In this work a global approach is chosen. The attitude of a spacecraft is parameterized by a unit quaternion. Consequently the configuration space is the unit sphere $S^3 = \{\boldsymbol{q} \in \mathbb{R}^4 : \boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} = 1\}$ and the motion is described on the cotangent bundle T^*S^3. An inclusion of T^*S^3 to $T^*\mathbb{R}^4$ is used to get the canonical form. The result is that the rotational motion of a rigid body is a function of the quaternion $\boldsymbol{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^{\mathrm{T}}$ and the conjugate momenta $\boldsymbol{p} = \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \end{bmatrix}^{\mathrm{T}}$. The idea adopted in this section was addressed earlier in celestial mechanics in the work of [6] and [7]. The authors studied a canonical transformation $y = f(x)$ of the state space $\boldsymbol{y} \in \mathbb{R}^{2n}$ to $\boldsymbol{x} \in \mathbb{R}^{2m}$ with $m > n$. The motion of the rigid body was a special case of this transformation for $m = 4,\ n = 3$. In other words the rigid body motion is no longer described locally in a 3 dimensional Euclidean space but rather globally in 4 dimensions. Following this idea the body angular velocity vector gets also an extra dimension, which is trivially 0 only on the unit sphere. This paper presents a new geometric insight which is necessary for formulation of a controlled canonical form and stability analysis addressed in the next section.

A. *Some Remarks on Tangent and Cotangent Bundle to Unit Sphere*

Consider an Euclidean space $\mathbb{R}^4$ with the standard bases $x_j,\ j = 0, \ldots, 3$. The quad $\boldsymbol{q} = [q_0\ q_1\ q_2\ q_3]^{\mathrm{T}}$ will stand for the coordinates of a point $\boldsymbol{\rho}$ in these bases. The basis vectors of the tangent space at the point $\boldsymbol{\rho}$ $T_\rho\mathbb{R}^4$ will be denoted by $\frac{\partial}{\partial x_j}$. For $\boldsymbol{q} \neq \boldsymbol{0}$ we shall introduce a linear one-to-one mapping $\boldsymbol{Q}(\boldsymbol{q}) : T_\rho\mathbb{R}^4 \rightarrow T_\rho\mathbb{R}^4$

$$\boldsymbol{Q}(\boldsymbol{q}) : \boldsymbol{v} \mapsto \boldsymbol{Q}(\boldsymbol{q})\boldsymbol{v}, \text{ where } \boldsymbol{Q}(\boldsymbol{q}) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \tag{1}$$

The matrix $\boldsymbol{Q}(\boldsymbol{q})$ is orthogonal on the subspace S^3. This can be concluded from the following inequality

$$\boldsymbol{Q}(\boldsymbol{q})\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q}) = \boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{Q}(\boldsymbol{q}) = \boldsymbol{q}^{\mathrm{T}}\boldsymbol{q}\boldsymbol{E}_{4\times4}, \tag{2}$$

where $\boldsymbol{E}_{4\times4}$ is the 4 by 4 identity matrix. In fact $\boldsymbol{Q}(\boldsymbol{q})$ corresponds to a rotation, such that the unit vector $[1\ 0\ 0\ 0]^{\mathrm{T}}$ is rotated to the normal to the unit sphere S^3 at $\boldsymbol{\rho}$. The mapping $\frac{1}{2}\boldsymbol{Q}(\boldsymbol{q})$ is used for a definition of new basis vectors Y_j of $T_\rho\mathbb{R}^4$

$$\frac{1}{2}\boldsymbol{Q}(\boldsymbol{q})Y_j(\boldsymbol{q}) = \frac{\partial}{\partial x_j}. \tag{3}$$

The basis vectors Y_j become orthogonal on S^3, due to orthogonality of $\boldsymbol{Q}(\boldsymbol{q})$. Furthermore for $\boldsymbol{q} \in S^3$ the vectors Y_1, Y_2, and Y_3 are bases of $T_\rho S^3$ and Y_0 complements to $T_\rho\mathbb{R}^4$.

Having the dual bases $\mathrm{d}x_j$ to $\frac{\partial}{\partial q_j}$ the dual bases $\mathrm{d}Y_j$ to Y_j are computed

$$2\ \mathrm{d}Y_j(\boldsymbol{q})\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q}) = \mathrm{d}q_j \tag{4}$$

since then $\delta(k,j) = \mathrm{d}x_k(\frac{\partial}{\partial x_j}) = \frac{1}{2}\boldsymbol{Q}(\boldsymbol{q})Y_k 2\mathrm{d}Y_j\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})$, which is equivalent to $\mathrm{d}Y_k(Y_j) = \delta(k,j)$. Now the covectors $\mathrm{d}Y_1$, $\mathrm{d}Y_2$, $\mathrm{d}Y_3$ are bases of the cotangent space at the point $\boldsymbol{\rho} \in S^3$, $T^*_\rho S^3$ and $\mathrm{d}Y_0$ complements $T^*_\rho\mathbb{R}^4$. It follows that the differential of a function $V(\boldsymbol{q})$ on the submanifold S^3 is

$$\mathrm{d}V(\boldsymbol{q}) = \sum_{i=1}^{3} \mathrm{d_j}V(\boldsymbol{q})\mathrm{d}Y_j, \tag{5}$$

where

$$\begin{bmatrix}\mathrm{d}_0V(\boldsymbol{q}) & \mathrm{d}_1V(\boldsymbol{q}) & \mathrm{d}_2V(\boldsymbol{q}) & \mathrm{d}_3V(\boldsymbol{q})\end{bmatrix} = \frac{1}{2}\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}\boldsymbol{Q}(\boldsymbol{q}) \tag{6}$$

At this point we shall establish a correspondence between Eq. (3) and the kinematics of a rigid body. Consider an integral curve $\boldsymbol{\rho}(t)$ of the vector field $\boldsymbol{X}_\omega = \sum_{i=0}^{3}\boldsymbol{\omega}_jY_j$. To get a global parameterization of the integral curve, it will be resolved on the bases x_j and the vector field $\boldsymbol{X}_\omega$ will be projected on $\frac{\partial}{\partial x_j}$. We shall use the equality (3) to compute the integral curve in the coordinates

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{Q}(\boldsymbol{q})\boldsymbol{\Omega}, \text{ where } \boldsymbol{\Omega} = \begin{bmatrix}\boldsymbol{\omega}_0 & \boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 & \boldsymbol{\omega}_3\end{bmatrix}^{\mathrm{T}} \tag{7}$$

Denoting the body angular velocity by $\boldsymbol{\omega} = \begin{bmatrix}\boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 & \boldsymbol{\omega}_3\end{bmatrix}^{\mathrm{T}}$ the kinematics of a rigid body motion takes the celebrated formula ([7] and [8])

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{Q}(\boldsymbol{q})i(\boldsymbol{\omega}), \tag{8}$$

where $i : \mathbb{R}^3 \to \mathbb{R}^4$ is the inclusion $\begin{bmatrix}\boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 & \boldsymbol{\omega}_3\end{bmatrix}^{\mathrm{T}} \mapsto \begin{bmatrix}0 & \boldsymbol{\omega}_1 & \boldsymbol{\omega}_2 & \boldsymbol{\omega}_3\end{bmatrix}^{\mathrm{T}}$. The integral curve $\boldsymbol{\rho}(t)$ of Eq. (8) remains always on the 3-sphere since the coordinate $\boldsymbol{\omega}_0$ of $\boldsymbol{\Omega}$ is always zero.

B. Kinetic Energy

In the remaining part of this section the lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q})$ and the hamiltonian $H(\boldsymbol{q}, \boldsymbol{p}) = \langle \boldsymbol{p}, \dot{\boldsymbol{q}}\rangle - L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ will be derived, where T is kinetic and U is potential energy. We shall first consider a simplified system for which $U = 0$.

The kinetic energy of a rigid body rotation is a function of the instantaneous angular velocity $\boldsymbol{\omega}$

$$\tilde{T} = \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{\omega}, \tag{9}$$

where $\boldsymbol{J}$ is the inertia tensor. Eq. (9) is equivalent to

$$\tilde{T} = \frac{1}{2}i^{\mathrm{T}}(\boldsymbol{\omega})\boldsymbol{J}^* i(\boldsymbol{\omega}), \tag{10}$$

where $\boldsymbol{J}^*$ is a block diagonal matrix

$$\boldsymbol{J}^* = \begin{bmatrix} J_0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{J} \end{bmatrix}. \tag{11}$$

The element J_0 takes in general an arbitrary nonsingular value. We shall consider in the sequel a system with kinetic energy

$$T = \frac{1}{2}\boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{J}^*\boldsymbol{\Omega} = \frac{1}{2}i^{\mathrm{T}}(\boldsymbol{\omega})\boldsymbol{J}^* i(\boldsymbol{\omega}) + \frac{1}{2}\boldsymbol{\omega}_0 J_0 \boldsymbol{\omega}_0, \tag{12}$$

Its configuration space is $\mathbb{R}^4$, however if the initial conditions are such that $\boldsymbol{\rho}(t_0) \in S^3$ and the coordinates of $\boldsymbol{X}_\omega$ are such that $\boldsymbol{\Omega}(t_0) = i(\boldsymbol{\omega}(t_0))$, i.e. $\boldsymbol{\omega}_0(t_0) = 0$ the lagrangians $\tilde{L} = \tilde{T}$ and $L = T$ give rise to equal paths $\tilde{\boldsymbol{q}}(t) = \boldsymbol{q}(t)$. This is true since locally the paths can be represented by the same coordinates ϕ_j such that $\dot{\phi}_j(t) = \boldsymbol{\omega}_j(t),\ j = 1, \ldots, 3$ and $\dot{\phi}_0(t) = 0$ for $t \geq t_0$.

Concluding the vector field $\boldsymbol{X}_\omega$ can be represented in the bases $\frac{\partial}{\partial x_j}$ and applied in the formula for the kinetic energy. This is done by substituting the inverse of Eq. (3) in Eq. (12)

$$T = 2\dot{\boldsymbol{q}}^{\mathrm{T}}(\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q}))^{-1}\boldsymbol{J}^*\boldsymbol{Q}^{-1}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{13}$$

C. Canonical Form

The canonical form will be calculated provided that the initial conditions are such that $\sum_{j=0}^{4} q_j(t_0)x_j \in S^3$ and $\sum_{j=0}^{4} \dot{q}_j(t_0)\frac{\partial}{\partial x_j} \in T_\rho S^3$ then using the orthogonality of $\boldsymbol{Q}(\boldsymbol{q})$ in Eq. (2) the kinetic energy is further simplified

$$T = 2\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{J}^*\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{14}$$

The conjugate momentum is then

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = \frac{\partial T}{\partial \dot{\boldsymbol{q}}} = 4\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{J}^*\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q}). \tag{15}$$

The hamiltonian for the rigid body motion is now

$$H(\boldsymbol{q}, \boldsymbol{p}) = \boldsymbol{p}^{\mathrm{T}}\dot{\boldsymbol{q}} - L(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{8}\boldsymbol{p}^{\mathrm{T}}\boldsymbol{Q}(\boldsymbol{q})\boldsymbol{J}^{*-1}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{p}. \tag{16}$$

Having the hamiltonian the canonical equations are calculated

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{1}{4}\boldsymbol{Q}(\boldsymbol{q})\boldsymbol{J}^{*-1}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{p} \\ \dot{\boldsymbol{p}} &= -\frac{1}{4}\boldsymbol{Q}(\boldsymbol{p})\boldsymbol{J}^{*-1}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{p})\boldsymbol{q} + \boldsymbol{M}_p, \end{aligned} \tag{17}$$

where $\boldsymbol{M}_p^{\mathrm{T}} = [M_0^p\ M_1^p\ M_2^p\ M_3^p]$ are the coordinates of the generalized moment $\boldsymbol{\mu} \in T_\rho^* S^3$ spanned on the bases $\mathrm{d}x_j$. The control torque $\boldsymbol{M}_c$ is generated on a spacecraft by a set of three independent actuators such as gas jets, momentum/reaction wheels, electromagnetic coils. To find the correspondence between the generalized moment and the control torque

$\boldsymbol{\mu}(\boldsymbol{\rho})$ has to be projected on the bases $\mathrm{d}Y_j,\ j = 1, \ldots, 3$. The bases $\mathrm{d}Y_j$ are related to $\mathrm{d}x_j$ by the formula (4) hence $\boldsymbol{M}_c \in \mathbb{R}^3$ and $\boldsymbol{M}_p \in \mathbb{R}^4$ are associated by

$$\boldsymbol{M}_p^{\mathrm{T}} = 2i^{\mathrm{T}}(\boldsymbol{M}_c)\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q}), \tag{18}$$

which is for $\boldsymbol{q} \in S^3$ equivalent to

$$i(\boldsymbol{M}_c) = \frac{1}{2}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{M}_p. \tag{19}$$

III. Control Synthesis

The classical energy shaping in [1] is formulated for a system in the canonical form

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial p},\ \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial q} + \boldsymbol{M}_p,$$

where the Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p}) = \langle \boldsymbol{p}, \dot{\boldsymbol{q}} \rangle - T(\boldsymbol{q}, \boldsymbol{p}) + U(\boldsymbol{q})$ and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in the Euclidean space $\mathbb{R}^n$. The feedback proposed is

$$\boldsymbol{M}_p = -\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}} + \boldsymbol{K}\dot{\boldsymbol{q}}, \tag{20}$$

where $\boldsymbol{K}$ is a negative definite matrix and $V(\boldsymbol{q})$ is a continuously differentiable scalar valued function. The term $\boldsymbol{M}_d = \boldsymbol{K}\dot{\boldsymbol{q}}$ is a dissipative force, and the time derivative of its work $\dot{W} = \langle \boldsymbol{M}_d, \dot{\boldsymbol{q}} \rangle$ is negative definite. The control law (20) makes the system asymptotically stable to the equilibrium point $(\boldsymbol{q}_0, \mathbf{0})$ if $\boldsymbol{q}_0$ is the minimum of the sum of the potential energy $U(\boldsymbol{q}) + V(\boldsymbol{q})$.

A. *Energy Shaping on 3-Sphere*

We shall apply the procedure (20) to the system (17). The immediate hindrance is however the fact that the moment $\boldsymbol{\mu}_v = -\sum_{k=0}^{3} \frac{\partial V(\boldsymbol{q})}{\partial q_k} \mathrm{d}q_k$ may not in general belong to the cotangent space of the 3-sphere as the generalized moment $\boldsymbol{\mu}$ does.

A solution proposed in this paper is to substitute $\boldsymbol{\mu}_v$ by its projection on $T_\rho^* S^3$. To do this $\boldsymbol{\mu}_v$ is spanned on $\mathrm{d}Y_k,\ k = 0, \ldots, 3$, on the other hand only $\mathrm{d}Y_1,\ \mathrm{d}Y_2,\ \mathrm{d}Y_3$ constitute bases for $T_\rho^* S^3$. Therefore the projection of $\boldsymbol{\mu}_v$ on the cotangent space is precisely the differential of the potential energy $V(\boldsymbol{q})$ on S^3 given in Eq. (5). Now asymptotic stability of the control $\boldsymbol{\mu} = -\mathrm{d}V + \boldsymbol{\mu}_d$, where $\boldsymbol{\mu}_d$ is the dissipation force in coordinate free settings follows from Theorem 2 in [3]. As mentioned in Section II-C the control torque $\boldsymbol{M}_c$ corresponds to $\boldsymbol{\mu}$ spanned on $\mathrm{d}Y_i$ bases, thus

$$i(\boldsymbol{M}_c) = \frac{1}{2}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{K}\dot{\boldsymbol{q}} - [0\ \mathrm{d}_1V\ \mathrm{d}_2V\ \mathrm{d}_3V]^{\mathrm{T}}, \tag{21}$$

where d_iV are defined in Eq. (6).

B. Potential Functions

The major effort in the construction of control algorithms with use of the energy shaping is spent on finding potential functions. A potential function has to be such that a desired equilibrium of the closed loop system becomes the attractor for the entire state space. It is reasonably easy to design a positive definite function on $\mathbb{R}^n$. Quadratic forms are frequent examples. It is however much more difficult to find a reasonable positive definite function on the 3-sphere. As a matter of fact especially one has gained a great attention in the literature of aerospace and robotics: $V(\boldsymbol{q}) = 1 - q_0$, e.g. [9].

The control procedure outlined above provides some other examples of the potential functions. For the control synthesis sketched the moment $\boldsymbol{\mu}_v$ has been calculated in $T^*_\rho\mathrm{R}^4$ and then projected on the cotangent space of the 3-sphere. Here we may design a potential function $V_R(\boldsymbol{q})$ in the Euclidean space with the minimum at the desired point $\boldsymbol{q}_e$ and then restrict it to the 3-sphere $V(\boldsymbol{q}) = V_R(\boldsymbol{q})\mid_{S^3}$. A possible choice is a quadratic form

$$V_R(\boldsymbol{q}) = \frac{1}{2}(\boldsymbol{Q}(\boldsymbol{q}_e)\boldsymbol{q} - \boldsymbol{e})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{Q}(\boldsymbol{q}_e)\boldsymbol{q} - \boldsymbol{e}) \tag{22}$$

where $\boldsymbol{P} > 0$ is a positive definite matrix and $\boldsymbol{e} = [1\ 0\ 0\ 0]^{\mathrm{T}}$ is the identity. The necessary condition for existence of extremes is $\mathrm{d}V(\boldsymbol{q}) = \boldsymbol{0}$, which is equivalent to saying that there exists a real k such that

$$\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}} = k\boldsymbol{q} \text{ and } \boldsymbol{q}^{\mathrm{T}}\boldsymbol{q} = 1 \tag{23}$$

since then

$$\frac{\partial V(\boldsymbol{q})}{\partial \boldsymbol{q}}\boldsymbol{Q}(\boldsymbol{q}) = \begin{bmatrix} k & 0 & 0 & 0 \end{bmatrix}, \tag{24}$$

which follows from the orthogonality of the matrix $\boldsymbol{Q}(\boldsymbol{q})$ and because $\boldsymbol{q}$ is the first column of $\boldsymbol{Q}(\boldsymbol{q})$, see Eq. (1). Using the definition of the differential in Eq. (5) it is seen that $\mathrm{d}V(\boldsymbol{q}) = 0$.

Eq. (23) provides two solutions for $k = 0$ and $k \neq 0$. From Eqs. (22) and (23) it is observed that the function $V(\boldsymbol{q})$ reaches minimum for $k = 0$ and $\boldsymbol{q} = \boldsymbol{q}_e$. The potential function $\mathrm{d}V(\boldsymbol{q})$ is continuous and S^3 is compact therefore both a minimum and a maximum of $V(\boldsymbol{q})$ exist on the 3-sphere. It was already shawn that the minimum is determined by $k = 0$. The maximum can be computed by solving Eq. (23) for $k \neq 0$.

For a particular choice of $\boldsymbol{K} = \boldsymbol{E}_{4\times 4}$ and $\boldsymbol{q}_e = \boldsymbol{e}$ the potential function $V(\boldsymbol{q}) = V_R(\boldsymbol{q}) \mid_{S^3}$, where $V_R(\boldsymbol{q})$ is defined in Eq. (22) is equivalent to celebrated $V(\boldsymbol{q}) = 1 - q_0$, which has the global minimum at the identity $\boldsymbol{e}$ and the global maximum at $-\boldsymbol{e}$.

IV. Spacecraft Attitude Control

The findings developed in the preceding sections are implemented to the three-axis attitude control in the inertial frame. The objective is to stabilize the spacecraft to the desired attitude given by $\boldsymbol{q}_e$. The potential energy employed is $V(\boldsymbol{q}) = V_R(\boldsymbol{q}) \mid_{S^3}$, where $V_R(\boldsymbol{q})$ is given by Eq. (22). The control law (21) takes the following form

$$i(\boldsymbol{M}_c) = \frac{1}{2}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{K}\boldsymbol{Q}(\boldsymbol{q})i(\boldsymbol{\omega}) - [0 \ \mathrm{d}_1 V \ \mathrm{d}_2 V \ \mathrm{d}_3 V]^{\mathrm{T}}, \tag{25}$$

where

$$[\mathrm{d}_0 V \ \mathrm{d}_1 V \ \mathrm{d}_2 V \ \mathrm{d}_3 V] = \frac{1}{2}(\boldsymbol{Q}(\boldsymbol{q}_e)\boldsymbol{q} - \boldsymbol{e})^{\mathrm{T}}\boldsymbol{P}\boldsymbol{Q}(\boldsymbol{q}_e)\boldsymbol{Q}(\boldsymbol{q}). \tag{26}$$

This seemingly a complex control law has an ordinary PD structure. To see this we shall consider an example in which the reference is the unit quaternion, the gains $\boldsymbol{K} = 4k_d\boldsymbol{E}_{4\times 4}$ and $\boldsymbol{P} = 2k_p\boldsymbol{E}_{4\times 4}$ then the differential $\mathrm{d}V(\boldsymbol{q})$ is

$$\begin{aligned} [\mathrm{d}_0 V \ \mathrm{d}_1 V \ \mathrm{d}_2 V \ \mathrm{d}_3 V] &= k_p\left(\boldsymbol{q}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q}) - \boldsymbol{e}\boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{q})\right) \\ &= k_p\left[1 - q_0 \quad q_1 \quad q_2 \quad q_3\right] \end{aligned} \tag{27}$$

and the control law reduces to the celebrated form

$$\boldsymbol{M}_c = -k_p[q_1 \ q_2 \ q_3]^{\mathrm{T}} + k_d\boldsymbol{\omega}. \tag{28}$$

This shows that the energy shaping approach presented in this paper is a generalization of the previous results on the three-axis attitude control summarized in [10].

V. CONCLUSION

The paper further enhanced the energy shaping method to be used for rotational motion control of a rigid body. The insight into the global canonical form representation of the spacecraft motion by the unit quaternion and its conjugate momentum was given. An elegant general scheme for control design of rigid body was proposed and implemented for three-axis attitude control.

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