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DYNAMIC ALGORITHM FOR LQGPC PREDICTIVE CONTROL

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Abstract. In this paper the optimal control law is derived for a multi-variable state space Linear Quadratic Gaussian Predictive Controller (LQGPC). A dynamic performance index is utilized resulting in an optimal steady state controller. Knowledge of future reference values is incorporated into the controller design and the solution is derived using the method of Lagrange multipliers. It is shown how the well-known GPC controller can be obtained as a special case of the LQGPC controller design. The important advantage of using the LQGPC framework for designing predictive controllers, e.g. GPC, is that LQGPC enables a systematic restriction of the design parameters to yield a stable closed loop system. The system model considered in this paper can be further extended to also include direct feed-through and knowledge about future external inputs.

Key Words. State-space design, multivariable control, Dynamic Predictive Control, Linear Quadratic Gaussian Predictive Control, Generalized Predictive Control

1. INTRODUCTION

An increasing popularity of Model Based Predictive Control algorithms may be noted over the recent years. Among predictive control schemes the Generalized Predictive Controller (GPC) is perhaps the best known and one of the most successful representatives. Several papers analyze the properties of GPC, e.g. [3], [11], [2]. The GPC is a *static* variance minimization algorithm, i.e. it separates the dynamic problem into individual steps and a solution is obtained for each such step. This does not necessarily give the optimal steady state solution. Furthermore GPC control formulae do not lend themselves easily to an analysis of the closed-loop stability and performance properties [1]. This is a problem one faces when tuning the GPC controller parameters.

As an alternative to static optimization in predictive controllers, dynamic optimization can be used. A range of methods are available to deal with dynamic variance minimization (see e.g. [4], [5], [6], [7]). These methods are derived using a frequency domain approach. A state-space dynamic optimization algorithm is proposed in [13]. The problem is formulated in state-space and a single input single output so called Dynamic Predictive Controller (DPC) is derived using a state feedback LQ formulation. The stochastic case, involving use of a Kalman filter is also considered. The multivariable version of DPC is outlined in [12].

This paper gives more detailed insight into the Linear Quadratic Gaussian approach to Predictive Control (LQGPC). The multivariable LQGPC controller is derived using Lagrange multipliers. The stability of the controller is discussed. The characteristic equation is obtained.

It splits into two parts for stochastic systems. It is shown explicitly how to incorporate the output and control horizons into the system equations. The relations between the proposed (LQGPC) controller and the state-space version of the GPC are discussed.

In LQGPC the optimal predictive control law is derived using a LQG approach. Working with a predictive control scheme like e.g. GPC in the LQGPC framework offers a systematic way of restricting the adjustable parameters to yield a stable closed loop system.

2. PROBLEM FORMULATION

2.1 PRELIMINARIES

Consider the linear system model in the ordinary discrete-time state-space form:

$$\begin{aligned} \boldsymbol{x}_{t+1}^{(0)} &= \boldsymbol{A}^{(0)}\boldsymbol{x}_t^{(0)} + \boldsymbol{B}^{(0)}\boldsymbol{u}_t + \boldsymbol{\xi}_{v,t} \\ \boldsymbol{y}_t^{(0)} &= \boldsymbol{C}^{(0)}\boldsymbol{x}_t^{(0)} + \boldsymbol{\xi}_{w,t} \end{aligned} \tag{1}$$

Here $\boldsymbol{x}_t^{(0)}$ is the system state vector with dimensions $(n_{x^{(0)}} \times 1)$. The vector of control signals $\boldsymbol{u}_t$ has the dimensions $(n_u \times 1)$. The vector of output signals $\boldsymbol{y}_t^{(0)}$ has the dimensions $(n_{y^{(0)}} \times 1)$. The process noise $\boldsymbol{\xi}_{v,t}$ and the measurement noise $\boldsymbol{\xi}_{w,t}$ have the dimensions $(n_{x^{(0)}} \times 1)$ and $(n_{y^{(0)}} \times 1)$ respectively. The system matrices $\boldsymbol{A}^{(0)}, \boldsymbol{B}^{(0)}$ and $\boldsymbol{C}^{(0)}$ are constant and can be obtained using any appropriate system identification method. Introducing integral action now define:

$$\boldsymbol{\Delta u}_t = \boldsymbol{u}_t - \boldsymbol{u}_{t-1} \tag{2}$$

Equation (1) and (2) can be combined defining an extended state vector $\boldsymbol{x}$:

$$\begin{aligned} \underbrace{\begin{bmatrix} \boldsymbol{x}_{t+1}^{(0)} \\ \boldsymbol{u}_t \end{bmatrix}}_{\boldsymbol{x}_{t+1}} &= \underbrace{\begin{bmatrix} \boldsymbol{A}^{(0)} & \boldsymbol{B}^{(0)} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} \boldsymbol{x}_t^{(0)} \\ \boldsymbol{u}_{t-1} \end{bmatrix}}_{\boldsymbol{x}_t} + \underbrace{\begin{bmatrix} \boldsymbol{B}^{(0)} \\ \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{B}} \boldsymbol{\Delta u}_t \\ &+ \underbrace{\begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{\xi}_{v,t}}_{\boldsymbol{v}_t} \\ \boldsymbol{y}_t &= \underbrace{\begin{bmatrix} \boldsymbol{C}^{(0)} & \boldsymbol{0} \end{bmatrix}}_{\boldsymbol{C}} \underbrace{\begin{bmatrix} \boldsymbol{x}_t^{(0)} \\ \boldsymbol{u}_{t-1} \end{bmatrix}}_{\boldsymbol{x}_t} + \underbrace{\boldsymbol{\xi}_{w,t}}_{\boldsymbol{w}_t} \end{aligned}$$

Hence, the state-space equations for the system can be written:

$$\begin{aligned} \boldsymbol{x}_{t+1} &= \boldsymbol{A}\boldsymbol{x}_t + \boldsymbol{B}\boldsymbol{\Delta u}_t + \boldsymbol{v}_t \\ \boldsymbol{y}_t &= \boldsymbol{C}\boldsymbol{x}_t + \boldsymbol{w}_t \end{aligned} \tag{3}$$

The systems of interest are subject to stochastic process noise ($\boldsymbol{v}$) and measurement noise ($\boldsymbol{w}$). However, from this point on it will be useful to consider the deterministic optimal control problem, assuming states are available for feedback. By invoking the *Separation Principle* of stochastic optimal control theory, the controller will utilize the deterministic control problem solution and a state estimator. The system model in prediction form is therefore given by:

$$\begin{aligned} \boldsymbol{x}_{t+k} &= \boldsymbol{A}\boldsymbol{x}_{t+k-1} + \boldsymbol{B}\boldsymbol{\Delta u}_{t+k-1} \\ \boldsymbol{y}_{t+k-1} &= \boldsymbol{C}\boldsymbol{x}_{t+k-1} \end{aligned} \tag{4}$$

The equations for the predicted future outputs can be written in a more compact block matrix form hereby obtaining the following equation:

$$\begin{aligned} \underbrace{\begin{bmatrix} \boldsymbol{y}_{t+1} \\ \boldsymbol{y}_{t+2} \\ \vdots \\ \boldsymbol{y}_{t+N+1} \end{bmatrix}}_{\boldsymbol{Y}_{t,N}} &= \underbrace{\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^N \end{bmatrix}}_{\boldsymbol{\Phi}_N} \boldsymbol{A}\boldsymbol{x}_t \\ &+ \underbrace{\begin{bmatrix} \boldsymbol{CB} & \boldsymbol{0} & \cdots\cdots & \boldsymbol{0} \\ \boldsymbol{CAB} & \boldsymbol{CB} & \boldsymbol{0} \ \ldots & \boldsymbol{0} \\ \cdots & \cdots & \cdots & \cdots \\ \boldsymbol{CA}^N\boldsymbol{B} & \boldsymbol{CA}^{N-1}\boldsymbol{B} & \cdots\cdots & \boldsymbol{CB} \end{bmatrix}}_{\boldsymbol{S}_N} \\ &\times \underbrace{\begin{bmatrix} \boldsymbol{\Delta u}_t \\ \boldsymbol{\Delta u}_{t+1} \\ \vdots \\ \boldsymbol{\Delta u}_{t+N} \end{bmatrix}}_{\boldsymbol{U}_{t,N}} \end{aligned} \tag{5}$$

To solve the infinite horizon optimization problem, knowledge about all future values of the reference signal would be required. However, as the prediction horizon is limited to N+1, only N+1 future references are assumed known at any particular time t. It is thus assumed that the future references may be evaluated from the N+1 first reference signals as follows:

$$\boldsymbol{R}_{t+1,N} = \boldsymbol{\Theta}_{R,N}\boldsymbol{R}_{t,N}$$

where:

$$\boldsymbol{R}_{t,N} = \begin{bmatrix} \boldsymbol{r}_{t+1} & \boldsymbol{r}_{t+2} & \ldots & \boldsymbol{r}_{t+N+1} \end{bmatrix}^T$$

Here $\boldsymbol{\Theta}_{R,N}$ represents the transition matrix for the reference signal. In order to incorporate the knowledge of future reference values the extended state vector is defined and the output equation is rewritten:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{x}_{t+1} \\ \boldsymbol{R}_{t+1,N} \end{bmatrix} = \boldsymbol{\chi}_{t+1} &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Theta}_{R,N} \end{bmatrix} \boldsymbol{\chi}_t + \begin{bmatrix} \boldsymbol{\beta} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{U}_{t,N} \\ &= \boldsymbol{\Lambda}\boldsymbol{\chi}_t + \boldsymbol{\Psi}\boldsymbol{U}_{t,N} \\ \boldsymbol{Y}_{t,N} &= \begin{bmatrix} \boldsymbol{\Phi}_N\boldsymbol{A} & \boldsymbol{0} \end{bmatrix} \boldsymbol{\chi}_t + \boldsymbol{S}_N\boldsymbol{U}_{t,N} \end{aligned} \tag{6}$$

where:

$$\boldsymbol{\beta} = [\boldsymbol{B} \quad \mathbf{0} \quad \ldots \quad \mathbf{0}]$$

Now the error vector $\boldsymbol{e}_{t,N}$ can be defined for the predicted error signals as:

$$\begin{aligned} \boldsymbol{e}_{t,N} &= \boldsymbol{Y}_{t,N} - \boldsymbol{R}_{t,N} \\ &= \underbrace{[\boldsymbol{\Phi}_N \boldsymbol{A} \quad -\boldsymbol{I}]}_{\boldsymbol{L}_N} \underbrace{\begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{R}_{t,N} \end{bmatrix}}_{\boldsymbol{\chi}_t} + \boldsymbol{S}_N \boldsymbol{U}_{t,N} \end{aligned} \quad (7)$$

Now consider for a while a so called *static* performance index of the type which is usually associated with predictive control problems:

$$\begin{aligned} J_t &= \sum_{j=0}^{N} [(\boldsymbol{y}_{t+j+1} - \boldsymbol{r}_{t+j+1})^T \boldsymbol{\Lambda}_e \\ &\quad \times (\boldsymbol{y}_{t+j+1} - \boldsymbol{r}_{t+j+1}) \\ &\quad + \boldsymbol{\Delta u}_{t+j}^T \boldsymbol{\Lambda}_u \boldsymbol{\Delta u}_{t+j}] \\ &= \sum_{j=0}^{N} \left[\boldsymbol{e}_{t+j+1}^T \boldsymbol{\Lambda}_e \boldsymbol{e}_{t+j+1} + \boldsymbol{\Delta u}_{t+j}^T \boldsymbol{\Lambda}_u \boldsymbol{\Delta u}_{t+j} \right] \end{aligned} \quad (8)$$

Using (7) this can be written:

$$J_t = \boldsymbol{e}_{t,N}^T \boldsymbol{\Lambda}_e \boldsymbol{e}_{t,N} + \boldsymbol{U}_{t,N}^T \boldsymbol{\Lambda}_u \boldsymbol{U}_{t,N} \quad (9)$$

Here it has been assumed, without loss of generality, the same interval $j = 0, 1, \ldots, N$ exists for the control signal (GPC: $k = 1, \ldots, N_u$) and for the output error signal (GPC: $l = N_1, N_1+1, \ldots, N_2$). The GPC design parameters N_1, N_2 and N_u can be incorporated in the LQGPC framework by proper adjustment of various matrices. This will be discussed later.

When using equation (9) as performance index the vector $\boldsymbol{U}_{t,N}$ of optimal control actions within the horizon N_u is calculated. However, only the first element $\boldsymbol{\Delta u}_t$ is applied and the procedure is repeated in the next step. Therefore the algorithm solves a static optimization problem in each step.

2.2 DYNAMIC OPTIMIZATION

Now, consider the dynamic optimization. The dynamic performance index will be defined as an infinite sum of the indices of the form as in (9):

$$\begin{aligned} J &= \lim_{T \to \infty} \frac{1}{T} \sum_{t=t_0}^{t_0+T} J_t \\ &= \lim_{T \to \infty} \frac{1}{T} \sum_{t=t_0}^{t_0+T} (\boldsymbol{e}_{t,N}^T \boldsymbol{\Lambda}_e \boldsymbol{e}_{t,N} \\ &\quad + \boldsymbol{U}_{t,N}^T \boldsymbol{\Lambda}_u \boldsymbol{U}_{t,N}) \end{aligned} \quad (10)$$

There are some special cases where the performance indices (9) and (10) yield the same optimal control solution, for instance if the problem can be transformed to the Åström minimum variance controller. However, apart from these special cases, the solutions obtained when minimizing (9) and (10) are different.

It is easy to observe that (9) corresponds to the situation when T = 1, it means static optimization, looking only one step ahead.

Starting from the same initial conditions the two performance indices will lead to two different control strategies and will settle on different steady state values. In steady state, the algorithm resulting from (10) will provide the minimum value of the performance indices (10) and (9). The value obtained from the algorithm resulting from (9) will be higher ([13]).

Using equations (5), (7) and (10) the LQGPC performance index can be written:

$$\begin{aligned} J &= \frac{1}{T} \sum_{t=t_0}^{t_0+T} \left[\boldsymbol{e}_{t,N}^T \boldsymbol{\Lambda}_e \boldsymbol{e}_{t,N} + \boldsymbol{U}_{t,N}^T \boldsymbol{\Lambda}_u \boldsymbol{U}_{t,N} \right] \\ &= \frac{1}{T} \sum_{t=t_0}^{t_0+T} [\boldsymbol{\chi}_t^T \underbrace{\boldsymbol{L}_N^T \boldsymbol{\Lambda}_e \boldsymbol{L}_N}_{\boldsymbol{Q}} \boldsymbol{\chi}_t \\ &\quad + \boldsymbol{\chi}_t^T \underbrace{\boldsymbol{L}_N^T \boldsymbol{\Lambda}_e \boldsymbol{S}_N}_{\boldsymbol{M}} \boldsymbol{U}_{t,N} \\ &\quad + \boldsymbol{U}_{t,N}^T \underbrace{\boldsymbol{S}_N^T \boldsymbol{\Lambda}_e \boldsymbol{L}_N}_{\boldsymbol{M}^T} \boldsymbol{\chi}_t \\ &\quad + \boldsymbol{U}_{t,N}^T \underbrace{\left(\boldsymbol{S}_N^T \boldsymbol{\Lambda}_e \boldsymbol{S}_N + \boldsymbol{\Lambda}_u \right)}_{\boldsymbol{R}} \boldsymbol{U}_{t,N}] \end{aligned} \quad (11)$$

Without loss of generality we may assume $\boldsymbol{\Lambda}_e$ and $\boldsymbol{\Lambda}_u$ to be symmetric and therefore $\boldsymbol{Q}$ and $\boldsymbol{R}$ to be symmetric.

The performance index J in (11) is to be minimized under the constraints (see (6)):

$$\begin{aligned} \boldsymbol{\chi}_{t+1} &= \boldsymbol{\Lambda} \boldsymbol{\chi}_t + \boldsymbol{\Psi} \boldsymbol{U}_{t,N} \\ t &= t_0, t_0+1, \ldots, t_0+T \end{aligned} \quad (12)$$

This is equivalent (see [10]) to minimizing the performance index:

$$J = \frac{1}{T}\sum_{t=t_0}^{t_0+T}\left[\boldsymbol{\chi}_t^T\hat{\boldsymbol{Q}}\boldsymbol{\chi}_t + \boldsymbol{V}_t^T\boldsymbol{R}\boldsymbol{V}_t\right] \tag{13}$$

under the constraints:

$$\begin{aligned}\boldsymbol{\chi}_{t+1} &= \hat{\boldsymbol{G}}\boldsymbol{\chi}_t + \boldsymbol{\Psi}\boldsymbol{V}_t\\ \boldsymbol{\chi}_{t_0} &= \boldsymbol{c}\\ t &= t_0, t_0+1, \ldots, t_0+T\end{aligned} \tag{14}$$

where:

$$\begin{aligned}\hat{\boldsymbol{Q}} &= \boldsymbol{Q} - \boldsymbol{M}\boldsymbol{R}^{-1}\boldsymbol{M}^T\\ \boldsymbol{V}_t &= \boldsymbol{R}^{-1}\boldsymbol{M}^T\boldsymbol{\chi}_t + \boldsymbol{U}_{t,N}\\ \hat{\boldsymbol{G}} &= \boldsymbol{\Lambda} - \boldsymbol{\Psi}\boldsymbol{R}^{-1}\boldsymbol{M}^T\end{aligned}$$

This is an equality constrained problem in two dimensions since the function to be minimized and the constraints are functions of two variables namely $\boldsymbol{\chi}$ and $\boldsymbol{U}$. Such a problem can be solved using the method of Lagrange multipliers.

3. SOLUTION

Minimizing J in (13) under the constraints in (14) corresponds (see [10]) to minimizing the following performance index:

$$\begin{aligned}L_0 = \;& \frac{1}{T}\sum_{t=t_0}^{t_0+T}[\left(\boldsymbol{\chi}_t^T\hat{\boldsymbol{Q}}\boldsymbol{\chi}_t + \boldsymbol{V}_t^T\boldsymbol{R}\boldsymbol{V}_t\right)\\ &+\boldsymbol{\lambda}_{t+1}\left(\hat{\boldsymbol{G}}\boldsymbol{\chi}_t + \boldsymbol{\Psi}\boldsymbol{V}_t - \boldsymbol{\chi}_{t+1}\right)\\ &+\left(\hat{\boldsymbol{G}}\boldsymbol{\chi}_t + \boldsymbol{\Psi}\boldsymbol{V}_t - \boldsymbol{\chi}_{t+1}\right)^T\boldsymbol{\lambda}_{t+1}]\end{aligned} \tag{15}$$

under the constraints:

$$\begin{aligned}\boldsymbol{\chi}_{t+1} &= \hat{\boldsymbol{G}}\boldsymbol{\chi}_t + \boldsymbol{\Psi}\boldsymbol{V}\\ \boldsymbol{\chi}_{t_0} &= \boldsymbol{c}\end{aligned}$$

where $\boldsymbol{c}$ is a constant vector. The vectors $\boldsymbol{\lambda}_{t_0+1}, \boldsymbol{\lambda}_{t_0+2}, \ldots, \boldsymbol{\lambda}_{t_0+T}$ are called Lagrange multipliers. The Lagrange multipliers are eliminated from the equations by assuming that they can be written in the following form:

$$\begin{aligned}\boldsymbol{\lambda}_t &= \boldsymbol{P}_t\boldsymbol{\chi}_t\\ &= \begin{bmatrix}\boldsymbol{P}_t^1 & \boldsymbol{P}_t^2\\ \boldsymbol{P}_t^{2^T} & \boldsymbol{P}_t^3\end{bmatrix}\boldsymbol{\chi}_t\end{aligned}$$

Now, solving the minimization problem with respect to $\boldsymbol{P}$ results in the block matrix Riccati equation:

$$\begin{aligned}\boldsymbol{P}_t = \;& \hat{\boldsymbol{Q}} + \hat{\boldsymbol{G}}^T\boldsymbol{P}_{t+1}\hat{\boldsymbol{G}}\\ &-\hat{\boldsymbol{G}}^T\boldsymbol{P}_{t+1}\boldsymbol{\Psi}\left(\boldsymbol{R} + \boldsymbol{\Psi}^T\boldsymbol{P}_{t+1}\boldsymbol{\Psi}\right)^{-1}\boldsymbol{\Psi}^T\boldsymbol{P}_{t+1}\hat{\boldsymbol{G}}\end{aligned}$$

After some algebra this can be simplified to the following block matrix Riccati equation:

$$\begin{aligned}\boldsymbol{P}_t = \;& \boldsymbol{Q} + \boldsymbol{\Lambda}^T\boldsymbol{P}_{t+1}\boldsymbol{\Lambda} - \left(\boldsymbol{M} + \boldsymbol{\Lambda}^T\boldsymbol{P}_{t+1}\boldsymbol{\Psi}\right)\\ &\times\left(\boldsymbol{R} + \boldsymbol{\Psi}^T\boldsymbol{P}_{t+1}\boldsymbol{\Psi}\right)^{-1}\left(\boldsymbol{M}^T + \boldsymbol{\Psi}^T\boldsymbol{P}_{t+1}\boldsymbol{\Lambda}\right)\end{aligned}$$

In order to be able to write the optimal control law in terms of $\boldsymbol{P}$'s (matrix-) elements this equation is decomposed, and the optimal control law can hereafter (see [12] or [8]) be written:

$$\begin{aligned}\boldsymbol{U}_{t,N} &= -\left(\boldsymbol{R} + \boldsymbol{\Psi}^T\boldsymbol{P}_{t+1}\boldsymbol{\Psi}\right)^{-1}\\ &\quad\times\left(\boldsymbol{M}^T + \boldsymbol{\Psi}^T\boldsymbol{P}_{t+1}\boldsymbol{\Lambda}\right)\boldsymbol{\chi}_t\\ &= -\boldsymbol{\Gamma}_{x,t}\boldsymbol{x}_t - \boldsymbol{\Gamma}_{R,t}\boldsymbol{R}_{t,N}\\ &= -\left(\boldsymbol{\Lambda}_u + \boldsymbol{S}_N^T\boldsymbol{\Lambda}_e\boldsymbol{S}_N + \boldsymbol{\beta}^T\boldsymbol{P}_{t+1}^1\boldsymbol{\beta}\right)^{-1}\\ &\quad\times\left(\boldsymbol{S}_N^T\boldsymbol{\Lambda}_e\boldsymbol{\Phi}_N\boldsymbol{A} + \boldsymbol{\beta}^T\boldsymbol{P}_{t+1}^1\boldsymbol{A}\right)\boldsymbol{x}_t\\ &\quad-\left(\boldsymbol{\Lambda}_u + \boldsymbol{S}_N^T\boldsymbol{\Lambda}_e\boldsymbol{S}_N + \boldsymbol{\beta}^T\boldsymbol{P}_{t+1}^1\boldsymbol{\beta}\right)^{-1}\\ &\quad\times\left(\boldsymbol{\beta}^T\boldsymbol{P}_{t+1}^2\boldsymbol{\Theta}_{R,N} - \boldsymbol{S}_N^T\boldsymbol{\Lambda}_e\right)\boldsymbol{R}_{t,N}\end{aligned}$$

where:

$$\begin{aligned}\boldsymbol{P}_t^1 &= \boldsymbol{A}^T\left(\boldsymbol{\Phi}_N^T\boldsymbol{\Lambda}_e\boldsymbol{\Phi}_N + \boldsymbol{P}_{t+1}^1\right)\boldsymbol{A}\\ &\quad-\boldsymbol{A}^T\left(\boldsymbol{\Phi}_N^T\boldsymbol{\Lambda}_e\boldsymbol{S}_N + \boldsymbol{P}_{t+1}^1\boldsymbol{\beta}\right)\\ &\quad\times\left(\boldsymbol{S}_N^T\boldsymbol{\Lambda}_e\boldsymbol{S}_N + \boldsymbol{\Lambda}_u + \boldsymbol{\beta}^T\boldsymbol{P}_{t+1}^1\boldsymbol{\beta}\right)^{-1}\\ &\quad\times\left(\boldsymbol{S}_N^T\boldsymbol{\Lambda}_e\boldsymbol{\Phi}_N + \boldsymbol{\beta}^T\boldsymbol{P}_{t+1}^1\right)\boldsymbol{A}\\ \boldsymbol{P}_t^2 &= \boldsymbol{A}^T\left(\boldsymbol{P}_{t+1}^2\boldsymbol{\Theta}_{R,N} - \boldsymbol{\Theta}_{R,N}^T\boldsymbol{\Lambda}_e\right)\\ &\quad-\boldsymbol{A}^T\left(\boldsymbol{\Phi}_N^T\boldsymbol{\Lambda}_e\boldsymbol{S}_N + \boldsymbol{P}_{t+1}^1\boldsymbol{\beta}\right)\\ &\quad\times\left(\boldsymbol{S}_N^T\boldsymbol{\Lambda}_e\boldsymbol{S}_N + \boldsymbol{\Lambda}_u + \boldsymbol{\beta}^T\boldsymbol{P}_{t+1}^1\boldsymbol{\beta}\right)^{-1}\\ &\quad\times\left(\boldsymbol{\beta}^T\boldsymbol{P}_{t+1}^2\boldsymbol{\Theta}_{R,N} - \boldsymbol{S}_N^T\boldsymbol{\Lambda}_e\right)\end{aligned}$$

Only the first (vector-) element of $\boldsymbol{U}_{t,N}$ is actually applied and can be written:

$$\begin{aligned}\boldsymbol{\Delta u}_t &= \underbrace{\left[\boldsymbol{I} \;\; \boldsymbol{0} \;\; \ldots \;\; \boldsymbol{0}\right]}_{\boldsymbol{\beta}_u}\boldsymbol{U}_{t,N}\\ &= -\underbrace{\boldsymbol{\beta}_u\boldsymbol{\Gamma}_{x,t}}_{\boldsymbol{L}_{x,t}}\boldsymbol{x}_t - \underbrace{\boldsymbol{\beta}_u\boldsymbol{\Gamma}_{R,t}}_{\boldsymbol{L}_{R,t}}\boldsymbol{R}_{t,N}\end{aligned}$$

Hence, the resulting controller is a 2 degree-of-freedom controller consisting of a feedback controller matrix $\boldsymbol{L}_{x,t}$ and a tracking controller matrix $\boldsymbol{L}_{R,t}$. In most practical situations, the state vector $\boldsymbol{x}$ is not measurable. However, it can be shown (see [8]) that the *Separation Theorem* holds for the problem of computing an optimal controller for the system defined by (3). Hence, in the optimal control law the state $\boldsymbol{x}_t$ can be substituted with $\hat{\boldsymbol{x}}_t$ resulting in:

$$\boldsymbol{\Delta u}_t = -\boldsymbol{L}_{x,t}\hat{\boldsymbol{x}}_t - \boldsymbol{L}_{R,t}\boldsymbol{R}_{t,N}$$

Next, the stability of the closed-loop system using the steady state solution is considered.

4. STABILITY USING THE SS SOLUTION INCLUDING OBSERVER

In this section it is shown how the steady-state stability of the overall closed-loop system including an observer can be evaluated. The approach taken here is the same as in [14].

Since the reference model is not part of a closed loop and is itself stable by construction, $\boldsymbol{R}_{t,N}$ can and will be set equal to zero in this section yielding the control law: $\boldsymbol{\Delta u}_t = -\boldsymbol{\Gamma}_{x,t}\hat{\boldsymbol{x}}_t$.

Remember that:

$$\hat{\boldsymbol{x}}_t = \begin{bmatrix} \hat{\boldsymbol{x}}_t^{(0)} \\ \hat{\boldsymbol{u}}_{t-1} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{x}}_t^{(0)} \\ \boldsymbol{u}_{t-1} \end{bmatrix}$$

The observer is given by:

$$\hat{\boldsymbol{x}}_t^{(0)} = \boldsymbol{A}^{(0)}\hat{\boldsymbol{x}}_t^{(0)} + \boldsymbol{B}^{(0)}\boldsymbol{u}_t + \boldsymbol{K}_{x^{(0)},t}\left(\boldsymbol{y}_t^{(0)} - \boldsymbol{C}^{(0)}\hat{\boldsymbol{x}}_t^{(0)}\right)$$

The observer error is given by:

$$\begin{aligned} \tilde{\boldsymbol{x}}_{t+1}^{(0)} &= \boldsymbol{x}_{t+1}^{(0)} - \hat{\boldsymbol{x}}_{t+1}^{(0)} \\ &= \left(\boldsymbol{A}^{(0)} - \boldsymbol{K}_{x^{(0)},t}\boldsymbol{C}^{(0)}\right)\tilde{\boldsymbol{x}}_t^{(0)} + \boldsymbol{v}_t - \boldsymbol{K}_{x^{(0)},t}\boldsymbol{w}_t \end{aligned}$$

Collecting the equation for the state $\boldsymbol{x}$ and the equation for the observer error $\tilde{\boldsymbol{x}}^{(0)}$ into a block matrix form yields:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{x}_{t+1} \\ \tilde{\boldsymbol{x}}_{t+1}^{(0)} \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} \boldsymbol{A}^{(0)} & \boldsymbol{B}^{(0)} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} & \boldsymbol{0} \\ \boldsymbol{0} & \left(\boldsymbol{A}^{(0)} - \boldsymbol{K}_{x^{(0)},t}\boldsymbol{C}^{(0)}\right) \end{bmatrix} \\ &\times \begin{bmatrix} \boldsymbol{x}_t \\ \tilde{\boldsymbol{x}}_t^{(0)} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} \boldsymbol{B}^{(0)} \\ \boldsymbol{I} \end{bmatrix} \\ \boldsymbol{0} \end{bmatrix} \underbrace{(-\boldsymbol{L}_{x,t}\hat{\boldsymbol{x}}_t)}_{\boldsymbol{\Delta u}_t} \\ &+ \begin{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{v}_t + \begin{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \\ -\boldsymbol{K}_{x^{(0)},t} \end{bmatrix} \boldsymbol{w}_t \end{aligned} \tag{16}$$

Now since:

$$\begin{aligned} \hat{\boldsymbol{x}}_t &= \begin{bmatrix} \hat{\boldsymbol{x}}_t^{(0)} \\ \boldsymbol{u}_{t-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_t^{(0)} - \tilde{\boldsymbol{x}}_t^{(0)} \\ \boldsymbol{u}_{t-1} \end{bmatrix} \\ \boldsymbol{L}_{x,t} &= \begin{bmatrix} \boldsymbol{L}_{x^{(0)},t} & \boldsymbol{L}_{u,t} \end{bmatrix} \end{aligned}$$

equation (16) can be rewritten:

$$\begin{aligned} \begin{bmatrix} \boldsymbol{x}_{t+1} \\ \tilde{\boldsymbol{x}}_{t+1}^{(0)} \end{bmatrix} &= \begin{bmatrix} \begin{bmatrix} \boldsymbol{A}^{(0)} & \boldsymbol{B}^{(0)} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} & \boldsymbol{0} \\ \boldsymbol{0} & \left(\boldsymbol{A}^{(0)} - \boldsymbol{K}_{x^{(0)},t}\boldsymbol{C}^{(0)}\right) \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_t \\ \tilde{\boldsymbol{x}}_t^{(0)} \end{bmatrix} \\ &+ \begin{bmatrix} \begin{bmatrix} \boldsymbol{B}^{(0)} \\ \boldsymbol{I} \end{bmatrix} \\ \boldsymbol{0} \end{bmatrix} \left(- \begin{bmatrix} \boldsymbol{L}_{x^{(0)},t} & \boldsymbol{L}_{u,t} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_t^{(0)} - \tilde{\boldsymbol{x}}_t^{(0)} \\ \boldsymbol{u}_{t-1} \end{bmatrix} \right) \\ &+ \begin{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{v}_t + \begin{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \\ -\boldsymbol{K}_{x^{(0)},t} \end{bmatrix} \boldsymbol{w}_t \\ &= \underbrace{\begin{bmatrix} \boldsymbol{\Omega}_{L,t} & \begin{bmatrix} \boldsymbol{B}^{(0)} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{L}_{x^{(0)},t} \\ \boldsymbol{0} & \boldsymbol{\Omega}_{K,t} \end{bmatrix}}_{\boldsymbol{\Omega}_t} \begin{bmatrix} \boldsymbol{x}_t \\ \tilde{\boldsymbol{x}}_t^{(0)} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{bmatrix} \\ \boldsymbol{I} \end{bmatrix} \boldsymbol{v}_t \\ &+ \begin{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \\ -\boldsymbol{K}_{x^{(0)},t} \end{bmatrix} \boldsymbol{w}_t \end{aligned}$$

where:

$$\begin{aligned} \boldsymbol{\Omega}_{L,t} &= \begin{bmatrix} \boldsymbol{A}^{(0)} - \boldsymbol{B}^{(0)}\boldsymbol{L}_{x^{(0)},t} & \boldsymbol{B}^{(0)}(\boldsymbol{I} - \boldsymbol{L}_{u,t}) \\ -\boldsymbol{L}_{x^{(0)},t} & \boldsymbol{I} - \boldsymbol{L}_{u,t} \end{bmatrix} \\ \boldsymbol{\Omega}_{K,t} &= \left(\boldsymbol{A}^{(0)} - \boldsymbol{K}_{x^{(0)},t}\boldsymbol{C}^{(0)}\right) \end{aligned}$$

The stability of the closed-loop steady-state solution including an observer can be evaluated by checking the location of the eigenvalues of the transition matrix $\boldsymbol{\Omega}_t$ using the steady-state solutions to the controller Riccati equation ($\boldsymbol{L}_{x^{(0)},\infty}$ and $\boldsymbol{L}_{u,\infty}$) and the steady-state solution to the observer Riccati equation ($\boldsymbol{K}_{x^{(0)},\infty}$). Denote by $\boldsymbol{\Omega}_\infty$ the value of $\boldsymbol{\Omega}_t$ obtained by inserting these steady-state matrices. The eigenvalues of $\boldsymbol{\Omega}_\infty$ are given as the solutions with respect to z of the characteristic equation:

$$\begin{gathered} det\left(z\boldsymbol{I} - \boldsymbol{\Omega}_\infty\right) = 0 \\ \Updownarrow \\ det\left(z\boldsymbol{I} - \boldsymbol{\Omega}_{L,\infty}\right) det\left(z\boldsymbol{I} - \boldsymbol{\Omega}_{K,\infty}\right) = 0 \end{gathered}$$

where:

$$\begin{aligned} \boldsymbol{\Omega}_{L,\infty} &= \begin{bmatrix} \boldsymbol{A}^{(0)} - \boldsymbol{B}^{(0)}\boldsymbol{L}_{x^{(0)},\infty} & \boldsymbol{B}^{(0)}(\boldsymbol{I} - \boldsymbol{L}_{u,\infty}) \\ -\boldsymbol{L}_{x^{(0)},\infty} & \boldsymbol{I} - \boldsymbol{L}_{u,\infty} \end{bmatrix} \\ \boldsymbol{\Omega}_{K,\infty} &= \left(\boldsymbol{A}^{(0)} - \boldsymbol{K}_{x^{(0)},\infty}\boldsymbol{C}^{(0)}\right) \end{aligned}$$

If all $\boldsymbol{\Omega}_\infty$'s eigenvalues lie inside the unit circle in the z-plane, the closed loop system is stable. It is seen that the controller eigenvalues and the observer eigenvalues can be chosen independently as they do not influence each other. The controller eigenvalues depend upon $\boldsymbol{L}_{x^{(0)},\infty}$ and $\boldsymbol{L}_{u,\infty}$ whereas the observer eigenvalues depend upon $\boldsymbol{K}_{x^{(0)},\infty}$.

5. GPC IN THE LQGPC FRAMEWORK

Consider now the GPC minimization problem assuming a model in the form (3):

$$\min_{\boldsymbol{\Delta u}_t, \boldsymbol{\Delta u}_{t+1}, \ldots, \boldsymbol{\Delta u}_{t+N_u-1}} E\left[J_t\right]$$

where:

$$\begin{aligned} J_t &= \sum_{l=N_1}^{N_2} \left[\left(\boldsymbol{y}_{t+l} - \boldsymbol{r}_{t+l}\right)^T \boldsymbol{\Lambda}_e \left(\boldsymbol{y}_{t+l} - \boldsymbol{r}_{t+l}\right)\right] \\ &\quad + \sum_{k=1}^{N_u} \left[\boldsymbol{\Delta u}_{t+k-1} \boldsymbol{\Lambda}_u \boldsymbol{\Delta u}_{t+k-1}\right] \\ \boldsymbol{x}_{t+1} &= \boldsymbol{A x}_t + \boldsymbol{B \Delta u}_t + \boldsymbol{v}_t \\ \boldsymbol{y}_t &= \boldsymbol{C x}_t + \boldsymbol{w}_t \end{aligned} \tag{17}$$

If $\boldsymbol{v}_t$ and $\boldsymbol{w}_t$ are independent Gaussian white noise sequences this minimization problem is equivalent to the deterministic minimization:

$$\min_{\boldsymbol{\Delta u}_t, \boldsymbol{\Delta u}_{t+1}, \ldots, \boldsymbol{\Delta u}_{t+N_u-1}} J_t$$

where:

$$\begin{aligned} J_t &= \sum_{l=N_1}^{N_2} \left[\left(\boldsymbol{y}_{t+l} - \boldsymbol{r}_{t+l}\right)^T \boldsymbol{\Lambda}_e \left(\boldsymbol{y}_{t+l} - \boldsymbol{r}_{t+l}\right)\right] \\ &\quad + \sum_{k=1}^{N_u} \left[\boldsymbol{\Delta u}_{t+k-1} \boldsymbol{\Lambda}_u \boldsymbol{\Delta u}_{t+k-1}\right] \\ \boldsymbol{x}_{t+1} &= \boldsymbol{A x}_t + \boldsymbol{B \Delta u}_t \\ \boldsymbol{y}_t &= \boldsymbol{C x}_t \end{aligned} \tag{18}$$

In GPC and LQGPC control the output and control intervals may be treated as tuning parameters. In order to incorporate the output interval $[N_1; N_2]$ and the control interval $[1; N_u]$ in the design, it is enough simply to cut the corresponding parts from the matrices in the output prediction equation (5), adjust various matrices accordingly and make certain substitutions. This will be shown in the following. First note that the output equation in (4) can be written in the following prediction form:

$$\boldsymbol{y}_{t+k} = \boldsymbol{C A}^k \boldsymbol{x}_t + \sum_{i=1}^{k} \boldsymbol{C A}^{k-i} \boldsymbol{B \Delta u}_{t+i-1}$$

If the output interval is $[N_1; N_2]$ and the control interval is $[1; N_u]$ like in (17) and (18) then the block matrix form in (5) is modified to:

$$\underbrace{\begin{bmatrix} \boldsymbol{y}_{t+N_1} \\ \boldsymbol{y}_{t+N_1+1} \\ \vdots \\ \boldsymbol{y}_{t+N_2} \end{bmatrix}}_{\boldsymbol{Y}_{t,N_1,N_2}} = \underbrace{\begin{bmatrix} \boldsymbol{C A}^{N_1-1} \\ \boldsymbol{C A}^{N_1} \\ \vdots \\ \boldsymbol{C A}^{N_2-1} \end{bmatrix}}_{\boldsymbol{\Phi}_{N_1,N_2}} \boldsymbol{A x}_t$$

$$+ \underbrace{\begin{bmatrix} \boldsymbol{C A}^{N_1-1}\boldsymbol{B} & \boldsymbol{C A}^{N_1-2}\boldsymbol{B} & \ldots & \boldsymbol{C A}^{N_1-N_u}\boldsymbol{B} \\ \boldsymbol{C A}^{N_1}\boldsymbol{B} & \boldsymbol{C A}^{N_1-1}\boldsymbol{B} & \ldots & \boldsymbol{C A}^{N_1-N_u+1}\boldsymbol{B} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{C A}^{N_2-1}\boldsymbol{B} & \boldsymbol{C A}^{N_2-2}\boldsymbol{B} & \ldots & \boldsymbol{C A}^{N_2-N_u} \end{bmatrix}}_{\boldsymbol{S}_{N_1,N_2}}$$

$$\times \underbrace{\begin{bmatrix} \boldsymbol{\Delta u}_t \\ \boldsymbol{\Delta u}_{t+1} \\ \vdots \\ \boldsymbol{\Delta u}_{t+N_u-1} \end{bmatrix}}_{\boldsymbol{U}_{t,N_u}}$$

Hence, the output equation in (6) is modified to:

$$\boldsymbol{Y}_{t,N_1,N_2} = \boldsymbol{\Phi}_{N_1,N_2} \boldsymbol{A x}_t + \boldsymbol{S}_{N_1,N_2} \boldsymbol{U}_{t,N_u}$$

The reference model is modified to:

$$\boldsymbol{R}_{t+1,N_1,N_2} = \boldsymbol{\Theta}_{R,N_1,N_2} \boldsymbol{R}_{t,N_1,N_2}$$

where:

$$\boldsymbol{R}_{t,N_1,N_2} = \begin{bmatrix} \boldsymbol{r}_{t+N_1} & \boldsymbol{r}_{t+N_1+1} & \ldots & \boldsymbol{r}_{t+N_2} \end{bmatrix}$$

Now the error vector in (7) is modified to:

$$\begin{aligned} \boldsymbol{e}_{t,N_1,N_2} &= \boldsymbol{Y}_{t,N_1,N_2} - \boldsymbol{R}_{t,N_1,N_2} \\ &= \underbrace{\begin{bmatrix} \boldsymbol{\Phi}_{N_1,N_2} \boldsymbol{A} & -\boldsymbol{I} \end{bmatrix}}_{\boldsymbol{L}_{N_1,N_2}} \underbrace{\begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{R}_{t,N_1,N_2} \end{bmatrix}}_{\boldsymbol{\chi}_{t,N_1,N_2}} \\ &\quad + \boldsymbol{S}_{N_1,N_2} \boldsymbol{U}_{t,N_u} \end{aligned}$$

Define the weighting matrices $\boldsymbol{\Lambda}_{e,N_1,N_2}$ and $\boldsymbol{\Lambda}_{u,N_u}$ with appropriate dimensions corresponding to $\boldsymbol{e}_{t,N_1,N_2}$ and $\boldsymbol{U}_{t,N_u}$ respectively. With these definitions the GPC performance index in (18) can be written:

$$\begin{aligned} J_t &= \boldsymbol{e}_{t,N_1,N_2}^T \boldsymbol{\Lambda}_{e,N_1,N_2} \boldsymbol{e}_{t,N_1,N_2} + \boldsymbol{U}_{t,N_u}^T \boldsymbol{\Lambda}_{u,N_u} \boldsymbol{U}_{t,N_u} \\ &= \left(\boldsymbol{L}_{N_1,N_2} \boldsymbol{\chi}_{t,N_1,N_2} - \boldsymbol{S}_{N_1,N_2} \boldsymbol{U}_{t,N_u}\right)^T \boldsymbol{\Lambda}_{e,N_1,N_2} \\ &\quad \times \left(\boldsymbol{L}_{N_1,N_2} \boldsymbol{\chi}_{t,N_1,N_2} - \boldsymbol{S}_{N_1,N_2} \boldsymbol{U}_{t,N_u}\right) \\ &\quad + \boldsymbol{U}_{t,N_u}^T \boldsymbol{\Lambda}_{u,N_u} \boldsymbol{U}_{t,N_u} \end{aligned} \tag{19}$$

Summing up over t yields the LQGPC performance index:

$$\begin{aligned} J &= \frac{1}{T} \sum_{t=t_0}^{t_0+T} \left[\left(\boldsymbol{L}_{N_1,N_2} \boldsymbol{\chi}_{t,N_1,N_2} - \boldsymbol{S}_{N_1,N_2} \boldsymbol{U}_{t,N_u}\right)^T\right. \\ &\quad \times \boldsymbol{\Lambda}_{e,N_1,N_2} \left(\boldsymbol{L}_{N_1,N_2} \boldsymbol{\chi}_{t,N_1,N_2} - \boldsymbol{S}_{N_1,N_2} \boldsymbol{U}_{t,N_u}\right) \\ &\quad \left. + \boldsymbol{U}_{t,u}^T \boldsymbol{\Lambda}_{u,N_u} \boldsymbol{U}_{t,N_u}\right] \end{aligned} \tag{20}$$

The structures of the performance indices (11) and (20) are equal and with the following substitutions:

$$\boldsymbol{L}_N = \boldsymbol{L}_{N_1,N_2} \;,\; \boldsymbol{\chi}_t = \boldsymbol{\chi}_{t,N_1,N_2} \;,\; \boldsymbol{S}_N = \boldsymbol{S}_{N_1,N_2}$$
$$\boldsymbol{U}_{t,N} = \boldsymbol{U}_{t,N_u} \;,\; \boldsymbol{\Lambda}_e = \boldsymbol{\Lambda}_{e,N_1,N_2} \;,\; \boldsymbol{\Lambda}_u = \boldsymbol{\Lambda}_{u,N_u}$$

exactly the same formulae as used in the common horizon case from (11) and forward can be used for deriving the optimal control law.

With time-varying $\boldsymbol{\Lambda}_e$ and/or $\boldsymbol{\Lambda}_u$ the stability properties of the closed-loop system can no longer be evaluated by checking the locations of $\boldsymbol{\Omega}_\infty$'s eigenvalues the way it was done in the previous using the steady-state solution. Bitmead et al have considered the analysis of receding horizon control schemes with time-varying weight matrices in the performance index ([1]).

6. COMPARISON OF GPC AND LQGPC

In both GPC and LQGPC, the vector $\boldsymbol{U}_{t,N_u}$ is to be found. For the GPC controller (performance index (19)) this can be obtained through straight forward static minimization. For the LQGPC controller (performance index (20)) a solution of the control Riccati equation is needed. Therefore, the LQGPC controller retains the stability features characteristic for LQG design (see [9]). In particular if the control algebraic Riccati equation has multiple solutions it is possible to select a stable solution, leading to a stable control system. In GPC design, a traditional approach to deal with instability of the system would be to increase the output horizon or to decrease the control horizon. The resulting controller may feature a slower, more sluggish response. Generally, the following features can be expected from a controller with a long output horizon and short control horizon:

- Relatively small activity of the controller.
- The controller reacts too early to anticipated changes in the reference signal in the future.

Therefore, it is beneficial to be able to operate with relatively low output horizons and relatively high control horizons. The constraint here is stability of the system. The LQGPC can provide this stable solution.

7. CONCLUSION

In this paper the multivariable state space LQGPC predictive control problem utilizing a dynamic performance index was formulated. The algorithm resulting from using the dynamic performance index yields the optimal steady-state value of both the static performance index and the dynamic performance index. The value of both the static and the dynamic performance indices will be higher when using the algorithm resulting from basing the controller design on the static performance index.

The solution to the optimal control problem was derived using the method of Lagrange multipliers. The resulting controller was shown to be a 2 degrees-of-freedom controller consisting of a feedback controller and a tracking controller. The optimal control signal is therefore a linear combination of the state vector and the future reference values. The formulae necessary for checking the steady-state stability of the closed loop system including an observer was given.

The GPC controller can be computed using the LQGPC framework. Making adjustments of certain matrices the GPC performance index can be rewritten into a LQGPC performance index. This can be done since GPC can be seen as a special case of LQGPC. When doing this it is possible to use the LQGPC derivations/formulae to systematically obtain GPC design parameters yielding a stable closed-loop system. This is an important advantage of LQGPC over e.g. GPC.

A comparison of GPC and LQGPC was made. The influence of the choice of control and output horizons on the controller behavior was described and general guidelines were given.

The control problem described in this paper can be extended to also include knowledge of future external input values, i.e. known signals from other parts of the plant, and direct feed through in the system model (see e.g. [12] or [8]). For state-space LQGPC these extensions can be made by extending the state vector and hereafter the state transition vector, input vector and output vector accordingly. In [12] such direct feed through terms and external inputs arose from generating a linearized model for a part of a larger system.

Simulation studies [15] have shown that GPC is sensitive towards the choice of horizons whereas LQGPC is almost invariant. Furthermore the LQGPC controller gives smoother responses with less overshoot than the GPC controller (see [12]).

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