Homogeneous hypersurfaces in Riemannian symmetric spaces

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HOMOGENEOUS HYPERSURFACES IN RIEMANNIAN SYMMETRIC SPACES

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To the brave people of Ukraine, whose strength and resilience has guided me even in the darkest of hours

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ABSTRACT

In this thesis, we study the geometry and congruence of homogeneous hypersurfaces in Riemannian symmetric spaces of compact and noncompact type and obtain a number of classification results.

Firstly, we prove that every multiplicity-preserving automorphism of the restricted root system of a real semisimple Lie algebra admits a natural lift to an automorphism of that Lie algebra and show when it can be further lifted to an isometry of an associated noncompact symmetric space.

Next, we extend the classification of homogeneous codimension-one foliations on irreducible Riemannian symmetric spaces of noncompact type obtained by Berndt and Tamaru to the reducible case, thus completing it for all noncompact symmetric spaces.

After that, we obtain a complete and explicit classification, up to orbit-equivalence, of cohomogeneity-one actions (and thus homogeneous hypersurfaces) on a number of irreducible noncompact Riemannian symmetric spaces of rank 2, namely on $\mathrm{SL}(3,\mathbb{H})/\mathrm{Sp}(3)$, $\mathrm{SO}(5,\mathbb{C})/\mathrm{SO}(5)$, and $\mathrm{Gr}^*(2,\mathbb{C}^{n+4}) = \mathrm{SU}(n+2,2)/\mathrm{S}(\mathrm{U}(n+2)\mathrm{U}(2))$, $n \geq 1$.

Finally, we study homogeneous complex hypersurfaces in irreducible Hermitian symmetric spaces. In the compact case, we make some progress on classification of such hypersurfaces up to congruence by using Konno's work on codimension-one embeddings of complex flag manifolds with $b_2 = 1$. In the noncompact case, we obtain a partial classification result: given an irreducible noncompact Hermitian symmetric space M realized as a simply connected solvable Lie group, we classify those complex hypersurfaces that are also Lie subgroups of M.

CONTENTS

Chapter 1

INTRODUCTION

The overarching theme of this thesis is the geometry, congruence, and ultimate classification of *homogeneous hypersurfaces* in *Riemannian symmetric spaces*. A *homogeneous submanifold* in a Riemannian manifold M is an orbit of an isometric Lie group action on M. A *Riemannian symmetric space* is a Riemannian manifold, each of whose points is an isolated fixed point of an involutive isometry, called the *geodesic symmetry* at that point.

Riemannian symmetric spaces, or simply *symmetric spaces*, are perhaps one of the most well-studied classes of Riemannian manifolds due to their exceptionally high degree of symmetry. Every symmetric space is a Riemannian homogeneous space, so it can be represented as a quotient of Lie groups and studied by means of Lie theory. But much more is true: the existence of geodesic symmetries enables one to translate most of the geometric properties of a symmetric space $M = G/K$ into statements about the Lie group G and its Lie algebra $\mathfrak{g}$. Various geometric quantities of M such as the curvature—be it the curvature endomorphism, the Ricci curvature, or the sectional curvatures—can be expressed in terms of the algebraic structure of $\mathfrak{g}$. As a result, many questions about the geometry of symmetric spaces can be rendered algebraic and thus greatly simplified. This puts symmetric spaces at the epicenter of a lot of quests in differential geometry (and especially differential geometry with symmetries) in the past century. To name a few:

- *Riemannian manifolds with positive sectional curvature.* There are very few known compact examples of such manifolds that are not homogeneous or cohomogeneity-one. Homogeneous positively-curved manifolds were essentially classified around 50 years ago by Bérard-Bergery, Wallach, etc. (see [WZ18] for a modern exposition). Those of cohomogeneity one were handled more recently (see a general survey [Zil14]). In either case, compact symmetric spaces of rank one (see[1] Definition 2.1.8) play a central role.

- *Homogeneous Einstein manifolds.* Irreducible symmetric spaces provide one of the most elementary—yet abundant—examples of homogeneous Einstein manifolds. They also serve as a guiding model for other such manifolds. For instance, every symmetric space of noncompact type (Definition 2.1.77)—which is Einstein when the metric is suitably normalized—can be realized as a simply connected solvable Lie group with a left-invariant Riemannian metric. In light of the recently resolved Alekseevskii conjecture ([BL23]), every homogeneous Einstein manifold of negative

[1]Precise definitions of many of the terms used in this introduction can be found in Chapter 2 and will be referenced when relevant.

scalar curvature is also of this form. We refer to the survey [Jab21] for further discussion.

- *Quaternion-Kähler manifolds.* These are Riemannian manifolds with holonomy contained in $\mathrm{Sp}(n)\mathrm{Sp}(1)$—one of the seven possible holonomy groups of non-locally-symmetric manifolds that appear in Berger's classification of Riemannian holonomy groups. In a certain sense, they are a generalization of hyperkähler manifolds, although these two types of spaces do not share so much in common. Each compact simple Lie group gives rise to a quaternion-Kähler symmetric space of positive Ricci curvature, known as a *Wolf space*. Remarkably, there are no known examples of compact quaternion-Kähler manifolds that are not locally symmetric space (or hyperkähler); this makes $\mathrm{Sp}(n)\mathrm{Sp}(1)$ the only special holonomy group without such examples. We refer to [Bes08, Chap. 14] for a detailed discussion of these spaces and further references.

- *Isometric polar actions.* A proper isometric Lie group action on a complete Riemannian manifold M is called *polar* if there exists a complete submanifold of M that intersects all the orbits and does so orthogonally. For many spaces, including all simply connected irreducible symmetric spaces, such actions are a generalization of the more well-known notion of *cohomogeneity-one action*. They are also related to *isoparametric submanifolds* (see, e.g., [PT87]). In [Dad85], Dadok proved that linear polar actions on Euclidean spaces have the same orbits as the restricted isotropy representations of semisimple symmetric spaces (to be defined in Proposition 2.1.95). (A more precise notion of *orbit-equivalence* will be introduced in Definition 2.3.18.) Polar actions on irreducible symmetric spaces of compact type were classified by Kollross, Lytchak, Podestà, Thorbergsson, and Gorodski ([Kol02, KL13, PT99, GK16]; see also a survey in [BCO16, Chap. 12]).

The close affinity of symmetric spaces with Lie groups and Lie algebras is also what ultimately led to their classification. Every symmetric space is covered by a simply connected symmetric space, which, in turn, decomposes as a product of a Euclidean space and a number of irreducible symmetric spaces. For simply connected irreducible symmetric spaces, the classification boils down to classifying either real simple Lie algebras or involutive automorphisms of compact real simple Lie algebras. This was carried out by Cartan—who was the first to study symmetric spaces systematically—almost a century ago in [Car26, Car27].

The other half of the focus in this thesis is on homogeneous hypersurfaces. Those are intimately related to isometric *cohomogeneity-one actions*: every such hypersurface can be realized as a nonsingular orbit of an isometric cohomogeneity-one action; and conversely, every nonsingular orbit of a such an action is a homogeneous hypersurface. Cohomogeneity-one actions have been in the limelight because they can be used to construct Riemannian metrics with special properties: for instance, Einstein metrics, metrics with special holonomy, or metrics of positive sectional curvature ([BB82, BS89, GWZ08]). Since most geometric structures on manifolds are governed by systems of partial differential equations, one can take advantage of the following principle: if a system of PDEs is invariant under a cohomogeneity-one action, it can be reduced to ODEs. Somewhat in a similar spirit, one can start with an isometric cohomogeneity-one action $G \curvearrowright M$ and alter the existing metric g along the orbits of G to produce a new G-invariant metric, which might retain some properties of g or even gain new ones—this method has been used to construct special metrics explicitly. Coming back to homogeneous hypersurfaces, one of their prominent features is that they have constant principal curvatures, and they also provide the chief example of what is known as *isoparametric hypersurfaces*. These hypersurfaces of Riemannian manifolds can be defined as regular level sets of certain functions, called *isoparametric functions*. They were introduced at the beginning of the twentieth century, motivated by questions in geometrical optics: they were conceived as a model for wavefronts traveling with constant uniform velocity at each moment of time. In space forms, they are the same as complete hypersurfaces with constant principal curvatures, but these two classes diverge in more complex Riemannian manifolds. One of the biggest—and still unresolved—conundrums in this area of research in the recent decades has been the classification of isoparametric hypersurfaces in spheres (see [BCO16, Sect. 2.9.6] for a short survey). As any other homogeneous submanifold, homogeneous hypersurfaces carry information about the geometry of the ambient manifold, as well as the structure of its isometry group.

There are several good reasons to study homogeneous hypersurfaces—and more generally, homogeneous submanifolds—specifically in the context of symmetric spaces. In general, studying and classifying homogeneous submanifolds is a grueling and perhaps unfeasible task, not least because the isometry group of a generic Riemannian homogeneous (let alone cohomogeneity-one) manifold is poorly understood. On the other hand, for a *semisimple symmetric space* M (which includes all irreducible symmetric spaces), the identity component $I^0(M)$ of the isometry group is semisimple, and M is the quotient of $I^0(M)$ by a compact subgroup of a very special type, called a *symmetric subgroup* (see Definition 2.1.20). This allows one to implement the extensive body of theory about subgroups of semisimple Lie groups (some seminal works[1] include [Dyn52b, Dyn52a, Mos61]) to study and ultimately classify various homogeneous objects in symmetric spaces, including homogeneous submanifolds and cohomogeneity-one actions. This approach is especially fruitful in the case of symmetric spaces of compact type, which led to the aforementioned classification of polar and cohomogeneity-one actions on such spaces—this is also the primary reason why *this thesis focuses predominantly on symmetric spaces of noncompact type*. Another reason why homogeneous submanifolds are usually considered within the framework of symmetric spaces is that they inherit some of the symmetry of the ambient space and often possess extra properties and special structures, which makes the theory richer and more intricate. Finally, the class of homogeneous submanifolds encompasses many other types of submanifolds commonly studied in the theory of symmetric spaces, such as totally geodesic submanifolds, reflective submanifolds, symmetric submanifolds, etc.

Returning to the opening sentence of this introduction, we must explain what we mean by classifying homogeneous hypersurfaces. Since we are chiefly interested in submanifold *geometry*, we do not wish to distinguish between two submanifolds of the same manifold M if there is an isometry of M mapping one onto the other. If this is the case, we say that these two submanifolds are *congruent*. Throughout the thesis, the problem of determining whether two submanifolds are congruent is referred to as *the problem of congruence*, or *the congruence problem*. The questions at the heart of this thesis are then,

[1] The two papers of Dynkin are ubiquitously available online, but they are in Russian. Both were translated by the AMS [Dyn57b, Dyn57a], but these versions are behind a harsher paywall and harder to come by.

> *What are the congruence classes of homogeneous hypersurfaces in a given symmetric space? In which ways do noncongruent homogeneous hypersurfaces differ geometrically?*

These questions are considered in Chapters 4, 5, and 6 of the thesis. They are preceded by Chapter 3, where we establish certain results that underpin the problem of congruence on symmetric spaces of noncompact type.

* * *

In Chapter 3, we lay some groundwork for the study of the congruence problem in the later chapters. Since our main object of interest is symmetric spaces of noncompact type, it is worth paying special attention to the congruence problem in this context. Symmetric spaces of noncompact type are intimately related to noncompact real semisimple Lie algebras. For this reason, they are frequently studied with tools taken from the theory of such Lie algebras: the restricted root space decomposition, the Iwasawa decomposition, the theory of parabolic subalgebras, etc. The first of these is arguably of highest importance, as it underpins most of the other tools and constructions. The relation between a real semisimple Lie algebra and its restricted root system bears a strong resemblance to its complex analog—the classical correspondence between complex semisimple Lie algebras and reduced root systems. Still, there are some notable differences: first, the restricted root system is only effective for examining *noncompact* semisimple Lie algebras; second, it may not be reduced; but most importantly, the dimension of a restricted root space, known as the *multiplicity* of the corresponding root, does not have to be equal to 1. In the complex case, the Isomorphism Theorem proves to be a very powerful tool; it asserts that an isomorphism between complex semisimple Lie algebras can be defined merely on the so-called *canonical generators*, provided that the Cartan matrix is preserved. Among other things, it allows to lift, in a certain sense, every symmetry of the Dynkin diagram of $\mathfrak{g}$ (or, more generally, every automorphism of the root system) to an automorphism of $\mathfrak{g}$. For real semisimple Lie algebras, this instrument is not available—there is no analog of canonical generators to begin with.

The central idea of Chapter 3 is to treat the restricted root multiplicities as a feature, not a bug, and incorporate them into the restricted root system itself. This leads naturally to the notions of a *weighted root system* and a *weighted Dynkin diagram*. It turns out that the noncompact part of a semisimple Lie algebra is completely determined by its weighted restricted root system (or, equivalently, its weighted Dynkin diagram). It would then be sensible to consider only those automorphisms of the restricted root system and the Dynkin diagram that preserve the root weights—we call them *weight-preserving automorphisms*. The main result of the chapter is the following

Theorem 1. *Every weight-preserving automorphism of the restricted root system of a real semisimple Lie algebra admits a lift to an automorphism of that Lie algebra.*

We will define exactly what we mean by a *lift* in Subsection 3.2.1. The proof of the theorem goes as follows. First, we show that it suffices to construct lifts only for the weight-preserving automorphisms of the Dynkin diagram. Next, we reduce the problem to simple noncompact Lie algebras (which correspond to irreducible root systems). After that, we handle the cases when $\mathfrak{g}$ is a complex simple Lie algebra or a split real form; here, the statement can be deduced from the Isomorphism Theorem. In the remaining cases, there is at most one nontrivial weight-preserving automorphism of the Dynkin diagram. Using some general theory of root systems and the classification of symmetric spaces, we show that this nontrivial automorphism equals the negative of the longest element in the Weyl group, whose lift is easy to construct explicitly. Some authors have stated partial versions of this theorem without references (see, e.g., [BT03, p. 11] or [Mur52, p. 111]).

We close the chapter with a reformulation of Theorem 1 in the language of symmetric spaces of noncompact type. Every such space M admits a natural Riemannian metric, called the *Killing metric*, and every other symmetric metric differs from the Killing one by rescaling by some positive real numbers along the de Rham factors of M. These numbers are called the *normalizing constants*. The isometry group of M embeds into the automorphism group of its isometry Lie algebra $\mathfrak{g}$ as an open subgroup. In general, the image of this embedding is a proper subgroup, but for some (generic) choices of normalizing constants it is the whole $\operatorname{Aut}(\mathfrak{g})$; if this is the case, we call the metric *almost Killing*. In the geometric reformulation of Theorem 1, we work out explicitly which weight-preserving automorphisms of the restricted root system of $\mathfrak{g}$ admit a lift to an isometry of M. The general statement is a bit involved (see Corollary 3.3.8), so we only give a simplified version here:

Theorem 2. *Let M be a symmetric space of noncompact type, and assume its metric is almost Killing. Then every weight-preserving automorphism of the restricted root system of the isometry Lie algebra of M admits a lift to an isometry of M.*

Let us illustrate how this result can be used in practice. Many geometric objects on symmetric spaces of noncompact type are constructed within the framework of the restricted root space decomposition and thus rely on the root data. For instance, to every subset of the set Λ of simple roots, one can associate a particular totally geodesic submanifold of M, called a *boundary component*, which will be of paramount importance to us throughout the thesis. It might happen that two subsets of Λ differ by a weight-preserving automorphism of the Dynkin diagram. As we will see in Proposition 3.3.9, the resulting boundary components will then be congruent, essentially due to Theorem 2.

* * *

In Chapter 4, we study the congruence problem in the context of homogeneous codimension-one foliations on reducible symmetric spaces of noncompact type. A *homogeneous foliation* on a Riemannian manifold M is the orbit foliation of a Lie group acting on M isometrically and without singular orbits. If M is a symmetric space of noncompact type, an isometric cohomogeneity-one action on M can have at most one singular orbit (see Proposition 2.3.43), so such actions split naturally into two categories: those that do and those that do not have a singular orbit. Every action in the first category gives rise to a homogeneous codimension-one foliation. On irreducible symmetric spaces of noncompact type, such foliations were classified by Berndt and Tamaru in [BT03]. They used the Iwasawa and restricted root space decompositions (we discuss both in Subsection 2.4.2) to build two different types of homogeneous codimension-one foliations. Let us briefly introduce their constructions.

Let M be a symmetric space of noncompact type, $G = I^0(M)$, and $\mathfrak{g} = \operatorname{Lie}(G)$. Pick any point $o \in M$ and let s_o be the geodesic symmetry of M at o. Then $\theta = \operatorname{Ad}(s_o)$ is a Cartan involution on $\mathfrak{g}$ (to be defined in Definition 2.1.69). If we write B for the Killing form of $\mathfrak{g}$, the form $B_\theta(X, Y) = -B(X, \theta Y)$ becomes an inner product on $\mathfrak{g}$. We decompose

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the +1- and −1-eigenspaces of θ and pick a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. The restricted root system Σ of $\mathfrak{g}$ lives in $\mathfrak{a}^*$; we denote its Dynkin diagram by DD. Any choice of positive roots in Σ gives rise to an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is a nilpotent subalgebra defined as the sum of all positive root spaces.

For any subalgebra of the solvable Lie algebra $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, the corresponding Lie subgroup induces a homogeneous foliation on M; we call any foliation that can be constructed in this way a *standard foliation*, and we refer to its leaf through o as the *base leaf*. For example, if we take any one-dimensional subspace $\ell \subseteq \mathfrak{a}$, its orthogonal complement $\mathfrak{s}_\ell$ in $\mathfrak{s}$ is a subalgebra, and it gives rise to a standard codimension-one foliation, denoted by $\mathcal{F}_\ell$. Similarly, if α_i is a simple root, removal of a one-dimensional subspace (does not matter which one) from the restricted root space $\mathfrak{g}_{\alpha_i}$ produces a subalgebra of $\mathfrak{s}$, whose corresponding standard foliation $\mathcal{F}_{\alpha_i}$ also has codimension 1. The notion of congruence easily extends to this context: two foliations of M are called *congruent* if there exists an isometry of M identifying their leaves (see Definition 2.3.18 for a precise formulation). For instance, the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\alpha_i}$ are never congruent to each other: the leaves of $\mathcal{F}_\ell$ are all pairwise congruent, whereas $\mathcal{F}_{\alpha_i}$ has a unique minimal leaf (the base leaf). Now suppose M is irreducible. The main results of [BT03] are as follows:

(a) The foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\alpha_i}$ exhaust the list of all homogeneous codimension-one foliations on M up to congruence.

(b) Given two lines $\ell, \ell' \subseteq \mathfrak{a}$ (resp., two simple roots α_i, α_j), the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ (resp., $\mathcal{F}_{\alpha_i}$ and $\mathcal{F}_{\alpha_j}$) are congruent if and only if ℓ and ℓ' (resp, α_i and α_j) differ by a weight-preserving automorphism of DD.

In (b), we tacitly use the fact that every weight-preserving automorphism of DD naturally extends to a linear operator on $\mathfrak{a}$. The authors used these results in their subsequent article [BDRT10] with Díaz-Ramos to obtain a classification result for the more general class of homogeneous hyperpolar foliations on all (possibly reducible) symmetric spaces of noncompact type. A homogeneous foliation on M is called *hyperpolar* if there exists a flat submanifold of M, called a *section*, that intersects all the leaves and does so orthogonally. Every homogeneous codimension-one foliation on M is hyperpolar: a section can be constructed by launching a geodesic from any point orthogonally to its leaf; it can be shown that such a geodesic will actually cross all the leaves orthogonally. When applied to codimension-one foliations, the main result of [BDRT10] ensures that every homogeneous codimension-one foliation on M is congruent to either some $\mathcal{F}_\ell$ or some $\mathcal{F}_{\alpha_i}$—except this time the space is allowed to be reducible. To complete the classification, one needs to tell when two foliations of the form $\mathcal{F}_\ell$ (or $\mathcal{F}_{\alpha_i}$) are mutually congruent. We achieve this by utilizing the notion of a weight-preserving automorphism conceived in Chapter 3:

Theorem 3. *Let M be a symmetric space of noncompact type whose Riemannian metric is almost Killing. Two homogeneous codimension-one foliations $\mathcal{F}_\ell, \mathcal{F}_{\ell'}$ (resp., $\mathcal{F}_{\alpha_i}, \mathcal{F}_{\alpha_j}$) on M are congruent if and only if ℓ and ℓ' (resp., α_i and α_j) differ by a weight-preserving automorphism of the Dynkin diagram of M.*

If M is irreducible, every automorphism of its Dynkin diagram is automatically weight-preserving (Theorem 3.2.10(2)), so Theorem 3 is a direct generalization of statement (b) above. Our proof combines geometric and algebraic methods and is an extension of Berndt and Tamaru's original proof of (b) in [BT03]. It is based on a painstaking analysis of the Lie bracket relations between the subalgebras $\mathfrak{a}$ and $\mathfrak{n}$, and it also relies on the fact that a symmetric space of noncompact type is determined by its weighted Dynkin diagram.

* * *

In Chapter 5, we obtain an explicit classification of proper isometric cohomogeneity-one actions (and thus homogeneous hypersurfaces) on a number of irreducible rank-two symmetric spaces of noncompact type, namely on $\mathrm{SL}(3,\mathbb{H})/\mathrm{Sp}(3)$, $\mathrm{SO}(5,\mathbb{C})/\mathrm{SO}(5)$, and $\mathrm{Gr}^*(2,\mathbb{C}^{n+4})$, $n \geq 1$.

Compared to the compact case, the theory of cohomogeneity-one actions on symmetric spaces of noncompact type is substantially more convoluted, which is why the saga of classification of these actions has been ongoing for over two decades. Let $H \curvearrowright M$ be such an action. Without loss of generality, we may assume that H is a connected closed subgroup of $I^0(M)$. As we mentioned earlier, H can have at most one singular orbit. If there is no singular orbit, the orbits form a homogeneous codimension-one foliation, the classification of which is discussed in Chapter 4. If the action has a singular orbit, there are two things that can happen. Let $H' \subset I^0(M)$ be a maximal proper connected Lie subgroup containing H. In [Mos61], Mostow showed that such a maximal subgroup is either reductive or the identity component of a parabolic subgroup of G (see Subsection 2.4.3 for a discussion of parabolic subalgebras and subgroups). In the first case, Berndt and Tamaru showed in [BT13] that H and H' have the same orbits, and the singular orbit must be totally geodesic. For irreducible spaces, cohomogeneity-one actions with a totally geodesic singular orbit were classified by the authors in [BT04]. They showed that, apart from five exceptional actions, the singular orbit must be a reflective submanifold (Definition 2.2.26). By relying on Leung's classification of such submanifolds in irreducible symmetric spaces of compact type ([Leu75, Leu79a]), they figured out which reflective submanifolds arise as singular orbits of cohomogeneity-one actions, which let them complete the classification. If H lies in a parabolic subgroup of G, its singular orbit may not be totally geodesic. A novel approach was needed to generate such actions.

In [BT13], the authors invented two new ways of constructing cohomogeneity-one actions on noncompact symmetric spaces. The first one is called the *canonical extension*, and it is a procedure that takes isometric actions on the boundary components of M and naturally extends them to global actions on M. The second one is known as the *nilpotent construction*, and it is arguably the more intricate of the two. This method concerns representations of certain reductive subgroups of G; we do not attempt to lay it out here and refer to Subsection 5.1.1 for details. The main result of [BT13] asserts that any cohomogeneity-one action on M with a non-totally-geodesic singular orbit arises via one of these two constructions.

In a recent paper [DRDVO23], Díaz-Ramos, Domínguez-Vázquez, and Otero managed to dispense with the irreducibility assumption on M altogether. In the above step where one takes a maximal proper subgroup of G, they used a result of Dynkin (Th. 15.1 in [Dyn52b] or [Dyn57b]) that states that every maximal proper subalgebra of a semisimple Lie algebra $\bigoplus_{i=1}^{k} \mathfrak{g}_i$ (here $\mathfrak{g}_i$ are simple) has to be of one of two forms: $\mathfrak{h}_i \oplus \bigoplus_{j \neq i} \mathfrak{g}_j$, where $\mathfrak{h}_i \subset \mathfrak{g}_i$, or $\mathfrak{h}_{i,j} \oplus \bigoplus_{l \neq i,j} \mathfrak{g}_l$, where $\mathfrak{g}_i \simeq \mathfrak{g}_j$ and $\mathfrak{h}_{i,j} \subset \mathfrak{g}_i \oplus \mathfrak{g}_j$ is a *diagonal* subalgebra. They also showed, roughly speaking, that the composition of a nilpotent construction with a canonical extension is a nilpotent construction, whereas Berndt and Tamaru showed in [BT13] that the composition of two canonical extensions is again a canonical extension. As a result of all these works, the search for cohomogeneity-one actions on a given symmetric space of noncompact type reduces to a problem in the representation theory of reductive Lie groups. The complexity and obscurity of that problem depends on the space in question but generally tends to grow along with the rank of a space. As of today, the nilpotent construction has only yielded two new actions on spaces of rank > 1 that do not arise via any other methods (both described in [BT13]). The primary result of Chapter 5 is the following

Theorem 4. *Every proper isometric cohomogeneity-one action with a non-totally-geodesic singular orbit on the symmetric spaces*

$$\mathrm{SL}(3, \mathbb{H})/\mathrm{Sp}(3), \quad \mathrm{SO}(5, \mathbb{C})/\mathrm{SO}(5), \quad \mathrm{Gr}^*(2, \mathbb{C}^{n+4}),\ n \geq 1,$$

arises via the canonical extension.

In other words, the nilpotent construction produces no new actions on these spaces. We consider the three of them individually in Sections 5.2 to 5.4, and the methods we use in the proofs are space-specific. For each of the spaces, we first give an explicit description of the actions with a totally geodesic singular orbit, as well as those arising by canonical extension, and then proceed to deal with the nilpotent construction—which, as we said before, is basically a problem in the representation theory of reductive groups. For $\mathrm{SL}(3, \mathbb{H})/\mathrm{Sp}(3)$, it involves the standard representation of $\mathrm{Sp}(2)\mathrm{Sp}(1)$ on $\mathbb{H}^2$, which we handle by using the notion of quaternion-Kähler angle and the theory developed in [DRDVRV21]. For $\mathrm{SO}(5, \mathbb{C})/\mathrm{SO}(5)$, we utilize some techniques established in [BDV15] specifically for the purposes of solving the nilpotent construction problem. Lastly, in the case of noncompact complex Grassmannians of two-planes, we first use some ad-hoc arguments to simplify the nilpotent construction, but then opt for a rather head-on approach. Here, the problem of congruence is particularly interesting because of the unique geometric characteristics of the space. Together with its compact dual (the duality for symmetric spaces will be discussed in Subsection 2.1.5), the noncompact complex Grassmannian of two-planes is the only semisimple symmetric space that is both Hermitian and quaternion-Kähler (see Subsection 2.5.1). The interplay between these two structures provides some rather fine tools for distinguishing between various submanifolds of M; we will use this to deduce that certain actions on M are not mutually congruent.

* * *

Finally, in Chapter 6, we examine the topic of the thesis through the lens of complex geometry. That is to say, we study homogeneous *complex* hypersurfaces in *Hermitian symmetric spaces* (which are simply symmetric spaces that are also Kähler manifolds). The property of being of real codimension 2 sets complex hypersurfaces quite far apart from their real counterparts.

Historically, one of the first results that motivated the study of homogeneous complex hypersurfaces is perhaps their classification in complex space forms by Smyth and Nomizu in [Smy68, NS68]: they proved that, up to congruence, the submanifolds $\mathbb{C}^{n-1} \subset \mathbb{C}^n$, $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$, and $\mathbb{C}P^{n-1}, Q^{n-1} \subset \mathbb{C}P^n$ exhaust the list of homogeneous complex hypersurfaces in simply connected complex space forms. Here Q^{n-1} is the standard nonsingular complex projective quadric. Another immediate example of such a hypersurface is a totally geodesic $Q^{n-1} \subset Q^n$. Further progress in this direction was made by a group of Japanese mathematicians: first, Sakane and Kimura discovered two more examples in [Sak85] and [Kim79], namely $\mathrm{Sp}(n)/\mathrm{Sp}(n-2)\mathrm{U}(2) \subset \mathrm{Gr}(2, \mathbb{C}^{2n})$

and $F_4/\mathrm{Spin}(7)\mathrm{U}(1) \subset E_6/\mathrm{Spin}(10)\mathrm{U}(1)$; shortly afterwords, Konno used methods from algebraic geometry to show that the above five examples (excluding $\mathbb{C}^{n-1} \subset \mathbb{C}^n$ and $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$) exhaust the list of all homogeneous complex hypersurfaces in complex flag manifolds with $b_2 = 1$ ([Kon88]). These spaces include all irreducible Hermitian symmetric spaces of compact type, and they can all be represented as a quotient of a complex simple Lie group by a parabolic subgroup. It follows a posteriori from Konno's result that if such a space admits a homogeneous complex hypersurface, then it is a Hermitian symmetric space. The only shortcoming of Konno's classification (for us) is that it is not up to congruence—he only shows when a complex flag manifold can be embedded into another such manifold with $b_2 = 1$ as a complex hypersurface. We refine his result in the case of homogeneous complex hypersurfaces in Q^n and show that they are all congruent to the standard totally geodesic $Q^{n-1} \subset Q^n$. We also set up the congruence problem for the remaining two spaces $\mathrm{Gr}(2, \mathbb{C}^{2n})$ and $E_6/\mathrm{Spin}(10)\mathrm{U}(1)$ and reformulate it in the language of representation theory.

Homogeneous complex hypersurfaces in Hermitian symmetric spaces of compact type are an interesting object of study from several geometric perspectives. To begin with, every such hypersurface can be realized as a singular orbit of an isometric cohomogeneity-one action on the ambient space. What is more, the principal orbits of these actions are what is known as *contact hypersurfaces*. In particular, the maximal holomorphic distribution of each such orbit is a contact structure. We discuss this briefly at the end of Subsection 6.2.4. Eventually, for those homogeneous complex hypersurfaces $S \subset M$ that are not totally geodesic, the other singular orbit S' of the corresponding cohomogeneity-one action (also known as the *focal manifold* of S) is a projective space over a normed real division algebra. The ambient space M can be described as the *complexification* of S'. We discuss this in Subsection 6.2.5.

In the second half of the chapter, we turn our attention toward homogeneous complex hypersurfaces in Hermitian symmetric spaces of noncompact type. This topic does not seem to have been studied to any notable degree. Similarly to the compact case, we have totally geodesic (and hence homogeneous) complex hypersurfaces $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$ and $\mathrm{Gr}^*(2, \mathbb{R}^{n+1}) \subset \mathrm{Gr}^*(2, \mathbb{R}^{n+2})$, $n \geq 3$. Using the notion of *index* of a symmetric space (discussed in Subsection 6.2.2), one can show that these are the only complete connected totally geodesic complex hypersurfaces in irreducible Hermitian symmetric spaces of noncompact type. By virtue of the canonical extension procedure from Chapter 5, one can use these two examples to generate more homogeneous complex hypersurfaces on any Hermitian symmetric space of noncompact type, provided it admits a complex boundary component isometric to $\mathbb{C}H^n$ or $\mathrm{Gr}^*(2, \mathbb{R}^{n+2})$; we show that such a boundary component always exists. Finally, we study homogeneous complex hypersurfaces within the framework of the Iwasawa and restricted root space decompositions. If $\mathfrak{h} \subset \mathfrak{s}$ is a subalgebra, it might happen that the o-orbit of its corresponding Lie subgroup is a complex hypersurface. In Subsection 6.3.2, we classify all such subalgebras. Equivalently, in the language of Chapter 4, we classify those homogeneous complex hypersurfaces that arise as the base leaf of a standard foliation on M. As it turns out, those are very scarce, and they are also closely related to the totally geodesic complex hypersurfaces above via the canonical extension:

Theorem 5. *Let M be an irreducible Hermitian symmetric space of noncompact type with restricted root system Σ. The number of congruence classes of standard codimension-2* *foliations on M with a complex base leaf is 2 if $\Sigma \simeq C_r$ and 1 if $\Sigma \simeq (BC)_r$. The base leaf of every such foliation can be obtained by canonical extension of a totally geodesic complex hypersurface in a boundary component of M isometric to $\mathbb{C}H^n$ or $\mathrm{Gr}^*(2, \mathbb{R}^{n+2})$, $n \geq 3$.*

The proof of this theorem is fleshed out in Subsections 6.3.2 and 6.3.3. To that end, we had to devise some novel techniques to study Hermitian symmetric spaces of noncompact type. The two key results underpinning the proof are:

(a) A relation between the almost complex structure of M and the restricted root space decomposition of its isometry Lie algebra (Theorem 6.3.12);

(b) A relation between the lift of the almost complex structure of M to the solvable Lie algebra $\mathfrak{s}$ and the Lie-algebraic structure of $\mathfrak{s}$ (Lemma 6.3.15).

We close the chapter with some conjectures and ideas for possible generalizations of Theorem 5.

Chapter 2

SYMMETRIC SPACES AND ISOMETRIC ACTIONS

This chapter serves as a collection of preliminaries for the rest of the thesis. We have opted for a more detailed and thorough exposition for a number of reasons. Firstly, this was driven by a desire to make the thesis more self-sustained and avoid excessive referencing. Second, in this chapter we establish the bulk of the notation and definitions that will be required later on. We also provide numerous examples to aid understanding. Finally, even though most of the material of this chapter can be found elsewhere in the literature, it appears to be scattered among many textbooks and papers. Some of the results discussed here do not seem to appear in other sources—at least according to our knowledge; most notably, this is Proposition 2.1.52 on holonomy-induced foliations on symmetric spaces, (a rigorous proof of) Proposition 2.1.60 on the isometry group of certain Riemannian products, and the property of having compact Euclidean part and its equivalent characterizations (Proposition 2.1.97). This chapter contains many references, but three sources really stand out:

- [Hel01] is our go-to reference for the general theory of symmetric spaces.
- [Kna02] covers most of the theory of Lie groups and Lie algebras underpinning this thesis. With its in-depth discussion of noncompact semisimple Lie algebras, it is also an excellent reference for the theory of symmetric spaces of noncompact type.
- [KN96a, KN96b] fully cover our needs when it comes to holonomy and homogeneous spaces. They also contain a thorough discussion of symmetric spaces, often complementary to [Hel01].

Since they are so ubiquitous, we will generally omit references to these textbooks in this chapter. Due to the preparatory nature of the chapter, we only give sporadic proofs and rely on references most of the time. The layout of the chapter is as follows:

- In Section 2.1, we go through the basics of symmetric space, paying special attention to Riemannian symmetric pairs and orthogonal symmetric Lie algebras, types of symmetric spaces, holonomy, and irreducibility.
- In Section 2.2, we review various types of submanifolds in symmetric spaces and discuss their properties. We also work out a handy formula for the second fundamental form of a homogeneous submanifold.
- In Section 2.3, we turn our attention to the theory of proper isometric actions and

homogeneous foliations, with a focus on polar, hyperpolar, and cohomogeneity-one actions.

- In Section 2.4, we home in on the primary object of interest in this thesis: symmetric spaces of noncompact type. After establishing their relation to noncompact real semisimple Lie algebras, we recall some basic tools used to study such Lie algebras: the restricted root space and Iwasawa decompositions as well as the theory of parabolic subalgebras and subgroups.

- Lastly, in Section 2.5, we discuss symmetric spaces endowed with extra geometric structures. The two types of spaces we are going to be interested in are Hermitian and quaternion-Kähler symmetric spaces.

2.1. Symmetric spaces

The first (and largest) section of this chapter is dedicated to the general theory of symmetric spaces. We go through the apparatus of Riemannian symmetric pairs and orthogonal symmetric Lie algebras and discuss various aspects of the theory such as type, rank, holonomy, irreducibility, duality, and eventually the classification. The primary reference for this section is [Hel01].

2.1.1. Symmetric and locally symmetric spaces

The bridge between Riemannian geometry and Lie theory—which is an indispensable component of the theory of symmetric spaces—begins with the isometry group of a Riemannian manifold. In the following proposition, we coalesce some elementary results about the isometry group and its Lie algebra. This is largely proven in [KN96a, Th.VI.3.4] and [DR08][1].

Proposition 2.1.1. *Let M be a Riemannian manifold with $|\pi_0(M)| < \infty$, and let $I(M)$ be its isometry group.*

(a) *When endowed with the compact-open topology, $I(M)$ is a Lie group, and its action on M is smooth. An isometric action of a Lie group G on M is smooth if and only if the corresponding morphism $G \to I(M)$ is smooth.*

(b) *Write $\boldsymbol{\mathfrak{i}(M)}$ for* $\mathrm{Lie}(I(M))$. *Given $X \in \mathfrak{i}(M)$, let $\widehat{X} \in \mathfrak{X}(M)$ be its corresponding fundamental vector field:*

$$\widehat{X}_p = \frac{d}{dt}\bigg|_{t=0} \exp(tX) \cdot p = d(\pi_p)_e(X),$$

where $\pi_p \colon I(M) \to M, g \mapsto g \cdot p$. Then $\widehat{X}$ lies in the Lie subalgebra $\mathcal{K}(M) \subseteq \mathfrak{X}(M)$ of Killing vector fields, and the map[2] $\mathfrak{i}(M) \to \mathcal{K}(M), X \mapsto \widehat{X}$, is an injective anti-homomorphism of Lie algebras. Moreover, if M is complete, this map is an anti-isomorphism.

[1]Note that the assumption on completeness of M in this paper is redundant.
[2]This is also known as the *infinitesimal generator* of the action $I(M) \curvearrowright M$.

(c) *Suppose M is connected and let $f, f' \in I(M)$ be such that $f(p) = f'(p)$ and $df_p = df'_p$ for some $p \in M$. Then $f = f'$. In particular, if an isometry f fixes some point p and $df_p = \mathrm{Id}_{T_pM}$, then $f = \mathrm{Id}_M$.*

(d) *Let $H \subseteq I(M)$ be a Lie subgroup. The action $H \curvearrowright M$ is proper if and only if H is a closed subgroup. In particular, the actions of $I(M)$ and $I^0(M)$ on M are proper and the stabilizers $I(M)_p$ and $I^0(M)_p$ of any point p are compact (thus $\pi_0(I(M))$ is also finite).*

(e) *If M is compact, then so is $I(M)$.*

We will call the elements of $I^0(M)$ ***inner isometries****.*

Let M be a connected Riemannian manifold and $p \in M$. Take $0 < r < \mathrm{inj}(p)$, where the latter is the injectivity radius of M at p. The exponential map $\exp_p\colon T_pM \to M$ restricts to a diffeomorphism between $B_r(0) \subset T_pM$ and $B_r(p) \subseteq M$. The linear isometry $v \mapsto -v$ of $B_r(0)$ exponentiates to a diffeomorphism $\exp(tv) \mapsto \exp(-tv)$ of $B_r(p)$, called a **local geodesic symmetry** of M at p.

Definition 2.1.2. A connected Riemannian manifold is called a (**Riemannian**) **locally symmetric space** if it satisfies the following equivalent conditions:

(i) For every $p \in M$, there exists $0 < r < \mathrm{inj}(p)$ such that the local geodesic symmetry of $B_r(p)$ is an isometry.

(ii) The curvature tensor of M is parallel: $\nabla R = 0$.

Definition 2.1.3. A connected Riemannian manifold M is called a (**Riemannian**) **symmetric space** if for every $p \in M$ there exists $s_p \in I(M)$ that fixes p and satisfies the following equivalent conditions:

(i) s_p is involutive and p is its isolated fixed point.

(ii) $d(s_p)_p = -\mathrm{Id}_{T_pM}$.

(iii) s_p reverses geodesics through p: if $\gamma_v(t) = \exp(tv)$, then $s_p \circ \gamma_v = \gamma_{-v}$ for every $v \in T_pM$.

If exists, such s_p is unique and it is called the (**global**) **geodesic symmetry** of M at p.

Some immediate examples of symmetric spaces include the Euclidean space $\mathbb{E}^n$, the sphere $\mathbb{S}^n$, and the real hyperbolic space $\mathbb{R}H^n$. We will see plenty more examples below (see Examples 2.1.35 to 2.1.38). Using the geodesic symmetries, one can show that

Proposition 2.1.4. *A symmetric space is a Riemannian homogeneous space. In particular, it is complete.*

If we already know that M is homogeneous, it suffices to check the existence of geodesic symmetries at just one point:

Proposition 2.1.5. *Let M be a connected Riemannian homogeneous space. Assume that M admits a geodesic symmetry at some point p. Then M is a symmetric space.*

If a Riemannian manifold (M, g) is symmetric (resp., homogeneous), we will sometimes express iy by saying that its *metric* g is *symmetric* (resp., *homogeneous*). Clearly, a symmetric space is locally symmetric, but the converse is not necessarily true. For example, a compact oriented surface M of genus $g \geq 2$ endowed with a metric of constant curvature is locally symmetric because it is locally isometric to $\mathbb{R}H^2$ (of some radius). But the isometry group $I(M)$ is finite, so M is not homogeneous, let alone symmetric. Nonetheless, we have the following result, which says, roughly speaking, that locally symmetric spaces are not far from being symmetric:

Proposition 2.1.6. *Let M be a locally symmetric space.*

(a) *For every point $p \in M$, there exists a neighborhood U and a symmetric space N such that U is isometric to some open subspace of N.*

(b) *If M is complete, its universal Riemannian covering space is a symmetric space. In particular, if M is also simply connected, then it is symmetric.*

Part (a) of the proposition justifies the term *locally symmetric space.* The property of being symmetric withstands some basic geometric constructions:

Proposition 2.1.7. *Let M be a Riemannian manifold.*

(a) *If $M = M_1 \times \cdots \times M_k$ is a Riemannian product, then M is a symmetric space if and only if each M_i is.*

(b) *If M is a symmetric space, then so is any Riemannian covering space of M.*

The most basic geometric invariant of a symmetric space, besides the dimension, is its rank.

Definition 2.1.8. The **rank** of a symmetric space M, denoted by $\mathbf{rk}(\boldsymbol{M})$, is the maximal dimension of a flat totally geodesic submanifold of M.

Agreement. Throughout the thesis, all submanifolds in smooth manifolds are assumed to be smooth and immersed, and all actions on smooth manifolds (resp., representations) are assumed to be smooth actions by (resp., representations of) Lie groups, unless otherwise stated.

We will discuss the definition of the rank in more detail in Subsection 2.2.1 (see Corollary 2.2.24). Here we just mention a few of its basic properties.

Proposition 2.1.9. *Let M be a symmetric space.*

(a) *If $M = M_1 \times \cdots \times M_k$ is a Riemannian product, then $\operatorname{rk}(M) = \sum_{i=1}^k \operatorname{rk}(M_i)$.*

(b) *If M' is a Riemannian covering space of M, then $\operatorname{rk}(M') = \operatorname{rk}(M)$.*

Digression: compact Lie algebras

Before we go further, we need to briefly discuss some relevant parts of Lie theory; we refer to [Kna02] for proofs and details. Let $\mathfrak{g}$ be a Lie algebra. (All Lie algebras, vector spaces, and representations in this thesis are going to be finite-dimensional over $\mathbb{R}$ by default). We will usually denote the Cartan-Killing form by B (or $B^{\mathfrak{g}}$ if there is a chance of ambiguity). We will write $\mathbf{Inn}(\boldsymbol{\mathfrak{g}})$ for the (possibly nonclosed) connected Lie subgroup of $\operatorname{Aut}(\mathfrak{g})$ corresponding to the subalgebra $\operatorname{ad}(\mathfrak{g})$ of $\operatorname{Der}(\mathfrak{g}) = \operatorname{Lie}(\operatorname{Aut}(\mathfrak{g}))$. If G is any connected Lie algebra with $\operatorname{Lie}(G) = \mathfrak{g}$, then $\operatorname{Inn}(\mathfrak{g}) = \operatorname{Ad}(G)$.

Definition 2.1.10. A real Lie algebra $\mathfrak{g}$ is called **compact** if the group $\operatorname{Inn}(\mathfrak{g})$ is compact.

Proposition 2.1.11. *The following conditions on a Lie algebra* $\mathfrak{g}$ *are equivalent:*

(i) $\mathfrak{g}$ *is compact.*

(ii) *There exists a compact Lie group* G *with* $\mathrm{Lie}(G) \simeq \mathfrak{g}$.

(iii) $\mathfrak{g}$ *is reductive and its semisimple part* $\mathfrak{g}_{\mathrm{ss}} = [\mathfrak{g}, \mathfrak{g}]$ *is compact.*

(iv) $\mathfrak{g}$ *admits an invariant inner product.*

(v) $\mathfrak{g}$ *is isomorphic to a Lie subalgebra of* $\mathfrak{so}(n)$ *for some* n.

If these conditions are satisfied, then the Killing form B *is negative semi-definite.*

Corollary 2.1.12. *Every subalgebra and quotient of a compact Lie algebra is also compact.*

Proposition 2.1.13. *The following conditions on a Lie algebra* $\mathfrak{g}$ *are equivalent:*

(i) $\mathfrak{g}$ *is compact semisimple.*

(ii) *Every connected Lie group* G *with* $\mathrm{Lie}(G) \simeq \mathfrak{g}$ *is compact.*

(iii) $\mathfrak{g}$ *is a direct sum of compact simple Lie algebras.*

(iv) B *is negative-definite.*

Let $\mathfrak{g}$ be any Lie algebra and $\mathfrak{k} \subseteq \mathfrak{g}$ a subalgebra. We write $\mathbf{Inn}_{\mathfrak{g}}(\mathfrak{k})$ for the connected Lie subgroup of $\mathrm{Inn}(\mathfrak{g})$ corresponding the subalgebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ of $\mathrm{ad}(\mathfrak{g})$. If G is any Lie group with $\mathrm{Lie}(G) = \mathfrak{g}$ and $K \subseteq G$ is the connected Lie subgroup corresponding to $\mathfrak{k}$, then $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k}) = \mathrm{Ad}_G(K)$.

Definition 2.1.14. A subalgebra $\mathfrak{k}$ of a Lie algebra $\mathfrak{g}$ is called **compactly embedded** if the group $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k})$ is compact.

Remark 2.1.15. In Definition 2.1.14, it does not matter whether we treat $\mathrm{Inn}_{\mathfrak{g}}(\mathfrak{k})$ as a Lie subgroup of $\mathrm{Inn}(\mathfrak{g})$ or of $\mathrm{Aut}(\mathfrak{g})$, because the resulting topology and smooth structure are the same.

Proposition 2.1.16. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{k} \subseteq \mathfrak{g}$ *a subalgebra. Consider the following conditions:*

(i) $\mathfrak{k}$ *is compactly embedded in* $\mathfrak{g}$.

(ii) *There exists a Lie group* G *with* $\mathrm{Lie}(G) \simeq \mathfrak{g}$ *such that the connected Lie subgroup* $K \subseteq G$ *corresponding to* $\mathfrak{k}$ *is compact.*

(iii) $\mathfrak{g}$ *admits a* $\mathfrak{k}$*-invariant inner product.*

(iv) *The restriction of* $B^{\mathfrak{g}}$ *to* $\mathfrak{k}$ *is negative semi-definite and its kernel coincides with* $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k}$.

(v) $\mathfrak{k}$ *is compact.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv),(v).

2.1.2. Riemannian symmetric pairs and orthogonal symmetric Lie algebras

Here, we lay out how symmetric spaces can be studied effectively by means of Lie theory.

Definition 2.1.17. A **$\mathbb{Z}/2\mathbb{Z}$-grading** on $\mathfrak{g}$ is a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ (indices taken mod 2). In other words, it means that

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_0, \quad [\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1.$$

In particular, $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$.

Observation 2.1.18. If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}/2\mathbb{Z}$-grading, the summands $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are orthogonal with respect to B.

Proposition 2.1.19. *The following pieces of data for $\mathfrak{g}$ are equivalent:*

(i) *A $\mathbb{Z}/2\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.*

(ii) *An involutive automorphism $\tau \in \mathrm{Aut}(\mathfrak{g})$.*

Under this correspondence, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are the $(+1)$- and (-1)-eigenspaces of τ, respectively.

Definition 2.1.20. A pair (G, K) consisting of a connected Lie group G and a closed subgroup $K \subseteq G$ is called a **Riemannian symmetric pair** if it satisfies the following two conditions:

(a) The subgroup $\mathrm{Ad}_G(K)$ of $\mathrm{Inn}(\mathfrak{g})$ is compact. (Here $\mathfrak{g} = \mathrm{Lie}(G)$.)

(b) K is a **symmetric** subgroup of G: there exists an involutive Lie group automorphism Θ of G such that $(G^\Theta)^0 \subseteq K \subseteq G$, or in other words, K is an open subgroup of G^Θ (the subgroup of fixed points of Θ).

Two Riemannian symmetric pairs (G, K) and (G', K') are called **isomorphic** if there exists an isomorphism $G \xrightarrow{\sim} G'$ sending K onto K'.

Let $\mathfrak{g} = \mathrm{Lie}(G), \mathfrak{k} = \mathrm{Lie}(K)$, and $\theta = \Theta_*$. Plainly, θ is an involutive automorphism of $\mathfrak{g}$. Condition (b) in Definition 2.1.20 simply means that $\mathfrak{k}$ coincides with the fixed point subalgebra of θ. The following is an infinitesimal version of Definition 2.1.20:

Definition 2.1.21. A pair $(\mathfrak{g}, \theta)$ consisting of a real Lie algebra $\mathfrak{g}$ and its involutive automorphism θ is called an **orthogonal symmetric Lie algebra** if the fixed point subalgebra $\mathfrak{k}$ of θ is compactly embedded in $\mathfrak{g}$. Two orthogonal symmetric Lie algebras $(\mathfrak{g}, \theta)$ and $(\mathfrak{g}', \theta')$ are called **isomorphic** if there exists a Lie algebra isomorphism $\varphi \colon \mathfrak{g} \to \mathfrak{g}'$ such that $\theta' \circ \varphi = \varphi \circ \theta$.

In essence, Riemannian symmetric pairs are a Lie-theoretic tool that allows to study symmetric spaces and their geometry globally, whereas orthogonal symmetric Lie algebras are designed for local investigation of symmetric spaces.

Let (G, K) be a Riemannian symmetric pair with a fixed involution Θ as in Definition 2.1.20. If we write $\mathfrak{g} = \mathrm{Lie}(G)$ and $\theta = \Theta_*$, then $(\mathfrak{g}, \theta)$ is clearly an orthogonal symmetric Lie algebra, and it does not depend on the choice of Θ (with K fixed) up to isomorphism. What is more, we will see in Proposition 2.1.25 that, under a mild assumption, Θ is unique. For this reason, we routinely omit the step of choosing Θ, allowing it to be any, and call $(\mathfrak{g}, \theta)$ *the orthogonal symmetric Lie algebra of* (G, K). If $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie algebra, it is customary to denote the $\mathbb{Z}/2\mathbb{Z}$-grading of θ by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, which we are going to do from now on.

Notation. Suppose $(\mathfrak{g}, \theta)$ an orthogonal symmetric Lie algebra. Given a vector $X \in \mathfrak{g}$, we are going to write $\boldsymbol{X_{\mathfrak{k}}} = \frac{1}{2}(X + \theta X) \in \mathfrak{k}$ and $\boldsymbol{X_{\mathfrak{p}}} = \frac{1}{2}(X - \theta X) \in \mathfrak{p}$ for its components with respect to the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We will sometimes refer to these components as the **$\mathfrak{k}$-part** and **$\mathfrak{p}$-part** of X. Likewise, given a subspace $V \subseteq \mathfrak{g}$, we will write $\boldsymbol{V_{\mathfrak{k}}} = \{X_{\mathfrak{k}} \mid X \in V\}$ and $\boldsymbol{V_{\mathfrak{p}}} = \{X_{\mathfrak{p}} \mid X \in V\}$.

Let (G, K) be a Riemannian symmetric pair and $M = G/K$ the corresponding homogeneous space. We will see shortly that any G-invariant Riemannian metric makes M into a symmetric space. To begin with, note that we have a Lie algebra anti-homomorphism $\mathfrak{g} \to \mathfrak{X}(M)$ that sends X to its corresponding fundamental vector field $\widehat{X}$. Let $o = eK \in M$ and write $\pi \colon G \twoheadrightarrow M, g \mapsto g \cdot o$, for the orbit map of the action $G \curvearrowright M$ at o. It gives rise to a linear map

$$d\pi_e \colon \mathfrak{g} \twoheadrightarrow T_oM, X \mapsto \widehat{X}_o,$$

which has $\mathfrak{k}$ as its kernel. If we have Θ fixed, then $d\pi_e$ restricts to an isomorphism between $\mathfrak{p}$ and T_oM. Throughout the thesis, we are going to tacitly identify $\mathfrak{p}$ with T_oM by means of this isomorphism. The adjoint action of K on $\mathfrak{g}$ preserves $\mathfrak{k}$ and $\mathfrak{p}$, which implies:

Corollary 2.1.22. *The splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a reductive decomposition for the homogeneous space $M = G/K$. The map $\mathfrak{p} \xrightarrow{\sim} T_oM$ is an isomorphism between the adjoint representation of K on $\mathfrak{p}$ and its isotropy representation on T_oM.*

Let us write $I \subseteq G$ for the *ineffectiveness kernel* of the action $G \curvearrowright M$, i.e., the subgroup of elements that act trivially on M. Clearly, $I \subseteq K$, and one can show that I is *the* maximal normal subgroup of G contained in K. In particular, $Z \cap K \subseteq I$, where $Z = Z(G)$. Thanks to Proposition 2.1.1(c), I can be alternatively described as the kernel of the isotropy representation $K \to GL(T_oM)$. If $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie algebra, we write $\mathfrak{i}$ for the kernel of the adjoint representation $\mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$. Similarly, this is *the* maximal ideal of $\mathfrak{g}$ contained in $\mathfrak{k}$. If $(\mathfrak{g}, \theta)$ comes from (G, K), we have $\mathfrak{i} = \mathrm{Lie}(I)$. To single out some better-behaving Riemannian symmetric pairs and orthogonal symmetric Lie algebras, we introduce the following

Definition 2.1.23. Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra. We call it

- **effective** if $\mathfrak{i} = \{0\}$;
- **weakly effective** if $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k} = \{0\}$.

If (G, K) is a Riemannian symmetric pair, we call it **infinitesimally (weakly) effective** if its corresponding orthogonal Lie algebra is (weakly) effective (this clearly does not depend on the choice of Θ). We call (G, K) **effective** if $I = \{e\}$.

If (G, K) is an infinitesimally effective Riemannian symmetric pair, then I is a discrete normal subgroup of G, so it must be central. Since $Z \cap K \subseteq I \subseteq K$, we deduce that $Z \cap K = I$ in this case.

Observation 2.1.24. Any Riemannian symmetric pair (G, K) (resp., orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$) gives rise to an effective one, namely $(G/I, K/I)$ (resp., $(\mathfrak{g}/\mathfrak{i}, \theta)$).

Proposition 2.1.25. *If $(\mathfrak{g}, \theta)$ is a weakly effective orthogonal symmetric Lie algebra, then θ is uniquely determined by $\mathfrak{k}$. Consequently, if (G, K) is an infinitesimally weakly effective Riemannian symmetric pair, then Θ in Definition 2.1.20 is unique.*

Proof. We know that $\mathfrak{p} \subseteq \mathfrak{k}^{\perp}$, where the orthogonal complement is taken with respect to B. By Proposition 2.1.16, the kernel of $B\big|_{\mathfrak{k} \times \mathfrak{k}}$ equals $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k}$ and hence is trivial. In other words, $\mathfrak{k} \cap \mathfrak{k}^\perp = \{0\}$, so $\mathfrak{p} = \mathfrak{k}^\perp$ is determined by $\mathfrak{k}$. □

Definition 2.1.26 (Isotropy). Let M be a connected Riemannian manifold and $p \in M$ any point.

- The **isotropy group** of M at p is the stabilizer $\widetilde{K} = I(M)_p$.
- The **restricted isotropy group** of M at p is the identity component $\widetilde{K}^0$.
- The (**restricted**) **isotropy representation** of M at p is the representation $\widetilde{K} \hookrightarrow \mathrm{O}(T_pM)$ (resp., $\widetilde{K}^0 \hookrightarrow \mathrm{SO}(T_pM)$), $k \mapsto dk_p$.
- The (**restricted**) **linear isotropy group** of M at p is the image $\overline{K} \subseteq \mathrm{O}(T_pM)$ (resp., $\overline{K}^0 \subseteq \mathrm{SO}(T_pM)$) of the (restricted) isotropy representation at p.
- If (G, K) is a Riemannian symmetric pair, its (**restricted**) **isotropy representation** is the representation $K \to \mathrm{GL}(T_oM)$ (resp., $K^0 \to \mathrm{GL}(T_oM)$), $k \mapsto dk_o$, where $M = G/K$ and $o = eK$.
- If M is a manifold, $H \curvearrowright M$ is an action, and $p \in M$, we will often call the stabilizer H_p the **isotropy subgroup** of H at p.

If M is a connected Riemannian homogeneous space, its isotropy groups at different points are conjugate, so we will sometimes drop the reference to a point and just say **isotropy group of M** if there is no ambiguity.

Take a Riemannian symmetric pair (G, K) and consider the homogeneous space $M = G/K$. We make the following simple but vital

Observation 2.1.27. The following pieces of data are in a natural 1-to-1 correspondence:

(i) A G-invariant inner product on M.

(ii) A K-invariant inner product on T_oM.

(iii) A K-invariant inner product on $\mathfrak{p}$.

If K is connected (e.g., if M is simply connected), these are the same as:

(iv) A $\mathfrak{k}$-invariant inner product on $\mathfrak{p}$.

By Definition 2.1.20, there exists[1] a K-invariant inner product on $\mathfrak{p}$, hence there exist G-invariant metrics on M. Picking such a metric turns M into a Riemannian homogeneous G-space. What is more, if we fix Θ on G as in Definition 2.1.20, it passes through the quotient $\pi\colon G \twoheadrightarrow M$ to a involutive isometry s_o of M: $s_o(gK) = \Theta(g)K$ or, in other words, $s_o \circ \pi = \pi \circ \Theta$. This isometry has o as its isolated fixed point and hence is a geodesic symmetry at o. By Proposition 2.1.5, M a symmetric space. By Proposition 2.1.1(c), any other choice of Θ leads to the same s_o. We conclude:

Corollary 2.1.28. *If (G, K) is a Riemannian symmetric pair, then there exist G-invariant metrics on $M = G/K$, and any of them makes M into a symmetric space. The geodesic symmetries of M do not depend on the choice of an invariant metric.*

[1] Actually, Definition 2.1.20 ensures that there exists a K-invariant inner product on the whole $\mathfrak{g}$ such that $\mathfrak{k} \perp \mathfrak{p}$. This translates to a left-invariant metric on G that is K-bi-invariant. The projection $\pi\colon G \twoheadrightarrow M$ becomes a Riemannian submersion, which can prove useful in certain situations.

Definition 2.1.29. Let (G, K) be a Riemannian symmetric pair. We say that it is **associated with** an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ if there is an isomorphism between $(\mathfrak{g}, \theta)$ and the orthogonal symmetric Lie algebra of (G, K). We say that (G, K) **represents** a symmetric space M if there exists a G-invariant metric on G/K that makes it isometric to M. Lastly, we say that an orthogonal symmetric Lie algebra **represents** a symmetric space M if it is associated with a Riemannian symmetric pair representing M.

Normally, when we say that (G, K) is associated with $(\mathfrak{g}, \theta)$, we implicitly assume an isomorphism as in the above definition has been fixed. Similarly, if (G, K) represents M, we assume a G-invariant metric on G/K and an isometry $G/K \simeq M$ have been fixed. In particular, this entails fixing a base point $o = eK$ in M.

Agreement. For concrete symmetric spaces, instead of writing "let M be represented by a Riemannian symmetric pair (G, K)", it is customary to simply write $M = G/K$. In fact, this is how most symmetric spaces are defined. We are going to use this shorthand as well.

Every symmetric space is represented by some Riemannian symmetric pair. Indeed, if we start with just M, take $G = I^0(M)$ and $K = G_o$, where $o \in M$ is any. Define an involutive automorphism Θ of G to be the conjugation C_{s_o}: $\Theta(g) = s_o g s_o$. Then K is an open subgroup of G^Θ and hence (G, K) is an effective Riemannian symmetric pair. The corresponding orthogonal symmetric Lie algebra is given by $(\mathfrak{i}(M), \mathrm{Ad}(s_o))$. The choice of the base point o is irrelevant:

Lemma 2.1.30. *Let M be a symmetric space, $o, o' \in M$ any two points, $G = I^0(M)$, and $K = G_o, K' = G_{o'}$. Then the Riemannian symmetric pairs (G, K) and (G, K') are isomorphic. Therefore, the corresponding orthogonal symmetric Lie algebras are isomorphic as well.*

Proof. If $g \in G$ is any isometry mapping o to o', $gKg^{-1} = K'$, so the conjugation C_g provides an isomorphism between (G, K) and (G, K'). $\square$

Remark 2.1.31. Note that the same choice $G = I^0(M), K = G_o$ allows to represent any connected Riemannian homogeneous space M by a pair (G, K) that satisfies condition (a) of Definition 2.1.20. It is condition (b) that distinguishes symmetric spaces as a very special subclass of Riemannian homogeneous spaces.

Definition 2.1.32. Let M be any symmetric space. Given any $o \in M$, we call $(I^0(M), I^0(M)_o)$ and $(\mathfrak{i}(M), \mathrm{Ad}(s_o))$ the **canonical** Riemannian symmetric pair and orthogonal symmetric Lie algebra of M, respectively.

Observation 2.1.33. Let M be a symmetric space represented by a Riemannian symmetric pair (G, K). Write $(\overline{G}, \overline{K})$ for the canonical Riemannian symmetric pair of M (at o). Then we have a morphism $G \to \overline{G}$. Write G' for the image of this morphism and $K' = G' \cap \overline{K}$ for the image of K. One can show that G' is a closed subgroup of $\overline{G}$ (in particular, K' is compact). Clearly, $G' \cong G/I, K' \cong K/I$. As we saw in Observation 2.1.24, any Θ on G as in Definition 2.1.20 passes to an involution Θ' on G', thus showing that (G', K') is an effective Riemannian symmetric pair. At the same time, the involution $\overline{\Theta} = C_{s_o}$ on $\overline{G}$ preserves G' and coincides with Θ' on G'. Essentially, this means that *every Riemannian symmetric pair representing M factors through a "subpair" of the canonical one.* If $(\mathfrak{g}', \theta')$ is the orthogonal symmetric Lie algebra of (G', K') and $(\overline{\mathfrak{g}}, \overline{\theta})$ is that of

$(\overline{G},\overline{K})$, then $\overline{\theta}$ preserves $\mathfrak{g}'$ and coincides with θ' on it, hence we have $\mathfrak{k}' \subseteq \overline{\mathfrak{k}}$, $\mathfrak{p}' \subseteq \overline{\mathfrak{p}}$. But both $\dim(\mathfrak{p}')$ and $\dim(\overline{\mathfrak{p}})$ equal $\dim(M)$, so we deduce that $\mathfrak{p}' = \overline{\mathfrak{p}}$.

Every orthogonal symmetric Lie algebra is associated with some Riemannian symmetric pair. Indeed, starting with $(\mathfrak{g},\theta)$, take $\widehat{G}$ to be a simply connected Lie group with Lie algebra $\mathfrak{g}$ and define $\widehat{\Theta}$ as the (unique) lift of θ to $\widehat{G}$. For any open subgroup $\widehat{K} \subseteq \widehat{G}^{\widehat{\Theta}}$, $(\widehat{G},\widehat{K})$ is a Riemannian symmetric pair, and its associated orthogonal Lie algebra (with $\widehat{\Theta}$ already defined) is $(\mathfrak{g},\theta)$. Note that if we take $\widehat{K} = (\widehat{G}^{\widehat{\Theta}})^0$, the corresponding space $M = \widehat{G}/\widehat{K}$ is going to be simply connected.

Proposition 2.1.34. *Let $(\mathfrak{g},\theta)$ be an orthogonal symmetric Lie algebra representing some symmetric space M. Let $(\widehat{G},\widehat{K})$ be a Riemannian symmetric pair associated with $(\mathfrak{g},\theta)$ with $\widehat{G}$ simply connected and $\widehat{K}$ connected. Then, equipped with a suitable (uniquely determined) $\widehat{G}$-invariant metric, $\widehat{G}/\widehat{K}$ is the universal Riemannian covering space of M.*

Informally, Proposition 2.1.34 says that an orthogonal symmetric Lie algebra represents a unique symmetric space up to Riemannian covering and a choice of an invariant metric. This encapsulates the idea that orthogonal symmetric Lie algebras are an infinitesimal version of symmetric spaces, designed to study their local properties.

Now we discuss some examples of symmetric spaces, most of which we will meet in profusion throughout the thesis. Things like *type* and *duality* will be defined later in the section.

Example 2.1.35 (Euclidean space). The Euclidean space $\mathbb{E}^n$ is a symmetric space of Euclidean type and rank n. Its isometry group is isomorphic to $\mathrm{O}(n) \ltimes \mathbb{R}^n$, so its canonical symmetric pair is $(\mathrm{SO}(n) \ltimes \mathbb{R}^n, \mathrm{SO}(n))$. But $\mathbb{E}^n$ can also be represented by a much smaller effective Riemannian symmetric pair $(\mathbb{R}^n, \{\mathrm{pt}\})$. We will see in Proposition 2.1.97 that this behavior is rather pathological and does not occur for "most" symmetric spaces. //

Example 2.1.36 (Rank-one symmetric spaces). Let[1] $n \in \mathbb{N}$, and let $\mathbb{F} \in \{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$[2]. If $\mathbb{F} = \mathbb{R}$, we require $n \geq 2$, and if $\mathbb{F} = \mathbb{O}$, we require $n = 2$. The projective space $\mathbb{F}P^n$ is an irreducible symmetric space of compact type and rank 1. Unless $\mathbb{F}$ is $\mathbb{R}$, it is simply connected. The real projective space $\mathbb{R}P^n$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$, and its universal Riemannian covering space is the round sphere $\mathbb{S}^n$. The dual of $\mathbb{F}P^n$ (or $\mathbb{S}^n$ if $\mathbb{F} = \mathbb{R}$) is the hyperbolic space $\mathbb{F}H^n$, which is an irreducible symmetric space of noncompact type and rank 1. In this thesis, whenever we say *projective* (resp., *hyperbolic*) *space*, we refer to any of $\mathbb{F}P^n$ (resp., $\mathbb{F}H^n$). These space are represented by the following almost effective (see Definition 2.4.7) Riemannian symmetric pairs:

$$\begin{aligned}
\mathbb{R}P^n &= \mathrm{SO}(n+1)/\mathrm{S}(\mathrm{O}(n)\mathrm{O}(1)), & & \\
\mathbb{S}^n &= \mathrm{SO}(n+1)/\mathrm{SO}(n), & \mathbb{R}H^n &= \mathrm{SO}^0(n,1)/\mathrm{SO}(n), \\
\mathbb{C}P^n &= \mathrm{SU}(n+1)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1)), & \mathbb{C}H^n &= \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1)), \\
\mathbb{H}P^n &= \mathrm{Sp}(n+1)/\mathrm{Sp}(n)\mathrm{Sp}(1), & \mathbb{H}H^n &= \mathrm{Sp}(n,1)/\mathrm{Sp}(n)\mathrm{Sp}(1), \\
\mathbb{O}P^2 &= F_4/\mathrm{Spin}(9), & \mathbb{O}H^2 &= F_4^{-20}/\mathrm{Spin}(9).
\end{aligned}$$

The case of octonions needs to be handled with extra care. Due to the nonassociativity of

[1] In this thesis, $\mathbb{N}$ starts with 1, and we write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

[2] These are the only finite-dimensional real division algebras that admit a multiplicative norm. (Many authors include multiplicativity of the norm in the definition of a normed division algebra.)

$\mathbb{O}$, the spaces $\mathbb{O}P^2$ and $\mathbb{O}H^2$ cannot be defined in terms of "octonionic lines" in $\mathbb{O}^3$—there is no such concept to begin with. Instead, the Cayley projective plane is normally defined as the set of rank-1 projectors in the exceptional Jordan algebra of Hermitian octonionic 3×3 matrices. (We will talk about this in more detail in Subsection 6.2.5.) The Cayley hyperbolic plane can then be defined as the *dual* of $\mathbb{O}P^2$. One conceptual reason why the only projective space over the octonions is a projective plane is that projective spaces of dimension ≥ 3 have to be *Desarguesian*, whereas the non-associativity of $\mathbb{O}$ causes this property to fail already for $\mathbb{O}P^2$ (see [Bae02, AB03, VY65]). For more on $\mathbb{O}P^2$, the octonions in general, as well as their relation to exceptional Lie groups, see [Bae02].

Together with $\mathbb{R}$ and $\mathbb{S}^1$, the above spaces exhaust the list of symmetric spaces of rank 1. When lumped together with the Euclidean spaces, they admit a number of alternative geometric characterizations. A connected Riemannian manifold M is called *two-point homogeneous* if for every $p_1, q_1, p_2, q_2 \in M$ such that $\operatorname{dist}(p_1, q_1) = \operatorname{dist}(p_2, q_2)$, there exists an isometry mapping p_1 to p_2 and q_1 to q_2. A connected Riemannian homogeneous space is called *isotropic* if its isotropy representation is transitive on the unit sphere (if $\dim(M) > 1$, this is the same as to say that the isotropy representation is of cohomogeneity one, see Definition 2.3.6). One can show that a Riemannian manifold is two-point homogeneous $\Leftrightarrow$ it is an isotropic Riemannian homogeneous space $\Leftrightarrow$ it is a Euclidean space or a rank-1 symmetric space (see [Wol11, Sect. 8.12]). Compact symmetric spaces of rank 1 are also characterized by the fact that all their geodesics are periodic, simple, and of the same length. //

Example 2.1.37 (Grassmannians). Generalizing on the previous example, let $k, n \geq 1$, and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$[1]. If $\mathbb{F}$ is $\mathbb{R}$, we require $n \geq 2$. The Grassmannian $\operatorname{Gr}(k, \mathbb{F}^{n+k})$ is an irreducible[2] symmetric space of compact type and rank $\min\{k, n\}$. Unless $\mathbb{F}$ is $\mathbb{R}$, it is simply connected. In case $\mathbb{F} = \mathbb{R}$, the Grassmannian $\operatorname{Gr}(k, \mathbb{R}^{n+k})$ has fundamental group $\mathbb{Z}/2\mathbb{Z}$, and its universal Riemannian covering space is the Grassmannian of oriented k-planes $\operatorname{Gr}^+(k, \mathbb{R}^{n+k})$. The dual of $\operatorname{Gr}(k, \mathbb{F}^{n+k})$ (or $\operatorname{Gr}^+(k, \mathbb{R}^{n+k})$ if $\mathbb{F} = \mathbb{R}$) is the noncompact Grassmannian $\operatorname{Gr}^*(k, \mathbb{F}^{n+k})$, defined as the set of k-dimensional $\mathbb{F}$-subspaces in $\mathbb{F}^{n+k}$ on which the restriction of the standard symmetric bilinear (resp., Hermitian $\mathbb{C}$-sesquilinear or q-Hermitian $\mathbb{H}$-sesquilinear) form of signature (n, k) is negative-definite. It is an irreducible (except for $k = n = 2$) symmetric space of noncompact type and rank $\min\{k, n\}$. Whenever we say *Grassmannian* (resp., *noncompact Grassmannian*), we refer to any of $\operatorname{Gr}(k, \mathbb{F}^{n+k})$ or $\operatorname{Gr}^+(k, \mathbb{R}^{n+k})$ (resp., $\operatorname{Gr}^*(k, \mathbb{F}^{n+k})$). These spaces are represented by the following almost effective Riemannian symmetric pairs:

$$\begin{aligned}
\operatorname{Gr}(k, \mathbb{R}^{n+k}) &= \mathrm{SO}(n+k)/\mathrm{S}(\mathrm{O}(n)\mathrm{O}(k)), & & \\
\operatorname{Gr}^+(k, \mathbb{R}^{n+k}) &= \mathrm{SO}(n+k)/\mathrm{SO}(n)\mathrm{SO}(k), & \operatorname{Gr}^*(k, \mathbb{R}^{n+k}) &= \mathrm{SO}^0(n,k)/\mathrm{SO}(n)\mathrm{SO}(k), \\
\operatorname{Gr}(k, \mathbb{C}^{n+k}) &= \mathrm{SU}(n+k)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(k)), & \operatorname{Gr}^*(k, \mathbb{C}^{n+k}) &= \mathrm{SU}(n,k)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(k)), \\
\operatorname{Gr}(k, \mathbb{H}^{n+k}) &= \mathrm{Sp}(n+k)/\mathrm{Sp}(n)\mathrm{Sp}(k), & \operatorname{Gr}^*(k, \mathbb{H}^{n+k}) &= \mathrm{Sp}(n,k)/\mathrm{Sp}(n)\mathrm{Sp}(k).
\end{aligned}$$ //

Example 2.1.38 (Compact Lie groups). Let G be a compact connected Lie group. By Proposition 2.1.11, $\mathfrak{g} = \operatorname{Lie}(G)$ is compact and thus admits an $\operatorname{Ad}(G)$-invariant inner product. This translates to a bi-invariant Riemannian metric on G, which clearly makes

[1] These are the only finite-dimensional associative real division algebras.

[2] The only exception is $\operatorname{Gr}(2, \mathbb{R}^4)$, which is reducible. Its universal Riemannian covering space $\operatorname{Gr}^+(2, \mathbb{R}^4)$ is isometric to $\mathbb{S}^2 \times \mathbb{S}^2$. The same goes for $\operatorname{Gr}^*(2, \mathbb{R}^4) \cong \mathbb{R}H^2 \times \mathbb{R}H^2$.

G into a Riemannian homogeneous space. The following fact is standard:

Proposition 2.1.39. *The Lie exponential map* $\exp\colon \mathfrak{g} \to G$ *coincides with its Riemannian exponential map at* e*. In particular, it is surjective.*

It follows from Proposition 2.1.39 that $s_e\colon g \mapsto g^{-1}$ is a geodesic symmetry of G at e. By Proposition 2.1.5, G is a symmetric space. In the literature, such spaces are occasionally said to be *of group type*. There is a natural choice of a Riemannian symmetric pair representing G. Indeed, observe that $G \times G$ acts isometrically on G by $(g, h) \cdot f = gfh^{-1}$. The isotropy subgroup of this action at e is the diagonal $\Delta_G = \{(g, g) \mid g \in G\}$. What is more, Δ_G is the fixed point subgroup of the involutive automorphism $\Theta = C_{s_e}, \Theta(g, h) = (h, g)$. This implies that $(G \times G, \Delta_G)$ is a Riemannian symmetric pair representing G. Note that it does not have to be effective (even infinitesimally), as $I = \Delta_Z = \Delta_G \cap (Z \times Z)$, where $Z = Z(G)$. For example, if G is abelian (i.e., a torus), then $I = \Delta_G$ and $(G \times G)/I \simeq G$. In any case, the corresponding orthogonal symmetric Lie algebra is given by $(\mathfrak{g} \oplus \mathfrak{g}, \theta)$ with $\theta(X, Y) = (Y, X)$, hence $\mathfrak{k} = \Delta_{\mathfrak{g}}$ and $\mathfrak{p} = \{(X, -X) \mid X \in \mathfrak{g}\}$. Observe that Δ_G is trivially isomorphic to G, and we can also identify $\mathfrak{p}$ with $\mathfrak{g}$ as $(X, -X) \leftrightarrow X$. The following observation is elementary but extremely important:

Proposition 2.1.40. *Under the identifications* $\Delta_G \simeq G$ *and* $\mathfrak{p} \simeq \mathfrak{g}$*, the isotropy representation of* $(G \times G, \Delta_G)$ *is equivalent to the adjoint representation of* G*.*

By representing G with $(\mathfrak{g} \oplus \mathfrak{g}, \theta)$ and using things like Proposition 2.1.40 and (2.1.5), one can derive many formulas and results that are specific to symmetric spaces of group type. We are not going to focus on that now but will see some examples later. //

Symmetric spaces as reductive homogeneous spaces

Now we discuss how various geometric properties and quantities of a symmetric space M can be described in the language of Riemannian symmetric pairs and orthogonal symmetric Lie algebras. We will see that many geometric objects associated to M do not actually depend on the choice of an invariant metric. To achieve this, we start with reductive homogeneous spaces and then see how symmetric spaces fit into the picture. We refer to [KN96a, Ch. II, Sect. 11] and [KN96b, Ch. X, Sect. 1-2] for details.

Let M be a reductive homogeneous space of a connected Lie group G. Pick $o \in M$ and write $K \subseteq G$ for the isotropy group of o and $\mathfrak{k} \subseteq \mathfrak{g}$ for its Lie algebra. Let us fix a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. As before, we have an isomorphism of K-representations $\mathfrak{p} \xrightarrow{\sim} T_oM$, $X \mapsto \widehat{X}_o$. Observe that the orbit map $\pi = \pi_o\colon G \twoheadrightarrow M$, $g \mapsto g \cdot o$, is a principal K-bundle, and its associated vector bundle $G \times_K \mathfrak{p} \twoheadrightarrow M$ is naturally isomorphic to the tangent bundle $TM \to M$. The left-invariant distribution on G determined by $\mathfrak{p}$ is a unique G-invariant connection on the K-bundle $G \twoheadrightarrow M$ that coincides with $\mathfrak{p}$ at $e \in G$. It is called a **canonical connection** on $G \twoheadrightarrow M$. The induced complete G-invariant affine connection ∇ on the associated bundle $TM \twoheadrightarrow M$ is called a **canonical affine connection** on M. If we pick another base point $g \cdot o$, we automatically get a reductive decomposition $\mathfrak{g} = \mathrm{Ad}(g)(\mathfrak{k}) \oplus \mathrm{Ad}(g)(\mathfrak{p})$ and thus a canonical connection on the principal

$C_g(K) \cong K$ bundle $\pi_{g\cdot o}\colon G \twoheadrightarrow M$. The commutative diagram

$$\begin{tikzcd} G \arrow[r, "C_g", "\sim"'] \arrow[d, two heads, "\pi_o"] & G \arrow[d, two heads, "\pi_{g\cdot o}"] \\ M \arrow[r, "g", "\sim"'] & M \end{tikzcd}$$

provides an isomorphism of K-bundles and identifies the two canonical connections. We see that a canonical (affine) connection does not depend on the choice of a base point up to isomorphism. It does, however, depend on the choice of $\mathfrak{p}$, so whenever we say *reductive homogeneous space*, we always assume a reductive decomposition (at some base point) has been fixed. Many properties of ∇ can be described in terms of G and $\mathfrak{g}$.

- To begin with, one can write down an explicit formula for ∇ thought of as a covariant derivative: given $X \in \mathfrak{p}$ and any $Y \in \mathfrak{X}(M)$, one has

$$\nabla_X Y = [\widehat{X}, Y]_o, \tag{2.1.1}$$

  where $\widehat{X} \in \mathfrak{X}(M)$ is, as usual, the fundamental vector field corresponding to X.

- Given $X \in \mathfrak{p}$, consider the curve $\gamma(t) = \exp_G(tX)\cdot o$ in M and its horizontal lift $\exp_G(tX)$ in G. For every $t_0 \in \mathbb{R}$, the parallel transport in $G \twoheadrightarrow M$ from $\pi^{-1}(\gamma(0)) = K$ to $\pi^{-1}(\gamma(t_0X))$ along γ is given by $L_{\exp_G(t_0X)}$. Therefore, the parallel transport in $TM \twoheadrightarrow M$ of $T_{\gamma(0)}M$ to $T_{\gamma(t_0X)}M$ along γ coincides with $d(\exp_G(t_0X))_o$. In particular, the velocity vector field of γ is parallel, so γ is a geodesic. We see that all geodesics of ∇ emanating from o are of the form $\exp_G(tX)\cdot o, X \in \mathfrak{p}$. We summarize this in the following commutative diagram[1]:

$$\begin{tikzcd} \mathfrak{p} \arrow[r, "\sim"] \arrow[d, "\exp_G"] & T_oM \arrow[d, "\exp_M"] \\ G \arrow[r, "\pi"] & M \end{tikzcd} \tag{2.1.2}$$

  We immediately get the following:

  Corollary 2.1.41. *For any geodesic γ of ∇, there exists a unique one-parameter subgroup $g(t)$ of G with the following property: for every $t_0, t_1 \in \mathbb{R}$,*

  (a) $\gamma(t_0 + t_1) = g(t_1)\cdot\gamma(t_0)$,

  (b) *The parallel transport in $G \twoheadrightarrow M$ from $\gamma(t_0)$ to $\gamma(t_0+t_1)$ along γ is given by $dg(t_1)_{\gamma(t_0)}$.*

  For every $t \in \mathbb{R}$, we call $g(t) \in G$ a **geodesic translation** along γ. The one-parameter subgroup $g(t)$ is called the **one-parameter subgroup of geodesic translations** along γ.

- Since parallelness of a tensor field can be checked only along geodesics, we arrive at the following

[1] This property lies in the heart of the definition of *naturally reductive spaces*—a class of Riemannian homogeneous spaces that generalizes symmetric spaces. See [BTV95, Ch. 2] for a precise definition and their relation to other classes of Riemannian homogeneous spaces.

Corollary 2.1.42. *Any tensor field on M invariant under geodesic translations is parallel with respect to the canonical affine connection. In particular, any G-invariant tensor field is parallel.*

- Since ∇ is G-invariant, so are its torsion and curvature. By Corollary 2.1.42, we have:

  Corollary 2.1.43. *The torsion and curvature of ∇ are parallel with respect to ∇.*

- Both torsion T and curvature R of ∇ admit simple expressions in terms of the reductive decomposition of $\mathfrak{g}$. For any $X, Y, Z \in \mathfrak{p}$, we have:

$$R_o(X,Y)Z = -[[X,Y]_{\mathfrak{k}}, Z], \tag{2.1.3}$$
$$T_o(X,Y) = -[X,Y]_{\mathfrak{p}}. \tag{2.1.4}$$

  Using (2.1.4), we arrive at the following conclusion:

  Corollary 2.1.44. *The following are equivalent for a reductive homogeneous space $M = G/K$:*

  (i) *The canonical affine connection on M is torsion-free*[1].

  (ii) $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

Now we can apply all this to symmetric spaces. Let (G, K) be a Riemannian symmetric pair. As we noted in Corollary 2.1.22, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a reductive decomposition. Endow $M = G/K$ with any G-invariant metric. Combining Corollaries 2.1.42 and 2.1.44 leads to the following:

Corollary 2.1.45. *The Levi-Civita connection of the symmetric space M coincides with the canonical affine connection and thus does not depend on the choice of a G-invariant metric. Consequently, the exponential map, parallel transport, curvature endomorphism, and Ricci curvature of M do not depend on that choice either.*

Using this corollary and what we know about the canonical connection, we can work out a handy Lie-algebraic expression for every type of curvature of a symmetric space.

- It follows from (2.1.3) that the curvature endomorphism of M at o is given by

$$R_o(X,Y)Z = -[[X,Y],Z], \quad (X, Y, Z \in \mathfrak{p}). \tag{2.1.5}$$

  Observe that the right-hand side does indeed lie in $\mathfrak{p}$. Another way to state (2.1.5) is that the curvature operator $R(X,Y)$ is given by $-\mathrm{ad}[X,Y]\big|_{\mathfrak{p}}$.

- One can use (2.1.5) to deduce that the Ricci curvature of M is in fact a multiple of the Killing form of $\mathfrak{g}$ (see [Bes08, Th. 7.73]):

$$Ric(X,Y) = -\frac{1}{2}B(X,Y) \quad (X, Y \in \mathfrak{p}). \tag{2.1.6}$$

- The curvature tensor and sectional curvatures of M do of course depend on the

[1] Every reductive homogeneous G-space admits a unique G-invariant torsion-free affine connection ∇^{tf} whose geodesics coincide with those of ∇. Condition (i) means that $\nabla^{\mathrm{tf}} = \nabla$.

choice of an invariant metric. Thanks to (2.1.5), they have the following expressions[1] at o:

$$Rm_o(X,Y,Z,W) = -\langle [[X,Y],Z] \,|\, W\rangle \qquad (X,Y,Z,W \in \mathfrak{p}),$$
$$K(X,Y) = -\langle [[X,Y],Y] \,|\, X\rangle \qquad (X,Y \in \mathfrak{p}, ||X|| = ||Y|| = 1, X \perp Y).$$

2.1.3. Holonomy and isometries

This part serves as a digression into a topic that underpins a lot of questions and results in the theory of symmetric spaces: Riemannian holonomy and its relation to isometries. First, we discuss the restricted holonomy representation of a Riemannian manifold and how its decomposition into irreducible subrepresentations yields local and global geometric decompositions of the manifold itself. Next, we prove an important and rather folklore structure result on the isometry group of Riemannian products that behave like the de Rham decomposition. Finally, we discuss a special type of isometries, called transvections, that respect the holonomy and parallel transport in a certain sense. Our general reference for this part is [KN96a].

Holonomy decompositions

Let M be a Riemannian manifold and $p \in M$ any point. Recall that the holonomy group $\mathrm{Hol}(M,p) \subseteq \mathrm{O}(T_pM)$ is defined as the group of parallel transports from p to itself along all piecewise-smooth loops based at p. Restricting this to only those loops that are contractible yields a subgroup $\mathbf{Hol^0(\boldsymbol{M},\boldsymbol{p})} \subseteq \mathrm{Hol}(M,p)$. It is well-known that $\mathrm{Hol}(M,p)$ is a (possibly non-closed) Lie subgroup of $\mathrm{O}(T_pM)$, $\mathrm{Hol}^0(M,p)$ is its identity component, and $\mathrm{Hol}^0(M,p)$ is actually a closed subgroup of $\mathrm{SO}(T_pM)$ (see [KN96a, Th. II.4.2, IV.5.5]).

Definition 2.1.46. Let M be Riemannian manifold and $p \in M$ any point. We call $\mathrm{Hol}^0(M,p)$ the **restricted holonomy group of $\boldsymbol{M}$ (at $\boldsymbol{p}$)**. We call its representation on T_pM the **restricted holonomy representation of $\boldsymbol{M}$ (at $\boldsymbol{p}$)**. If M is connected, we say that it is **reducible** if so is its restricted holonomy representation at some point. We say that M is **irreducible** if it is not reducible and not flat.

Remark 2.1.47. If M is connected and $p,q \in M$ are any two points, then the holonomy groups at p and q are isomorphic by means of parallel transport along any piecewise smooth curve from p to q; with respect to any such isomorphism, the holonomy representations at p and q become equivalent. Consequently, if M is irreducible, then its restricted holonomy representation at *any* point is irreducible. When there is no ambiguity, we will sometimes write $\mathrm{Hol}(M)$ or $\mathrm{Hol}^0(M)$ without reference to any specific point. Note that M being flat is equivalent to $\mathrm{Hol}^0(M)$ being trivial. The non-flatness assumption in the definition of irreducibility rules out precisely the cases where M is one-dimensional.

Observation 2.1.48. Let M be connected and $\pi\colon \widetilde{M} \twoheadrightarrow M$ its universal Riemannian covering. Take $p \in M$ and any $\widetilde{p} \in \widetilde{M}$ lying over p. Note that $\mathrm{Hol}^0(\widetilde{M},\widetilde{p}) = \mathrm{Hol}(\widetilde{M},\widetilde{p})$. We have an isomorphism $d\pi_{\widetilde{p}}\colon T_{\widetilde{p}}\widetilde{M} \xrightarrow{\sim} T_pM$, which induces an isomorphism $\mathrm{Hol}(\widetilde{M},\widetilde{p}) \xrightarrow{\sim} \mathrm{Hol}^0(M,p)$. This is due to the fact that the contractible loops at p are precisely those whose lift with initial point $\widetilde{p}$ ends also at $\widetilde{p}$. We deduce that the restricted holonomy

[1] Throughout the thesis, we usually denote Riemannian metrics and inner products by $\langle - | - \rangle$, unless otherwise stated.

representation of M is naturally equivalent to the holonomy representation of $\widetilde{M}$. This means that local results using the full holonomy group work in the not simply connected setting if one passes to the restricted holonomy group. More generally, the restricted holonomy representation (and hence irreducibility) is preserved under Riemannian covering maps.

Let $V^p \subseteq T_pM$ be a subrepresentation of $\mathrm{Hol}^0(M,p)$. Fix a (relatively) simply connected neighborhood W of p. Given any $q \in W$, carry V to T_qM by means of parallel transport along any piecewise-smooth curve from p to q lying in W. By design, the resulting subspace $V^q \subseteq T_qM$ does not depend on the curve chosen. This gives a distribution $V = \bigcup_{q\in W} V^q$ on W. If M is simply connected, we can take $W = M$ and thus obtain a global distribution. The following is proven in [KN96a, Prop. IV.5.1]:

Proposition 2.1.49. *The distribution V is smooth and parallel (meaning, $\nabla_X Y \in \Gamma(V)$ for any $X \in \mathfrak{X}(M), Y \in \Gamma(V)$). In particular, V is involutive. The corresponding foliation has all its leaves totally geodesic.*

We need to introduce one more vital, albeit technical notion, taken from [KN96a, Sect. IV.5].

Definition 2.1.50. Let M be a Riemannian manifold and $p \in M$ any point. A direct sum decomposition $T_pM = V_0^p \oplus V_1^p \oplus \cdots \oplus V_k^p$ is called a **canonical decomposition of $\boldsymbol{T_pM}$** if:

(a) The summands V_i^p, $0 \le i \le k$, are mutually orthogonal,

(b) V_0^p is the subspace of $\mathrm{Hol}^0(M,p)$-invariants in T_pM, and

(c) Each of the summands V_i^p, $1 \le i \le k$, is an irreducible $\mathrm{Hol}^0(M,p)$-subrepresentation of T_pM.

It is easy to show inductively that canonical decompositions exist.

Proposition 2.1.51 (Holonomy decompositions). *Let M be a Riemannian manifold and $p \in M$ any point.*

(a) *There is a unique canonical decomposition $T_pM = V_0^p \oplus V_1^p \oplus \cdots \oplus V_k^p$ up to reordering of the irreducible summands.*

(b) *The restricted holonomy group decomposes as a product $\mathrm{Hol}^0(M,p) = G_1 \times \cdots \times G_k$ of its closed connected normal subgroups such that G_i acts irreducibly on V_i^p and trivially on every other $V_j^p, j \neq i$.*

Fix a (relatively) simply connected open neighborhood W of p and write $V_0, V_1, \dots, V_k$ for the autoparallel distributions on W determined by $V_0^p, V_1^p, \dots, V_k^p$.

(c) *The leaves of V_0 are flat.*

(d) *For every $0 \le i \le k$, there exists an open neighborhood U_i of p in the leaf of V_i through p such that the embedding $U_0 \cup U_1 \cup \cdots \cup U_k \hookrightarrow M$ extends to an isometry of the Riemannian product $U_0 \times U_1 \times \cdots \times U_k$ onto an open neighborhood U of p (here each U_i embeds into $U_0 \times U_1 \times \cdots \times U_k$ in the obvious way as a slice with a constant coordinate p in all the other factors).*

(e) *(de Rham decomposition) If M is complete and simply connected, we can take*

$W = M$ and each U_i to be the whole leaf M_i of V_i through p, in which case $U = M$ and we have an isometry $M = M_0 \times M_1 \times \cdots M_k$. Moreover, here:

(1) *M_0 is isometric to a Euclidean space.*

(2) *M_i is a complete, simply connected, and irreducible for $1 \leq i \leq p$.*

(3) *G_i is naturally isomorphic to $\mathrm{Hol}(M_i, p)$.*

For a proof of Proposition 2.1.51, see [KN96a, Sect. IV.5]. It is worth pointing out that the authors define a canonical decomposition with respect to the full holonomy group and prove some of these results only in the case when M is simply connected, but the proofs remains valid here due to Observation 2.1.48. For symmetric spaces, Proposition 2.1.51 admits a refinement.

Proposition 2.1.52. *Let M be a symmetric space and and $T_pM = V_0^p \oplus V_1^p \oplus \cdots \oplus V_k^p$ the canonical decomposition at any $p \in M$.*

(a) *For each i, the parallel transport of V_i^p to any other point of M does not depend on the choice of a curve, and thus V_i^p extends uniquely to a smooth global parallel distribution V_i. Consequently, we have $TM = V_0 \oplus V_1 \oplus \cdots \oplus V_k$. The distributions V_i are pairwise orthogonal.*

(b) *Each V_i is involutive, and the leaves of the corresponding foliation $\mathcal{F}_i$ are totally geodesic and properly embedded (hence complete). If U_i stands for the leaf of $\mathcal{F}_i$ through p, then G_i in Proposition 2.1.51(b) is naturally isomorphic to $\mathrm{Hol}^0(U_i, p)$.*

(c) *Each foliation $\mathcal{F}_i$ is homogeneous (see Definition 2.3.15) and the action of $I^0(M)$ on M interchanges its leaves. In particular, all the leaves of $\mathcal{F}_i$ are congruent to one another (see Definition 2.3.18).*

(d) *The leaves of every $\mathcal{F}_i$ are symmetric spaces in the induced metric. For $i = 0$, they are flat, whereas for $1 \leq i \leq k$, they are irreducible.*

(e) *If M is simply connected, $M = M_0 \times M_1 \times \cdots \times M_k$ is its de Rham decomposition, and π_j is the projection of M onto M_j, then $V_0 = \pi_0^*(TM_0)$ and (up to permutation) $V_i \cong \pi_i^*(TM_i)$. For any i, the leaves of V_i are isometric to M_i by means of π_i.*

Definition 2.1.53. Let M by any symmetric space. The distributions V_i (resp., foliations $\mathcal{F}_i$) as in Proposition 2.1.52 are called **de Rham distributions** (resp., **foliations**) of M. Both V_0 and $\mathcal{F}_0$ are called **Euclidean** (or **flat**), and any leaf of $\mathcal{F}_0$ is called the **Euclidean** (or **flat**) **part** of M. For each $i = 1, \ldots, k$, V_i and $\mathcal{F}_i$ are said to be **irreducible** and any leaf of $\mathcal{F}_i$ is called an **irreducible part** of M.

Example 2.1.54. Let G be a compact connected Lie group endowed with a bi-invariant metric. Since $\mathfrak{g}$ is compact, it splits as $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}$, where $\mathfrak{g}_{\mathrm{ss}} = [\mathfrak{g}, \mathfrak{g}]$ is compact semisimple. Let $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be the decomposition of $\mathfrak{g}_{\mathrm{ss}}$ into the sum of its simple compact ideals. Then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is the canonical decomposition of G at e. The flat part of G is $Z(G)^0$, whereas the irreducible parts are the compact topologically simple subgroups corresponding to $\mathfrak{g}_i$. //

With Proposition 2.1.51, we can give the following geometric description of irreducibility:

Proposition 2.1.55. *Let M be a connected Riemannian manifold.*

(a) *If M is reducible, it is isometric to a nontrivial Riemannian product in a neighborhood of each of its points.*

(b) *If M is complete, it is reducible if and only if its universal Riemannian covering space is isometric to a nontrivial Riemannian product.*

(c) *Suppose the metric on M is real-analytic (e.g., M a Riemannian homogeneous space). If M is irreducible, no open subset of M is isometric to a nontrivial Riemannian product.*

Part (c) here can be deduced from [KN96a, Th. II.10.8].

Definition 2.1.56. A connected Riemannian is said to **have** (resp., **not have**) a **flat local factor** if its restricted holonomy representation has (resp., does not have) nontrivial invariants.

Proposition 2.1.57. *Let M be a connected Riemannian manifold.*

(a) *If M has a flat local factor, it is locally (around each of its points) isometric to a Riemannian product with a nontrivial flat factor.*

(b) *If M is complete, it has a flat local factor if and only if its universal Riemannian covering space is isometric to a Riemannian product with a nontrivial flat factor.*

(c) *If the metric on M is analytic and M does not have a flat local factor, no open subset of M is isometric to a Riemannian product with a nontrivial flat factor.*

Obviously, a symmetric space does not have a flat local factor if and only if its Euclidean part is trivial.

The isometry group of a Riemannian product

Here we prove a result known colloquially as "an isometry of a Riemannian product must permute its isometric factors". Even though it is fairly simple and intuitively clear, this result will prove of great importance to us in Chapters 3 and 4.

Definition 2.1.58. Let M be a connected Riemannian manifold. A Riemannian product decomposition $M = M_0 \times M_1 \times \cdots \times M_k$ is called **de Rham-like** if

(a) M_0 is flat, and

(b) M_i is irreducible for $1 \leq i \leq k$.

Example 2.1.59. The de Rham decomposition of a simply connected complete Riemannian manifold is de Rham-like (hence the name). //

Let $M = M_0 \times M_1 \times \cdots \times M_k$ be a de Rham-like decomposition. We have an obvious embedding of Lie groups $I(M_0) \times I(M_1) \times \cdots \times I(M_k) \hookrightarrow I(M)$. This does not have to be an isomorphism though, as some of the factors might be isometric, so there might be additional isometries that interchange those. Let S_k be the symmetric group on k elements, and let us introduce its subgroup

$$\boldsymbol{S_k^{\simeq}} = \left\{ \sigma \in S_k \mid M_i \simeq M_{\sigma(i)}\ \forall\, i = 1, \ldots, k \right\},$$

where $\simeq$ means isometric. For any pair of indices $i, j \in \{1, \ldots, k\}$ such that $M_i \simeq M_j$, pick an isometry $\varphi_{ij} \colon M_i \xrightarrow{\sim} M_j$ in such a way that if we have $M_i \simeq M_j \simeq M_l$, then $\varphi_{jl} \circ \varphi_{ij} = \varphi_{il}$. This gives an embedding

$$S_k^{\simeq} \hookrightarrow I(M), \sigma \mapsto \varphi_\sigma, \text{ where}$$
$$\varphi_\sigma(p_1, \ldots, p_k) = (\varphi_{\sigma(1)1}(p_{\sigma(1)}), \ldots, \varphi_{\sigma(k)k}(p_{\sigma(k)}))$$

(to be precise, this is an injective group anti-homomorphism). Surely, this embedding depends on the choice of φ_{ij}'s.

Proposition 2.1.60 (Isometry group of a Riemannian product). *Let $M = M_0 \times M_1 \times \cdots \times M_k$ be a de Rham-like decomposition. Then the isometry group $I(M)$ decomposes as a semidirect product*

$$I(M) = [I(M_0) \times I(M_1) \times \cdots \times I(M_k)] \rtimes S_k^{\simeq}. \tag{2.1.7}$$

In particular, $I(M_0) \times I(M_1) \times \cdots \times I(M_k)$ is an open normal subgroup of $I(M)$. The corresponding action of $S_k^{\simeq}$ on it is given by $\sigma \cdot (g_0, (g_s)) = (g_0, (\varphi_{\sigma(s)s} \circ g_{\sigma(s)} \circ \varphi_{\sigma(s)s}^{-1}))$.

Proof. The subgroups $I(M_0) \times I(M_1) \times \cdots \times I(M_k)$ and $S_k^{\simeq}$ clearly do not intersect inside $I(M)$, so we need only show that their product is the whole isometry group.

Pick any point $p = (p_0, p_1, \ldots, p_k) \in M$. Then

$$T_pM = T_{p_0}M_0 \oplus T_{p_1}M_1 \oplus \cdots \oplus T_{p_k}M_k$$

is a canonical decomposition of T_pM (see Definition 2.1.50). This follows from the fact that the restricted holonomy group at p splits naturally as $\mathrm{Hol}^0(M, p) = \mathrm{Hol}^0(M_1, p_1) \times \cdots \times \mathrm{Hol}^0(M_k, p_k)$, where the action of $\mathrm{Hol}^0(M_i, p_i)$ on $T_{p_j}M_j$ is trivial if $i \neq j$ and is simply the restricted holonomy representation of M_i at p_i if $i = j$ (this agrees with Proposition 2.1.51(b)).

Let $g \in I(M)$ be any isometry and write $g(p) = q = (q_0, q_1, \ldots, q_k)$. Since isometries commute with parallel transport, one can show that the differential $dg_p \colon T_pM \xrightarrow{\sim} T_qM$ must send the canonical decomposition of T_pM to that of T_qM. In other words, due to Proposition 2.1.51(a), $dg_p(T_{p_0}M_0) = T_{q_0}M_0$, and for every $i \in \{1, \ldots, k\}$, there exists $i' \in \{1, \ldots, k\}$ such that $dg_p(T_{p_i}M_i) = T_{q_{i'}}M_{i'}$. Write $\sigma \in S_k$ for the permutation sending i to its corresponding i'. Let us write $M_{0,p}$ for $M_0 \times \{(p_1, \ldots, p_k)\}$ and $M_{i,p}$ for $\{(p_0, p_1, \ldots, p_{i-1})\} \times M_i \times \{(p_{i+1}, \ldots, p_k)\}$ for any $i = 1, \ldots, k$ (and the same at q). These are totally geodesic submanifolds of M, and we have obvious isometries $M_{0,p} \cong M_0 \cong M_{0,q}$, $M_{i,p} \cong M_i \cong M_{i,q}$. Since isometries commute with the exponential map, g must send $M_{0,p}$ onto $M_{0,q}$ and $M_{i,p}$ onto $M_{\sigma(i),q}$. This implies that $M_i \simeq M_{\sigma(i)}$, so σ actually lies in the subgroup $S_k^{\simeq}$ of S_k. We also have the isometries $M_0 \cong M_{0,p} \xrightarrow{g} M_{0,q} \cong M_0$ and $M_i \cong M_{i,p} \xrightarrow{g} M_{\sigma(i),q} \cong M_{\sigma(i)}$, $1 \le i \le k$, which we denote by g_0 and g_i, respectively. By construction, the isometry

$$(g_0, g_{\sigma^{-1}(1)} \circ \varphi_{1\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(k)} \circ \varphi_{k\sigma^{-1}(k)}) \circ \sigma^{-1}$$

lies in the product of $I(M_0) \times I(M_1) \times \cdots \times I(M_k)$ and $S_k^{\simeq}$ and coincides with g on $M_{0,p} \cup \bigcup_{1 \le i \le k} M_{i,p}$. But then the differentials of these two isometries at p must coincide as well. Since an isometry of a connected Riemannian manifold is uniquely determined by its value at a point and its differential at that point, the constructed isometry coincides with g, which finishes the proof. □

Corollary 2.1.61. *Let $M = M_0 \times M_1 \times \cdots \times M_k$ be a de Rham-like decomposition. Then $I^0(M) = I^0(M_0) \times I^0(M_1) \times \cdots \times I^0(M_k)$.*

Remark 2.1.62. Proposition 2.1.60 allows a slightly different reformulation if we group the isometric factors together. Namely, let $M = M_0 \times M_1^{l_1} \times \cdots \times M_k^{l_k}$ be a de Rham-like decomposition where $M_i \not\simeq M_j$ for $i \neq j$ and $M_i^{l_i}$ simply means the product of l_i copies of M_i. Then each φ_{ij} used in the construction of $S_k^{\simeq} \hookrightarrow I(M)$ would have to be an isometry between two copies of M_s for some $s \in \{1, \ldots, k\}$, so we can take it to be the identity. The group $S_l^{\simeq}$ (here $l = \sum_{i=1}^{k} l_i$) then consists of those permutations that shuffle the first l_1 elements with each other, the next l_2 elements with each other, and so on. Hence, $S_l^{\simeq} \simeq S_{l_1} \times \cdots \times S_{l_k}$. The embedding $S_l^{\simeq} \hookrightarrow I(M)$ then looks like

$$\sigma(p_0, p_1, \ldots, p_l) = (p_0, p_{\sigma(1)}, \ldots, p_{\sigma(l)}),$$

and decomposition (2.1.7) becomes

$$I(M) = \left[I(M_0) \times I(M_1)^{l_1} \times \cdots \times I(M_k)^{l_k}\right] \rtimes S_l^{\simeq}.$$

Remark 2.1.63. The argument used in the proof of Proposition 2.1.60 can also be used to prove the uniqueness property of the decomposition $M = M_0 \times M_1 \times \cdots \times M_k$. Namely, assume we have another de Rham-like decomposition $M = M_o' \times M_1' \times \cdots \times M_s'$. Then $k = s$ and there exist a permutation $\sigma \in S_k$, an isometry $\varphi_0 \colon M_0 \xrightarrow{\sim} M_0'$, and a collection of isometries $\varphi_i \colon M_i \xrightarrow{\sim} M'_{\sigma^{-1}(i)}$, such that the resulting isometry

$$M_0 \times M_1 \times \cdots \times M_k \xrightarrow{\sim} M_0' \times M_1' \times \cdots \times M_l'$$

is of the form

$$(p_0, p_1, \ldots, p_k) \mapsto (\varphi_0(p_0), \varphi_{\sigma(1)}(p_{\sigma(1)}), \ldots, \varphi_{\sigma(k)}(p_{\sigma(k)})).$$

As a special case, we get the well-known uniqueness property of the de Rham decomposition.

The relation between holonomy and isometries

In symmetric spaces, there is a remarkable interplay between parallel transport and isometries. Since the former only uses the connection and not the metric, we need to broaden the scope of our consideration temporarily. If M is a Riemannian manifold, we can treat it as an affine manifold—endowed with the Levi-Civita connection. Its group of isometries then becomes a subgroup of an a priori larger group of affine transformations.

More generally, let M be a connected smooth manifold with an affine connection ∇. The group $A(M)$ of affine transformations of M is a Lie group in the compact-open topology (see [KN96a, Th.VI.1.5]). Let P stand for the frame bundle of M. Then P is foliated by the holonomy bundles of ∇: given $u \in P$, its leaf $P(u)$ consists of all points of P reachable from u by a piecewise-smooth horizontal curve. The affine group $A(M)$ acts naturally on P and permutes the leaves.

Definition 2.1.64. An affine transformation f of (M, ∇) is called a **transvection** if it preserves some ($\Leftrightarrow$ every) holonomy bundle $P(u) \subseteq P$. The group of all transvections of (M, ∇) is denoted by $\mathbf{Tr}(\boldsymbol{M})$. An affine space (M, ∇) is called **affine reductive** if $\mathrm{Tr}(M)$ acts transitively on some ($\Leftrightarrow$ every) holonomy bundle $P(u)$.

Clearly, $\mathrm{Tr}(M)$ is a normal subgroup of $A(M)$. An affine transformation f is a transvection if for some ($\Leftrightarrow$ every) $p \in M$, there exists a piecewise-smooth curve γ from p to $f(p)$ such that $df_p \colon T_pM \xrightarrow{\sim} T_{f(p)}M$ coincides with the parallel transport along γ. For example, in a reductive homogeneous space G/H equipped with the canonical connection, every geodesic translation is a transvection. On the other hand, M is an affine reductive space if and only if for every $p, q \in M$ and every piecewise-smooth curve γ from p to q, there exists a transvection f mapping p to q such that df_p coincides with the parallel transport along γ.

It is proven in [Kow79] that an affine space (M, ∇) is affine reductive if and only if M can be expressed as a reductive homogeneous space G/K so that ∇ coincides with the canonical affine connection (hence the name). Using Corollary 2.1.45 and the results of [Kow79], one can show the following:

Proposition 2.1.65. *Any symmetric space M is affine reductive. If we write (G, K) for its canonical Riemannian symmetric pair, then $\mathrm{Tr}(M)$ is contained in G and is in fact a connected closed normal Lie subgroup of G. Its corresponding ideal of $\mathfrak{g}$ is $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$. In particular, $\mathrm{Tr}(M)$ acts transitively on M, contains all one-parameter subgroups of geodesic translations, and is in fact generated by geodesic translations.*

Remark 2.1.66. Let M be a symmetric space, (G, K) its canonical Riemannian symmetric pair, and $X \in \mathfrak{p}$. Then the geodesic translation $\exp_G(tX)$ can be expressed as the composition of two geodesic symmetries, namely $s_{\exp_M(\frac{t}{2}X)} \circ s_o$. Consequently, $\mathrm{Tr}(M)$ is contained in the subgroup of $I(M)$ generated by all the geodesic symmetries.

We can use Proposition 2.1.65 to draw some conclusions about the holonomy group of a symmetric space. For a general Riemannian manifold M, the only relation that always exists between its isometries and holonomy is that the holonomy groups are preserved under isometries: given $f \in I(M)$ and $p \in M$, df_p induces an isomorphism between $\mathrm{O}(T_pM)$ and $\mathrm{O}(T_{f(p)}M)$ under which $\mathrm{Hol}(M, p)$ gets identified with $\mathrm{Hol}(M, f(p))$. In particular, the full linear isotropy group $\overline{K}$ at p normalizes $\mathrm{Hol}(M, p)$. For symmetric spaces, however, more is true, as implied by Proposition 2.1.65:

Proposition 2.1.67. *Let M be a symmetric space, $o \in M$ any point, and let $\overline{K} \subseteq \mathrm{O}(T_oM)$ be the full linear isometry group. Then $\mathrm{Hol}(M, o) \subseteq \overline{K}$ is a normal subgroup. If $(\mathfrak{g}, \theta)$ is the canonical orthogonal symmetric Lie algebra of M at o, then the holonomy Lie algebra at o is given by $\mathfrak{hol}(M, o) = [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.*

The last statement of Proposition 2.1.67 follows from Proposition 2.1.65 and the Ambrose-Singer theorem (and is true for any affine reductive space). Later we will see that a stronger version of this result holds for a special subclass of symmetric spaces (see Proposition 2.1.97). If M is represented by an arbitrary orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$, the holonomy Lie algebra representation at o is given (modulo the kernel) by the restriction of the adjoint representation $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ to $[\mathfrak{p}, \mathfrak{p}]$.

Before we go further, we mention one more crucial result that gives a somewhat intrinsic description of the linear isotropy group of a simply connected symmetric space. Recall that if we have a vector space V, the representation of $\mathrm{GL}(V)$ (resp., $\mathfrak{gl}(V)$) on V extends uniquely to one on the full tensor algebra $TV = \bigoplus_{p,q\geq 0} T^{(p,q)}V$ by algebra automorphisms (resp., derivations) such that, on V^*, it coincides with the dual representation. This extension preserves the bi-degree and commutes with all contractions. The following is proven in [Hel01, p. 227, Ex. A6] (see p. 564 for the solution):

Proposition 2.1.68. *Let M be a simply connected symmetric space and $o \in M$ any point. The full linear isotropy group $\overline{K}$ at o (resp., its Lie algebra $\bar{\mathfrak{k}}$) consists precisely of those elements of $\mathrm{GL}(T_oM)$ (resp., $\mathfrak{gl}(T_oM)$) that preserve the inner product[1] $g_o \in T^{(0,2)}T_oM$ and the curvature endomorphism $R_o \in T^{(1,3)}T_oM$.*

Essentially, Proposition 2.1.68 allows one to extend an operator on a tangent space to M to a global isometry of M if certain conditions are satisfied. We will use it repeatedly throughout the thesis.

2.1.4. Types of symmetric spaces

Now, we introduce the three types of symmetric spaces and discuss how they are the building blocks for all symmetric spaces.

Let $\mathfrak{g}$ be a real semisimple Lie algebra. The Killing form B of $\mathfrak{g}$ is nondegenerate, but it can be of mixed signature.

Definition 2.1.69. Let θ be an involutive automorphism of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding $\mathbb{Z}/2\mathbb{Z}$-grading. We call θ a **Cartan involution** and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its **Cartan decomposition** if B is negative-definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{p}$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra with a fixed Cartan involution θ and the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Consider a symmetric bilinear form on $\mathfrak{g}$ given by $\boldsymbol{B}_\theta(X,Y) = -B(X, \theta Y)$. One can readily see that B_θ coincides with B on $\mathfrak{p}$ and equals $-B$ on $\mathfrak{k}$. In particular, B_θ is positive-definite. The property of B_θ being positive-definite can be taken as an alternative definition of a Cartan involution.

Example 2.1.70. Let $\mathfrak{g}$ be a transpose-invariant semisimple subalgebra of $\mathfrak{sl}(n,\mathbb{R})$. Then $\theta(X) = -X^t$ is a Cartan involution on $\mathfrak{g}$. //

Example 2.1.70 essentially exhausts all examples of Cartan involutions:

Proposition 2.1.71. *Let $\mathfrak{g}$ be a real semisimple Lie algebra with a Cartan involution θ. Then $\mathfrak{g}$ is isomorphic to a transpose-invariant subalgebra of $\mathfrak{sl}(n,\mathbb{R})$ such that θ becomes $X \mapsto -X^t$.*

Proof. Since we have an inner product B_θ on $\mathfrak{g}$, every $A \in \mathfrak{gl}(\mathfrak{g})$ has an adjoint operator A^*. Define an operator $\dagger_\theta$ on $\mathfrak{gl}(\mathfrak{g})$ by $\dagger_\theta(X) = -X^*$. One can readily see that $\dagger_\theta$ is an involutive automorphism of $\mathfrak{gl}(\mathfrak{g})$. Moreover, it preserves $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g}) \subseteq \mathfrak{sl}(\mathfrak{g})$, and the isomorphism $\mathrm{ad}\colon \mathfrak{g} \xrightarrow{\sim} \mathrm{ad}(\mathfrak{g})$ identifies θ with $\dagger_\theta$, i.e., $\mathrm{ad}(\theta X) = \mathrm{ad}(X)^*$. After choosing an orthonormal basis for $\mathfrak{g}$, $\dagger_\theta$ becomes the negative transpose of a matrix. □

Cartan involutions are designed to study noncompact Lie algebras due to the following

[1]This condition cuts out precisely $\mathrm{O}(T_oM)$ (resp., $\mathfrak{so}(T_oM)$).

Proposition 2.1.72. *The following conditions on a real semisimple Lie algebra $\mathfrak{g}$ are equivalent:*

(i) $\mathfrak{g}$ *is compact.*

(ii) $\mathrm{Id}_{\mathfrak{g}}$ *is a Cartan involution.*

In this case, $\mathrm{Id}_{\mathfrak{g}}$ *is the only Cartan involution on* $\mathfrak{g}$*.*

Proposition 2.1.73 (Cartan involutions). *Let $\mathfrak{g}$ be a real semisimple Lie algebras.*

(a) *There exists a Cartan involution on* $\mathfrak{g}$*.*

(b) *Any two Cartan involutions on $\mathfrak{g}$ are conjugate via* $\mathrm{Inn}(\mathfrak{g})$*.*

An important property of a Cartan involution is that it makes $\mathfrak{g}$ into an orthogonal symmetric Lie algebra:

Proposition 2.1.74. *If θ is a Cartan involution on $\mathfrak{g}$, then $\mathfrak{k}$ is a compactly embedded subalgebra of $\mathfrak{g}$. In particular, $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie algebra.*

In fact, $\mathfrak{k}$ is a maximal compactly embedded subalgebra of $\mathfrak{g}$, as can be shown from Proposition 2.4.1(d). We are now ready to define the types of symmetric spaces. We start on the level of Lie algebras.

Let $\mathfrak{g}$ be a compact semisimple Lie algebra and $\theta \in \mathrm{Aut}(\mathfrak{g})$ any nontrivial involution. Then $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie algebra. Indeed, if we take G to be a simply connected Lie group with Lie algebra $\mathfrak{g}$, then G is compact by Proposition 2.1.13. If we lift θ to an involution Θ on G, G^Θ will be a closed subgroup of G and hence compact. By Proposition 2.1.16, $\mathfrak{k}$ is compactly embedded. We call $(\mathfrak{g}, \theta)$ an orthogonal symmetric Lie algebra **of compact type**. Note that it is automatically weakly effective.

Let $\mathfrak{g}$ be a noncompact real semisimple Lie algebra with a Cartan involution θ. By Proposition 2.1.74, $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie algebra, and we say that it is **of noncompact type**. It is also automatically weakly effective.

From Proposition 2.1.73, we immediately get:

Corollary 2.1.75. *An orthogonal symmetric Lie algebra of noncompact type is completely determined by its underlying Lie algebra up to isomorphism.*

Finally, let $\mathfrak{p}$ be a finite-dimensional real vector space and $\mathfrak{k} \subseteq \mathfrak{gl}(\mathfrak{p})$ the Lie algebra of some compact subgroup of $\mathrm{GL}(\mathfrak{p})$. Then we can treat $\mathfrak{p}$ as an abelian Lie algebra and form a semidirect sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. It is easy to see that this is a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathfrak{g}$, hence by Proposition 2.1.19 it yields an involution $\theta \in \mathrm{Aut}(\mathfrak{g})$. By design, $\mathfrak{k}$ is compactly embedded in $\mathfrak{g}$, and thus $(\mathfrak{g}, \theta)$ is an effective orthogonal symmetric Lie algebra. Note that $\mathfrak{p}$ is an abelian ideal in $\mathfrak{g}$. Inspired by this example, we say that a weakly effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ is of **Euclidean type** if $\mathfrak{p}$ is an abelian ideal in $\mathfrak{g}$ (it suffices to ask that $\mathfrak{p}$ is a subalgebra).

Observation 2.1.76. The notion of type of an orthogonal symmetric Lie algebra is clearly respected by isomorphisms.

Definition 2.1.77. A Riemannian symmetric pair (G, K) is said to be **of compact**, **noncompact**, or **Euclidean type** if so is its orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. A symmetric space M is said to be **of compact**, **noncompact**, or **Euclidean type** if so is its canonical Riemannian symmetric pair.

Remark 2.1.78. In the literature on the subject, it is common to refer to symmetric spaces of (non)compact type simply as *(non)compact symmetric spaces*. We will also sometimes do this if there is no chance of ambiguity.

Observation 2.1.79. The notion of type of a Riemannian symmetric pair is also respected by isomorphisms. Therefore, the notion of type of a symmetric space is well defined (by Lemma 2.1.30) and respected by isometries.

Proposition 2.1.80. *Let X stand for "compact", "noncompact", or "Euclidean".*

(a) *If a Riemannian symmetric pair is of type X, then so is its corresponding symmetric space G/K for any choice of a G-invariant metric. In other words, if M is represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ of type X, then M is of type X.*

(b) *Conversely, if a symmetric space M is of type X, then so is every infinitesimally weakly effective Riemannian symmetric pair and weakly effective orthogonal symmetric Lie algebra that represents M.*

Definition 2.1.77 together with Proposition 2.1.13 immediately implies the following

Corollary 2.1.81. *Let (G, K) be a Riemannian symmetric pair. If it is of compact (resp., noncompact) type, then G is compact (resp., noncompact) semisimple.*

Example 2.1.82. Recall from Example 2.1.38 that a compact connected Lie group G endowed with a bi-invariant metric is a symmetric space. Now, G is represented by a Riemannian symmetric pair $(G \times G, \Delta_G)$, so if G is compact semisimple, it is a symmetric space of compact type. //

Example 2.1.83. The Cartan involution $\theta(X) = -X^t$ on $\mathfrak{sl}(n, \mathbb{R})$ admits a lift to an involution $\Theta(A) = (A^t)^{-1}$ on $\mathrm{SL}(n, \mathbb{R})$. This gives rise to a Riemannian symmetric pair $(\mathrm{SL}(n, \mathbb{R}), \mathrm{SO}(n))$ and thus a symmetric space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ of noncompact type. This space is special due to the following fact: every symmetric space M of noncompact type can be realized as a totally geodesic submanifold of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ for some n. This can be deduced relatively easily from Proposition 2.1.71 and Proposition 2.2.12.

Now we discuss why symmetric spaces of the above three types are the building blocks for all symmetric spaces.

Proposition 2.1.84. *Let $(\mathfrak{g}, \theta)$ be a weakly effective orthogonal symmetric Lie algebra. Then there exist ideals $\mathfrak{g}_0, \mathfrak{g}_{\mathrm{c}}, \mathfrak{g}_{\mathrm{nc}}$ in $\mathfrak{g}$ such that:*

(a) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\mathrm{c}} \oplus \mathfrak{g}_{\mathrm{nc}}$.

(b) *Each of the three ideals is invariant under θ. In particular, we have $\theta = (\theta_0, \theta_{\mathrm{c}}, \theta_{\mathrm{nc}})$.*

(c) $(\mathfrak{g}_0, \theta_0)$ *is an orthogonal symmetric Lie algebra of Euclidean type.*

(d) $(\mathfrak{g}_{\mathrm{c}}, \theta_{\mathrm{c}})$ *is an orthogonal symmetric Lie algebra of compact type.*

(e) $(\mathfrak{g}_{\mathrm{nc}}, \theta_{\mathrm{nc}})$ *is an orthogonal symmetric Lie algebra of noncompact type.*

The corresponding subspaces $\mathfrak{p}_0, \mathfrak{p}_c, \mathfrak{p}_{nc}$ of $\mathfrak{p}$ are uniquely determined and do not depend on the choice of $\mathfrak{g}_0, \mathfrak{g}_c$, and $\mathfrak{g}_{nc}$. If $(\mathfrak{g}, \theta)$ is effective, then $\mathfrak{g}_0, \mathfrak{g}_c$, and $\mathfrak{g}_{nc}$ are uniquely determined.

Definition 2.1.85. If $(\mathfrak{g}, \theta)$ is an effective orthogonal symmetric Lie algebra, the ideals $\mathfrak{g}_0, \mathfrak{g}_c, \mathfrak{g}_{nc}$ are called its **Euclidean**, **compact**, and **noncompact part**, respectively.

A proof of Proposition 2.1.84 can be found in [Hel01, Th.V.1.1]. Let us now discuss the geometric equivalent of Proposition 2.1.84. Given a symmetric space M, let $V_0, V_1, \dots, V_k$ and $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_k$ be its de Rham distributions and foliations, respectively. Let $\boldsymbol{V}_{\mathbf{c}}$ (resp., $\boldsymbol{V}_{\mathbf{nc}}$) be the sum of all V_i whose corresponding irreducible part (leaf of $\mathcal{F}_i$) is of compact (resp., noncompact) type.

Proposition 2.1.86. (a) *The distributions V_c and V_{nc} are parallel (hence involutive), and we have an orthogonal decomposition $TM = V_0 \oplus V_c \oplus V_{nc}$.*

(b) *Let $\mathcal{F}_c$ and $\mathcal{F}_{nc}$ stand for the foliations corresponding to V_c and V_{nc}, respectively. Each of these two foliations has properly embedded totally geodesic leaves that are all congruent to each other by means of the action of $I^0(M)$. The leaves of $\mathcal{F}_c$ (resp., $\mathcal{F}_{nc}$) are symmetric spaces of compact (resp., noncompact) type.*

(c) *Suppose M is represented by a weakly effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_c \oplus \mathfrak{g}_{nc}$ as in Proposition 2.1.84. Then $\mathfrak{p}_0 = (V_0)_o, \mathfrak{p}_c = (V_c)_o$, and $\mathfrak{p}_{nc} = (V_{nc})_o$. Moreover, $(\mathfrak{g}_0, \theta_0), (\mathfrak{g}_c, \theta_c)$, and $(\mathfrak{g}_{nc}, \theta_{nc})$ naturally represent the leaves of $\mathcal{F}_0, \mathcal{F}_c$, and $\mathcal{F}_{nc}$, respectively.*

(d) *If M is simply connected, it naturally decomposes as a Riemannian product $M = M_0 \times M_c \times M_{nc}$. In terms of the de Rham decomposition of M, M_c (resp., M_{nc}) is the product of all the irreducible de Rham factors of compact (resp., noncompact) type.*

*We call V_c and $\mathcal{F}_c$ (resp, V_{nc} and $\mathcal{F}_{nc}$) the **compact** (resp., **noncompact**) **distribution** and **foliation** of M, respectively. Any leaf of $\mathcal{F}_c$ (resp., $\mathcal{F}_{nc}$) is called the **compact** (resp., **noncompact**) **part** of M.*

Example 2.1.87 (The unitary group). Consider the group $\mathrm{U}(n)$ ($n \geq 2$) endowed with a bi-invariant metric. Its Euclidean part if isometric to the circle and coincides with the center $Z(\mathrm{U}(n)) = \left\{e^{i\lambda}E \mid \lambda \in \mathbb{R}\right\}$. The compact part is irreducible and given by $\mathrm{SU}(n) \subseteq \mathrm{U}(n)$, and the noncompact part is trivial. Note that $\mathrm{U}(n)$ does not split as the product of its Euclidean and compact parts because they intersect at n points (the intersection is the subgroup of n-th roots of unity inside $Z(\mathrm{U}(n))$). //

It turns out that the noncompact part of a symmetric space always splits off as a Riemannian factor:

Proposition 2.1.88. *Any symmetric space M decomposes as a Riemannian product of its noncompact part M_{nc} and a symmetric space M' with a trivial noncompact part*[1]*.*

Conceptually, Proposition 2.1.88 owes to the fact that a symmetric space of noncompact type is Hadamard (Proposition 2.1.92) and its group of inner isometries is centerless

[1] In other words, the universal Riemannian covering space of M' is the Riemannian product of a Euclidean space with a symmetric space of compact type.

(Corollary 2.4.5). Using expression (2.1.5), one can relate the type of a symmetric spaces to its curvature:

Proposition 2.1.89 (Type vs curvature). *Let M be a symmetric space.*

(a) *M is of Euclidean type $\Leftrightarrow$ M is flat.*

(b) *M is of compact type $\Leftrightarrow$ M does not have a flat local factor and is of nonnegative sectional curvature.*

(c) *M is of noncompact type $\Leftrightarrow$ M does not have a flat local factor and is of nonpositive sectional curvature.*

Corollary 2.1.90. *A symmetric space is of type X if and only if its universal Riemannian covering space is of type X.*

Using Proposition 2.1.89, one can derive some basic geometric properties of the three types of symmetric spaces.

Proposition 2.1.91. *A symmetric space of Euclidean type is a Riemannian product of a Euclidean space and a flat torus.*

Regarding symmetric spaces of noncompact type, one can prove that they are always simply connected (see Proposition 2.4.1). Since they are also of nonpositive sectional curvature, we have:

Proposition 2.1.92. *A symmetric space M of noncompact type is a Hadamard manifold. In particular, M is diffeomorphic to a Euclidean space. In fact, for every $p \in M$, $\exp_p \colon T_pM \to M$ is a diffeomorphism.*

Finally, a symmetric space of compact type is a quotient of a compact connected semisimple Lie group by a compact subgroup. Using the long exact sequence of homotopy groups, one obtains:

Proposition 2.1.93. *A symmetric space of compact type is compact and has a finite fundamental group.*

Remark 2.1.94. Symmetric spaces of compact type have positive Ricci curvature by Proposition 2.1.89(b). The assertion of Proposition 2.1.93 then also follows from Myers's theorem.

The following proposition singles out a special class of symmetric spaces:

Proposition 2.1.95 (Semisimplicity criteria). *Let M be a symmetric space. The following are equivalent:*

(a) *M does not have a flat local factor.*

(b) *The Euclidean part of M is trivial.*

(c) *M is a Riemannian product of symmetric spaces of compact and noncompact type.*

(d) *The isometry group $I(M)$ is semisimple*[1]*.*

(e) *For some ($\Leftrightarrow$ any) infinitesimally weakly effective Riemannian symmetric pair (G, K) representing M, G is semisimple.*

[1] We do not require semisimple Lie groups to be connected in general.

(f) *For some (⇔ any) weakly effective orthogonal symmetric Lie algebra* $(\mathfrak{g}, \theta)$ *representing* M*,* $\mathfrak{g}$ *is semisimple.*

Definition 2.1.96 (Semisimplicity). A symmetric space is called **semisimple** if it satisfies the conditions in Proposition 2.1.95. A Riemannian symmetric pair (G, K) (resp., an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$) is called **semisimple** if G (resp., $\mathfrak{g}$) is semisimple.

Suppose a symmetric space M is represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ (resp., a Riemannian symmetric pair (G, K)). If the latter is semisimple, then so is M. The converse is not true: provided M is semisimple, $(\mathfrak{g}, \theta)$ (resp., (G, K)) is semisimple if and only if it is (infinitesimally) weakly effective.

Agreement. Whenever a semisimple symmetric space is represented by an orthogonal symmetric Lie algebra (or a Riemannian symmetric pair), we are going to assume by default that the latter is semisimple, unless otherwise stated.

Together with compact connected Lie groups, semisimple symmetric spaces belong to a larger class of better-behaving symmetric spaces that admits a number of characterizations.

Proposition 2.1.97. *The following are equivalent for a symmetric space* M*:*

(a) *The Euclidean part of* M *is compact.*

(b) M *is a Riemannian product of a compact symmetric space with a symmetric space of noncompact type.*

(c) *The canonical orthogonal symmetric Lie algebra* $(\mathfrak{g}, \theta)$ *of* M *has* $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$*.*

(d) *For every Riemannian symmetric pair* (G, K) *representing* M*, the morphism* $G \to I^0(M)$ *is surjective. In other words,* $G/I \cong I^0(M)$*.*

(e) *For some (⇔ every)* $p \in M$*,* $\mathrm{Hol}^0(M, p)$ *coincides with the restricted linear isotropy group* $\overline{K}^0$ *at* p*. In other words,* $\mathrm{Hol}(M, p)$ *is an open subgroup of the linear isotropy group* $\overline{K}$*.*

(f) $\mathrm{Tr}(M) = I^0(M)$*.*

If these conditions are satisfied, we say that M ***has compact Euclidean part****. In this case, there is a unique effective Riemannian symmetric pair (resp., orthogonal symmetric Lie algebra) representing* M *up to isomorphism—the canonical one.*

Sketch of the proof. To begin with, (c) is equivalent to (e) by Proposition 2.1.67, to (f) by Proposition 2.1.65, and implies (d) by Observation 2.1.33. For any symmetric space M, $(\mathrm{Tr}(M), \mathrm{Tr}(M)_o)$ is a Riemannian symmetric pair representing M, so (d) implies (f). Next, (a) is equivalent to (b) by Proposition 2.1.88. The equivalence of (a) and (c) can be shown by passing to the universal Riemannian covering space. One of the main steps is to show that if $(\mathfrak{g}, \theta)$ is an effective semisimple orthogonal symmetric Lie algebra, then $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$. In the noncompact case, this is the content of problems 22-25 in [Kna02, Sect. VI] (see p. 558 for a solution). In general, this follows from the proof of Proposition 2.1.84 in [Hel01, Th.V.1.1]. □

Example 2.1.98. By Proposition 2.1.95(b), any semisimple symmetric space has compact Euclidean part. //

Example 2.1.99. A compact connected Lie group G with a bi-invariant metric has compact Euclidean part, which follows trivially by Proposition 2.1.97(b). As we saw in Example 2.1.38, G can be represented by the Riemannian symmetric pair $(G \times G, \Delta_G)$, whose inefficiency kernel is given by $I = \Delta_Z$. It then follows from Proposition 2.1.97(d) that $I^0(G) \cong (G \times G)/\Delta_Z$. //

Remark 2.1.100. Part (e) of Proposition 2.1.97 means that for a symmetric space M with compact Euclidean part, the restricted holonomy and isotropy representations are the same thing. If M is represented by a Riemannian symmetric pair (G, K), they are both given by the adjoint representation of K^0 on $\mathfrak{p}$ (modulo the kernel).

2.1.5. Irreducibility, duality, and the classification

In this final part of the section, we discuss the classification of symmetric spaces. To that end, we first need to talk about two more crucial ingredients in this theory: irreducibility and duality. Once again, the primary reference here is [Hel01].

Irreducibility

The property of being irreducible has a number of useful reformulations for symmetric spaces.

Proposition 2.1.101 (Irreducibility criteria). *The following are equivalent for a symmetric space M:*

(a) *M is irreducible.*

(b) *M is not flat and its restricted isotropy representation is irreducible.*

(c) *For some ($\Leftrightarrow$ any) Riemannian symmetric pair (G, K) representing M, (G, K) is not of Euclidean type, and its restricted isotropy representation is irreducible.*

(d) *For some ($\Leftrightarrow$ any) orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ representing M, $(\mathfrak{g}, \theta)$ is not of Euclidean type, and the adjoint representation $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ is irreducible.*

Definition 2.1.102. A Riemannian symmetric pair (G, K) is called **irreducible** if it is not of Euclidean type and the representation of K^0 on $\mathfrak{p}$ is irreducible. An orthogonal symmetric Lie algebra is called **irreducible** if it is not of Euclidean type and the representation of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible.

Proposition 2.1.101 essentially means that the three notions of irreducibility agree: if $(\mathfrak{g}, \theta)$ is an orthogonal symmetric Lie algebra associated with a Riemannian symmetric pair (G, K), and they represent a symmetric space M, then $(\mathfrak{g}, \theta)$ is irreducible $\Leftrightarrow$ (G, K) is irreducible $\Leftrightarrow$ M is irreducible. From Proposition 2.1.84, we also have:

Corollary 2.1.103. *An irreducible symmetric space (or a weakly effective orthogonal symmetric Lie algebra, or an infinitesimally weakly effective Riemannian symmetric pair) is semisimple and in fact of either compact or noncompact type.*

In compliance with our agreement on page 43, *if M is an irreducible symmetric space, any orthogonal symmetric Lie algebra (or Riemannian symmetric pair) representing it is assumed to be (infinitesimally) weakly effective by default.*

Proposition 2.1.104. *An orthogonal symmetric Lie algebra* $(\mathfrak{g}, \theta)$ *is irreducible if and only if* $\mathfrak{k}$ *is a maximal proper subalgebra of* $\mathfrak{g}$*. A Riemannian symmetric pair* (G, K) *is irreducible if and only if* K^0 *is a maximal proper connected Lie subgroup of* G*.*

Proof. We need only prove the first statement. If $\mathfrak{k} \subsetneq \mathfrak{h} \subsetneq \mathfrak{g}$ is a larger proper subalgebra, then $\mathfrak{h} \cap \mathfrak{p} \subsetneq \mathfrak{p}$ is a nontrivial proper subrepresentation of $\mathfrak{k}$. Conversely, if $V \subsetneq \mathfrak{p}$ is such a subrepresentation, consider the subalgebra $\mathfrak{h} = \mathfrak{k} \oplus V$. □

Orthogonal symmetric Lie algebras can be decomposed into irreducible parts, which can be regarded as the infinitesimal version of the de Rham decomposition for symmetric spaces:

Proposition 2.1.105. *Let* $(\mathfrak{g}, \theta)$ *be a weakly effective orthogonal symmetric Lie algebra. There exist ideals* $\mathfrak{g}_i, 0 \leq i \leq k$*, such that:*

(a) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$.

(b) θ *preserves* $\mathfrak{g}_i$ *for every* $0 \leq i \leq k$*. In particular,* θ *can be written* $(\theta_0, \theta_1, \ldots, \theta_k)$.

(c) *Each* $(\mathfrak{g}_i, \theta_i)$ *is a weakly effective orthogonal symmetric Lie algebra.*

(d) $(\mathfrak{g}_0, \theta_0)$ *is of Euclidean type, and* $(\mathfrak{g}_i, \theta_i)$ *is irreducible for* $1 \leq i \leq k$.

If we write $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$*, then* $\mathfrak{p}_i$*'s are uniquely determined and do not depend on the choice of* $\mathfrak{g}_i$*'s. Moreover, if* $(\mathfrak{g}, \theta)$ *is effective, the ideals* $\mathfrak{g}_i$ *are unique.*

Remark 2.1.106. Let $(\mathfrak{g}, \theta)$ be an effective orthogonal symmetric Lie algebra decomposed as $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ as in Proposition 2.1.105. Then the Euclidean part of $\mathfrak{g}$ is $\mathfrak{g}_0$, while its compact (resp., noncompact) part is the sum of all $\mathfrak{g}_i$ such that $(\mathfrak{g}_i, \theta_i)$ is of compact (resp., noncompact) type.

Proposition 2.1.105 is closely related to the de Rham distributions:

Proposition 2.1.107. *Let* M *be a symmetric space represented by a weakly effective orthogonal symmetric Lie algebra* $(\mathfrak{g}, \theta)$*, and let* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ *be as in Proposition 2.1.105. Then* $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$ *is the canonical decomposition of* $\mathfrak{p} \cong T_oM$ *(in particular, its summands are pairwise orthogonal). If we let* $\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_k$ *stand for the corresponding de Rham foliations, then* $(\mathfrak{g}_i, \theta_i)$ *represents any leaf of* $\mathcal{F}_i$*.*

The normalizing constants

Having discussed irreducibility, we can now talk about the degree of freedom one has when choosing an invariant metric on a symmetric space and what role this choice plays. Before doing that, we need to say a few words about Schur's lemma and its validity over $\mathbb{R}$.

The easy part of Schur's lemma works trivially over any field: any nonzero morphism between two irreducible representations of a group is an isomorphism. The "hard" part—which asserts that the space of such morphisms is at most one-dimensional—is only applicable over algebraically closed fields. Nevertheless, there is a substitute if the ground field is $\mathbb{R}$. Indeed, let V be a real irreducible representation of a group G. Thanks to the easy part of Schur's lemma, the space $\mathrm{End}_G(V)$ is a finite-dimensional associative unital division algebra over $\mathbb{R}$. By the Frobenius theorem, it must be isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. In the latter two cases, there exists an invariant complex (resp., quaternionic) structure on V. Fix such a structure; in case of $\mathbb{H}$, pick a basis J_1, J_2, J_3 of the quaternionic structure and thus turn V into a left $\mathbb{H}$-module (see Subsection 2.5.2 for all the necessary definitions). Now assume that G is a compact Lie group and the representation is smooth. By a standard averaging argument, V admits an invariant Euclidean, Hermitian, or quaternion-Hermitian (depending on the dimension of $\operatorname{End}_G(V)$) inner product h. The real part g of h is an invariant symmetric $\mathbb{R}$-bilinear form on V. The space $\operatorname{Bil}_G(V)$ of invariant $\mathbb{R}$-bilinear forms on V is canonically isomorphic to $\operatorname{Hom}_G(V, V^*)$, hence it has the same dimension as $\operatorname{End}_G(V)$. It is easy to see that

$$\operatorname{Bil}_G(V) \text{ is spanned by } \begin{cases} g & \text{if } \operatorname{End}_G(V) \simeq \mathbb{R}, \\ g, \omega & \text{if } \operatorname{End}_G(V) \simeq \mathbb{C}, \\ g, \omega_1, \omega_2, \omega_3 & \text{if } \operatorname{End}_G(V) \simeq \mathbb{H}, \end{cases}$$

where $\omega(v, w) = g(Iv, w)$, I is the complex structure on V, and $\omega_i(v, w) = g(J_i v, w)$. As the forms ω and ω_i are skew-symmetric, we see that the space of invariant symmetric $\mathbb{R}$-bilinear forms on V is 1-dimensional and spanned by g. We deduce:

Corollary 2.1.108. *Let V be an irreducible real representation of a compact Lie group G. There exists a unique—up to rescaling by a positive constant—G-invariant Euclidean inner product on V.*

Corollary 2.1.109. *If (G, K) is an irreducible Riemannian symmetric pair, then there is a unique G-invariant Riemannian metric on $M = G/K$ up to rescaling by a positive constant.*

Now, let $(\mathfrak{g}, \theta)$ be a weakly effective irreducible orthogonal symmetric Lie algebra. There is a natural $\mathfrak{k}$-invariant inner product on $\mathfrak{p}$:

$$\langle - | - \rangle_B = \begin{cases} -B\big|_{\mathfrak{p}\times\mathfrak{p}} & \text{if } (\mathfrak{g}, \theta) \text{ is of compact type,} \\ B\big|_{\mathfrak{p}\times\mathfrak{p}} = B_\theta\big|_{\mathfrak{p}\times\mathfrak{p}} & \text{if } (\mathfrak{g}, \theta) \text{ is of noncompact type.} \end{cases} \tag{2.1.8}$$

Owing to Corollary 2.1.108, any other $\mathfrak{k}$-invariant inner product on $\mathfrak{p}$ is proportional to $\langle - | - \rangle_B$. More generally, let $(\mathfrak{g}, \theta)$ be a weakly effective orthogonal symmetric Lie algebra, and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be as in Proposition 2.1.105. The restriction of B to $\mathfrak{p}$ has kernel $\mathfrak{p}_0$, the other summands $\mathfrak{p}_i$ are pairwise orthogonal with respect to it, and we have $B\big|_{\mathfrak{p}\times\mathfrak{p}} = B_1\big|_{\mathfrak{p}_1\times\mathfrak{p}_1} + \cdots + B_k\big|_{\mathfrak{p}_k\times\mathfrak{p}_k}$. Combining these arguments with Proposition 2.1.107, we arrive at the following important

Corollary 2.1.110. *Let $(\mathfrak{g}, \theta)$ be a weakly effective orthogonal symmetric Lie algebra, and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be as in Proposition 2.1.105. For any $\mathfrak{k}$-invariant inner product $\langle - | - \rangle$ on $\mathfrak{p}$, the summands $\mathfrak{p}_i$ are pairwise orthogonal, and we have*

$$\langle - | - \rangle = \langle - | - \rangle_0 + \lambda_1 \langle - | - \rangle_{B_1} + \cdots + \lambda_k \langle - | - \rangle_{B_k}, \tag{2.1.9}$$

where $\langle - | - \rangle_0$ is a $\mathfrak{k}_0$-invariant inner product on $\mathfrak{p}_0$, and $\lambda_i > 0$. The constants λ_i do not depend on the choice of such a decomposition of $(\mathfrak{g}, \theta)$.

In particular, if M is a symmetric space, one can represent it by some weakly effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ and decompose g_o as in (2.1.9). It turns out that the resulting constants λ_i are invariants of M.

Proposition 2.1.111. *Let M be a symmetric space represented by a weakly effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. Decompose the $\mathfrak{k}$-invariant inner product g_o on $\mathfrak{p}$ as in (2.1.9) and let $\lambda_1, \dots, \lambda_k > 0$ be the resulting constants. These constants depend neither on the choice of a weakly effective orthogonal symmetric Lie algebra representing M, nor on the choice of a base point. If M' is another space isometric to M, it has the same constants up to reordering.*

A general proof of this result is complicated by the fact that M may not be simply connected. To deal with this, one has to prove some structure results on the deck transformation group of the universal Riemannian covering of M. We will prove this statement in the special case when M is simply connected and semisimple.

Proof. Let M be a simply connected semisimple symmetric space represented by a weakly effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. By Proposition 2.1.95, $\mathfrak{g}$ is semisimple, so it splits as a sum of two ideals $\mathfrak{i} \oplus \mathfrak{g}'$, where $\mathfrak{i}$ is the ineffectiveness kernel. The involution θ respects this decomposition and is trivial on $\mathfrak{i}$. The Killing form of $\mathfrak{g}'$ is the restriction of that of $\mathfrak{g}$. This means that we can replace the initial orthogonal symmetric Lie algebra with an effective one, which, by Proposition 2.1.97, is simply the canonical one. So we take $G = I^0(M)$ and $K = G_o$. Let $M = M_1 \times \cdots \times M_k$ be the de Rham decomposition. By Corollary 2.1.61, $G = G_1 \times \cdots \times G_k$, where $G_i = I^0(M_i)$. The isometry Lie algebra splits accordingly as $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, and we have $g_o = \lambda_1 \langle - | - \rangle_{B_1} + \cdots + \lambda_k \langle - | - \rangle_{B_k}$. We need to prove that if M' is a space isomorphic to M with analogous decompositions at some o', then λ_i and λ'_i coincide up to reordering.

Let $f\colon M \xrightarrow{\sim} M'$ be an isometry. We may assume it maps o to o'. The uniqueness of de Rham decomposition (Remark 2.1.63) implies that M' has the same number of de Rham factors (say, $M' = M'_1 \times \cdots \times M'_k$), and f is of the form $(p_i)_{i=1}^k \mapsto (f_{\sigma^{-1}(i)}(p_{\sigma^{-1}(i)}))_{i=1}^k$, where $\sigma \in S_k$ and $f_i\colon M_i \xrightarrow{\sim} M'_{\varphi(i)}$ is an isometry. Let $F_i\colon G_i \xrightarrow{\sim} G'_{\varphi(i)}$ stand for $g \mapsto f_i \circ g \circ f_i^{-1}$ and $\varphi_i\colon \mathfrak{g}_i \xrightarrow{\sim} \mathfrak{g}'_{\varphi(i)}$ for $(F_i)_*$. We have the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{p}_i \arrow[r, "f_i", "\sim"'] \arrow[d, "\wr"'] & \mathfrak{p}'_{\sigma(i)} \arrow[d, "\wr"] \\
T_{o_i} M_i \arrow[r, "d(\varphi_i)_{o_i}", "\sim"'] & T_{o'_{\sigma(i)}} M'_{\sigma(i)}
\end{tikzcd}$$

The top arrow is an isometry with respect to the inner products $\langle - | - \rangle_{B_i}$ and $\langle - | - \rangle_{B'_{\sigma(i)}}$, whereas the bottom one is an isometry with respect to $(g_i)_{o_i} = \lambda_i \langle - | - \rangle_{B_i}$ and $(g'_{\sigma(i)})_{o'_{\sigma(i)}} = \lambda'_{\sigma(i)} \langle - | - \rangle_{B'_{\sigma(i)}}$. This implies that $\lambda_i = \lambda'_{\sigma(i)}$, which concludes the proof. □

Definition 2.1.112. The constants $\lambda_1, \dots, \lambda_k$ defined above are called the **normalizing constants** of M. If M is semisimple and all the normalizing constants are equal to 1, the metric on M is called **Killing**.

It is important to point out that if we have a symmetric space M represented by a Riemannian symmetric pair (G, K), and g_o is decomposed as in (2.1.9), we can rescale the normalizing constants however we want, and the resulting inner product will still be K-invariant, so it would give rise to another G-invariant metric on M. Roughly speaking, this procedure amounts to dilating the initial metric g by some positive constants along each of the irreducible de Rham foliations $\mathcal{F}_1, \ldots, \mathcal{F}_k$. We shall call this procedure **rescaling the normalizing constants** of M. One can show that the rescaled metric has the same group of inner isometries (although the full isometry group might change). On a semisimple symmetric space, the Killing metric is unique by Proposition 2.1.111. In a sense, it is the *canonical* metric defined purely algebraically, and the normalizing constants tell how much the Riemannian metric of M differs from the Killing one.

Another thing we can do in the semisimple case is to extend the Killing metric to an invariant inner product on the whole isometry Lie algebra. Let $(\mathfrak{g}, \theta)$ be a semisimple orthogonal symmetric Lie algebra, and let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be as in Proposition 2.1.105. Then we can define an inner product $\langle -|-\rangle_{B_i}$ on each $\mathfrak{g}_i$:

$$\langle -|-\rangle_{B_i} = \begin{cases} -B_i & \text{if } (\mathfrak{g}, \theta) \text{ is of compact type,} \\ (B_i)_{\theta_i} & \text{if } (\mathfrak{g}, \theta) \text{ is of noncompact type.} \end{cases} \tag{2.1.10}$$

Note that this agrees with (2.1.8). Adding these up and letting the ideals $\mathfrak{g}_i$ be mutually orthogonal leads to a $\mathfrak{k}$-invariant inner product[1] $\boldsymbol{\langle -|-\rangle_B} = \langle -|-\rangle_{B_1} + \cdots + \langle -|-\rangle_{B_k}$ on $\mathfrak{g}$. It is not hard to show that $\langle -|-\rangle_B$ does not depend on the choice of a decomposition of $\mathfrak{g}$ as above. This inner product proves especially useful in the noncompact type (where it can be written simply as B_θ). If there is no ambiguity, we will sometimes drop the subscript B and write this inner product simply as $\langle -|-\rangle$.

Warning. Suppose a semisimple orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ represents a symmetric space M. Then, there are two a priori distinct inner products on $\mathfrak{p}$: $\langle -|-\rangle_B\big|_{\mathfrak{p}\times\mathfrak{p}}$ and g_o. By definition, they coincide precisely when the metric is Killing. In the presence of M, we will normally write g_o as $\langle -|-\rangle_o$ or just $\langle -|-\rangle$, so one cannot drop the subscript of $\langle -|-\rangle_B$ in this case.

Remark 2.1.113. When we have a specific symmetric space M represented by a Riemannian symmetric pair (G, K), we never impose any restrictions on the choice of a G-invariant metric on M, unless otherwise stated. If M is irreducible, such a metric is unique up to a constant by Corollary 2.1.109, but in general, there exists a host of such metrics. For example, the Grassmannian $\mathrm{Gr}(2, \mathbb{R}^4) = \mathrm{SO}(4)/\mathrm{S}(\mathrm{O}(2)\mathrm{O}(2))$ admits a 2-dimensional family of $\mathrm{SO}(4)$-invariant symmetric metrics. So when we say that another symmetric space M_1 is isometric to M, we mean isometric with respect to *some* G-invariant metric. One needs to be cautious when given yet another space M_2 "isometric to M": unless the isometries $M_1 \simeq M$ and $M_2 \simeq M$ are with respect to the same metric on M, M_1 and M_2 may not be mutually isometric. (In the irreducible case, they would be homothetic.)

Recall from (2.1.6) that the Ricci curvature of a symmetric space is given by $Ric_o = -\frac{1}{2}B\big|_{\mathfrak{p}\times\mathfrak{p}}$. The normalizing constants can thus be used to formulate when a symmetric space is Einstein:

Proposition 2.1.114 (Einsteinness criterion). *A symmetric space M is Einstein if and only if it is of Euclidean ($Ric = 0$), compact ($Ric > 0$), or noncompact ($Ric < 0$) type, and in the latter two cases its normalizing constants need to be all equal to each other (so the metric has to be proportional to the Killing one). In particular, irreducible symmetric spaces are Einstein.*

[1] This notation might be a little ambiguous as it does not capture the dependence of $\langle -|-\rangle_B$ on θ.

Duality

There is a 1-to-1 correspondence between (simply connected) symmetric spaces of compact and noncompact type. This remarkable feature is called the *duality*. A symmetric space M and its dual M^* share a lot in common, including rank, irreducibility, holonomy, the set of totally geodesic submanifolds, etc. As we will witness repeatedly throughout the thesis, results obtained for M often carry over essentially for free to M^*.

Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra. Inside the complexification $\mathfrak{g}_{\mathbb{C}}$, consider the subspace $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$. It is straightforward to verify that this is a $\mathbb{Z}/2\mathbb{Z}$-graded subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let us denote the resulting involutive automorphism $(X + iY \mapsto X - iY)$ by θ^*.

Proposition 2.1.115. *Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra.*

(a) *$(\mathfrak{g}^*, \theta^*)$ is an orthogonal symmetric Lie algebra. It is called the* ***dual*** *of $(\mathfrak{g}, \theta)$.*

(b) *If $(\mathfrak{g}, \theta)$ and $(\mathfrak{g}', \theta')$ are isomorphic, then so are $(\mathfrak{g}^*, \theta^*)$ and $(\mathfrak{g}'^*, \theta'^*)$.*

(c) *If $(\mathfrak{g}, \theta)$ is of compact type, then $(\mathfrak{g}^*, \theta^*)$ is of noncompact type, and vice versa.*

Now we carry the dualization construction over to symmetric spaces. Since the Euclidean case is of little interest, we confine our attention to semisimple symmetric spaces. Let M be a simply connected semisimple symmetric space. Take any orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$ representing M and consider its dual $(\mathfrak{g}^*, \theta^*)$. Take a simply connected Lie group G^* with Lie algebra $\mathfrak{g}^*$, lift θ^* to an involutive automorphism Θ^* of G^*, and take $K^* = (G^{*\Theta^*})^0$ to be the connected Lie subgroup of G^* corresponding to $\mathfrak{k} \subset \mathfrak{g}^*$. If $\langle - \,|\, - \rangle_o$ stands for the Riemannian metric of M at o, then define an inner product on $i\mathfrak{p}$ by the formula $\langle iX \,|\, iY \rangle^* = \langle X \,|\, Y \rangle_o$. It is clearly K^*-invariant, so it makes $M^* = G^*/K^*$ into a simply connected semisimple symmetric space, which we call the **dual** of M. Let us denote the base point eK^* of M^* by o^*. Note that we have a natural isometric isomorphism $T_oM \cong \mathfrak{p} \cong i\mathfrak{p} \cong T_{o^*}M^*$. The following are some basic properties of duality:

Proposition 2.1.116 (Properties of duality). *Let M be a simply connected semisimple symmetric space.*

(a) *M^* does not depend on the choice of $o \in M$ up to isometry.*

(b) *$M^{**} \simeq M$.*

(c) *If N is another simply connected semisimple symmetric space isometric to M, then*[1] *$N^* \simeq M^*$.*

(d) $\dim(M^*) = \dim(M)$.

(e) $\mathrm{rk}(M^*) = \mathrm{rk}(M)$.

(f) *If $M = M_1 \times \cdots \times M_k$ is the de Rham decomposition of M, then $M^* = M_1^* \times \cdots \times M_k^*$ is the de Rham decomposition of M^*. In particular, M is irreducible if and only if M^* is.*

(g) *Under the identification $T_oM \cong T_{o^*}M^*$, the linear isotropy groups $\overline{K} \subseteq \mathrm{O}(T_oM)$ and*

[1] One can say that the duality is an involutive functor from the groupoid of simply connected semisimple symmetric spaces to itself.

$\overline{K}^* \subseteq \mathrm{O}(T_{o^*}M^*)$ *coincide*[1], *and hence so do the holonomy groups* $\mathrm{Hol}(M,o)$ *and* $\mathrm{Hol}(M^*,o^*)$.

(h) *Under the identification* $T_oM \cong T_{o^*}M^*$, *the curvatures of* M *and* M^**—be it* R, Rm, Ric, S, *or* K*—are of opposite signs.*

(i) *If* M *of compact type, then* M^* *is of noncompact type, and vice versa.*

The classification of symmetric spaces

In this final part of the section, we discuss how the introduction of Lie theory to the theory of symmetric spaces ultimately leads to their classification. The first step is to observe how irreducibility of a symmetric space is related to whether its isometry Lie algebra is simple.

Observation 2.1.117. Let $\mathfrak{g}$ be a semisimple Lie algebra with a Cartan involution θ, and let $\mathfrak{g} = \bigoplus_{\mu=1}^{k} \mathfrak{g}_\mu \oplus \bigoplus_{\nu=k+1}^{n} \mathfrak{g}_\nu$ be its decomposition into simple ideals, where the first k ideals are compact (whose sum we call the **compact part** of $\mathfrak{g}$) and the rest are noncompact (whose sum we call the **noncompact part**[2] of $\mathfrak{g}$). Then, by Proposition 2.1.72, θ respects this decomposition and is the identity on the compact ideals. Consequently, an orthogonal symmetric Lie algebra $(\mathfrak{g},\theta)$ of noncompact type is effective if and only if $\mathfrak{g}$ has no nontrivial compact ideals. In particular, the isometry Lie algebra of a symmetric space of noncompact type has no nontrivial compact ideals.

Combining Observation 2.1.117 with Proposition 2.1.107, we obtain:

Proposition 2.1.118. *An effective orthogonal symmetric Lie algebra* $(\mathfrak{g},\theta)$ *of noncompact type is irreducible if and only if* $\mathfrak{g}$ *is simple. Consequently, a symmetric space of noncompact type is irreducible if and only if the Lie group* $I^0(M)$ *is topologically simple (*$\Leftrightarrow$ *the isometry Lie algebra* $\mathfrak{i}(M)$ *is simple).*

Later, we will see that for M irreducible of noncompact type, $I^0(M)$ is actually simple in the group-theoretic sense (see Corollary 2.4.5). However, this is no longer true in the compact type, even on the level of Lie algebras. For example, a compact topologically simple Lie group G with a bi-invariant metric is an irreducible compact symmetric space, but its isometry Lie algebra $\mathfrak{g}\oplus\mathfrak{g}$ is a sum of two simple ideals. Fortunately, this is the only thing that can happen, as the following proposition shows. Recall that a complex Lie algebra that is simple over $\mathbb{C}$ is also simple over $\mathbb{R}$ ([Kna02, Prop. 6.95]). In this case, we are going to say that it is simple without specifying the ground field.

Proposition 2.1.119. *Let* $(\mathfrak{g},\theta)$ *be an effective irreducible orthogonal symmetric Lie algebra. Then exactly one of the following holds:*

(I) $\mathfrak{g}$ *is compact simple.*

(II) $\mathfrak{g}$ *is compact and it splits as a sum of two isomorphic simple ideals interchanged by* θ.

(III) $\mathfrak{g}$ *is noncompact simple and it does not admit a complex structure making it into a complex Lie algebra* $\Leftrightarrow$ $\mathfrak{g}$ *is noncompact and* $\mathfrak{g}_\mathbb{C}$ *is simple.*

[1] Note that this follows directly from Proposition 2.1.68

[2] If $(\mathfrak{g},\theta)$ is an effective semisimple orthogonal symmetric Lie algebra, then the compact and noncompact parts of $\mathfrak{g}$ as defined here coincide with those defined in Proposition 2.1.84.

(IV) $\mathfrak{g}$ *is noncompact simple and it admits a complex structure making it into a complex Lie algebra* $\Leftrightarrow$ $\mathfrak{g}$ *is noncompact and* $\mathfrak{g}_{\mathbb{C}}$ *splits (over* $\mathbb{C}$*) as a sum of two isomorphic simple ideals.*

Depending on the case, we are going to say that $(\mathfrak{g}, \theta)$ *is* ***of type I****,* ***II****,* ***III****, or* ***IV****, respectively.*

Definition 2.1.120. Let (G, K) be an infinitesimally effective irreducible Riemannian symmetric pair. We say that (G, K) is of type is **of type I**, **II**, **III**, or **IV** if so is its orthogonal symmetric Lie algebra. An irreducible symmetric space M is said to be **of type I**, **II**, **III**, or **IV** if so is some ($\Leftrightarrow$ any) infinitesimally effective Riemannian symmetric pair representing it.

The four types behave well with respect to duality:

Proposition 2.1.121. *Let* $(\mathfrak{g}, \theta)$ *be an effective irreducible orthogonal symmetric Lie algebra. Then:*

- $(\mathfrak{g}, \theta)$ *is of type I* $\Leftrightarrow$ $(\mathfrak{g}^*, \theta^*)$ *is of type III.*
- $(\mathfrak{g}, \theta)$ *is of type II* $\Leftrightarrow$ $(\mathfrak{g}^*, \theta^*)$ *is of type IV.*

The same is true for simply connected irreducible symmetric spaces and their duals.

Example 2.1.122. Recall from Example 2.1.82 that a compact connected semisimple Lie group G endowed with a bi-invariant metric is a symmetric space of compact type. From Proposition 2.1.40 we see that G is irreducible if and only if it is topologically simple. In this case, it is clearly of type II. //

It is clear that the simply connected irreducible symmetric spaces of type II are exhausted by simply connected compact topologically simple Lie groups. But can such a group have a symmetric quotient that is no longer a Lie group? This possibility is ruled out by the following (see [Hel01, Prop. X.1.2]):

Proposition 2.1.123 (Type II). *Any irreducible symmetric space of type II is isometric to a compact topologically simple Lie group with a bi-invariant metric.*

With this in mind, we have the following global description of types I-IV:

Proposition 2.1.124. *Let* M *be an irreducible symmetric space represented by an infinitesimally effective Riemannian symmetric pair* (G, K)*.*

(I) M *is of type I precisely when* G *is a compact topologically simple Lie group.*

(II) M *is of type II precisely when it is a compact topologically simple Lie group with a bi-invariant metric.*

(III) M *is of type III precisely when* G *is a noncomplex noncompact topologically simple Lie group.*

(IV) M *is of type IV precisely when* G *is a complex topologically simple Lie group.*

In (III) and (IV), G *can be chosen simple.*

Up to the question of coverings, the classification of symmetric spaces boils down to the classification of irreducible simply connected symmetric spaces—thanks to the de Rham decomposition. This is equivalent to classifying effective irreducible orthogonal symmetric

Lie algebras. By duality, it suffices to focus on the noncompact or compact type only. For type IV, this is simply the matter of classifying complex simple Lie algebras—which is classically done by means of root systems and Dynkin diagrams. This is also equivalent to classifying compact simple Lie algebras (type II). To deal with type III, one needs to classify all real simple (non-complex) Lie algebras—that is, classify real forms of all complex simple Lie algebras (see, e.g., [Kna02, Th. 6.105]). Equivalently, to settle type I, one needs to classify involutions of all compact simple Lie algebras. Thanks to Proposition 2.1.104, the classification of irreducible symmetric spaces can also be derived from Dynkin's classification of maximal subalgebras of semisimple Lie algebras ([Dyn52b]). For the full list of irreducible simply connected symmetric spaces, see [Hel01, Ch. X, Sect. 6] or [BCO16, pp. 414-417]. For the noncompact ones, see also [BCO16, pp. 336-340].

Definition 2.1.125. Irreducible symmetric spaces can be divided into two categories: M is called **classical** (resp., **exceptional**) if $I^0(M)$ is a classical (resp., exceptional) Lie group—or a product thereof.

For a background on exceptional Lie groups (including various embeddings between them), see [Yok09]. Here are some examples of exceptional symmetric spaces:

- Exceptional simply connected symmetric spaces of type II: E_6, E_7, E_8, F_4, G_2.
- Exceptional irreducible symmetric spaces of type IV:

$$E_6(\mathbb{C})/E_6,\ E_7(\mathbb{C})/E_7,\ E_8(\mathbb{C})/E_8,\ F_4(\mathbb{C})/F_4,\ G_2(\mathbb{C})/G_2.$$

- The Cayley projective and hyperbolic planes $\mathbb{O}P^2 = F_4/\mathrm{Spin}(9)$ and $\mathbb{O}H^2 = F_4^{-20}/\mathrm{Spin}(9)$.

2.2. Submanifold theory in symmetric spaces

In this section, we discuss some aspects of the submanifold theory in the context of symmetric spaces. We are mainly pursuing two goals: first, we will look at various classes of submanifolds in symmetric spaces and go through some of their properties; second, we will derive a convenient formula for the second fundamental form of a homogeneous submanifold that will prove useful later in the thesis. Our main reference for this part is [BCO16].

2.2.1. Types of submanifolds in symmetric spaces

Most types of submanifolds we are interested in follow a common pattern: they can be defined in more general Riemannian manifolds that are not necessarily symmetric, but they possess nice additional properties when the ambient space is symmetric. Perhaps, the most important such property that most of them acquire in the presence of symmetry is being *homogeneous*.

Homogeneous submanifolds

Definition 2.2.1. Let M be a Riemannian manifold. A complete submanifold $S \subseteq M$ is called

(a) (**extrinsically**) **homogeneous** if it is an orbit of an isometric Lie group action on M,

(b) **intrinsically homogeneous** if it is a Riemannian homogeneous space in the induced metric.

Remark 2.2.2. We will usually refer to extrinsically homogeneous submanifolds as just homogeneous. This notion is clearly stronger than being intrinsically homogeneous. Both of these types of submanifolds are automatically complete.

If $S \subseteq M$ is a homogeneous submanifold, its second fundamental forms at different points are essentially the same. In particular, if S is a hypersurface, it has constant principal curvatures and constant mean curvature.

Lemma 2.2.3. *Let M be a Riemannian manifold with $\pi_0(M)$ finite, and let $S \subseteq M$ be a properly embedded submanifold. Assume that for every $p, q \in S$, there exists an isometry of M that preserves S and maps p to q. Then S is a homogeneous submanifold. If, in addition, S is connected, then it is an orbit of an isometric action on M by a connected Lie group.*

Proof. Define

$$I(M, S) = \{f \in I(M) \mid f(S) = S\}.$$

Since S is closed, $I(M, S)$ is a closed ($\Rightarrow$ Lie) subgroup of $I(M)$. By assumption, it has S as one of its orbits. If S is connected, then the identity component $I^0(M, S)$ of $I(M, S)$ still acts transitively on S. □

In symmetric spaces, homogeneous submanifolds are one of the most natural types of submanifolds to study, since they retain some of the symmetry of the ambient space and can be studied by means of Lie theory. We make a useful observation regarding homogeneous submanifolds that we will be using repeatedly in the sequel.

Proposition 2.2.4. *Let M be a symmetric space represented by a Riemannian symmetric pair (G, K) and $H \subseteq G$ a Lie subgroup. Write $\mathfrak{h} = \mathrm{Lie}(H) \subseteq \mathfrak{g}$ and $S = H \cdot o$. Then, under the identification $T_oM \simeq \mathfrak{p}$, $T_oS = \mathrm{pr}_p(\mathfrak{h})$.*

Proof.

$$T_oS = \{\widehat{X}_o \mid X \in \mathfrak{h}\} = \{X_\mathfrak{p} \mid X \in \mathfrak{h}\} = \mathrm{pr}_\mathfrak{p}(\mathfrak{h}).$$

□

Totally geodesic submanifolds

Totally geodesic submanifolds are one of the most fundamental and well-known classes of Riemannian submanifolds. A generic Riemannian manifold admits no totally geodesic submanifolds of dimension greater than one—not even locally (see [MW19]). Things begin to change when the ambient space acquires a sufficient degree of symmetry. The extreme case of this is, of course, symmetric spaces, which do indeed admit an abundance of higher-dimensional totally geodesic submanifolds. At the same time, in a symmetric space, such a submanifold can be fully reduced to a (deceptively) simple piece of algebraic data, called a *Lie triple system*. In low rank, that data is manageable enough to allow a classification of totally geodesic submanifolds. But let us start from the beginning.

Observation 2.2.5. If M is a Riemannian manifold and $S \subseteq M$ is a complete connected totally geodesic submanifold, then for any $p \in S$, we have $S = \exp(T_pS)$. This observation, however trivial, will prove highly useful as we go along.

Lemma 2.2.6. *Let M be a symmetric space and $S \subseteq M$ a complete connected submanifold. The following are equivalent:*

(i) *S is totally geodesic.*

(ii) *For every $p \in S$, $s_p(S) = S$.*

Proof. (i) $\Rightarrow$ (ii). According to Observation 2.2.5, given any $p \in S$, $S = \exp(T_pS)$. As the geodesic symmetry at a point reverses geodesics through that point, s_p preserves S.
(ii) $\Rightarrow$ (i). We need to show that the second fundamental form $I\!I$ of S vanishes. Take an arbitrary $p \in S$ and any $X, Y \in T_pS$. We compute:

$$-I\!I(X,Y) = d(s_p)(I\!I(X,Y)) = I\!I(d(s_p)(X), d(s_p)(Y)) = I\!I(-X,-Y) = I\!I(X,Y),$$

hence $I\!I = 0$. □

Corollary 2.2.7. *A complete connected totally geodesic submanifold of a symmetric space is a symmetric space in its own right in the induced metric.*

Proof. Indeed, for every $p \in S$, s_p restricts to a geodesic symmetry of S at p. □

We will now see how all totally geodesic submanifolds of a symmetric space can be constructed solely in terms of Lie-theoretic data.

Definition 2.2.8. Let M be a Riemannian manifold and $p \in M$ any point. A subspace $V \subseteq T_pM$ is called **curvature-invariant** if $R(V,V)V \subseteq V$.

Example 2.2.9. Let $S \subseteq M$ be a totally geodesic submanifold. Then for any $p \in S$, T_pS is a curvature-invariant subspace of T_pM. This follows from the Gauss formula. //

Definition 2.2.10. Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra. A subset $V \subseteq \mathfrak{p}$ is called a **Lie triple system** if $[[V,V],V] \subseteq V$.

Thanks to the curvature formula (2.1.5) for symmetric spaces, we have the following:

Corollary 2.2.11. *Let M be a symmetric space represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. Under the identification $\mathfrak{p} \cong T_oM$, Lie triple systems in $\mathfrak{p}$ correspond precisely to curvature-invariant subspaces of T_oM.*

It is not true for general Riemannian manifolds that every curvature-invariant subspace is the tangent space of a totally geodesic submanifold (see [BCO16, Th. 10.3.3]). However, we are about to see that this is the case for symmetric spaces.

Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called **θ-stable** if it is preserved by θ, or equivalently if $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})$. The importance of θ-stable subalgebras stems from their intimate relationship with totally geodesic submanifolds.

Proposition 2.2.12. *Let M be a symmetric space represented by a Riemannian symmetric pair (G, K), $\mathfrak{h} \subseteq \mathfrak{g}$ a θ-stable subalgebra, and $H \subseteq G$ the connected Lie subgroup corresponding to $\mathfrak{h}$. Then the orbit $H \cdot o$ is a complete connected totally geodesic submanifold of M whose tangent space at o is $\mathfrak{h} \cap \mathfrak{p}$.*

Proof. Being a homogeneous submanifold, $S = H \cdot o$ is complete. According to Proposition 2.2.4, $T_o S = \mathrm{pr}_{\mathfrak{p}}(\mathfrak{h}) = \mathfrak{h} \cap \mathfrak{p}$, so we need only prove that S is totally geodesic. It suffices to show its second fundamental form $I\!I$ vanishes at o. Take any nonzero $X \in \mathfrak{h} \cap \mathfrak{p}$ and observe that $\exp_G(tX)$ is a one-parameter subgroup in H. Therefore, the curve $\exp_G(tX) \cdot o$ lies in S. But according to (2.1.2), this curve is a geodesic in M. In particular, $I\!I_o$ vanishes on X and thus $I\!I_o = 0$. □

It turns out that essentially every totally geodesic submanifold arises in this way. Indeed, let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra and $V \subseteq \mathfrak{p}$ a Lie triple system. It follows from the definition of a Lie triple system that $\mathfrak{h} = [V, V] \oplus V$ is a Lie subalgebra of $\mathfrak{g}$. Proposition 2.2.12 then implies:

Corollary 2.2.13. *Let M be a symmetric space.*

(a) *Every curvature-invariant subspace $V \subseteq T_p M$ is the tangent space of a unique complete connected totally geodesic submanifold of M, namely of* $\exp(V)$*.*

(b) *Every connected totally geodesic submanifold of M is an open part of a (unique) complete connected totally geodesic submanifold.*

(c) *Every complete connected totally geodesic submanifold of M is a homogeneous submanifold.*

We can summarize the discussion so far with the following commutative diagram:

$$
\begin{tikzcd}
\left\{\begin{matrix}\text{Lie triple}\\ \text{systems in } \mathfrak{p}\end{matrix}\right\} \arrow[r, hook, "V \mapsto [V,V]\oplus V"] \arrow[rr, bend right, "V \mapsto \exp_M(V)", "\sim"'] & \left\{\begin{matrix}\theta\text{-stable}\\ \text{subalgebras of } \mathfrak{g}\end{matrix}\right\} \arrow[r, two heads, "\mathfrak{h} \mapsto H\cdot o"] & \left\{\begin{matrix}\text{complete connected}\\ \text{t.g. submanifolds of } M\\ \text{passing through } o\end{matrix}\right\} \arrow[ll, bend right, "T_o S \mapsfrom S"', "\sim"]
\end{tikzcd}
$$

Remark 2.2.14. Let M be a symmetric space represented by a Riemannian symmetric pair (G, K). Let $V \subseteq \mathfrak{p}$ be a Lie triple system and $S = \exp_M(V)$ its corresponding complete totally geodesic submanifold. As we know from Corollary 2.2.7, S is a symmetric space in its own right. Consider the normalizer $N_{\mathfrak{k}}(V)$. By design, $[V, V] \subseteq N_{\mathfrak{k}}(V)$ and $N_{\mathfrak{k}}(V) \oplus V$ is a θ-stable subalgebra of $\mathfrak{g}$. It is not hard to show that $(N_{\mathfrak{k}}(V) \oplus V, \theta)$ is an orthogonal symmetric Lie algebra representing S.

Example 2.2.15 (T. g. submanifolds of $\mathbb{S}^n$). One can show that every subspace V of a tangent space to $\mathbb{S}^n$ is curvature-invariant. If $\dim(V) = k$, the corresponding complete totally geodesic submanifold is congruent (Definition 2.3.18) to the equatorial sphere $\mathbb{S}^k \subseteq \mathbb{S}^n$. An analogous statement is true for $\mathbb{R}P^n, \mathbb{R}H^n$, and symmetric spaces of Euclidean type. These are the only symmetric spaces exhibiting such a property. By a result of Iwahori, they can also be characterized as the only symmetric spaces with one de Rham factor admitting totally geodesic hypersurfaces (see [Iwa66] or else [BCO16, Th. 11.1.6]), as well as the only symmetric spaces of constant sectional curvature. //

Example 2.2.16 (T. g. submanifolds of compact Lie groups). Let G be a compact connected Lie group equipped with a bi-invariant metric, and let $H \subseteq G$ be a connected Lie subgroup. Then H is a complete totally geodesic submanifold. Indeed, it is plainly a homogeneous submanifold, and it is preserved by the geodesic symmetric $s_e(g) = g^{-1}$ at e. This easily implies that H is preserved by the geodesic symmetry at any of its points, so it is totally geodesic by Lemma 2.2.6. If we write $\mathfrak{h} = \mathrm{Lie}(H)$, then $\mathfrak{h}$ becomes the Lie triple system of $H \subseteq G$ under the standard identification $\mathfrak{p} \cong \mathfrak{g}$, $(X, -X) \leftrightarrow X$. //

Totally geodesic subspaces behave well with respect to duality. Let M be a simply connected semisimple symmetric space represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. Then the dual M^* is represented by $(\mathfrak{g}^*, \theta^*)$, $\mathfrak{g}^* = \mathfrak{k} \oplus i\mathfrak{p}$, and we have an obvious isomorphism between $\mathfrak{p}$ and $\mathfrak{p}^* = i\mathfrak{p}$.

Proposition 2.2.17 (Duality for t. g. submanifolds). *Let M be a simply connected semisimple symmetric space and M^* its dual.*

(a) *Under the isomorphism $\mathfrak{p} \cong \mathfrak{p}^*$, the Lie triple systems in $\mathfrak{p}$ correspond precisely to those in $i\mathfrak{p}$.*

(b) *There is a natural 1-to-1 correspondence between the set of complete connected totally geodesic submanifolds in M passing through o and the set of those in M^* passing through o^*, namely $\exp_o(V) \leftrightarrow \exp_{o^*}(iV)$.*

(c) *The correspondence in (b) induces a 1-to-1 correspondence between the set of congruence classes of complete connected totally geodesic submanifolds in M and the set of those in M^*.*

The latter statement in Proposition 2.2.17 can be deduced using Proposition 2.1.116.

Definition 2.2.18. If M is a simply connected semisimple symmetric space and $S \subseteq M$ a connected complete totally geodesic submanifold, we denote the corresponding complete connected totally geodesic submanifold of M^* by S^* and call it the **dual**[1] of S.

One of the big long-standing problems in the theory of symmetric spaces has been classification of totally geodesic submanifolds. Since each such submanifold is itself a symmetric space of rank not greater than that of the ambient space by Corollary 2.2.7, a sort of inductive procedure is possible, so it is reasonable to confine oneself to classifying *maximal* proper totally geodesic submanifolds first. Moreover, by duality, it suffices to restrict to either compact ot noncompact type. Still, finding all maximal Lie triple systems for a given symmetric space is a very complicated algebraic problem, so totally geodesic submanifolds have only been classified in symmetric space that are relatively simple in one way or another: in rank one due to Wolf ([Wol63]); in the irreducible spaces of rank two due to Chen, Nagano, and Klein ([CN77, CN78], [Kle10], as well as the previous 3 articles of Klein mentioned in the latter); in products or rank-one spaces due to Rodríguez-Vázquez ([RV22]); there is a classification of maximal nonsemisimple totally geodesic submanifolds due to Berndt and Olmos ([BO16]); finally, there is a classification

[1] It might happen that S, being a symmetric space, is itself simply connected semisimple, in which case there is possible ambiguity between its dual as of a totally geodesic submanifold and as of a symmetric space. The difference is insignificant though, as the latter is always going to be the universal Riemannian covering space of the former. In such a situation, we will always mean the former, unless otherwise stated.

of maximal totally geodesic submanifolds in exceptional symmetric spaces due to Kollross and Rodríguez-Vázquez ([KRV23]). See also the discussion on the index of a symmetric space at the beginning of Subsection 6.2.2.

We are not going to delve into the details and methods of the above papers but will prove one important result pertaining to maximal totally geodesic submanifolds. First of all, it is not hard to show that the following are equivalent for a symmetric space M:

(i) The Euclidean part of M is simply connected.

(ii) The Euclidean part of M is isometric to a Euclidean space.

(iii) The fundamental group of M is finite.

Recall that even in a symmetric space, a totally geodesic submanifold does not have to be embedded; for instance, in any symmetric space of compact type and rank greater than 1, one could take a dense geodesic in a maximal flat (see Definition 2.2.20). With the above equivalent conditions in mind, we can prove the following

Proposition 2.2.19. *Let M be a symmetric space whose Euclidean part is simply connected. Then every maximal connected proper totally geodesic submanifold of M is automatically properly embedded.*

Proof. Let M be represented by an effective Riemannian symmetric pair (G, K). Let $S \subset M$ be a maximal ($\Rightarrow$ complete) connected proper totally geodesic submanifold and $V = T_oS \subseteq \mathfrak{p}$ the corresponding Lie triple system. As we noticed in Remark 2.2.14, S is represented by $(\mathfrak{h}, \theta)$, where $\mathfrak{h} = N_{\mathfrak{k}}(V) \oplus V$. We claim that $\mathfrak{h}$ is self-normalizing. Indeed, if $X \in N_{\mathfrak{g}}(\mathfrak{h})$ and $Y \in \mathfrak{h}$, then

$$[\theta X, Y] = \theta[X, \theta Y] \in \theta([X, \mathfrak{h}]) \subseteq \theta(\mathfrak{h}) = \mathfrak{h},$$

so $N_{\mathfrak{g}}(\mathfrak{h})$ is itself θ-stable. The intersection of $N_{\mathfrak{g}}(\mathfrak{h})$ with $\mathfrak{p}$ is a Lie triple system containing V, so by maximality it must be either V or the whole $\mathfrak{p}$. First, consider the former case. The intersection of $N_{\mathfrak{g}}(\mathfrak{h})$ with $\mathfrak{k}$ then has to be $N_{\mathfrak{k}}(V)$ and thus $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. Let H be the connected Lie subgroup of G corresponding to $\mathfrak{h}$. This is a closed subgroup because $H = N_G^0(\mathfrak{h})$, and its orbit through o is S by Proposition 2.2.12. As we will discuss in Remark 2.3.3, closed subgroups of G have properly embedded orbits. Let now $N_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{k}$ be $\mathfrak{p}$. In this case, for any one-dimensional subspace $\ell \subseteq \mathfrak{p}$ not lying in V, the sum $V \oplus \ell$ is a larger Lie triple system, which implies that V must be a hyperplane in $\mathfrak{p}$. But then S is a totally geodesic hypersurface in M. If we write $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ as in Proposition 2.1.105, then, by maximality, we must have $V = V_i \oplus \bigoplus_{j \neq i} \mathfrak{p}_i$ with $V_i \subset \mathfrak{p}_i$ for some $0 \leq i \leq k$. We can thus assume M is irreducible or flat. As we mentioned in Example 2.2.15—and with our assumption on M in mind—M must be isometric to $\mathbb{S}^n, \mathbb{R}P^n, \mathbb{R}H^n$, or $\mathbb{E}^n$. But then S has to be a great hypersphere or a projective/hyperbolic/affine hyperplane, respectively; in each of these cases, S is clearly properly embedded. $\square$

Flats

Flats are a special type of totally geodesic submanifolds in symmetric spaces that are intimately related to the notion of rank.

Definition 2.2.20. Let M be a symmetric space. A **flat** in M is a complete connected flat totally geodesic submanifold. A **maximal flat** is a flat that is not contained in any larger flat.

Lemma 2.2.21. *Let M be a symmetric space represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. A complete connected totally geodesic submanifold $S \subseteq M$ passing through o is flat if and only if its corresponding Lie triple system $V = T_pS$ is an abelian subspace of $\mathfrak{p}$.*

Proof. If $\mathfrak{p}$ is abelian, the curvature of S is zero due to (2.1.5). Conversely, if S is flat, then it is a symmetric space of Euclidean type as follows from Corollary 2.2.7 and Proposition 2.1.89. But as we saw in Remark 2.2.14, S is represented by the orthogonal symmetric Lie algebra $(N_{\mathfrak{k}}(V) \oplus V, \theta)$. Now, thanks to Proposition 2.1.80, $(N_{\mathfrak{k}}(V) \oplus V, \theta)$ is also of Euclidean type, which means that V is abelian. □

We can thus draw a flat version of diagram on page 55:

$$\left\{\begin{matrix}\text{abelian}\\ \text{subspaces in } \mathfrak{p}\end{matrix}\right\} \underset{T_oS \leftarrow\!\shortmid S}{\overset{V \mapsto \exp_M(V)}{\rightleftarrows}} \left\{\begin{matrix}\text{flats in } M\\ \text{passing through } o\end{matrix}\right\}$$

Suppose M is represented by a Riemannian symmetric pair (G, K). If V is an abelian subspace of $\mathfrak{p}$, it is a subalgebra and its corresponding connected abelian Lie subgroup of G is $H = \exp_G(V)$. As we know from Proposition 2.2.12, the corresponding flat is the orbit $H \cdot o$.

Let us turn attention to maximal flats now. Can two maximal flats have different dimensions? The following proposition rules out this possibility:

Proposition 2.2.22. *Let M be a symmetric space represented by a Riemannian symmetric pair (G, K).*

(a) *G acts transitively on the set of pointed maximal flats in M, i.e., for any two maximal flats $S, S' \subseteq M$ and any $p \in S, p' \in S'$, there exists $g \in G$ mapping S onto S' and p to p'.*

(b) *K^0 acts transitively on the set of maximal abelian subspaces in $\mathfrak{p}$.*

(c) *Every tangent vector to M is tangent to some maximal flat.*

(d) *Every maximal flat in M is properly embedded.*

Proof. To begin with, (a) immediately follows from (b) and the fact that G acts transitively on M. Also, (c) is trivial because every vector in $\mathfrak{p}$ lies in some maximal abelian subspace of $\mathfrak{p}$. So we only need to prove (b) and (d). For (d), let $S \subseteq M$ be a maximal flat. We may assume it passes through o; let $\mathfrak{a} \subseteq \mathfrak{p}$ correspond to $T_oS \subseteq T_oM$, and let A be the connected Lie subgroup of G with Lie algebra $\mathfrak{a}$. This subgroup is abelian and Θ-stable, and the restriction of Θ to it is just the inverse map. The same must then be true for the closure $\overline{A}$. In other words, the Lie algebra $\overline{\mathfrak{a}}$ of $\overline{A}$ is abelian and contained in $\mathfrak{p}$. Since it also contains $\mathfrak{a}$, we must have $\mathfrak{a} = \overline{\mathfrak{a}}$, which implies that A is closed. Being an orbit of A, S has to be properly embedded by Remark 2.3.3.

We now proceed to prove (b), which is essentially a problem in Lie theory. It is proven in [Hel01, Lem.V.6.3] in case M is of compact or noncompact type, so we only show how to reduce the general case to that. First of all, quotienting by I, we may assume (G, K) is effective. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\mathrm{c}} \oplus \mathfrak{g}_{\mathrm{nc}}$ be the decomposition of $\mathfrak{g}$ into its Euclidean and compact/noncompact parts as in Proposition 2.1.84. Let $\widehat{G}$ be the universal covering Lie group of G and $\widehat{\Theta}$ the lift of θ to $\widehat{G}$. Take $\widehat{K} = (\widehat{G}^{\widehat{\Theta}})^0$. Then we need to prove that $\widehat{K}$ acts transitively on the set of maximal abelian subspaces of $\mathfrak{p}$, as this action factors through the action of K^0. But $\widehat{G}$ splits as $\widehat{G}_0 \times \widehat{G}_{\mathrm{c}} \times \widehat{G}_{\mathrm{nc}}$, and we have $\widehat{\Theta} = (\widehat{\Theta}_0, \widehat{\Theta}_{\mathrm{c}}, \widehat{\Theta}_{\mathrm{nc}})$, hence $\widehat{K} = \widehat{K}_0 \times \widehat{K}_{\mathrm{c}} \times \widehat{K}_{\mathrm{nc}}$. At the same time, every maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_{\mathrm{c}} \oplus \mathfrak{p}_{\mathrm{nc}}$ is trivially of the form $\mathfrak{p}_0 \oplus \mathfrak{a}_{\mathrm{c}} \oplus \mathfrak{a}_{\mathrm{nc}}$, where $\mathfrak{a}_{\mathrm{c}}$ is maximal abelian in $\mathfrak{p}_{\mathrm{c}}$ and $\mathfrak{a}_{\mathrm{nc}}$ is such in $\mathfrak{p}_{\mathrm{nc}}$. It then suffices to show that $\widehat{K}_{\mathrm{c}}$ (resp., $\widehat{K}_{\mathrm{nc}}$) acts transitively on the set of maximal abelian subspaces of $\mathfrak{p}_{\mathrm{c}}$ (resp., $\mathfrak{p}_{\mathrm{nc}}$). But $(\widehat{G}_{\mathrm{c}}, \widehat{K}_{\mathrm{c}})$ and $(\widehat{G}_{\mathrm{nc}}, \widehat{K}_{\mathrm{nc}})$ are of compact and noncompact type, respectively, so we are done. $\square$

Remark 2.2.23. In the setting of the proof of Proposition 2.2.22 we can go further and decompose $(\mathfrak{g}, \theta)$ into its irreducible parts as in Proposition 2.1.105: $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Then every maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$ must be of the form $\mathfrak{p}_0 \oplus \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$, where $\mathfrak{a}_i$ is maximal abelian in $\mathfrak{p}_i$.

Corollary 2.2.24. *All the maximal flats in a symmetric space M have the same dimension. If M is represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$, then $\mathrm{rk}(M)$ coincides with the dimension of any maximal abelian subspace of $\mathfrak{p}$.*

Example 2.2.25 (Flats in compact Lie groups). Let G be a compact connected Lie group equipped with a bi-invariant metric. It follows from (2.1.5) that G is flat if and only if it is abelian. In particular, a connected Lie subgroup $H \subseteq G$ is a flat if and only if it is abelian. It turns out that every flat in G passing through o arises in this way. Indeed, let $\mathfrak{h} \subseteq \mathfrak{g}$ be any subspace. Let us write $V \subseteq \mathfrak{p}$ for the subspace corresponding to $\mathfrak{h}$ under $\mathfrak{g} \cong \mathfrak{p}$. Then V is abelian if and only if

$$[V, V] = \operatorname{span}\{([X, Y], [X, Y]) \mid X, Y \in \mathfrak{h}\} = \{0\},$$

which happens precisely when $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$. In this case, the flat corresponding to V is the connected abelian Lie subgroup corresponding to $\mathfrak{g}$. As a consequence, maximal flats in G passing through e are nothing but maximal tori. In particular, the rank of G as a symmetric space coincides with its rank as a Lie group. //

Reflective submanifolds

Reflective submanifolds are another special subclass of totally geodesic submanifolds that, as the name implies, are related to reflections. They were studied in depth by Leung, who obtained the classification of reflective submanifolds in simply connected irreducible symmetric spaces of compact type ([Leu75, Leu79a]). By duality, that result also yields their classification in irreducible symmetric spaces of noncompact type (see Remark 2.2.38).

Definition 2.2.26. Let M be a Riemannian manifold. A connected submanifold $S \subseteq M$ is called **reflective** if there exists an involutive isometry σ of M such that S is a connected component of the fixed point set M^{σ}.

Example 2.2.27. Let M be a symmetric space and $o \in M$ any point. Then s_o is an involutive isometry. The connected components of M^{s_o} other than o are called **polars** of M. In the compact type case, they carry deep information about the geometry and topology of M (see, e.g., [Nag88] and the following papers in that series). //

Proposition 2.2.28. *Let M be a Riemannian manifold and $S \subseteq M$ a reflective submanifold.*

(a) *S is properly embedded.*

(b) *S is totally geodesic.*

(c) *The involution σ in Definition 2.2.26 is unique, provided M is connected.*

Proof. Let us prove that every connected component of M^σ is a properly embedded submanifold. Indeed, given $p \in M^\sigma$, $T_pM = V_+ \oplus V_-$, where $V_\pm$ is a $\pm$-eigenspace of $d\sigma_p$. Hence, in a normal neighborhood of p, M^σ is given by $\exp(V_+ \cap B_r(0))$, where $B_r(0)$ is a small enough ball in T_pM. This proves that the connected component of M^σ containing p is embedded and totally geodesic. It is properly embedded because M^σ is closed. Note that the other connected components may have different dimensions. We are left to show (c). Given $p \in S$, $d\sigma_p$ must be[1] E on T_pS and $-E$ on[2] N_pS, so $d\sigma_p$ is uniquely determined. Now everything follows from Proposition 2.1.1(c). □

Proposition 2.2.28 suggests that there should be an intrinsic way to describe the involution σ in terms of S. For simplicity, we do it under the assumption that M is complete. We first define the notion of reflection in a submanifold.

Lemma 2.2.29. *Let M be a complete connected Riemannian manifold and $S \subseteq M$ a properly embedded submanifold. Then for every $p \in M$, there exists a closest point $q \in S$ to p, i.e., $\operatorname{dist}(p, q) = \operatorname{dist}(p, S)$. Moreover, any minimizing geodesic segment from p to q intersects S orthogonally at q.*

Definition 2.2.30. Let M be a complete Riemannian manifold and $S \subseteq M$ a properly embedded connected submanifold. Take any $p \in M$ and let q be a closest point to q in S (see Lemma 2.2.29). Take a geodesic γ with initial point q such that $\gamma(T) = p$ for some $T \geq 0$ and $\gamma|_{[0,T]}$ is minimizing. Consider the point $\gamma(-T) \in M$. If $\gamma(-T)$ does not depend on the choice of a closest point q, then we say that p is **reflectable** in S and denote $\gamma(-T)$ by $\boldsymbol{r_S(p)}$. If every point of M is reflectable in S, we call $\boldsymbol{r_S}\colon M \to M$ the **geodesic reflection of $\boldsymbol{M}$ in $\boldsymbol{S}$** and say that $\boldsymbol{r_S}$ **is well-defined**.

The following essentially sums up what we have discussed so far:

Corollary 2.2.31. *Let M be a complete Riemannian manifold and $S \subseteq M$ a connected properly embedded submanifold. The following are equivalent:*

(i) *S is reflective.*

(ii) *The geodesic reflection of M in S is well defined and is an isometry.*

[1]Throughout the thesis, we often denote the identity operator on a vector space and the identity element of a linear Lie group by E.

[2]Here and elsewhere in the thesis, if S is a submanifold, NS stands for its normal bundle and N_pS for the normal space at $p \in S$.

If the above conditions are satisfied, then σ *in Definition 2.2.26 coincides with* r_S. *In particular,* S *is a connected component of* M^{r_S}. *Also,* S *is complete.*

Before we move to the case when the ambient space is symmetric, we mention one more important property of reflective submanifolds in general.

Definition 2.2.32. Let M be a Riemannian manifold and $p \in M$ any point. A subspace $V \subseteq T_pM$ is called **strongly curvature-invariant** if both V and $V^\perp$ are curvature-invariant. A submanifold $S \subseteq M$ is called (**strongly**) **curvature-invariant** if T_pS is (strongly) curvature-invariant for every $p \in S$.

Example 2.2.33. As we saw in Example 2.2.9, totally geodesic submanifolds are curvature-invariant. The converse, however, does not always hold (see [Nai00]). //

Proposition 2.2.34. *Let* S *be a reflective submanifold in a Riemannian manifold* M. *Then* S *is strongly curvature-invariant.*

Proof. Thanks to Proposition 2.2.28 and Example 2.2.33, we need only show that the normal spaces to S are curvature-invariant. Let $X, Y, Z \in N_pS$. We compute:

$$d\sigma_p(R(X,Y)Z) = R(d\sigma_p(X), d\sigma_p(Y))d\sigma_p(Z) = R(-X,-Y)(-Z) = -R(X,Y)Z,$$

hence $R(X,Y)Z \in N_pS$. □

In symmetric spaces, reflective submanifolds enjoy a special extra property: they always come in pairs. Let M be a symmetric space and $S \subseteq M$ a reflective submanifold. Take any $p \in S$. As we showed in Proposition 2.2.34, N_pS is curvature-invariant. Hence, according to Corollary 2.2.13, $\boldsymbol{S_p^\perp} = \exp(N_pS)$ is a complete connected totally geodesic submanifold of M.

Proposition 2.2.35. *The submanifold* $S_p^\perp$ *is reflective.*

Proof. Consider the composition $\sigma = s_o \circ r_S$. This is an isometry of M such that[1] $d\sigma_p$ is E on N_pS and $-E$ on T_pS. Since $d\sigma_p$ is involutive, so is σ itself (by Proposition 2.1.1(c)). Using an argument similar to the one we used in the proof of Proposition 2.2.28, it is easy to see that $S_p^\perp$ is one of the connected components of M^σ. □

Observation 2.2.36. Let $S \subseteq M$ be reflective and $p, q \in S$ any two points. Thanks to Corollary 2.2.7, S is itself a symmetric spaces, hence it is a homogeneous submanifold. Take $f \in I(M,S)$ that maps p to q. We then have $f(S_p^\perp) = S_q^\perp$. So the congruence class of $S_p^\perp$ in M does not depend on p. We will refer to it as the **orthogonal complement** of S and denote it simply by $\boldsymbol{S^\perp}$ if there is no ambiguity.

We finish our discussion of reflective submanifolds with one more important result. Observe that curvature-invariant subspaces can be regarded as the infinitesimal version of totally geodesic submanifolds, and, in a similar spirit, strongly curvature-invariant subspaces can be regarded as the infinitesimal version of reflective submanifolds. Now, in a symmetric

[1] A submanifold S such that for every $p \in S$ there exists an isometry of M that preserves S and whose differential at p satisfies these two properties is called *symmetric*. Symmetric submanifolds are symmetric spaces in their own right. In symmetric spaces, reflective submanifolds are symmetric, although in general this is not true. A complete connected totally geodesic submanifold of a symmetric space is symmetric if and only if it is reflective.

space, as we know from Corollary 2.2.13, every curvature-invariant subspace comes from a totally geodesic submanifold. Is the analogous statement true for strongly curvature-invariant subspaces? It turns out, the answer is affirmative[1], at least in the simply connected case.

Proposition 2.2.37. *Let M be a simply connected symmetric space, $o \in M$, and $V \subseteq T_oM$ a strongly curvature-invariant subspace. Then there exists a (unique) reflective submanifold $S \subseteq M$ passing through o such that $T_oS = V$, namely $S = \exp(V)$.*

Proof. Let M be represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. We need to show that there exists $f \in I(M)$ that fixes o such that df_o is E on V and $-E$ on $V^\perp$. When thought of as subspaces of $\mathfrak{p}$, both V and $V^\perp$ are Lie triple systems by assumption. Let $T \in \mathrm{GL}(\mathfrak{p})$ be an operator that is E on V and $-E$ on $V^\perp$. Thanks to Proposition 2.1.68, it suffices to show that T preserves the inner product and curvature tensor at o. The former is obvious. For the latter, take $X, Y, Z \in \mathfrak{p}$. By (2.1.5), we need to show that

$$T[[X,Y],Z] = [[TX,TY],TZ]. \tag{2.2.1}$$

By linearity, we may assume each of X, Y, Z lies either in V or $V^\perp$. If X, Y, Z all lie in V, so does $[[X,Y],Z]$, so both sides of (2.2.1) equal $[[X,Y],Z]$. Similarly, if $X, Y, Z \in V^\perp$, both sides of (2.2.1) equal $-[[X,Y],Z]$. Let us consider the less trivial case when $X, Y \in V$ but $Z \in V^\perp$. Then $[[TX,TY],TZ] = -[[X,Y],Z]$, so we need to show that $[[X,Y],Z]$ lies in $V^\perp$—i.e., that $\mathrm{ad}[X,Y]$ preserves $V^\perp$—for (2.2.1) to hold. The inner product on $\mathfrak{p}$ is $\mathfrak{k}$-invariant, which means that the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}$ is by skew-symmetric operators. One such operator is $\mathrm{ad}[X,Y]$, as $[X,Y] \in \mathfrak{k}$. Since $\mathrm{ad}[X,Y]$ preserves V, it must also preserve $V^\perp$. Now assume $X, Z \in V, Y \in V^\perp$. Then we need to have $[[X,Y],Z] \in V^\perp$ for (2.2.1) to hold. In other words, for every $W \in V$, we want $\langle [[X,Y],Z] | W \rangle = 0$. But

$$\langle [[X,Y],Z] | W \rangle = -Rm(X,Y,Z,W) = -Rm(Z,W,X,Y) = \langle [[Z,W],X] | Y \rangle,$$

which is zero because $[[Z,W],X] \in V$. The other cases can be proven in a similar way using the symmetries of the curvature tensor. $\square$

Remark 2.2.38. Since reflective submanifolds are totally geodesic, they can be dualized (see Proposition 2.2.17). Let M be a simply connected semisimple symmetric space and M^* its dual. Let $S \subseteq M$ be a complete connected totally geodesic submanifold and $S^* \subseteq M^*$ its dual in M^*. Then, thanks to Proposition 2.2.34, Proposition 2.2.17(a), and Proposition 2.2.37, S is reflective if and only if S^* is.

Austere submanifolds

The last class in our list is that of *austere submanifolds*, which, in a sense, occupy the middle ground between totally geodesic and minimal submanifolds. They are the only submanifolds in the list that are not automatically homogeneous.

Definition 2.2.39. A submanifold S of a Riemannian manifold M is called **austere** if for every $p \in S$ and $\xi \in N_pS$, the spectrum of the shape operator A_ξ—when taken

[1] A proof of this statement can be found in [Leu73], but we believe it is incomplete, as the author seems to omit the case when the submanifold lies diagonally within the de Rham decomposition.

with multiplicities—is invariant under multiplication by -1. In other words, if λ is an eigenvalue of A_ξ of multiplicity m, then so is $-\lambda$.

Observation 2.2.40 (Austerity vs minimality). Another way to express the definition of austerity is that A_ξ and $-A_\xi$ have the same spectrum when counted with multiplicities. If S is austere, its shape operators are clearly trace-free. If H stands for the mean curvature vector field of S, then $\langle H_p \,|\, \xi \rangle = \operatorname{tr}(A_\xi) = 0$, so austere submanifolds are minimal. For surfaces in M, the notions of austerity and minimality coincide. But for higher-dimensional submanifolds, austerity is a stronger notion in general.

Note that if S is austere and odd-dimensional, its shape operators must have nontrivial kernels. We are not going to study austere submanifolds in detail but will discuss two related results that will be relevant in the sequel. The first one concerns cohomogeneity-one actions, which will be discussed in more detail in Section 2.3 (see Definition 2.3.6 and Subsection 2.3.3).

Proposition 2.2.41. *Let M be a Riemannian manifold and $H \curvearrowright M$ a proper isometric action of cohomogeneity 1 by some Lie group H. Then the singular orbits of H are austere.*

Proof. Let S be a singular orbit of H. Take any $p \in S$ and $\xi \in N_pS$. The action of H_p on N_pS is of cohomogeneity 1 by Proposition 2.3.14, so there exists $h \in H_p$ such that $dh_p(\xi) = -\xi$. Since h is an isometry that preserves S and p, we have, on T_pS:

$$dh_p \circ A_\xi = A_{dh_p(\xi)} \circ dh_p = A_{-\xi} \circ dh_p.$$

This implies that A_ξ and $A_{-\xi}$ have the same spectrum when counted with multiplicities. But $A_{-\xi} = -A_\xi$, so S is austere. □

The second result pertains to austere submanifolds in the complex setting.

Proposition 2.2.42. *Let M be a Kähler manifold and $S \subseteq M$ a complex submanifold. Then S is austere.*

Proof. Since the almost complex structure I is parallel, the Levi-Civita connection on M, thought of as a map $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, is $\mathbb{C}$-linear in the second argument. Therefore, if $S \subseteq M$ is a complex submanifold, its second fundamental form $I\!I(X,Y) = (\nabla_X \widetilde{Y})^\perp$ is also $\mathbb{C}$-linear in the second argument. Since $I\!I$ is symmetric, it is $\mathbb{C}$-bilinear. But then the shape operators are $\mathbb{C}$-antilinear. Indeed, given $\xi \in N_pS$ and $X, Y \in T_pS$, we have

$$\begin{aligned}
\langle A_\xi(IX) \,|\, Y \rangle &= \langle I\!I(IX, Y) \,|\, \xi \rangle \\
&= \langle I\!I(X, IY) \,|\, \xi \rangle && (I\!I \text{ is } \mathbb{C}\text{-bilinear}) \\
&= \langle A_\xi(X) \,|\, IY \rangle \\
&= \langle -IA_\xi(X) \,|\, Y \rangle, && (I \text{ is skew-symmetric})
\end{aligned}$$

so $A_\xi \circ I = -I \circ A_\xi$ on T_pS. Consequently, if X is an eigenvector of A_ξ with eigenvalue λ, then IX is an eigenvector with eigenvalue $-\lambda$. This implies that S is austere. □

2.2.2. The second fundamental form of a homogeneous submanifold

Here we derive a relatively simple Lie algebraic formula for the second fundamental form and shape operators of an arbitrary (extrinsically) homogeneous submanifold of a symmetric space.

Let M be a Riemannian symmetric space represented by a Riemannian symmetric pair (G, K). Let $S \subseteq M$ is a homogeneous submanifold (we may assume $o \in S$). Let $H \subseteq G$ be a Lie subgroup having S as one of its orbits. We write $\mathfrak{h} \subseteq \mathfrak{g}$ for the Lie algebra of H.

Proposition 2.2.43. *The second fundamental form of S at o is given by*

$$I\!I(X, Y) = \mathrm{pr}_{N_oS}([X^{\mathfrak{h}}_{\mathfrak{k}}, Y]), \tag{2.2.2}$$

where $X, Y \in T_oS$ and $X^{\mathfrak{h}} \in \mathfrak{h}$ is any vector whose $\mathfrak{p}$-part is X (and $X^{\mathfrak{h}}_{\mathfrak{k}}$ is its $\mathfrak{k}$-part as usual). For any $Z \in N_oS$, we have

$$\langle A_Z X \,|\, Y \rangle = \langle I\!I(X, Y) \,|\, Z \rangle = \langle [X^{\mathfrak{h}}_{\mathfrak{k}}, Y] \,|\, Z \rangle, \tag{2.2.3}$$

where A_Z is the shape operator of S at o corresponding to Z.

Implicit in this proposition is the fact that the right hand sides of (2.2.2) and (2.2.3) do not depend on the choice of $X^{\mathfrak{h}}$—which can also be easily verified directly: given another lift $\widetilde{X}^{\mathfrak{h}} \in \mathfrak{h}$, $X^{\mathfrak{h}}_{\mathfrak{k}} - \widetilde{X}^{\mathfrak{h}}_{\mathfrak{k}} = X^{\mathfrak{h}} - \widetilde{X}^{\mathfrak{h}} \in \mathfrak{h} \cap \mathfrak{k}$, so $[X^{\mathfrak{h}}_{\mathfrak{k}} - \widetilde{X}^{\mathfrak{h}}_{\mathfrak{k}}, Y] \in T_oS$. There are various ways to prove this proposition, some of them shorter than the proof we are giving here (see, e.g., Remark 2.2.45). We have opted for this proof to highlight that this is a rather elementary result that requires only basic techniques and computations.

Proof. Equations (2.2.2) and (2.2.3) are clearly equivalent, so it suffices to prove the latter. The first equality in (2.2.3) is essentially the definition of A_Z, hence we need only prove the second one. The idea is to extend X, Y and Z to vector fields in a clever way and then use the Koszul formula. The extensions of X and Z can be arbitrary, so we just take them to be the corresponding Killing vector fields $\hat{X}$ and $\hat{Z}$. We could have taken the extension of Y to be the Killing vector field $\hat{Y}$, but it is not in general tangent to S over S (unless S is totally geodesic). Instead, pick $Y^{\mathfrak{h}}$ in $\mathfrak{h}$ whose $\mathfrak{p}$-component is Y and write $Y^{\mathfrak{h}}_{\mathfrak{k}}$ for its $\mathfrak{k}$-component as usual. Observe that $(\hat{Y}^{\mathfrak{h}})_o = d(\pi_o)_e(Y^{\mathfrak{h}}) = Y$, and $\hat{Y}^{\mathfrak{h}}$ is the infinitesimal generator of the flow $\Psi_t(p) = \exp_G(tY^{\mathfrak{h}}) \cdot p$, so it is everywhere tangent to the orbits of H and in particular to S. We have:

$$\begin{aligned}
\langle I\!I(X, Y) \,|\, Z \rangle = \langle (\nabla^M_X \hat{Y}^{\mathfrak{h}})^{\perp} \,|\, Z \rangle &= \langle \nabla^M_X \hat{Y}^{\mathfrak{h}} \,|\, Z \rangle \\
&= \frac{1}{2}\left(X\langle \hat{Y}^{\mathfrak{h}} \,|\, \hat{Z} \rangle + Y\langle \hat{Z} \,|\, \hat{X} \rangle - Z\langle \hat{X} \,|\, \hat{Y}^{\mathfrak{h}} \rangle \right) \\
&\quad + \frac{1}{2}\left(-\langle X \,|\, [\hat{Y}^{\mathfrak{h}}, \hat{Z}]_o \rangle + \langle Y \,|\, [\hat{Z}, \hat{X}]_o \rangle + \langle Z \,|\, [\hat{X}, \hat{Y}^{\mathfrak{h}}]_o \rangle \right),
\end{aligned}$$

where in the last equality we apply the Koszul formula. Let us denote the contents of the parentheses above by $(*)$ and $(**)$, respectively.

We first deal with $(**)$ since it is much easier. Recall that sending an element of $\mathfrak{g}$ to its corresponding Killing vector field on M is an antihomomorphism of Lie algebras, so we

have
$$(**) = \langle X \,|\, \widehat{[Y^{\mathfrak{h}}, Z]}_o \rangle - \langle Y \,|\, \widehat{[Z, X]}_o \rangle - \langle Z \,|\, \widehat{[X, Y^{\mathfrak{h}}]}_o \rangle.$$
Since the value of a Killing vector field at o is just the projection of the corresponding element of $\mathfrak{g}$ to $\mathfrak{p}$, we see that $\widehat{[Z, X]}_o = 0$ and

$$\begin{aligned}(**) &= \langle X \,|\, \mathrm{pr}_{\mathfrak{p}}[Y^{\mathfrak{h}}, Z] \rangle - \langle Z \,|\, \mathrm{pr}_{\mathfrak{p}}[X, Y^{\mathfrak{h}}] \rangle \\ &= \langle X \,|\, [Y^{\mathfrak{h}}_{\mathfrak{k}}, Z] \rangle - \langle Z \,|\, [X, Y^{\mathfrak{h}}_{\mathfrak{k}}] \rangle \\ &= \langle X \,|\, [Y^{\mathfrak{h}}_{\mathfrak{k}}, Z] \rangle + \langle [Y^{\mathfrak{h}}_{\mathfrak{k}}, X] \,|\, Z \rangle = 0,\end{aligned}$$

where for the last equality we use the fact that the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}$ is orthogonal. Now we proceed to computing $(*)$. We will need the following technical

Lemma 2.2.44. *Let $U, V, W \in \mathfrak{g}$. Then one has*

$$U_{\mathfrak{p}} \langle \hat{V} \,|\, \hat{W} \rangle = -\langle [U_{\mathfrak{p}}, V_{\mathfrak{k}}] \,|\, W_{\mathfrak{p}} \rangle - \langle V_{\mathfrak{p}} \,|\, [U_{\mathfrak{p}}, W_{\mathfrak{k}}] \rangle.$$

Proof of the lemma. We begin by computing the function $\langle \hat{V} \,|\, \hat{W} \rangle$. At a point $g \cdot o \in M, g \in G$, we have:

$$\begin{aligned}\langle \hat{V} \,|\, \hat{W} \rangle (g \cdot o) &= \langle d(\pi_{g \cdot o})_e V \,|\, d(\pi_{g \cdot o})_e W \rangle \\ &= \langle d(g^{-1})_{g \cdot o} \circ d(\pi_{g \cdot o})_e(V) \,|\, d(g^{-1})_{g \cdot o} \circ d(\pi_{g \cdot o})_e(W) \rangle,\end{aligned}$$

where $\pi_{g \cdot o} \colon G \to M$ is the orbit map at $g \cdot o$. The second equality follows from the fact that g^{-1} is an isometry. Observe that

$$d(g^{-1})_{g \cdot o} \circ d(\pi_{g \cdot o})_e = d(g^{-1} \circ \pi_{g \cdot o})_e = d(\pi_o \circ C_{g^{-1}})_e = d(\pi_o)_e \circ \mathrm{Ad}(g^{-1}),$$

where $C_{g^{-1}}$ is the conjugation of G by g^{-1}. We deduce:

$$\langle \hat{V} \,|\, \hat{W} \rangle (g \cdot o) = \langle \mathrm{pr}_{\mathfrak{p}}(\mathrm{Ad}(g^{-1})(V)) \,|\, \mathrm{pr}_{\mathfrak{p}}(\mathrm{Ad}(g^{-1})(W)) \rangle.$$

Define a smooth function $f_{V,W}$ on G by the same formula:

$$f_{V,W}(g) = \langle \mathrm{pr}_{\mathfrak{p}}(\mathrm{Ad}(g^{-1})(V)) \,|\, \mathrm{pr}_{\mathfrak{p}}(\mathrm{Ad}(g^{-1})(W)) \rangle.$$

Plainly, $f_{V,W}$ is the lift of the function $\langle \hat{V} \,|\, \hat{W} \rangle$ to G along $\pi_o \colon G \to M$. For this reason,

$$U_{\mathfrak{p}} \langle \hat{V} \,|\, \hat{W} \rangle = U f_{V,W} = \left. \frac{d}{dt} \right|_{t=o} f_{V,W}(\exp_G(tU)).$$

Now,

$$\begin{aligned}f_{V,W}(\exp_G(tU)) &= \langle \mathrm{pr}_{\mathfrak{p}}(\mathrm{Ad}(\exp_G(-tU))(V)) \,|\, \mathrm{pr}_{\mathfrak{p}}(\mathrm{Ad}(\exp_G(-tU))(W)) \rangle \\ &= \langle \mathrm{pr}_{\mathfrak{p}}(e^{-t \,\mathrm{ad}(U)} V) \,|\, \mathrm{pr}_{\mathfrak{p}}(e^{-t \,\mathrm{ad}(U)} W) \rangle\end{aligned}$$

But $\mathrm{pr}_{\mathfrak{p}}(e^{-t \,\mathrm{ad}(U)} V) = \mathrm{pr}_{\mathfrak{p}}(V - t[U, V] + O(t^2)) = V_{\mathfrak{p}} - t([U_{\mathfrak{k}}, V_{\mathfrak{p}}] + [U_{\mathfrak{p}}, V_{\mathfrak{k}}]) + O(t^2)$, and the same is true for W. It follows that

$$f_{V,W}(\exp_G(tU)) = \langle V_{\mathfrak{p}} \,|\, W_{\mathfrak{p}} \rangle$$

$$- t\big(\langle V_{\mathfrak{p}} \,|\, [U_{\mathfrak{k}}, W_{\mathfrak{p}}] + [U_{\mathfrak{p}}, W_{\mathfrak{k}}]\rangle + \langle [U_{\mathfrak{k}}, V_{\mathfrak{p}}] + [U_{\mathfrak{p}}, V_{\mathfrak{k}}] \,|\, W_{\mathfrak{p}}\rangle\big) + O(t^2),$$

and, differentiating at $t = 0$, we obtain:

$$\begin{aligned}
U_{\mathfrak{p}}\langle \hat{V} \,|\, \hat{W}\rangle &= \frac{d}{dt}\bigg|_{t=o} f_{V,W}(\exp_G(tU)) \\
&= -\langle V_{\mathfrak{p}} \,|\, [U_{\mathfrak{k}}, W_{\mathfrak{p}}]\rangle - \langle V_{\mathfrak{p}} \,|\, [U_{\mathfrak{p}}, W_{\mathfrak{k}}]\rangle - \langle [U_{\mathfrak{k}}, V_{\mathfrak{p}}] \,|\, W_{\mathfrak{p}}\rangle - \langle [U_{\mathfrak{p}}, V_{\mathfrak{k}}] \,|\, W_{\mathfrak{p}}\rangle \\
&= -\langle [U_{\mathfrak{p}}, V_{\mathfrak{k}}] \,|\, W_{\mathfrak{p}}\rangle - \langle V_{\mathfrak{p}} \,|\, [U_{\mathfrak{p}}, W_{\mathfrak{k}}]\rangle - \big(\langle [U_{\mathfrak{k}}, V_{\mathfrak{p}}] \,|\, W_{\mathfrak{p}}\rangle + \langle V_{\mathfrak{p}} \,|\, [U_{\mathfrak{k}}, W_{\mathfrak{p}}]\rangle\big) \\
&= -\langle [U_{\mathfrak{p}}, V_{\mathfrak{k}}] \,|\, W_{\mathfrak{p}}\rangle - \langle V_{\mathfrak{p}} \,|\, [U_{\mathfrak{p}}, W_{\mathfrak{k}}]\rangle,
\end{aligned}$$

which was to be proved. □

Now we can apply Lemma 2.2.44 to compute $(*)$:

$$\begin{aligned}
X\langle \hat{Y}^{\mathfrak{h}} \,|\, \hat{Z}\rangle &= -\langle [X, Y^{\mathfrak{h}}_{\mathfrak{k}}] \,|\, Z\rangle, \\
Y\langle \hat{Z} \,|\, \hat{X}\rangle &= 0, \\
Z\langle \hat{X} \,|\, \hat{Y}^{\mathfrak{h}}\rangle &= -\langle X \,|\, [Z, Y^{\mathfrak{h}}_{\mathfrak{k}}]\rangle,
\end{aligned}$$

hence

$$\begin{aligned}
(*) &= X\langle \hat{Y}^{\mathfrak{h}} \,|\, \hat{Z}\rangle + Y\langle \hat{Z} \,|\, \hat{X}\rangle - Z\langle \hat{X} \,|\, \hat{Y}^{\mathfrak{h}}\rangle \\
&= -\langle [X, Y^{\mathfrak{h}}_{\mathfrak{k}}] \,|\, Z\rangle + \langle X \,|\, [Z, Y^{\mathfrak{h}}_{\mathfrak{k}}]\rangle \\
&= 2\langle [Y^{\mathfrak{h}}_{\mathfrak{k}}, X] \,|\, Z\rangle.
\end{aligned}$$

Putting it all together, we arrive at:

$$\langle I\!I(X,Y) \,|\, Z\rangle = \frac{1}{2}((*) + (**)) = \langle [Y^{\mathfrak{h}}_{\mathfrak{k}}, X] \,|\, Z\rangle.$$

Since the second fundamental form is symmetric, this coincides with the desired expression $\langle [X^{\mathfrak{h}}_{\mathfrak{k}}, Y] \,|\, Z\rangle$. □

Remark 2.2.45. There is an alternative, shorter way to prove Lemma 2.2.44. Recall from (2.1.1) (see also [Zil10, Prop. 6.34(a)]) that, given $X \in \mathfrak{p}$ and $Y \in \mathfrak{X}(M)$, one has $(\nabla_{\hat{X}} Y)_o = [\hat{X}, Y]_o$. Going back to the setting of Lemma 2.2.44, using this formula, we compute:

$$\begin{aligned}
U_{\mathfrak{p}}\langle \hat{V} \,|\, \hat{W}\rangle &= (\hat{U}_{\mathfrak{p}}\langle \hat{V} \,|\, \hat{W}\rangle)_o \\
&= \langle \nabla_{\hat{U}_{\mathfrak{p}}} \hat{V} \,|\, \hat{W}\rangle_o + \langle \hat{V} \,|\, \nabla_{\hat{U}_{\mathfrak{p}}} \hat{W}\rangle_o \\
&= \langle [\hat{U}_{\mathfrak{p}}, \hat{V}]_o \,|\, \hat{W}_o\rangle + \langle \hat{V}_o \,|\, [\hat{U}_{\mathfrak{p}}, \hat{W}]_o\rangle \\
&= -\langle [U_{\mathfrak{p}}, V]_{\mathfrak{p}} \,|\, W_{\mathfrak{p}}\rangle - \langle V_{\mathfrak{p}} \,|\, [U_{\mathfrak{p}}, W]_{\mathfrak{p}}\rangle \\
&= -\langle [U_{\mathfrak{p}}, V_{\mathfrak{k}}] \,|\, W_{\mathfrak{p}}\rangle - \langle V_{\mathfrak{p}} \,|\, [U_{\mathfrak{p}}, W_{\mathfrak{k}}]\rangle.
\end{aligned}$$

Remark 2.2.46. If S is given as an orbit of some isometric Lie group action $H \curvearrowright M$ and H does not lie in G a priori, we can always assume H is connected and replace it with its image in $I^0(M)$. We can then take $G = I^0(M)$ and $K = G_o$.

2.3. Polar and cohomogeneity-one actions

In this section, we discuss some aspects of the theory of isometric actions on general Riemannian manifolds as well as on symmetric spaces. After some brief recap, we focus on our main object of interest: polar and cohomogeneity-one actions. Our main references for this section are [Mic08, Ch.VI] and [BCO16, Sect. 2.1-2.3].

2.3.1. Proper actions

We will be working almost exclusively with proper actions—as they are generally better behaved. Before we begin, let us mention the most basic property of such actions (see [Lee13, Prop. 21.7, 21.8]):

Proposition 2.3.1. *If H is a Lie group acting properly on a smooth manifold M, then all its orbits are properly embedded submanifolds of M, all its stabilizers are compact subgroups, and the orbit space M/H is Hausdorff*[1].

Example 2.3.2 (Linear actions). A linear action of a Lie group on a vector space is the same as a representation on that vector space. Such an action is proper if and only if the group is compact. //

Remark 2.3.3. Virtually all the properties of proper actions discussed below (incl. Propositions 2.3.5 and 2.3.14) are satisfied by a larger class of actions $H \curvearrowright M$, namely such that $H/I \curvearrowright M$ is proper, where I is the ineffectiveness kernel. For example, if (G, K) is a Riemannian symmetric pair, the action $G \curvearrowright M = G/K$ does not have to be proper: consider, for instance, the real hyperbolic plane $\mathbb{R}H^2 \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ represented as $\widetilde{\mathrm{SL}}(2,\mathbb{R})/K$, where $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ and K is a subgroup of it isomorphic to $\mathbb{R}$. In fact, the action $G \curvearrowright M$ proper if and only if K is compact. However, this action factors through G/I, whose action on M is proper by Observation 2.1.33 and Proposition 2.1.1(d). Moreover, the action of any closed subgroup $H \subseteq G$ on M becomes proper when factored through $H/(H \cap I)$. In particular, such H has all its orbits properly embedded.

Let H be a Lie group acting properly on a smooth manifold M. Given an orbit S of H, the set $\{H_p \mid p \in S\}$ forms precisely one conjugacy class of subgroups of H, called the **isotropy type** of S. At the same time, the set of conjugacy classes of subgroups of H is partially ordered by reverse inclusion: $[K_1] \leq [K_2] \Leftrightarrow \exists K_2' \in [K_2]$ such that $K_2' \subseteq K_1$. This induces a preorder on the orbit space M/H: $H \cdot p \leq H \cdot q \Leftrightarrow H_q$ is conjugate to a subgroup of H_p. In particular, this gives rise to an equivalence relation on M/H: $H \cdot p \sim H \cdot q \Leftrightarrow H \cdot p \leq H \cdot q$ and $H \cdot q \leq H \cdot p \Leftrightarrow H_p$ is conjugate to H_q. We denote the equivalence class of $H \cdot p$ by $[\boldsymbol{H \cdot p}]$ and call it its (**orbit**) **type**. Let $\boldsymbol{\mathcal{O}}$ stand for the set of all orbit types of the action. The preorder $\leq$ induces a partial order on $\mathcal{O}$: $[H \cdot p] \leq [H \cdot q] \Leftrightarrow H_q$ is conjugate to a subgroup of H_p.

Definition 2.3.4. An orbit $H \cdot p$ is called **principal** if it is locally a maximal element of M/H: there exists a neighborhood U of p such that H_p is conjugate to a subgroup of H_q for any $q \in U$. A point $p \in M$ is called **principal** if its orbit is principal. If M is a vector space and the action is linear, we also say "principal vector". The sets of principal orbits and principal points are denoted by $(M/H)_{\mathrm{reg}} \subseteq M/H$ and $M_{\mathrm{reg}} \subseteq M$, respectively.

[1]One can even show that M/H is completely regular in this case (see [Mic08, Cor. 6.29]).

Proposition 2.3.5. *Let H be a Lie group acting properly on a smooth manifold M. Then:*

(a) *Every $p \in M$ has a neighborhood U such that $[H \cdot p] \leq [H \cdot q]$ for every $q \in U$. In other words, the orbit type is locally non-decreasing.*

(b) *$(M/H)_{\mathrm{reg}}$ (resp., M_{reg}) is an open dense subset of M/H (resp., M). In particular, principal orbits exist. The orbit type is locally constant on $(M/H)_{\mathrm{reg}}$.*

(c) *Suppose the orbit space M/H is connected (e.g., M is connected). Then $(M/H)_{\mathrm{reg}}$ and M_{reg} are connected. Consequently, all principal orbits have the same type and thus the same dimension. The principal orbit type is the maximal element of $\mathcal{O}$.*

(d) *If M is compact, then $\mathcal{O}$ is finite.*

Thanks to part (c) of this proposition, we can introduce the following

Definition 2.3.6. Let H be a Lie group acting properly on a smooth manifold M in such a way that M/H is connected. The **cohomogeneity** of $H \curvearrowright M$ is the codimension of a principal orbit. An orbit is called **singular** if its dimension is less than that of a principal orbit. An orbit is called **exceptional** if it is neither singular nor principal. A point (or vector—if M is a vector space and the action is linear) $p \in M$ is called **singular** or **exceptional** if so is its orbit.

It is customary to refer to actions of cohomogeneity one simply as **C1-actions**. We will also use this shorthand when we work with cohomogeneity-two actions and refer to them as **C2-actions**. Similarly, we will refer to foliations of codimension one or two as **C1-** or **C2-foliations**.

Example 2.3.7 (Actions of cohomogeneity 0). An action $H \curvearrowright M$ is of cohomogeneity 0 if and only if it has an open orbit. If it is proper and M/H is connected, then it has to be transitive. In general, however, this does not have to be the case. For example, the standard representation of $\mathrm{GL}(n, \mathbb{R})$ on $\Lambda^2(\mathbb{R}^n)^*$ is of cohomogeneity 0 but of course not transitive. //

The nonsingular points are precisely those whose stabilizers have the lowest dimension possible. Among them, the principal points are those whose stabilizers have the lowest number of connected components possible. Note that parts (a) and (b) of Proposition 2.3.5 imply that the sets of singular and exceptional orbits (or points) are both closed and nowhere dense.

Agreement. Whenever we talk about principal, singular, or exceptional orbits or cohomogeneity, the action in question is tacitly assumed to be proper with connected orbit space.

We will be working with proper actions in the context of Riemannian geometry, i.e., with proper isometric actions on Riemannian manifolds. It is worth pointing out that this does not lead to any loss of generality:

Proposition 2.3.8. *Suppose we have a proper action of a Lie group H on a smooth manifold M. Then M admits a H-invariant Riemannian metric.*

Remark 2.3.9. For isometric actions, the assumption of being proper is essentially equivalent to an assumption that the orbits are properly embedded. Indeed, given an action $H \curvearrowright M$ with such orbits, the closure of the image of H in $I(M)$ acts properly and has the same orbits ([DR08, Th. 5]).

Example 2.3.10 (Orthogonal representations). A special case of an isometric action is an orthogonal representation of a Lie group on a Euclidean space. //

Remark 2.3.11. An orbit of an isometric Lie group action on a Riemannian manifold is by definition the same as a homogeneous submanifold (see Definition 2.2.1). What is more, an orbit of a proper isometric action is the same as a properly embedded homogeneous submanifold (see the proof of Lemma 2.2.3 and Proposition 2.1.1(d)).

One immediate property of proper isometric actions is that their orbits are *equidistant*. Indeed, let $H \curvearrowright M$ be such on action with M complete and connected, and let $S, S' \subseteq M$ be two orbits. Given any $p, q \in S$, we claim that $\operatorname{dist}(p, S') = \operatorname{dist}(q, S')$. Indeed, let p' be a closest point to p on S'. Using the action, one can show that p is a closest point to p' on S. Let γ be a minimizing geodesic segment between p and p'. We have

$$L(\gamma) = \operatorname{dist}(p, p') = \operatorname{dist}(p, S') = \operatorname{dist}(p', S).$$

By Lemma 2.2.29, γ intersects both S and S' orthogonally. Now, if $g \in H$ maps p to q, then $q' = g(p')$ is a closest point to q on S' and $g \circ \gamma$ is a minimizing geodesic segment between q and q', hence

$$\operatorname{dist}(q, S') = \operatorname{dist}(q, q') = L(g \circ \gamma) = L(\gamma) = \operatorname{dist}(p, S').$$

One of the very important tools in the theory of proper isometric actions is the special version of the tubular neighborhood theorem adapted to homogeneous submanifolds:

Proposition 2.3.12. *Let S be a properly embedded homogeneous submanifold of a Riemannian manifold M. Then there exists $\varepsilon > 0$ small enough such that the normal exponential map of S is defined on $N^\varepsilon S = \{v \in NS \mid ||v|| < \varepsilon\}$ and is a diffeomorphism from $N^\varepsilon S$ onto a neighborhood $U^\varepsilon(S)$.*

Whenever $S \subseteq M$ is as in Proposition 2.3.12 and we write $N^\varepsilon S$ or $U^\varepsilon(S)$, we always assume ε is small enough to satisfy the assertion of the proposition. Assume S is an orbit of a proper isometric action $H \curvearrowright M$. Given $p \in S$, the submanifold $S_p^\varepsilon = \exp(N_p^\varepsilon S)$, where $N_p^\varepsilon S = N^\varepsilon S \cap N_p S$, is a so-called *slice* of the action $H \curvearrowright M$ at p. It is preserved by H_p, and the action $H_p \curvearrowright S_p^\varepsilon$, in a sense, encapsulates all information about the action of H in an invariant neighborhood $(U^\varepsilon(S))$ of S. We are not going to discuss slices and define them in general and refer to [Mic08, Th. 6.26] instead. We will, however, define their linear version.

Let $H \curvearrowright M$ be a proper isometric action. Given $p \in M$, note that H_p acts linearly and orthogonally on T_pM by $g \mapsto dg_p$ (this is often called the isotropy representation of $H \curvearrowright M$ at p). The tangent space $T_p(H \cdot p)$ to the orbit $H \cdot p$ is a subrepresentation of T_pM and thus so is the normal space $N_p(H \cdot p)$.

Definition 2.3.13. The representation of H_p on $N_p(H \cdot p)$ is called the **slice representation** of the action $H \curvearrowright M$ (or simply of H) at p.

The slice representation encodes the action of H in a neighborhood of the orbit and helps to detect orbit type.

Proposition 2.3.14. *Suppose we have a proper isometric action $H \curvearrowright M$ and let $p \in M$.*

(a) *The map* $\exp\colon N_p^\varepsilon S \xrightarrow{\sim} S_p^\varepsilon$ *is H_p-equivariant.*

(b) *The cohomogeneity of the slice representation at p coincides with the cohomogeneity of $H \curvearrowright M$.*

(c) *The orbit $H \cdot p$ is principal if and only if the slice representation at p is trivial.*

We need to discuss the special case of isometric actions when there are no singular orbits.

Definition 2.3.15. A foliation $\mathcal{F}$ on a Riemannian manifold M is said to be **homogeneous** if there exists an isometric Lie group action $H \curvearrowright M$ whose orbits are precisely the leaves of $\mathcal{F}$. If H is specified, $\mathcal{F}$ is also called the **orbit foliation**[1] of H. The **codimension** of a foliation $\mathcal{F}$ is the codimension of its leaves.

Note that H in Definition 2.3.15 can be taken connected. If the leaves of $\mathcal{F}$ are properly embedded, we can also assume the action of H is proper (using the same ideas as in the proof of Lemma 2.2.3). In this thesis, we will occasionally use the following shorthand: a foliation is called **proper** if all of its leaves are properly embedded.

Observation 2.3.16. Let H be a connected Lie group acting isometrically on a Riemannian manifold M. Then the orbits of H form a homogeneous foliation if and only if there are no singular orbits.

The following result will prove useful in Chapter 4 when we study homogeneous foliations on symmetric spaces of noncompact type (see [BDRT10, Prop. 2.1] for a proof):

Proposition 2.3.17. *Let M be a Hadamard manifold and H a connected Lie group acting properly, isometrically, and without singular orbits on M. Then every orbit of H is principal (i.e., there are no exceptional orbit).*

We now introduce a suitable notion of equivalence for all the homogeneous objects we have seen so far.

Definition 2.3.18. Two isometric actions $H_1 \curvearrowright M_1$ and $H_2 \curvearrowright M_2$ are called **orbit-equivalent** if there exists an isometry $f\colon M_1 \xrightarrow{\sim} M_2$ such that $f(H_1 \cdot p) = H_2 \cdot f(p)$ for every $p \in M_1$. If $H_1 \curvearrowright M_1$ and $H_2 \curvearrowright M_2$ are orthogonal representations, we additionally require f to be linear. Two submanifolds S_1 and S_2 (resp., foliations $\mathcal{F}_1$ and $\mathcal{F}_2$) of Riemannian manifolds M_1 and M_2, respectively, are called (**isometrically**) **congruent** if there exists an isometry $f\colon M_1 \xrightarrow{\sim} M_2$ such that $f(S_1) = S_2$ (resp., $f(S)$ is a leaf of $\mathcal{F}_2$ for every leaf S of $\mathcal{F}_1$). Two isometric actions on (resp., submanifolds or foliations of) a Riemannian manifold M are called **strongly orbit-equivalent** (resp., **strongly** (**isometrically**) **congruent**) if f as above can be chosen in $I^0(M)$.

If there is no ambiguity and the Riemannian context is clear, we will drop the word "isometric" and simply say *congruent*. Note that orbit-equivalence preserves the cohomogeneity of an action and the sets of principal, exceptional, and singular orbits, whereas congruence of submanifolds preserves the property of being (intrinsically or extrinsically) homogeneous. Similarly, congruence of foliations preserves the property of being homogeneous. The question of whether two given submanifolds of M (resp., isometric actions on M) are

[1]Some authors use this term even when the orbits do not form a foliation by extending the notion of a foliation and allowing it to be singular. For the general framework of singular Riemannian foliations, see, for example, [Mol88] or [Lyt10].

congruent (resp., orbit-equivalent) can often be highly nontrivial and features prominently in this thesis. We will often refer to this questions as the **congruence problem**.

Remark 2.3.19. Suppose we have a homogeneous foliation $\mathcal{F}_i$ on M given as the orbit foliation of an isometric action $H_i \curvearrowright M$ for $i = 1, 2$. Then a congruence between $\mathcal{F}_1$ and $\mathcal{F}_2$ is the same as an orbit-equivalence between $H_1 \curvearrowright M$ and $H_2 \curvearrowright M$.

Observation 2.3.20. Suppose M is a symmetric space represented by a Riemannian symmetric pair (G, K) such that $G \to I^0(M)$ is surjective (this is always the case if M has compact Euclidean part). We claim that any isometric action on M by a connected Lie group is orbit-equivalent to an action of a connected Lie subgroup of G. Indeed, given $H \curvearrowright M$, consider the image of H in $I^0(M)$ and let $H' \subseteq G$ be the preimage of that under $G \twoheadrightarrow I^0(M)$. Clearly, H' is a Lie subgroup of G and it has the same orbits as H, hence so does $(H')^0$. Moreover, if $H \curvearrowright M$ is proper, then $(H')^0$ is closed.

2.3.2. Polar and hyperpolar actions

Now, we proceed to a subclass of proper isometric actions with special geometric properties, called *polar actions.*

Definition 2.3.21. Suppose we have a *proper*[1] isometric action of a Lie group H on a complete Riemannian manifold M. A complete connected submanifold $\Sigma \subseteq M$ is called a **section** of the action $H \curvearrowright M$ if it intersects all the orbits and does so orthogonally[2]:

(a) $\Sigma \cap (H \cdot p) \neq \varnothing$ for all $p \in M$.

(b) $T_p\Sigma$ and $T_p(H \cdot p)$ are orthogonal subspaces of T_pM for all $p \in \Sigma$.

The action $H \curvearrowright M$ is called **polar** if it admits a section. An orthogonal representation of a Lie group H on a Euclidean space V is called **polar** if it is polar as an action $H \curvearrowright V$.

When working with polar actions, one usually restricts to connected groups, due to the following fact, which can be easily deduced from Lemma 2.2.29:

Proposition 2.3.22. *Let H be a Lie group acting properly and isometrically on a complete Riemannian manifold M in such a way that M/H is connected. Then H acts polarly if and only if H^0 does.*

We list some basic properties of polar actions:

Proposition 2.3.23 (Properties of polar actions). *Let H be a Lie group, M a complete Riemannian manifold, and $H \curvearrowright M$ a polar action.*

(a) *The dimension of any section of $H \curvearrowright M$ equals the cohomogeneity of the action.*

(b) *For every $p \in M$, there exists a section passing through it.*

(c) *If $p \in M$ is a nonsingular point of the action and $\Sigma \subseteq M$ is a section passing through it, then $T_p\Sigma = N_p(H \cdot p)$.*

(d) *Every section of $H \curvearrowright M$ is a totally geodesic submanifold of M.*

[1]Although we add properness as an assumption, in many cases it is automatically satisfied (see [Lyt10, Cor. 1.3]).

[2]Some authors also add the assumption that Σ is embedded or even properly embedded. We will not assume this by default and mention it whenever required.

(e) *If $p \in M$ is a nonsingular point of the action, then there exists a unique section passing through it, and it is given by* $\exp(N_p(H \cdot p))$.

(f) *All sections of $H \curvearrowright M$ are mutually congruent via H. Namely, let Σ and Σ' be two sections, $p \in \Sigma$ any nonsingular point of the action, $q \in \Sigma' \cap (H \cdot p)$, and $g \in H$ any element mapping p to q. Then $g(\Sigma) = \Sigma'$.*

(g) *For any $p \in M$, the slice representation $H_p \curvearrowright N_p(H \cdot p)$ is polar. A subspace $V \subseteq N_p(H \cdot p)$ is a section of the slice representation at p if and only if it is the tangent space of a section of $H \curvearrowright M$ passing through p.*

(h) *If M is simply connected, then there are no exceptional orbits.*

Proofs of most of these statements can be found in [Mic08, Ch.VI]. Most available proofs of part (d) are incomplete, see [LNS22] for details. For (h), see [Lyt10, Cor. 1.3].

Given a complete connected Riemannian manifold M, a proper isometric action $H \curvearrowright M$, and a principal point p, one can show that the set $\exp(N_p(H \cdot p))$ intersects all the orbits. So what makes polar actions special is that this set is a submanifold and it always intersects the orbits orthogonally.

Corollary 2.3.24. *All sections of a polar representation $H \to \mathrm{O}(V)$ are linear subspaces of V.*

Proof. By Proposition 2.3.23(d), any section has to be totally geodesic, hence an affine subspace in V. By definition, it has to pass through 0. □

Example 2.3.25 (Polar actions on $\mathbb{S}^n$). Since $I(\mathbb{S}^n) = \mathrm{O}(n+1) \subset I(\mathbb{R}^{n+1})$, an isometric cohomogeneity-k action of a Lie group G on $\mathbb{S}^n$ is the same as its orthogonal representation of cohomogeneity $k+1$ on $\mathbb{R}^{n+1}$. What is more, such an action $G \curvearrowright \mathbb{S}^n$ is polar if and only if the corresponding representation $G \curvearrowright \mathbb{R}^{n+1}$ is. A section of the latter is simply the affine cone over a section of the former. //

Part (g) of Proposition 2.3.23 asserts that all slice representations of a polar action are polar. The converse is not true in general, which leads to another, more general class of actions.

Definition 2.3.26. A proper isometric action of a Lie group G on a complete Riemannian manifold M is called **infinitesimally polar** if all of its slice representations are polar.

Example 2.3.27 (Cohomogeneity-2 actions). Let $G \curvearrowright M$ be an isometric action of cohomogeneity two and $p \in M$ any point. By Proposition 2.3.14(b), the slice representation of the action at p has cohomogeneity two. As we observed in Example 2.3.25, that representation induces a cohomogeneity-one action on the unit sphere. In the next subsection, we will see that cohomogeneity-one actions on symmetric spaces are polar. But then the slice representation itself is polar. We deduce that *isometric cohomogeneity-two actions are infinitesimally polar.* //

The importance of infinitesimally polar actions comes from the following result of Lytchak and Thorbergsson ([LT10]): a proper isometric action $G \curvearrowright M$ is infinitesimally polar if and only if the orbit space M/G, with its quotient metric space structure, is a Riemannian orbifold. Now we go in the opposite direction and define a special *sub*class of polar actions.

Definition 2.3.28. A polar action is called **hyperpolar**[1] if its sections are flat.

We do not define hyperpolar representations as that would be redundant: being a linear subspace, every section of a polar representation is automatically flat. On symmetric spaces, there is an obvious upper bound on the cohomogeneity of a hyperpolar action. Indeed, we know from Proposition 2.3.23(a) that it coincides with the dimension of a section. But a section of a hyperpolar action is a flat totally geodesic submanifold. From the very definition of rank, we get:

Corollary 2.3.29. *Let M be a symmetric space and $H \curvearrowright M$ a hyperpolar action (resp., $\mathcal{F}$ a hyperpolar foliation on M). Then the cohomogeneity of $H \curvearrowright M$ (resp., the codimension of $\mathcal{F}$) cannot exceed* $\mathrm{rk}(M)$.

Here is an crucial example of a hyperpolar action and a polar representation.

Example 2.3.30. Let M be a semisimple Riemannian symmetric space represented by a Riemannian symmetric pair (G, K). Then the action $K^0 \curvearrowright M$ is hyperpolar and thus the restricted isotropy representation $K^0 \to \mathrm{SO}(T_oM)$ is polar by Proposition 2.3.23(g). The sections of $K^0 \curvearrowright M$ are precisely the maximal flats in M passing through p. The sections of $K^0 \curvearrowright \mathfrak{p} \cong T_oM$ are precisely the maximal abelian subspaces of $\mathfrak{p}$. //

Corollary 2.3.31. *The rank of a semisimple symmetric space coincides with the cohomogeneity of its isotropy representation.*

Definition 2.3.32. The restricted isotropy representation of a semisimple Riemannian symmetric pair is called an **s-representation**.

It turns out that s-representations essentially exhaust all polar representations. The following was proven by Dadok in [Dad85]:

Theorem 2.3.33 (Classification of polar representations). *Every polar representation of a connected Lie group is orbit-equivalent to an s-representation.*

Definition 2.3.34. A homogeneous foliation $\mathcal{F}$ on a complete Riemannian manifold M is called **polar** (resp., **hyperpolar**) if it is the orbit foliation of a polar (resp., hyperpolar) isometric action $H \curvearrowright M$ without singular orbits. In this case, a section of $H \curvearrowright M$ is also said to be a **section** of $\mathcal{F}$.

Notice that a polar homogeneous foliation is proper by definition.

2.3.3. Cohomogeneity-one actions and homogeneous hypersurfaces

Last but not least, we discuss some aspects of the theory of cohomogeneity-one actions. We start off with the following collection of results concerning the orbit spaces of such actions.

[1] There is yet another class of isometric actions closely related to hyperpolar actions: $G \curvearrowright M$ is called *variationally complete* if, roughly speaking, it produces enough Jacobi fields along transversal geodesics to determine the multiplicity of focal points to the orbits (see [BS58, Def. 6.8] for a precise definition). This notion was introduced by Bott and Samelson in their study of the topology of loop spaces of symmetric spaces. Variationally complete actions are infinitesimally polar ([ABT13, Th. 6.3]). In [Con71], Conlon showed that hyperpolar actions are variationally complete. In many cases, the converse is also true; for instance, this is the case when $G \curvearrowright M$ is an orthogonal representation ([DSO01]), or when M is nonnegatively curved (e.g., a compact symmetric space; see [LT07]).

Proposition 2.3.35 (Orbit spaces of C1-actions). *Let H be a connected Lie group, M a complete connected Riemannian manifold, and $H \curvearrowright M$ a proper isometric action of cohomogeneity 1.*

(a) *The orbit space M/H is a 1-dimensional topological manifold with or without boundary, hence it is homeomorphic to $\mathbb{R}, \mathbb{S}^1, [0,1]$, or $[0,1)$.*

(b) *The interior points of M/H are precisely the principal orbits.*

(c) *If all the orbits are principal (e.g., if $M/H \simeq \mathbb{S}^1$ or $\mathbb{R}$), then $M \twoheadrightarrow M/H$ is a fiber bundle.*

(d) *If M is compact, then so is M/H, hence it is homeomorphic to either $\mathbb{S}^1$ or $[0,1]$.*

(e) *If M is simply connected, then so is M/H, hence it is homeomorphic to $\mathbb{R}, [0,1]$, or $[0,1)$. Moreover, there are no exceptional orbits in this case.*

(f) *If M is Hadamard, then M/H is noncompact, hence it is homeomorphic to either $\mathbb{R}$ or $[0,1)$.*

Proofs of these results can be found in [Mos57, BB82, Lyt10], as well as [BB01].

Example 2.3.36. Consider the isometric C1-action of $\mathrm{SO}(2)$ on $\mathbb{S}^2$ given by rotation around the z-axis. The orbit space of this action is homeomorphic to $[0,1]$, where the endpoints correspond to the singular orbits, which are just the South and North poles. This action factors through the Riemannian covering map $\pi\colon \mathbb{S}^2 \to \mathbb{R}P^2$, $x \mapsto [x]$, to produce an isometric C1-action on $\mathbb{R}P^2$. The orbit space of this action is also homeomorphic to a closed interval, which we can identify with $[0,1/2]$. The effect that π has on the orbit spaces is that it folds $\mathbb{S}^2/\mathrm{SO}(2) \simeq [0,1]$ in half. The endpoint $0 \in [0,1/2] \simeq \mathbb{R}P^2/\mathrm{SO}(2)$ is still singular, but the new endpoint $1/2$ is an exceptional orbit (the image of the equator under π) with isotropy $\mathbb{Z}/2\mathbb{Z}$. //

One useful property of C1-actions is that they are often hyperpolar. Indeed, let $H \curvearrowright M$ be such an action with M complete and connected. Given a nonsingular point $p \in M$, the normal space $N_p(H \cdot p)$ is 1-dimensional, so $\Sigma = \exp(N_p(H \cdot p))$ is just the image of a geodesic emanating from p orthogonally to $H \cdot p$ (if there is no ambiguity, we simply call it a *normal geodesic at* p). As we mentioned earlier, Σ intersects all the orbits. In theory, it might have self-intersections. Assume this is not the case and Σ is an immersed submanifold. One can then show that Σ always intersects the orbits orthogonally (see [DRK11, Lem. 5]). We deduce:

Corollary 2.3.37. *Let M be a complete connected Riemannian manifold and $H \curvearrowright M$ an isometric cohomogeneity-one action. Assume that for some nonsingular point p, the image of a normal geodesic at p is a submanifold. Then $H \curvearrowright M$ is hyperpolar.*

The most obvious example of a space that would satisfy the assumption of Corollary 2.3.37 is a space where the image of every geodesic is a submanifold. As we know from (2.1.2), this includes all symmetric spaces.

Remark 2.3.38. As we mentioned earlier, some authors require the sections to be (properly) embedded as part of the definition of a polar action. In this case, Corollary 2.3.37 would not hold in general. Still, there are many spaces where the image of every geodesic is properly embedded: for example, Hadamard manifolds, which includes the Euclidean spaces and all symmetric spaces of noncompact type. Even though this is no longer true for symmetric spaces of compact type, one can show the following: given a simply connected symmetric space of compact type M and an isometric C1-action $H \curvearrowright M$, any geodesic that is normal to some orbit of H is closed (see [TT95, Th. 1.6(a)]).

One of the big advances in the theory of polar and C1-actions was their classification on irreducible symmetric spaces of compact type. For $\mathbb{S}^n$ and $\mathbb{R}P^n$, the problem can be reduced—according to Example 2.3.25—to classifying polar representations, so it follows from Dadok's result Theorem 2.3.33. For the other projective spaces, a classification was essentially obtained by Podestà and Thorbergsson in [PT99] (they overlooked one action on $\mathbb{O}P^2$, which was later discovered by Kollross and Gorodski in [GK16]). In [Kol02], Kollross classified hyperpolar and C1-actions on all irreducible symmetric spaces of compact type. Among other things, he showed that every hyperpolar action on such a space is either of cohomogeneity one or else orbit-equivalent to a so-called Hermann action. Later, in a series of articles [Kol07, Kol09, Lyt14, KL13], Kollross and Lytchak proved that polar actions (that have positive-dimensional orbits) on irreducible compact symmetric spaces of rank ≥ 2 are hyperpolar, which completed the classification. Note that the rank condition is essential: every symmetric space of compact type and rank 1 (and $\dim > 2$) admits a polar action (that has a positive-dimensional orbit) that is not hyperpolar. We refer to [BCO16, Ch. 12] for further details.

Finally, we discuss the relation between cohomogeneity-one actions and homogeneous hypersurfaces, which will prove of great importance in Chapter 5. If $H \curvearrowright M$ is a proper isometric action of cohomogeneity 1, then its principal orbits are properly embedded homogeneous hypersurfaces. Conversely, given such a hypersurface S, Remark 2.3.11 implies that there exists a proper isometric action $H \curvearrowright M$ having S as an orbit (e.g., $H = I(M, S)$). We want to show that any single orbit of a C1-action determines all the other orbits.

Definition 2.3.39. Let M be a complete Riemannian manifold and $S \subseteq M$ a properly embedded submanifold. Given $r > 0$, define $\boldsymbol{S_r} = \{\exp(v) \mid v \in NS, ||v|| = r\}$. Suppose some connected component S_r' of S_r is an embedded submanifold of M. Then we call it

- an **equidistant hypersurface** to S if both it and S are hypersurfaces,
- a **tube** of radius r around S if it is a hypersurface but S had codimension greater than 1,
- a **focal manifold** of S if it has codimension greater than 1.

Observation 2.3.40. Let M and S be as in Definition 2.3.39.

(a) If S has codimension greater than 1, then each S_r is connected. However, if S is a hypersurface, S_r may have two connected components if the normal bundle NS is trivial. For example, if $H^1(M, \mathbb{Z}/2\mathbb{Z}) = 0$, then $M \setminus S$ has 2 connected components by a generalized version of the Jordan–Brouwer separation theorem, so S_r must have 2 connected components—at least for r small enough. Note that these two components may both happen to be submanifolds but of different dimension (take, for instance, $M = \mathbb{R}^3$ and S a cylinder of radius r).

(b) If S is compact or a homogeneous submanifold and r is small enough, then S_r is a properly embedded hypersurface consisting of all points of M with distance r to S—this follows from the tubular neighborhood theorem or its homogeneous analog

Proposition 2.3.12. In general, however, points in S_r can have distance to S less than r. For example, if $M = \mathbb{S}^2$ with its round metric of diameter π and S is the equator, then $S_\pi = S$.

Proposition 2.3.41. *Let M be a complete connected Riemannian manifold and $S \subseteq M$ an orbit of an isometric C1-action on M by a connected Lie group H. Then for every $r > 0$, S_r is an orbit of H (or a union of two orbits if S_r is disconnected). In particular, any orbit of H uniquely determines all the other orbits (and thus the actions itself up to orbit-equivalence).*

Proposition 2.3.41 implies that classifying C1-actions on a given Riemannian manifold is essentially the same as classifying homogeneous hypersurfaces in it:

Corollary 2.3.42 (Homogeneous hypersurfaces vs C1-actions). *A connected homogeneous properly embedded hypersurface S in a complete connected Riemannian manifold M is an orbit of an isometric C1-action on M by a connected Lie group H, all of whose orbits are determined by S. If two such hypersurfaces are congruent, then their corresponding C1-actions are orbit-equivalent. We thus have the following diagram:*

$$\left\{\begin{matrix}\textit{congruence classes of}\\ \textit{connected homogeneous}\\ \textit{properly embedded}\\ \textit{hypersurfaces in } M\end{matrix}\right\} \xrightarrow{S \mapsto I^0(M,S) \curvearrowright M} \left\{\begin{matrix}\textit{orbit-equivalence classes}\\ \textit{of isometric C1-actions}\\ H \curvearrowright M \textit{ with } H \textit{ connected}\end{matrix}\right\},$$

and if $[S], [S']$ are mapped to the same orbit-equivalence class of C1-actions, then S' is congruent to an equidistant hypersurface of S, and they are both nonsingular orbits of the corresponding action.

In Hadamard manifolds, the orbits of a C1-action have particularly simple topology. In combination with Proposition 2.3.41 and Proposition 2.3.35, this can be formulated in the following proposition, which will come in handy in Chapter 5 (see [BB01, Prop. 1] for a proof):

Proposition 2.3.43. *Assume M is a Hadamard manifold of dimension n and $H \curvearrowright M$ is an isometric C1-action with H connected. Then exactly one of the following is true:*

(a) *All the orbits of H are principal, and for any orbit S, the other orbits are precisely the equidistant hypersurfaces to S. Each orbit is diffeomorphic to $\mathbb{R}^{n-1}$.*

(b) *There exists a unique singular orbit, and it is diffeomorphic to $\mathbb{R}^k$ for some $k < n-1$. All the other orbits are principal and diffeomorphic to $\mathbb{R}^k \times \mathbb{S}^{n-k-1}$, and they are precisely the tubes around the singular orbit.*

2.4. Symmetric spaces of noncompact type

In this section, we hone in on the spaces of most prominence in this thesis: symmetric spaces of noncompact type. Although any such space M is topologically trivial (Proposition 2.1.92), its isometry Lie algebra carries special structures such as the Iwasawa and restricted root space decompositions. These structures allow to develop a rich theory of parabolic subalgebras and subgroups—which play a central role in the theory of noncompact symmetric spaces. Together with the Iwasawa decomposition, parabolic subgroups allow for numerous geometric constructions on M, some of which we will encounter in Chapter 5. One of the prime ways in which noncompact symmetric spaces differ from their compact counterparts is that their isometry groups admit way more subgroups, which leads to more homogeneous objects such as homogeneous foliations and submanifolds. We begin our discussion with the Cartan decomposition and its special properties, which will allow us to deepen our understanding of this type of symmetric spaces.

2.4.1. The Cartan decomposition

The following structure result underpins most of the theory of symmetric spaces of noncompact type. (For a proof, see [Kna02, Th. 6.31].)

Proposition 2.4.1. *Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra of noncompact type and G any connected Lie group with Lie algebra $\mathfrak{g}$.*

(a) *θ admits a unique lift to an automorphism Θ of G, and Θ is involutive.*

(b) *The fixed point subgroup $K = G^{\Theta}$ is connected and* $\mathrm{Lie}(K) = \mathfrak{k}$*. The pair (G, K) is a Riemannian symmetric pair associated with $(\mathfrak{g}, \theta)$.*

(c) *The center Z of G is contained in K.*

(d) *K is compact if and only if Z is finite. In this case, K is a maximal compact subgroup of G.*

(e) *The map $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \exp(X)$ is a diffeomorphism.*

This result immediately yields the following description of symmetric spaces of noncompact type:

Proposition 2.4.2. *Any symmetric space of noncompact type can be represented by a Riemannian symmetric pair (G, K), where G is a semisimple Lie group and K its maximal compact subgroup. Moreover, M is irreducible if and only if G can be chosen topologically simple.*

Remark 2.4.3. It follows from Proposition 2.4.1(d) that if Z is finite, then any compact subgroup of G fixes some point in the symmetric space $M = G/K$. This also follows from the Cartan fixed point theorem, as the symmetric space $M = G/K$ is nonpositively curved by Proposition 2.1.89(c).

It turns out that the group of inner isometries of a noncompact symmetric space is centerless. To see this, we need the following

Lemma 2.4.4. *Suppose $\mathfrak{g}$ is a Lie algebra with trivial center. Then* $\mathrm{Inn}(\mathfrak{g})$ *has trivial center as well. If G is a connected Lie group with Lie algebra $\mathfrak{g}$ and Z is its center, then G/Z has trivial center, hence Z is the largest discrete normal subgroup of G.*

Proof. Since $G/Z \cong \mathrm{Inn}(G)$, we only need to prove the first assertion. If g lies in the center of $\mathrm{Inn}(\mathfrak{g})$, then it must commute with every element of $\mathrm{ad}(\mathfrak{g})$. Take $X, Y \in \mathfrak{g}$ and compute:

$$[X, gY] = \mathrm{ad}(X) \circ g(Y) = g \circ \mathrm{ad}(X)(Y) = g[X, Y] = [gX, gY],$$

so $X - gX \in \mathfrak{z}(\mathfrak{g})$ and thus $gX = X$. Since X was arbitrary, we conclude that $g = e$. □

Somewhat informally, Lemma 2.4.4 says that if $\mathfrak{g}$ is a centerless Lie algebra, then $\mathrm{Inn}(\mathfrak{g})$ is the smallest connected Lie group with Lie algebra isomorphic to $\mathfrak{g}$.

Corollary 2.4.5. *If (G, K) is an infinitesimally effective Riemannian symmetric pair of noncompact type, then $Z = I$. In particular, if (G, K) is effective, then G has trivial center. In other words, if M is a symmetric space of noncompact type, the group $I^0(M)$ is centerless. If M is irreducible, $I^0(M)$ is simple.*

Proof. As we know, for any infinitesimally effective Riemannian symmetric pair $Z \cap K = I$. But $Z \subseteq K$ by Proposition 2.4.1(c), so $Z = I$. The other assertions follow trivially. The last statement follows from Proposition 2.1.118. □

The fact that $I^0(M)$ is centerless is essentially the reason why symmetric spaces of noncompact type are simply connected. Note that if M is of compact type, $I^0(M)$ may in general have nontrivial finite center. Corollary 2.4.5 suggests a simple description of the identity component of the isometry group of a symmetric space of noncompact type:

Corollary 2.4.6. *Let M by a symmetric space of noncompact type represented by an infinitesimally effective Riemannian symmetric pair (G, K). Then $I^0(M) \cong G/Z \cong \mathrm{Ad}(G) \cong \mathrm{Inn}(\mathfrak{g})$.*

We will strengthen this corollary in the next chapter (see Proposition 3.3.4).

We can give a precise formulation of the correspondence between symmetric spaces of noncompact type and noncompact semisimple Lie algebras. Let us say that two symmetric spaces M and M' of noncompact type are *equivalent* if they become isometric after a suitable rescaling of their normalizing constants. For irreducible spaces, this is the same as being homothetic, but in general, this notion of equivalence is weaker. We can now formulate the aforementioned correspondence:

$$\left\{\begin{array}{c}\text{equivalence classes}\\ \text{of symmetric spaces}\\ \text{of noncompact type}\end{array}\right\} \xrightarrow[\sim]{M \mapsto \mathfrak{i}(M)} \left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of real semisimple Lie algebras}\\ \text{without nonzero compact ideals}\end{array}\right\} \tag{2.4.1}$$

If we start with a real semisimple Lie algebra $\mathfrak{g}$ without nonzero compact ideals, its corresponding equivalence class of symmetric spaces of noncompact type is given by any symmetric space represented by $(\mathfrak{g}, \theta)$, where θ is any Cartan involution on $\mathfrak{g}$. Consequently, a symmetric space M of noncompact type is determined by its isometry Lie algebra up to equivalence. Notice that this is no longer true in the compact type: for example, the isometry Lie algebra of the Grassmann manifold $\mathrm{Gr}(k, \mathbb{R}^n)$ is isomorphic to $\mathfrak{so}(n)$ regardless of k.

In practice, symmetric spaces are usually represented by Riemannian symmetric pairs that are not effective but have finite I. If (G, K) is infinitesimally effective of compact type, this is automatically the case. But in the noncompact type, this is not necessarily true: recall the example from Remark 2.3.3, where $\mathbb{R}H^2$ was represented as $\widetilde{\mathrm{SL}}(2, \mathbb{R})/K$ with $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ the universal covering group of $\mathrm{SL}(2, \mathbb{R})$ and K a subgroup of it isomorphic to $\mathbb{R}$. To avoid such pathologies, we introduce the following

Definition 2.4.7. A Riemannian symmetric pair (G, K) is called **almost effective** if I is finite.

An almost effective Riemannian symmetric pair is infinitesimally effective. Conversely, if (G, K) is an infinitesimally effective Riemannian symmetric pair of noncompact type, it is almost effective $\Leftrightarrow$ $Z(G)$ is finite $\Leftrightarrow$ K is compact (by Corollary 2.4.5 and Proposition 2.4.1(d)). In this case, K is a maximal compact subgroup of G. Virtually all Riemannian symmetric pairs we will be encountering in practice throughout the thesis will be almost effective. We thus make the following

Agreement. Whenever we introduce a symmetric space as $M = G/K$, we tacitly assume that (G, K) is an almost effective Riemannian symmetric pair representing M—unless otherwise specified.

This shorthand is especially common for concrete symmetric spaces: we will often write things like "consider $M = E_6/\mathrm{Spin}(10)\mathrm{U}(1)$", tacitly assuming that a choice of an E_6-invariant (and thus symmetric) metric has been made.

2.4.2. The restricted root space and Iwasawa decompositions

Now that we know that noncompact symmetric spaces are essentially the same as noncompact semisimple Lie algebras, we need to spend some time on the theory of such Lie algebras. The importance of this subsection really can not be overstated, as pretty much most of the thesis relies on the results we are about to lay out. The exposition here closely follows [Kna02, Ch. VI, Sect. 2-5].

Let $\mathfrak{g}$ be a real semisimple Lie algebra and θ a Cartan involution on $\mathfrak{g}$. Recall that we have an inner product on $\mathfrak{g}$ given by $B_\theta(X, Y) = -B(X, \theta Y)$. We fix it as our default inner product on $\mathfrak{g}$. Note that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to both B and B_θ. Let us write $\mathrm{Inn}(\mathfrak{g})^\theta$ for the subgroup of inner automorphisms of $\mathfrak{g}$ that commute with θ (this is a standard shorthand for $\mathrm{Inn}(\mathfrak{g})^{C_\theta}$). Since $\mathfrak{g}$ is semisimple, $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Aut}^0(\mathfrak{g})$, and we can identify $\mathfrak{g}$ with the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ by means of the adjoint representation: $\mathrm{ad}\colon \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$. Under this identification, we have

$$\begin{aligned}
\mathrm{Lie}(\mathrm{Inn}(\mathfrak{g})^\theta) &= \mathrm{Der}(\mathfrak{g})^{\mathrm{Ad}(\theta)} = \mathrm{Der}(\mathfrak{g})^\theta \\
&\cong \{X \in \mathfrak{g} \mid \mathrm{ad}(X) \circ \theta = \theta \circ \mathrm{ad}(X)\} \\
&= \{X \in \mathfrak{g} \mid \mathrm{ad}(X) = \mathrm{ad}(\theta X)\} \\
&= \{X \in \mathfrak{g} \mid \theta(X) = X\} = \mathfrak{k}.
\end{aligned}$$

The action of $\mathrm{Inn}(\mathfrak{g})^\theta$ on $\mathfrak{g}$ is orthogonal with respect to B_θ. Note that $\mathrm{Inn}(\mathfrak{g})^\theta$ preserves the Cartan decomposition and thus acts on $\mathfrak{p}$. By Proposition 2.4.1(b), $(\mathrm{Inn}(\mathfrak{g}), \mathrm{Inn}(\mathfrak{g})^\theta)$ is a Riemannian symmetric pair. It then follows from Proposition 2.2.22(b) that:

Proposition 2.4.8. *Any two maximal abelian subspaces of $\mathfrak{p}$ differ by some element of* $\mathrm{Inn}(\mathfrak{g})^\theta$.

Fix a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and write $r = \dim(\mathfrak{a})$. Given any nonzero $\alpha \in \mathfrak{a}^*$, define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \forall\, H \in \mathfrak{a}, [H, X] = \langle \alpha, H \rangle X\}\,.$$

This may be a zero subspace. For this reason, we define

$$\boldsymbol{\Sigma} = \{\alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq \{0\}\}\,, \quad \boldsymbol{\Sigma_0} = \Sigma \cup \{0\}\,.$$

The restriction $B_\theta\big|_{\mathfrak{a}\times\mathfrak{a}} = B\big|_{\mathfrak{a}\times\mathfrak{a}}$, being nondegenerate, determines an isomorphism $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$. If $\xi \in \mathfrak{a}^*$, we denote the corresponding element of $\mathfrak{a}$ by $\boldsymbol{H_\xi}$ (thus, $\langle \xi, H\rangle = \langle H_\xi \,|\, H\rangle$ for any $H \in \mathfrak{a}$). We carry the inner product from $\mathfrak{a}$ to[1] $\mathfrak{a}^*$ along this isomorphism: $\langle \xi \,|\, \xi'\rangle = \langle H_\xi \,|\, H_{\xi'}\rangle$.

Notation. Given an inner product space V and a subspace $U \subseteq V$, we write $V \ominus U$ for the orthogonal complement of U in V. Furthermore, if we have a direct sum decomposition $V = U \oplus W$ such that U and W are orthogonal to each other, we sometimes stress it by writing $V = U \oplus^\perp W$.

We list a few properties of the objects defined so far (see [Kna02, Ch.VI, Sect. 4-5] for proofs):

Proposition 2.4.9 (Restricted root space decomposition). *Let $\mathfrak{g}$ be a real semisimple Lie algebra with a fixed Cartan involution θ and a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$.*

(a) *With the inner product defined above, $(\mathfrak{a}^*, \Sigma)$ is a (possible reduced) root system. In particular, Σ spans $\mathfrak{a}^*$.*

(b) $\mathfrak{g} = \bigoplus_{\alpha\in\Sigma_0} \mathfrak{g}_\alpha = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$.

(c) *The summands of the decomposition in* (b) *are pairwise orthogonal.*

(d) *For any $\alpha, \beta \in \Sigma_0$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.*

(e) *$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus^\perp \mathfrak{a}$, where $\boldsymbol{\mathfrak{k}_0} = Z_{\mathfrak{k}}(\mathfrak{a}) = N_{\mathfrak{k}}(\mathfrak{a})$.*

(f) *For any $\alpha \in \Sigma_0$, $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$.*

(g) *For any $\alpha \in \Sigma$ and any $X, Y \in \mathfrak{g}_\alpha$, $[X, \theta Y] = B(X, \theta Y) H_\alpha = -\langle X \,|\, Y\rangle H_\alpha$. In particular, if $X \neq 0$, then $[X, \theta X]$ is a nonzero multiple of H_α.*

Definition 2.4.10. Σ is called the **restricted root system** of $\mathfrak{g}$. The decomposition in Proposition 2.4.9(b) is called the **restricted root space decomposition** of $\mathfrak{g}$ and each $\mathfrak{g}_\alpha$ is called a (**restricted**) **root space**. We call $\dim(\mathfrak{g}_\alpha)$ the **multiplicity** of the root α and denote it by $\mathbf{mult}(\boldsymbol{\alpha})$. The Dynkin diagram of Σ is denoted by **DD**.

There are several other diagrams one can associate to a real semisimple Lie algebra—for instance, the Satake and Vogan diagrams. These diagrams actually determine the Lie algebra up to isomorphism, whereas the Dynkin diagram of the restricted root system is only suitable for studying noncompact Lie algebras, and it only determines the Lie algebra after some modifications (we will discuss this in detail in Chapter 3). The main reason why the restricted root system is preferred in the context of noncompact symmetric spaces is because it better reflects the geometry of the underlying space (as we will witness repeatedly throughout the thesis). Notice also that, in contrast to the complex semisimple case, the root system Σ does not have to be reduced. We will talk more about this in

[1]This is our default choice of an inner product on $\mathfrak{a}^*$. Even in the presence of a symmetric space M represented by $(\mathfrak{g}, \theta)$, we stick with this inner product rather than one that can be induced by $g_o\big|_{\mathfrak{a}\times\mathfrak{a}}$. The same goes for the isomorphism $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$.

Section 3.2. For a full list of simple noncompact Lie algebras, their restricted root systems, and corresponding noncompact symmetric spaces, see [BCO16, pp. 336-340].

Given $\alpha \in \Sigma$, define

$$\begin{aligned} \mathfrak{k}_\alpha &= (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{k} = \{X + \theta X \mid X \in \mathfrak{g}_\alpha\}, \\ \mathfrak{p}_\alpha &= (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{p} = \{X - \theta X \mid X \in \mathfrak{g}_\alpha\}. \end{aligned}$$

Note that we have $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha$. It follows immediately from Proposition 2.4.9(c) that $\mathfrak{k}_\alpha \perp \mathfrak{k}_\beta$ and $\mathfrak{p}_\alpha \perp \mathfrak{p}_\beta$ for $\alpha \neq \beta$. We also have $\mathfrak{k}_\alpha \perp \mathfrak{k}_0$, $\mathfrak{p}_\beta \perp \mathfrak{a}$, and $\mathfrak{k}_\alpha \perp \mathfrak{p}_\beta$ for any α, β.

Let us denote the Weyl group of Σ by $\mathbf{W(\Sigma)} \subseteq \mathrm{O}(\mathfrak{a}^*)$. Write $N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})$ for the normalizer of $\mathfrak{a}$ in $\mathrm{Inn}(\mathfrak{g})^\theta$ with respect to the action $\mathrm{Inn}(\mathfrak{g})^\theta \curvearrowright \mathfrak{p}$. Let us define a Lie group homomorphism $\mathbf{\Omega}\colon N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a}) \to \mathrm{GL}(\mathfrak{a}^*), \varphi \mapsto (\varphi\big|_{\mathfrak{a}}{}^*)^{-1}$. This map will be of great importance to us and we will study it in more detail in Section 3.2. For the time being, we just state the following

Proposition 2.4.11. (a) *For any $\varphi \in N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})$, $\Omega(\varphi)$ preserves Σ. Given $\alpha \in \Sigma_0$, $\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\Omega(\varphi)(\alpha)}$. Informally, φ shuffles the summands of the restricted root space decomposition in a way that agrees with $\Omega(\varphi)$.*

(b) *For any $\varphi \in N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})$, $\Omega(\varphi) \in \mathrm{W}(\Sigma)$.*

(c) *Given $\alpha \in \Sigma$ and any nonzero $X \in \mathfrak{g}_\alpha$ normalized so that $||X|| = \frac{\sqrt{2}}{||\alpha||}$, $\exp(\mathrm{ad}\,\frac{\pi}{2}(X + \theta X))$ lies in $N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})$ and its image under Ω is the reflection s_α in the root hyperplane $\alpha^\perp$. Consequently, $\mathrm{Im}(\Omega) = \mathrm{W}(\Sigma)$.*

Pick a Weyl chamber $D \subset \mathfrak{a}^*$ and denote the corresponding subsets of positive and simple roots by Σ^+ and $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, respectively. Note that $H_{\alpha_1}, \ldots, H_{\alpha_r}$ is a basis for $\mathfrak{a}$. We have another basis, namely the dual basis of $\alpha_1, \ldots, \alpha_r$, which we denote by $\boldsymbol{H^1}, \ldots, \boldsymbol{H^r}$. By definition, $\langle H_i, \alpha_j \rangle = \langle H^i | H_{\alpha_j} \rangle = \delta_{ij}$. Since the Weyl group acts transitively on the set of Weyl chambers, it follows from Proposition 2.4.11(c) that:

Corollary 2.4.12. *For any two choices Σ_1^+, Σ_2^+ of positive roots for Σ, there exists $\varphi \in N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})$ such that $\Omega(\varphi)(\Sigma_1^+) = \Sigma_2^+$.*

Because of this result as well as Propositions 2.1.73 and 2.4.8, the choices of $\theta, \mathfrak{a}$, and Σ^+ are irrelevant. For this reason, given a real semisimple Lie algebra, we will sometimes omit that step and assume such a choice has already been fixed implicitly. Together with some standard theory of root systems, Proposition 2.4.9(d) yields the following useful result: two simple roots $\alpha, \beta \in \Lambda$ are orthogonal (that is, not connected by an edge in DD) $\Leftrightarrow$ their sum is not a root $\Leftrightarrow$ the subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ commute.

We can relate the Cartan and restricted root space decompositions as follows:

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{k}_\alpha, \quad \mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{p}_\alpha. \tag{2.4.2}$$

All the summands in these decompositions are pairwise orthogonal. Next, define $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. This is a nilpotent subalgebra of $\mathfrak{n}$.

Proposition 2.4.13 (Iwasawa decomposition). *The Lie algebra $\mathfrak{g}$ decomposes as the* *direct sum of its subspaces* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$.

If we combine Propositions 2.1.73 and 2.4.8 and Corollary 2.4.12 one more time, we see that any two Iwasawa decompositions of $\mathfrak{g}$ differ by an inner automorphism. The semidirect sum $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ is a solvable subalgebra of $\mathfrak{g}$. The subalgebras $\mathfrak{s}$ and $\mathfrak{n}$ are commonly referred to as the *solvable* and *nilpotent parts of the Iwasawa decomposition*, respectively. Note that $\mathfrak{s}$ and $\mathfrak{n}$ are graded Lie algebras:

$$\mathfrak{s} = \bigoplus_{k=0}^{m} \mathfrak{s}^k, \quad \mathfrak{n} = \bigoplus_{k=1}^{m} \mathfrak{n}^k, \quad \text{where} \quad \mathfrak{s}^0 = \mathfrak{a}, \quad \mathfrak{s}^k = \mathfrak{n}^k = \bigoplus_{\mathrm{ht}(\alpha)=k} \mathfrak{g}_\alpha. \tag{2.4.3}$$

Here ht is the height function on Σ^+ and m is the height of the highest root $\delta \in \Sigma^+$.

If we compare the Iwasawa and Cartan decompositions, we see that the projection to $\mathfrak{p}$ along $\mathfrak{k}$ establishes a linear isomorphism $\mathfrak{s} \xrightarrow{\sim} \mathfrak{p}$. This map will be of great relevance to us in Chapter 6.

Example 2.4.14 (The hyperbolic spaces). Let $\mathfrak{g}$ be the isometry Lie algebra of the hyperbolic space $\mathbb{F}H^n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $n \geq 2$. Since the rank of $\mathbb{F}H^n$ is 1, the subspace $\mathfrak{a} \subset \mathfrak{p}$ is one-dimensional. If $\mathbb{F}$ is $\mathbb{R}$, the restricted root system Σ is A_1, but in the other three cases Σ is $(BC)_1$. The positive roots are $\alpha, 2\alpha$ if $\mathbb{F} \neq \mathbb{R}$ and just α if $\mathbb{F} = \mathbb{R}$. The short root α has multiplicity $\dim_{\mathbb{R}}(\mathbb{F})(n-1)$, so $\mathfrak{g}_\alpha$ can be identified with $\mathbb{F}^{n-1}$. The long root 2α has multiplicity $\dim_{\mathbb{R}}(\mathbb{F}) - 1$, so we can identify the sum $\mathfrak{a} \oplus \mathfrak{g}_{2\alpha}$ with $\mathbb{F}$, where $\mathfrak{a}$ corresponds to $\mathbb{R}$ and $\mathfrak{g}_{2\alpha}$ to $\mathrm{Im}(\mathbb{F})$. The subalgebra $\mathfrak{n}$ is 2-step nilpotent with center $\mathfrak{g}_{2\alpha}$ if $\mathbb{F} \neq \mathbb{R}$ and abelian if $\mathbb{F} = \mathbb{R}$. //

Example 2.4.15 (Split real forms). Every complex semisimple Lie algebra has two (unique up to an inner automorphism) special real forms: the compact real form and the split real form. The latter is always noncompact and admits a number of intrinsic characterizations:

Proposition 2.4.16. *The following are equivalent for a real semisimple Lie algebra* $\mathfrak{g}$*:*

(i) *Every root in* Σ *has multiplicity 1.*

(ii) *Every root in* Λ *has multiplicity 1.*

(iii) $\mathfrak{k}_0 = \{0\}$.

(iv) $\mathfrak{a}$ *is a maximal abelian subalgebra of* $\mathfrak{g}$.

(v) $\mathfrak{a}$ *is a Cartan subalgebra of* $\mathfrak{g}$.

(vi) $\mathfrak{a}_{\mathbb{C}}$ *is a Cartan subalgebra of* $\mathfrak{g}_{\mathbb{C}}$.

If these conditions are satisfied, $\mathfrak{g}$ *is called* ***split****.*

We will call a semisimple symmetric space **split** if its isometry Lie algebra is split (it has to be of noncompact type). As we will see in Section 3.2, the restricted root system of a split real semisimple Lie algebra is isomorphic to the root system of its complexification. In particular, it is always reduced. //

Observation 2.4.17. Every complex semisimple Lie algebra $\mathfrak{g}$ produces two natural noncompact real semisimple Lie algebras: the realification $\mathfrak{g}$ and a split real form $\mathfrak{g}_0 \subset \mathfrak{g}$. In Section 3.2, we will see that the restricted root systems of $\mathfrak{g}$ and $\mathfrak{g}_0$ are isomorphic to the root system Δ of $\mathfrak{g}$ thought of as a complex semisimple Lie algebra. Suppose $\mathfrak{g}$ is simple—which is equivalent to Δ being irreducible. It might happen that there are no more real simple Lie algebras with restricted root system isomorphic to Δ; this is the case for $\Delta = B_r\,(r \geq 2), D_r\,(r \geq 4), E_6, E_7, E_8, G_2$.

We can also lift the Iwasawa decomposition to the level of Lie groups. Let (G, K) be a Riemannian symmetric pair associated with $(\mathfrak{g}, \theta)$. Defined $\boldsymbol{A}$ and $\boldsymbol{N}$ to be the connected Lie subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$.

Proposition 2.4.18 (Global Iwasawa decomposition). *The multiplication map $K \times A \times N \to G$ is a diffeomorphism. In particular, A and N are simply connected closed subgroups of G.*

Since A normalizes N inside G, they form a semidirect product AN, which is a simply connected closed solvable subgroup of G corresponding to $\mathfrak{s}$. The exponential map of A is clearly a diffeomorphism, and one can show (e.g., by means of [OV94, Ch. 2, Th. 6.4]) that the same is true for N and AN. Now let M be a symmetric space of noncompact type represented by (G, K). The orbit $A \cdot o$ is a maximal flat in M. The orbit $N \cdot o$ is diffeomorphic to $\mathbb{R}^{n-r}$ and is known as a **horocycle**[1] (here $n = \dim(M)$). It follows from the Iwasawa decomposition that:

Corollary 2.4.19. *The solvable group AN acts simply transitively on M. The pullback of the Riemannian metric of M along the orbit map $AN \xrightarrow{\sim} M, an \mapsto an \cdot o$, is a left-invariant metric on AN. Consequently, every symmetric space of noncompact type is isometric to a simply connected solvable Lie group with a left-invariant metric.*

As a consequence, we see that the groups A and N do not depend on the choice of (G, K) up to isomorphism.

2.4.3. Parabolic subalgebras and subgroups

Here we introduce the theory of parabolic subalgebras and subgroups and discuss its relation to the geometry of symmetric spaces of noncompact type. The objects introduced here are an indispensable tool for studying cohomogeneity-one actions on symmetric spaces of noncompact type and will be of great use to us in Chapter 5. As always, we start on the level of Lie algebras. (See [Kna02, Ch.VII, Sect. 7] and [BCO16, Sect. 13.2] for a detailed exposition).

Definition 2.4.20. A maximal solvable subalgebra of a Lie algebra is called a **Borel subalgebra**. A subalgebra of a complex semisimple Lie algebra is called **parabolic** if it contains a Borel subalgebra.

In a complex semisimple Lie algebra $\mathfrak{g}$, all Borel subalgebras are conjugate by inner automorphisms and can be described as $\mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{h}$ is a suitable Cartan subalgebra and $\mathfrak{n}$ is the sum of all positive root spaces (for some choice of a Weyl chamber). Parabolic subalgebras of a complex semisimple Lie algebra are parametrized by subsets of the set of simple roots (see [Kna02, Prop. 5.90]). We are not going to delve any deeper into the complex case and proceed directly to real semisimple Lie algebras. If $\mathfrak{g}$ is such a Lie algebra, then its complexification $\mathfrak{g}_{\mathbb{C}}$ is complex semisimple.

[1] If M has rank 1, its horocycles are hypersurfaces and are usually called *horospheres*. In general, a horosphere in a Hadamard manifold is a level set of a Busemann function (see [Ebe96, Sect. 1.10]). If $M = \mathbb{R}H^n$, its horospheres are actually flat and thus isometric to $\mathbb{E}^{n-1}$.

Definition 2.4.21. A subalgebra $\mathfrak{q}$ of real semisimple Lie algebra $\mathfrak{g}$ is called **parabolic** if its complexification $\mathfrak{q}_{\mathbb{C}}$ is parabolic in $\mathfrak{g}_{\mathbb{C}}$.

Let $\mathfrak{g}$ be a real semisimple Lie algebra with a fixed choice of a Cartan involution θ and a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$.

Proposition 2.4.22 (Classification of parabolic subalgebras I). *Let $\mathfrak{g}$ be a real semisimple Lie algebra with a fixed choice of a Cartan involution θ and a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$.*

(a) $\boldsymbol{\mathfrak{q}_0} = \mathfrak{g}_0 \oplus \mathfrak{n} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ *is a minimal (by inclusion) parabolic subalgebra of $\mathfrak{g}$.*

(b) *All minimal parabolic subalgebras of $\mathfrak{g}$ are conjugate via* $\mathrm{Inn}(\mathfrak{g})$*. Consequently, any parabolic subalgebra of $\mathfrak{g}$ is conjugate via* $\mathrm{Inn}(\mathfrak{g})$ *to one containing $\mathfrak{q}_0$.*

The parabolic subalgebras containing $\mathfrak{q}_0$ admit an explicit description in terms of the restricted root space decomposition. Fix a set of positive roots $\Sigma^+ \subseteq \Sigma$. Let $\Phi \subseteq \Lambda$ be any subset, write $\boldsymbol{r_\Phi}$ for its cardinality. Let us write $\boldsymbol{\Sigma_\Phi}$ for the root subsystem of Σ spanned by Φ. Clearly, the choice of positive roots in Σ induces a choice of positive roots in Σ_Φ : $\boldsymbol{\Sigma_\Phi^+} = \Sigma_\Phi \cap \Sigma^+$, and the corresponding set of simple roots is just Φ. Before we proceed, we need a simple but crucial result on subalgebras of $\mathfrak{g}$ that are θ-stable.

Lemma 2.4.23. *Every θ-stable subalgebra of $\mathfrak{g}$ is reductive.*

Proof. Let $\mathfrak{h}$ be θ-invariant in $\mathfrak{g}$. By Proposition 2.1.71, we can assume that $\mathfrak{h}$ is a transpose-invariant subalgebra of $\mathfrak{sl}(n, \mathbb{R})$. It then has a representation on $\mathbb{R}^n$, whose induced invariant symmetric bilinear form $\mathrm{tr}(XY)$ is nondegenerate: given nonzero $X \in \mathfrak{h}$, we have $\mathrm{tr}(XX^t) = ||X||^2 > 0$. A real (or complex) Lie algebra admitting a representation whose induced invariant bilinear is nondegenerate has to be reductive (see [Kir08, Th. 5.48]). As a side result, we see that the restriction of B to any θ-stable subalgebra of $\mathfrak{g}$ is nondegenerate. □

We can now explain how Φ gives rise to a parabolic subalgebra and study its properties.

- Define $\boldsymbol{\mathfrak{l}_\Phi} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha$. This is a θ-stable subalgebra of $\mathfrak{g}$, hence it is reductive.
- Define $\boldsymbol{\mathfrak{n}_\Phi} = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_\Phi^+} \mathfrak{g}_\alpha$. This is a subalgebra of $\mathfrak{n}$, hence it is nilpotent.
- The subalgebras $\mathfrak{l}_\Phi$ and $\mathfrak{n}_\Phi$ do not intersect and we have $[\mathfrak{l}_\Phi, \mathfrak{n}_\Phi] \subseteq \mathfrak{n}_\Phi$, so we can form a semidirect sum $\boldsymbol{\mathfrak{q}_\Phi} = \mathfrak{l}_\Phi \inplus \mathfrak{n}_\Phi$. Clearly, $\mathfrak{q}_\Phi$ contains $\mathfrak{q}_0$, so it is parabolic. The decomposition $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \inplus \mathfrak{n}_\Phi$ is called the **Chevalley decomposition** of $\mathfrak{q}_\Phi$. Note that $\mathfrak{q}_\Phi$ is self-normalizing.

Example 2.4.24. Let $\Phi = \varnothing$. Then $\mathfrak{l}_\varnothing = \mathfrak{g}_0, \mathfrak{n}_\varnothing = \mathfrak{n}$, and $\mathfrak{q}_\varnothing = \mathfrak{q}_0$. //

Proposition 2.4.25 (Classification of parabolic subalgebras II). *Let $\mathfrak{g}$ be a real semisimple Lie algebra.*

(a) *Every parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{q}_0$ is of the form $\mathfrak{q}_\Phi$ for some subset $\Phi \subseteq \Lambda$. Every parabolic subalgebra of $\mathfrak{g}$ is conjugate via* $\mathrm{Inn}(\mathfrak{g})$ *to $\mathfrak{q}_\Phi$ for a unique $\Phi \subseteq \Lambda$.*

(b) *Given two subsets $\Phi_1, \Phi_2 \subseteq \Lambda$, the parabolic subalgebras $\mathfrak{q}_{\Phi_1}$ and $\mathfrak{q}_{\Phi_2}$ are conjugate via* $\mathrm{Aut}(\mathfrak{g})$ *if and only if the subsets Φ_1 Φ_2 differ by a weight-preserving automorphism*[1]

[1]We will define and study this later in Section 3.2 (for a precise definition, see Definition 3.2.4).

of the Dynkin diagram DD.

Example 2.4.26. Let $j \in \{1, \ldots, r\}$ be any, and let $\boldsymbol{\Phi_j} = \Lambda \setminus \{\alpha_j\}$. In this case, let us simplify the notation and write j instead of Φ_j in subscripts and superscripts (here and further). It follows from Proposition 2.4.25 that $\mathfrak{q}_j$ is a maximal proper parabolic subalgebra. What is more, every maximal proper parabolic subalgebra of $\mathfrak{g}$ is conjugate via $\mathrm{Inn}(\mathfrak{g})$ to $\mathfrak{q}_j$ for a unique j. //

We carry on with our study of $\mathfrak{q}_\Phi$ and its subalgebras.

- Let $\boldsymbol{\mathfrak{a}_\Phi} = \bigcap_{\alpha\in\Phi} \mathrm{Ker}(\alpha) = \bigoplus_{j\in\Lambda\setminus\Phi} \mathbb{R}H^j$. It is a subspace of $\mathfrak{a}$ of dimension $r - r_\Phi$. One can verify that $\mathfrak{l}_\Phi = Z_\mathfrak{g}(\mathfrak{a}_\Phi) = N_\mathfrak{g}(\mathfrak{a}_\Phi)$.
- Define $\boldsymbol{\mathfrak{a}^\Phi} = \mathfrak{a} \ominus \mathfrak{a}_\Phi = \bigoplus_{\alpha\in\Phi} \mathbb{R}H_\alpha$. This is an r_Φ-dimensional subspace of $\mathfrak{a}$.
- Next, define $\boldsymbol{\mathfrak{m}_\Phi} = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi = \mathfrak{k}_0 \oplus \mathfrak{a}^\Phi \oplus \bigoplus_{\alpha\in\Sigma_\Phi} \mathfrak{g}_\alpha$. This is another θ-stable subalgebra of $\mathfrak{g}$, hence it is also reductive. Plainly, the Lie algebra $\mathfrak{l}_\Phi$ decomposes as a direct sum $\mathfrak{l}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi$. Plugging this into the Chevalley decomposition yields

$$\mathfrak{q}_\Phi = (\mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi) \inplus \mathfrak{n}_\Phi =: \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \inplus \mathfrak{n}_\Phi,$$

  which is called the **Langlands decomposition** of $\mathfrak{q}_\Phi$.
- Write $\boldsymbol{\mathfrak{z}_\Phi} = \mathfrak{z}(\mathfrak{m}_\Phi)$. One can easily see that $\mathfrak{z}_\Phi \subseteq \mathfrak{k}_0$ and that $\mathfrak{z}(\mathfrak{l}_\Phi) = \mathfrak{z}_\Phi \oplus \mathfrak{a}_\Phi$.
- Let us write $\mathfrak{m}_\Phi = \mathfrak{z}_\Phi \oplus \boldsymbol{\mathfrak{g}_\Phi}$ for the (unique) Levi decomposition of $\mathfrak{m}_\Phi$. Here $\mathfrak{g}_\Phi = [\mathfrak{m}_\Phi, \mathfrak{m}_\Phi] = [\mathfrak{l}_\Phi, \mathfrak{l}_\Phi]$ is semisimple. One readily sees that $\mathfrak{g}_\Phi = (\mathfrak{k}_0 \ominus \mathfrak{z}_\Phi) \oplus \mathfrak{a}^\Phi \oplus \bigoplus_{\alpha\in\Sigma_\Phi} \mathfrak{g}_\alpha$ and that it is a θ-stable subalgebra of $\mathfrak{g}$. What is more, θ restricts to a Cartan involution of $\mathfrak{g}_\Phi$, hence $(\mathfrak{g}_\Phi, \theta)$ is an orthogonal symmetric Lie algebra of noncompact type. Note that it may not be effective (even if $(\mathfrak{g}, \theta)$ is) because $\mathfrak{g}_\Phi$ might have nonzero compact ideals.
- We denote $\boldsymbol{\mathfrak{b}_\Phi} = \mathfrak{g}_\Phi \cap \mathfrak{p}$. This is a Lie triple system in $\mathfrak{p}$. Note that $\mathfrak{a}^\Phi$ is a maximal abelian subspace of $\mathfrak{b}_\Phi$ and the corresponding restricted root system for $\mathfrak{g}_\Phi$ is Σ_Φ. Observe that the sum $\mathfrak{b}_\Phi \oplus \mathfrak{a}_\Phi$ is also a Lie triple system.
- We write $\mathbf{DD}_{\boldsymbol{\Phi}}$ for the subdiagram of DD determined by Φ: the vertices of DD_Φ are the simple roots in Φ, while the edges of DD_Φ are the edges in DD, both of whose endpoints lie in Φ. It is not hard to see that DD_Φ is the Dynkin diagram of $\mathfrak{g}_\Phi$.
- Define $\boldsymbol{\mathfrak{k}_\Phi} = \mathfrak{q}_\Phi \cap \mathfrak{k} = \mathfrak{l}_\Phi \cap \mathfrak{k} = \mathfrak{m}_\Phi \cap \mathfrak{k} = \mathfrak{k}_0 \oplus \bigoplus_{\alpha\in\Sigma_\Phi} \mathfrak{k}_\alpha$. This is a compactly embedded subalgebra of $\mathfrak{g}$. Note that $\mathfrak{z}_\Phi \subseteq \mathfrak{k}_\Phi$.
- Finally, write $\boldsymbol{\mathfrak{g}'_\Phi}$ for the noncompact part of $\mathfrak{g}_\Phi$. It is also θ-stable and reductive, and, by Observation 2.1.117, $(\mathfrak{g}'_\Phi, \theta)$ is an effective orthogonal symmetric Lie algebra of noncompact type. In particular, its Cartan decomposition is given by $\mathfrak{g}'_\Phi = [\mathfrak{b}_\Phi, \mathfrak{b}_\Phi] \oplus \mathfrak{b}_\Phi$. The restricted root system and Dynkin diagram of $\mathfrak{g}'_\Phi$ are the same as those of $\mathfrak{g}_\Phi$: Σ_Φ and DD_Φ, respectively. One can show (see [Sol23, Prop. 2.3, Rem. 2.4]) that the restricted root space decomposition of $\mathfrak{g}'_\Phi$ is given by $\mathfrak{g}'_\Phi = (\mathfrak{g}'_\Phi \cap \mathfrak{k}_0) \oplus \mathfrak{a}^\Phi \oplus \bigoplus_{\alpha\in\Sigma_\Phi} \mathfrak{g}_\alpha$ (so $\mathfrak{g}'_\Phi$ and $\mathfrak{g}_\Phi$ only differ within $\mathfrak{k}_0$) and, as a subalgebra of $\mathfrak{g}$, it is generated by $\mathfrak{g}_\alpha$ as α runs through Σ_Φ. The compact part of $\mathfrak{g}_\Phi$ is orthogonal to $\mathfrak{g}'_\Phi$ with respect to B_θ and is given by $Z_{\mathfrak{k}_0}(\mathfrak{b}_\Phi) \ominus \mathfrak{z}_\Phi$.

Now we lift all these subalgebras to the level of Lie groups. Here we assume that $(\mathfrak{g}, \theta)$

is effective, or equivalently, $\mathfrak{g}$ does not have nonzero compact ideals. Let (G, K) be an almost effective Riemannian symmetric pair associated with $(\mathfrak{g}, \theta)$. First things first, we have the simply connected closed Lie subgroups $\boldsymbol{A_\Phi} \subseteq A$ and $\boldsymbol{N_\Phi} \subseteq N$ corresponding to $\mathfrak{a}_\Phi$ and $\mathfrak{n}_\Phi$, respectively. By design, A_Φ is abelian and N_Φ is nilpotent. Define $\boldsymbol{L_\Phi} = Z_G(\mathfrak{a}_\Phi)$. This is a (possibly disconnected) closed reductive Lie subgroup of G with Lie algebra $\mathfrak{l}_\Phi$. The product $\boldsymbol{Q_\Phi} = L_\Phi N_\Phi \subseteq G$ is a closed Lie subgroup with Lie algebra $\mathfrak{q}_\Phi$. The following—admittedly ad-hoc—definition will be improved upon considerably in Proposition 2.4.36

Definition 2.4.27. A Lie subgroup of G is called parabolic if it is given by Q_Φ for some choice of θ, $\mathfrak{a} \subseteq \mathfrak{p}$, $\Sigma^+ \subseteq \Sigma$, and $\Phi \subseteq \Lambda$.

It is not hard to see that L_Φ normalizes N_Φ and they intersect trivially, so we actually have a semidirect product $Q_\Phi = L_\Phi \ltimes N_\Phi$, called the **Chevalley decomposition** of Q_Φ. Using the fact that $\mathfrak{q}_\Phi$ is self-normalizing, one can show that $Q_\Phi = N_G(\mathfrak{q}_\Phi)$. Next, let $\boldsymbol{G_\Phi}$ and $\boldsymbol{G'_\Phi}$ be the connected Lie subgroups corresponding to $\mathfrak{g}_\Phi$ and $\mathfrak{g}'_\Phi$, respectively. These are closed semisimple[1] subgroups of G, and they are both normal in L_Φ. Write $\boldsymbol{K_\Phi} = L_\Phi \cap K$, and define $\boldsymbol{M_\Phi} = K_\Phi G_\Phi$. These are both closed reductive subgroups of G, $\mathrm{Lie}(K_\Phi) = \mathfrak{k}_\Phi$, and $\mathrm{Lie}(M_\Phi) = \mathfrak{m}_\Phi$. We have inclusions $M_\Phi \subseteq L_\Phi \subseteq Q_\Phi$, and one can show that K_Φ is a maximal compact subgroup in any of these three groups. Finally, let $\boldsymbol{Z_\Phi}$ stand for the center of M_Φ. This is a compact subgroup of K_Φ with Lie algebra $\mathfrak{z}_\Phi$. The subgroups M_Φ and A_Φ commute and intersect trivially, hence we have a direct product decomposition $L_\Phi = M_\Phi \times A_\Phi$, which, when plugged into the global Chevalley decomposition, induces the **Langlands decomposition** of Q_Φ:

$$Q_\Phi = (M_\Phi \times A_\Phi) \ltimes N_\Phi =: M_\Phi \times A_\Phi \ltimes N_\Phi.$$

With respect to this decomposition, the multiplication in Q_Φ is given by:

$$(m, a, n) \cdot (m', a', n') = (mm', aa', (m'a')^{-1} n (m'a') n').$$

Eventually, we look at the what these subgroups mean for the geometry of symmetric spaces of noncompact type. Let M be such a space represented by the pair (G, K). It follows from Proposition 2.2.4 that the parabolic subgroup Q_Φ acts on M with an open orbit. Since Q_Φ is a closed subgroup, Proposition 2.1.1(d) implies that it acts transitively on M. The isotropy subgroup of Q_Φ at o is K_Φ. The subgroups A_Φ and N_Φ produce orbits $A_\Phi \cdot o \simeq \mathbb{E}^{r - r_\Phi}$, which is a flat lying in the maximal flat $A \cdot o$, and $N_\Phi \cdot o$, which a properly embedded submanifold of the horocycle $N \cdot o$ also diffeomorphic to a Euclidean space. We have a Lie triple system $\mathfrak{b}_\Phi \subseteq \mathfrak{p}$, which corresponds to a properly embedded totally geodesic submanifold $\boldsymbol{B_\Phi} = G_\Phi \cdot o = M_\Phi \cdot o \subseteq M$ often called a **boundary component** of M in the context of the maximal Satake compactification of M (see, e.g., [BJ06]). The submanifold B_Φ is itself a symmetric space of noncompact type. It has rank r_Φ and can be represented, for example, by the Riemannian symmetric pair (M_Φ^0, K_Φ^0). A somewhat better choice of a Riemannian symmetric pair representing B_Φ would be $(G'_\Phi, G'_\Phi \cap K)$, which is almost effective thanks to the argument following Definition 2.4.7. In particular, we have $\mathfrak{g}'_\Phi \cong \mathfrak{i}(B_\Phi)$. We also have a totally geodesic submanifold $\boldsymbol{F_\Phi} = L_\Phi \cdot o \cong L_\Phi / K_\Phi \cong B_\Phi \times (A_\Phi \cdot o)$ corresponding to the Lie triple system $\mathfrak{b}_\Phi \oplus \mathfrak{a}_\Phi$ (note that F_Φ is a simply connected symmetric space and $F_\Phi \simeq B_\Phi \times (A_\Phi \cdot o)$ is

[1] In many situations, semisimple Lie subgroups are automatically closed, see [Mos50, Sect. 6].

its decomposition into the Riemannian product of its noncompact and Euclidean parts). Finally, we can form a commutative diagram

$$\begin{tikzcd} M_\Phi \times A_\Phi \ltimes N_\Phi \arrow[r, "\sim"] \arrow[d, two heads] & Q_\Phi \arrow[d, two heads] \arrow[dr, two heads] & \\ B_\Phi \times A_\Phi \times N_\Phi \arrow[r, "\sim"] & Q_\Phi / K_\Phi \arrow[r, "\sim"] & M \end{tikzcd}$$

The diffeomorphism $B_\Phi \times A_\Phi \times N_\Phi \simeq M$ is called a **horospherical decomposition** of M. Note that $F_\Phi \cong B_\Phi \times (A_\Phi \cdot o)$ is an isometry, but $M \cong F_\Phi \times (N_\Phi \cdot o)$ is just a diffeomorphism. The submanifolds F_Φ and $N_\Phi \cdot o$ of M intersect orthogonally at o. Topologically, the horospherical decomposition is far from being noteworthy, as each of the factors is diffeomorphic to a Euclidean space. Its significance, however, is justified by the following fact. With respect to the Langlands and horospherical decompositions, the action of Q_Φ on M can be written as:

$$\begin{aligned} &M_\Phi \times A_\Phi \ltimes N_\Phi \curvearrowright B_\Phi \times A_\Phi \times N_\Phi, \\ &(m, a, n) \cdot (m' \cdot o, a', n') = ((mm') \cdot o, aa', (m'a')^{-1} n (m'a') n'). \end{aligned} \tag{2.4.4}$$

The Langlands and horospherical decomposition are going to be the main machinery underlying the canonical extension construction, to be introduced in Chapter 5.

2.4.4. Singular vectors and points at infinity

The aspects of the theory of noncompact symmetric spaces we have discussed so far in this section have been largely rooted in Lie theory. In this part, we talk about two more topics that are entirely geometric in nature. The first one has to do with telling apart tangent vectors to a symmetric space and will come in handy when we deal with the congruence problem for cohomogeneity-one actions in Chapter 5. The second one will allow us to give an alternative geometric definition of parabolic subgroups.

Singular and regular vectors

Tangent vectors to a Riemannian manifold can sometimes be distinguished by how they interact with the curvature tensor. Let M be a Riemannian manifold and $p \in M$. For any $v \in T_pM$, the curvature endomorphism of M gives rise to a linear map $R_p(v, -)\colon T_pM \to \mathfrak{so}(T_pM)$.

Definition 2.4.28. The **nullity**[1] of $v \in T_pM$, denoted by **null(v)**, is the dimension of $\operatorname{Ker}(R_p(v, -)) = \{Y \in T_pM \mid R(X, Y) = 0\}$.

Obviously, the nullity of a vector is preserved under isometries. The concept of nullity thus helps to distinguish between different tangent vectors, which can be extremely useful, for instance, when one tries to tell whether two given submanifolds are congruent.

Now let M be a symmetric space, and let $(\mathfrak{g}, \theta)$ be any effective orthogonal symmetric Lie algebra representing it. The effectiveness assumption ensures that the adjoint representation $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ is faithful. Owing to the curvature formula (2.1.5), under the identification

[1] This is different from (but related to) the notion of *nullity of M at p* (see, e.g., [DSOV22]).

$T_oM \cong \mathfrak{p}$, we have $R_o(X,-) = \mathrm{ad}[X,-]\big|_{\mathfrak{p}}$, which implies that

$$R_o(X,Y) = 0 \Leftrightarrow [X,Y] = 0, \text{ so } \mathrm{Ker}(R_o(X,-)) = Z_{\mathfrak{p}}(X). \tag{2.4.5}$$

In fact, (2.4.5) holds even if $(\mathfrak{g},\theta)$ is just weakly effective. If $\mathfrak{a} \subseteq \mathfrak{p}$ is a maximal abelian subspace containing X, then $\mathfrak{a} \subseteq Z_{\mathfrak{p}}(X)$, so $\mathrm{null}(X) \geq r$, where $r = \mathrm{rk}(M)$.

Definition 2.4.29. A tangent vector X to a symmetric space M is called **regular** if $\mathrm{null}(X) = r$. Otherwise, it is called **singular**.

One can show the following:

Proposition 2.4.30. *Let M be a symmetric pair represented by a weakly effective orthogonal symmetric Lie algebra $(\mathfrak{g},\theta)$.*

(a) *For any point p, the set of regular vectors in T_pM is a connected open dense cone invariant under the isotropy representation. Consequently, the set of regular vectors forms a connected open dense subset of TM invariant under the action of $I(M)$.*

(b) *A vector $X \in \mathfrak{p} \cong T_oM$ is regular $\Leftrightarrow$ it is contained in a unique maximal abelian subspace of $\mathfrak{p}$ $\Leftrightarrow$ it is tangent to a unique maximal flat of M passing through o.*

(c) *If X is regular, the maximal abelian subspace of $\mathfrak{p}$ containing it is given by $Z_{\mathfrak{p}}(X)$, and the maximal flat passing through o to which X is tangent is given by $\exp_M(Z_{\mathfrak{p}}(X))$.*

We can say more about regular and singular vectors if M is semisimple.

Proposition 2.4.31. *Let M be a semisimple symmetric space represented by a Riemannian symmetric pair (G,K).*

(a) *A vector $X \in T_oM$ is singular (resp., regular) in the sense of Definition 2.4.29 if and only if it is singular (resp., principal) with respect to the restricted isotropy representation $K^0 \to \mathrm{SO}(T_oM)$ (Definition 2.3.6).*

(b) *If $X \in T_oM$ is a regular vector, the unique abelian subspace $Z_{\mathfrak{p}}(X)$ of $\mathfrak{p}$ containing X is the (unique) section of the polar representation $K^0 \to \mathrm{SO}(T_oM)$ passing through X. Similarly, the unique maximal flat $\exp_M(Z_{\mathfrak{p}}(X))$ passing through o tangentially to X is the (unique) section of the hyperpolar action $K^0 \curvearrowright M$.*

Finally, if we specialize further to symmetric spaces of noncompact type, we have an explicit description of regular and singular vectors in terms of restricted roots.

Proposition 2.4.32. *Let M be a symmetric space of noncompact type represented by an orthogonal symmetric Lie algebra $(\mathfrak{g},\theta)$. Fix a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$ and a set of positive roots $\Sigma^+ \subseteq \Sigma$. Given $X \in \mathfrak{a}$, its nullity is given by*

$$\mathrm{null}(X) = r + \sum_{\substack{\alpha\in\Sigma^+\\ \alpha(X)=0}} \dim(\mathfrak{g}_\alpha).$$

In particular, X is regular if and only if no root in Σ vanishes on it. In other words,

$$\mathfrak{a}_{\mathbf{reg}} := \mathfrak{a} \cap \mathfrak{p}_{\mathrm{reg}} = \mathfrak{a} \setminus \bigcup_{\alpha\in\Sigma} \mathrm{Ker}(\alpha).$$

Points at infinity

Recall that every noncompact symmetric space is diffeomorphic to an open Euclidean ball. For instance, for the hyperbolic plane $\mathbb{R}H^2$, this can be done via the Poincaré disk model. The boundary circle of $\mathbb{R}H^2$ is often called the ideal boundary and it can be attached to $\mathbb{R}H^2$ in a sensible way that takes into account the geometry of the hyperbolic plane. The same can be done in general. (For details and proofs, see [Ebe96].)

Definition 2.4.33. Let M be a Hadamard manifold. Two unit speed geodesics γ_1, γ_2 in M are called **asymptotic** if there exists $C > 0$ such that $\operatorname{dist}(\gamma_1(t), \gamma_2(t)) \leq C$ for t large enough. Clearly, being asymptotic is an equivalence relation on the set of unit speed geodesics in M. The equivalence class of γ is denoted by $\boldsymbol{\gamma(\infty)}$ and is called a **point at infinity** for M. The set of all points at infinity for M is denoted by $\boldsymbol{M(\infty)}$.

Note that isometries send asymptotic geodesics to asymptotic ones, so the action of $I(M)$ on M extends naturally to an action on $\overline{\boldsymbol{M}} = M \cup M(\infty)$. Here are a few basic properties of $M(\infty)$:

Proposition 2.4.34. *Let M be a Hadamard manifold. There is a natural topology on $\overline{M}$ called a cone topology. With respect to this topology:*

(a) *$\overline{M}$ is homeomorphic to a closed ball $\bar{\mathbb{B}}^n$. Moreover, $\operatorname{Int}(\overline{M}) = M$ and $\partial\overline{M} = M(\infty)$.*

(b) *The topology on M induced from $\overline{M}$ is the original one.*

(c) *For every $p \in M$, the map $S^1_p M \to M(\infty), v \mapsto \gamma_v(\infty)$, is a homeomorphism. Here $S^1_p M$ is the unit sphere in $T_p M$ and γ_v is the geodesic emanating from p with initial velocity v.*

(d) *The action $I(M) \curvearrowright \overline{M}$ is continuous.*

Example 2.4.35. Consider the real hyperbolic space $\mathbb{R}H^n$ in the ball model. The geodesics in $\mathbb{R}H^n$ are precisely the intersections of $\mathbb{R}H^n$ with circles in $\mathbb{R}^n$ that intersect the boundary sphere $\mathbb{S}^n$ orthogonally, as well as its intersections with lines through the origin. Two geodesics are asymptotic if and only if they meet the boundary sphere at the same point (as $t \to +\infty$). This means that $\mathbb{R}H^n(\infty)$ can be identified with the boundary sphere $\mathbb{S}^n$, also called the *ideal boundary* of $\mathbb{R}H^n$, and this identification is a homeomorphism. //

Our main reason for introducing points at infinity is to give a geometric description of parabolic subgroups.

Proposition 2.4.36. *Let $M = G/K$ be a symmetric space of noncompact type. Consider the induced continuous action $G \curvearrowright M(\infty)$.*

(a) *For every $x \in M(\infty)$, the stabilizer G_x is a parabolic subgroup.*

(b) *Conversely, every proper parabolic subgroup of G is the stabilizer of some $x \in M(\infty)$.*

Let us discuss a proof of Proposition 2.4.36 on the level of Lie algebras. The following is proven in [Ebe96, Prop. 2.7.13(1)]:

Proposition 2.4.37. *Let $x \in M(\infty)$, and let $X \in T_o M$ be such that $\gamma_X(\infty) = x$. Assume* *that* $\mathfrak{a}$ *contains* X*. Then the Lie algebra* $\mathfrak{g}_x$ *of* G_x *is given by*

$$\mathfrak{g}_x = \mathfrak{g}_0 \oplus \bigoplus_{\substack{\alpha\in\Sigma \\ \alpha(X)\geq 0}} \mathfrak{g}_\alpha .$$

Having fixed $\mathfrak{a}$, let us now show that

$$\{\mathfrak{g}_x \mid X \in \mathfrak{a}\setminus\{0\}, x = \gamma_X(\infty)\} = \left\{\mathfrak{q}_\Phi \mid \Sigma^+ \subseteq \Sigma, \Phi \subseteq \Lambda\right\}.$$

Given $\mathfrak{g}_\Phi$, take $X \in \mathfrak{a}_\Phi$ such that all the simple roots that are not in Φ are positive on X and write $x = \gamma_X(\infty)$. It then easily follows that $\mathfrak{g}_x = \mathfrak{q}_\Phi$. Conversely, given a nonzero $X \in \mathfrak{a}$ and the corresponding x at infinity, let ξ be the vector in $\mathfrak{a}^*$ corresponding to X under the isomorphism $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$. The ξ lies in the closure of some (maybe not unique) Weyl chamber C. If we write Λ for the corresponding set of simple roots, then all of these roots are nonnegative on X by construction. Let $\Phi = \{\alpha \in \Lambda \mid \alpha(X) = 0\}$. It is straightforward to check that $\mathfrak{q}_\Phi = \mathfrak{g}_x$.

Observation 2.4.38. Let $\Phi \subseteq \Lambda$, $X \in \mathfrak{a}_\Phi$, and $x = \gamma_X(\infty)$ such that $\mathfrak{q}_\Phi = \mathfrak{g}_x$ as above. Recall that the semisimple part $\mathfrak{l}_\Phi$ of $\mathfrak{q}_\Phi$ can be given as the centralizer of $\mathfrak{a}_\Phi$ in $\mathfrak{g}$. It is easy to verify that this actually coincides with $Z_\mathfrak{g}(X)$. Similarly, the subgroup L_Φ of G, initially given as $Z_G(\mathfrak{a}_\Phi)$, can also be described as $Z_G(X)$. Note the difference between this and the whole parabolic subgroup Q_Φ, which does not fix X but does fix the corresponding point x at infinity.

Corollary 2.4.39. *Let* M *be a symmetric space of noncompact type and* $g \in I(M)$ *any isometry. By Proposition 2.4.34, the extension of* g *to* $\overline{M}$ *is a continuous transformation of a closed Euclidean ball. By Brouwer's fixed point theorem, it has a fixed point* p*. If* p *is an interior point, then* g *has a fixed point in* M*. If* p *is a point at infinity, then* g *lies in the parabolic subgroup* G_p*.*

2.5. Hermitian and quaternionic Kähler symmetric spaces

In the last section of the chapter, we briefly discuss symmetric spaces equipped with additional geometric structures. In this thesis, we are primarily interested in complex and quaternionic structures, which lead to the notions of *Hermitian* and *quaternion-Kähler symmetric spaces*. Since this is a vast topic, we only cover those aspects that will be relevant to us later in the thesis. Our primary references here are [Hel01, Ch. VIII] and [Bes08, Ch. 14].

2.5.1. Hermitian symmetric spaces

Intuitively, a Hermitian symmetric space is a symmetric space that is also a complex manifold, except we want the Riemannian and complex structures to agree. As the following result shows, this is equivalent to asking that "sufficiently many" isometries are holomorphic.

Proposition 2.5.1. *Let* M *be a symmetric space endowed an orthogonal almost complex* *structure I. The following are equivalent:*

(i) *I is integrable and M is Kähler $\Leftrightarrow$ I is parallel $\Leftrightarrow$ $\mathrm{Hol}(M, o) \subseteq \mathrm{U}(T_oM)$.*

(ii) *Every geodesic symmetry of M is holomorphic with respect to I.*

If these conditions are satisfied, M is called a ***Hermitian*** *symmetric space.*

Sketch of the proof. It is a standard fact that the almost complex structure of an almost Hermitian manifold is parallel if and only if it is integrable and the manifold is Kähler, so the equivalences in (i) are clear. To show that (ii) implies (i), note that every geodesic translation of M, being a composition of two geodesic symmetries, is holomorphic. By Corollary 2.1.42, I is parallel. For the opposite direction, first note that I being parallel implies that every transvection of M is holomorphic. Since $\mathrm{Tr}(M)$ acts transitively on M, it suffices to show that s_o is holomorphic. This is done in the proof of Proposition VIII.4.2 in [Hel01]. □

As is always the case with parallel tensor fields, the complex structure of a Hermitian symmetric space can be constructed in the isotropy representation and then extended to the whole space in an invariant manner. Proposition 2.5.1 suggests a way to do this:

Corollary 2.5.2. *Let M be a symmetric space represented by a Riemannian symmetric pair (G, K). Suppose I_o is an orthogonal K-invariant complex structure on T_oM. Then it extends to a unique G-invariant complex structure on M making it into a Hermitian symmetric space. Every complex structure making M Hermitian arises in this way for a suitable choice of (G, K).*

Given a Hermitian symmetric space M, let us write $I_{\mathrm{hol}}(M)$ for the subgroup of $I(M)$ consisting of holomorphic isometries. It is straightforward to check that this is a closed subgroup of $I(M)$. As shown in the proof of Proposition 2.5.1, $I_{\mathrm{hol}}(M)$ contains $\mathrm{Tr}(M)$ and thus acts transitively on M. Moreover, for every $o \in M$, $I_{\mathrm{hol}}(M)$ is preserved by $\Theta = C_{s_o}$, so $(I^0_{\mathrm{hol}}(M), I^0_{\mathrm{hol}}(M)_o)$ is a Riemannian symmetric pair representing M. It is tacitly agreed upon that *Hermitian symmetric spaces are represented by default by Riemannian symmetric pairs (G, K) such that G acts by holomorphic isometries.*

Example 2.5.3 (Complex Grassmannians). With its standard complex structure, the Grassmannian $\mathrm{Gr}(k, \mathbb{C}^{n+k})$ (as well as its dual $\mathrm{Gr}^*(k, \mathbb{C}^{n+k})$) is a Hermitian symmetric space. Its group of inner isometries is $\mathrm{PSU}(n+k)$ (resp., $\mathrm{PSU}(n, k)$), so $I^0_{\mathrm{hol}}(M) = I^0(M)$ in this case. //

Example 2.5.4. Every symmetric space of Euclidean type and of even dimension is a quotient of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ by a discrete subgroup, so it is Hermitian. For example, $\mathbb{C}^n$ itself is a Hermitian symmetric space. In this case, $I(M) = \mathrm{SO}(2n) \ltimes \mathbb{C}^n$, but the subgroup $I_{\mathrm{hol}}(M)$ is much smaller: $I_{\mathrm{hol}}(M) = \mathrm{U}(n) \ltimes \mathbb{C}^n$. //

In most interesting cases, however, the above requirement on Riemannian symmetric pairs is redundant. Indeed, it follows from Proposition 2.1.97(f) that:

Corollary 2.5.5. *If a Hermitian symmetric space M has compact Euclidean part, then $I^0_{\mathrm{hol}}(M) = I^0(M)$ and thus $I_{\mathrm{hol}}(M)$ is an open subgroup of $I(M)$. In particular, for any Riemannian symmetric pair (G, K) representing M, G acts on M by holomorphic isometries.*

Hermitian symmetric spaces behave well with respect to products:

Proposition 2.5.6. *A Riemannian product of Hermitian symmetric spaces is again Hermitian. Conversely, if $M = M_0 \times M_1 \times \cdots \times M_k$ is a de Rham-like decomposition and M is a Hermitian symmetric space, then so is every M_i.*

Given a Hermitian symmetric space M and a complex submanifold $S \subseteq M$, S is itself Kähler. If S is a symmetric space in the induced metric (e.g., if S is complete, connected, and totally geodesic), then it is a Hermitian symmetric space in its own right thanks to Proposition 2.5.1. We need to introduce one more type of submanifolds that will be of great importance later in the thesis.

Lemma 2.5.7. *Let V be a Hermitian vector space. Given a real subspace $U \subseteq V$, the following conditions are equivalent:*

(i) *U is isotropic with respect to the symplectic form of V.*

(ii) *$I(U) \perp U$ (here I is the complex structure).*

For this reason, isotropic subspaces of Hermitian vector spaces are sometimes called **totally real**. Similarly, isotropic submanifolds in Kähler manifolds are occasionally called **totally real**. Given such a submanifold $S \subset M$, the complex structure of M maps the tangent bundle TS into the normal bundle NS. If M has complex dimension n, then S is of dimension at most n, and $\dim(S) = n$ if and only if S is Lagrangian.

Example 2.5.8. For $0 \leq k \leq n$, $\mathbb{R}P^k$ (resp., $\mathbb{R}H^k$) is a totally real totally geodesic submanifold of $\mathbb{C}P^n$ (resp., $\mathbb{C}H^n$)—when embedded in an obvious way. //

Now we proceed to semisimple and irreducible Hermitian symmetric spaces.

Proposition 2.5.9. *A semisimple Hermitian symmetric space M is automatically simply connected*[1]*.*

Proof. By Proposition 2.1.95, a semisimple symmetric space decomposes as a product $M = M_{\mathrm{c}} \times M_{\mathrm{nc}}$ of its compact and noncompact parts. If M is Hermitian, so are both of the factors, which can be shown by applying Proposition 2.5.6 to the universal Riemannian covering space of M. The noncompact part is simply connected by Proposition 2.1.92. For the compact part, see [Hel01, Th.VIII.4.6]. □

One can show that every Hermitian symmetric space of noncompact type can be represented as an open bounded subset of $\mathbb{C}^n$ with a certain natural Kähler metric called the *Bergman metric* (see, e.g., [Hel01, Ch.VIII, §3,7]); such subsets are known as *bounded symmetric domains*. For example, for $\mathbb{C}H^1 \simeq \mathbb{R}H^2$, such an embedding can be given by the Poincaré disk and upper half-plane models.

Corollary 2.5.2 implies that Hermitian symmetric spaces behave well with respect to duality:

Corollary 2.5.10. *Let M be a semisimple Hermitian symmetric space. Then the dual M^* carries a natural complex structure making it into a Hermitian symmetric space.*

[1] Using this result, one can show that a Hermitian symmetric space with compact Euclidean factor decomposes as the Riemannian product of its Euclidean and irreducible parts (see Proposition 2.1.88 and [Bes08, 8.97]).

In the Hermitian setting, there is an additional relation between dual spaces: a Hermitian symmetric space of noncompact type admits a natural holomorphic embedding into its dual as an open domain. This is known as the *Borel embedding*. For instance, for $\mathbb{C}H^1$, this is the composition of the Poincaré disk or upper half-plane model $\mathbb{C}H^1 \simeq \mathbb{R}H^2 \hookrightarrow \mathbb{C}$ with an affine chart $\mathbb{C} \hookrightarrow \mathbb{C}P^1$. We refer to [Wol72] for details.

Another crucial property of semisimple Hermitian symmetric spaces is that their complex structure comes from the isotropy representation. For any Hermitian symmetric space, observe that I_o lies in both $\mathrm{SO}(T_oM)$ and $\mathfrak{so}(T_oM)$.

Proposition 2.5.11. *Let M be a semisimple Hermitian symmetric space represented by a Riemannian symmetric pair (G, K).*

(a) *I_o lies in the center of the linear isotropy Lie algebra $\overline{\mathfrak{k}} \subseteq \mathfrak{so}(T_oM)$. Moreover, there exists $Z \in \mathfrak{z}(\mathfrak{k})$ such that $\mathrm{ad}(Z)\big|_{\mathfrak{p}} = I_o$.*

(b) *I_o lies in the center of the restricted linear isotropy group $\overline{K}^0 \subseteq \mathrm{SO}(T_oM)$. Moreover, there exists $k \in Z(K)^0$ such that $\mathrm{Ad}(k)\big|_{\mathfrak{p}} = I_o$. If Z is fixed as in* (a)*, k can be given as $\exp(\frac{\pi}{2}Z)$.*

(c) *The geodesic symmetry s_o is given by $\exp_G(\pi Z)$.*

Proof. Thanks to Remark 2.1.100, $\mathrm{ad}(\mathfrak{k})\big|_{\mathfrak{p}} = \overline{\mathfrak{k}}$, so we just need to prove that I_o lies in $\overline{\mathfrak{k}}$. By Proposition 2.1.68, it suffices to show that I_o preserves that curvature tensor (since it is already orthogonal. This is done in [Hel01, Th.VIII.4.5(i)]. For (b) and (c), consider the subgroup $\mathrm{U}(T_oM) \subseteq \mathrm{SO}(T_oM)$ and its center $\mathbb{T} = \{\alpha E \mid \lambda \in \mathbb{C}, |\lambda| = 1\}$. On the level of Lie algebras, we have $\mathfrak{t} \subseteq \mathfrak{u}(T_oM) \subseteq \mathfrak{so}(T_oM)$, where $\mathfrak{t} = \mathrm{Lie}(\mathbb{T}) = \{\lambda E \mid \lambda \in i\mathbb{R}\}$. Consider the 1-dimensional subalgebra $\mathfrak{h} \subseteq \mathfrak{z}(\mathfrak{k})$ spanned by Z and the corresponding Lie subgroup $H \subseteq Z(K)^0$. We have the restriction of the isotropy representation $f\colon \mathfrak{h} \xrightarrow{\sim} \mathfrak{t}$ and the induced morphism $F\colon H \twoheadrightarrow \mathbb{T}$. Given $t > 0$, we have $F(\exp_H(tZ)) = e^{tI_o} = e^{itE}$, which equals $iE = I_o$ for $t = \pi/2$ and $-E$ for $t = \pi$. In the latter case, $k = \exp_H(tZ)$ gives an isometry fixing o with differential $-E$ at o, so it has to coincide with s_o by Proposition 2.1.1(c). ☐

Corollary 2.5.12. *If M is a semisimple Hermitian symmetric space represented by a Riemannian symmetric pair (G, K), then for any $p \in M$ and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, there exists $k \in G$ that fixes p and whose differential at p is the multiplication by λ in T_pM.*

When combined with Corollary 2.5.2, Proposition 2.5.11 also implies:

Corollary 2.5.13. *If an irreducible symmetric space admits an almost complex structure making it into a Hermitian symmetric space, then that structure is unique up to a sign.*

Recall that, in contrast to the noncompact type, a symmetric space M of compact type can have $I^0(M)$ with nontrivial center. For example, the center of $I^0(\mathbb{S}^{2n+1}) = \mathrm{SO}(2n+2)$ is $\{\pm E\}$. It is worth noting that this does not happen for *Hermitian* symmetric spaces of compact type (see [Hel01, Ch.VIII, Th. 6.1]).

For irreducible symmetric spaces, discerning whether a space is Hermitian is very easy. Indeed, if M is irreducible Hermitian, the center $Z(K)$ in Proposition 2.5.11 has to be 1-dimensional, which follows easily from Schur's lemma. Conversely, if (G, K) is an irreducible Riemannian symmetric pair and K has nondiscrete center, then $Z(K)$ has to contain a circle subgroup, which, in turn, has an element Z of order 4. By irreducibility, the isometry Z^2 has to coincide with s_o, so Z gives an isotropy-invariant complex structure at o, which turns M into a Hermitian symmetric space by Corollary 2.5.2. A posteriori, we see that $Z(K)$ is 1-dimensional. From our description Proposition 2.1.119 of the 4 types of irreducible symmetric spaces, we deduce (see also [Hel01, Ch.VIII, §6]):

Corollary 2.5.14. *An irreducible symmetric space M represented by a Riemannian symmetric pair (G, K) admits a complex structure making it a Hermitian symmetric space if and only if K has nondiscrete center. If this is the case, then $Z(K)$ is actually 1-dimensional, and M can only be of type I or III. If, in addition, (G, K) is effective, then $Z(K)$ is isomorphic to the circle group.*

Let M be an irreducible Hermitian symmetric space and $\widetilde{K}$ the isotropy subgroup of $I(M)$ at $o \in M$. The adjoint action of $\widetilde{K}$ preserves $\mathfrak{z}(\mathfrak{k})$, so each element of $\widetilde{K}$ acts on $\mathfrak{z}(\mathfrak{k})$ either trivially or as the multiplication by -1. Since $\widetilde{K}$ meets every connected component of $I(M)$ and $I^0_{\mathrm{hol}}(M) = I^0(M)$, we deduce that every isometry of M is either holomorphic or anti-holomorphic. But more is true: there always exists an anti-holomorphic isometry, so $I^0_{\mathrm{hol}}(M)$ is a proper subgroup ([Leu79b]). We deduce:

Corollary 2.5.15. *If M is an irreducible Hermitian symmetric space, then $I_{\mathrm{hol}}(M)$ is a subgroup of $I(M)$ of index 2, and its other coset consists of anti-holomorphic isometries.*

This corollary implies that changing the complex structure of an irreducible Hermitian symmetric space from I to $-I$ results in a Hermitian symmetric space holomorphically isometric to the original one. From this and Corollary 2.5.13, one can see that for any semisimple symmetric space, if there exists a complex structure making it into a Hermitian symmetric space, then it is unique up to holomorphic isometry.

We will work with Hermitian symmetric spaces more in Chapters 5 and 6. In particular, see Table 6.4 for the complete list of irreducible Hermitian symmetric spaces.

2.5.2. Quaternion-Kähler symmetric spaces

In this part, we discuss the quaternionic analog of Hermitian symmetric spaces. Intuitively, it would make sense to ask that every tangent space T_pM has a fixed structure of an $\mathbb{H}$-module—which would lead to the notion of a hyperkähler manifold. Unfortunately, this is not a very useful idea in the context of symmetric spaces: a hyperkähler manifold is Ricci-flat, and Ricci flat homogeneous spaces are flat. The key is to let a quaternionic structure to be only locally trivial but not necessarily globally.

By a **quaternionic vector space** we will mean a left $\mathbb{H}$-module of finite rank. The model example is the space $\mathbb{H}^n$ with multiplication given by $p \cdot v := v\overline{p}$, where $p \in \mathbb{H}, v \in \mathbb{H}^n$, and the multiplication on the right-hand side is coordinate-wise. More loosely, we will say that a **quaternionic structure** on a real vector space V is a choice of a unital $\mathbb{R}$-subalgebra $\mathcal{H} \subseteq \mathrm{End}(V)$ isomorphic to $\mathbb{H}$. Note that $\mathbb{H} = \mathbb{R} \oplus \mathrm{Im}(\mathbb{H})$, and the first summand is $Z(\mathbb{H})$. The unit 2-sphere in $\mathrm{Im}(\mathbb{H})$ is precisely the set of quaternions whose square equals -1. Therefore, a quaternionic structure on a vector space V can be written as $\mathcal{H} = \mathbb{R}E \oplus \mathcal{J}$, where $\mathcal{J}$ is span of elements with square -1. Apart from $\mathcal{J}$, many other things in $\mathcal{H}$ do not depend on the choice of an isomorphism $\mathbb{H} \xrightarrow{\sim} \mathcal{H}$: the inner product, the conjugation, the orientation of $\mathcal{J}$, the 3-sphere $\mathbb{S}^3_{\mathcal{H}}$ of unit-length elements, and of course the 2-sphere $\mathbb{S}^2_{\mathcal{H}} = \mathbb{S}^3_{\mathcal{H}} \cap \mathcal{J}$. Every element of $\mathbb{S}^2_{\mathcal{H}}$ is a complex structure on V. Note that $\mathbb{S}^3_{\mathcal{H}}$ is a

Lie subgroup of $\mathrm{GL}(V)$; and its Lie algebra can be identified with $\mathcal{J}$. For any $\mathbb{H} \xrightarrow{\sim} \mathcal{H}$, it corresponds to the group of unit-length quaternions $\mathrm{Sp}(1) \subseteq \mathbb{H}^\times$. An isomorphism $\mathbb{H} \xrightarrow{\sim} \mathcal{H}$ is the same as a triple $J_1, J_2, J_3 \in \mathcal{J}$ satisfying the quaternionic relations $J_1^2 = J_2^2 = J_3^2 = J_1J_2J_3 = -E$, which, in turn, is the same as an oriented orthonormal basis for $\mathcal{J}$. We call such a triple a **canonical basis** of the quaternionic structure on V. The group $\mathrm{GL}_{\mathbb{H}}(V)$ of $\mathbb{H}$-linear automorphisms of V also does not depend on the choice of $\mathbb{H} \xrightarrow{\sim} \mathcal{H}$. Observe that $\mathrm{GL}_{\mathbb{H}}(V) = Z_{\mathrm{GL}(V)}(\mathbb{S}^3_{\mathcal{H}})$, whereas the normalizer of $\mathbb{S}^3_{\mathcal{H}}$ in $\mathrm{GL}(V)$ is given by the product $\mathrm{GL}_{\mathbb{H}}(V) \cdot \mathbb{S}^3_{\mathcal{H}} \cong (\mathrm{GL}_{\mathbb{H}}(V) \times \mathbb{S}^3_{\mathcal{H}})/\{\pm E\}$. If we pick a canonical basis $J_1, J_2, J_3 \in \mathbb{S}^2_{\mathcal{H}}$ and an $\mathbb{H}$-basis $e_1, \dots, e_n$ for V, we get an $\mathbb{H}$-isomorphism $\mathbb{H}^n \xrightarrow{\sim} V$ given by $(\overline{p}_k)_{k=1}^n \mapsto \sum_{k=1}^n \varphi(p_i)e_i$, where $\varphi(a+bi+cj+dk) = aE + bJ_1 + cJ_2 + dJ_3$. Under this isomorphism, the action $\mathrm{GL}_{\mathbb{H}}(V) \curvearrowright V$ corresponds to the normal matrix multiplication $\mathrm{GL}(n, \mathbb{H}) \curvearrowright \mathbb{H}^n$.

Definition 2.5.16. Let V be an $\mathbb{H}$-vector space. A **quaternion-Hermitian inner product** on V is an $\mathbb{R}$-bilinear map $H \colon V \times V \to \mathbb{H}$ that is:

(a) QUATERNION-HERMITIAN: $H(v, w) = \overline{H(w, v)}$ ($\forall\, v, w \in V$),

(b) $\mathbb{H}$-SESQUILINEAR: $H(pv, w) = pH(v, w)$, $H(v, pw) = H(v, w)\overline{p}$ ($\forall\, p \in \mathbb{H}$, $\forall\, v, w \in V$),

(c) POSITIVE-DEFINITE[1]: $H(v, v) > 0$ ($\forall\, v \in V \setminus \{0\}$).

Example 2.5.17. The formula $H(v, w) = \sum_{i=1}^n \overline{v}_i w_i$ defines a quaternion-Hermitian inner product on $\mathbb{H}^n$. //

We will often use a prefix "q-" as a shorthand for "quaternion". Given an $\mathbb{H}$-vector space with a q-Hermitian inner product H, observe that the real part g of H is a Euclidean inner product. What is more, g fully determines H: for any canonical basis $J_1, J_2, J_3 \in \mathrm{Sp}(1)$ (e.g., i, j, k), we have:

$$H(v, w) = g(v, w) - \sum_{i=1}^{3} J_i g(J_i v, w). \tag{2.5.1}$$

With respect to g, we have $\mathbb{S}^3_{\mathcal{H}} \subseteq \mathrm{SO}_g(V)$ and $\mathcal{J} \subseteq \mathfrak{so}_g(V)$. We have the group of q-unitary transformations $\mathrm{Sp}_H(V) \subseteq \mathrm{GL}_{\mathbb{H}}(V)$ consisting of those $\mathbb{H}$-linear operators that preserve H. One can show that $\mathrm{Sp}_H(V) = \mathrm{GL}_{\mathbb{H}}(V) \cap \mathrm{O}_g(V)$. Similar to the above, we have

$$\begin{aligned} Z_{\mathrm{O}_g(V)}(\mathbb{S}^3_{\mathcal{H}}) &= \mathrm{Sp}_H(V), \\ N_{\mathrm{O}_g(V)}(\mathbb{S}^3_{\mathcal{H}}) &= \mathrm{Sp}_H(V) \cdot \mathbb{S}^3_{\mathcal{H}} \cong (\mathrm{Sp}_H(V) \times \mathbb{S}^3_{\mathcal{H}})/\{\pm E\}. \end{aligned} \tag{2.5.2}$$

An $\mathbb{H}$-basis for V is H-orthonormal if and only if it is g-orthonormal. With respect to such a basis and the induced isomorphism $\mathbb{H}^n \xrightarrow{\sim} V$, H becomes the model inner product as in Example 2.5.17, while the groups $\mathrm{Sp}_H(V)$ and $\mathrm{Sp}_H(V) \cdot \mathbb{S}^3_{\mathcal{H}}$ correspond to $\mathrm{Sp}(n) \subseteq \mathrm{GL}(n, \mathbb{H})$ and $\mathrm{Sp}(n)\mathrm{Sp}(1) \subseteq \mathrm{SO}(4n)$, respectively. The action of the latter on $\mathbb{H}^n \simeq \mathbb{R}^{4n}$ is given by $(A, q) \cdot v = Av\overline{q}$, where v is a column-vector of quaternions. As a subgroup of $\mathrm{GL}(V)$, $\mathrm{Sp}_H(V) \cdot \mathbb{S}^3_{\mathcal{H}}$ allows a useful description in terms of its action on the bases. Namely, it acts transitively on the set of orthonormal $\mathbb{H}$-bases for V, as well as on $\mathbb{S}^2_{\mathcal{H}}$ (by conjugation). It thus acts on the set of pairs consisting of an orthonormal $\mathbb{H}$-basis for V and a canonical basis of $\mathcal{H}$, and one can show that this actions is simply transitive.

[1]Just like in the Euclidean and Hermitian cases, this condition can be relaxed to just nondegeneracy, in which case H has a signature (p, q) such that $p + q = \dim_{\mathbb{H}}(V)$.

Conversely, an element $A \in \mathrm{GL}(V)$ lies in $\mathrm{Sp}_H(V) \cdot \mathbb{S}^3_{\mathcal{H}}$ if and only if it preserves both the set of orthonormal $\mathbb{H}$-bases and the set of canonical bases.

Definition 2.5.16 would not make sense if we only had a vector space V with a quaternionic structure $\mathcal{H}$: we need a fixed isomorphism $\mathbb{H} \xrightarrow{\sim} \mathcal{H}$. However, suppose we have a Euclidean inner product g on V such that $\mathbb{S}^3_{\mathcal{H}} \subseteq \mathrm{O}_g(V)$, or equivalently, $\mathcal{J} \subseteq \mathfrak{so}_g(V)$ (by a slight abuse of terminology, we will call such g **quaternion-Hermitian**). Then, if we pick an isomorphism $\varphi\colon \mathbb{H} \xrightarrow{\sim} \mathcal{H}$, formula (2.5.1) defines a q-Hermitian inner product H on V with real part g. Any other isomorphism $\mathbb{H} \xrightarrow{\sim} \mathcal{H}$ is of the form $\varphi \circ C_p$, where C_p is the conjugation of $\mathbb{H}$ by some $p \in \mathrm{Sp}(1)$. The resulting q-Hermitian inner product on V will then be $H'(v, w) = \overline{p}H(v, w)p$. Whatever the choice is, the group $\mathrm{Sp}_g(V) := \mathrm{Sp}_H(V) = \mathrm{O}_g(V) \cap \mathrm{GL}_{\mathbb{H}}(V)$ does not depend on it. Now we can proceed to define quaternion-Kähler manifolds and symmetric spaces.

Definition 2.5.18. A smooth $4n$-manifold is called **almost**[1] **quaternionic** if the following equivalent pieces of data are set:

(a) A quaternionic structure $\mathcal{H}_p$ on each T_pM such that the resulting rank-4 subbundle $\mathcal{H} \subseteq \mathrm{End}(TM)$ ($\Leftrightarrow$ the rank-3 subbundle $\mathcal{J} \subseteq \mathrm{End}(TM)$) is smooth.

(b) A reduction of the structure group of TM to $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1) \subseteq \mathrm{GL}(4n, \mathbb{R})$.

Observe that an almost quaternionic manifold M comes equipped with an $\mathbb{S}^3$-bundle $Y \twoheadrightarrow M, Y_p = \mathbb{S}^3_p := \mathbb{S}^3_{\mathcal{H}_p}$, and an $\mathbb{S}^2$-bundle $Z \twoheadrightarrow M, Z_p = \mathbb{S}^2_p := \mathbb{S}^2_{\mathcal{H}_p}$. A **canonical local frame** on M is a smooth local frame J_1, J_2, J_3 for the bundle $\mathcal{J} \twoheadrightarrow M$ over some open subset U such that for every $p \in U$, J_{1p}, J_{2p}, J_{3p} is a canonical basis of $\mathcal{H}_p$.

Definition 2.5.19. An almost quaternionic manifold M is called **quaternion-Hermitian** if the following equivalent pieces of date are set:

(a) A Riemannian metric g on M such that $Y \subseteq \mathrm{O}(TM)$ (or equivalently, $\mathcal{J} \subseteq \mathfrak{so}(TM)$).

(b) A common reduction of the structure groups $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$ and $\mathrm{SO}(4n)$ of TM to $\mathrm{Sp}(n)\mathrm{Sp}(1)$.

Definition 2.5.20. A **quaternion-Kähler manifold** is a quaternion-Hermitian manifold satisfying the following equivalent conditions:

(a) The subbundle $\mathcal{J} \subseteq \mathrm{End}(TM)$ is parallel.

(b) For every $p \in M$, $\mathrm{Hol}(M, p) \subseteq \mathrm{Sp}(T_pM) \cdot \mathbb{S}^3_p$.

A symmetric space that is also a quaternion-Kähler manifold is called a **quaternion-Kähler symmetric space**.

The property of being q-Kähler can be described purely in terms of holonomy:

Proposition 2.5.21. *A connected Riemannian manifold M admits an almost quaternionic structure making it into a quaternion-Kähler manifold if and only if its holonomy is contained in* $\mathrm{Sp}(n)\mathrm{Sp}(1)$.

Since $\mathrm{Sp}(n)\mathrm{Sp}(1)$ is contained in $\mathrm{SO}(4n)$, a q-Kähler manifold must be orientable. We also have $\mathrm{Sp}(1)\mathrm{Sp}(1) = \mathrm{SO}(4)$, which means that every orientable Riemannian manifold

[1] Here we are following Salamon ([Sal80, Sal82]), who reserves the term *quaternionic* for those almost quaternionic manifolds that admit a torsion-free $\mathrm{GL}(n, \mathbb{H})\mathrm{Sp}(1)$-connection.

of dimension 4 is q-Kähler. For this reason, *it is generally assumed by default* (and we adhere to this as well) *that q-Kähler manifolds are of dimension* ≥ 8. For M q-Kähler, the $\mathbb{S}^2$-bundle $Z \twoheadrightarrow M$ is called its *twistor bundle* and it has remarkable properties that allow one to study M by means of complex geometry. We will not go in this direction but will mention a few standard properties of q-Kähler manifolds.

Proposition 2.5.22 (Properties of q-Kähler manifolds). *Let M be a complete quaternion-Kähler manifold.*

(a) *M is Einstein. In particular, if M has positive Ricci curvature, then it is compact and has finite fundamental group.*

(b) *The Einstein constant of M is zero (i.e., M is Ricci-flat) $\Leftrightarrow$ M is locally hyperkähler (i.e., $\mathrm{Hol}^0(M) \subseteq \mathrm{Sp}(n)$). If this is the case and M is Riemannian homogeneous, then it is flat.*

(c) *If M is not Ricci-flat, it is irreducible*[1] *and* $\mathrm{Hol}(M)$ *contains the* $\mathrm{Sp}(1)$ *factor of* $\mathrm{Sp}(n)\mathrm{Sp}(1)$.

(d) *If the Einstein constant is positive, M is simply connected. If, in addition, M is Riemannian homogeneous, then it is symmetric.*

The second assertion in Proposition 2.5.22(b) follows from the classical fact that a Ricci-flat Riemannian homogeneous space must be flat.

Let M be a q-Kähler manifold. Note that every local section of Z over some U is an almost complex structure on U, and the Riemannian metric g is Hermitian with respect to it. However, unless M is Ricci-flat, this almost complex structure is not integrable and does not extend to a global section. This sets q-Kähler manifolds apart from their hyperkähler counterparts.

In a q-Kähler manifold, one can distinguish several types of submanifolds based on how they interact with the quaternionic structure.

Definition 2.5.23. Let V be a vector space endowed with a quaternionic structure $\mathcal{H}$ and a q-Hermitian Euclidean inner product. A subspace $U \subseteq V$ is called

- **quaternionic**, if every $p \in \mathcal{H}$ preserves U.
- **totally complex**, if there exists $p \in \mathcal{J} \setminus \{0\}$ that preserves U, and for every $q \in \mathcal{J}$ orthogonal[2] to p we have $q(U) \perp U$.
- **totally real**, if for every $p \in \mathcal{J}$, $p(U) \perp U$.

A submanifold S of a quaternion-Kähler manifold M is called **quaternionic**, **totally complex**, or **totally real** if so is T_pS in T_pM for every $p \in S$.

If $U \subseteq V$ is a totally complex subspace preserved by $J \in \mathbb{S}^2_{\mathcal{H}}$, then U is a complex subspace of the complex vector space (V, J) and a totally real subspace of (V, J') for any $J' \in \mathbb{S}^2_{\mathcal{H}}$, $J' \perp J$. The following criterion is straightforward:

[1] This means that we cannot take products in the category of q-Kähler manifolds.

[2] Here we mean orthogonal with respect to the intrinsic inner product on $\mathcal{H}$ that comes from any isomorphism $\mathbb{H} \xrightarrow{\sim} \mathcal{H}$.

Proposition 2.5.24. *Let M be a quaternion-Kähler manifold and $S \subseteq M$ an embedded submanifold. Then S is*

- *quaternionic $\Leftrightarrow$ for every $p \in S$, there exists a canonical local frame J_1, J_2, J_3 over a neighborhood U of p such that $J_i(T_qS) = T_qS$ for each i and each $q \in S \cap U$,*
- *totally complex $\Leftrightarrow$ for every $p \in S$, there exists a canonical local frame J_1, J_2, J_3 over a neighborhood U of p such that $J_1(T_qS) = T_qS$ and $J_2(T_qS), J_3(T_qS) \perp T_qS$ for each $q \in S \cap U$,*
- *totally real $\Leftrightarrow$ for every $p \in S$, there exists a canonical local frame J_1, J_2, J_3 over a neighborhood U of p such that $J_i(T_qS) \perp T_qS$ for each i and each $q \in S \cap U$.*

Fortunately, the property of being quaternionic, totally complex, or totally real is preserved under isometric congruence, thanks to the following result. It can be deduced from Proposition 2.5.22(c) (see also [AM93]).

Proposition 2.5.25. *Let M be a complete quaternion-Kähler manifold of nonzero scalar curvature. Every isometry $g \in I(M)$ preserves the quaternionic structure $\mathcal{H}$ (and thus $\mathcal{J}$) of M. In other words, for every $p \in M$ and every $H \in \mathcal{H}_p$, there exists $H' \in \mathcal{H}_{g(p)}$ such that $dg_p \circ H = H' \circ dg_p$.*

Finally, we make a couple of remarks about q-Kähler symmetric spaces. If M is such a space and it is not flat, it is irreducible and simply connected by Proposition 2.5.22(3, 4). In light of (2.5.2), Proposition 2.5.22 also asserts that $\mathrm{Hol}(M, o) = \mathrm{Hol}^0(M, o)$ has a normal subgroup isomorphic to $\mathrm{Sp}(1)$, whose representation on T_oM is equivalent to standard representation of $\mathrm{Sp}(1)$ on $\mathbb{H}^n \simeq \mathbb{R}^{4n}$. But since M is semisimple, $\mathrm{Hol}(M, o)$ coincides with the restricted linear isotropy group at o. Conversely, let M be a simply connected irreducible symmetric space represented by a Riemannian symmetric pair (G, K), and suppose K has a normal subgroup K_0 isomorphic to $\mathrm{Sp}(1)$ whose representation on T_oM is equivalent to $\mathrm{Sp}(1) \curvearrowright \mathbb{H}^n$. This means that K_0 gives rise to a quaternionic structure $\mathcal{H}$ on T_oM. Since the isotropy representation is orthogonal, the inner product g_o is q-Hermitian. By (2.5.2), the image of K in $\mathrm{O}(T_oM)$ is contained in $\mathrm{Sp}_{g_o}(T_oM) \cdot \mathbb{S}^3_{\mathcal{H}}$. Since this image coincides with $\mathrm{Hol}(M, o) = \mathrm{Hol}^0(M, o)$, we see that the quaternionic structure on T_oM extends to the whole M and makes M into a q-Kähler symmetric space. Using this criterion, one can figure out which of the irreducible symmetric spaces are q-Kähler (see [Bes08, 14.52]). By examining the list of irreducible symmetric spaces, one can actually deduce that the assumption on the representation of K_0 is redundant:

Proposition 2.5.26. *Let M be a simply connected irreducible symmetric space whose restricted isotropy group (at some $o \in M$) has a normal subgroup K_0 isomorphic to* $\mathrm{Sp}(1)$*. Then the representation of K_0 on T_oM is equivalent to* $\mathrm{Sp}(1) \curvearrowright \mathbb{H}^n$*. Consequently, M is quaternion-Kähler.*

Our discussion yields an immediate

Corollary 2.5.27. *Let M be a simply connected irreducible symmetric space.*

(a) *There exists at most one almost quaternionic structure on M making it into a quaternion-Kähler manifold.*

(b) *M is quaternion-Kähler if and only if M^* is.*

(c) *M can only be quaternion-Kähler if it is of type I or III.*

Example 2.5.28. The quaternionic projective and hyperbolic spaces $\mathbb{H}P^n$ and $\mathbb{H}H^n$ are q-Kähler symmetric spaces because their restricted isotropy group is $\mathrm{Sp}(n)\mathrm{Sp}(1)$. In fact, their restricted isotropy representation is isomorphic to the standard representation of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ on $\mathbb{H}^n$. //

Curiously, for every simple Lie group G, there exists precisely one q-Kähler symmetric space M of type I such that $I^0(M)$ is isomorphic to G. This can be seen from the classification, but it can also be proven directly by taking a simple Lie group G and constructing a subgroup K (using the root data) such that G/K is a q-Kähler symmetric space. This was originally done by Wolf in [Wol65], for which reason q-Kähler symmetric spaces of compact type are often called *Wolf spaces*.

Finally, let us remark that the Grassmannians $\mathrm{Gr}(2, \mathbb{C}^{n+2})$ and $\mathrm{Gr}^*(2, \mathbb{C}^{n+2})$ $(n \geq 1)$ are the only non-flat symmetric spaces that are both Hermitian and q-Kähler. It should be noted, however, that on any of them, the complex structure I does not lie in $\mathcal{J}$ at any point. This stems from the fact that in the isotropy Lie algebra $\mathfrak{k} = \mathfrak{su}(n) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, the summand $\mathfrak{su}(2)$ gives rise to the quaternionic structure, whereas the complex structure arises from $\mathfrak{u}(1)$. We will work with these space much more closely in Chapter 5.

Chapter 3

AUTOMORPHISMS OF REAL SEMISIMPLE LIE ALGEBRAS AND THEIR RESTRICTED ROOT SYSTEMS

In this first research chapter of the thesis, we launch our investigation of homogeneous hypersurfaces in symmetric spaces. Since we are mostly interested in symmetric spaces of noncompact type, it is worth spending some time studying the problem of congruence on such spaces. Recall that noncompact symmetric spaces are interlinked with noncompact real semisimple Lie algebras (see (2.4.1)). It is reasonable to expect that automorphisms of such a Lie algebra can be translated into isometries of the corresponding space—we will make this into a formal statement in Proposition 3.3.4. If we had a complex semisimple Lie algebra, we could then move further to the corresponding root system and Dynkin diagram, which encode the same amount of information; and thanks to the Isomorphism Theorem, their automorphisms can be *lifted* to automorphisms of the Lie algebra. For real semisimple Lie algebras, the natural analog of this is the restricted root system. Unfortunately, this is no longer a 1-to-1 correspondence. Consider, for instance, the Lie algebras $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{sl}(n, \mathbb{C})$. They both have A_{n-1} as their restricted root system, although they are clearly not isomorphic. This also means that we may not always be able to lift an automorphism of the restricted root system back to the level of the Lie algebra: the restricted root system $A_{n-1} \sqcup A_{n-1}$ of $\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{C})$ has an obvious automorphism that interchanges its irreducible components, and its lift would have to interchange the simple summands of the Lie algebra, which is impossible (we will define the notion of lift formally in Subsection 3.2.1).

To make up for this loss of information, we adorn the restricted root system of a real semisimple Lie algebra with an additional piece of data: to each restricted root, we attach the dimension of the corresponding root space (in contrast to the complex case, it does not have to be equal to one). This leads to the notions of a *weighted root system* and *weight-preserving root system automorphisms*. It follows a posteriori from the classification of real semisimple Lie algebras that every such Lie algebra is determined by its weighted root system up to isomorphism. The main result of this chapter is Theorem 3.2.10 (called Theorem 1 in the introduction), which states that every weight-preserving automorphism of the restricted root system Σ of $\mathfrak{g}$ can be lifted to an automorphism of $\mathfrak{g}$. At the end of the chapter, we use these results to deepen our understanding of the correspondence between real semisimple Lie algebras and noncompact symmetric spaces. We also obtain a precise criterion for when a weight-preserving automorphism of the restricted root system Σ of $\mathfrak{g}$ can be *lifted* to an isometry of the corresponding space M (Corollary 3.3.8). This criterion will prove highly useful when we deal with the congruence problem in the next two chapters. As a special case, we get Theorem 2 announced in the introduction. It should be noted that the exposition here follows closely the author's preprint [Sol22]. Here is the layout of the chapter:

- In Section 3.1, we recall some basic properties of isomorphisms between root systems. After that, we review the classical correspondence between complex semisimple Lie algebras and reduced root systems and look at it through the lens of root system isomorphisms.
- In Section 3.2, we introduce the notion of weight-preserving isomorphism between root systems and prove the main theorem.
- In Section 3.3, we discuss applications of the theory developed in this chapter to symmetric spaces of noncompact type.

3.1. Some aspects of the theory of root systems

This section mostly serves as preparation for the substantive part of the chapter. Practically everything discussed here can be found in [Kna02, Ch. II] or [Oni04, §1,4].

3.1.1. Root system isomorphisms

We begin by reviewing some aspects of the theory of root systems. Let (V, Δ) be a root system. Here V is a finite-dimensional Euclidean real vector space and $\Delta \subseteq V$ is the root system itself. We are not assuming Δ to be reduced or irreducible. First, we recall the notion of isomorphism of root systems.

Definition 3.1.1. Let (V', Δ') be another root system. A linear isomorphism $f\colon V \xrightarrow{\sim} V'$ is called a **(root system) isomorphism between (V, Δ) and (V', Δ')** (or **between Δ and Δ'**, for brevity) if the following two conditions are satisfied:

(a) $f(\Delta) = \Delta'$.

(b) f preserves the root integers, i.e., $n_{f(\alpha)f(\beta)} = n_{\alpha\beta}$ for all $\alpha, \beta \in \Delta$ (here $n_{\alpha\beta} = \frac{2\langle \alpha | \beta \rangle}{||\beta||^2}$).

If $V' = V$ and $\Delta' = \Delta$, we call f an **automorphism of (V, Δ)** (or **of Δ**, for brevity). The (finite) group of all automorphisms of Δ is denoted by **$\mathrm{Aut}(\Delta)$** $\subseteq \mathrm{GL}(V)$.

Note that in Definition 3.1.1, condition (ii) follows from (i) automatically if f is conformal (i.e., a scalar multiple of an isometric isomorphism). Although a root system isomorphism does not have to be conformal in general, we are going to prove that it cannot stray too far from being one (see Proposition 3.1.3). To this end, we need the following simple

Lemma 3.1.2. *Let (V, Δ) be a root system. There exists a unique (up to reordering) orthogonal decomposition $V = \bigoplus_{i=1}^k V_i$ such that $\Delta = \bigsqcup_{i=1}^k \Delta_i$, where $\Delta_i = \Delta \cap V_i$, and (V_i, Δ_i) is an irreducible root system. Two roots $\alpha, \beta \in \Delta$ lie in the same component Δ_i if and only if there exists a chain of roots $\lambda_0, \lambda_1, \dots, \lambda_s \in \Delta$ with $\lambda_0 = \alpha, \lambda_s = \beta$, such that $\langle \lambda_{i-1} | \lambda_i \rangle \neq 0$ for $1 \leq i \leq s$.*

Naturally, we call each (V_i, Δ_i) an **irreducible component of (V, Δ)** and the decomposition $V = \bigoplus_{i=1}^k V_i, \Delta = \bigsqcup_{i=1}^k \Delta_i$ the **decomposition of (V, Δ) into its irreducible components**.

Proof of the lemma. Introduce an equivalence relation on Δ: two roots $\alpha, \beta \in \Delta$ are equivalent if and only if they can be connected by a chain of roots $\lambda_0, \lambda_1, \ldots, \lambda_s \in \Delta$ as above. This is clearly an equivalence relation, so we can write $\Delta = \bigsqcup_{i=1}^{k} \Delta_i$ for the decomposition of Δ into the equivalence classes. Define V_i to be the linear span of Δ_i. Since Δ spans V, we have $V = \sum_{i=1}^{k} V_i$. By construction, given $i, j \in \{1, \ldots, k\}, i \neq j$, every root $\alpha \in \Delta_i$ is orthogonal to every root $\beta \in \Delta_j$, so $V_i \perp V_j$. Therefore, we have an orthogonal decomposition $V = \bigoplus_{i=1}^{k} V_i$. In particular, this implies that $\Delta_i = \Delta \cap V_i$ for each $i \in \{1, \ldots, k\}$. Trivially, for every subspace $W \subseteq V$, $(W, \Delta \cap W)$ is a root system, hence so is each (V_i, Δ_i). Note that each Δ_i is irreducible by design. Let $V = \bigoplus_{i=1}^{k'} V_i'$ be another decomposition of V as in the lemma. It follows from what we have already proven that all roots in $\Delta \cap V_i'$ are equivalent to each other for each $i \in \{1, \ldots, k'\}$. On the other hand, if $i, j \in \{1, \ldots, k'\}, i \neq j$, no root in $\Delta \cap V_i'$ can be equivalent to any root in $\Delta \cap V_j'$. Consequently, the decomposition $V = \bigoplus_{i=1}^{k'} V_i'$ coincides with our constructed decomposition up to reordering of the factors, which completes the proof. □

Now we can prove the following result, which asserts that root system isomorphisms are "almost" conformal maps.

Proposition 3.1.3. *Let (V, Δ) and (V', Δ') be root systems and $f\colon V \xrightarrow{\sim} V'$ an isomorphism between them. Write $V = \bigoplus_{i=1}^{k} V_i, \Delta = \bigsqcup_{i=1}^{k} \Delta_i$ and $V' = \bigoplus_{i=1}^{k'} V_i', \Delta' = \bigsqcup_{i=1}^{k'} \Delta_i'$ for the decompositions of (V, Δ) and (V', Δ') into their irreducible components. Then $k = k'$ and, after reordering V_i's if needed, $f(V_i) = V_i'$ and $f(\Delta_i) = \Delta_i'$ for each $i \in \{1, \ldots, k\}$. Moreover, for each i, $f\big|_{V_i}\colon V_i \xrightarrow{\sim} V_i'$ is a conformal map, i.e., there exists $a_i > 0$ such that $a_i f\big|_{V_i}\colon V_i \xrightarrow{\sim} V_i'$ is an isometry.*

Proof. To begin with, observe that $\alpha \perp \beta \Leftrightarrow n_{\alpha\beta} = 0$, so f must preserve root orthogonality. From this it easily follows that f preserves the equivalence relation on roots described in the proof of Lemma 3.1.2, which, in turn, implies the first assertion. For the remainder of the proof, we may assume that both (V, Δ) and (V', Δ') are irreducible, and we need to proof that f is conformal. Pick any $\alpha_0 \in \Delta$ and define $a = \frac{||f(\alpha_0)||}{||\alpha_0||} > 0$. We will prove that $a^{-1}f$ is an isometry. Since $a^{-1}f$ already preserves the root integers, it suffices to show that it preserves the length of each root. Note that $\frac{n_{\alpha\beta}}{n_{\beta\alpha}} = \frac{||\alpha||^2}{||\beta||^2}$ whenever $\langle \alpha | \beta \rangle \neq 0$. Hence, f preserves the length-ratio of any pair of non-orthogonal roots. Pick any $\beta \in \Delta$. According to Lemma 3.1.2, there exists a chain of roots $\lambda_0, \lambda_1, \ldots, \lambda_s$ with $\lambda_0 = \alpha_0, \lambda_s = \beta$, such that $\langle \lambda_{i-1} | \lambda_i \rangle \neq 0$ for $1 \leq i \leq s$. We can compute:

$$\frac{||f(\alpha_0)||}{||f(\beta)||} = \frac{||f(\lambda_0)||}{||f(\lambda_1)||} \frac{||f(\lambda_1)||}{||f(\lambda_2)||} \cdots \frac{||f(\lambda_{s-1})||}{||f(\lambda_s)||} = \frac{||\lambda_0||}{||\lambda_1||} \frac{||\lambda_1||}{||\lambda_2||} \cdots \frac{||\lambda_{s-1}||}{||\lambda_s||} = \frac{||\alpha_0||}{||\beta||},$$

i.e $\frac{||f(\beta)||}{||\beta||} = \frac{||f(\alpha_0)||}{||\alpha_0||} = a$ or, in other words, $||a^{-1}f(\beta)|| = ||\beta||$. Consequently, $a^{-1}f$ preserves the lengths of all the roots and is an isometry, which means that f is conformal. □

Corollary 3.1.4. *If (V, Δ) is an irreducible root system, then $\mathrm{Aut}(\Delta) \subseteq \mathrm{O}(V)$.*

Proof. Take any automorphism $f \in \mathrm{Aut}(\Delta)$. By Proposition 3.1.3, there exists $a > 0$ such that af is an isometry. Assume that f is not orthogonal, i.e., $a \neq 1$. By replacing f with f^{-1} if needed, we may assume $a > 1$, i.e., f increases the length of any nonzero vector.

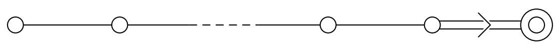
But Δ is finite, hence so is the set of lengths of all roots in Δ. Since $f(\Delta) = \Delta$, we arrive at a contradiction. $\square$

Let (V, Δ) be any root system. Let us look more closely at the automorphism group $\mathrm{Aut}(\Delta)$. Recall that we have the Weyl group $\mathrm{W}(\Delta)$ generated by the reflections s_α in the root hyperplanes $\alpha^\perp$, $\alpha \in \Delta$. Each s_α is orthogonal and preserves Δ, hence it is an automorphism of Δ. We deduce that $\mathrm{W}(\Delta) \subseteq \mathrm{Aut}(\Delta)$. In fact, it is a normal subgroup, which can be easily checked on its generators: $f \in \mathrm{Aut}(\Delta), \alpha \in \Delta \Rightarrow f s_\alpha f^{-1} = s_{f(\alpha)}$. The short exact sequence of groups $\mathrm{W}(\Delta) \hookrightarrow \mathrm{Aut}(\Delta) \twoheadrightarrow \mathrm{Aut}(\Delta)/\mathrm{W}(\Delta)$ splits, albeit not canonically. In order to split it, one first has to make a choice of positive roots (the same standard procedure we already carried out in Subsection 2.4.2). Pick a Weyl chamber $D \subseteq V$, let $\Delta^+ \subseteq \Delta$ be the corresponding subset of positive roots and $\Lambda = \{\alpha_1, \dots, \alpha_r\} \subseteq \Delta^+$ the subset of simple roots. We denote the corresponding Dynkin diagram by **DD**. If a simple root α_i has the property that $2\alpha_i$ is also a root, the corresponding vertex of the Dynkin diagram is represented by two concentric circles.

Definition 3.1.5. Let (V', Δ') be another root system with a fixed choice of simple roots $\Lambda' \subseteq \Delta'^+ \subseteq \Delta'$ and the corresponding Dynkin diagram DD'. A bijection $s\colon \Lambda \xrightarrow{\sim} \Lambda'$ is called a (**diagram**) **isomorphism between DD and DD′** if it a graph isomorphism that preserves edge directions, the number of lines an edge consists of, and the number of circles a vertex consists of. If $V' = V, \Delta' = \Delta$, and $\Lambda' = \Lambda$, we call s an **automorphism of DD**. The group of all automorphisms of DD is denoted by **Aut(DD)**.

The chief example of diagram isomorphisms comes from root system isomorphisms. Suppose that $f\colon V \xrightarrow{\sim} V'$ is an isomorphism between Δ and Δ' such that $f(\Lambda) = \Lambda'$. Then $s = f\big|_\Lambda \colon \Lambda \xrightarrow{\sim} \Lambda'$ is clearly an isomorphism between DD and DD'. This also explains why the Dynkin diagram of a root system is well-defined in the first place and does not depend on the choice of a Weyl chamber: if $\Lambda' \subseteq \Delta$ is another set of simple roots, then there exists $w \in \mathrm{W}(\Delta) \subseteq \mathrm{Aut}(\Delta)$ mapping Λ onto Λ', so the corresponding Dynkin diagrams DD and DD' are isomorphic. This construction $(f \mapsto f\big|_\Lambda)$ actually exhausts[1] all Dynkin diagram isomorphisms between DD and DD'. Although this is a standard fact in the theory of root systems (see, for example, [Kna02, Prop. 2.66]), we will reprove it for our own purposes in Proposition 3.1.7 below.

Recall that for each $r \geq 1$ there exists only one irreducible nonreduced root system of rank r up to isomorphism (see [Kna02, Prop. 2.92]). It is denoted by $(BC)_r$ and its Dynkin diagram looks like this:



Remark 3.1.6. Some authors who work only with reduced root systems ask $s\colon \Lambda \xrightarrow{\sim} \Lambda'$ in Definition 3.1.5 to preserve the Cartan matrix instead. This is equivalent to our definition for reduced root systems, as the Cartan matrix and the Dynkin diagram encode the same amount of data for such systems. However, for nonreduced root systems, our definition is stronger because the Dynkin diagram carries more (in fact, all) information about the root system in this case. For instance, the Cartan matrices of B_r and $(BC)_r$ are the same, whereas their Dynkin diagrams are not—the difference is precisely the vertex represented by two concentric circles.

[1]This implies that a root system is fully determined by its Dynkin diagram up to isomorphism.

It is very straightforward to compute the group $\operatorname{Aut}(\mathrm{DD})$ for all irreducible root systems by looking at their classification:

$$\operatorname{Aut}(\mathrm{DD}) \simeq \begin{cases} S_3 & \text{if } \Delta \simeq D_4, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } \Delta \simeq A_n\,(n \geq 2), D_n\,(n \geq 5), \text{ or } E_6, \\ \{e\} & \text{otherwise.} \end{cases}$$

Since the set of simple roots forms a basis for the underlying space of a root system, every diagram isomorphism $s\colon \Lambda \xrightarrow{\sim} \Lambda'$ between DD and DD$'$ extends uniquely to a linear isomorphism $V \xrightarrow{\sim} V'$, which we denote by the same letter. In particular, we have a natural group embedding $\operatorname{Aut}(\mathrm{DD}) \subseteq \mathrm{GL}(V)$ (once again, this embedding only makes sense after we fix the set of simple roots). Before we relate diagram isomorphisms to root system isomorphisms, we make a few observations. First off, note that $\operatorname{Aut}(\Delta)$ acts naturally on the set of Weyl chambers of (V, Δ). Second, let $V = \bigoplus_{i=1}^{k} V_i, \Delta = \bigsqcup_{i=1}^{k} \Delta_i$ be the decomposition of (V, Δ) into its irreducible components. It is easy to see that for each $i \in \{1, \ldots, k\}$, $\Delta_i^+ = \Delta_i \cap \Delta^+$ is a set of positive roots for Δ_i. Consequently, we have $\Lambda = \bigsqcup_{i=1}^{k} \Lambda_i$ and $D = \prod_{i=1}^{k} D_i$, where $\Lambda_i = \Lambda \cap \Delta_i^+$ is a set of simple roots for Δ_i and $D_i = D \cap V_i$ is the corresponding Weyl chamber. This implies that for each i, the Dynkin diagram DD_i of Δ_i is a connected component of DD, and we have $\mathrm{DD} = \bigsqcup_{i=1}^{k} \mathrm{DD}_i$. Finally, note that $\mathrm{W}(\Delta) = \prod_{i=1}^{k} \mathrm{W}(\Delta_i)$.

Proposition 3.1.7. *Let (V, Δ) and (V', Δ') be root systems with fixed choices of simple roots $\Lambda \subseteq \Delta$ and $\Lambda' \subseteq \Delta'$.*

(a) *Given any diagram isomorphism $s\colon \Lambda \xrightarrow{\sim} \Lambda'$ between* DD *and* DD$'$*, its linear extension $s\colon V \xrightarrow{\sim} V'$ is an isomorphism between Δ and Δ'. An isomorphism $V \xrightarrow{\sim} V'$ between Δ and Δ' comes from a diagram isomorphism $\Lambda \xrightarrow{\sim} \Lambda'$ precisely when it maps Λ onto Λ'.*

(b) $\operatorname{Aut}(\mathrm{DD}) \subseteq \operatorname{Aut}(\Delta)$. *In terms of the action of* $\operatorname{Aut}(\Delta)$ *on the set of Weyl chambers,* $\operatorname{Aut}(\mathrm{DD})$ *is the stabilizer of D.*

(c) $\operatorname{Aut}(\Delta) = \mathrm{W}(\Delta) \rtimes \operatorname{Aut}(\mathrm{DD})$.

Proof. Let $s\colon \Lambda \xrightarrow{\sim} \Lambda'$ be a diagram isomorphism between DD and DD$'$. Recall that the Weyl group of a root system is generated by the simple reflections with respect to any choice of simple roots: $\mathrm{W}(\Delta)$ is generated by $\{s_\alpha \mid \alpha \in \Lambda\}$ and the same is true for $\mathrm{W}(\Delta')$. Since $s(\Lambda) = \Lambda'$, we deduce that $s\,\mathrm{W}(\Delta)s^{-1} = \mathrm{W}(\Delta')$. On the other hand, it is well known that every root in a root system is simple (or double of a simple one) for a suitable choice of a Weyl chamber ([Kna02, Prop. 2.62]). To simplify the notation, let us write $\overline{\Lambda}$ for $\Lambda \cup (2\Lambda \cap \Delta)$. Since the Weyl group acts transitively on the set of Weyl chambers, we deduce that $\Delta = \mathrm{W}(\Delta) \cdot \overline{\Lambda}$ (the same is true for Δ'). We know that for any $\alpha \in \Lambda$, 2α is a root if and only if $2s(\alpha)$ is one. Altogether, we have:

$$s(\Delta) = s(\mathrm{W}(\Delta) \cdot \overline{\Lambda}) = s\,\mathrm{W}(\Delta)s^{-1} \cdot s(\overline{\Lambda}) = \mathrm{W}(\Delta') \cdot \overline{\Lambda}' = \Delta',$$

so s satisfies condition (i) of Definition 3.1.1. As for condition (ii), observe that s provides a bijection between the connected components of DD and those of DD$'$. Thus, for each $i \in \{1, \ldots, k\}$, there exists $j \in \{1, \ldots, k'\}$ (clearly, $k' = k$) such that $s(\mathrm{DD}_i) = \mathrm{DD}'_j$, which means that $s(\Lambda_i) = \Lambda'_j$ and thus $s(V_i) = V'_j$ and $s(\Delta_i) = \Delta'_j$. Take any $\alpha \in \Lambda_i$

and let $a = \frac{||s(\alpha)||}{||\alpha||}$. We want to show that for every other $\beta \in \Lambda_i$, $\frac{||s(\beta)||}{||\beta||} = a$. Assume that β is connected to α by an edge. Consider the root systems $\Delta \cap \operatorname{span}_{\mathbb{R}}\{\alpha, \beta\}$ and $\Delta' \cap \operatorname{span}_{\mathbb{R}}\{s(\alpha), s(\beta)\}$. They are both of rank 2 and we have an isomorphism between their Dynkin diagrams provided by s. Since there are just five root systems of rank 2 up to isomorphism, it is straightforward to see that two such root systems with isomorphic Dynkin diagrams are isomorphic. What it means for us is that $\frac{||\beta||}{||\alpha||} = \frac{||s(\beta)||}{||s(\alpha)||}$, hence $\frac{||s(\beta)||}{||\beta||} = \frac{||s(\alpha)||}{||\alpha||} = a$. Since DD_i is connected, it follows by induction that s increases the lengths of all simple roots in Λ_i by the same factor of a. As we already know that it preserves the Cartan integers, we deduce that it is conformal on V_i ($a^{-1}s\colon V_i \xrightarrow{\sim} V'_j$ is an isometry). But this, together with condition (i) of Definition 3.1.1, implies that it preserves the root integers between all the roots in Δ_i (and not only between the simple ones). Since the root integers between roots lying in different components of Δ are all zero, we see that s is a root system isomorphism, which was to be proven. The second assertion in part (a) of the proposition is trivial.

Part (b) follows from part (a), as $s \in \operatorname{Aut}(\Delta)$ preserves D if and only if it preserves Λ.

Part (c) hinges on the fact that $\mathrm{W}(\Delta)$ acts simply transitively on the set of Weyl chambers. It is clear from (b) that $\mathrm{W}(\Delta)$ and $\operatorname{Aut}(\mathrm{DD})$ do not intersect. On the other hand, let $f \in \operatorname{Aut}(\Delta)$. There exists $w \in \mathrm{W}(\Delta)$ such that $w(f(D)) = D$. But then $s = wf$ fixes D and thus lies in $\operatorname{Aut}(\mathrm{DD})$. Therefore, we have a decomposition $f = w^{-1}s, w^{-1} \in \mathrm{W}(\Sigma), s \in \operatorname{Aut}(\mathrm{DD})$. This completes the proof of part (c). $\square$

3.1.2. Root systems and complex semisimple Lie algebras

Here we recall some aspects of the correspondence between reduced root systems and complex semisimple Lie algebras. Let $\mathfrak{g}$ be a (finite-dimensional) complex semisimple Lie algebra. Pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We have the corresponding set of roots $\Delta \subset \mathfrak{h}^*$ and the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$. The restriction of the Killing form B of $\mathfrak{g}$ to $\mathfrak{h}$ is nondegenerate, so it induces a $\mathbb{C}$-linear isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$. Write $\mathfrak{h}^*(\mathbb{R}) \subset \mathfrak{h}^*$ for the real span of Δ and $\mathfrak{h}(\mathbb{R}) \subset \mathfrak{h}$ for its preimage under $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$. It is a standard fact that $\mathfrak{h}(\mathbb{R}) = \{h \in \mathfrak{h} \mid f(h) \in \mathbb{R}\ \forall f \in \mathfrak{h}^*(\mathbb{R})\}$ (hence $\mathfrak{h}^*(\mathbb{R})$ is the real dual of $\mathfrak{h}(\mathbb{R})$), and we have $\mathfrak{h} = \mathfrak{h}(\mathbb{R}) \oplus_{\mathbb{R}} i\mathfrak{h}(\mathbb{R})$ and $\mathfrak{h}^* = \mathfrak{h}^*(\mathbb{R}) \oplus_{\mathbb{R}} i\mathfrak{h}^*(\mathbb{R})$. The restriction of B to $\mathfrak{h}(\mathbb{R})$ is positive definite and we can carry it along the isomorphism $\mathfrak{h}(\mathbb{R}) \xrightarrow{\sim} \mathfrak{h}^*(\mathbb{R})$ to an inner product on $\mathfrak{h}^*(\mathbb{R})$. This makes $(\mathfrak{h}^*(\mathbb{R}), \Delta)$ into a reduced root system. Note that this inner product on $\mathfrak{h}^*(\mathbb{R})$ is natural and does not require any additional choices, for it comes from the Killing form, which is fully determined by Lie algebra structure of $\mathfrak{g}$.

Now we make a choice of positive roots $\Delta^+ \subset \Delta$ and let $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$ be the corresponding set of simple roots. Write $H_i \in \mathfrak{h}(\mathbb{R})$ for the preimage of α_i under the isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ and let $h_i = \frac{2}{||\alpha_i||^2} H_i$. Finally, make a choice of canonical generators $e_i \in \mathfrak{g}_{\alpha_i}, f_i \in \mathfrak{g}_{-\alpha_i}$. It follows from the definition of h_i's that

$$[h_i, e_j] = n_{\alpha_j\alpha_i} e_j, \quad [h_i, f_j] = -n_{\alpha_j\alpha_i} f_j.$$

The Isomorphism Theorem asserts that if $\mathfrak{g}'$ is another complex semisimple Lie algebra with a fixed choice of $\mathfrak{h}'$, $\Lambda' = \{\alpha'_1, \ldots, \alpha'_r\} \subset \Delta'$, and $e'_i \in \mathfrak{g}_{\alpha'_i}$, $f'_i \in \mathfrak{g}_{-\alpha'_i}$, $1 \le i \le r$, such that the Cartan matrices $A = (n_{\alpha_i\alpha_j})_{i,j=1}^r$ and $A' = (n_{\alpha'_i\alpha'_j})_{i,j=1}^r$ coincide, then there exists a unique Lie algebra isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}'$ sending h_i to h'_i, e_i to e'_i, and f_i to f'_i for

$1 \le i \le r$.

Let $F\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}'$ be a Lie algebra isomorphism mapping $\mathfrak{h}$ onto $\mathfrak{h}'$ and write $f = (F\big|_{\mathfrak{h}}{}^{*})^{-1}\colon \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}'^*$. It is a matter of simple computation that $f(\Delta) = \Delta'$ and for any $\alpha \in \Delta$, $F(\mathfrak{g}_\alpha) = \mathfrak{g}'_{f(\alpha)}$. In particular, we have $f(\mathfrak{h}^*(\mathbb{R})) = \mathfrak{h}'^*(\mathbb{R})$. We will slightly abuse the notation and use the same letter f for the restriction $f\big|_{\mathfrak{h}^*(\mathbb{R})}\colon \mathfrak{h}^*(\mathbb{R}) \xrightarrow{\sim} \mathfrak{h}'^*(\mathbb{R})$. As F respects the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}'$, it follows that f is an isometry and thus a root system isomorphism[1] between Δ and Δ'. The Isomorphism Theorem ensures that every isomorphism $\mathfrak{h}^*(\mathbb{R}) \xrightarrow{\sim} \mathfrak{h}'^*(\mathbb{R})$ between Δ and Δ' arises in this way (this fact is essentially equivalent to the Isomorphism Theorem and is proven directly in [Kna02, Th. 2.108]). As a consequence, every such isomorphism is an isometry.

If we let $\mathfrak{g}' = \mathfrak{g}$ and $\mathfrak{h}' = \mathfrak{h}$, we get a surjective Lie group homomorphism $\boldsymbol{\Psi}\colon N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h}) \twoheadrightarrow \mathrm{Aut}(\Delta)$, $F \mapsto f$, where $N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h})$ is the normalizer of $\mathfrak{h}$ in $\mathrm{Aut}(\mathfrak{g})$. This induces an isomorphism $N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h})/Z_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h}) \cong \mathrm{Aut}(\Delta)$. Using the Isomorphism Theorem, it is not hard to show that $Z_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h})$ is the connected Lie subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\mathfrak{h}$; in particular, $Z_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h}) = Z_{\mathrm{Inn}(\mathfrak{g})}(\mathfrak{h})$. Inside $N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h})$, there is a subgroup $N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{n})$, where $\mathfrak{n}$ is the sum of positive root spaces. Its image under Ψ is $\mathrm{Aut}(\mathrm{DD})$, and we get an isomorphism[2] $N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{n})/Z_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h}) \cong N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{n})/Z_{\mathrm{Inn}(\mathfrak{g})}(\mathfrak{h}) \cong \mathrm{Aut}(\mathrm{DD})$.

For each $\alpha \in \Delta$, an automorphism $\eta \in N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h})$ such that $\Psi(\eta) = s_\alpha$ can be constructed explicitly (here s_α is the reflection of $\mathfrak{h}^*(\mathbb{R})$ in the hyperplane $\alpha^\perp$). Namely, if $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ are such that $[e_\alpha, f_\alpha] = h_\alpha$ ($\Leftrightarrow B(e_\alpha, f_\alpha) = \frac{2}{||\alpha||^2}$), then $\eta = \exp(\mathrm{ad}\,\frac{\pi}{2}(e_\alpha - f_\alpha)) \in N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h})$ and $\Psi(\eta) = s_\alpha$ (see [GG78, p. 210]). Observe also that the Isomorphism Theorem allows to construct a section[3] $\mathrm{Aut}(\mathrm{DD}) \hookrightarrow N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h})$ of Ψ over $\mathrm{Aut}(\mathrm{DD})$: take $s \in \mathrm{Aut}(\mathrm{DD})$ and send it to $\hat{s} \in N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h})$ given by $\hat{s}(h_i) = h_{s(i)}$, $\hat{s}(e_i) = e_{s(i)}$, $\hat{s}(f_i) = f_{s(i)}$ (here we think of s as a permutation of $\{1, \dots, r\} \simeq \{\alpha_1, \dots, \alpha_r\}$).

Eventually, we make the following observation. Recall that a root system isomorphism is not in general an isometry (or even a conformal map). However, as we have seen, if $\mathfrak{g}$ and $\mathfrak{g}'$ are complex semisimple Lie algebras with Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'$, respectively, then every root system isomorphism $\mathfrak{h}^*(\mathbb{R}) \xrightarrow{\sim} \mathfrak{h}'^*(\mathbb{R})$ between Δ and Δ' is an isometry. The reason for this is that the inner products on $\mathfrak{h}^*(\mathbb{R})$ and $\mathfrak{h}'^*(\mathbb{R})$ come from the Killing forms of $\mathfrak{g}$ and $\mathfrak{g}'$. We can make the following definition. Let (V, Δ) be any reduced root system. Take a complex semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that the root system $(\mathfrak{h}^*(\mathbb{R}), \Delta_{\mathfrak{g}})$ is isomorphic to (V, Δ). Pick any isomorphism $\varphi\colon V \xrightarrow{\sim} \mathfrak{h}^*(\mathbb{R})$ between Δ and $\Delta_{\mathfrak{g}}$ and carry the inner product from $\mathfrak{h}^*(\mathbb{R})$ to V along φ (as we know from Proposition 3.1.3, it simply amounts to renormalizing the existing inner product on V by some conformal factors on the irreducible components of (V, Δ)). Suppose we have another pair $(\mathfrak{g}', \mathfrak{h}')$ and an isomorphism $\varphi'\colon V \xrightarrow{\sim} \mathfrak{h}'^*(\mathbb{R})$ between Δ and $\Delta_{\mathfrak{g}'}$. We claim that the inner product on V pulled back from $\mathfrak{h}'^*(\mathbb{R})$ along φ' is the same. Indeed, if we write $f = \varphi' \circ \varphi^{-1}\colon \mathfrak{h}^*(\mathbb{R}) \xrightarrow{\sim} \mathfrak{h}'^*(\mathbb{R})$, then f is an isomorphism between $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{g}'}$. As we discussed above, each such isomorphism is an isometry, which proves the claim. We

[1] This construction also shows that different choices of a Cartan subalgebra of $\mathfrak{g}$ lead to isomorphic root systems, as any two Cartan subalgebras differ by an inner automorphism of $\mathfrak{g}$. The map sending $\mathfrak{g}$ to $(\mathfrak{h}^*(\mathbb{R}), \Delta)$ is a 1-to-1 correspondence between the isomorphism classes of complex semisimple Lie algebras on the one hand and of reduced root systems on the other.

[2] In fact, the embedding $N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{n}) \hookrightarrow \mathrm{Aut}(\mathfrak{g})$ induces an isomorphism $N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{n})/Z_{\mathrm{Inn}(\mathfrak{g})}(\mathfrak{h}) \xrightarrow{\sim} \mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g}) = \mathrm{Out}(\mathfrak{g})$, so we end up with $\mathrm{Out}(\mathfrak{g}) \cong \mathrm{Aut}(\mathrm{DD})$ (see [Kna02, Th. 7.8]).

[3] We thus have $N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{n}) \cong Z_{\mathrm{Inn}(\mathfrak{g})}(\mathfrak{h}) \rtimes \mathrm{Aut}(\mathrm{DD})$ and $\mathrm{Aut}(\mathfrak{g}) \cong \mathrm{Inn}(\mathfrak{g}) \rtimes \mathrm{Aut}(\mathrm{DD})$.

call the inner product on V constructed above **Killing**.

Corollary 3.1.8. *Let* (V, Δ) *and* (V', Δ') *be reduced root systems with Killing inner products. Then every isomorphism* $V \xrightarrow{\sim} V'$ *between* Δ *and* Δ' *is an isometry. In particular,* $\mathrm{Aut}(\Delta) \subseteq \mathrm{O}(V)$.

3.2. Real semisimple Lie algebras and their restricted root systems

In this section, we resume our investigation of real semisimple Lie algebras and their restricted root systems (initiated in Subsection 2.4.2) and prove the main results. See [Kna02, Ch.VI] and [Oni04, §2,3] for a detailed exposition of the theory of real semisimple Lie algebras.

3.2.1. Weight-preserving isomorphisms

Let $\mathfrak{g}$ be a real semisimple Lie algebra with a fixed Cartan involution θ and maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{p}$. There are three main differences between the restricted root system Σ of $\mathfrak{g}$ and the root system of a complex semisimple Lie algebra (apart from the obvious difference in how the root system is constructed):

(a) Σ does not have to be reduced.

(b) Σ loses all information about the compact ideals of $\mathfrak{g}$.

(c) The dimensions of the restricted root spaces $\mathfrak{g}_\alpha$ do not have to be equal to 1.

Let us address these points individually. To begin with, (a) is not really an issue, since we know the classification of all—not necessarily reduced—root systems up to isomorphism. As we mentioned in Subsection 3.1.1, the root systems $(BC)_r$, $r \geq 1$, exhaust the list of all irreducible nonreduced root systems up to isomorphism. Regarding (b), we already observed in Proposition 2.1.72 and Observation 2.1.117 that θ respects the decomposition of $\mathfrak{g}$ into simple ideals and is the identity precisely on the compact ideals. Later we will see that the compact part of $\mathfrak{g}$ is the only information lost by Σ.

Arguably, (c) is the most important difference with the complex semisimple case. Recall that $\dim(\mathfrak{g}_\alpha)$ is said to be the multiplicity of the restricted root α. The fact that root multiplicities do not have in be 1 turns out to be a feature, not a bug. The idea is that we incorporate the multiplicities into the root system Σ itself, as they are precisely the information one needs to add to Σ to fully encode the noncompact part of $\mathfrak{g}$. To this end, we make the following definition: a **weighted root system** is a root system in which every root is assigned a positive integer, called the multiplicity of the root. Of course, the assignment of multiplicities to the roots in Σ is far from random. Since $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, we know that $\mathrm{mult}(\alpha) = \mathrm{mult}(-\alpha)$ for any $\alpha \in \Sigma$.

Notation. To avoid unnecessary ambiguity, we will use the symbol Σ when talking about weighted root systems and reserve Δ for regular root systems.

Definition 3.2.1. Let $\mathfrak{g}'$ be another real semisimple Lie algebra with a Cartan involution θ' and a maximal abelian subspace $\mathfrak{a}' \subseteq \mathfrak{p}'$ fixed, and let $\Sigma' \subseteq \mathfrak{a}'^*$ be the corresponding restricted root system. We call a root system isomorphism $f\colon \mathfrak{a}^* \xrightarrow{\sim} \mathfrak{a}'^*$ between Σ and

Σ' **weight-preserving** if it preserves the root multiplicities: $\mathrm{mult}(f(\alpha)) = \mathrm{mult}(\alpha)$ for every $\alpha \in \Sigma$. We say that $(\mathfrak{a}^*, \Sigma)$ and $(\mathfrak{a}'^*, \Sigma')$ are **weighted-isomorphic** if there exists a weight-preserving isomorphism between them. Finally, we call a weight-preserving isomorphism from $(\mathfrak{a}^*, \Sigma)$ to itself a **weight-preserving automorphism of $(\mathfrak{a}^*, \Sigma)$** (or **of Σ**, for short). The group of all weight-preserving automorphisms of $(\mathfrak{a}^*, \Sigma)$ will be denoted by $\mathbf{Aut^w(\Sigma)} \subseteq \mathrm{Aut}(\Sigma)$.

Our goal in this section is to relate weight-preserving root system isomorphisms to Lie algebra isomorphisms. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be as above and suppose that $F\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}'$ is an isomorphism such that $F \circ \theta = \theta' \circ F$ (hence $F(\mathfrak{k}) = \mathfrak{k}', F(\mathfrak{p}) = \mathfrak{p}'$) and $F(\mathfrak{a}) = \mathfrak{a}'$. Consider $F\big|_{\mathfrak{a}}\colon \mathfrak{a} \xrightarrow{\sim} \mathfrak{a}'$ and define $f = (F\big|_{\mathfrak{a}}^*)^{-1}\colon \mathfrak{a}^* \xrightarrow{\sim} \mathfrak{a}'^*$. Similarly to what we did in the complex semisimple case, it is easy to check that $f(\Sigma) = \Sigma'$ and

$$F(\mathfrak{g}_\alpha) = \mathfrak{g}'_{f(\alpha)} \text{ for every } \alpha \in \Sigma. \tag{3.2.1}$$

It is also clear that F is an isometry with respect to the inner products B_θ and $B'_{\theta'}$, so f is an isometry as well. All this implies that f is a weight-preserving isomorphism between Σ and Σ'.

We can apply this construction to Lie algebra automorphisms. Consider the Lie group $\mathrm{Aut}(\mathfrak{g})$. We have a distinguished element of this group fixed, namely the Cartan involution $\theta \in \mathrm{Aut}(\mathfrak{g})$.

Lemma 3.2.2. *The following conditions on $\varphi \in \mathrm{Aut}(\mathfrak{g})$ are equivalent:*

(i) *φ commutes with θ.*

(ii) *φ preserves the Cartan decomposition.*

(iii) *φ preserves $\mathfrak{k}$.*

(iv) *φ preserves $\mathfrak{p}$.*

(v) *φ is orthogonal with respect to B_θ.*

We denote the subgroup of elements satisfying these conditions by $\mathbf{Aut(\mathfrak{g})^\theta}$.

Proof. The equivalence of (i) and (ii) is obvious. Their equivalence to (iii) and (iv) follows from the fact that $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal with respect to B, and $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{O}_B(\mathfrak{g})$. For (v), see [Gün10, Lem. 2.2]. ☐

Note that (i) means that $\mathrm{Aut}(\mathfrak{g})^\theta$ is the fixed point subgroup of the involutive automorphism C_θ of $\mathrm{Aut}(\mathfrak{g})$. At the same time, (v) means that this subgroup is compact and its representation on $\mathfrak{g}$ is orthogonal. Since $\mathfrak{g}$ is semisimple, $\mathrm{Aut}^0(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$, so we have an open subgroup $\mathrm{Inn}(\mathfrak{g})^\theta \subseteq \mathrm{Aut}(\mathfrak{g})^\theta$. Later we will show that $\mathrm{Aut}(\mathfrak{g})^\theta$ is actually a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Inn}(\mathfrak{g})^\theta = (\mathrm{Aut}(\mathfrak{g})^\theta)^0$, but we do not need that now. As we showed in Subsection 2.4.2, under the identification $\mathrm{ad}\colon \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$, the Lie algebra of $\mathrm{Aut}(\mathfrak{g})^\theta$ is $\mathfrak{k}$.

We are especially interested in the subgroup $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$ of $\mathrm{Aut}(\mathfrak{g})^\theta$. This is also a compact subgroup. Under the identification $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^\theta) \cong \mathfrak{k}$, the Lie algebra of $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$ is

$N_{\mathfrak{k}}(\mathfrak{a}) = \mathfrak{k}_0$. According to the above discussion, we have a well-defined map[1]

$$\boldsymbol{\Omega}\colon N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}) \to \mathrm{Aut}^{\mathrm{w}}(\Sigma),\, \varphi \mapsto (\varphi\big|_{\mathfrak{a}}{}^*)^{-1}.$$

This is easily seen to be a Lie group homomorphism. Using some standard facts from the theory of real semisimple Lie algebras, we can prove the following:

Proposition 3.2.3. *If $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic real semisimple Lie algebras, then their restricted root systems are weighted-isomorphic for any choices of $\theta, \theta', \mathfrak{a}$, and $\mathfrak{a}'$. In particular, the restricted root system of $\mathfrak{g}$ does not depend on the choice of θ and $\mathfrak{a}$ (up to a weight-preserving isomorphism).*

Proof. Let $F\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}'$ be any isomorphism. By Proposition 2.1.73, we may assume without loss of generality that $F \circ \theta = \theta' \circ F$. By Proposition 2.4.8, we may assume $F(\mathfrak{a}) = \mathfrak{a}'$. But now the discussion after Definition 3.2.1 implies that F induces a weight-preserving isomorphism between Σ and Σ'. □

In the context of automorphisms, equation (3.2.1) takes on the following form:

$$\varphi(\mathfrak{g}_\alpha) = \mathfrak{g}_{\Omega(\varphi)(\alpha)} \quad (\forall\, \varphi \in N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}),\, \alpha \in \Sigma). \tag{3.2.2}$$

We have a normal subgroup $N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a}) \trianglelefteq N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$. As we know from Proposition 2.4.11, $\Omega(N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})) = \mathrm{W}(\Sigma)$. This implies that $\mathrm{W}(\Sigma) \subseteq \mathrm{Aut}^{\mathrm{w}}(\Sigma)$. As a consequence, restricted roots lying in the same orbit of $\mathrm{W}(\Sigma)$ have the same multiplicities.

We make a choice of simple roots $\Lambda \subseteq \Sigma^+ \subseteq \Sigma$ and write $D \subseteq \mathfrak{a}^*$ for the positive Weyl chamber. The Dynkin diagram DD has a natural number attached to its every vertex, namely the multiplicity of the corresponding simple root. If a vertex consists of two circles, i.e., corresponds to a simple root α such that 2α is also a root, we assign to it not just one number but the ordered pair $(\mathrm{mult}(\alpha), \mathrm{mult}(2\alpha))$. With these numbers attached, we call DD a **weighted Dynkin diagram**.

Definition 3.2.4. Let $\mathfrak{g}'$ be another real semisimple Lie algebra with a Cartan involution θ', a maximal abelian subspace $\mathfrak{a}' \subseteq \mathfrak{p}'$, and a choice of positive roots $\Sigma'^+ \subseteq \Sigma'$ fixed. We call a diagram isomorphism $s\colon \Lambda \xrightarrow{\sim} \Lambda'$ between DD and DD$'$ **weight-preserving** if it preserves the vertex weights: $\mathrm{mult}(s(\alpha)) = \mathrm{mult}(\alpha)$ (and $\mathrm{mult}(2s(\alpha)) = \mathrm{mult}(2\alpha)$ in case 2α is a root) for every $\alpha \in \Lambda$. We say that DD and DD$'$ are **weighted-isomorphic** if there exists a weight-preserving isomorphism between them. Finally, we call a weight-preserving isomorphism $\Lambda \xrightarrow{\sim} \Lambda$ from DD to itself a **weight-preserving automorphism of DD**. The group of all weight-preserving automorphisms of DD will be denoted by $\mathbf{Aut^w(DD)} \subseteq \mathrm{Aut(DD)}$.

Since $\mathrm{W}(\Sigma) \subseteq \mathrm{Aut}^{\mathrm{w}}(\Sigma)$ and $\mathrm{W}(\Sigma)$ acts transitively on the set of Weyl chambers in $\mathfrak{a}^*$, we immediately get the following:

Proposition 3.2.5. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be real semisimple Lie algebras with restricted root systems Σ and Σ', respectively. If Σ and Σ' are weighted-isomorphic (in particular, if $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic), then their Dynkin diagrams are weighted-isomorphic as well for*

[1] In Subsection 2.4.2, we defined this map only on $N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})$, but now we are extending it to a (possibly) larger group.

any choices of Σ^+ *and* Σ'^+*. In particular, the Dynkin diagram of* Σ *does not depend on the choice of* Σ^+ *(up to a weight-preserving isomorphism).*

Using the results of Subsection 3.1.2, we can prove the converse:

Proposition 3.2.6. *Let* $\mathfrak{g}$ *and* $\mathfrak{g}'$ *be real semisimple Lie algebras with restricted root systems* Σ *and* Σ' *and Dynkin diagrams* DD *and* DD'*, respectively. If* DD *and* DD' *are weighted-isomorphic, then so are* Σ *and* Σ'*. More specifically, if* $s\colon \Lambda \xrightarrow{\sim} \Lambda'$ *is a weight-preserving isomorphism between* DD *and* DD'*, then its unique linear extension* $s\colon \mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$ *is a weight-preserving isomorphism between* Σ *and* Σ'*. In particular,* $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}) \subseteq \mathrm{Aut}^{\mathrm{w}}(\Sigma)$*.*

Proof. We already know from Proposition 3.1.7(a) that $s\colon \mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$ is an isomorphism between Σ and Σ', so we only need to prove that it preserves the root multiplicities. We also know from the proof of Proposition 3.1.7 that $s\,\mathrm{W}(\Sigma)s^{-1} = \mathrm{W}(\Sigma')$ and $\mathrm{W}(\Sigma)\cdot\overline{\Lambda} = \Sigma$. Let $\alpha \in \Sigma$ be any root. Take $w \in \mathrm{W}(\Sigma)$ such that $w(\alpha) \in \overline{\Lambda}$, and write $w' = sws^{-1} \in \mathrm{W}(\Sigma')$. We have:

$$s(\alpha) = sw^{-1}(w(\alpha)) = w'^{-1}s(w(\alpha)).$$

Since $\mathrm{mult}(s(w(\alpha))) = \mathrm{mult}(w(\alpha))$ and elements of the Weyl group preserve root multiplicities, we get $\mathrm{mult}(s(\alpha)) = \mathrm{mult}(\alpha)$, so s is a weight-preserving root system isomorphism. □

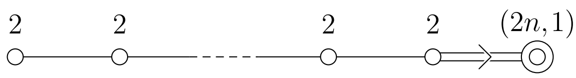
Since both $\mathrm{W}(\Sigma)$ and $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$ are contained in $\mathrm{Aut}^{\mathrm{w}}(\Sigma)$ and $\mathrm{W}(\Sigma)$ acts transitively on the set of Weyl chambers, we immediately get the following weighted analog of Proposition 3.1.7(c):

Corollary 3.2.7. $\mathrm{Aut}^{\mathrm{w}}(\Sigma) = \mathrm{W}(\Sigma) \rtimes \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$.

To recapitulate, we know that a weighted-isomorphism class of restricted root systems yields a weighted-isomorphism class of Dynkin diagrams, and it is determined by that class. Similarly, an isomorphism class of real semisimple Lie algebras yields a weighted-isomorphism class of restricted root systems. It cannot be determined by that class though, since adding a compact semisimple summand to the Lie algebra does not change the restricted root system. But it turns out that this is the only obstacle: if $\mathfrak{g}$ has no nonzero compact ideals, it is determined up to isomorphism by its (weighted) restricted root system—and thus by its (weighted) Dynkin diagram. The standard proof of this fact, however, is rather roundabout. One usually first classifies real semisimple Lie algebras—compact or not—by some other means like Satake or Vogan diagrams, and then computes explicitly the restricted root system of every Lie algebra in the classification list. It turns out that non-isomorphic real semisimple Lie algebras (without nonzero compact ideals) have non-weighted-isomorphic restricted root systems.

The list of all noncompact simple Lie algebras together with their weighted restricted root systems can be found in [BCO16, pp. 336-340]. Note that it is given there in the equivalent context of irreducible symmetric spaces of noncompact type (we will discuss this equivalence more in Section 3.3).

Example 3.2.8. The restricted root system of $\mathfrak{su}(r, r+n), n \geq 1$, is isomorphic to $(BC)_r$, and its Dynkin diagram looks like this:



Here the number above a vertex is its weight. As we pointed out earlier, keeping track of vertex weights is crucial: the Lie algebra $\mathfrak{sp}(r, r+n), n \geq 1$, also has $(BC)_r$ as its restricted root system, but the multiplicities are different. //

Let us look at the homomorphism $\Omega\colon N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}) \to \mathrm{Aut}^{\mathrm{w}}(\Sigma)$ through the lens of the semidirect product decomposition $\mathrm{Aut}^{\mathrm{w}}(\Sigma) = \mathrm{W}(\Sigma) \rtimes \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$. We know two things:

$$\mathrm{Ker}(\Omega) = Z_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}), \quad \Omega(N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})) = \mathrm{W}(\Sigma).$$

In particular, $N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})/Z_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a}) \cong \mathrm{W}(\Sigma)$.

Observation 3.2.9. It follows from Proposition 2.4.9(e) that the Lie algebra of $Z_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$ is also $\mathfrak{k}_0$, so it is actually an open subgroup of $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$. By (3.2.2), $Z_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$ preserves $\mathfrak{g}_\alpha$ for each $\alpha \in \Sigma$. As a result, every restricted root space $\mathfrak{g}_\alpha$ becomes an orthogonal representation of $Z_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$. These representations are going to be of great importance later in the thesis.

Consider another normalizer subgroup of $\mathrm{Aut}(\mathfrak{g})^\theta$ given by $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})$. Any element $k \in N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})$ commutes with θ and thus preserves $\theta\mathfrak{n}$. Since k is orthogonal with respect to B_θ, it must preserve $\mathfrak{g} \ominus (\mathfrak{n} \oplus \theta\mathfrak{n}) = \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$. But $\mathfrak{k}_0 \subseteq \mathfrak{k}$, whereas $\mathfrak{a} \subseteq \mathfrak{p}$, so k preserves both $\mathfrak{k}_0$ and $\mathfrak{a}$. We conclude that $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n}) \subseteq N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$. Since $k(\mathfrak{g}_\alpha) = \mathfrak{g}_{\Omega(k)(\alpha)}$, and $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$ consists precisely of those weight-preserving automorphisms of Σ that preserve the set of positive roots, it follows that $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n}) = \Omega^{-1}(\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}))$. In view of Observation 3.2.9, this subgroup contains the kernel $Z_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$ of Ω. We can draw the following commutative diagram:

$$\begin{tikzcd} & N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n}) \arrow[dr, hook] & \\ Z_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a}) \arrow[ur, hook] \arrow[dr, hook] & & N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}) \\ & N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a}) \arrow[ur, hook] & \end{tikzcd}$$

This is actually a pullback diagram, i.e., $N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{n}) = Z_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})$. Indeed, for every $\varphi \in N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{n})$, $\Omega(\varphi)$ is an element of the Weyl group that preserves the positive Weyl chamber, hence it is trivial.

3.2.2. The lifting theorem

Now we are in a position to prove the main result of this chapter.

Theorem 3.2.10. *Let $\mathfrak{g}$ be a real semisimple Lie algebra with $\theta, \mathfrak{a}$, and Σ^+ fixed. Then:*

(a) $\Omega(N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})) = \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$, *and hence Ω is surjective.*

(b) *If $\mathfrak{g}$ is simple, then* $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}) = \mathrm{Aut}(\mathrm{DD})$, *and hence* $\mathrm{Aut}^{\mathrm{w}}(\Sigma) = \mathrm{Aut}(\Sigma)$.

Informally, (a) means that every weight-preserving automorphism of Σ can be lifted to an automorphism of $\mathfrak{g}$. Also note that (b) might fail in case $\mathfrak{g}$ is not simple. For instance, if $\mathfrak{g} = \mathfrak{su}(r, r+n) \oplus \mathfrak{sp}(r, r+n)$, $n \geq 1$, then $\Sigma = (BC)_r \sqcup (BC)_r$, so $\mathrm{Aut}(\mathrm{DD}) = \mathbb{Z}/2\mathbb{Z}$. But the two connected components of DD are not weighted-isomorphic, which means that $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$ is trivial. Partial versions of Theorem 3.2.10 appeared in the literature without proofs or references (see, e.g., [BT03, p. 11] or [Mur52, p. 111]).

We will prove Theorem 3.2.10 by first reducing it to the simple case and then (mostly) to the theory of complex semisimple Lie algebras. We need to take account of two things: first, $\mathfrak{g}$ might have compact ideals, which make no contribution to the restricted root system, and second, $\mathfrak{g}$ might have isomorphic noncompact simple ideals.

We start by writing $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, where $\mathfrak{g}_c$ is the sum of all compact ideals and each $\mathfrak{g}_i$ is a noncompact simple ideal. As we observed earlier in Observation 2.1.117, the Cartan involution θ respects this decomposition and is trivial on $\mathfrak{g}_c$, so we can write $\theta = (\mathrm{Id}_{\mathfrak{g}_c}, \theta_1, \ldots, \theta_k)$. Here θ_i is a Cartan involution of $\mathfrak{g}_i$. All the objects related to the restricted root system necessarily decompose accordingly:

- $\mathfrak{k} = \mathfrak{g}_c \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_k$, where $\mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{g}_i$.
- $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_k$, where $\mathfrak{p}_i = \mathfrak{p} \cap \mathfrak{g}_i$.
- $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$, where $\mathfrak{a}_i = \mathfrak{a} \cap \mathfrak{p}_i$ is maximal abelian in $\mathfrak{p}_i$.
- $\mathfrak{a}^* = \mathfrak{a}_1^* \oplus \cdots \oplus \mathfrak{a}_k^*$.
- All these decompositions are orthogonal with respect to B and B_θ (or the induced inner product on $\mathfrak{a}^*$).
- $B = B_c + B^1 + \cdots + B^k$, where B_c is the Killing form of $\mathfrak{g}_c$ and B^i is that of $\mathfrak{g}_i$.
- $B_\theta = -B_c + B^1_{\theta_1} + \cdots + B^k_{\theta_k}$.
- $\Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_k$, where $\Sigma_i = \Sigma \cap \mathfrak{a}_i^*$ is the restricted root system of $\mathfrak{g}_i$.

Before going further, we need to figure out when the restricted root system is irreducible. If $\mathfrak{g}$ has more than one noncompact simple ideal, then Σ is reducible—as follows from the discussion above. The converse is also true:

Lemma 3.2.11. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, and assume that its restricted root system Σ is reducible. Then $\mathfrak{g}$ can be written as a direct sum of two noncompact ideals.*

Proof. Let $\mathfrak{a}^* = V_1 \oplus^\perp V_2$ such that $\Sigma = \Sigma_1 \sqcup \Sigma_2$, where $\Sigma_i = \Sigma \cap V_i$, and either of Σ_i is nonempty. Let $\mathfrak{a}_i \subseteq \mathfrak{a}$ correspond to V_i under $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$. Consider the subspaces $\mathfrak{p}_i = \mathfrak{a}_i \oplus \bigoplus_{\alpha \in \Sigma_i^+} \mathfrak{p}_\alpha$. By (2.4.2), $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, and it follows from Proposition 2.4.9 that $[\mathfrak{p}_1, \mathfrak{p}_2] = \{0\}$. If we let $\mathfrak{k}_i = [\mathfrak{p}_i, \mathfrak{p}_i] \subseteq \mathfrak{k}$, then the intersection $\mathfrak{k}_1 \cap \mathfrak{k}_2$ must be trivial. Indeed, we have $[\mathfrak{k}_1, \mathfrak{p}_2] = [\mathfrak{k}_2, \mathfrak{p}_1] = \{0\}$, so the intersection $\mathfrak{k}_1 \cap \mathfrak{k}_2$ must act trivially on $\mathfrak{p}$ and hence on $[\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p}$. But the latter is an ideal of $\mathfrak{g}$ and thus itself a semisimple Lie algebra. Being contained in its center, $\mathfrak{k}_1 \cap \mathfrak{k}_2$ is trivial. Now, define $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$. We already know that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ commute and intersect trivially. If they do not span $\mathfrak{g}$, then we replace $\mathfrak{g}_1$ with $\mathfrak{g}_1' = \mathfrak{g}_2^\perp = \mathfrak{g}_1 \oplus ([\mathfrak{p}, \mathfrak{p}] \oplus \mathfrak{p})^\perp$, where the orthogonal complement is taken with respect to the Killing form. In the end, we obtain $\mathfrak{g} = \mathfrak{g}_1' \oplus \mathfrak{g}_2$, which is the desired decomposition. ☐

Corollary 3.2.12. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and Σ its restricted root system. Then Σ is irreducible if and only if $\mathfrak{g}$ has at most one noncompact simple ideal. In particular, if $\mathfrak{g}$ has no nonzero compact ideals, then Σ is irreducible if and only if $\mathfrak{g}$ is simple.*

Now we can return to our decompositions:

- Each Σ_i is irreducible, and $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_k$ is the decomposition of Σ into its irreducible components.
- $\Sigma^+ = \Sigma_1^+ \sqcup \ldots \sqcup \Sigma_k^+$, where $\Sigma_i^+ = \Sigma^+ \cap \Sigma_i$.
- $\Lambda = \Lambda_1 \sqcup \ldots \sqcup \Lambda_k$, where $\Lambda_i = \Lambda \cap \Sigma_i^+$.
- $D = D_1 \times \cdots \times D_k$, where $D_i = D \cap \mathfrak{a}_i^*$.
- $\mathrm{DD} = \mathrm{DD}_1 \sqcup \cdots \sqcup \mathrm{DD}_k$, where DD_i is the Dynkin diagram of $\mathfrak{g}_i$ arising from Λ_i.

Next, we need to understand how these decompositions lift to the level of automorphisms. Similarly to what we did in Proposition 2.1.60, define a finite group[1]

$$\boldsymbol{S_k^{\sim}} = \left\{\sigma \in S_k \mid \mathfrak{g}_i \simeq \mathfrak{g}_{\sigma(i)}\ \forall\, i = 1, \ldots, k\right\},$$

For each $i, j \in \{1, \ldots, k\}$ such that $\mathfrak{g}_i \simeq \mathfrak{g}_j$, we pick an isomorphism $f_{ij} \colon \mathfrak{g}_i \xrightarrow{\sim} \mathfrak{g}_j$ such that the following conditions are satisfied:

(a) $f_{ij} \circ \theta_i = \theta_j \circ f_{ij}$ (hence $f_{ij}(\mathfrak{k}_i) = \mathfrak{k}_j, f_{ij}(\mathfrak{p}_i) = \mathfrak{p}_j$).

(b) $f_{ij}(\mathfrak{a}_i) = \mathfrak{a}_j$.

(c) Whenever $\mathfrak{g}_i \simeq \mathfrak{g}_j \simeq \mathfrak{g}_l$, we have $f_{jl} \circ f_{ij} = f_{il}$.

Such a choice is possible by virtue of Propositions 2.1.73 and 2.4.8. For each f_{ij}, consider the induced weight-preserving isomorphism $F_{ij} = (f_{ij}\big|_{\mathfrak{a}_i}{}^*)^{-1}$ between Σ_i and Σ_j. We obtain embeddings

$$\begin{gathered} S_k^{\sim} \hookrightarrow \mathrm{Aut}(\mathfrak{g}), \sigma \mapsto f_\sigma, \text{ where} \\ f_\sigma(X_0, X_1, \ldots, X_k) = (X_0, f_{\sigma(1)1}(X_{\sigma(1)}), \ldots, f_{\sigma(k)k}(X_{\sigma(k)})), \end{gathered}$$

and

$$\begin{gathered} S_k^{\sim} \hookrightarrow \mathrm{Aut}^{\mathrm{w}}(\Sigma), \sigma \mapsto F_\sigma, \text{ where} \\ F_\sigma(v_1, \ldots, v_k) = (F_{\sigma(1)1}(v_{\sigma(1)}), \ldots, F_{\sigma(k)k}(v_{\sigma(k)})). \end{gathered}$$

(to be precise, these are injective group anti-homomorphisms). Owing to our choice of f_{ij}'s, the image of the former embedding actually lies in the subgroup $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$. We also have obvious subgroups $\mathrm{Aut}(\mathfrak{g}_c) \times \mathrm{Aut}(\mathfrak{g}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{g}_k) \subseteq \mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Aut}^{\mathrm{w}}(\Sigma_1) \times \cdots \times \mathrm{Aut}^{\mathrm{w}}(\Sigma_k) \subseteq \mathrm{Aut}^{\mathrm{w}}(\Sigma)$.

Proposition 3.2.13. (a) *The group* $\mathrm{Aut}(\mathfrak{g})$ *decomposes as a semidirect product*

$$\mathrm{Aut}(\mathfrak{g}) = [\mathrm{Aut}(\mathfrak{g}_c) \times \mathrm{Aut}(\mathfrak{g}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{g}_k)] \rtimes S_k^{\sim}.$$

In particular, we have $\mathrm{Inn}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g}_c) \times \mathrm{Inn}(\mathfrak{g}_1) \times \cdots \times \mathrm{Inn}(\mathfrak{g}_k)$.

(b) *The group* $\mathrm{Aut}(\mathfrak{g})^\theta$ *decomposes as a semidirect product*

$$\mathrm{Aut}(\mathfrak{g})^\theta = \left[\mathrm{Aut}(\mathfrak{g}_c) \times \mathrm{Aut}(\mathfrak{g}_1)^{\theta_1} \times \cdots \times \mathrm{Aut}(\mathfrak{g}_k)^{\theta_k}\right] \rtimes S_k^{\sim}.$$

[1] The notational difference between $S_k^{\sim}$ defined here and $S_k^{\simeq}$ defined before Proposition 2.1.60 will become clear in Section 3.3.

In particular, we have $\mathrm{Inn}(\mathfrak{g})^\theta = \mathrm{Inn}(\mathfrak{g}_c) \times \mathrm{Inn}(\mathfrak{g}_1)^{\theta_1} \times \cdots \times \mathrm{Inn}(\mathfrak{g}_k)^{\theta_k}$.

(c) *The group* $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$ *decomposes as a semidirect product*

$$N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}) = \left[\mathrm{Aut}(\mathfrak{g}_c) \times N_{\mathrm{Aut}(\mathfrak{g}_1)^{\theta_1}}(\mathfrak{a}_1) \times \cdots \times N_{\mathrm{Aut}(\mathfrak{g}_k)^{\theta_k}}(\mathfrak{a}_k)\right] \rtimes S_k^{\sim}.$$

In particular, we have $N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a}) = \mathrm{Inn}(\mathfrak{g}_c) \times N_{\mathrm{Inn}(\mathfrak{g}_1)^{\theta_1}}(\mathfrak{a}_1) \times \cdots \times N_{\mathrm{Inn}(\mathfrak{g}_k)^{\theta_k}}(\mathfrak{a}_k)$.

(d) *The group* $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})$ *decomposes as a semidirect product*

$$N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n}) = \left[\mathrm{Aut}(\mathfrak{g}_c) \times N_{\mathrm{Aut}(\mathfrak{g}_1)^{\theta_1}}(\mathfrak{n}_1) \times \cdots \times N_{\mathrm{Aut}(\mathfrak{g}_k)^{\theta_k}}(\mathfrak{n}_k)\right] \rtimes S_k^{\sim}.$$

(e) *The group* $\mathrm{Aut}^{\mathrm{w}}(\Sigma)$ *decomposes as a semidirect product*

$$\mathrm{Aut}^{\mathrm{w}}(\Sigma) = [\mathrm{Aut}^{\mathrm{w}}(\Sigma_1) \times \cdots \times \mathrm{Aut}^{\mathrm{w}}(\Sigma_k)] \rtimes S_k^{\sim}.$$

(f) *The group* $\mathrm{W}(\Sigma)$ *decomposes as a product*

$$\mathrm{W}(\Sigma) = \mathrm{W}(\Sigma_1) \times \cdots \times \mathrm{W}(\Sigma_k).$$

(g) *The group* $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$ *decomposes as a semidirect product*

$$\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}) = [\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}_1) \times \cdots \times \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}_k)] \rtimes S_k^{\sim}.$$

(h) *With respect to the decompositions (c) and (e), the homomorphism* $\Omega\colon N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}) \to \mathrm{Aut}^{\mathrm{w}}(\Sigma)$ *decomposes as*

$$\Omega = \left(E, \Omega_1, \ldots, \Omega_k, \mathrm{Id}_{S_k^{\sim}}\right),$$

where E *is the trivial homomorphism* $\mathrm{Aut}(\mathfrak{g}_c) \to \{e\}$, $\Omega_i\colon N_{\mathrm{Aut}(\mathfrak{g}_i)^{\theta_i}}(\mathfrak{a}_i) \to \mathrm{Aut}^{\mathrm{w}}(\Sigma_i)$, *and the last component* $\mathrm{Id}_{S_k^{\sim}}$ *formally means that the following diagram commutes:*

$$\begin{tikzcd}
 & N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}) \arrow[dd, "\Omega"] \\
S_k^{\sim} \arrow[ur, hook] \arrow[dr, hook] & \\
 & \mathrm{Aut}^{\mathrm{w}}(\Sigma)
\end{tikzcd}$$

We omit the proof, as it is similar to that of Proposition 2.1.60, except it is much easier because any automorphism of $\mathfrak{g}$ must obviously preserve $\mathfrak{g}_c$ and permute the remaining noncompact simple ideals, and the same is true for weight-preserving automorphisms of $\Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_k$.

Part (h) of Proposition 3.2.13 implies that if $\Omega_i(N_{\mathrm{Aut}(\mathfrak{g}_i)^{\theta_i}}(\mathfrak{n}_i)) = \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}_i)$ for each i, then $\Omega(N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})) = \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$. Consequently, in order to prove Theorem 3.2.10, we may restrict to the case when $\mathfrak{g}$ is simple and noncompact. We will actually show that $\Omega(N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})) = \mathrm{Aut}(\mathrm{DD})$ in this case, thus proving both parts (a) and (b) of the theorem.

We will consider three different scenarios. To begin with, we can immediately cast aside all those simple Lie algebras where $\mathrm{Aut}(\mathrm{DD})$ (and hence $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$) is trivial. This leaves us with those Lie algebras where $\Sigma = A_n\,(n \geq 2), D_n\,(n \geq 4)$, or E_6.

Each complex semisimple Lie algebra $\mathfrak{g}$ gives rise to at least two noncompact real ones: the realification and the split real form of $\mathfrak{g}$. These are going to be our first two scenarios. As a matter of fact, here we do not require the Lie algebra to be simple, and we will only use this assumption in the third scenario. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{h}, \Delta, \Lambda$, and $\{h_i, e_i, f_i\}_{i=1}^{r}$ be as on page 105. It follows from the Isomorphism Theorem that there exists a unique automorphism θ of $\mathfrak{g}$ as of a real Lie algebra that is $\mathbb{C}$-antilinear and satisfies

$$\theta(h_i) = -h_i, \quad \theta(e_i) = -f_i, \quad \theta(f_i) = -e_i. \tag{3.2.3}$$

This automorphism is involutive and is in fact a compact real structure and a Cartan involution[1] (hence the notation). In particular, $\mathfrak{p} = i\mathfrak{k}$. Every Cartan involution on $\mathfrak{g}$ is of this form for some choice of $\mathfrak{h}, \Lambda$, and canonical generators. We can introduce two more involutive automorphisms of $\mathfrak{g}$: the Weyl involution ω and the split real structure τ. The Weyl involution is given on the canonical generators by the same formula (3.2.3) but is $\mathbb{C}$-linear, whereas the split real form fixes all the canonical generators but is $\mathbb{C}$-antilinear. Once again, the existence and uniqueness of both of these automorphisms follow from the Isomorphism Theorem. Clearly, the three automorphisms commute pairwise and the product of any two of them equals the third one. The fixed point (real) subalgebra $\mathfrak{g}^{\tau}$ is the split real form of $\mathfrak{g}$, and every split real form is of this form for some choice of $\mathfrak{h}, \Lambda$, and canonical generators.

Scenario 1: the realification. Here we assume that our real semisimple Lie algebra is the realification of a complex one and use the notation established in the previous paragraph. In particular, the Cartan involution θ is given on the canonical generators by (3.2.3). Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition. Since θ is $\mathbb{C}$-antilinear and $\theta(h_i) = -h_i$ for each i, we have $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{h}(\mathbb{R})$, $\mathfrak{h} \cap \mathfrak{k} = i\mathfrak{h}(\mathbb{R})$. We claim that $\mathfrak{h}(\mathbb{R})$ is a maximal abelian subspace of $\mathfrak{p}$. Indeed, let $\mathfrak{b} \subset \mathfrak{p}$ be an abelian subspace containing $\mathfrak{h}(\mathbb{R})$. If we think of $\mathfrak{g}$ as a real Euclidean vector space (with respect to the inner product B_θ), then all operators of the form $\mathrm{ad}(X)$, $X \in \mathfrak{p}$, are self-adjoint, as is evident from the proof of Proposition 2.1.71. Since such operators are $\mathbb{C}$-linear, they are also self-adjoint with respect to the Hermitian inner product $\langle X \,|\, Y \rangle = B_\theta(X, Y) + iB_\theta(X, iY)$ and hence diagonalizable over $\mathbb{C}$. It follows that $\mathfrak{b}_{\mathbb{C}} = i\mathfrak{b} \oplus \mathfrak{b}$ (here $i\mathfrak{b} \subseteq i\mathfrak{p} = \mathfrak{k}$) is an abelian complex subalgebra of $\mathfrak{g}$ consisting of semisimple elements and containing $\mathfrak{h}$, so we must have $\mathfrak{b}_{\mathbb{C}} = \mathfrak{h}$ and thus $\mathfrak{b} = \mathfrak{h}(\mathbb{R})$. As a result, we can write $\mathfrak{a} = \mathfrak{h}(\mathbb{R})$. In this case, $\mathfrak{a}^* = \mathfrak{h}^*(\mathbb{R})$, and the root system Δ of $\mathfrak{g}$ as of a complex semisimple Lie algebra coincides with the restricted root system Σ of $\mathfrak{g}$ as of a real semisimple Lie algebra (for our specific choice of θ and $\mathfrak{a}$). Moreover, the root space decomposition and the restricted root space decomposition coincide as well. Note that $\mathfrak{g}_0 = \mathfrak{h} = i\mathfrak{a} \oplus \mathfrak{a}$ and $\mathfrak{k}_0 = i\mathfrak{a}$. Each restricted root space $\mathfrak{g}_\alpha$, $\alpha \in \Sigma$, thus has real dimension 2, i.e., all of the root multiplicities equal 2. As we know from Subsection 3.1.2, every (not necessarily weight-preserving) automorphism $s \in \mathrm{Aut}(\mathrm{DD})$ can be lifted to a (complex) automorphism $\hat{s}$ of $\mathfrak{g}$ given by the rule $\hat{s}(h_i) = h_{s(i)}, \hat{s}(e_i) = e_{s(i)}, \hat{s}(f_i) = f_{s(i)}$. This automorphism satisfies $\hat{s}(\mathfrak{a}) = \mathfrak{a}$ and

[1] On a complex semisimple Lie algebra, a compact real form is the same as a Cartan involution (see [Oni04, Sec. 5]).

$\Psi(\hat{s}) = (\hat{s}\big|_{\mathfrak{a}}{}^{*})^{-1} = s$. It is easily seen on the canonical generators that $\hat{s}$ commutes with θ and preserves $\mathfrak{n}$. Consequently, $\hat{s} \in N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})$ and $\Omega(\hat{s}) = \Psi(\hat{s}) = s$, which finishes the proof in this scenario.

Scenario 2: the split real form. Now suppose that our real semisimple Lie algebra is split. We denote its complexification by $\mathfrak{g}$ and use the same notation as above. Then our real semisimple Lie algebra can be described as $\mathfrak{g}^\tau$. One can easily see that $\mathfrak{g}^\tau$ is the real Lie subalgebra of $\mathfrak{g}$ generated by e_i, f_i, and $h_i, 1 \leq i \leq r$. The automorphism θ of $\mathfrak{g}$ clearly preserves $\mathfrak{g}^\tau$. Since $\mathfrak{g}^\tau$ is a real form of $\mathfrak{g}$, the Killing form B of $\mathfrak{g}$ is the $\mathbb{C}$-bilinear extension of the Killing form B^τ of $\mathfrak{g}^\tau$. This implies that the restriction of θ to $\mathfrak{g}^\tau$ is a Cartan involution on $\mathfrak{g}^\tau$. Moreover, if we write the corresponding Cartan decomposition as $\mathfrak{g}^\tau = \mathfrak{k} \oplus \mathfrak{p}$, then $\mathfrak{h}(\mathbb{R})$ lies in $\mathfrak{p}$ and is a maximal abelian subspace of it. Indeed, for each $X \in \mathfrak{p}$, the operator $\mathrm{ad}_{\mathfrak{g}^\tau}(X)$ is diagonalizable over $\mathbb{R}$, hence its $\mathbb{C}$-linear extension $\mathrm{ad}_{\mathfrak{g}}(X)$ is diagonalizable over $\mathbb{C}$ as an operator on $\mathfrak{g}$. Therefore, in the same fashion as above, the existence of a larger abelian subspace of $\mathfrak{p}$ would lead to a toral subalgebra of $\mathfrak{g}$ larger than $\mathfrak{h}$, hence a contradiction. Once again, we can take $\mathfrak{a} = \mathfrak{h}(\mathbb{R})$, in which case $\mathfrak{a}^* = \mathfrak{h}^*(\mathbb{R})$ and the restricted root system Σ of $\mathfrak{g}^\tau$ coincides with Δ. Just as before, every diagram automorphism $s \in \mathrm{Aut}(\mathrm{DD})$ can be lifted to the complex automorphism $\hat{s} \in N_{\mathrm{Aut}(\mathfrak{g})}(\mathfrak{h})$ of $\mathfrak{g}$ such that $\Psi(\hat{s}) = s$, and it can be seen from the defining formula for $\hat{s}$ that it commutes with both θ and τ. In particular, it preserves $\mathfrak{g}^\tau$ and the restriction $\hat{s}\big|_{\mathfrak{g}^\tau}$ lies in $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})$. We have $\Omega(\hat{s}\big|_{\mathfrak{g}^\tau}) = \Psi(\hat{s}) = s$, which finishes the proof in scenario 2.

Scenario 3: the rest. Now we go back to our assumption that $\mathfrak{g}$ is simple and Aut(DD) is nontrivial. An examination of the list of all real simple noncompact Lie algebras ([BCO16, pp. 336-340]) reveals that if $\mathfrak{g}$ is neither split nor complex, it has to be isomorphic to either $\mathfrak{sl}(n, \mathbb{H})$ $(n \geq 3)$ or $\mathfrak{e}_6^{-26}$. The restricted root systems of these Lie algebras are A_{n-1} and A_2, respectively. In both cases, $\mathrm{Aut}(\mathrm{DD}) \cong \mathbb{Z}/2\mathbb{Z}$, and there is only one nontrivial diagram automorphism that we want to lift to $N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})$. Recall that we have a distinguished automorphism θ of $\mathfrak{g}$ fixed. Plainly, $\theta \in N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})$ and $\Omega(\theta) = -\mathrm{Id}_{\mathfrak{a}^*}$. The weight-preserving root system automorphism $-\mathrm{Id}_{\mathfrak{a}^*}$ can be decomposed as $-\mathrm{Id}_{\mathfrak{a}^*} = w_0 s$, where $w_0 \in \mathrm{W}(\Sigma)$ and $s \in \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$. Here $s(D) = D$ and $-\mathrm{Id}_{\mathfrak{a}^*}(D) = -D$, so $w_0(D) = -D$. This uniquely determines w_0 and also shows that it is the longest element of $\mathrm{W}(\Sigma)$ with respect to the system of generators $s_{\alpha_1}, \ldots, s_{\alpha_r}$. The diagram automorphism $s = -w_0$ may or may not be trivial, depending on Σ. Note that this construction does not really rely on $\mathfrak{g}$, nor does it use root multiplicities, so it can be carried out for any root system (V, Δ): pick Δ^+ and decompose $-\mathrm{Id}_V \in \mathrm{Aut}(\Delta)$ as $-\mathrm{Id}_V = w_0 s$ with respect to the semidirect product decomposition $\mathrm{Aut}(\Delta) = \mathrm{W}(\Delta) \rtimes \mathrm{Aut}(\mathrm{DD})$. It was shown in [Oni04, §4, Prop. 4] that, provided that Δ is irreducible, s is a nontrivial diagram automorphism precisely when $\Delta = A_n\,(n \geq 2), D_{2n+1}(n \geq 2)$, or E_6. This covers both of our cases $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{H})$ $(n \geq 3)$ and $\mathfrak{g} = \mathfrak{e}_6^{-26}$. Now, since w_0 is an element of the Weyl group, it is the image of some $\varphi \in N_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})$ under Ω. We have $\varphi\theta \in N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})$ and $\Omega(\varphi\theta) = w_0^2 s = s$. In other words, the only nontrivial element of Aut(DD) lies in the image of Ω and so $\Omega(N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})) = \mathrm{Aut}(\mathrm{DD})$, which concludes the proof of Theorem 3.2.10.

Remark 3.2.14. Part (b) of Theorem 3.2.10 can also be proven on a case by case basis by examining the classification of simple noncompact Lie algebras and the list of their weighted Dynkin diagrams. Indeed, for any such diagram, if there are two vertices that differ by a (not necessarily weight-preserving) diagram automorphism, then they happen to have the same multiplicity, which means that every diagram automorphism is weight-preserving.

Corollary 3.2.15. *Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then we have:*

$$N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a})/Z_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}) \cong \mathrm{Aut}^{\mathrm{w}}(\Sigma),$$
$$N_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{n})/Z_{\mathrm{Aut}(\mathfrak{g})^\theta}(\mathfrak{a}) \cong \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}).$$

Note that, unlike in the complex semisimple case, we do not in general have an isomorphism between $\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$ and $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$. For instance, $\mathrm{Out}(\mathfrak{so}(4,4)) \simeq S_4$ ([Gün10, Prop. 2.14]), but the restricted root system $\Sigma \simeq D_4$ of $\mathfrak{so}(4,4)$ has $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}) = \mathrm{Aut}(\mathrm{DD}) \simeq S_3$. Still, at least in the simple case, the short exact sequence $\mathrm{Inn}(\mathfrak{g}) \hookrightarrow \mathrm{Aut}(\mathfrak{g}) \twoheadrightarrow \mathrm{Out}(\mathfrak{g})$ splits, so we have $\mathrm{Aut}(\mathfrak{g}) \simeq \mathrm{Inn}(\mathfrak{g}) \rtimes \mathrm{Out}(\mathfrak{g})$ (see [Gün10]).

Corollary 3.2.16. *Let $\mathfrak{g}, \mathfrak{g}'$ be real semisimple Lie algebras with $\theta, \theta', \mathfrak{a}, \mathfrak{a}', \Sigma^+$, and Σ'^+ fixed.*

(a) *Every weight-preserving isomorphism $f\colon \mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$ between Σ and Σ' is an isometry. In particular, $\mathrm{Aut}^{\mathrm{w}}(\Sigma) \subseteq \mathrm{O}(\mathfrak{a}^*)$.*

(b) *Now assume that neither $\mathfrak{g}$ nor $\mathfrak{g}'$ have nonzero compact ideals. Then for every weight-preserving isomorphism $f\colon \mathfrak{a}^* \xrightarrow{\sim} \mathfrak{a}'^*$ between Σ and Σ', there exists a Lie algebra isomorphism $F\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}'$ such that $F \circ \theta = \theta' \circ F$, $F(\mathfrak{a}) = \mathfrak{a}'$, and the induced weight-preserving isomorphism $\mathfrak{a}^* \xrightarrow{\sim} \mathfrak{a}'^*$ between Σ and Σ' coincides with f. In particular, for every weight-preserving diagram isomorphism $s\colon \Lambda \xrightarrow{\sim} \Lambda'$ between DD and DD', there exists such $F\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}'$ that the induced diagram isomorphism $f\big|_\Lambda\colon \Lambda \xrightarrow{\sim} \Lambda'$ coincides with s.*

Proof. For part (b), we know from Subsection 3.2.1 that there exists some Lie algebra isomorphism $\widetilde{F}\colon \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}'$. As explained in the proof of Proposition 3.2.3, we may assume $\widetilde{F} \circ \theta = \theta' \circ \widetilde{F}$ and $\widetilde{F}(\mathfrak{a}) = \mathfrak{a}'$. Let $\widetilde{f}\colon \mathfrak{a}^* \xrightarrow{\sim} \mathfrak{a}'^*$ be the induced weight-preserving isomorphism between Σ and Σ'. Then $f \circ \widetilde{f}^{-1} \in \mathrm{Aut}^{\mathrm{w}}(\Sigma')$. According to Theorem 3.2.10, there exists $\varphi' \in N_{\mathrm{Aut}(\mathfrak{g}')^{\theta'}}(\mathfrak{a}')$ such that $\Omega'(\varphi') = f \circ \widetilde{f}^{-1}$. Then the weight-preserving root system isomorphism between Σ and Σ' induced by $F = \varphi' \circ \widetilde{F}$ is $\Omega'(\varphi') \circ \widetilde{f} = f$. For part (a), write $\mathfrak{g} = \mathfrak{g}_{\mathrm{c}} \oplus \mathfrak{g}_{\mathrm{nc}}$, where $\mathfrak{g}_{\mathrm{c}}$ is the maximal compact ideal and $\mathfrak{g}_{\mathrm{nc}}$ is the complementary noncompact ideal, and do the same for $\mathfrak{g}'$: $\mathfrak{g}' = \mathfrak{g}'_{\mathrm{c}} \oplus \mathfrak{g}'_{\mathrm{nc}}$. The restricted root systems of $\mathfrak{g}$ and $\mathfrak{g}'$ coincide with those of $\mathfrak{g}_{\mathrm{nc}}$ and $\mathfrak{g}'_{\mathrm{nc}}$, respectively, so every weight-preserving isomorphism between them is an isometry by part (b) and the observation after Definition 3.2.1. □

3.3. Applications to symmetric spaces of noncompact type

The results of the previous section have a useful interpretation in terms of the theory of symmetric spaces of noncompact type and allow to deepen the link between such spaces and noncompact real semisimple Lie algebras.

Suppose M is a symmetric space of noncompact type, and let (G, K) be its canonical Riemannian symmetric pair at some point o. Define $\widetilde{G} = I(M)$ and $\widetilde{K} = \widetilde{G}_o$. We know that K is a maximal compact subgroup of G. But $\widetilde{K}$ is compact, and $\widetilde{G}/\widetilde{K} \cong M$ is contractible, so $\widetilde{K}$ is a maximal compact subgroup of $\widetilde{G}$ ([Ant12]). In particular, K is the identity component of $\widetilde{K}$.

Let $M = M_1 \times \cdots \times M_k$ be the de Rham decomposition of M. If we write $o = (o_1, \ldots, o_k)$, then we can introduce the same groups as above for M_i at its point o_i: we get $\widetilde{G}_i, G_i, \widetilde{K}_i$, and K_i. According to Corollary 2.1.61, $G = G_1 \times \cdots \times G_k$, and hence $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Let $\lambda_1, \ldots, \lambda_k$ be the normalizing constants of M corresponding to this decomposition. The geodesic symmetry s_o splits as $(s_{o_1}, \ldots, s_{o_k})$, where s_{o_i} is the geodesic symmetry of M_i at o_i. Therefore, the involution Θ splits as $(\Theta_1, \ldots, \Theta_k)$, and thus the Cartan involution $\theta = \mathrm{Ad}(s_o)$ on $\mathfrak{g}$ splits as $(\theta_1, \ldots, \theta_k)$, where $\theta_i = \mathrm{Ad}(s_{o_i})$. From this, we see that the Cartan decomposition and all the other objects related to the restricted root system Σ of $\mathfrak{g}$ decompose in agreement with the de Rham decomposition, as on page 112 (we will use the notation from there). As we know, each $\mathfrak{g}_i$ is noncompact.

Recall that two Riemannian manifolds (N, h) and (N', h') are called **homothetic**—denoted as $\boldsymbol{N \sim N'}$—if there exists a diffeomorphism $\varphi\colon N \xrightarrow{\sim} N'$ such that $\varphi^* h' = ah$ for some $a > 0$. In other words, N and N' are conformally equivalent with a constant conformal factor.

Proposition 3.3.1. *Given $1 \leq i, j \leq k$, consider the following conditions:*

(i) *M_i is isometric to M_j.*

(ii) *M_i is homothetic to M_j.*

(iii) *$\mathfrak{g}_i \simeq \mathfrak{g}_j$.*

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii)*. Moreover,* (ii) $\Rightarrow$ (i) *if and only if $\lambda_i = \lambda_j$.*

Proof. Clearly, (i) $\Rightarrow$ (ii) and, since rescaling the Riemannian metric does not change the isometry Lie algebra, (ii) $\Rightarrow$ (iii). To show (iii) $\Rightarrow$ (ii), recall that M_i can be recovered from $\mathfrak{g}_i$ up to isometry by taking a Riemannian symmetric pair $(\widehat{G}_i, \widehat{K}_i)$ associated with $(\mathfrak{g}_i, \theta_i)$ with $\widehat{G}_i$ simply connected and endowing $\widehat{G}_i/\widehat{K}_i$ with a (unique up to rescaling) $\widehat{G}_i$-invariant metric. If $\mathfrak{g}_i \simeq \mathfrak{g}_j$, we can always find an isomorphism $F\colon \widehat{G}_i \xrightarrow{\sim} \widehat{G}_j$ such that $F(\widehat{K}_i) = \widehat{K}_j$, which induces a homothety $M_i \to M_j$. The equivalence (ii) $\Leftrightarrow$ (iii) essentially follows from (2.4.1).

The proof of the last assertion is very similar to the last bit of the proof of Proposition 2.1.111, except now we are proving the converse. Assume (ii) and start with any homothety $\varphi\colon M_i \to M_j$. By composing it with a suitable isometry of M_j, we can replace φ with a homothety $\varphi'\colon M_i \to M_j$ mapping o_i to o_j. Note that φ' is an isometry if and only if φ is. We have an isomorphism $F\colon G_i \xrightarrow{\sim} G_j, \psi \mapsto \varphi' \circ g \circ \varphi'^{-1}$, mapping s_{o_i} to s_{o_j}. If we write $\varphi = F_*\colon \mathfrak{g}_i \xrightarrow{\sim} \mathfrak{g}_j$, then $f(\mathfrak{p}_i) = \mathfrak{p}_j$, and the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{p}_i \arrow[r, "f", "\sim"'] \arrow[d, "\wr"'] & \mathfrak{p}_j \arrow[d, "\wr"'] \\
T_{o_i} M_i \arrow[r, "d\varphi'_{o_i}", "\sim"'] & T_{o_j} M_j
\end{tikzcd}$$

The top arrow is an isometry with respect to the inner products $\langle - | - \rangle_{B_i}$ and $\langle - | - \rangle_{B_j}$. Consequently, the bottom arrow is an isometry (that is, φ' is an isometry) with respect to $(g_i)_{o_i} = \lambda_i\langle -|-\rangle_{B_i}$ on $T_{o_i}M_i$ and $(g_j)_{o_j} = \lambda_j\langle -|-\rangle_{B_j}$ on $T_{o_j}M_j$ if and only if $\lambda_i = \lambda_j$, which completes the proof. □

As we know from the previous section, condition (iii) in Proposition 3.3.1 is also equivalent to Σ_i and Σ_j (or DD_i and DD_j) being weighted-isomorphic. Thus, the symmetric space M is fully determined up to isometry by the (weighted) Dynkin diagram DD of $\mathfrak{g}$ together with the normalizing constants $\lambda_1, \dots, \lambda_k$ (which we could assign as weights to the connected components $\mathrm{DD}_1, \dots, \mathrm{DD}_k$ of DD). Recall that we have finite groups

$$\begin{aligned} S_k^{\simeq} &= \left\{\sigma \in S_k \mid M_i \simeq M_{\sigma(i)}\ \forall\, i = 1, \dots, k\right\}, \\ S_k^{\sim} &= \left\{\sigma \in S_k \mid \mathfrak{g}_i \simeq \mathfrak{g}_{\sigma(i)}\ \forall\, i = 1, \dots, k\right\}, \end{aligned}$$

such that, according to Propositions 2.1.60 and 3.2.13,

$$\begin{aligned} \widetilde{G} &\simeq [\widetilde{G}_1 \times \cdots \times \widetilde{G}_k] \rtimes S_k^{\simeq}, \\ \mathrm{Aut}(\mathfrak{g}) &\simeq [\mathrm{Aut}(\mathfrak{g}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{g}_k)] \rtimes S_k^{\sim}. \end{aligned}$$

In view of Proposition 3.3.1, the group $S_k^{\sim}$ can be alternatively described as

$$S_k^{\sim} = \left\{\sigma \in S_k \mid M_i \sim M_{\sigma(i)}\ \forall\, i = 1, \dots, k\right\},$$

which justifies the notation. Proposition 3.3.1 also implies that $S_k^{\simeq} \subseteq S_k^{\sim}$, and we have the following immediate

Corollary 3.3.2. *The following conditions are equivalent:*

(i) $\lambda_i = \lambda_j$ *whenever* $M_i \sim M_j$.

(ii) $S_k^{\simeq} = S_k^{\sim}$.

If these conditions are satisfied, we call the Riemannian metric g on M ***almost Killing****.*

Plainly, the Killing metric is almost Killing. Note that the Riemannian metric on M is automatically almost Killing if M is irreducible. More generally, if no two distinct de Rham factors of M are homothetic, then its Riemannian metric is almost Killing.

Remark 3.3.3. Since the isometry group is not affected by constant rescaling of the metric, Proposition 2.1.60 tells us that by rescaling the normalizing constants, we might gain or lose some connected components of $\widetilde{G}$, whereas G always stays the same. From this perspective, the almost Killing condition on the Riemannian metric ensures precisely that the isometry group is as large as possible, namely $\widetilde{G} \simeq [\widetilde{G}_1 \times \cdots \times \widetilde{G}_k] \rtimes S_k^{\sim}$.

Since $S_k^{\simeq}$ is a subgroup of $S_k^{\sim}$, we have an open subgroup[1]

$$\begin{aligned} \boldsymbol{\mathrm{Aut}(\mathfrak{g})_M} &= [\mathrm{Aut}(\mathfrak{g}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{g}_k)] \rtimes S_k^{\simeq} \\ &\subseteq [\mathrm{Aut}(\mathfrak{g}_1) \times \cdots \times \mathrm{Aut}(\mathfrak{g}_k)] \rtimes S_k^{\sim} = \mathrm{Aut}(\mathfrak{g}). \end{aligned}$$

By construction, these groups coincide if and only if the metric is almost Killing. Note that the definition of $\mathrm{Aut}(\mathfrak{g})_M$ does not depend on the choice of f_{ij}'s used to construct the embedding $S_k^{\sim} \hookrightarrow \mathrm{Aut}(\mathfrak{g})$, although the embedding itself surely does. By passing

[1] It may be preferable to write $\mathrm{Aut}(\mathfrak{g})_{(M,g)}$ to avoid ambiguity, especially in the presence of another metric $\widetilde{g}$ obtained from g by rescaling the normalizing constants.

from $\mathrm{Aut}(\mathfrak{g})$ to $\mathrm{Aut}(\mathfrak{g})_M$, we prohibit those automorphisms of $\mathfrak{g}$ that permute isomorphic simple ideals whose corresponding normalizing constants do not coincide.

We are now in a position to prove the following result, which relates the isometry group of M with the automorphism group of $\mathfrak{g}$:

Proposition 3.3.4. *Let M be a symmetric space of noncompact type. The adjoint map $\mathrm{Ad}\colon \widetilde{G} \to \mathrm{Aut}(\mathfrak{g})$ is an open embedding of Lie groups with image $\mathrm{Aut}(\mathfrak{g})_M$. Moreover, Ad is an isomorphism if and only if the Riemannian metric of M is almost Killing. In particular, we always have $\mathrm{Ad}\colon G \xrightarrow{\sim} \mathrm{Inn}(\mathfrak{g})$.*

Proof. To begin with, observe that Ad is a local isomorphism. Indeed, its induced morphism of Lie algebras is $\mathrm{ad}\colon \mathfrak{g} \xrightarrow{\sim} \mathrm{Der}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$.

Next we prove that Ad is injective. Assume that $\varphi \in \mathrm{Ker}(\mathrm{Ad})$. We first show that $\varphi(o) = o$. We have $(C_\varphi)_* = \mathrm{Id}_\mathfrak{g}$, i.e., $C_\varphi\big|_G = \mathrm{Id}_G$, which is the same as to say that φ commutes with every element of G. In particular, it commutes with every element of K, which implies that K stabilizes $\varphi(o)$. Suppose $\varphi(o) \neq o$. Since the exponential map of M at a point is a diffeomorphism, K must fix every point of a geodesic γ emanating from o and passing through $\varphi(o)$. Let $v = \dot\gamma(0) \in T_oM$. We see that v is an invariant of the restricted isotropy representation of M at o. But M does not have a flat local factor by Proposition 2.1.95, so the only such invariant is 0. We deduce that $\varphi \in K$. But then $d\varphi_o = \mathrm{Ad}(\varphi)\big|_\mathfrak{p} = \mathrm{Id}_\mathfrak{p}$, so $\varphi = e$ by Proposition 2.1.1(c). This, together with the previous paragraph, implies that Ad embeds $\widetilde{G}$ into $\mathrm{Aut}(\mathfrak{g})$ as an open subgroup.

We are left to prove that $\mathrm{Im}(\mathrm{Ad}) = \mathrm{Aut}(\mathfrak{g})_M$. Fix $o \in M$ and consider the subgroup $\mathrm{Aut}(\mathfrak{g})^\theta \subseteq \mathrm{Aut}(\mathfrak{g})$, where $\theta = \mathrm{Ad}(s_o)$. We want to show that $\mathrm{Aut}(\mathfrak{g})^\theta$ intersects every connected component of $\mathrm{Aut}(\mathfrak{g})$. Take any $\eta \in \mathrm{Aut}(\mathfrak{g})$. The automorphism $\eta\theta\eta^{-1}$ is also a Cartan involution. Since all Cartan involutions are conjugate by inner automorphisms, there exists $\delta \in \mathrm{Inn}(\mathfrak{g}) = \mathrm{Aut}^0(\mathfrak{g})$ such that $\delta\eta\theta\eta^{-1}\delta^{-1} = \theta$, i.e., $\delta\eta \in \mathrm{Aut}(\mathfrak{g})^\theta$. As $\delta\eta$ and η lie in the same connected component of $\mathrm{Aut}(\mathfrak{g})$, we are done.

It then suffices to show that $\mathrm{Im}(\mathrm{Ad})$ contains

$$\mathbf{Aut}(\mathfrak{g})^\theta_M = \mathrm{Aut}(\mathfrak{g})^\theta \cap \mathrm{Aut}(\mathfrak{g})_M.$$

Take any element η of this subgroup. It preserves the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, so we can write $T = \eta\big|_\mathfrak{p} \in \mathrm{GL}(\mathfrak{p}) \cong \mathrm{GL}(T_oM)$. We claim that T lies in the linear isotropy subgroup at o. Thanks to Proposition 2.1.68, it suffices to show that it is orthogonal and preserves the curvature tensor. As an automorphism of $\mathfrak{g}$, η is orthogonal with respect to B, so T is orthogonal with respect to $\langle - | - \rangle_B$. By construction, for every $i \in \{1, \dots, k\}$, if we write $T(\mathfrak{p}_i) = \mathfrak{p}_j$, then $\lambda_i = \lambda_j$, so $T\big|_{\mathfrak{p}_i} : \mathfrak{p}_i \xrightarrow{\sim} \mathfrak{p}_j$ is an isometry with respect to the inner products $(g_i)_{o_i}$ and $(g_j)_{o_j}$, which implies that T is orthogonal with respect to g_o as well. The fact that T preserves the curvature tensor at o follows immediately from (2.1.5). We deduce that there exists $k \in \widetilde{K}$ such that $\mathrm{Ad}(k)$ and η coincide on $\mathfrak{p}$. Since they are both Lie algebra automorphisms, they have to also coincide on $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ and thus on the whole $\mathfrak{g}$ by Proposition 2.1.97(c). $\square$

Looking at the above proof, we derive the following isotropy version of Proposition 3.3.4:

Corollary 3.3.5. *In the notation of Proposition 3.3.4, the map* $\mathrm{Ad}_{\widetilde{G}}\colon \widetilde{K} \hookrightarrow \mathrm{Aut}(\mathfrak{g})^{\theta}$ *is an open embedding of Lie groups with image* $\mathrm{Aut}(\mathfrak{g})^{\theta}_{M}$*. Moreover, its image is the whole* $\mathrm{Aut}(\mathfrak{g})^{\theta}$ *if and only if the Riemannian metric is almost Killing. In particular, we always have* $\mathrm{Ad}_{\widetilde{G}}\colon K \xrightarrow{\sim} \mathrm{Inn}(\mathfrak{g})^{\theta}$ *and* $\widetilde{K} = \widetilde{G}^{\Theta}$*.*

Proof. We need only prove the very last assertion. We compute:

$$\begin{aligned}
\mathrm{Ad}_{\widetilde{G}}(\widetilde{G}^{\Theta}) &= \mathrm{Ad}(\widetilde{G})^{\theta} && (\mathrm{Ad}_{\widetilde{G}} \circ \Theta = C_{\theta} \circ \mathrm{Ad}_{\widetilde{G}})\\
&= \mathrm{Aut}(\mathfrak{g})_M \cap \mathrm{Aut}(\mathfrak{g})^{\theta} && (\text{by Proposition 3.3.4})\\
&= \mathrm{Aut}(\mathfrak{g})^{\theta}_M\\
&= \mathrm{Ad}_{\widetilde{G}}(\widetilde{K}) && (\text{by the first assertion}),
\end{aligned}$$

so $\widetilde{G}^{\Theta} = \widetilde{K}$. □

We can use Proposition 3.3.4 to prove the following simple result, which tells how one can recover M from its isometry Lie algebra $\mathfrak{g}$ in an invariant fashion. Define $\mathcal{C} \subseteq \mathrm{Aut}(\mathfrak{g})$ to be the set of Cartan involutions on $\mathfrak{g}$. It is an immersed submanifold of $\mathrm{Aut}(\mathfrak{g})$, as it is an orbit of the adjoint action of $\mathrm{Aut}(\mathfrak{g})$ on itself (one can easily check it is a closed orbit).

Proposition 3.3.6. *The map* $\Xi\colon M \to \mathrm{Aut}(\mathfrak{g}), p \mapsto \mathrm{Ad}(s_p)$*, establishes a diffeomorphism between* M *and* $\mathcal{C}$*.*

This proposition essentially means that for noncompact symmetric spaces, choosing a base point $o \in M$ is *the same* as fixing a Cartan decomposition of $\mathfrak{g}$. Note that we do not assume the Riemannian metric to be almost Killing here.

Proof. We know that each $\mathrm{Ad}(s_p)$ is indeed a Cartan involution. If we identify $\widetilde{G}$ with a subgroup of $\mathrm{Aut}(\mathfrak{g})$ by means of Ad, Ξ is easily seen to be $\widetilde{G}$-equivariant. As both M and $\mathcal{C}$ are smooth homogeneous $\widetilde{G}$-spaces, Ξ is smooth and surjective. Since we already know that Ad is injective, it suffices to show that the map $M \to \widetilde{G}, p \mapsto s_p$, is injective. This follows from the fact that $\exp_p\colon T_pM \to M$ is a diffeomorphism and thus p is the only[1] fixed point of the geodesic symmetry s_p. □

Corollary 3.3.7. *Let* $\mathfrak{g}$ *be a real semisimple Lie algebra. For any Cartan involution* θ *on* $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})^{\theta}$ *is a maximal compact subgroup of* $\mathrm{Aut}(\mathfrak{g})$*.*

Proof. The group $\mathrm{Aut}(\mathfrak{g})$ acts transitively on the space $\mathcal{C}$ of Cartan involutions by conjugations, and $\mathrm{Aut}(\mathfrak{g})^{\theta}$ is precisely the stabilizer of θ under this action. By Proposition 3.3.6, $\mathcal{C}$ is diffeomorphic to a symmetric space of noncompact type (represented by $(\mathfrak{g}, \theta)$) and hence contractible. The group $\mathrm{Aut}(\mathfrak{g})^{\theta}$ is compact as a closed subgroup of $\mathrm{O}_{B_{\theta}}(\mathfrak{g})$. The assertion then follows from [Ant12]. □

Eventually, we can reformulate the results of Section 3.2 in the language of symmetric spaces. We let M be any symmetric space of noncompact type and use the notation

[1] As we mentioned in Example 2.2.27, this property does not hold at all for symmetric spaces of compact type.

established at the beginning of this section. In a similar fashion to $\mathrm{Aut}(\mathfrak{g})_M$, we can define:

$$\mathbf{Aut^w(\Sigma)_M} := \prod_{i=1}^{k} \mathrm{Aut}(\Sigma_i) \rtimes S_k^{\simeq} \subseteq \prod_{i=1}^{k} \mathrm{Aut}(\Sigma_i) \rtimes S_k^{\sim} \simeq \mathrm{Aut^w}(\Sigma),$$
$$\mathbf{Aut^w(DD)_M} := \prod_{i=1}^{k} \mathrm{Aut}(\mathrm{DD}_i) \rtimes S_k^{\simeq} \subseteq \prod_{i=1}^{k} \mathrm{Aut}(\mathrm{DD}_i) \rtimes S_k^{\sim} \simeq \mathrm{Aut^w}(\mathrm{DD}).$$

We are implicitly using Theorem 3.2.10(b) here by writing $\mathrm{Aut}(\Sigma_i)$ and $\mathrm{Aut}(\mathrm{DD}_i)$ instead of $\mathrm{Aut^w}(\Sigma_i)$ and $\mathrm{Aut^w}(\mathrm{DD}_i)$, respectively.

Just as $\mathrm{Aut}(\mathfrak{g})_M$ can be described as $\mathrm{Im}(\mathrm{Ad}_{\widetilde{G}})$ thanks to Proposition 3.3.4, the groups $\mathrm{Aut^w}(\Sigma)_M$ and $\mathrm{Aut^w}(\mathrm{DD})_M$ allow a neat alternative description as well. Indeed, note that we could endow $\mathfrak{a}^*$ with an alternative inner product by considering $g_o\big|_{\mathfrak{a}\times\mathfrak{a}}$ and carrying it to $\mathfrak{a}^*$ along the induced isomorphism $\mathfrak{a} \xrightarrow{\sim} \mathfrak{a}^*$. Let us denote the corresponding orthogonal group by $\mathrm{O}_{g_o}(\mathfrak{a}^*)$. It follows by a straightforward computation that:

$$\mathrm{Aut^w}(\Sigma)_M = \mathrm{Aut^w}(\Sigma) \cap \mathrm{O}_{g_o}(\mathfrak{a}^*),$$
$$\mathrm{Aut^w}(\mathrm{DD})_M = \mathrm{Aut^w}(\mathrm{DD}) \cap \mathrm{O}_{g_o}(\mathfrak{a}^*).$$

Theorem 3.2.10 together with Proposition 3.2.13 imply that:

$$\begin{aligned}
&\mathrm{W}(\Sigma) \subseteq \mathrm{Aut^w}(\Sigma)_M,\\
&\mathrm{Aut^w}(\Sigma)_M = \mathrm{W}(\Sigma) \rtimes \mathrm{Aut^w}(\mathrm{DD})_M,\\
&\Omega(N_{\mathrm{Aut}(\mathfrak{g})_M^{\theta}}(\mathfrak{a})) = \mathrm{Aut^w}(\Sigma)_M,\\
&\Omega(N_{\mathrm{Aut}(\mathfrak{g})_M^{\theta}}(\mathfrak{n})) = \mathrm{Aut^w}(\mathrm{DD})_M.
\end{aligned}$$

Consider the adjoint representation of $\widetilde{K}$ on $\mathfrak{g}$ and the normalizer $N_{\widetilde{K}}(\mathfrak{a})$ together with its subgroups $N_K(\mathfrak{a})$ and $N_{\widetilde{K}}(\mathfrak{n})$. It easily follows from Corollary 3.3.5 that:

$$\begin{aligned}
\mathrm{Ad}(N_{\widetilde{K}}(\mathfrak{a})) &= N_{\mathrm{Aut}(\mathfrak{g})_M^{\theta}}(\mathfrak{a}),\\
\mathrm{Ad}(N_K(\mathfrak{a})) &= N_{\mathrm{Inn}(\mathfrak{g})^{\theta}}(\mathfrak{a}),\\
\mathrm{Ad}(N_{\widetilde{K}}(\mathfrak{n})) &= N_{\mathrm{Aut}(\mathfrak{g})_M^{\theta}}(\mathfrak{n}).
\end{aligned} \tag{3.3.1}$$

We arrive at the following result, which can be regarded as the geometric version of Theorem 3.2.10(a):

Corollary 3.3.8. *Let M be a symmetric space of noncompact type. For every $f \in \mathrm{Aut^w}(\Sigma)_M$, there exists an isometry $k \in N_{\widetilde{K}}(\mathfrak{a})$ such that $\mathrm{Ad}(k)\big|_{\mathfrak{a}}^* = f$. If $f \in \mathrm{Aut^w}(\mathrm{DD})_M$, then k necessarily lies in $N_{\widetilde{K}}(\mathfrak{n})$, and if $f \in \mathrm{W}(\Sigma)$, k can be chosen in $N_K(\mathfrak{a})$.*

We finish off with the following application. Corollary 3.3.8 proves especially useful when studying the congruence problem on symmetric spaces of noncompact type. Among other things, it can be applied to boundary components—a class of submanifolds of particular salience in this thesis.

Proposition 3.3.9. *Let M be a symmetric space of noncompact type. Let $\Phi_1, \Phi_2 \subseteq \Lambda$,* *and assume that there exists* $s \in \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$ *such that* $s(\Phi_1) = \Phi_2$. *Then the boundary components* B_{Φ_1} *and* B_{Φ_2} *are congruent.*

Proof. According to Corollary 3.3.8, there exists some $k \in N_{\widetilde{K}}(\mathfrak{n})$ such that $\Omega(\mathrm{Ad}(k)) = (\mathrm{Ad}(k)\big|_{\mathfrak{a}}{}^{*})^{-1} = s$. Since $\mathrm{Ad}(k)(\mathfrak{g}_{\alpha}) = \mathfrak{g}_{\Omega(\mathrm{Ad}(k))(\alpha)} = \mathfrak{g}_{s(\alpha)}$, we must have $\mathrm{Ad}(k)(\mathfrak{g}_{\Phi_1}) = \mathfrak{g}_{\Phi_2}$, which implies $kG_{\Phi_1}k^{-1} = G_{\Phi_2}$ and thus $k(B_{\Phi_1}) = B_{\Phi_2}$. This completes the proof. □

Chapter 4

HOMOGENEOUS CODIMENSION-ONE FOLIATIONS ON REDUCIBLE NONCOMPACT SYMMETRIC SPACES

By their very definition, homogeneous hypersurfaces are intimately related to isometric cohomogeneity-one actions (see Proposition 2.3.41 for a precise statement). This means that, for the most part, the objective of this thesis reduces to studying cohomogeneity-one actions on symmetric spaces and classifying them up to orbit-equivalence. Among those, there is a special subclass consisting of actions without singular orbits, which are essentially the same as homogeneous codimension-one foliations. Recall that every such foliation on a symmetric space is hyperpolar and, in particular, polar (see Corollary 2.3.37 and Remark 2.3.38). It is known that irreducible symmetric spaces of compact type admit no such nontrivial foliations (see, e.g., [PT99, Lem. 1A.2]), so we confine our attention to symmetric spaces of noncompact type.

As a motivation, consider the real hyperbolic plane $\mathbb{R}H^2$. There are two obvious examples of homogeneous codimensions-one foliations on $\mathbb{R}H^2$: by a family of geodesics all asymptotic to each other, and by horocycles all perpendicular to the same fixed geodesic. In the upper half-plane model, they can be given as the foliations by vertical and horizontal lines, respectively. If $I^0(\mathbb{R}H^2) = KAN$ stands for an Iwasawa decomposition, then these foliations can be described as the orbit foliations of A and N, respectively. It is relatively straightforward to generalize this construction to any hyperbolic space $M = \mathbb{F}H^n$: if $I^0(M) = KAN$ is an Iwasawa decomposition, then the orbits of N form a homogeneous codimension-one foliation by horospheres. As for the other foliation, write $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$ and let $\ell_\alpha \subseteq \mathfrak{g}_\alpha$ be any one-dimensional subspace. Then $\mathfrak{s}_\alpha = \mathfrak{s} \ominus \ell_\alpha$ is a subalgebra of $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, and its corresponding Lie subgroup S_α gives rise to a homogeneous codimension-one foliation on M, which does not depend on the choice of ℓ_α up to congruence.

In [BT03], Berndt and Tamaru studied homogeneous codimension-one foliations on symmetric spaces of noncompact type. It follows from their work that, up to congruence, the only two such foliations on a hyperbolic space $\mathbb{F}H^n$ are the orbit foliations of N and S_α. But they actually did much more: for every noncompact symmetric space M, they construct a homogeneous codimension-one foliation for every line $\ell \subseteq \mathfrak{a}$, and one for every simple root $\alpha_i \in \Lambda$—these are denoted as $\mathcal{F}_\ell$ and $\mathcal{F}_{\alpha_i}$, respectively. For M irreducible, they showed that these foliations exhaust the list of all homogeneous codimension-one foliations on M up to congruence. Moreover, they proved that two foliations $\mathcal{F}_\ell, \mathcal{F}_{\ell'}$ (resp., $\mathcal{F}_{\alpha_i}, \mathcal{F}_{\alpha_j}$) are congruent if and only if ℓ and ℓ' (resp., α_i and α_j) differ by an automorphism of the Dynkin diagram of M, whereas $\mathcal{F}_\ell$ and $\mathcal{F}_{\alpha_i}$ are never mutually congruent. (Recall from Chapter 3 that $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$ acts naturally on $\mathfrak{a}^*$ and thus on $\mathfrak{a}$.)

Building on that work, Berndt, Tamaru, and Díaz-Ramos obtained a classification result for the much wider class of homogeneous hyperpolar foliations on all noncompact symmetric spaces in [BDRT10]. When applied to the codimension-one case, their findings imply that the first part of the above classification holds true for M reducible: $\mathbb{P}(\mathfrak{a}) \sqcup \Lambda$ is still a parameter space for all homogeneous codimension-one foliations on M. In the previous chapter, we learned that the group responsible for the diagram-induced congruence on M is $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$. It is then natural to ask whether the moduli space of homogeneous codimension-one foliations on M is given by $(\mathbb{P}(\mathfrak{a}) \sqcup \Lambda)/\,\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$. The main result of this chapter is Theorem 3, in which we verify this hypothesis in case the Riemannian metric on M is almost Killing. The chapter, which is based on the author's preprint [Sol21], is structured as follows:

- In Section 4.1, we introduce the works of Berndt, Tamaru, and Díaz-Ramos on codimension-one and hyperpolar homogeneous foliations. We adopt a slightly more abstract approach in order to formulate the results in the true moduli space spirit.
- In Section 4.2, we prove the main theorem.

4.1. Homogeneous hyperpolar and C1-foliations

Chronologically, the classification of homogeneous C1-foliations ([BT03]) precedes and underpins that of homogeneous hyperpolar foliations ([BDRT10]). However, we are going to present these results in an anachronistic manner, as it makes the exposition clearer. In [BDRT10], the authors devised a method for constructing homogeneous hyperpolar foliations on an arbitrary symmetric space of noncompact type from those on Euclidean and hyperbolic spaces. The starting point of their construction is a special type of boundary component that splits as a product of hyperbolic spaces. Let M be a symmetric space of noncompact type represented by its canonical Riemannian symmetric pair (G, K). Writing θ for the Cartan involution of $\mathfrak{g}$ at $o \in M$, let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and $\Sigma \subset \mathfrak{a}^*$ the restricted root system. Pick a Weyl chamber D and let Λ be the set of simple roots.

Definition 4.1.1. A subset $\Phi \subseteq \Lambda$ is called **orthogonal** if all roots in it are pairwise orthogonal or, in other words, no two roots in Φ are connected by an edge in the Dynkin diagram DD.

Fix an orthogonal subset $\Phi \subseteq \Lambda$. For any $\alpha, \beta \in \Phi, \alpha \neq \beta$, and any $k, l \neq 0$, we have $\mathfrak{g}_{k\alpha + l\beta} = \{0\}$ and hence $[\mathfrak{g}_{k\alpha}, \mathfrak{g}_{l\beta}] = \{0\}$. Since $\mathfrak{g}'_\Phi$ is generated by $\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha$, we have a direct sum decomposition

$$\mathfrak{g}'_\Phi = \bigoplus_{\alpha \in \Phi} \mathfrak{g}'_{\{\alpha\}}.$$

Consequently, the multiplication map $\prod_{\alpha \in \Phi} \widetilde{G}_{\{\alpha\}} \twoheadrightarrow \widetilde{G}_\Phi$ is a local isomorphism, and it passes to a Riemannian covering $\prod_{\alpha \in \Phi} B_{\{\alpha\}} \twoheadrightarrow B_\Phi$. But B_Φ is simply connected, so we get an isometric decomposition

$$B_\Phi \cong \prod_{\alpha \in \Phi} B_{\{\alpha\}}.$$

Each $B_{\{\alpha\}}$ is a symmetric space of noncompact type and rank 1 and is thus isometric to a hyperbolic space over $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. Recall that we have a horospherical decomposition

$$M \cong B_\Phi \times A_\Phi \times N_\Phi \cong \left(\prod_{\alpha\in\Phi} B_{\{\alpha\}}\right) \times A_\Phi \times N_\Phi. \tag{4.1.1}$$

Every nontrivial homogeneous hyperpolar[1] foliation on $B_{\{\alpha\}}$ is of codimension one by Corollary 2.3.29. As $B_{\{\alpha\}}$ is irreducible, the classification of homogeneous codimension-one foliations obtained in [BT03] (to be formulated below, see Theorem 4.1.14) tells us that there are exactly two such foliations on this space up to congruence: one of them is a foliation by horospheres all congruent to each other, while the other, whose congruence class we denote by $\left[\mathcal{F}_{\{\alpha\}}\right]$, has a unique minimal leaf. On the other hand, the orbit $A_\Phi \cdot o$ is a flat of dimension $r - r_\Phi$, so it is isometric to $\mathbb{E}^{r-r_\Phi}$. We can also think of it as the Euclidean space $\mathfrak{a}_\Phi$ endowed with the inner product $g_o\big|_{\mathfrak{a}_\Phi\times\mathfrak{a}_\Phi}$ or the abelian group A_Φ endowed with the left-invariant metric corresponding to this inner product on $\mathfrak{a}_\Phi$. We have the isometries $\exp_{A_\Phi} : \mathfrak{a}_\Phi \xrightarrow{\sim} A_\Phi$ and $A_\Phi \xrightarrow{\sim} A_\Phi \cdot o$, $g \mapsto g \cdot o$. Every homogeneous hyperpolar foliation on $\mathfrak{a}_\Phi$ (which is the same as polar in this case) is a foliation by affine subspaces parallel to a fixed linear subspace $V \subseteq \mathfrak{a}_\Phi$: $\mathcal{F}_V = \left\{x + V \mid x \in V^\perp \subseteq \mathfrak{a}_\Phi\right\}$. We denote the corresponding foliation on A_Φ (or $A_\Phi \cdot o$) by the same symbol $\mathcal{F}_V$.

Going back to the horospherical decomposition (4.1.1), pick a representative $\mathcal{F}_{\{\alpha\}}$ in each class $\left[\mathcal{F}_{\{\alpha\}}\right]$ and consider the product foliation

$$\left(\prod_{\alpha\in\Phi} \mathcal{F}_{\{\alpha\}}\right) \times \mathcal{F}_V \times N_\Phi \tag{4.1.2}$$

on M, where the last factor is simply the trivial foliation on N_Φ consisting of just one leaf. It is not hard to show that the congruence class of this foliation does not depend on the choice of representatives in the classes $\left[\mathcal{F}_{\{\alpha\}}\right]$. We will denote the congruence class of (4.1.2) by $[\boldsymbol{\mathcal{F}_{\Phi,V}}]$. The main result of [BDRT10] is:

Theorem 4.1.2. *Let M be a symmetric space of noncompact type whose Riemannian metric is Killing. Then:*

(a) *For every orthogonal subset $\Phi \subseteq \Lambda$ and linear subspace $V \subseteq \mathfrak{a}_\Phi$, $\mathcal{F}_{\Phi,V}$ is a homogeneous hyperpolar foliation on M.*

(b) *Every homogeneous hyperpolar foliation on M lies in $[\mathcal{F}_{\Phi,V}]$ for some choice of $o \in M, \mathfrak{a} \subset \mathfrak{p}, \Sigma^+ \subset \Sigma$, an orthogonal subset $\Phi \subseteq \Lambda$, and a linear subspace $V \subseteq \mathfrak{a}_\Phi$.*

Before we build a Lie subgroup of G whose orbit foliation represents $[\mathcal{F}_{\Phi,V}]$, it is useful to introduce the following general construction. Let $\mathfrak{h} \subseteq \mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$ be any Lie subalgebra, and let $H \subseteq AN$ be the corresponding connected Lie subgroup. (Here the Riemannian symmetric pair (G, K) is allowed to be almost effective, although this does not add any generality.) As we mentioned in Subsection 2.4.2, the exponential map of AN is a diffeomorphism. It means that this map identifies $\mathfrak{h}$ and H, and hence H is a closed subgroup. This argument shows that every connected Lie subgroup of AN is closed. The action of H on M does not have singular orbits. Indeed, under the diffeomorphism

[1] Recall that we defined polar actions to be proper by default, so a homogeneous polar foliation has all its leaves properly embedded.

$AN \xrightarrow{\sim} M$, the orbits of H in M simply become its right cosets in AN. As a result, the orbits of H in M form a homogeneous foliation with properly embedded leaves. By Proposition 2.3.17, every orbit is principal. This type of foliations plays a significant role in the theory of symmetric spaces of noncompact type, so it deserves a special name.

Definition 4.1.3. Let M be a symmetric space of noncompact type. A foliation on M is called **standard** if it is the orbit foliation of some connected Lie subgroup of AN for some choice of $o \in M$, $\mathfrak{a} \subseteq \mathfrak{p}$, and $\Sigma^+ \subseteq \Sigma$. If such Lie subgroup $H \subseteq AN$ is set, the leaf $H \cdot o$ is called a **base leaf**.

By design, a standard foliation is homogeneous, has all its leaves properly embedded, and is the orbit foliation of a solvable group of isometries. Notice that for a given standard foliation, there might be more than one choice of $o, \mathfrak{a}$, and Σ^+ in Definition 4.1.3. In particular, a base leaf might not be unique.

Now we can return to Theorem 4.1.2. Pick an r_Φ-dimensional linear subspace $\ell_\Phi \subseteq \mathfrak{g}$ (recall that r_Φ stands for $|\Phi|$) such that $\dim(\ell_\Phi \cap \mathfrak{g}_\alpha) = 1$ for each $\alpha \in \Phi$ and consider the subspace

$$\boldsymbol{\mathfrak{s}_{\Phi,V}} = (\mathfrak{a}^\Phi \oplus V) \oplus (\mathfrak{n} \ominus \ell_\Phi) \subseteq \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{s}. \tag{4.1.3}$$

One can easily check that this is a Lie subalgebra. The corresponding connected closed Lie subgroup $\boldsymbol{S_{\Phi,V}} \subset G$ induces a standard foliation on M. This foliation turns out to be hyperpolar and lies in $[\mathcal{F}_{\Phi,V}]$. For this reason, we will denote the orbit foliation of $S_{\Phi,V}$ by $\boldsymbol{\mathcal{F}_{\Phi,V}}$. In this notation, the choice of ℓ_Φ is implicit, but it is also not really important[1]: a different choice would lead to a subgroup congruent to $S_{\Phi,V}$ by an inner isometry and thus to a strongly congruent foliation (we will essentially show this as part of Lemma 4.1.9 below). This argument shows that $[\mathcal{F}_{\Phi,V}]$ can be represented by a standard foliation. Below we will see that the property of being standard is preserved by congruence, so every foliation in $[\mathcal{F}_{\Phi,V}]$ is standard.

We will now show that the assumption on the Riemannian metric in Theorem 4.1.2 can be removed. The proof is rather straightforward but does require one to keep track of notation, which makes it a bit cumbersome.

Proposition 4.1.4. *The Killing assumption on the Riemannian metric in Theorem 4.1.2 is redundant.*

Proof. Let M be a symmetric space of noncompact type whose metric g is not necessarily Killing. First, we verify that part (a) of Theorem 4.1.2 holds true for (M, g) by showing that rescaling the normalizing constants does not change the set of homogeneous hyperpolar foliations on M. Let g' be a metric obtained from g by some rescaling of the normalizing constants. As we know from Proposition 3.3.4, $I(M, g)$ and $I(M, g')$ are both naturally open subgroups of $\operatorname{Aut}(\mathfrak{g})$. In particular, they share the same identity component $G = I^0(M) \cong \operatorname{Inn}(\mathfrak{g})$. Let $H \subseteq I(M, g)$ be a closed connected subgroup inducing a homogeneous hyperpolar foliation $\mathcal{F}$ on (M, g). As $H \subseteq I(M, g')$, $\mathcal{F}$ is also a homogeneous foliation on (M, g'). Rescaling the normalizing constants does not change the exponential map and hence the set of totally geodesic submanifolds. Moreover, it does not change the set of maximal flats either, since every maximal flat in (M, g) is the product of those in its de Rham factors. Altogether, we deduce that the set of flats is

[1] In a situation like this, it is common not to distinguish notationally between a congruence class of foliations and its specific representatives.

the same for (M, g) and (M, g'), which implies that sections of $\mathcal{F}$ in (M, g) are also its sections in (M, g'), and vice versa. We see that $\mathcal{F}$ is a homogeneous hyperpolar foliation on (M, g'). As we can apply this argument to the case in which g' is Killing and $\mathcal{F} = \mathcal{F}_{\Phi,V}$, part (a) readily follows.

To prove part (b), we temporarily refine the notation established above. This time, let g' be the Killing metric on M. We define two sets:

$$\begin{aligned}
\mathcal{C} = \big\{(o, \mathfrak{a}, \Sigma^+, \Phi, V, \ell_\Phi) \mid\, & o \in M, \\
& \mathfrak{a} \subset \mathfrak{p} \text{ a maximal abelian subspace}, \\
& \Sigma^+ \subset \Sigma \text{ a set of positive roots}, \\
& \Phi \subseteq \Lambda \subseteq \Sigma^+ \text{ an orthogonal subset}, \\
& V \subseteq \mathfrak{a}_\Phi \text{ a linear subspace}, \\
& \ell_\Phi \subset \mathfrak{g} \text{ s.th. } \dim(\ell_\Phi) = r_\Phi \text{ and } \dim(\ell_\Phi \cap \mathfrak{g}_\alpha) = 1\ \forall\, \alpha \in \Phi\big\}, \\
\mathcal{D} = \big\{(o, \mathfrak{a}, \Sigma^+, \mathfrak{h}) \mid\, & o \in M, \\
& \mathfrak{a} \subset \mathfrak{p} \text{ a maximal abelian subspace}, \\
& \Sigma^+ \subset \Sigma \text{ a set of positive roots}, \\
& \mathfrak{h} \subseteq \mathfrak{s} \text{ a subalgebra}\,\big\}.
\end{aligned}$$

Here $\mathfrak{p}$ is determined by o as the (-1)-eigenspace of the Cartan involution $\mathrm{Ad}(s_o)$. Similarly, the root system Σ is fully determined as soon as we fix $\mathfrak{a}$. Let us write $\mathcal{L}$ for the set of homogeneous foliations on M with properly embedded leaves, and let $\mathcal{L}_h$ be its subset of homogeneous hyperpolar foliations. As we know from the first part of the proof, the sets $\mathcal{L}$ and $\mathcal{L}_h$ do not depend on the choice of normalizing constants. The same is true for $\mathcal{C}$ and $\mathcal{D}$.

We have two maps $\mathcal{C} \xrightarrow{\iota} \mathcal{D} \xrightarrow{\vartheta} \mathcal{L}$ defined as:

$$\begin{aligned}
\iota &\colon (o, \mathfrak{a}, \Sigma^+, \Phi, V, \ell_\Phi) \mapsto (o, \mathfrak{a}, \Sigma^+, (\mathfrak{a}^\Phi \oplus V) \oplus (\mathfrak{n} \ominus \ell_\Phi)), \\
\vartheta &\colon (o, \mathfrak{a}, \Sigma^+, \mathfrak{h}) \mapsto \mathcal{F}_\mathfrak{h}.
\end{aligned}$$

Here $\mathcal{F}_\mathfrak{h}$ is the orbit foliation of the connected Lie subgroup $H \subseteq AN$ corresponding to $\mathfrak{h}$. As we know, the composition $\vartheta \circ \iota$ takes values in the subset $\mathcal{L}_h$ of $\mathcal{L}$. Let us denote this map by $\vartheta_h \colon \mathcal{C} \to \mathcal{L}_h$. The group $I(M, g') \cong \mathrm{Aut}(\mathfrak{g})$ acts on $\mathcal{L}$ in the obvious way, and this action preserves $\mathcal{L}_h$.

Lemma 4.1.5. *There exist natural actions of $I(M, g')$ on $\mathcal{C}$ and $\mathcal{D}$ with respect to which the maps ι, ϑ, and ϑ_h are $I(M, g')$-equivariant.*

Proof of the lemma. Take any $\varphi \in I(M, g')$ and $(o, \mathfrak{a}, \Sigma^+, \Phi, V, \ell_\Phi) \in \mathcal{C}$. It is easy to see that $\mathrm{Ad}(\varphi)(\mathfrak{p})$ is the (-1)-eigenspace of the Cartan involution $\mathrm{Ad}(s_{\varphi(o)})$. Moreover, $\mathrm{Ad}(\varphi)(\mathfrak{a})$ is a maximal abelian subspace of $\mathrm{Ad}(\varphi)(\mathfrak{p})$, hence we have the induced restricted root system in $\mathrm{Ad}(\varphi)(\mathfrak{a})^*$. Consider the restriction $\mathrm{Ad}(\varphi)\big|_\mathfrak{a} \colon \mathfrak{a} \xrightarrow{\sim} \mathrm{Ad}(\varphi)(\mathfrak{a})$ and write $\varphi_\mathfrak{a} = (\mathrm{Ad}(\varphi)\big|_\mathfrak{a}{}^*)^{-1} \colon \mathfrak{a}^* \xrightarrow{\sim} \mathrm{Ad}(\varphi)(\mathfrak{a})^*$. A straightforward computation shows that the restricted root system in $\mathrm{Ad}(\varphi)(\mathfrak{a})^*$ coincides with $\varphi_\mathfrak{a}(\Sigma)$. By (3.2.1), we have

$$\mathrm{Ad}(\varphi)(\mathfrak{g}_\alpha) = \mathfrak{g}_{\varphi_\mathfrak{a}(\alpha)} \text{ for each } \alpha \in \Sigma. \tag{4.1.4}$$

Plainly, $\varphi_{\mathfrak{a}}(\Sigma^+)$ is a choice of positive roots for $\varphi_{\mathfrak{a}}(\Sigma)$, and the corresponding set of simple roots is nothing but $\varphi_{\mathfrak{a}}(\Lambda)$. Furthermore, $\varphi_{\mathfrak{a}}(\Phi)$ is an orthogonal subset of $\varphi_{\mathfrak{a}}(\Lambda)$. One can check that $\mathrm{Ad}(\varphi)(\mathfrak{a})_{\varphi_{\mathfrak{a}}(\Phi)}$ coincides with $\mathrm{Ad}(\varphi)(\mathfrak{a}_\Phi)$. Finally, it follows from (4.1.4) that

$$\mathrm{Ad}(\varphi)(\ell_\Phi) \cap \mathfrak{g}_{\varphi_{\mathfrak{a}}(\alpha)} = \mathrm{Ad}(\varphi)(\ell_\Phi) \cap \mathrm{Ad}(\mathfrak{g}_\alpha) = \mathrm{Ad}(\ell_\Phi \cap \mathfrak{g}_\alpha).$$

We can thus define

$$\varphi \cdot (o, \mathfrak{a}, \Sigma^+, \Phi, V, \ell_\Phi) = (\varphi(o), \mathrm{Ad}(\varphi)(\mathfrak{a}), \varphi_{\mathfrak{a}}(\Sigma^+), \varphi_{\mathfrak{a}}(\Phi), \mathrm{Ad}(\varphi)(V), \mathrm{Ad}(\varphi)(\ell_\Phi)).$$

It is easy to check that this formula defines an action of $I(M, g')$ on $\mathcal{C}$. Its action on $\mathcal{D}$ is defined in a similar fashion. If we write $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}' = \bigoplus_{\alpha \in \varphi_{\mathfrak{a}}(\Sigma^+)} \mathfrak{g}_\alpha$, then $\mathrm{Ad}(\varphi)(\mathfrak{n}) = \mathfrak{n}'$ by virtue of (4.1.4). If $\mathfrak{h}$ is a subalgebra of $\mathfrak{s}$, then $\mathrm{Ad}(\varphi)(\mathfrak{h})$ is a subalgebra of $\mathrm{Ad}(\varphi)(\mathfrak{s}) = \mathrm{Ad}(\varphi)(\mathfrak{a}) \oplus \mathfrak{n}'$. The formula

$$\varphi \cdot (o, \mathfrak{a}, \Sigma^+, \mathfrak{h}) = (\varphi(o), \mathrm{Ad}(\varphi)(\mathfrak{a}), \varphi_{\mathfrak{a}}(\Sigma^+), \mathrm{Ad}(\varphi)(\mathfrak{h}))$$

defines an action of $I(M, g')$ on $\mathcal{D}$. Given $(o, \mathfrak{a}, \Sigma^+, \Phi, V, \ell_\Phi)$, we denote $(\mathfrak{a}^\Phi \oplus V) \oplus (\mathfrak{n} \ominus \ell_\Phi)$ temporarily as $\mathfrak{s}(o, \mathfrak{a}, \Sigma^+, \Phi, V, \ell_\Phi)$. To show that ι is $I(M, g')$-equivariant, it suffices to verify that

$$\mathrm{Ad}(\varphi)(\mathfrak{s}(o, \mathfrak{a}, \Sigma^+, \Phi, V, \ell_\Phi)) = \mathfrak{s}(\varphi \cdot (o, \mathfrak{a}, \Sigma^+, \Phi, V, \ell_\Phi)). \tag{4.1.5}$$

This follows from (4.1.4) and two simple facts. First, $\mathrm{Ad}(\varphi)(\mathfrak{a})^{\varphi_{\mathfrak{a}}(\Phi)} = \mathrm{Ad}(\varphi)(\mathfrak{a}^\Phi)$. Second, $\mathrm{Ad}(\varphi)$ is an isometry between $(\mathfrak{g}, B_\theta)$ and $(\mathfrak{g}, B_{\theta'})$, where $\theta' = \mathrm{Ad}(s_{\varphi(o)}) = \mathrm{Ad}(\varphi)\theta\,\mathrm{Ad}(\varphi)^{-1}$. Now, we show that ϑ is equivariant. Given $(o, \mathfrak{a}, \Sigma^+, \mathfrak{h})$, the foliation $\vartheta(\varphi \cdot (o, \mathfrak{a}, \Sigma^+, \mathfrak{h}))$ is the orbit foliation of the connected Lie subgroup of G with Lie algebra $\mathrm{Ad}(\varphi)(\mathfrak{h})$. On the other hand, $\varphi(\mathcal{F}_{\mathfrak{h}})$ is the orbit foliation of the group $\varphi \cdot \exp_G(\mathfrak{h}) \cdot \varphi^{-1}$, which is also the connected Lie subgroup of G with Lie algebra $\mathrm{Ad}(\varphi)(\mathfrak{h})$. The equivariance of ϑ_h follows from those of ι and ϑ. This concludes the proof of the lemma. □

The fact that ϑ_h is $I(M, g')$-equivariant means that it induces a map $\mathcal{C}/I(M, g') \to \mathcal{L}_h/I(M, g')$. Part (b) of Theorem 4.1.2 asserts that this map is surjective. But then—again, by equivariance—ϑ_h must be surjective itself. If we now restrict the actions on $\mathcal{C}$ and $\mathcal{L}_h$ to the subgroup $I(M, g) \subseteq I(M, g')$, then the induced map $\mathcal{C}/I(M, g) \to \mathcal{L}_h/I(M, g)$ has to be surjective as well, because so is ϑ_h. This means exactly that part (b) of Theorem 4.1.2 holds true for (M, g), which completes the proof. □

Corollary 4.1.6. *Let M be a symmetric space of noncompact type.*

(a) *For every orthogonal subset $\Phi \subseteq \Lambda$ and linear subspace $V \subseteq \mathfrak{a}_\Phi$, $\mathcal{F}_{\Phi,V}$ is a homogeneous hyperpolar foliation on M.*

(b) *Every homogeneous hyperpolar foliation on M is congruent to $\mathcal{F}_{\Phi,V}$ for some choice of $o \in M, \mathfrak{a} \subset \mathfrak{p}, \Sigma^+ \subset \Sigma$, an orthogonal subset $\Phi \subseteq \Lambda$, and a linear subspace $V \subseteq \mathfrak{a}_\Phi$.*

Observation 4.1.7. The subset of $\mathcal{L}$ consisting of standard foliations is precisely the image of $\vartheta\colon \mathcal{D} \to \mathcal{L}$. Since ϑ is $I(M, g')$-equivariant, its image is preserved by $I(M, g')$. In other words, if some foliation is congruent to a standard foliation, then it is itself standard. As a consequence, every homogeneous hyperpolar foliation on M is standard.

The main advantage of the formalism established in the proof of Proposition 4.1.4 is that it gives a useful moduli space perspective on homogeneous hyperpolar foliations. In order to give a less clumsy description of the moduli space $\mathcal{L}_h/I(M,g)$, we first show that the foliation $\mathcal{F}_{\Phi,V}$ does not depend on the choice of $o, \mathfrak{a}$, and Σ^+ up to congruence. Fix $o \in M, \mathfrak{a} \subset \mathfrak{p}$, and $\Sigma^+ \subset \Sigma$. Form now on, we write $\widetilde{G}$ for $I(M)$ and $\widetilde{K}$ for its isotropy subgroup at o. Consider the subset $\mathcal{C}_{o,\mathfrak{a},\Sigma^+}$ of $\mathcal{C}$ consisting of those 6-tuples whose first three components are $o, \mathfrak{a}$, and Σ^+, respectively. Although this subset is not preserved under the action $\widetilde{G} \curvearrowright \mathcal{C}$, it is preserved by the subgroup $N_{\widetilde{K}}(\mathfrak{n}) \subset \widetilde{G}$. In fact, $N_{\widetilde{K}}(\mathfrak{n})$ consists precisely of those elements of $\widetilde{G}$ that preserve $\mathcal{C}_{o,\mathfrak{a},\Sigma^+}$.

Lemma 4.1.8. *The map $\mathcal{C}_{o,\mathfrak{a},\Sigma^+}/N_{\widetilde{K}}(\mathfrak{n}) \to \mathcal{C}/\widetilde{G}$ induced by the inclusion $\mathcal{C}_{o,\mathfrak{a},\Sigma^+} \hookrightarrow \mathcal{C}$ is a bijection.*

Proof. This can be seen by consecutively applying the facts that G acts transitively on M, K acts transitively on the set of maximal abelian subspaces of $\mathfrak{p}$ (Proposition 2.4.8, Corollary 3.3.5), and $N_K(\mathfrak{a})$ acts transitively on the set of Weyl chambers of Σ (Corollary 2.4.12, (3.3.1)). □

Now we can dispense with the need to choose ℓ_Φ. Define

$$\mathcal{C}'_{o,\mathfrak{a},\Sigma^+} = \{(\Phi, V) \mid \Phi \subseteq \Lambda \text{ orthogonal subset}, V \subseteq \mathfrak{a}_\Phi \text{ linear subspace}\},$$

and let $\mathcal{C}_{o,\mathfrak{a},\Sigma^+} \twoheadrightarrow \mathcal{C}'_{o,\mathfrak{a},\Sigma^+}$ be the projection map. Recall that we have an open normal subgroup $Z_{\widetilde{K}}(\mathfrak{a}) \trianglelefteq N_{\widetilde{K}}(\mathfrak{n})$ that acts trivially on $\mathfrak{a}$ and preserves each restricted root space $\mathfrak{g}_\alpha$. The projection map is clearly constant on the orbits of the action $Z_{\widetilde{K}}(\mathfrak{a}) \curvearrowright \mathcal{C}_{o,\mathfrak{a},\Sigma^+}$.

Lemma 4.1.9. *The map $\mathcal{C}_{o,\mathfrak{a},\Sigma^+}/Z_{\widetilde{K}}(\mathfrak{a}) \twoheadrightarrow \mathcal{C}'_{o,\mathfrak{a},\Sigma^+}$ is a bijection.*

Proof. Recall that the boundary component B_Φ can be represented by an almost effective Riemannian symmetric pair (G'_Φ, K^Φ), where we temporarily denote $K^\Phi = G'_\Phi \cap K$. We have a local isomorphism $G'_\Phi \twoheadrightarrow I^0(B_\Phi)$ with finite kernel. The Cartan decomposition of $\mathfrak{g}'_\Phi$ is given by $\mathfrak{g}'_\Phi = [\mathfrak{b}_\Phi, \mathfrak{b}_\Phi] \oplus \mathfrak{b}_\Phi$. Moreover, $\mathfrak{a}^\Phi$ is a maximal abelian subspace of $\mathfrak{b}_\Phi$, so we can consider the normalizer $N_{K^\Phi}(\mathfrak{a}^\Phi)$ and its identity component $N^0_{K^\Phi}(\mathfrak{a}^\Phi) = Z^0_{K^\Phi}(\mathfrak{a}^\Phi)$. As we established at the beginning of the section, $\mathfrak{g}'_\Phi = \bigoplus_{\alpha\in\Phi} \mathfrak{g}'_{\{\alpha\}}$, and the multiplication map $\prod_{\alpha\in\Phi} G'_{\{\alpha\}} \twoheadrightarrow G'_\Phi$ is a local isomorphism. Each boundary component $B_{\{\alpha\}}$ can be represented by an almost effective Riemannian symmetric pair $(G'_{\{\alpha\}}, K^{\{\alpha\}})$. It is easy to see that the multiplication map $\prod_{\alpha\in\Phi} K^{\{\alpha\}} \twoheadrightarrow K^\Phi$ is a local isomorphism, and so is $\prod_{\alpha\in\Phi} Z^0_{K^{\{\alpha\}}}(\mathfrak{a}^{\{\alpha\}}) \twoheadrightarrow Z^0_{K^\Phi}(\mathfrak{a}^\Phi)$. Let $\ell_\Phi, \ell'_\Phi \subseteq \mathfrak{g}$ be of dimension r_Φ such that both $\ell_\Phi \cap \mathfrak{g}_\alpha$ and $\ell'_\Phi \cap \mathfrak{g}_\alpha$ are 1-dimensional for each $\alpha \in \Phi$. As it will be shown in Lemma 4.2.5 below, the adjoint action of $Z^0_{K^{\{\alpha\}}}(\mathfrak{a}^{\{\alpha\}})$ on $\mathfrak{g}_\alpha$ is of cohomogeneity one for each $\alpha \in \Phi$. It means that for every $\alpha \in \Phi$, there exists $k_\alpha \in Z^0_{K^{\{\alpha\}}}(\mathfrak{a}^{\{\alpha\}})$ such that $\mathrm{Ad}(k_\alpha)(\ell_\Phi \cap \mathfrak{g}_\alpha) = \ell'_\Phi \cap \mathfrak{g}_\alpha$. What is more, since $[\mathfrak{g}_{\{\alpha\}}, \mathfrak{g}_{\{\beta\}}] = \{0\}$ for $\alpha, \beta \in \Phi, \alpha \neq \beta$, the action of $Z^0_{K^{\{\alpha\}}}(\mathfrak{a}^{\{\alpha\}})$ on $\mathfrak{g}_\beta$ is trivial. We deduce that the image of $(k_\alpha)_{\alpha\in\Phi}$ under $\prod_{\alpha\in\Phi} Z^0_{K^{\{\alpha\}}}(\mathfrak{a}^{\{\alpha\}}) \twoheadrightarrow Z^0_{K^\Phi}(\mathfrak{a}^\Phi)$ sends ℓ_Φ onto ℓ'_Φ. But this means exactly that two points of $\mathcal{C}_{o,\mathfrak{a},\Sigma^+}$ lying in the same fiber over $\mathcal{C}'_{o,\mathfrak{a},\Sigma^+}$ must differ by some element of $Z^0_{K^\Phi}(\mathfrak{a}^\Phi)$. It remains to prove that $Z^0_{K^\Phi}(\mathfrak{a}^\Phi)$ is a subgroup of $Z_{\widetilde{K}}(\mathfrak{a})$. Since the former is connected, it suffices to show that this holds on the level of Lie algebras. But we know from Subsection 2.4.3 that those are $\mathfrak{k}_0 \cap \mathfrak{g}'_\Phi$ and $\mathfrak{k}_0$, respectively. This completes the proof of the lemma. □

Note that we have shown incidentally that $Z_{\widetilde{K}}(\mathfrak{a})$ has the same orbits on $\mathcal{C}_{o,\mathfrak{a},\Sigma^+}$ as its identity component $Z^0_{\widetilde{K}}(\mathfrak{a})$. From Lemma 4.1.9 and the fact that $N_{\widetilde{K}}(\mathfrak{n})/Z_{\widetilde{K}}(\mathfrak{a}) \cong \operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M$ (Corollary 3.2.15, (3.3.1)), it easily follows that we have an induced map $\mathcal{C}_{o,\mathfrak{a},\Sigma^+}/N_{\widetilde{K}}(\mathfrak{n}) \twoheadrightarrow \mathcal{C}'_{o,\mathfrak{a},\Sigma^+}/\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M$, and it is a bijection. Combining all of the above, we arrive at:

$$\mathcal{C}'_{o,\mathfrak{a},\Sigma^+}/\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M \cong \mathcal{C}_{o,\mathfrak{a},\Sigma^+}/N_{\widetilde{K}}(\mathfrak{n}) \xrightarrow{\sim} \mathcal{C}/\widetilde{G} \twoheadrightarrow \mathcal{L}_h/\widetilde{G}.$$

We sum up the above discussion in a separate

Corollary 4.1.10. *Let M be a symmetric space of noncompact type. Pick $o \in M, \mathfrak{a} \subseteq \mathfrak{p}$, and $\Sigma^+ \subseteq \Sigma$. Then the map*

$$\mathcal{C}'_{o,\mathfrak{a},\Sigma^+}/\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M \twoheadrightarrow \mathcal{L}_h/\widetilde{G},\ [(\Phi, V)] \mapsto [\mathcal{F}_{\Phi,V}] \tag{4.1.6}$$

is well-defined and surjective.

In other words, the moduli space $\mathcal{L}_h/\widetilde{G}$ of homogeneous hyperpolar foliations on M is a certain quotient of $\mathcal{C}'_{o,\mathfrak{a},\Sigma^+}/\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M$. Here $\mathcal{C}'_{o,\mathfrak{a},\Sigma^+}$ is a disjoint union of Grassmannians of various dimensions, and $\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M$ is a finite group acting smoothly on it. Given an automorphism $s \in \operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M \subset \operatorname{Aut}^{\mathrm{w}}(\Sigma)_M$, its action on $\mathcal{C}'_{o,\mathfrak{a},\Sigma^+}$ is given by

$$s \cdot (\Phi, V) = (s(\Phi), (s^*)^{-1}(V)).$$

Unfortunately, Corollary 4.1.10 does not give a complete classification of homogeneous hyperpolar foliations, as it does not tell whether the map $\mathcal{C}'_{o,\mathfrak{a},\Sigma^+}/\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M \twoheadrightarrow \mathcal{L}_h/\widetilde{G}$ is bijective. In other words, the foliation $\mathcal{F}_{\Phi,V}$ might in theory be congruent to $\mathcal{F}_{\Phi',V'}$ even if (Φ, V) and (Φ', V') lie in different orbits of the action of $\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M$ on $\mathcal{C}'_{o,\mathfrak{a},\Sigma^+}$. However, there are no such known examples, which naturally leads to the following

Conjecture 4.1.11. *For any symmetric space M of noncompact type, the map* (4.1.6) *is a bijection.*

Remark 4.1.12. We can also restrict the notion of equivalence to strong congruence. It is clear from the above discussion that we have a surjective map $\mathcal{C}'_{o,\mathfrak{a},\Sigma^+} \twoheadrightarrow \mathcal{L}_h/G$ given by the same formula as in (4.1.6). We can then ask whether this map is bijective, which is similar but not directly equivalent to Conjecture 4.1.11.

In general, Conjecture 4.1.11 seems far from being resolved. All the progress toward its resolution has so far been confined to proving that (4.1.6) is bijective over some specific parts of the moduli space $\mathcal{L}_h/\widetilde{G}$. For example, as part of their study of isoparametric hypersurfaces in [DVSL23], Domínguez-Vázquez and Sanmartin-López investigated the subset of $\mathcal{C}'_{o,\mathfrak{a},\Sigma^+}$ consisting of pairs $(\varnothing, V)$, $V \subseteq \mathfrak{a}$. They obtained the following partial result: if two such pairs $(\varnothing, V)$ and $(\varnothing, V')$ give rise to congruent foliations, then V and V' differ by some $\operatorname{Aut}^{\mathrm{w}}(\Sigma)_M$. In this chapter, we are interested in the part of $\mathcal{L}_h/\widetilde{G}$ consisting of proper homogeneous C1-foliations.

Agreement. For the rest of the chapter all homogeneous C1-foliations are assumed proper by default.

Recall that such foliations are hyperpolar by Corollary 2.3.37. The codimension of the model foliation $\mathcal{F}_{\Phi,V}$ can be easily calculated from its construction: $\operatorname{codim}(\mathcal{F}_{\Phi,V}) =$

$|\Phi| + \operatorname{codim}_{\mathfrak{a}_\Phi}(V)$. Therefore, there are two types of proper homogeneous C1-foliations:

(a) $\Phi = \varnothing$, $V = \mathfrak{a} \ominus \ell$, where $\ell \subseteq \mathfrak{a}$ is a one-dimensional linear subspace[1]. In this case, we write $\boldsymbol{\mathcal{F}_\ell}$ instead of $\mathcal{F}_{\Phi,V}$ for simplicity.

(b) $\Phi = \{\alpha_i\}$, $V = \mathfrak{a}_\Phi$. In this case, we denote[2] $\mathcal{F}_{\Phi,V}$ by $\boldsymbol{\mathcal{F}_{\alpha_i}}$.

The subalgebra $\mathfrak{s}_{\Phi,V}$ (resp., subgroup $S_{\Phi,V}$) in this case will be denoted simply by $\boldsymbol{\mathfrak{s}_\ell}$ (resp., $\boldsymbol{S_\ell}$) or $\boldsymbol{\mathfrak{s}_{\alpha_i}}$ (resp., $\boldsymbol{S_{\alpha_i}}$). Note that $\mathfrak{s}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ and $\mathfrak{s}_{\alpha_i} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell_i)$, where $\ell_i \subseteq \mathfrak{g}_{\alpha_i}$ is a line. Berndt and Tamaru gave a detailed geometric description of these foliations in [BT03]. The foliation $\mathcal{F}_\ell$ has all its leaves congruent to each other[3]. If ℓ lies in the closure of the positive Weyl chamber, the leaves of $\mathcal{F}_\ell$ are horospheres (see [DVSLT21, Rem. 5.4]). The foliation $\mathcal{F}_{\alpha_i}$ has a unique minimal leaf, namely the base leaf $S_{\alpha_i} \cdot o$. In particular, $\mathcal{F}_\ell$ is never congruent to $\mathcal{F}_{\alpha_i}$. What is more, there exists a congruence of $\mathcal{F}_{\alpha_i}$ with itself that preserves the base leaf and, for each $t > 0$, interchanges the two leaves at distance t from $S_{\alpha_i} \cdot o$. Any two distinct leaves of $\mathcal{F}_{\alpha_i}$ have different (constant) mean curvatures, except for the case when they are at the same distance from $S_{\alpha_i} \cdot o$.

Corollary 4.1.10 yields the following immediate

Corollary 4.1.13. *Let M be a symmetric space of noncompact type with a fixed choice of $o \in M, \mathfrak{a} \subset \mathfrak{p}$, and $\Sigma^+ \subset \Sigma$.*

(a) *For every one-dimensional subspace $\ell \subseteq \mathfrak{a}$ and simple root $\alpha_i \in \Lambda$, the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\alpha_i}$ are homogeneous and have codimension one.*

(b) *Every homogeneous codimension-one foliation on M is congruent to either $\mathcal{F}_\ell$ (for some $\ell \in \mathbb{P}\mathfrak{a}$) or $\mathcal{F}_{\alpha_i}$ (for some $\alpha_i \in \Lambda$).*

(c) *If $\ell, \ell' \in \mathbb{P}\mathfrak{a}$ (resp., $\alpha_i, \alpha_j \in \Lambda$) differ by some $s \in \operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M$, then the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ (resp., $\mathcal{F}_{\alpha_i}$ and $\mathcal{F}_{\alpha_j}$) are congruent.*

Notice that the action of $\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M$ on the set of foliations $\mathcal{F}_\ell$, $\ell \in \mathbb{P}\mathfrak{a}$, is via its representation on $\mathfrak{a}$ (dual to the one on $\mathfrak{a}^*$). In the irreducible case, the above result was originally obtained in [BT03]. They also showed the converse to part (c), thereby completing the classification of homogeneous C1-foliations for irreducible spaces:

Theorem 4.1.14. *Let M be an irreducible symmetric space of noncompact type, and assume that the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ (resp., $\mathcal{F}_{\alpha_i}$ and $\mathcal{F}_{\alpha_j}$) are congruent. Then there exists $s \in \operatorname{Aut}(\mathrm{DD})$ mapping ℓ onto ℓ' (resp., α_i to α_j). Consequently, the moduli space of proper homogeneous codimension-one foliations on M is isomorphic to*

$$(\mathbb{R}P^{r-1} \sqcup \{1, \ldots, r\}) / \operatorname{Aut}(\mathrm{DD}).$$

Here we are writing Aut(DD) because in the irreducible case every metric is almost Killing and every Dynkin diagram automorphism is weight-preserving, hence $\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M = \operatorname{Aut}(\mathrm{DD})$. This also explains why the Killing assumption on the Riemannian metric in [BT03] is not essential and can be dropped.

[1]The orthogonal complement $V = \mathfrak{a} \ominus \ell$ is taken with respect to $\langle - | - \rangle_B = B\big|_{\mathfrak{p}\times\mathfrak{p}}$, not g_o.

[2]This foliation is denoted by $\mathcal{F}_i$ in [BT03], but our notation will prove less ambiguous in the reducible case, so we stick with it.

[3]This is an example of a standard foliation whose base leaf is not unique. As a matter of fact, every leaf of $\mathcal{F}_\ell$ is a base leaf.

4.2. The congruence theorem

In this section we prove the following result, which extends Theorem 4.1.14 to the reducible case:

Theorem 4.2.1. *Let M be a symmetric space of noncompact type whose metric is almost Killing, and assume that the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ (resp., $\mathcal{F}_{\alpha_i}$ and $\mathcal{F}_{\alpha_j}$) are congruent. Then there exists $s \in \operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})$ mapping ℓ onto ℓ' (resp., α_i to α_j). Consequently, the moduli space of proper homogeneous codimension-one foliations on M is isomorphic to*

$$(\mathbb{R}P^{r-1} \sqcup \{1, \ldots, r\}) / \operatorname{Aut}^{\mathrm{w}}(\mathrm{DD}).$$

First, we deal with the discrete part of the moduli space. In this case, we actually do not need the assumption on the Riemannian metric.

Proposition 4.2.2. *Let M be a symmetric space of noncompact type. Suppose the foliations $\mathcal{F}_{\alpha_i}$ and $\mathcal{F}_{\alpha_j}$ are congruent for some $\alpha_i, \alpha_j \in \Lambda$. Then there exists $s \in \operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M$ mapping α_i to α_j.*

Proof. Let $M = M_1 \times \cdots \times M_k$ be the de Rham decomposition of M. We use the notation established at the beginning Section 3.3. In this section, we will denote the solvable group AN in the Iwasawa decomposition by S. The Lie algebra $\mathfrak{s}$ splits as $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$, hence we also have $S = S_1 \times \cdots \times S_k$. To keep track of the de Rham factors, let us write $\Lambda_l = \{\alpha_l^1, \ldots, \alpha_l^{r_l}\}$. To comply with this notation, we write α_i^p and α_j^q instead of α_i and α_j. Pick some lines $\ell_i^p \subseteq \mathfrak{g}_{\alpha_i^p}$ and $\ell_j^q \subseteq \mathfrak{g}_{\alpha_j^q}$. The foliations $[\mathcal{F}_{\alpha_i^p}]$ and $[\mathcal{F}_{\alpha_j^q}]$ are the orbit foliations of the groups $S_{\alpha_i^p}$ and $S_{\alpha_j^q}$, respectively. Observe that

$$\begin{aligned} S_{\alpha_i^p} &= S_1 \times \cdots \times S_{i,\alpha_i^p} \times \cdots \times S_k, \text{ so} \\ S_{\alpha_i^p} \cdot o &= M_1 \times \cdots \times (S_{i,\alpha_i^p} \cdot o) \times \cdots \times M_k, \end{aligned} \tag{4.2.1}$$

where S_{i,α_i^p} is the connected Lie subgroup of G_i with Lie algebra $\mathfrak{s}_{i,\alpha_i^p} = \mathfrak{s}_i \ominus \ell_i^p$. The same is true for $S_{\alpha_j^q}$ and its orbit $S_{\alpha_j^q} \cdot o$. Let $g \in \widetilde{G}$ be a congruence identifying the orbit foliations of $S_{\alpha_i^p}$ and $S_{\alpha_j^q}$ and thus their minimal leaves $S_{\alpha_i^p} \cdot o$ and $S_{\alpha_j^q} \cdot o$. Without loss of generality, we may assume that g fixes o. Looking at (4.2.1) and Proposition 2.1.60, we see that g must send $M_{i,o} = G_i \cdot o$ onto $M_{j,o} = G_j \cdot o$ and thus provide a congruence between the orbit foliations of S_{i,α_i^p} on $M_{i,o} \cong M_i$ and S_{j,α_j^q} on $M_{j,o} \cong M_j$ (in particular, M_i and M_j are isometric). It easily follows from Theorem 4.1.14 that there exists an isomorphism s between the Dynkin diagrams DD_i and DD_j sending α_i^p to α_j^q. Since M_i and M_j are irreducible, every isomorphism between their Dynkin diagrams is automatically weight-preserving by Theorem 3.2.10. These two Dynkin diagrams can be regarded as connected components of DD. Therefore, we can extend s to a weight-preserving automorphism $\tilde{s}$ of DD by letting it be s^{-1} on DD_j and the identity on all the components other than DD_i and DD_j. This gives an element of $\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})$ sending α_i^p to α_j^q. As $M_i \simeq M_j$, the normalizing constants λ_i and λ_j coincide, so $\tilde{s}$ actually lies in the subgroup $\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})_M \subseteq \operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})$, which completes the proof. □

The congruence problem for $\mathcal{F}_\ell$'s is more subtle in the reducible case, and a similar approach would not work, for ℓ does not have to be positioned nicely with respect to the de Rham decomposition. More precisely, if we write $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$, then ℓ does not have to be contained in any of the summands. Nonetheless, we have:

Proposition 4.2.3. *Let M be a symmetric space of noncompact type whose Riemannian metric is almost Killing. Suppose the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ are congruent for some $\ell, \ell' \in \mathfrak{a}$. Then there exists $s \in \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$ mapping ℓ to ℓ'.*

Combining the above two propositions with Corollary 4.1.10 yields a proof of Theorem 4.2.1. So we are only left to prove Proposition 4.2.3. Let us first discuss why this problem is more subtle and how it was tackled in [BT03].

Let $k \in \widetilde{G}$ be a congruence between $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$. Without loss of generality, we may assume that k preserves o and hence lies in $\widetilde{K}$. The primary difficulty is that k may not preserve $\mathfrak{a}$, let alone $\mathfrak{n}$, so it does not produce an element of $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}) \cong N_{\widetilde{K}}(\mathfrak{n})/Z_{\widetilde{K}}(\mathfrak{a})$ to begin with. In their original proof in the irreducible case, Berndt and Tamaru bypassed this problem by establishing an isomorphism $\mathfrak{s}_\ell \xrightarrow{\sim} \mathfrak{s}_{\ell'}$ that respects certain natural gradings on these Lie algebras. The main hindrance in their approach is that this isomorphism does not in general extend to an automorphism of $\mathfrak{g}$ and thus does not directly produce an automorphism of DD. Yet, the authors managed to prove—purely algebraically—that the existence of such an isomorphism implies that ℓ and ℓ' differ by Aut(DD) (see [BT03, pp. 9-20]). The proof is quite complicated and involves a case-by-case consideration of all possible irreducible root systems from A_r to G_2 and $(BC)_r$. We will use their ideas and extend their approach to the reducible case.

Given $\ell \subseteq \mathfrak{a}$, note that the solvable Lie algebra $\mathfrak{s}_\ell$ inherits the grading (2.4.3) from $\mathfrak{s}$:

$$\mathfrak{s}_\ell = \bigoplus_{i=0}^{m} \mathfrak{s}_\ell^i, \quad \text{where} \quad \mathfrak{s}_\ell^0 = \mathfrak{a}_\ell = \mathfrak{a} \ominus \ell, \quad \mathfrak{s}_\ell^i = \mathfrak{n}^i = \bigoplus_{\mathrm{ht}(\alpha)=i} \mathfrak{g}_\alpha.$$

In particular, $\mathfrak{s}_\ell^1 = \mathfrak{n}^1 = \bigoplus_{\alpha \in \Lambda} \mathfrak{g}_\alpha$. If we denote $L^k = \bigoplus_{i=k}^m \mathfrak{s}_\ell^i$, it follows from Proposition 2.4.9(d) that $[\mathfrak{s}_\ell, \mathfrak{s}_\ell] = L^1 = \mathfrak{n}$ and $[L^1, L^k] = L^{k+1}$ for $k \geq 0$. Note that $\mathfrak{s}_\ell$ is completely solvable[1], i.e., $\mathrm{ad}(\mathfrak{s}_\ell) \subset \mathfrak{gl}(\mathfrak{s}_\ell)$ consists of upper-triangular matrices in a suitable basis for $\mathfrak{s}_\ell$. Indeed, first take a basis for $\mathfrak{s}_\ell^m$, then for $\mathfrak{s}_\ell^{m-1}$, and so on, and then combine all these bases together. We refer to [BT03] for more details.

Proof of Proposition 4.2.3. We will be using the notation established in the proof of Proposition 4.2.2. Let ℓ and ℓ' be one-dimensional subspaces of $\mathfrak{a}$ such that the foliations $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ are congruent by some $k \in \widetilde{G}$. Since all the leaves of $\mathcal{F}_\ell$ are congruent to each other, we may assume that $k \in \widetilde{K}$. First, suppose that $\ell \subseteq \mathfrak{a}_i$ for some i. In this case, the leaf $S_\ell \cdot o$ contains $M_{l,o}$ for all $l \neq i$. By Proposition 2.1.60, $k(M_{i,o}) = M_{j,o}$ for some j, and thus the leaf $S_{\ell'} \cdot o = k(S_\ell \cdot o)$ has to contain $M_{l,o}$ for all $l \neq j$. In particular, we must have $\ell' \subseteq \mathfrak{a}_j$ and $M_i \simeq M_j$. Arguing in a similar vein as in the proof of Proposition 4.2.2, we see that there is a weight-preserving isomorphism $\mathrm{DD}_i \xrightarrow{\sim} \mathrm{DD}_j$ that extends to $s \in \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$ sending ℓ onto ℓ'. Therefore, we may assume that ℓ (and hence ℓ') does not lie in any of the $\mathfrak{a}_i$'s.

In this case, one can show that $\mathfrak{a}_\ell$ and $\mathfrak{a}_{\ell'}$ are Cartan subalgebras of $\mathfrak{s}_\ell$ and $\mathfrak{s}_{\ell'}$, respectively (see [BT03, Lem. 3.3]). Recall that both S_ℓ and $S_{\ell'}$ are connected and completely solvable (this just means that their Lie algebras are completely solvable). By means of the

[1] For complex Lie algebras, the notions of solvability and complete solvability coincide by Lie's theorem.

congruence k, we can regard both of these groups as Lie subgroups of $I(S_\ell \cdot o)$. It follows from [Ale71] that any two connected completely solvable transitive Lie groups of isometries of a connected Riemannian manifold are conjugate in the isometry group of that manifold. Therefore, S_ℓ and $S_{\ell'}$ are isomorphic, and so are $\mathfrak{s}_\ell$ and $\mathfrak{s}_{\ell'}$. Let $F\colon \mathfrak{s}_\ell \xrightarrow{\sim} \mathfrak{s}_{\ell'}$ be an isomorphism.

The first step is to adjust F to make it into a graded isomorphism. Observe that $F(\mathfrak{a}_\ell)$ is a Cartan subalgebra of $\mathfrak{s}_{\ell'}$. Every two Cartan subalgebras of a solvable Lie algebra are conjugate by an inner automorphism, so we may assume $F(\mathfrak{a}_\ell) = \mathfrak{a}_{\ell'}$. Note also that $\mathfrak{n} = [\mathfrak{s}_\ell, \mathfrak{s}_\ell] = [\mathfrak{s}_{\ell'}, \mathfrak{s}_{\ell'}]$, so $F(\mathfrak{n}) = \mathfrak{n}$. Writing X^i for the i-th graded component of a vector X, define a map $\mathfrak{s}_\ell \to \mathfrak{s}_{\ell'}, X \mapsto \sum_{i=0}^m (F(X^i))^i$. It is easy to check that this map is a graded Lie algebra isomorphism (see [BT03, Th. 3.4] for an argument). We continue to denote it by F.

The main idea of the proof is to use the Lie bracket relations between $\mathfrak{a}_\ell$ (or $\mathfrak{a}_{\ell'}$) and $\mathfrak{n}$ to show that F must permute the restricted root spaces in $\mathfrak{n}$ in a way that induces a weight-preserving automorphism of DD. For each pair $\alpha, \beta \in \Lambda, \alpha \neq \beta$, let $L_{\alpha\beta}$ stand for the hyperplane in $\mathfrak{a}$ consisting of such vectors Z that the eigenvalue of $\operatorname{ad}(Z)$ on $\mathfrak{g}_\alpha$ coincides with that on $\mathfrak{g}_\beta$. If we write $Z = \sum_{\gamma\in\Lambda} Z_\gamma H^\gamma$, then $L_{\alpha\beta} = \{Z \in \mathfrak{a} \mid Z_\alpha = Z_\beta\} = \mathfrak{a} \ominus \mathbb{R}(H_\alpha - H_\beta)$.

First, we consider the generic choice of ℓ such that $\mathfrak{a}_\ell \neq L_{\alpha\beta}$ for any pair $\alpha, \beta \in \Lambda$. In this case, $\bigcup_{\substack{\alpha,\beta\in\Lambda \\ \alpha\neq\beta}} (\mathfrak{a}_\ell \cap L_{\alpha\beta})$ is the union of finitely many hyperplanes in $\mathfrak{a}_\ell$, so its complement $\mathfrak{a}_\ell^\circ$ is open and dense in $\mathfrak{a}_\ell$. Pick any $Z \in \mathfrak{a}_\ell^\circ$. By design, all the coordinates Z_γ of Z with respect to the basis $(H^\gamma)_{\gamma\in\Lambda}$ of $\mathfrak{a}$ are pairwise distinct. Since Z_γ is the eigenvalue of $\operatorname{ad}(Z)$ on $\mathfrak{g}_\gamma$, $\operatorname{ad}(Z)$ has the maximal possible number ($= r$) of distinct eigenvalues on $\mathfrak{n}^1$ among all vectors in $\mathfrak{a}$. Now, given X in a simple root space $\mathfrak{g}_\gamma$, one has

$$[F(Z), F(X)] = F[Z, X] = F(Z_\gamma X) = Z_\gamma F(X),$$

which means that F maps the eigenspaces of $\operatorname{ad}(Z)$ in $\mathfrak{n}^1$ onto the eigenspaces of $\operatorname{ad}(F(Z))$ in $\mathfrak{n}^1$ corresponding to the same eigenvalues. Since there are r such eigenvalues, F must permute the root spaces in $\mathfrak{n}^1$. Formally, there exists a bijection $s\colon \Lambda \xrightarrow{\sim} \Lambda$ such that

$$F(\mathfrak{g}_\gamma) = \mathfrak{g}_{s(\gamma)} \ (\forall\, \gamma \in \Lambda) \quad \text{and} \quad F(Z) = \sum_{\gamma\in\Lambda} Z_\gamma H^{s(\gamma)}.$$

We see that ℓ is generic in the sense explained above if and only if ℓ' is, and we also have $F(\mathfrak{a}_\ell^\circ) = \mathfrak{a}_{\ell'}^\circ$. Since it is a permutation of a basis of $\mathfrak{a}^*$, s extends uniquely to a linear operator on $\mathfrak{a}^*$, which we denote by the same letter. Assume for a moment that $s \in \operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})$. In this case, the induced operator $(s^*)^{-1}$ on $\mathfrak{a}$ sends H^γ to $H^{s(\gamma)}$. Therefore, we have $(s^*)^{-1}(Z) = \sum_{\gamma\in\Lambda} Z_\gamma H^{s(\gamma)}$. We conclude that the restrictions of $(s^*)^{-1}$ and F to $\mathfrak{a}_\ell^\circ$ coincide. Since $\mathfrak{a}_\ell^\circ$ is dense in $\mathfrak{a}_\ell$, these two operators coincide on the whole $\mathfrak{a}_\ell$. But our assumption implies that s is orthogonal with respect to the inner product on $\mathfrak{a}^*$ induced by B, hence $(s^*)^{-1}$ is orthogonal with respect to $B\big|_{\mathfrak{a}\times\mathfrak{a}}$. This means that $(s^*)^{-1}$ sends $\ell = \mathfrak{a} \ominus \mathfrak{a}_\ell$ onto $\mathfrak{a} \ominus (s^*)^{-1}(\mathfrak{a}_\ell) = \mathfrak{a} \ominus \mathfrak{a}_{\ell'} = \ell'$, which was to be proved. So we are left to show that s does indeed lie in $\operatorname{Aut}^{\mathrm{w}}(\mathrm{DD})$.

Suppose that $\alpha, \beta \in \Lambda$ are connected by an edge in DD, i.e., the angle between them is greater than $\frac{\pi}{2}$. This is equivalent to asking that $\mathfrak{g}_{\alpha+\beta} = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \neq \{0\}$. We compute:

$$\mathfrak{g}_{s(\alpha)+s(\beta)} = [\mathfrak{g}_{s(\alpha)}, \mathfrak{g}_{s(\beta)}] = [F(\mathfrak{g}_\alpha), F(\mathfrak{g}_\beta)] = F([\mathfrak{g}_\alpha, \mathfrak{g}_\beta]) = F(\mathfrak{g}_{\alpha+\beta}) \neq \{0\}\,, \tag{4.2.2}$$

which means that $s(\alpha)$ and $s(\beta)$ are also connected by an edge in the Dynkin diagram. Applying the same argument to s^{-1}, we see that s preserves adjacency between the vertices. Consequently, s permutes the connected components of DD: there exists $\sigma \in S_k$ such that $s(\mathrm{DD}_i) = \mathrm{DD}_{\sigma(i)}$ for each $i \in \{1, \ldots, k\}$. Note that we then have $F(\mathfrak{n}_i^1) = \mathfrak{n}_{\sigma(i)}^1$ and, since $\mathfrak{n}_i^1$ generates $\mathfrak{n}_i$ for each i, $F(\mathfrak{n}_i) = \mathfrak{n}_{\sigma(i)}$. We will now show that s provides an isomorphism between DD_i and $\mathrm{DD}_{\sigma(i)}$.

To start with, note that DD_i and $\mathrm{DD}_{\sigma(i)}$ must have the same number of vertices. Also, s preserves the degrees and multiplicities of the vertices. Next, it is a standard fact from the theory of root spaces that any positive root η can be expressed as $\gamma_{l_1} + \gamma_{l_2} + \cdots + \gamma_{l_s}$, where each summand is a simple root and each partial sum $\gamma_{l_1} + \cdots + \gamma_{l_t}, t \in \{2, \ldots, t-1\}$, is also a root. This, together with a computation similar to (4.2.2), implies that a linear combination $\sum_{j=1}^{r_i} n_j \alpha_i^j$ is a root in Σ_i^+ if and only if $\sum_{j=1}^{r_i} n_j s(\alpha_i^j)$ is a root in $\Sigma_{\sigma(i)}^+$. Among other things, this implies that $|\Sigma_i| = |\Sigma_{\sigma(i)}|$ and that Σ_i is reduced if and only if $\Sigma_{\sigma(i)}$ is. By looking at the list of the Dynkin diagrams of all possible irreducible symmetric spaces of noncompact type ([BCO16, pp. 336-340]), one deduces that DD_i and $\mathrm{DD}_{\sigma(i)}$ must be weighted-isomorphic, and one such weight-preserving isomorphism is provided by s. By Proposition 3.3.1, as well as our assumption that the metric is almost Killing, $\mathrm{DD}_i \simeq \mathrm{DD}_{\sigma(i)}$ is equivalent to M_i and $M_{\sigma(i)}$ being isometric. We deduce that $s \in \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$, which completes the proof for the generic choice of ℓ.

Remark 4.2.4. To tell the diagrams B_r and C_r apart, one might need to use the fact that, if we denote the two adjacent vertices of nonequal lengths by α_{r-1} and α_r, the sum $\alpha_{r-1} + 2\alpha_r$ is a root for B_r but not for C_r.

We are left to consider the situation when $\mathfrak{a}_\ell = L_{\alpha\beta}$ for some $\alpha, \beta \in \Lambda, \alpha \neq \beta$, which simply means that ℓ is spanned by $H_\alpha - H_\beta$. Since we assume that ℓ does not lie in any of the $\mathfrak{a}_l$'s, the roots α and β must lie in different components of DD: $\alpha \in \Lambda_i, \beta \in \Lambda_j, i \neq j$. In this case, each $L_{\alpha'\beta'}$ with $(\alpha', \beta') \neq (\alpha, \beta)$ intersects $\mathfrak{a}_\ell$ by a hyperplane. Let $\mathfrak{a}_\ell^\circ \subseteq \mathfrak{a}_\ell$ stand for the complement to the union of all such hyperplanes (for all $(\alpha', \beta') \neq (\alpha, \beta)$). For every $Z \in \mathfrak{a}_\ell^\circ$, $\mathrm{ad}(Z)$ has $r-1$ distinct eigenvalues on $\mathfrak{n}^1$. The only two restricted root spaces in $\mathfrak{n}^1$ with the same eigenvalue (equal to $Z_\alpha = Z_\beta$) are $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$. As we have already seen, $\mathrm{ad}(F(Z))$ must have the same eigenvalues on $\mathfrak{n}^1$, and F must send the eigenspaces of $\mathrm{ad}(Z)$ onto the corresponding eigenspaces of $\mathrm{ad}(F(Z))$. This implies that there exist some $\alpha', \beta' \in \Lambda$ such that $\mathfrak{a}_{\ell'} = L_{\alpha'\beta'}$ and thus ℓ' is spanned by $H_{\alpha'} - H_{\beta'}$. Since ℓ' cannot lie in any of the $\mathfrak{a}_l$'s, we have $\alpha' \in \Lambda_{i'}, \beta' \in \Lambda_{j'}, i' \neq j'$. Arguing in a similar spirit to what we did in the proof of Proposition 4.2.2, note that $S_\ell \cdot o = (\prod_{l \neq i,j} M_l) \times (S_i \times S_j)_\ell \cdot o$ and $S_{\ell'} \cdot o = (\prod_{l \neq i',j'} M_l) \times (S_{i'} \times S_{j'})_{\ell'} \cdot o$. Here we write $(S_i \times S_j)_\ell$ for the connected Lie subgroup of $G_i \times G_j$ corresponding to the Lie subalgebra $(\mathfrak{s}_i \oplus \mathfrak{s}_j) \ominus \ell$, and the same for $(S_{i'} \times S_{j'})_{\ell'}$. In view of Proposition 2.1.60, we see that the congruence k between $\mathcal{F}_\ell$ and $\mathcal{F}_{\ell'}$ must send $M_{i,o}$ onto either $M_{i',o}$ or $M_{j',o}$ and $M_{j,o}$ onto the other of the two. Consequently—and slightly informally—the unordered pair (M_i, M_j) is isometric to the pair $(M_{i'}, M_{j'})$. This means that there exists an weight-preserving isomorphism of Dynkin diagrams $\mathrm{DD}_i \sqcup \mathrm{DD}_j \xrightarrow{\sim} \mathrm{DD}_{i'} \sqcup \mathrm{DD}_{j'}$, which extends easily—just like we did at the end of the proof of Proposition 4.2.2—to some $s \in \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$. We will prove that all four of the spaces $M_i, M_j, M_{i'}, M_{j'}$ have rank 1. If this is the case, then s sends $\Lambda_i \sqcup \Lambda_j = \{\alpha, \beta\}$ onto $\Lambda_{i'} \sqcup \Lambda_{j'} = \{\alpha', \beta'\}$ and thus $H_\alpha - H_\beta \in \ell$ to $\pm(H_{\alpha'} - H_{\beta'}) \in \ell'$, which will complete the proof.

It follows from our argument involving the eigenspaces of $\mathrm{ad}(Z)$ and $\mathrm{ad}(F(Z))$ that F establishes a bijection

$$\{\mathfrak{g}_{\alpha_1}, \ldots, \widehat{\mathfrak{g}}_\alpha, \ldots, \widehat{\mathfrak{g}}_\beta, \ldots, \mathfrak{g}_{\alpha_r}, \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta\} \xrightarrow{\sim} \{\mathfrak{g}_{\alpha_1}, \ldots, \widehat{\mathfrak{g}}_{\alpha'}, \ldots, \widehat{\mathfrak{g}}_{\beta'}, \ldots, \mathfrak{g}_{\alpha_r}, \mathfrak{g}_{\alpha'} \oplus \mathfrak{g}_{\beta'}\}$$

and sends the subspaces on the left isomorphically onto the corresponding subspaces on the right (here a hat over a subspace means that it is omitted from the list). Before we proceed further, we prove two Lie-theoretic lemmas of separate interest.

Recall from Observation 3.2.9 that we have an orthogonal representation of the compact Lie group $Z_{\widetilde{K}}(\mathfrak{a})$ on each restricted root space $\mathfrak{g}_\alpha$ in $\mathfrak{g}$. The identity component $Z^0_{\widetilde{K}}(\mathfrak{a}) = Z^0_K(\mathfrak{a})$ is the connected Lie subgroup of G with Lie algebra $\mathfrak{k}_0$, so we denote it by $\boldsymbol{K_0}$.

Lemma 4.2.5. *For every $\mu \in \Sigma$ of multiplicity greater than one, the adjoint representation $K_0 \to \mathrm{SO}(\mathfrak{g}_\alpha)$ is of cohomogeneity 1. In other words, K_0 (and thus $Z_{\widetilde{K}}(\mathfrak{a})$) acts transitively on the unit sphere in $\mathfrak{g}_\mu$.*

Proof. Note that this statement admits a purely algebraic reformulation: under the assumption on μ, we need to show that the representation of $Z^0_{\mathrm{Inn}(\mathfrak{g})^\theta}(\mathfrak{a})$ on $\mathfrak{g}_\alpha$ has cohomogeneity 1. If μ is a simple root or the double of a simple root, this is the content of [Hel08, Ch. III, Pr. D.2] (see p. 585 for a solution). But it is a well-known fact that every root in a root system is either a simple root or the double of one under a suitable choice of a Weyl chamber. □

Lemma 4.2.6. *Let $\mu, \nu \in \Sigma$ be any two roots such that $\mu + \nu$ is also a root (e.g., they can be simple roots connected by an edge in the Dynkin diagram). Then the pairing $\mathfrak{g}_\mu \times \mathfrak{g}_\nu \to \mathfrak{g}_{\mu+\nu}$ given by the Lie bracket is nondegenerate*[1]*. In other words, for every $X \in \mathfrak{g}_\mu$ (resp., $Y \in \mathfrak{g}_\nu$), there exists $Y \in \mathfrak{g}_\nu$ (resp., $X \in \mathfrak{g}_\mu$) such that $[X, Y] \neq 0$.*

Proof. Assume the converse: there exists $X \in \mathfrak{g}_\mu$ such that $[X, \mathfrak{g}_\nu] = \{0\}$. For each $k \in K_0$, we have $[\mathrm{Ad}(k)X, \mathfrak{g}_\nu] = [\mathrm{Ad}(k)X, \mathrm{Ad}(k)(\mathfrak{g}_\nu)] = \mathrm{Ad}(k)[X, \mathfrak{g}_\nu] = \{0\}$. But, according to Lemma 4.2.5, every vector in $\mathfrak{g}_\mu$ is a multiple of $\mathrm{Ad}(k)X$ for a suitable choice of $k \in K_0$. Therefore, $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] = \{0\}$, which contradicts the fact that $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] = \mathfrak{g}_{\mu+\nu} \neq \{0\}$ (Proposition 2.4.9(d)). □

For an alternative proof using the representation theory of $\mathfrak{sl}(2, \mathbb{C})$, see [Kna02, Lem, 7.75]. Now we can go back to the proof of the theorem. We have to consider two cases.

Case 1: $F(\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta) = \mathfrak{g}_\gamma$ for some $\gamma \neq \alpha', \beta'$. Suppose there is $\gamma' \in \Lambda$ connected to γ by an edge in DD. Note that $F^{-1}(\mathfrak{g}_{\gamma'})$ must lie within a single restricted root space, which we denote by $\mathfrak{g}_{\gamma''}, \gamma'' \in \Lambda$. By Lemma 4.2.6, for any nonzero $X \in \mathfrak{g}_\alpha$, there exists $Y \in \mathfrak{g}_{\gamma'}$ with $[F(X), Y] \neq 0$, which implies that $[X, F^{-1}(Y)] \neq 0$, hence γ'' must be connected to α in DD. But we can apply this argument to a nonzero $X \in \mathfrak{g}_\beta$ to deduce that γ'' must be connected to β as well. Since α and β lie in different connected components of DD,

[1]An analogous result in the complex semisimple case is well known and is just a reformulation of the fact that $[\mathfrak{g}_\mu, \mathfrak{g}_\nu] = \mathfrak{g}_{\mu+\nu}$, for all the root spaces are one-dimensional over $\mathbb{C}$.

we arrive at a contradiction. Consequently, the connected component of DD containing γ consists of nothing but γ. But then the same must hold for α and β: $\Lambda_i = \{\alpha\}, \Lambda_j = \{\beta\}$. Indeed, otherwise we would have γ' connected to, say, α, meaning that $[\mathfrak{g}_\alpha, \mathfrak{g}_{\gamma'}] \neq \{0\}$, whereas $F([\mathfrak{g}_\alpha, \mathfrak{g}_{\gamma'}]) = [F(\mathfrak{g}_\alpha), F(\mathfrak{g}_{\gamma'})] \subseteq [\mathfrak{g}_\gamma, F(\mathfrak{g}_{\gamma'})] = \{0\}$. We conclude that Λ_i and Λ_j are singletons, i.e., M_i and M_j have rank 1. The same considerations can be applied to F^{-1} to deduce that $M_{i'}$ and $M_{j'}$ are also of rank 1. This finishes the proof in Case 1.

Case 2: $F(\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta) = \mathfrak{g}_{\alpha'} \oplus \mathfrak{g}_{\beta'}$. It suffices to show that $F(\mathfrak{g}_\alpha)$ is one of the subspaces $\mathfrak{g}_{\alpha'}, \mathfrak{g}_{\beta'}$, while $F(\mathfrak{g}_\beta)$ is the other one. Indeed, if this is the case, then we get a permutation s of Λ, which—as we have already seen in the proof for a generic choice of ℓ—must be an element of $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})$ mapping ℓ onto ℓ'. Arguing by contradiction, we may assume, without loss of generality, that the projections of $F(\mathfrak{g}_\alpha)$ to both $\mathfrak{g}_{\alpha'}$ and $\mathfrak{g}_{\beta'}$ are nonzero. We also do not lose generality if we let the projection of $F(\mathfrak{g}_\beta)$ to $\mathfrak{g}_{\beta'}$ be nonzero. By invoking Lemma 4.2.6, we see, just as we did for γ in Case 1, that $\Lambda_{j'} = \{\beta'\}$. There might be two possibilities.

Subcase 2.1: $F(\mathfrak{g}_\beta) \subseteq \mathfrak{g}_{\beta'}$. In this case, we can write $\mathfrak{g}_\alpha = U \oplus V$, where $F(U) = \mathfrak{g}_{\alpha'}$ and $F(V) \oplus F(\mathfrak{g}_\beta) = \mathfrak{g}_{\beta'}$. Mimicking the argument from the proof in Case 1, we immediately see that $\Lambda_j = \{\beta\}$. If there exists $\gamma \in \Lambda$ connected to α in DD, then, by Lemma 4.2.6, for any nonzero $X \in V$, there exists $Y \in \mathfrak{g}_\gamma$ such that $[X, Y] \neq 0$. But then $[F(X), \mathfrak{g}_{\beta'}] \neq \{0\}$, which implies that $F(\mathfrak{g}_\gamma)$ must be a root space whose corresponding root is connected to β, which is a contradiction. Therefore, $\Lambda_i = \{\alpha\}$. Arguing in the same fashion, we deduce that $\Lambda_{i'} = \{\alpha'\}$, so all four of $M_i, M_j, M_{i'}, M_{j'}$ are of rank 1, and we are done.

Subcase 2.2: The projection of $F(\mathfrak{g}_\beta)$ to $\mathfrak{g}_{\alpha'}$ is nonzero. Just like we did for $\Lambda_{j'}$, we see that $\Lambda_{i'} = \{\alpha'\}$. Arguing in a similar manner, we deduce that $\Lambda_i = \{\alpha\}$ and $\Lambda_j = \{\beta\}$. Once again, all four of $M_i, M_j, M_{i'}, M_{j'}$ are of rank 1. This completes the proof of Subcase 2.2 and thus Proposition 4.2.3. $\square$

Chapter 5

CLASSIFICATION OF HOMOGENEOUS HYPERSURFACES IN SOME NONCOMPACT SYMMETRIC SPACES OF RANK TWO

Having dealt with the classification of homogeneous codimension-one foliations in Chapter 4, we can now proceed to those isometric cohomogeneity-one actions that have a singular orbit. Once again we focus on noncompact symmetric spaces, as an explicit classification of such actions on irreducible compact symmetric spaces was obtained by Kollross in [Kol02]. By a cruel twist of fate, classifying homogeneous objects on noncompact symmetric spaces is often much harder than in the compact case: this is because, informally speaking, noncompact semisimple Lie groups admit way more subgroups than compact ones. There does exist a general classification scheme for cohomogeneity-one actions on irreducible symmetric spaces of noncompact type due to Berndt and Tamaru; recently, it has been extended to the general reducible case by Díaz-Ramos, Domínguez-Vázquez, and Otero. We will talk about it in detail in Section 5.1. Sadly, unlike the result of Kollross, this classification does not give an explicit list of actions on such spaces—rather, it reduces the search for cohomogeneity-one actions on a given space to a certain intricate problem in the representation theory of reductive Lie groups. In order to take a glimpse into the complexity of that problem, let us first describe it in an abstract setting and then see what it looks like for complex hyperbolic spaces.

Let G be a reductive Lie group, $K \subseteq G$ a maximal compact subgroup, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a $\mathbb{Z}/2\mathbb{Z}$-grading such that $\mathfrak{k} = \mathrm{Lie}(K)$. Suppose V is a real representation of G. We are looking for subspaces $\mathfrak{v} \subseteq V$ such that a) the projection of $N_{\mathfrak{g}}(\mathfrak{v})$ to $\mathfrak{p}$ along $\mathfrak{k}$ is onto, and b) $N_K(\mathfrak{v})$ acts on $\mathfrak{v}$ with cohomogeneity one. For example, for the space $\mathbb{C}H^{n+1}$, we have $G = K = \mathrm{U}(n)$, and $V = \mathbb{C}^n$ is the tautological representation. Notice how special this situation is: the representation is particularly simple, the group G is compact, and, as a consequence, condition (a) holds vacuously for any subspace $\mathfrak{v}$. Yet, the corresponding problem leads to some nontrivial geometry. A real subspace $\mathfrak{v} \subseteq \mathbb{C}^n$ is said to have *constant Kähler angle* if, given a nonzero $v \in \mathfrak{v}$, the Euclidean angle between Iv and $\mathfrak{v}$ does not depend on the choice of $\mathfrak{v}$; in that case, that angle is called the *Kähler angle* of $\mathfrak{v}$. In [BB01], Berndt and Brück proved that a real subspace $\mathfrak{v}$ of $\mathbb{C}^n$ satisfies condition (b) above if and only if it has constant Kähler angle; they found out that such a subspace is determined by its Kähler angle and dimension up to the action of $\mathrm{U}(n)$, but such subspaces exist for every Kähler angle $\varphi \in [0, \frac{\pi}{2}]$. For any space M of rank greater than 1, the corresponding representation problem has G noncompact (hence condition (a) becomes nontrivial), and the representation V becomes much more complicated. In general, the complexity of the representation grows along with the rank of a space. To make the matters worse, there are virtually no general results on this problem that would not be case-specific. (Perhaps, the only exception is [BDV15, Prop. 3.4].)

As it stands, classifying cohomogeneity-one actions on noncompact symmetric spaces in bulk is rarely a feasible task. In most cases, one has to consider individual spaces, usually of low rank. In this chapter, we classify cohomogeneity-one actions—and thus homogeneous hypersurfaces—in the rank-2 symmetric spaces $\mathrm{SL}(3,\mathbb{H})/\mathrm{Sp}(3)$, $\mathrm{SO}(5,\mathbb{C})/\mathrm{SO}(5)$, and $\mathrm{Gr}^*(2,\mathbb{C}^{n+4})$, $n \geq 1$. In particular, we prove Theorem 4. The exposition is based on the author's article [Sol23], and the layout is as follows:

- In Section 5.1, we present the classification scheme for cohomogeneity-one actions on noncompact symmetric spaces and discuss the key steps of the proof.
- In Sections 5.2 to 5.4, we classify cohomogeneity-one actions on the aforementioned symmetric spaces.

5.1. Cohomogeneity-one actions on noncompact symmetric spaces

In this section, we present the 6 types of isometric cohomogeneity-one actions on symmetric spaces of noncompact type. Among them, we introduce two important procedures allowing one to build new cohomogeneity-one actions either by extending them from lower-rank spaces, or by means of representation theory: the canonical extension and the nilpotent construction. Then we formulate the main classification result and discuss how it yields a classification of homogeneous hypersurfaces in these spaces. After that, we suggest a representation-theoretic generalization of the nilpotent construction method. Finally, we prove a technical result pertaining to cohomogeneity-one actions on hyperbolic spaces that will allow us to formulate the primary results of the chapter more neatly. The exposition here is built upon [BT13, DRDVO23, BT04, BDV15], and we refer to these papers for proofs and further details.

5.1.1. Classes of cohomogeneity-one actions

Here we describe the 6 different types of isometric C1-actions that appear in the classification scheme.

Agreement. All actions in this chapter are assumed to be isometric and by a connected Lie group by default. Recall also that we agreed in Section 2.3 that whenever we mention cohomogeneity, the action in question is assumed to be proper.

Let $M = G/K$ be a symmetric space of noncompact type. Pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and a Weyl chamber D for the restricted root system $\Sigma \subset \mathfrak{a}^*$, and write $\Lambda = \{\alpha_1, \ldots, \alpha_r\} \subseteq \Sigma^+ \subset \Sigma$ for the corresponding set of simple roots. Recall from Proposition 2.3.43 that a C1-action on M can have either 0 or 1 singular orbit. The first two types of C1-actions are those without singular orbits. By virtue of Observation 2.3.16, the orbits of such an action form a proper homogeneous C1-foliation, the classification of which was discussed in the previous chapter.

Foliations of horospherical type

By a foliation of horospherical type we mean the foliation $\mathcal{F}_\ell$ (for some line $\ell \subseteq \mathfrak{a}$) introduced in Chapter 4. This name was coined in [DRDVO23] and comes from the fact that for some choices of ℓ, the leaves of $\mathcal{F}_\ell$ are horospheres (see the discussion before Theorem 4.1.14). Let us do a quick recap.

Let $\ell \subseteq \mathfrak{a}$ be a one-dimensional subspace. The connected Lie subgroup S_ℓ of G with Lie algebra $\mathfrak{s}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ is closed, and its action on M has cohomogeneity one and no singular orbits. In other words, its orbits form a proper homogeneous C1-foliation. Moreover, all of its orbits are congruent to each other. If $s \in \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$, then the actions of S_ℓ and $S_{s(\ell)}$ are orbit-equivalent. If the metric on M is almost Killing, the converse is true: if ℓ and ℓ' are one-dimensional subspaces of $\mathfrak{a}$ such that the actions of S_ℓ and $S_{\ell'}$ are orbit-equivalent, then there exists $s \in \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$ mapping ℓ onto ℓ'.

Foliations of solvable type

By a foliation of solvable type we mean the foliation $\mathcal{F}_{\alpha_i}$ (for some $\alpha_i \in \Lambda$) introduced in Chapter 4. This term also comes from [DRDVO23], although it is not particularly descriptive: as we know from Chapter 4, every proper homogeneous C1-foliation on M is the orbit foliation of a solvable group of isometries. Here is a brief summary of the construction and classification of these foliations.

Let $\alpha_i \in \Lambda$ be any simple root and $\ell_{\alpha_i} \subseteq \mathfrak{g}_{\alpha_i}$ a one-dimensional subspace. The connected Lie subgroup S_{α_i} of G with Lie algebra $\mathfrak{s}_{\alpha_i} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell_{\alpha_i})$ is closed and its action on M has cohomogeneity one and no singular orbits, so the orbits form a proper homogeneous C1-foliation. This foliation has a unique minimal leaf—the one passing through o. Moreover, for every $t > 0$, the two leaves at the distance t from the minimal one are congruent to each other and to no other leaf. Were we to choose another line ℓ'_{α_i} in $\mathfrak{g}_{\alpha_i}$, the resulting action would be strongly orbit-equivalent to that for ℓ. Given two simple roots α_i and α_j, the actions of S_{α_j} and S_{α_j} are orbit-equivalent if and only if there exists $s \in \mathrm{Aut}^{\mathrm{w}}(\mathrm{DD}_M)$ sending α_i to α_j.

Remark 5.1.1. Since we are discussing the classification of C1-actions up to orbit-equivalence, we might occasionally say things like "an action orbit-equivalent to a foliation of horospherical or solvable type".

Totally geodesic singular orbit

Here we describe the C1-actions that have a totally geodesic singular orbit. More specifically, we present two special types of such actions: one is when M is irreducible, and the other is when M is reducible of rank 2 and its de Rham factors are homothetic. It turns out that any C1-action with a totally geodesic singular orbit arises from these two by means of the canonical extension procedure, to be defined below. The primary reason behind this fact lies in Mostow's description of maximal subalgebras of semisimple Lie algebras ([Mos61]).

First things first, a totally geodesic orbit of a C1-action is automatically singular, with one exception: when $M = \mathbb{R}H^n$ and the orbit is a totally geodesic hypersurface (see

Example 2.2.15). In this exceptional case, the action is orbit-equivalent to the (unique) foliation of solvable type on $\mathbb{R}H^n$.

Now, suppose M is irreducible. In this case, C1-actions on M with a totally geodesic singular orbit are intimately related to reflective submanifolds of M (see Subsection 2.2.1). Let $F \subseteq M$ be a reflective submanifold and consider its orthogonal complement $F^\perp$ (see Observation 2.2.36). It was shown in [BT04] that F is a singular orbit of some C1-action on M if and only if $F^\perp$ has rank 1. The proof is geometric in nature, and the idea behind it is rather simple: if F is given as an orbit of a C1-action $H \curvearrowright M$, the slice representation of H at a point of F has cohomogeneity 1 by Proposition 2.3.14(b), which easily implies that the isotropy representation of $F^\perp$ has cohomogeneity 1, and so the rank[1] of $F^\perp$ is 1. Conversely, if $F^\perp$ has rank 1, let $V = T_oF$ be the Lie triple system of F (we may assume $o \in F$), and consider the orthogonal symmetric Lie algebra $(N_{\mathfrak{k}}(V) \oplus V, \theta)$ representing F (see Remark 2.2.14). It is easy to show that the connected Lie subgroup of G corresponding to $N_{\mathfrak{k}}(V) \oplus V$ is closed and its slice representation at o has cohomogeneity 1. This means that this group acts on M with cohomogeneity 1.

Now let $H \curvearrowright M$ be any C1-action with a totally geodesic singular orbit F. In [BT04], Berndt and Tamaru showed that F has to be reflective—apart from 5 exceptions (up to orbit-equivalence). Somewhat mysteriously, each of those 5 exceptions is related to the exceptional Lie group G_2 (see [BT04, Th. 4.2]). We see that the classification of C1-actions with a totally geodesic singular orbit reduces to that of reflective submanifolds with orthogonal complement of rank 1. Note that such an action is defined by its singular orbit, as the other orbits have to be the tubes around the singular one (see Proposition 2.3.41). What is more, the congruence class of a reflective submanifold is determined by its isometry class (see [Leu79a]):

Proposition 5.1.2. *If two reflective submanifolds in a simply connected irreducible symmetric space are isometric, then they are congruent.*

Leung classified reflective submanifolds in simply connected irreducible symmetric spaces of compact type ([Leu75, Leu79a]). Using duality (Remark 2.2.38), Berndt and Tamaru translated that into a classification of reflective submanifolds in irreducible symmetric spaces of noncompact type and calculated which of those have rank-1 orthogonal complements (see [BT04, Th. 3.1, 3.3]). This concludes the classification of C1-actions with a totally geodesic singular orbit in the irreducible case.

The second type of such actions was introduced in [DRDVO23], and it appears when the space is reducible but of a very specific form. Suppose M has only two de Rham factors, $M = M_1 \times M_2$, and write $G = G_1 \times G_2$ and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Assume, in addition, that M_1 and M_2 are homothetic, which, as we know from Proposition 3.3.1, is equivalent to $\mathfrak{g}_1$ and $\mathfrak{g}_2$ being isomorphic. Pick an isomorphism $\varphi\colon \mathfrak{g}_1 \xrightarrow{\sim} \mathfrak{g}_2$ and consider the *diagonal* subalgebra $\mathfrak{g}_\varphi = \{X + \varphi(X) \mid X \in \mathfrak{g}_1\}$. Let $G_\varphi \subseteq G_1 \times G_2$ be the corresponding connected Lie subgroup. Then G_φ is closed and acts on M hyperpolarly with cohomogeneity equal to $\operatorname{rk}(M_i)$. What is more, it has a unique singular orbit, which is reflective (hence totally geodesic) and homothetic to M_i. Finally, the action of G_φ does not depend on the choice of φ up to orbit-equivalence. If M_i are of rank 1—and thus homothetic to a hyperbolic space over $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$,—then this construction yields a C1-action with a totally geodesic

[1] One has to rule out the possibility that $F^\perp$ is flat of dimension > 1. This can be easily done, for instance, by using the restricted root space decomposition and Proposition 2.2.43.

singular orbit. We call this a **diagonal C1-action** on $M_1 \times M_2$.

The canonical extension

The canonical extension construction was introduced in [BT13, Sect. 4.1], and it is not so much a type of C1-actions but a general procedure that allows one to extend homogeneous objects defined on the boundary components of M to the whole M.

Fix a subset $\Phi \subseteq \Lambda$. Recall that we have a boundary component B_Φ, which is a totally geodesic submanifold of M and itself a symmetric space of noncompact type and rank $r_\Phi = |\Phi|$. It is represented by an almost effective Riemannian symmetric pair $(G'_\Phi, G'_\Phi \cap K)$. In particular, the isometry Lie algebra of B_Φ is naturally isometric to $\mathfrak{g}'_\Phi$ (see Subsection 2.4.3 for more detail). Suppose we have a proper isometric action $H_\Phi \curvearrowright B_\Phi$. By Observation 2.3.20, we may assume H_Φ is a closed connected subgroup of G'_Φ. Let $\mathfrak{h}_\Phi \subseteq \mathfrak{g}'_\Phi$ be its Lie algebra. We know that $\mathfrak{g}'_\Phi \subseteq \mathfrak{m}_\Phi$ and $G'_\Phi \subseteq M_\Phi$. Recall that we have the Langlands decompositions $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \inplus \mathfrak{n}_\Phi$ and $Q_\Phi = M_\Phi \times A_\Phi \ltimes N_\Phi$. Therefore, we can form a new subalgebra and its corresponding connected Lie subgroup:

$$\mathfrak{h}^\Lambda_\Phi = \mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \inplus \mathfrak{n}_\Phi \subseteq \mathfrak{q}_\Phi \quad \rightsquigarrow \quad H^\Lambda_\Phi = H_\Phi \times A_\Phi \ltimes N_\Phi \subseteq Q_\Phi.$$

The subgroup H^Λ_Φ is closed and thus acts properly on M. This action is called the **canonical extension** of $H_\Phi \curvearrowright B_\Phi$. We also refer to the Lie algebra $\mathfrak{h}^\Lambda_\Phi$ and Lie group H^Λ_Φ as the **canonical extensions** of $\mathfrak{h}_\Phi$ and H_Φ, respectively. We list some elementary properties of this construction. Recall that we have the horospherical decomposition $M = B_\Phi \times A_\Phi \times N_\Phi$.

(a) If $S \subseteq B_\Phi$ is an orbit of H_Φ, then $S \times A_\Phi \times N_\Phi \subseteq M$ is an orbit of H^Λ_Φ. This follows directly from (2.4.3). We refer to the latter orbit as the *orbit extended from* S.

(b) The codimension of any orbit $S \subseteq B_\Phi$ of H_Φ is equal to the codimension of the orbit of H^Λ_Φ in M extended from S. In particular, the former is singular if and only if the latter is.

(c) The cohomogeneity of $H_\Phi \curvearrowright B_\Phi$ is equal to that of $H^\Lambda_\Phi \curvearrowright M$.

(d) Let $S \subseteq B_\Phi$ be an orbit of H_Φ and $b \in S$ any point. Consider any point of the form (b, a, n) on the orbit of H^Λ_Φ in M extended from S. The isotropy subgroup of H_Φ at b is the same as that of H^Λ_Φ at (b, a, n). Similarly, the normal space to S at b coincides with the normal space to the extended orbit at (b, a, n). Consequently, the slice representations of H_Φ at b and of H^Λ_Φ at (b, a, n) coincide.

(e) $H_\Phi \curvearrowright B_\Phi$ is (hyper)polar if and only if $H^\Lambda_\Phi \curvearrowright M$ is.

(f) If $H'_\Phi \curvearrowright B_\Phi$ is another proper isometric action that is *strongly* orbit-equivalent to $H_\Phi \curvearrowright B_\Phi$, then the canonical extensions of these two actions are (strongly) orbit-equivalent. This follows from (2.4.3) and the fact that any inner isometry of B_Φ comes from $G'_\Phi \subseteq G$.

(g) The canonical extension construction does not depend on the choice of initial data: for any other $o' \in M, \mathfrak{a}' \subset \mathfrak{p}', \Lambda' \subset \Sigma'$, and $\Phi' \subseteq \Lambda'$, the canonical extension of any action on the boundary component $B_{\Phi'}$ is *strongly* orbit-equivalent to the canonical extension of an action on B_Φ for some subset $\Phi \subseteq \Lambda$.

(h) The canonical extensions construction can be carried out inductively: if we have two subsets $\Psi \subset \Phi \subset \Lambda$ and $H_\Psi \subseteq G'_\Psi$, then $(H_\Psi^\Phi)^\Lambda = H_\Psi^\Lambda$. Consequently, for any action on B_Ψ, if we first canonically extend it to an action on B_Φ and then to one on M, the result is going to be orbit-equivalent to the canonical extension directly from B_Ψ to M.

In (f), the assumption that the orbit-equivalence is strong is essential: if the actions of H_Φ and H'_Φ on B_Φ are merely orbit-equivalent, their canonical extensions may fail to be orbit-equivalent (see the example on p. 139 in [BT13]).

Observation 5.1.3. The canonical extension construction in not limited to just actions: it can be applied to other homogeneous objects on a boundary component like homogeneous foliations or homogeneous hypersurfaces. In order to do that, one first passes from such an object to a suitable group action on B_Φ and then canonically extends that action. For example, any foliation of solvable type on M is the canonical extension of a homogeneous foliation (also of solvable type) on some rank-1 boundary component. Many of the above properties can be adapted to these other contexts.

Remark 5.1.4. It can be seen from our discussion in Subsection 2.4.3 that $\mathfrak{m}_\Phi$ splits as $Z_{\mathfrak{k}_0}(\mathfrak{b}_\Phi) \oplus \mathfrak{g}'_\Phi$. The first summand is precisely the kernel of the representation of $\mathfrak{m}_\Phi$ on $\mathfrak{b}_\Phi$. This means that, in the canonical extension construction, we can take the Lie algebra $\mathfrak{h}_\Phi^\Lambda$ to be not $\mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ but $\widehat{\mathfrak{h}}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, where $\widehat{\mathfrak{h}}_\Phi \subseteq \mathfrak{m}_\Phi$ is any Lie subalgebra satisfying the following two properties:

(a) The image of the projection of $\widehat{\mathfrak{h}}_\Phi$ to $\mathfrak{g}'_\Phi$ along $Z_{\mathfrak{k}_0}(\mathfrak{b}_\Phi)$ is $\mathfrak{h}_\Phi$.

(b) The connected Lie subgroup of M_Φ corresponding to $\mathfrak{g}'_\Phi$ is closed.

For instance, we can take $\widehat{\mathfrak{h}}_\Phi = Z_{\mathfrak{k}_0}(\mathfrak{b}_\Phi) \oplus \mathfrak{h}_\Phi$. The resulting action on M will still have the same orbits as that of H_Φ^Λ. This observation will let us describe canonically extended actions later in the chapter more neatly.

The nilpotent construction

The sixth and final type of C1-actions arises from a construction rooted in the representation theory of reductive Lie groups, known as the **nilpotent construction**. It was conceived in [BT13] and further studied in [DRDVO23, BDV15]; we refer to these papers for a more detailed exposition and proofs.

Once again, fix $\Phi \subseteq \Lambda$. The nilpotent Lie algebra $\mathfrak{n}_\Phi$ is graded:

$$\mathfrak{n}_\Phi = \bigoplus_{\nu=1}^{k} \mathfrak{n}_\Phi^\nu, \quad \mathfrak{n}_\Phi^\nu = \bigoplus_{\alpha(H^\Phi)=\nu} \mathfrak{g}_\alpha, \quad k = \delta(H^\Phi), \quad H^\Phi = \sum_{\alpha_i \in \Lambda \setminus \Phi} H^i,$$

where $\delta \in \Sigma^+$ is the highest root. Note that, unless DD_Φ is a union of some connected components of DD, this grading is not the one inherited from $\mathfrak{n}$. The adjoint action of L_Φ on $\mathfrak{n}_\Phi$ respects this grading. Let $\mathfrak{v} \subseteq \mathfrak{n}_\Phi^1$ be a linear subspace of dimension at least 2. One can easily see that $\mathfrak{n}_{\Phi,\mathfrak{v}} = \mathfrak{n}_\Phi \ominus \mathfrak{v}$ is a Lie subalgebra. Its corresponding connected Lie subgroup of N_Φ is closed and will be denoted by $N_{\Phi,\mathfrak{v}}$. We also have $N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = N_{\mathfrak{m}_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) \oplus \mathfrak{a}_\Phi$, which implies $N_{L_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = N_{M_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) \times A_\Phi$. Furthermore, $N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = \theta N_{\mathfrak{l}_\Phi}(\mathfrak{v})$ and thus $N^0_{L_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = \Theta N^0_{L_\Phi}(\mathfrak{v})$, and the same remains true if $\mathfrak{l}_\Phi$

and L_Φ are replaced with $\mathfrak{m}_\Phi$ and M_Φ, respectively. Define a Lie subalgebra and its corresponding connected Lie subgroup:

$$\boldsymbol{\mathfrak{h}_{\Phi,\mathfrak{v}}} = N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) \niplus \mathfrak{n}_{\Phi,\mathfrak{v}} \subseteq \mathfrak{l}_\Phi \niplus \mathfrak{n}_\Phi = \mathfrak{q}_\Phi,$$
$$\boldsymbol{H_{\Phi,\mathfrak{v}}} = N^0_{L_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) \ltimes N_{\Phi,\mathfrak{v}} \subseteq L_\Phi \ltimes N_\Phi = Q_\Phi.$$

Since both factors in its semidirect product decomposition are closed, $H_{\Phi,\mathfrak{v}}$ is a closed subgroup. It was shown in [BDV15] and [BT13] that the following assumptions are equivalent:

(i) $F_\Phi \subseteq H_{\Phi,\mathfrak{v}} \cdot o$.

(ii) $N^0_{L_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}})$ acts transitively on F_Φ.

(iii) $N^0_{M_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}})$ acts transitively on B_Φ.

(iv) The image of the projection of $N_{\mathfrak{l}_\Phi}(\mathfrak{v})$ to $\mathfrak{p}$ along $\mathfrak{k}$ equals $\mathfrak{b}_\Phi \oplus \mathfrak{a}_\Phi$.

(v) The image of the projection of $N_{\mathfrak{m}_\Phi}(\mathfrak{v})$ to $\mathfrak{p}$ along $\mathfrak{k}$ equals $\mathfrak{b}_\Phi$.

If $\mathfrak{v}$ is chosen randomly, there is no reason for the action of $H_{\Phi,\mathfrak{v}}$ to be of cohomogeneity 1. This means that special subspaces of $\mathfrak{n}^1_\Phi$ have to be singled out. If $\mathfrak{v}$ satisfies the following two assumptions, we call it **admissible** and **protohomogeneous**, respectively:

(a) Admissible: The image of the projection of $N_{\mathfrak{m}_\Phi}(\mathfrak{v})$ to $\mathfrak{p}$ along $\mathfrak{k}$ equals $\mathfrak{b}_\Phi$;

(b) Protohomogeneous: $N_{K_\Phi}(\mathfrak{n}_{\Phi,\mathfrak{v}}) = N_{K_\Phi}(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$.

The main virtue of the nilpotent construction is the following

Proposition 5.1.5. *Suppose $\mathfrak{v} \subseteq \mathfrak{n}^1_\Phi$ is a protohomogenenous and admissible subspace of dimension ≥ 2. Then, the subgroup $H_{\Phi,\mathfrak{v}}$ acts on M with cohomogeneity one, and its orbit though o is singular of codimension equal to* $\dim \mathfrak{v}$*. If $\mathfrak{v}_1, \mathfrak{v}_2 \subseteq \mathfrak{n}^1_\Phi$ are two subspace of dimension ≥ 2 that differ by* $\mathrm{Ad}(k)$ *for some $k \in K_\Phi$, then:*

(a) *$\mathfrak{v}_1$ is protohomogenenous or admissible if and only if $\mathfrak{v}_2$ is.*

(b) *If $\mathfrak{v}_i$ are both protohomogeneous and admissible, then* $\mathrm{Ad}(k)(\mathfrak{h}_{\Phi,\mathfrak{v}_1}) = \mathfrak{h}_{\Phi,\mathfrak{v}_2}$*, and thus the actions of $H_{\Phi,\mathfrak{v}}$ and $H_{\Phi,\mathfrak{v}'}$ on M are strongly orbit-equivalent by means of k.*

As was the case with the canonical extension, the nilpotent construction does not depend on the choice of initial data (namely, on o, $\mathfrak{a}$, and Λ) up to strong orbit-equivalence.

Remark 5.1.6. As we will see below, the nilpotent construction is only relevant for maximal proper parabolic subgroups, i.e., when Φ is of the form $\Phi_j = \Lambda \setminus \{\alpha_j\}$. In this case—just like we agreed in Example 2.4.26—we will simplify the notation by replacing Φ_j with j in all subscripts and superscripts.

5.1.2. The classification

Now we formulate the main classification result for C1-actions on symmetric spaces of noncompact type. In the present form, this result was established in [DRDVO23], although that paper is largely underpinned by [BT13].

Theorem 5.1.7. *Let $M = G/K$ be a symmetric space of noncompact type and rank r. Let H be a connected Lie group acting properly and isometrically on M with cohomogeneity one. Then one of the following statements holds:*

(a) *The orbits of H form a Riemannian foliation, and the action of H is orbit-equivalent to exactly one of the following:*

  (1) *The action of H_ℓ for some one-dimensional linear subspace $\ell \in \mathbb{P}\mathfrak{a}/\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$. If the metric on M is almost Killing, such ℓ is unique.*

  (2) *The action of H_{α_i} for a unique $\alpha_i \in \Lambda/\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$.*

(b) *There exists exactly one singular orbit, and the action of H is orbit-equivalent to one of the following:*

  (1) *The canonical extension of a C1-action with a totally geodesic singular orbit on an irreducible boundary component B_Φ of M.*

  (2) *The canonical extension of a diagonal C1-action on a reducible rank-2 boundary component B_Φ of M whose de Rham factors are homothetic.*

  (3) *The action of $H_{j,\mathfrak{v}}$ for some $j \in \{1, \dots, r\}$ obtained by nilpotent construction applied to some protohomogeneous admissible subspace $\mathfrak{v} \subseteq \mathfrak{n}_j^1$ with $\dim \mathfrak{v} \geq 2$.*

Notice that we do not make any restrictions on the normalizing constants of M (see [DRDVO23, Rem. 2.5]). Also, if we weaken parts (a)-(1) and (a)-(2) and allow ℓ (resp., α_i) to be just some (not necessarily unique) element of $\mathbb{P}\mathfrak{a}$ (resp., Λ), then we can insist that the orbit equivalence in the theorem is a *strong orbit equivalence* (see [DRDVO23, Rem. 2.3, 2.4]).

Comments on the classification

Issues with the canonical extension. As we mentioned earlier, two orbit-equivalent actions on a boundary component can give rise to non-orbit-equivalent canonical extensions, unless they are strongly orbit-equivalent to begin with. At the same time, the classifications of C1-actions with a totally geodesic singular orbit on irreducible spaces (in [BT04]) and of diagonal C1-actions on products of hyperbolic spaces (in [DRDVO23]) are up to just orbit-equivalence. Let us go through these two situations separately.

Let M_1 and M_2 be homothetic symmetric space of noncompact type and rank 1, and let $\varphi, \varphi' \colon \mathfrak{g}_1 \xrightarrow{\sim} \mathfrak{g}_2$ be two isomorphisms. It was proven in [DRDVO23] that the actions of G_φ and $G_{\varphi'}$ on $M_1 \times M_2$ are in fact strongly orbit-equivalent if the automorphism $\varphi' \circ \varphi^{-1}$ is inner. The group $\mathrm{Aut}(\mathfrak{g}_i)$—which is isomorphic to the isometry group of the hyperbolic space M_i by Proposition 3.3.4—has at most two connected components and is trivial if $M_i \sim \mathbb{H}P^n\,(n > 1)$ or $\mathbb{O}P^2$ (see [Gün10]). This means that there are at most two strong orbit-equivalence classes of diagonal C1-actions on $M_1 \times M_2$, and thus (b)-(2) can produce at most two orbit-equivalence classes of C1-actions on M for each such B_Φ.

The situation in (b)-(1) is grimmer. Let M be irreducible and $H \curvearrowright M$ a proper isometric action with a totally geodesic singular orbit. We know that the action $H \curvearrowright M$ is orbit-equivalent to one of the actions described in [BT04, Th. 3.1, 3.3, 4.2] (or rather, the singular orbit of H is congruent to one of the submanifolds described there, but the action is fully determined by it up to orbit-equivalence). The component group

$I(M)/I^0(M)$ is a finite group, and a very small one, in fact. It is S_4 if $M = \mathrm{Gr}^*(4, \mathbb{R}^8)$, D_4 if $M = \mathrm{Gr}^*(2r, \mathbb{R}^{4r})$ $(r > 2)$, and in all the other cases it is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, or trivial. In any case, this means that the orbit-equivalence class of $H \curvearrowright M$ can split into a small number of strong orbit-equivalence classes. The problem here is that those classes have not been identified. To resolve this issue, one needs to refine Leung's classification of reflective submanifolds and make it up to strong congruence. Until then, each such action on a boundary component in (b)-(1) can potentially give rise to several orbit-equivalence classes of C1-actions on M. The only cases when we know for sure this does not happen are when $I(B_\Phi)$ is connected or when M is irreducible and $B_\Phi = M$ (in which case, no canonical extension is needed).

The problem of congruence. Any action $H \curvearrowright M$ arising from (b)-(1) or (b)-(3) has the following property: the H-orbit of every $p \in M$ contains the p-leaf of all but one de Rham foliations on M. In (b)-(2), this is the case if and only if both roots in Φ lie in the same component of the Dynkin diagram DD. If Φ does not satisfy this property, the resulting action $H \curvearrowright M$ cannot be orbit-equivalent to any action arising from (b)-(1) or (b)-(3). As far as we are aware, this is the only known general result on congruence of the three types of C1-actions with a singular orbit. For instance, an action arising from the nilpotent construction may happen to have a totally geodesic singular orbit (for M irreducible, that action would also come from (b)-(1) when $\Phi = \Lambda$). In a nutshell, this means that Theorem 5.1.7 is an existence but not uniqueness result and it leaves the congruence problem wide open.

Classifying C1-actions in practice. Given a concrete symmetric space M of noncompact type, Theorem 5.1.7 reduces the problem of classification of C1-actions on M to the following:

(1) Compute all actions arising from (b)-(1) and (b)-(2) and check which of them are orbit-equivalent.

(2) Compute all actions arising by nilpotent construction for each $j \in \{1, \ldots, r\}$ and check which of them are orbit-equivalent.

(3) Find out if any of the actions in (1) are orbit-equivalent to those in (2).

Since the nilpotent construction is so cumbersome, it might be easier to compute all possible C1-actions arising by canonical extension first—even when the singular orbit of the action on a boundary component is not totally geodesic—and then investigate whether the nilpotent construction produces any actions not yet accounted for. This is the approach we are going to adopt in the chapter. Again, note that if two actions on a boundary component are not strongly orbit-equivalent, one will have to check if their canonical extensions are orbit-equivalent.

Making use of symmetries of the Dynkin diagram. If the group $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$ is not trivial, it can alleviate the classification of C1-actions on M. Indeed, suppose $\Phi, \Phi' \subseteq \Lambda$ differ by some element of $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$. Owing to Proposition 3.3.9, the boundary components B_Φ and $B_{\Phi'}$ are congruent by means of some $k \in N_{\widetilde{K}}(\mathfrak{n})$. One can easily show that the actions arising from B_Φ by canonical extension via (b)-(1) or (b)-(2) are congruent to those arising from $B_{\Phi'}$, and the congruence can be given by k. The same is true for the nilpotent construction: if $\alpha_i, \alpha_j \in \Lambda$ differ by some element of $\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$, it suffices to deal with either i or j in (b)-(3).

Congruence in the nilpotent construction. In the context of the nilpotent construction, it is usually fairly straightforward to compute what the Lie algebras $\mathfrak{m}_j$ (or $\mathfrak{l}_j$) and $\mathfrak{k}_j$ and hence the groups M_j^0 (or L_j^0) and K_j^0 look like. It may be harder, however, to do so for the full groups M_j (or L_j) and K_j. In practical terms, it means that it may be easier to find all protohomogeneous and admissible subspaces $\mathfrak{v} \subseteq \mathfrak{n}_j^1$ up to K_j^0-congruence but not K_j-congruence. To make things worse, even if two subspaces $\mathfrak{v}, \mathfrak{v}' \subseteq \mathfrak{n}_j^1$ are not K_j-congruent, they might, in theory, produce orbit-equivalent actions, so we may end up with a larger list of actions and have to discard some of them. However, this hardly poses an actual problem: in reality, whenever the nilpotent construction produces an action, chances are it is orbit-equivalent to an already known C1-action on M. In fact, for irreducible symmetric spaces of noncompact type and rank > 1, there are only two actions known so far that were obtained by the nilpotent construction and not any other method (see [BT13]). Once again, both of them are related to the exceptional Lie group G_2.

The rank-one case. Finally, if M has rank 1, then $\Lambda = \{\alpha_1\}$ and the only proper boundary component is a point. For this reason, the canonical extension method does not produce any C1-actions. Moreover, every $\mathfrak{v} \subseteq \mathfrak{n}_1^1$ is admissible. Historically, the rank-1 case had mostly been dealt with before the advent of the canonical extension and nilpotent construction methods (see [BB01, BT07]), although the case of $\mathbb{H}H^n$ stayed unresolved until recently (see [DRDVRV21]).

The classification of homogeneous hypersurfaces

Thanks to Proposition 2.3.41, Theorem 5.1.7 yields a classification result for connected homogeneous properly embedded hypersurfaces in symmetric spaces of noncompact type. Namely, every such hypersurface S in M is a principal orbit of a C1-action on M, and the orbit-equivalence class of the action is determined by S. Each H_ℓ ($\ell \in \mathbb{P}\mathfrak{a}/\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$) in (a)-(1) gives rise to precisely one congruence class of such hypersurfaces, each H_{α_i} ($\alpha_i \in \Lambda/\mathrm{Aut}^{\mathrm{w}}(\mathrm{DD})_M$) in (a)-(2) yields a one-parameter family of congruence classes parametrized by $t \geq 0$, and each action with a singular orbit gives a one-parameter family parametrized by $t > 0$.

Discussion of the proof

Before going further, we would like to discuss some key steps in the proof of Theorem 5.1.7. The actions in the theorem all have cohomogeneity one, so one only needs to prove that they exhaust the list of C1-actions up to orbit-equivalence. To this end, we need to introduce a structure result of Mostow concerning maximal subalgebras of semisimple Lie algebras. We begin with some brief preliminaries, most of which can be found in [OV94] and [Che55, Ch. 5, §4].

Let V be a real or complex vector space and $\mathfrak{h} \subseteq \mathfrak{gl}(V)$ a Lie subalgebra. We say that $\mathfrak{h}$ is a *reductive subalgebra* if its radical consists of semisimple endomorphisms (or equivalently, of semisimple elements of $\mathfrak{gl}(V)$). This is equivalent to asking that the trace-form of $\mathfrak{gl}(V)$ is nondegenerate on $\mathfrak{h}$. One has to tread carefully: a reductive subalgebra is necessarily reductive as a Lie algebra, but the converse is not generally true (see [Kol11, Sect. 3]). We say that $\mathfrak{h}$ is *algebraic* if it is the Lie algebra of some algebraic subgroup of $\mathrm{GL}(V)$. Assume that $\mathfrak{h}$ is algebraic. The *unipotent radical* of $\mathfrak{h}$, denoted by $\mathrm{rad}_u(\mathfrak{h})$, is the largest ideal of $\mathfrak{h}$ consisting of nilpotent endomorphisms (or equivalently nilpotent elements of $\mathfrak{gl}(V)$).

One can show that $\mathrm{rad}_u(\mathfrak{h})$ is well-defined and coincides with the set of all elements of $\mathrm{rad}(\mathfrak{h})$ that are nilpotent in $\mathfrak{gl}(V)$. It follows from Engel's theorem that the unipotent radical is nilpotent and thus contained in the nilradical $\mathfrak{n}(\mathfrak{h})$ of $\mathfrak{h}$. One should be careful[1] with these definitions because the nilradical admits a description dangerously similar to that of $\mathrm{rad}_u(\mathfrak{h})$: it is the set of all *nilpotent elements of* $\mathrm{rad}(\mathfrak{h})$ (that is, nilpotent in the Lie algebra $\mathrm{rad}(\mathfrak{h})$). The unipotent radical of $\mathfrak{h}$ is trivial if and only if $\mathfrak{h}$ is a reductive subalgebra. The idea behind this last statement is actually rather simple: the radical of $\mathfrak{h}$ is always algebraic, so it contains the semisimple and nilpotent parts of each of its elements ([Che51, Ch. 2, §14]).

All these notions make sense in the semisimple setting. Let $\mathfrak{g}$ be any real or complex semisimple Lie algebra. We can identify $\mathfrak{g}$ with a subalgebra of $\mathfrak{gl}(\mathfrak{g})$ by means of the adjoint representation. We say that a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is *algebraic* if so is $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h}) \subset \mathfrak{gl}(\mathfrak{g})$. Assuming $\mathfrak{h}$ is algebraic, its *unipotent radical* $\mathrm{rad}_u(\mathfrak{h})$ is its largest ideal consisting of nilpotent elements of $\mathfrak{g}$, or equivalently the set of all nilpotent elements of $\mathfrak{g}$ lying in $\mathrm{rad}(\mathfrak{h})$. With all this in mind, reductive subalgebras now allow a number of equivalent definitions:

Proposition 5.1.8 (Characterization of reductive subalgebras). *Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a subalgebra. The following conditions are equivalent:*

(i) *The radical of $\mathfrak{h}$ consists of semisimple elements of $\mathfrak{g}$.*

(ii) *The unipotent radical of $\mathfrak{h}$ is trivial.*

(iii) *The restriction of $B^{\mathfrak{g}}$ to $\mathfrak{h}$ is nondegenerate.*

If these conditions are satisfied, we say that $\mathfrak{h}$ is a ***reductive subalgebra*** *of $\mathfrak{g}$ (or simply that it is* ***reductive in $\mathfrak{g}$****). If $\mathfrak{g}$ is complex, the above conditions are also equivalent to the following:*

(iv) *The adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ is completely reducible.*

If $\mathfrak{g}$ is real, then $\mathfrak{h}$ is reductive in $\mathfrak{g}$ if and only if $\mathfrak{h}(\mathbb{C})$ is such in $\mathfrak{g}(\mathbb{C})$.

The significance of reductive subalgebras becomes evident in light of the *generalized Mostow-Karpelevich theorem* ([OV94, Ch. VI, Th. 3.6]), which asserts that an algebraic subalgebra $\mathfrak{h}$ of a real semisimple Lie algebra $\mathfrak{g}$ is reductive in $\mathfrak{g}$ if and only if it is θ-stable with respect to some Cartan involution θ on $\mathfrak{g}$.

We are now ready to formulate Mostow's result. Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a maximal proper subalgebra. One can show that $\mathfrak{h}$ must be an algebraic subalgebra (e.g., it follows from [OV94, Ch. I, Th. 6.2]). In [Mos61, Th. 3.1, 3.2], Mostow proved that there are two possibilities:

[1] To make the matter even worse, there is another ideal called the *nilpotent radical* and denoted by $\mathrm{rad}_n(\mathfrak{h})$, which can be defined for any Lie algebra $\mathfrak{h}$ as the intersection of the kernels of all irreducible representations of $\mathfrak{h}$, or equivalently as $\mathrm{rad}(\mathfrak{h}) \cap [\mathfrak{h}, \mathfrak{h}]$. This radical is contained in $\mathfrak{n}(\mathfrak{h})$ and, in case $\mathfrak{h} \subseteq \mathfrak{gl}(V)$ is algebraic, contains $\mathrm{rad}_u(\mathfrak{h})$. Since the term *nilpotent radical* is already taken, we had no choice but call $\mathrm{rad}_u(\mathfrak{h})$ the *unipotent radical.* If the base field is complex and $H \subseteq \mathrm{GL}(V)$ is the connected algebraic subgroup corresponding to $\mathfrak{h}$, then the *unipotent radical* $\mathrm{Rad}_u(H)$ of H (the largest normal subgroup consisting of unipotent endomorphisms) has Lie algebra $\mathrm{rad}_u(\mathfrak{h})$.

(a) $\mathfrak{h}$ is unimodular $\Rightarrow$ its unipotent radical is trivial[1].

(b) $\mathfrak{h}$ is not unimodular $\Rightarrow$ it is parabolic.

In the former case, $\mathfrak{h}$ is reductive in $\mathfrak{g}$ by virtue of Proposition 5.1.8. Combining this with the generalized Mostow-Karpelevich theorem and Proposition 2.4.25, we obtain the following[2]

Corollary 5.1.9. *Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a maximal proper subalgebra. Then $\mathfrak{h}$ is algebraic, and there exists a Cartan involution θ on $\mathfrak{g}$ such that exactly one of the following is satisfied:*

(a) *$\mathfrak{h}$ is a θ-stable reductive subalgebra.*

(b) *$\mathfrak{h}$ is a parabolic subalgebra, and it coincides with $\mathfrak{q}_j$ for some choices of $\mathfrak{a} \subset \mathfrak{p}, \Lambda \subset \Sigma$, and $\alpha_j \in \Lambda$.*

Now we can go back to discussing the proof of Theorem 5.1.7. Suppose $M = G/K$ is a symmetric space of noncompact type, and let $H \curvearrowright M$ be an isometric C1-action. If H has no singular orbits, it is bound to be orbit-equivalent to one of the actions described in part (a) of Theorem 5.1.7 in view of Corollary 4.1.13. We may thus assume that H has a singular orbit, which is unique by Proposition 2.3.43. Since we are only interested in classification up to orbit-equivalence, we do not lose generality by assuming that $H \subset G$, thanks to Observation 2.3.20. According to our agreement, H is connected; it is also a closed subgroup because its action is proper. Let F be a maximal proper connected Lie subgroup of G containing H; its Lie algebra $\mathfrak{f}$ is then a maximal proper subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. One can show F is a closed subgroup. There are two possibilities: either F acts with cohomogeneity one and thus has the same orbits as H, or its action is transitive. We write $M = M_1 \times \cdots \times M_k$ for the de Rham decomposition and adopt the notation established on p. 112. By a result of Dynkin (Th. 15.1 in[3] [Dyn52b] or [Dyn57b]), $\mathfrak{f}$ must be of the following form:

$$\mathfrak{f} = \mathfrak{f}_i \oplus \bigoplus_{j \neq i} \mathfrak{g}_j \text{ for some } i, \text{or } \mathfrak{f} = \mathfrak{f}_{i,j,\varphi} \oplus \bigoplus_{l \neq i,j} \mathfrak{g}_l \text{ for some } i \neq j.$$

Here $\mathfrak{f}_i \subset \mathfrak{g}_i$ and $\mathfrak{f}_{i,j,\varphi} = \{X + \varphi(X) \mid X \in \mathfrak{g}_i\}$ for some isomorphism $\varphi \colon \mathfrak{g}_i \xrightarrow{\sim} \mathfrak{g}_j$. In the latter case, $F = F_{i,j,\varphi} \times \prod_{l \neq i,j} G_l$, where $F_{i,j,\varphi} \subset G_i \times G_j$ is the connected Lie subgroup corresponding to $\mathfrak{f}_{i,j,\varphi}$. The action of F can thus be described as the product of the transitive actions $G_l \curvearrowright M_l$, $l \neq i, j$, and the diagonal action $F_{i,j,\varphi} \curvearrowright M_i \times M_j$ (already introduced in Subsection 5.1.1). In particular, it cannot be transitive, so it has the same

[1]Mostow also showed that the radical of $\mathfrak{h}$ is compactly embedded in $\mathfrak{g}$ ($\Leftrightarrow$ in $\mathfrak{h}$) in this case, but we do not need that.

[2]In [BT13], Berndt and Tamaru arrived at the same conclusion (see Theorem 3.2 there), also relying on Mostow's result, but their argument is incorrect. First, they apparently mistook the unipotent radical for the nilradical and thought that Mostow had proved that the nilradical of a unimodular maximal proper subalgebra vanishes. This would imply that the radical vanishes as well and the subalgebra is semisimple, which of course does not have to be the case—consider, for instance, the maximal proper subalgebra $\mathfrak{s}(\mathfrak{u}(n-k) \oplus \mathfrak{u}(k,1))$ of $\mathfrak{su}(n,1)$. Second, they applied the Mostow-Karpelevich theorem to an algebraic subalgebra that happens to be reductive as a Lie algebra. As we explained above, this latter condition is generally weaker than being reductive as a subalgebra, which is necessary for the Mostow-Karpelevich theorem to work. In any case, Corollary 5.1.9 still holds, so no other results in [BT13] are affected by this issue.

[3]In these papers, the author works over $\mathbb{C}$, but the proof of Theorem 15.1 applies verbatim over $\mathbb{R}$.

orbits as the original action of H. Since $M_i \times M_j$ is a boundary component of M, this action can also be described as the canonical extension of $F_{i,j,\varphi} \curvearrowright M_i \times M_j$, which is accounted for in part (b)-(2) of Theorem 5.1.7.

Now, assume $\mathfrak{f} = \mathfrak{f}_i \oplus \bigoplus_{j \neq i} \mathfrak{g}_j$. Thanks to Corollary 5.1.9, $\mathfrak{f}$ is either θ-stable for some Cartan involution θ on $\mathfrak{g}$, or else a parabolic subalgebra. Let us first deal with the former. By Proposition 2.2.12, the orbit $F \cdot o$ is totally geodesic. Since $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ (Proposition 2.1.97(c)), this orbit cannot be the whole M, for that would contradict the fact that $\mathfrak{f}$ is a proper subalgebra. We see that F must act with cohomogeneity one and thus have the same orbits as H. Since F has the form $F_i \times \prod_{j \neq i} G_j$, the action of H can be described as the canonical extension of a C1-action with a totally geodesic singular orbit on the (irreducible) boundary component M_i of M (namely, of $F_i \curvearrowright M_i$). This action was taken into account in part (b)-(1) of Theorem 5.1.7.

Finally, suppose $\mathfrak{f}$ is a parabolic subalgebra of $\mathfrak{g}$, and thus the actions $F \curvearrowright M$ is transitive. We may assume $F = Q_j$ for some $1 \leq j \leq r$. In [BT13, Th. 5.8], Berndt and Tamaru proved that the action $H \curvearrowright M$ is then orbit-equivalent to either a nilpotent construction action of $H_{j,\mathfrak{v}}$ for some $\mathfrak{v} \subseteq \mathfrak{n}_j^1$, or else to the canonical extension of some C1-action on the boundary component B_j. The former was accounted for in part (b)-(3) of Theorem 5.1.7, so we are only left to deal with the canonical extension. Recall that the canonical extension is an inductive procedure (see (h) on p. 144). If the action on B_j itself arises via canonical extension from some boundary component of B_j, we can replace B_j with that boundary component. By performing this procedure a sufficient number of times, we end up with a C1-action of $H' \subset G'_\Phi$ on some boundary component B_Φ that does not arise via further canonical extension, and whose canonical extension $H'^{\Lambda}_{\Phi} \curvearrowright M$ is orbit-equivalent to the original action $H \curvearrowright M$. By repeating the above argument involving a maximal proper subalgebra, we see that there are three possibilities:

(a) The boundary component B_Φ is irreducible, and the action $H' \curvearrowright B_\Phi$ has a totally geodesic singular orbit. Then the action of H is described in part (b)-(1) of Theorem 5.1.7.

(b) The boundary component B_Φ is reducible of rank 2, and the action $H' \curvearrowright B_\Phi$ is diagonal. In this case, the action of H is taken into account in part (b)-(2) of the theorem.

(c) Finally, it might happen that the action $H' \curvearrowright B_\Phi$ arises via nilpotent construction. Formally, it is strongly orbit-equivalent to the action of $H_{\Phi,j,\mathfrak{v}} \subseteq G'_\Phi$ for some $\alpha_j \in \Phi$ and $\mathfrak{v} \subseteq (\mathfrak{g}'_\Phi \cap \mathfrak{n})_j^1$. It was shown in [DRDVO23, Lem. 4.3] that the canonical extension of such an action arises via nilpotent construction performed directly on M: it has the same orbits as the action $H_{j,\mathfrak{v}} \curvearrowright M$ (among other things, this means that $\mathfrak{v} \subseteq \mathfrak{n}_j^1$). Therefore, this case is described in part (b)-(3) of the theorem.

5.1.3. Generalizing the nilpotent construction problem.

Due to its complexity, the nilpotent construction is arguably one of the two big conundrums in the quest for a complete and explicit classification of C1-actions on symmetric spaces of noncompact type (the other one being the problem of congruence). For this reason, it is worth looking at this construction from a slightly more general perspective.

Let G be a reductive Lie group with Lie algebra $\mathfrak{g}$, a maximal compact subgroup $K \subseteq G$, and a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (we are using the notation from [Kna02, Sect. VII.2]). Let V be a finite-dimensional real representation of G, and let $\mathfrak{v} \subseteq V$ be a subspace with $\dim \mathfrak{v} \geq 2$. Inspired by the nilpotent construction, we call $\mathfrak{v}$ **admissible** if the projection of $N_{\mathfrak{g}}(\mathfrak{v})$ to $\mathfrak{p}$ along $\mathfrak{k}$ is onto, i.e., its image equals the whole $\mathfrak{p}$. We call $\mathfrak{v}$ **protohomogeneous** if there exists a subgroup of K preserving $\mathfrak{v}$ and acting on $\mathfrak{v}$ with cohomogeneity one[1]. The problem, which we call **the generalized nilpotent construction problem**, is to classify subspaces of V that are both protohomogeneous and admissible up to the action of K. One readily sees how this generalizes the nilpotent construction problem by taking $G = M_j$ (or L_j), $K = K_j$, $\mathfrak{g} = \mathfrak{m}_j = \mathfrak{k}_j \oplus \mathfrak{b}_j$ (or $\mathfrak{l}_j = \mathfrak{k}_j \oplus (\mathfrak{b}_j \oplus \mathfrak{a}_j)$), and $V = \mathfrak{n}_j^1$ (in this case, we are going to drop the word *generalized*). Note that every subspace of V is automatically admissible if G is compact.

This problem has been solved for a few representations. For instance, Berndt and Brück handled the cases $(G, V) = (\mathrm{SO}(n), \mathbb{R}^n)$, $(\mathrm{U}(n), \mathbb{C}^n)$, $(\mathrm{Spin}(7), \mathbb{R}^8_{\mathrm{spin}})$ in [BB01], while Díaz-Ramos et al. dealt with $(G, V) = (\mathrm{Sp}(n)\mathrm{Sp}(1), \mathbb{H}^n)$ in [DRDVRV21]. See also [BT13] and [BDV15] for some examples with G noncompact. The latter of these two papers contains also a general result on the nilpotent construction problem that facilitates the search for admissible and protohomogeneous subspaces for certain noncompact symmetric spaces and choices of j (see Proposition 5 there). It may, in theory, be possible to extend that result to the generalized nilpotent construction problem. When dealing with the nilpotent construction on some rank-2 spaces later in the chapter, we will solve this problem for a few more representations. We want to stress, however, that almost all attempts to solve the problem have so far been ad-hoc, and it does not seem feasible to proceed with this approach in the future—the more complex the representation of G on V is, the more difficult the problem becomes. For instance, it took the authors of [DRDVRV21] an entire paper to solve the problem for just one representation $\mathrm{Sp}(n)\mathrm{Sp}(1) \curvearrowright \mathbb{H}^n$. We believe that a more holistic and general approach is required to solve the nilpotent construction problem for all symmetric spaces of noncompact type.

5.1.4. The rank-one case

Here we prove a small technical result concerning C1-actions on rank-1 noncompact symmetric spaces, i.e., on hyperbolic spaces over $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$. It will allow us to describe actions on rank-2 spaces canonically extended from their boundary components in a relatively nice and uniform way. The idea is that whenever we represent a C1-action on M by a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, we want $\mathfrak{h}$ to be positioned nicely with respect to the Iwasawa decomposition of $\mathfrak{g}$: $\mathfrak{h} = \mathfrak{h}^{\mathfrak{k}} \oplus \mathfrak{h}^{\mathfrak{a}} \oplus \mathfrak{h}^{\mathfrak{n}}$, where $\mathfrak{h}^{\mathfrak{k}} = \mathfrak{h} \cap \mathfrak{k}$, and so on. If this is the case, we call the subalgebra $\mathfrak{h}$ **Iwasawa-adapted**[2]. Notice that if the rank of M is 1, we have $\Lambda = \{\alpha_1\}$ and $\mathfrak{m}_1 = \mathfrak{k}_1 = \mathfrak{k}_0$, so the subgroup M_1 is compact, and every subspace of $\mathfrak{n}_1^1$ is automatically admissible.

Proposition 5.1.10. *Let M be a symmetric space of noncompact type and rank 1. Every C1-action on M is orbit-equivalent to the action of a closed connected subgroup $H \subseteq G$ whose Lie algebra $\mathfrak{h}$ is Iwasawa-adapted. Moreover, we can take H such that $\mathfrak{h}^{\mathfrak{k}} = N_{\mathfrak{k}}(\mathfrak{h}^{\mathfrak{a}} \oplus \mathfrak{h}^{\mathfrak{n}})$ (in this case, we have a semidirect sum decomposition $\mathfrak{h} = \mathfrak{h}^{\mathfrak{k}} \inplus (\mathfrak{h}^{\mathfrak{a}} \oplus \mathfrak{h}^{\mathfrak{n}})$),*

[1] We can always pick a K-invariant inner product on V, in which case this condition is equivalent to asking that $N_K(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$.

[2] This is somewhat akin to how we presented congruence classes of homogeneous hyperpolar foliations on M by standard foliations in Chapter 4.

except when H has a totally geodesic singular orbit F given by

$$F = \begin{cases} \mathbb{R}H^n & \text{if } M = \mathbb{C}H^n, \\ \mathbb{C}H^n & \text{if } M = \mathbb{H}H^n, \\ \mathbb{H}H^2 & \text{if } M = \mathbb{O}H^2. \end{cases} \tag{5.1.1}$$

In (5.1.1), the first two embeddings are standard, whereas the last one comes from an embedding $\mathrm{Sp}(2,1) \hookrightarrow F_4^{-20}$ by means of duality (see [Yok09, Sect. 2.11]).

Proof. The proof relies on the classification of C1-actions on symmetric spaces of noncompact type and rank 1 ([BB01, BT07, DRDVRV21]). Every C1-action on M falls into one of the following three categories: it induces a homogeneous foliation, or it has a totally geodesic singular orbit, or it arises via nilpotent construction. Moreover, the only actions with a totally geodesic singular orbit that do not arise via nilpotent construction are those with a fixed point, as well as those in (5.1.1). Let us go through these three cases separately.

If an action has no singular orbits, we know it is orbit-equivalent to the action of H_{α_1} or $H_{\mathfrak{a}} = N$. In both cases, the Lie algebra is Iwasawa-adapted. If we replace $\mathfrak{h}_{\alpha_1}$ (resp., $\mathfrak{n}$) with a larger Lie algebra $N_{\mathfrak{k}}(\mathfrak{h}_{\alpha_1}) \oplus \mathfrak{h}_{\alpha_1}$ (resp., $N_{\mathfrak{k}}(\mathfrak{n}) \oplus \mathfrak{n}$), the resulting action will have cohomogeneity at most 1 but the same orbit through o, so it has to have the same orbits.

Next, if an action on M arises by nilpotent construction, the corresponding Lie algebra is of the form $\mathfrak{h}_{1,\mathfrak{v}} = N_{\mathfrak{k}_0}(\mathfrak{n}_{1,\mathfrak{v}}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{1,\mathfrak{v}}$. Since the representation of $\mathfrak{k}_0$ on $\mathfrak{a}$ is trivial, $\mathfrak{h}_{1,\mathfrak{v}}$ is contained in the Lie algebra $N_{\mathfrak{k}}(\mathfrak{a} \oplus \mathfrak{n}_{1,\mathfrak{v}}) \oplus \mathfrak{a} \oplus \mathfrak{n}_{1,\mathfrak{v}}$. Again, the corresponding connected Lie group acts with cohomogeneity at most 1 but has the same orbit through o, so it must have the same orbits as $H_{1,\mathfrak{v}}$.

Finally, assume we have an action with a totally geodesic singular orbit F that is either a point or as given in (5.1.1). If F is a point, then the action can be realized by the restricted isotropy group at that point. Without loss of generality, we can take the point to be o, in which case we have the action $K \curvearrowright M$. The Lie algebra $\mathfrak{k}$ of K is obviously Iwasawa-adapted, and we also have $N_{\mathfrak{k}}(\mathfrak{h}^{\mathfrak{a}} \oplus \mathfrak{h}^{\mathfrak{n}}) = N_{\mathfrak{k}}(0) = \mathfrak{k}$. The problem with the actions whose totally geodesic singular orbit is given in (5.1.1) is that they cannot be realized by a subgroup of the form $N_K^0(H)H$, where H is a connected Lie subgroup of AN. The first part of the proposition still works though. To see this, observe that—up to orbit-equivalence—these actions can be given by

$$G' = \begin{cases} \mathrm{SO}^0(n,1) \subseteq \mathrm{SU}(n,1) & \text{if } F = \mathbb{R}H^n \subseteq \mathbb{C}H^n, \\ \mathrm{U}(n,1) \subseteq \mathrm{Sp}(n,1) & \text{if } F = \mathbb{C}H^n \subseteq \mathbb{H}H^n, \\ \mathrm{Sp}(2,1) \times \mathrm{SU}(2) \subseteq F_4^{-20} & \text{if } F = \mathbb{H}H^2 \subseteq \mathbb{O}H^2. \end{cases}$$

In each of these cases, the embedding $G' \subset G$ is the standard one (see [BB01, Th. 1]). The subgroup G' is Θ-stable and, what is more, its Iwasawa decomposition is induced by that of G: $G' = K'A'N'$, where $K' \subseteq K$, $A' \subseteq A$, and $N' \subseteq N$. This means that the Lie algebra $\mathfrak{g}'$ of G' is Iwasawa-adapted, which concludes the proof. □

Remark 5.1.11. In the last part of the proof above, we can change $\mathfrak{g}'$ to a larger subalgebra $\mathfrak{g}'' = N_{\mathfrak{k}}(\mathfrak{p}') \oplus \mathfrak{p}'$, which induces a C1-action with the same orbits. This subalgebra $\mathfrak{g}''$ is also Iwasawa-adapted: $\mathfrak{g}'' = N_{\mathfrak{k}}(\mathfrak{p}') \oplus \mathfrak{a}' \oplus \mathfrak{n}'$.

5.2. Classification of cohomogeneity-one actions on $\mathrm{SL}(3,\mathbb{H})/\mathrm{Sp}(3)$

In this section we classify, up to orbit-equivalence, cohomogeneity-one actions on the symmetric space $\mathrm{SL}(3,\mathbb{H})/\mathrm{Sp}(3)$.

The symmetric space $M = \mathrm{SL}(3,\mathbb{H})/\mathrm{Sp}(3)$ is irreducible and of noncompact type; it has rank 2 and dimension 14, and its restricted root system is A_2. It is the quaternionic analog of the A_2-type spaces $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ and $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$, an explicit classification of cohomogeneity-one actions on which has already been obtained in [BT13, BDV15]. Write $\Lambda = \{\alpha_1, \alpha_2\}$. Then $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$, and all the roots have multiplicity 4. The Lie subalgebra $\mathfrak{k}_0$ is isomorphic to $\mathfrak{sp}(1)^{\oplus 3}$.

Theorem 5.2.1. *Let H be a connected Lie group acting properly and isometrically on $M = \mathrm{SL}(3,\mathbb{H})/\mathrm{Sp}(3)$ with cohomogeneity 1. Then its action is orbit-equivalent to exactly one of the following:*

(a) *The action of the connected Lie subgroup H_ℓ of G with Lie algebra*

$$\mathfrak{h}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n},$$

where ℓ is a one-dimensional linear subspace of $\mathfrak{a}$ determined uniquely up to the action of $\mathrm{Aut}(DD) \cong \mathbb{Z}/2\mathbb{Z}$ on $\mathfrak{a}$. The orbits of H_ℓ are all congruent to each other and form a Riemannian foliation on M.

(b) *The action of the connected Lie subgroup H_{α_1} of G with Lie algebra*

$$\mathfrak{h}_{\alpha_1} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell_{\alpha_1}),$$

where ℓ_{α_1} is any one-dimensional linear subspace of $\mathfrak{g}_{\alpha_1}$. Its orbits form a Riemannian foliation on M, and there is exactly one minimal orbit.

(c) *The action of the subgroup $L_1 = L_1^0$ of G. It has a 6-dimensional totally geodesic singular orbit $F_1 \simeq \mathbb{R}H^5 \times \mathbb{E}$.*

(d) *The action of the subgroup $\mathrm{SL}(3,\mathbb{C})$ of G embedded in a standard way. It has an 8-dimensional totally geodesic singular orbit isometric to $\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$.*

(e) *The action of the connected Lie subgroup $H^\Lambda_{1,k}$, $k \in \{0,1,2,3\}$, of G with Lie algebra*

$$\mathfrak{h}^\Lambda_{1,k} = N_{\mathfrak{k}_1}(\mathfrak{w}) \oplus \mathfrak{w} \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1,$$

where $\mathfrak{w}$ is a k-dimensional subspace of $\mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_2}$ containing $\mathfrak{a}^1$ (unless $k = 0$). Here $N_{\mathfrak{k}_1}(\mathfrak{w}) \simeq \mathfrak{sp}(1) \oplus \mathfrak{so}(5-k) \oplus \mathfrak{so}(k)$, where the first summand is $Z_{\mathfrak{k}_0}(\mathfrak{b}_1)$ and the rest is the normalizer of $\mathfrak{w}$ in the Lie algebra $\mathfrak{g}'_1 \cap \mathfrak{k} \simeq \mathfrak{so}(5)$ of the isotropy group of the boundary component $B_1 \simeq \mathbb{R}H^5$. This action has a minimal[1] singular orbit of codimension $5-k$ and can be obtained by canonical extension of the cohomogeneity-one action on B_1 with $\mathbb{R}H^k$ as a totally geodesic singular orbit.

[1] By Proposition 2.2.41.

Proof. We consider the different cases of Theorem 5.1.7. If the orbits of H form a foliation, we get the actions in (a) and (b). No boundary component of M is reducible, so part (b)-(2) of Theorem 5.1.7 is not applicable. The actions in (c) and (d) are the only ones that have a totally geodesic singular orbit according to [BT04]. Here we need to check that the slice representations of L_1 and $\mathrm{SL}(3,\mathbb{C})$ are transitive on spheres to make sure that these two groups do indeed act on M with cohomogeneity one. It is not hard to show that $K_1 \simeq \mathrm{Sp}(2) \times \mathrm{Sp}(1)$ (hence L_1 is connected) and that the slice representation of L_1 at o is equivalent to the standard representation of $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ on $\mathbb{H}^2$, which is transitive on the unit sphere. Modulo $\mathbb{Z}/2\mathbb{Z}$, this is the isotropy representation of the quaternionic hyperbolic plane, which should not be surprising because $F_1^\perp \simeq \mathbb{H}H^2$. As for $\mathrm{SL}(3,\mathbb{C})$, its stabilizer at o is $\mathrm{SU}(3)$, and its slice representation at o is easily seen to be of cohomogeneity 1. In fact, it is isomorphic to the tautological representation of $\mathrm{SU}(3)$ on $\mathbb{C}^3$, which indicates that $(\mathrm{SL}(3,\mathbb{C}) \cdot o)^\perp \simeq \mathbb{C}H^3$.

Now we determine the actions induced by canonical extension. Since the root system of M is A_2, the boundary components B_1 and B_2 are congruent by Proposition 3.3.9, so it suffices to consider only actions arising from B_1. The boundary component is isometric to $\mathrm{SL}(2,\mathbb{H})/\mathrm{Sp}(2) \simeq \mathrm{SO}^0(5,1)/\mathrm{SO}(5) \cong \mathbb{R}H^5$. There are, up to strong orbit-equivalence, exactly 4 cohomogeneity-one actions with a singular orbit on $\mathbb{R}H^5$. According to Proposition 5.1.10 and Remark 5.1.4, the actions in (e) are the canonical extensions of these 4 actions on B_1. None of them are congruent to the actions in (c) and (d) because their singular orbits have different dimensions.

We are left to investigate the actions arising from the nilpotent construction. Again, we need only consider one of the indices $\{1,2\}$, but this time we choose $j = 2$ (to render the congruence problem a little less burdensome later on). Recall that for $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{H})$ with $\theta(X) = -X^*$, the standard choice of $\mathfrak{a}$ is the subspace of real diagonal traceless matrices. If we write $\varepsilon_i \colon \mathfrak{a} \to \mathbb{R}$, $X \mapsto X_{ii}$, then $\Sigma = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ and $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{H}E_{ij} \simeq \mathbb{H}$. We also choose $\Sigma^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$, in which case $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$. We have $\mathfrak{n}_2 = \mathfrak{n}_2^1 = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \simeq \mathbb{H}^2$ and $K_2 \simeq \mathrm{Sp}(2) \times \mathrm{Sp}(1)$. The representation of K_2 on $\mathfrak{n}_2$ is equivalent to the standard representation of $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ on $\mathbb{H}^2$:

$$(A, q) \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A \begin{bmatrix} xq^{-1} \\ yq^{-1} \end{bmatrix},$$

where A is a quaternion-unitary 2×2 matrix.

We first filter out those subspaces of $\mathfrak{n}_2$ that are not protohomogeneous. This problem was recently solved in greater generality for the standard action of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ on $\mathbb{H}^n$ by Díaz-Ramos, Domínguez-Vázquez, and Rodríguez-Vázquez in [DRDVRV21] (they also coined the term *protohomogeneous*). The authors explicitly classified protohomogeneous subspaces of $\mathbb{H}^n$ up to the action of $\mathrm{Sp}(n)\mathrm{Sp}(1)$ in terms of their quaternionic Kähler angle—the quaternionic analog of the Kähler angle of a real subspace of a Hilbert space first introduced by Berndt and Brück in [BB01]. We recall its definition via the following lemma, which was essentially proven in [BB01]:

Lemma 5.2.2. *Let V be a vector space endowed with a quaternionic structure $\mathcal{H}$ and a q-Hermitian Euclidean inner product. Let $\mathfrak{v} \subseteq V$ be a real subspace and $v \in \mathfrak{v}$ be a nonzero vector. There exists a canonical basis (J_1, J_2, J_3) of the quaternionic structure $\mathcal{H}$ and a* *uniquely defined triple* $(\varphi_1, \varphi_2, \varphi_3) \in [0, \frac{\pi}{2}]^3$ *such that:*

(a) φ_i *is the Kähler angle of* v *with respect to* J_i *(that is, the angle between* $J_i v$ *and* $\mathfrak{v}$*) for each* $i \in \{1,2,3\}$*,*

(b) $\langle \mathrm{pr}_{\mathfrak{v}} \circ J_i(v) \,|\, \mathrm{pr}_{\mathfrak{v}} \circ J_j(v) \rangle = 0$ *for each* $i \neq j$*, where* $\mathrm{pr}_{\mathfrak{v}}$ *is the orthogonal projector onto* $\mathfrak{v}$ *in* V*,*

(c) $\varphi_1 \leq \varphi_2 \leq \varphi_3$*,*

(d) φ_1 *is minimal and* φ_3 *is maximal among the Kähler angles of* v *with respect to all elements of* $\mathbb{S}^2_{\mathcal{H}}$*.*

In fact, (J_1, J_2, J_3) *is an orthonormal basis for* $\mathcal{J}$ *that diagonalizes the symmetric bilinear form*

$$\mathcal{J} \times \mathcal{J} \to \mathbb{R}, \quad (J, J') \mapsto \langle \mathrm{pr}_{\mathfrak{v}} \circ J(v) \,|\, \mathrm{pr}_{\mathfrak{v}} \circ J'(v) \rangle$$

with $\cos^2(\varphi_i)||v||^2$, $i \in \{1,2,3\}$ *, on the diagonal.*

Definition 5.2.3. Let V be a vector space endowed with a quaternionic structure $\mathcal{H}$ and a q-Hermitian Euclidean inner product. Let $\mathfrak{v} \subseteq V$ be a real subspace and $v \in \mathfrak{v}$ be a nonzero vector. The triple $(\varphi_1, \varphi_2, \varphi_3)$ as in Lemma 5.2.2 is called the **quaternionic Kähler angle** of $\mathfrak{v}$ with respect to v.

As with everything quaternionic, we are going to say *q-Kähler angle* for the sake of brevity. The q-Kähler angle of $\mathfrak{v}$ in general depends on v. However, we have the following fact (see [DRDVRV21, Lem. 2.4] for a proof):

Lemma 5.2.4. *If* $\mathfrak{v}$ *is a protohomogeneous subspace of* $\mathbb{H}^n$*, then it has constant q-Kähler angle, i.e., the angle does not depend on the choice of* $v \in \mathfrak{v} \setminus \{0\}$*. In this case, we call* $(\varphi_1, \varphi_2, \varphi_3)$ *simply the* ***quaternionic Kähler angle*** *of* $\mathfrak{v}$ *and denote it by* $\boldsymbol{\angle_q(\mathfrak{v})}$*.*

Example 5.2.5 (Q-Kähler angles of subspaces). Let V be a vector space endowed with a quaternionic structure $\mathcal{H}$ and a q-Hermitian Euclidean inner product, and let $\mathfrak{v} \subseteq V$ be a real subspace. Then:

(a) $\mathfrak{v}$ is quaternionic $\Leftrightarrow$ it is protohomogeneous[1] of q-Kähler angle $(0,0,0)$.

(b) $\mathfrak{v}$ is totally complex $\Leftrightarrow$ it is protohomogeneous of q-Kähler angle $(0, \frac{\pi}{2}, \frac{\pi}{2})$.

(c) $\mathfrak{v}$ is totally real $\Leftrightarrow$ it is protohomogeneous of q-Kähler angle $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$. //

The classification of protohomogeneous subspaces of $\mathbb{H}^n$ obtained in [DRDVRV21, Th. A] shows that a k-dimensional protohomogeneous subspace $\mathfrak{v} \subseteq \mathbb{H}^n$ is "almost always" determined by its q-Kähler angle up to $\mathrm{Sp}(n)\mathrm{Sp}(1)$: the only exception is when $k \leq n$ is equal to 3 or congruent to 0 (mod 4), in which case there may be (at most) two $\mathrm{Sp}(n)\mathrm{Sp}(1)$-noncongruent protohomogeneous subspaces of dimension k with the same q-Kähler angle. Since in our case $n = 2$, this uncertainty does not concern us. Using this classification, we see that there are the following possibilities for $\mathfrak{v} \subseteq \mathfrak{n}_2$:

- CASE 1: $\dim \mathfrak{v} = 2$ and the q-Kähler angle of $\mathfrak{v}$ is $(\varphi, \frac{\pi}{2}, \frac{\pi}{2})$ for some $\varphi \in [0, \frac{\pi}{2}]$.

[1] Here protohomogeneity is understood with respect to the representation of $\mathrm{Sp}(V) \cdot \mathbb{S}^3_{\mathcal{H}}$ on V.

- Case 2: $\dim \mathfrak{v} = 3$ and the q-Kähler angle of $\mathfrak{v}$ is $(\varphi, \varphi, \frac{\pi}{2})$ for some $\varphi \in \{0, \frac{\pi}{3}\}$.
- Case 3: $\dim \mathfrak{v} = 4$ and the q-Kähler angle of $\mathfrak{v}$ is $(0, \varphi, \varphi)$ for some $\varphi \in [0, \frac{\pi}{2}]$.

We start our investigation with

Case 1: $\dim \mathfrak{v} = 2$, $\angle_q(\mathfrak{v}) = (\varphi, \frac{\pi}{2}, \frac{\pi}{2})$, $\varphi \in [0, \frac{\pi}{2}]$. First, assume that $\varphi = 0$. In that case, $\mathfrak{v}$ is a totally complex subspace of $\mathfrak{n}_2$ and we can take it to be $\mathbb{C}jE_{23}$ so that $\mathfrak{n}_{2,\mathfrak{v}} = \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathbb{C}E_{23}$. Now we need to check whether $\mathfrak{v}$ is admissible. To this end, we have to compute $N_{\mathfrak{m}_2}(\mathfrak{v})$ and see if its projection to $\mathfrak{p}$ is the whole $\mathfrak{b}_2$ or a proper subspace of it. Observe that

$$\mathfrak{m}_2 = \left\{ \begin{pmatrix} p_{11} & p_{12} & 0 \\ p_{21} & p_{22} & 0 \\ 0 & 0 & q \end{pmatrix} \middle| \operatorname{Re}(p_{11}) + \operatorname{Re}(p_{22}) = 0, \operatorname{Re}(q) = 0 \right\} \simeq \mathfrak{sl}(2, \mathbb{H}) \oplus \mathfrak{sp}(1).$$

By commuting such matrices with jE_{23} and kE_{23}, one readily sees that

$$N_{\mathfrak{m}_2}(\mathfrak{v}) = \left\{ \begin{pmatrix} p_{11} & 0 & 0 \\ p_{21} & p_{22} & 0 \\ 0 & 0 & q \end{pmatrix} \middle| \operatorname{Re}(p_{11}) + \operatorname{Re}(p_{22}) = 0, p_{22} \in \mathbb{C}, q \in \mathbb{R}i \right\}.$$

The projection of this to $\mathfrak{p}$ is clearly equal to $\mathfrak{b}_2$, so $\mathfrak{v} = \mathbb{C}jE_{23} \subseteq \mathfrak{n}_2$ produces a cohomogeneity-one action. The corresponding Lie algebra is

$$\mathfrak{h}_{2,\mathfrak{v}} = (\mathbb{R}iE_{11} \oplus \operatorname{Im}(\mathbb{H})E_{22} \oplus \mathbb{R}iE_{33}) \oplus \mathfrak{a} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathbb{C}E_{23} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}, \tag{5.2.1}$$

where the first summand in the parentheses is $\mathfrak{h}_{2,\mathfrak{v}} \cap \mathfrak{k} = \mathfrak{h}_{2,\mathfrak{v}} \cap \mathfrak{k}_0$. We claim that this action is orbit-equivalent to the canonical extension of a cohomogeneity-one action on B_1 with a singular orbit of codimension 2. Indeed, in the notation of (e), we take $k = 3$ and $\mathfrak{w} = \mathfrak{a}^1 \oplus \mathbb{C}E_{23}$. Then we have:

$$\mathfrak{h}^{\Lambda}_{1,3} = N_{\mathfrak{k}_1}(\mathfrak{w}) \oplus \mathfrak{a} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathbb{C}E_{23} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}. \tag{5.2.2}$$

Observe that $\mathfrak{h}_{2,\mathfrak{v}}$ and $\mathfrak{h}^{\Lambda}_{1,3}$ are Lie subalgebras of the parabolic subalgebra $\mathfrak{q}_1$, and they both sit nicely within the Langlands decomposition $\mathfrak{q}_1 = \mathfrak{m}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1$: $\mathfrak{h}_{2,\mathfrak{v}} = (\mathfrak{h}_{2,\mathfrak{v}} \cap \mathfrak{m}_1) \oplus (\mathfrak{h}_{2,\mathfrak{v}} \cap \mathfrak{a}_1) \oplus (\mathfrak{h}_{2,\mathfrak{v}} \cap \mathfrak{n}_1)$, and the same for $\mathfrak{h}^{\Lambda}_{1,3}$. It follows that, in terms of the horospherical decomposition $M = B_1 \times A_1 \times N_1$, the singular orbits of the actions of $H_{2,\mathfrak{v}}$ and $H^{\Lambda}_{1,3}$ are $((H_{2,\mathfrak{v}} \cap M_1) \cdot o) \times A_1 \times N_1$ and $((H^{\Lambda}_{1,3} \cap M_1) \cdot o) \times A_1 \times N_1$, respectively. The first factors here are the singular orbits of the cohomogeneity-one actions of $H_{2,\mathfrak{v}} \cap M_1$ and $H^{\Lambda}_{1,3} \cap M_1$ on B_1, respectively. Looking at the decompositions (5.2.1) and (5.2.2), it is clear that these singular orbits coincide, since they both correspond to the Lie triple system $\mathfrak{a}^1 \oplus \{\lambda E_{23} + \bar{\lambda} E_{32} \mid \lambda \in \mathbb{C}\}$. Since the singular orbits of $H_{2,\mathfrak{v}}$ and $H^{\Lambda}_{1,3}$ coincide, these groups have the same orbits, for all the other orbits are just equidistant tubes around the singular one.

Next, assume that $\varphi = \frac{\pi}{2}$. Then $\mathfrak{v}$ is a totally real subspace of $\mathfrak{n}_2$, so, acting by K_2 if necessary, we may assume $\mathfrak{v} = \mathbb{R}E_{13} \oplus \mathbb{R}E_{23}$. Simple computations reveal that a matrix $X \in \mathfrak{m}_2$ normalizing $\mathfrak{v}$ must have $p_{12}, p_{21} \in \mathbb{R}$, which means that the image of the projection of $N_{\mathfrak{m}_2}(\mathfrak{v})$ to $\mathfrak{p}$ will be a proper subspace of $\mathfrak{b}_2$, so such $\mathfrak{v}$ is not an admissible subspace.

We are left to consider the case $\varphi \in (0, \frac{\pi}{2})$. Here $\mathfrak{v}$ is a subspace of constant Kähler angle φ inside a totally complex subspace of $\mathfrak{n}_2$ of real dimension 4. Without loss of generality, we choose the latter to be $\mathbb{C}E_{13} \oplus \mathbb{C}E_{23}$. Then we can take $\mathfrak{v}$ to be the real span of E_{23} and $i\cos\varphi E_{23} + i\sin\varphi E_{13}$ (see [BB01, Prop. 7]). By commuting elements of $\mathfrak{m}_2$ with these two vectors and solving simple systems of linear equations, one gets, among other things, that $\mathrm{Re}(p_{12}) = 0$ and $\mathrm{Re}(p_{21}) = 2\mathrm{Re}(p_{11})\cot\varphi$. It implies that the image of the projection of $N_{\mathfrak{m}_2}(\mathfrak{v})$ to $\mathfrak{p}$ is contained in $\mathfrak{b}_2 \cap \{\mathrm{Re}(p_{11})\cot\varphi - \mathrm{Re}(p_{12}) = 0\}$, which is a linear hyperplane in $\mathfrak{b}_2$. Therefore, this $\mathfrak{v}$ is not admissible either.

Case 2: $\dim\mathfrak{v} = 3$, $\angle_q(\mathfrak{v}) = (\varphi, \varphi, \frac{\pi}{2})$, $\varphi \in \{0, \frac{\pi}{3}\}$. First, suppose $\varphi = 0$. Such $\mathfrak{v}$ can be described as $\mathrm{Im}(\mathbb{H})v$ for some nonzero $v \in \mathfrak{n}_2$. We may assume without loss of generality that $v = E_{23}$. One then easily computes:

$$N_{\mathfrak{m}_2}(\mathfrak{v}) = \left\{ \begin{pmatrix} p_{11} & 0 & 0 \\ p_{21} & p_{22} & 0 \\ 0 & 0 & q \end{pmatrix} \middle| \, \mathrm{Re}(p_{11}) + \mathrm{Re}(p_{22}) = 0, q = \mathrm{Im}(p_{22}) \right\}.$$

The image of the projection of that to $\mathfrak{p}$ is the whole of $\mathfrak{b}_2$, so this $\mathfrak{v}$ is admissible. We have:

$$\mathfrak{h}_{2,\mathfrak{v}} = (\mathrm{Im}(\mathbb{H})E_{11} \oplus \mathrm{Im}(\mathbb{H})(E_{22} + E_{33})) \oplus \mathfrak{a} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathbb{R}E_{23} \oplus \mathfrak{g}_{\alpha_1+\alpha_2},$$

where the first summand in the parentheses is $\mathfrak{h}_{2,\mathfrak{v}} \cap \mathfrak{k} = \mathfrak{h}_{2,\mathfrak{v}} \cap \mathfrak{k}_0 \simeq \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. In a similar vein to what we did in case 1 with $\varphi = 0$, one can show that the orbits of $H_{2,\mathfrak{v}}$ coincide with the orbits of $H^{\Lambda}_{1,2}$ if we take $\mathfrak{w} = \mathfrak{a}^1 \oplus \mathbb{R}E_{23}$. Therefore, this $\mathfrak{v}$ does not produce a new action.

Now let $\mathfrak{v}$ be of q-Kähler angle $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{2})$. It follows from [DRDVRV21, Prop. 5.3, Rem. 5.4] that $\mathfrak{v} = \mathrm{span}\{E_{23}, iE_{23} + i\sqrt{3}E_{13}, jE_{23} - j\sqrt{3}E_{13}\}$ does the trick. Simple calculations reveal that an element of $\mathfrak{m}_2$ normalizing $\mathfrak{v}$ must have $\mathrm{Re}(p_{11}) = \mathrm{Re}(p_{22}) = 0$, so the image of the projection of $N_{\mathfrak{m}_2}(\mathfrak{v})$ to $\mathfrak{p}$ will be a proper subspace of $\mathfrak{b}_2$, hence this $\mathfrak{v}$ is not admissible.

Case 3: $\dim\mathfrak{v} = 4$, $\angle_q(\mathfrak{v}) = (0, \varphi, \varphi)$, $\varphi \in [0, \frac{\pi}{2}]$. First, let $\varphi = 0$, that is, let $\mathfrak{v}$ be a quaternionic line in $\mathfrak{m}_2$. Without loss of generality, we choose $\mathfrak{v} = \mathbb{H}E_{23} = \mathfrak{g}_{\alpha_2}$. One immediately sees that $N_{\mathfrak{m}_2}(\mathfrak{v}) = \mathfrak{m}_2 \ominus \mathfrak{g}_{\alpha_1}$, whose projection to $\mathfrak{p}$ is the whole $\mathfrak{b}_2$, so this $\mathfrak{v}$ is admissible. We have:

$$\mathfrak{h}_{2,\mathfrak{v}} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}.$$

But this Lie subalgebra coincides with $\mathfrak{h}^{\Lambda}_{1,1}$ if we take $\mathfrak{w} = \mathfrak{a}^1$. Consequently, $\mathfrak{v}$ does not give a new action.

Now let the q-Kähler angle of $\mathfrak{v}$ be $(0, \frac{\pi}{2}, \frac{\pi}{2})$. Then $\mathfrak{v}$ is a totally complex subspace, hence we may assume $\mathfrak{v} = \mathbb{C}E_{13} \oplus \mathbb{C}E_{23}$. Any element of $\mathfrak{m}_2$ normalizing such a subspace must have $p_{12}, p_{21} \in \mathbb{C}$, so the image of the projection of $N_{\mathfrak{m}_2}(\mathfrak{v})$ to $\mathfrak{p}$ is smaller than $\mathfrak{b}_2$ and $\mathfrak{v}$ is not admissible.

Finally, let $\mathfrak{v}$ be of q-Kähler angle $(0, \varphi, \varphi)$, $\varphi \in (0, \frac{\pi}{2})$. According to [BB01] (see the discussion before Theorem 5 there), we can take $\mathfrak{v}$ to be spanned by $E_{23}, iE_{23}, j\cos\varphi E_{23} +$

$j \sin \varphi E_{13}$, and $k \cos \varphi E_{23} + k \sin \varphi E_{13}$. In a similar fashion to what we had in case 1, by solving the system of linear equations defining $N_{\mathfrak{m}_2}(\mathfrak{v})$, one gets—among other things—the same two linear dependencies $\mathrm{Re}(p_{12}) = 0$ and $\mathrm{Re}(p_{21}) = 2\mathrm{Re}(p_{11}) \cot \varphi$. Therefore, the projection of $N_{\mathfrak{m}_2}(\mathfrak{v})$ to $\mathfrak{p}$ is yet again contained in $\mathfrak{b}_2 \cap \{\mathrm{Re}(p_{11}) \cot \varphi - \mathrm{Re}(p_{12}) = 0\}$ and $\mathfrak{v}$ fails to be admissible. This shows that the nilpotent construction yields no new actions for M and finishes the proof of Theorem 5.2.1. □

5.3. Classification of cohomogeneity-one actions on $\mathrm{SO}(5, \mathbb{C})/\mathrm{SO}(5)$

In this section we classify, up to orbit-equivalence, cohomogeneity-one actions on the noncompact dual of the compact Lie group Spin(5).

The symmetric space $M = \mathrm{SO}(5, \mathbb{C})/\mathrm{SO}(5)$ is irreducible and of noncompact type; it has rank 2 and dimension 10, and its restricted root system is B_2. As we noticed in scenario 1 in the proof of Theorem 3.2.10, the restricted root system (resp., root space decomposition) of $\mathfrak{so}(5, \mathbb{C})$ coincides with the root system (resp., root space decomposition) of $\mathfrak{so}(5, \mathbb{C})$ considered as a complex simple Lie algebra. As a result, all the root multiplicities are equal to 2. Write $\Lambda = \{\alpha_1, \alpha_2\}$, where α_1 is the long root. Then $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$. We also have $\mathfrak{k}_0 = i\mathfrak{a} = \mathbb{R}iH_{\alpha_1} \oplus^\perp \mathbb{R}iH^2 = \mathbb{R}iH^1 \oplus^\perp \mathbb{R}iH_{\alpha_2} \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(1)$, where i the complex structure of $\mathfrak{so}(5, \mathbb{C})$. The fact that α_1 and α_2 have different lengths implies that the corresponding boundary components $B_1 \simeq \mathbb{R}H^3$ and $B_2 \simeq \mathbb{R}H^3$ have different sectional curvatures ($-||\alpha_1||^2$ and $-||\alpha_2||^2$, respectively) and thus are not congruent (which reflects the asymmetry of the Dynkin diagram B_2).

Theorem 5.3.1. *Let H be a connected Lie group acting properly and isometrically on $M = \mathrm{SO}(5, \mathbb{C})/\mathrm{SO}(5)$ with cohomogeneity 1. Then its action is orbit-equivalent to exactly one of the following:*

(a) *The action of the connected Lie subgroup H_ℓ of G with Lie algebra*

$$\mathfrak{h}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n},$$

where ℓ is a one-dimensional linear subspace of $\mathfrak{a}$. The orbits of H_ℓ are all congruent to each other and form a Riemannian foliation on M.

(b) *The action of the connected Lie subgroup $H_{\alpha_i}, i \in \{1, 2\}$, of G with Lie algebra*

$$\mathfrak{h}_{\alpha_i} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell_{\alpha_i}),$$

where ℓ_{α_i} is any one-dimensional linear subspace of $\mathfrak{g}_{\alpha_i}$. Its orbits form a Riemannian foliation on M and there is exactly one minimal orbit.

(c) *The action of the subgroup $\mathrm{SO}(4, \mathbb{C})$ of G embedded in a standard way. It has a 6-dimensional totally geodesic singular orbit isometric to $\mathrm{SO}(4, \mathbb{C})/\mathrm{SO}(4) \simeq \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2) \times \mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2) \simeq \mathbb{R}H^3 \times \mathbb{R}H^3$.*

(d) *The action of the connected Lie subgroup $H^\Lambda_{j,0}, j \in \{1, 2\}$, of G with Lie algebra*

$$\mathfrak{h}^\Lambda_{j,0} = \mathfrak{k}_j \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j.$$

This action has a minimal singular orbit of codimension 3 *and can be obtained by canonical extension of the cohomogeneity-one action on* B_j *with a single point as a singular orbit.*

(e) *The action of the connected Lie subgroup* $H^{\wedge}_{j,1}$, $j \in \{1,2\}$, *of* G *with Lie algebra*

$$\mathfrak{h}^{\wedge}_{j,1} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}_j,$$

This action has a minimal singular orbit of codimension 2 *and can be obtained by canonical extension of the cohomogeneity-one action on* B_j *with a geodesic as a singular orbit.*

Proof. We consider the different cases of Theorem 5.1.7. If the orbits of H form a foliation, we get the actions in (a) and (b). As M is irreducible of rank 2, part (b)-(2) of Theorem 5.1.7 does not apply. The action in (c) is the only one that has a totally geodesic singular orbit according to [BT04]. The stabilizer of $\mathrm{SO}(4,\mathbb{C})$ at o is $\mathrm{SO}(4)$, whose slice representation at o is equivalent to the tautological one (which reflects the fact that $(\mathrm{SO}(4,\mathbb{C}) \cdot o)^{\perp} \simeq \mathbb{R}H^4$), hence $\mathrm{SO}(4,\mathbb{C})$ does indeed act with cohomogeneity one. It is also worth noting that we can take

$$\mathfrak{so}(4,\mathbb{C}) = \mathfrak{g}_0 \oplus \bigoplus_{\substack{\alpha\in\Sigma \\ \text{long root}}} \mathfrak{g}_\alpha.$$

The long roots in B_2 form a root subsystem isomorphic to $A_1 \sqcup A_1$, whose irreducible components A_1 correspond to the two de Rham factors $\mathbb{R}H^3$ of $\mathrm{SO}(4,\mathbb{C})/\mathrm{SO}(4)$. As a consequence, these two hyperbolic spaces have the same curvature.

Now we determine the actions arising via the canonical extension method. Each boundary component B_j is isometric to the real hyperbolic space $\mathbb{R}H^3$ (with different curvatures depending on j), which has precisely two cohomogeneity-one actions with a singular orbit up to strong orbit-equivalence. One of them is the action of the restricted isotropy group of $\mathbb{R}H^3$, which has a single point as a singular orbit and whose canonical extension is described in (d). The other one has a geodesic as a singular orbit and, in view of Proposition 5.1.10, is given by the connected Lie subgroup corresponding to $N_{\mathfrak{k}_j}(\mathfrak{a}^j) \oplus \mathfrak{a}^j = (\mathfrak{k}_j)_0 \oplus \mathfrak{a}^j$. Thanks to Remark 5.1.4, its canonical extension is given in (e). Observe that the actions in (d) and (e) are not orbit-equivalent to each other because the normal spaces of their singular orbits are tangent to the corresponding boundary components and thus have different sectional curvatures.

Now we proceed to the main part, namely the nilpotent construction. Since the Dynkin diagram B_2 is asymmetric, we have to consider two cases.

Nilpotent construction with $j = 2$. In this case, we have:

$$\begin{aligned}
\mathfrak{n}_2^1 &= \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \simeq \mathbb{C}^2,\\
\mathfrak{l}_2 &= \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} = \mathfrak{g}_2 \oplus \mathfrak{z}_2 \oplus \mathfrak{a}_2 \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \simeq \mathfrak{gl}(2,\mathbb{C}),\\
\mathfrak{k}_2 &= \mathfrak{k}_0 \oplus \mathfrak{k}_{\alpha_1} = (\mathfrak{g}_2 \cap \mathfrak{k}) \oplus \mathfrak{z}_2 \simeq \mathfrak{su}(2) \oplus \mathfrak{u}(1) \simeq \mathfrak{u}(2).
\end{aligned}$$

The adjoint representation of $\mathfrak{g}_2$ on $\mathfrak{n}_2^1$ is a nontrivial complex representation, hence it is equivalent to the irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^2$. The adjoint representation of $\mathfrak{a}_2 \oplus \mathfrak{z}_2 = \mathbb{C}H^2$ on $\mathfrak{n}_2^1$ is the standard complex representation by scalars because H^2 acts on $\mathfrak{n}_2^1$ as multiplication by $\langle \alpha_2, H^2 \rangle = 1$. It follows that the adjoint representation of $\mathfrak{l}_2$ (resp., $\mathfrak{k}_2$) on $\mathfrak{n}_2^1$ is equivalent to the tautological representation of $\mathfrak{gl}(2,\mathbb{C})$ (resp., $\mathfrak{u}(2)$) on $\mathbb{C}^2$. Since $\mathfrak{l}_2$ is a θ-stable Lie subalgebra of $\mathfrak{g}$, we may assume θ corresponds to the standard Cartan involution on $\mathfrak{gl}(2,\mathbb{C})$. Therefore, the problem of finding admissible and protohomogeneous subspaces of the L_2^0-module $\mathfrak{n}_2^1$ is equivalent to the analogous problem for the tautological representation of $\mathrm{GL}(2,\mathbb{C})$.

Let $\mathfrak{v} \subseteq \mathfrak{n}_2^1$ be a linear subspace of dimension at least 2. According to [BB01, Lem. 1, Prop. 7], $\mathfrak{v}$ is protohomogeneous if and only if it has constant Kähler angle. In particular, it must be even-dimensional. If $\mathfrak{v} = \mathfrak{n}_2^1$, then it is trivially admissible and we have $\mathfrak{h}_{2,\mathfrak{v}} = \mathfrak{l}_2 \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} = \mathfrak{so}(4,\mathbb{C}) \ominus \mathfrak{g}_{-\alpha_1-2\alpha_2}$. Since we know that $H_{2,\mathfrak{v}}$ acts with cohomogeneity one, it follows that it has the same orbits as $\mathrm{SO}(4,\mathbb{C})$, so its action has already been accounted for (c). Now suppose $\mathfrak{v}$ has real dimension 2. It was shown in [BDV15, Th. 6] that such a subspace is admissible if and only if its Kähler angle is zero, i.e., if it is a complex subspace. Up to the action of $K_2^0 \simeq \mathrm{U}(2)$, we may assume $\mathfrak{v} = \mathfrak{g}_{\alpha_2}$. We have

$$\mathfrak{h}_{2,\mathfrak{v}} = \mathfrak{l}_2 \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_1.$$

If we look back at (e), we see that $\mathfrak{h}_{1,1}^{\wedge}$ is a Lie subalgebra of $\mathfrak{h}_{2,\mathfrak{v}}$. Since $H_{1,1}^{\wedge}$ acts with cohomogeneity one, its orbits coincide with the orbits of $H_{2,\mathfrak{v}}$. Altogether, we see that the nilpotent construction method with $j = 2$ does not give rise to any new actions.

Nilpotent construction with $j = 1$. In this case we have:

$$\begin{aligned}
&\mathfrak{n}_1^1 = \mathfrak{n}_1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \simeq \mathbb{C}^3,\\
&\mathfrak{l}_1 = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} = \mathfrak{g}_1 \oplus \mathfrak{z}_1 \oplus \mathfrak{a}_1 \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \simeq \mathfrak{gl}(2,\mathbb{C}),\\
&\mathfrak{k}_2 = \mathfrak{k}_0 \oplus \mathfrak{k}_{\alpha_2} = (\mathfrak{g}_1 \cap \mathfrak{k}) \oplus \mathfrak{z}_1 \simeq \mathfrak{su}(2) \oplus \mathfrak{u}(1) \simeq \mathfrak{u}(2).
\end{aligned}$$

We fix an isomorphism between $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{g}_1$ by sending the standard basis e, f, h of the former to $X, \theta X, h_{\alpha_2} \in \mathfrak{g}_1$, where $X \in \mathfrak{g}_{\alpha_2}, \theta X \in \mathfrak{g}_{-\alpha_2}$, and $h_{\alpha_2} = \frac{2}{||\alpha_2||^2} H_{\alpha_2}$. Observe that $\mathfrak{n}_1$ is an α_2-string, so it is an irreducible complex representation of $\mathfrak{g}_1 \simeq \mathfrak{sl}(2,\mathbb{C})$. Since it is 3-dimensional, it is isomorphic to the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$. Now, $\mathfrak{sl}(2,\mathbb{C})$ is simple over $\mathbb{C}$ and hence over $\mathbb{R}$, which means that the representation of $\mathfrak{g}_1$ on $\mathfrak{n}_1$ is irreducible as a real representation. An alternative description of this representation is the second symmetric power of the tautological representation of $\mathfrak{sl}(2,\mathbb{C})$. The adjoint representation of $\mathfrak{a}_1 \oplus \mathfrak{z}_1 = \mathbb{C}H^1$ on $\mathfrak{n}_1^1$ is the standard complex representation by scalars because H^1 acts on $\mathfrak{n}_1$ as multiplication by $\langle \alpha_1, H^1 \rangle = 1$. Consequently, we can describe the adjoint representation of $\mathfrak{l}_1 \simeq \mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$ (resp., $\mathfrak{k}_1 \simeq \mathfrak{su}(2) \oplus \mathfrak{u}(1)$) on $\mathfrak{n}_1$ as the exterior tensor product $\rho_3 \otimes \sigma$, where ρ_3 is the irreducible 3-dimensional complex representation of $\mathfrak{sl}(2,\mathbb{C})$ (resp., of $\mathfrak{su}(2)$) and σ is the tautological representation of $\mathbb{C}$ (resp., $\mathfrak{u}(1)$) on $\mathbb{C}$.

Now we need to determine—up to $K_1^0 \simeq \mathrm{U}(2)$—all subspaces $\mathfrak{v}$ of $\mathfrak{n}_1$ that are both protohomogeneous and admissible. We begin by borrowing an argument from [BDV15, Prop. 6]. The case $\mathfrak{v} = \mathfrak{n}_1$ can be excluded straight away, since $\mathrm{U}(2)$ cannot act transitively on the 5-sphere. According to [BDV15, Prop. 5], we may assume that $N_{\mathfrak{m}_1}(\mathfrak{v}) = \theta N_{\mathfrak{m}_1}(\mathfrak{n}_{1,\mathfrak{v}})$ is contained in $(\mathfrak{g}_1 \cap (\mathfrak{g}_0 \oplus \mathfrak{n})) \oplus \mathfrak{z}_1 = \mathfrak{k}_0 \oplus \mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_2}$. In this case, $N_{\mathfrak{k}_1}(\mathfrak{v})$ is contained in $\mathfrak{k}_0 \simeq \mathfrak{u}(1) \oplus \mathfrak{u}(1)$. But then $\mathfrak{v}$ must be 2-dimensional, for $K_0 \simeq \mathrm{U}(1) \times \mathrm{U}(1)$ cannot act transitively on any sphere of dimension greater than one. We will show that—under the assumption $N_{\mathfrak{m}_1}(\mathfrak{v}) \subseteq \mathfrak{k}_0 \oplus \mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_2}$—the only option for $\mathfrak{v}$ is $\mathfrak{g}_{\alpha_1+2\alpha_2}$.

Take nonzero vectors $e_j \in \mathfrak{g}_{\alpha_1+j\alpha_2}, j \in \{0,1,2\}$, such that $e_j = \mathrm{ad}(X)e_{j-1}$. Let $v = \sum_{j=0}^{2} z_j e_j \in \mathfrak{v}$ be a nonzero vector. An arbitrary element of $\mathfrak{k}_0 \oplus \mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_2}$ can be written as

$$Y = (a+ib)X + ch_{\alpha_2} + i(dh_{\alpha_2} + eH^1),$$

where a, b, c, d, and e are some real numbers. If $\mathfrak{v}$ lies in $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$, then a and b must be zero for Y to normalize $\mathfrak{v}$, which means that the projection of $N_{\mathfrak{m}_1}(\mathfrak{v})$ to $\mathfrak{p}$ will be smaller than $\mathfrak{b}_1$ in that case. Hence, we may assume that $z_2 \neq 0$ and, for the sake of contradiction, that either z_0 or z_1 is also nonzero. Observe that $\mathrm{ad}(h_{\alpha_2})$ acts diagonally on $\mathfrak{n}_1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}$ with eigenvalues $-2, 0$, and 2, whereas $\mathrm{ad}(H^1)$ acts as the identity on the whole $\mathfrak{n}_1$. Consequently, $i(dh_{\alpha_2} + eH^1)$ acts on $\mathfrak{n}_1$ diagonally with eigenvalues $i(e-2d), ie, i(e+2d)$. It implies that $N_{\mathfrak{k}_1}(\mathfrak{v}) \subseteq \mathfrak{k}_0$ must be one-dimensional (otherwise it would be the whole $\mathfrak{k}_0$ and $\mathfrak{v}$ would have to be of dimension at least 3). In this situation, $N_{\mathfrak{k}_1}(\mathfrak{v})$ is spanned by some $i(d_0 h_{\alpha_2} + e_0 H^1)$ and $\mathfrak{v} = \mathbb{R}v \oplus \mathbb{R}\,\mathrm{ad}(i(d_0 h_{\alpha_2} + e_0 H^1))v$. In order for $\mathrm{ad}(Y)$ to normalize $\mathfrak{v}$, $\mathrm{ad}(Y)v$ must be a linear combination of v and $\mathrm{ad}(i(d_0 h_{\alpha_2} + e_0 H^1))v$, which boils down to the following system of equations:

$$\begin{cases} (-2c + i(e-2d))z_1 = \lambda z_1 + i\mu(e_0 - 2d_0)z_1, \\ (a+ib)z_1 + iez_2 = \lambda z_2 + i\mu e_0 z_2, \\ (a+ib)z_2 + (2c + i(e+2d))z_3 = \lambda z_3 + i\mu(e_0 + 2d_0)z_3. \end{cases}$$

First assume that $z_1 \neq 0$. The first equation then implies $\lambda = -2c$, which, when substituted into the second one, gives

$$a + ib = \frac{z_2}{z_1}(-2c + i(\mu e_0 - e)).$$

By writing $\dfrac{z_2}{z_1} = f + ig$, we obtain

$$\begin{cases} a = -2fc - g(\mu e_0 - e), \\ b = -2gc + f(\mu e_0 - e). \end{cases}$$

Whatever f and g are, we get a linear dependency on a, b, e, which means that $\mathfrak{v}$ cannot be admissible. In case $z_1 = 0$ but $z_2 \neq 0$, the second equation of our system implies $\lambda = 0$, which transforms the third equation into

$$a + ib = \frac{z_3}{z_2}(-2c + i(\mu(e_0 + 2d_0) - e - 2d)).$$

In a similar way, we get a linear dependency on a, b, e.

The upshot of the above argument is that, up to K_1^0, $\mathfrak{v}$ must be equal to $\mathfrak{g}_{\alpha_1+2\alpha_2}$. Observe that this is indeed an admissible and protohomogeneous subspace. Since the representation of $\mathrm{SU}(2) \subseteq K_1^0$ on $\mathfrak{n}_1$ is equivalent to the 3-dimensional complex irreducible representation of $\mathrm{SU}(2)$, there exists an element of K_1^0 that maps $\mathfrak{g}_{\alpha_1+2\alpha_2}$ onto $\mathfrak{g}_{\alpha_1}$ (in terms of the representation on the space of quadratic polynomials, we can take the special unitary matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, which induces $y^2 \mapsto x^2$). We have:

$$\mathfrak{n}_{1,\mathfrak{g}_{\alpha_1}} = \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2},$$

$$N_{\mathfrak{m}_1}(\mathfrak{n}_{1,\mathfrak{g}_{\alpha_1}}) = \theta N_{\mathfrak{m}_1}(\mathfrak{g}_{\alpha_1}) = \mathfrak{k}_0 \oplus \mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_2},$$
$$\mathfrak{h}_{1,\mathfrak{g}_{\alpha_1}} = N_{\mathfrak{m}_1}(\mathfrak{n}_{1,\mathfrak{g}_{\alpha_1}}) \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_{1,\mathfrak{g}_{\alpha_1}} = \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} = \mathfrak{g}_0 \oplus \mathfrak{n}_2.$$

But looking at the actions given in (e), one sees that $\mathfrak{h}_{1,\mathfrak{g}_{\alpha_1}} = \mathfrak{h}^{\wedge}_{2,1}$, which means that the actions of $H^{\wedge}_{2,1}$ and $H_{1,\mathfrak{g}_{\alpha_1}}$ have the same orbits. We conclude that the nilpotent construction method does not give any new actions for M, which completes the proof. □

5.4. Classification of cohomogeneity-one actions on the noncompact complex two-plane Grassmannians

In this section we classify, up to orbit-equivalence, cohomogeneity-one actions on the noncompact complex Grassmannians $\mathrm{Gr}^*(2,\mathbb{C}^{n+4}) = \mathrm{SU}(n+2,2)/\mathrm{S}(\mathrm{U}(n+2)\mathrm{U}(2)), n \geq 1$.

The symmetric space $M = \mathrm{SU}(n+2,2)/\mathrm{S}(\mathrm{U}(n+2)\mathrm{U}(2))$ is irreducible and of noncompact type; it has rank 2 and dimension $4n+8$, and its restricted root system is $(BC)_2$. We have added the restriction $n \geq 1$ because $\mathrm{Gr}^*(2,\mathbb{C}^4) \simeq \mathrm{Gr}^*(2,\mathbb{R}^6)$ and $\mathrm{Gr}^*(2,\mathbb{C}^3) = \mathbb{C}H^2$, both of which are spaces of different type. The space M is the complex analog of the symmetric space $\mathrm{Gr}^*(2,\mathbb{R}^{n+4}) = \mathrm{SO}^0(n+2,2)/\mathrm{SO}(n+2)\mathrm{SO}(2)$ of type B_2, a classification of cohomogeneity-one actions on which was obtained by Berndt and Domínguez-Vázquez in [BDV15]. If we choose simple roots α_1 and α_2 so that $2\alpha_2$ is also a root, we have $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_2, \alpha_1+2\alpha_2, 2\alpha_1+2\alpha_2\}$, where α_1 and $\alpha_1+2\alpha_2$ have multiplicity 2, α_2 and $\alpha_1+\alpha_2$ have multiplicity $2n$, and $2\alpha_2$ and $2\alpha_1+2\alpha_2$ have multiplicity 1. The Lie algebra $\mathfrak{k}_0$ is isomorphic to $\mathfrak{u}(n) \oplus \mathfrak{u}(1)$. As we mentioned in Subsection 2.5.2, M is the only symmetric space of noncompact type that is both Hermitian and q-Kähler. The interplay between these two structures will aid us to deal with the congruence problem. The boundary components B_1 and B_2 are isometric to $\mathbb{C}H^{n+1}$ and $\mathbb{R}H^3$, respectively. Therefore, for the first time, we are encountering a rank-2 symmetric space of noncompact type containing a boundary component not isometric to the real hyperbolic space. The reason why this is special is because, unlike real hyperbolic spaces, complex hyperbolic spaces have a nondiscrete moduli space of C1-actions with a singular orbit, so the canonical extension method, when applied to B_1, will produce a one-parameter family of C1-actions on M.

In order to formulate the theorem, we need to know what the almost complex structure I on M looks like in terms of the restricted root space decomposition of $\mathfrak{g}$. With respect to I_o, $\mathfrak{p}_{\alpha_2}$ and $\mathfrak{p}_{\alpha_1+\alpha_2}$ are complex subspaces of $\mathfrak{p}$. Moreover, I_o swaps $\mathfrak{p}_{\alpha_1}$ and $\mathfrak{p}_{\alpha_1+2\alpha_2}$, so each of them is a totally real subspace. Finally, I_o sends $\mathbb{R}H_{\alpha_2}$ to $\mathfrak{p}_{2\alpha_2}$ and $\mathbb{R}H^1$ to $\mathfrak{p}_{2\alpha_1+2\alpha_2}$. (In the next chapter, we will prove a generalization of this relation to all Hermitian symmetric spaces, see Theorem 6.3.12.) We pull the complex structure I_o back to $\mathfrak{a} \oplus \mathfrak{n}$ along the isomorphism $\mathfrak{a} \oplus \mathfrak{n} \xrightarrow{\sim} \mathfrak{p}$. Note that B_1 is a complex submanifold of M, whereas B_2 is a totally real.

Theorem 5.4.1. *Let H be a connected Lie group acting properly and isometrically on $M = \mathrm{SU}(n+2,2)/\mathrm{S}(\mathrm{U}(n+2)\mathrm{U}(2))$ with cohomogeneity 1. Then its action is orbit-equivalent to exactly one of the following:*

(a) *The action of the connected Lie subgroup* H_ℓ *of* G *with Lie algebra*

$$\mathfrak{h}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n},$$

where ℓ *is a one-dimensional linear subspace of* $\mathfrak{a}$*. The orbits of* H_ℓ *are all isometrically congruent to each other and form a Riemannian foliation on* M*.*

(b) *The action of the connected Lie subgroup* $H_{\alpha_i}, i \in \{1, 2\}$*, of* G *with Lie algebra*

$$\mathfrak{h}_{\alpha_i} = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell_{\alpha_i}),$$

where ℓ_{α_i} *is any one-dimensional linear subspace of* $\mathfrak{g}_{\alpha_i}$*. Its orbits form a Riemannian foliation on* M *and there is exactly one minimal orbit.*

(c) *The action of the subgroup* $\mathrm{SU}(n+1, 2)$ *of* G *embedded in a standard way. It has a totally geodesic singular orbit of codimension* 4 *isometric to* $\mathrm{Gr}^*(2, \mathbb{C}^{n+3})$*. This orbit is a complex (resp., quaternionic) submanifold with respect to the complex (resp., quaternion-Kähler) structures of* M*.*

(d) *The action of the subgroup* $\mathrm{SU}(n+2, 1)$ *of* G *embedded in a standard way. It has a totally geodesic singular orbit of dimension* $2n+4$ *isometric to* $\mathbb{C}H^{n+2}$*. This orbit is a complex (resp., totally complex) submanifold with respect to the complex (resp., quaternion-Kähler) structures of* M*.*

(e) *In case* $n = 2m$*, the action of the subgroup* $\mathrm{Sp}(m+1, 1)$ *of* G *embedded in a standard way. It has a totally geodesic singular orbit of dimension* $2n+4$ *isometric to* $\mathbb{H}H^{m+1}$*. This orbit is a totally real (resp., quaternionic) submanifold with respect to the complex (resp., quaternion-Kähler) structures of* M*.*

(f) *The action of the connected Lie subgroup* $H^\wedge_{2,k}, k \in \{0, 1\}$*, of* G *with Lie algebra*

$$\mathfrak{h}^\wedge_{2,0} = \mathfrak{k}_2 \oplus \mathfrak{a}_2 \oplus \mathfrak{n}_2, \quad \mathfrak{h}^\wedge_{2,1} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}_2.$$

This action has a minimal singular orbit of codimension $3-k$ *and can be obtained by canonical extension of the cohomogeneity-one action on* $B_2 \simeq \mathbb{R}H^3$ *with a single point* $(k=0)$ *or geodesic* $(k=1)$ *as a singular orbit.*

(g) *The action of the connected Lie subgroup* $H^\wedge_{1,(\varphi,k)}$ *of* G *with Lie algebra*

$$\mathfrak{h}^\wedge_{1,(\varphi,k)} = N_{\mathfrak{k}_1}(\mathfrak{w}) \oplus \mathfrak{w} \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1,$$

where $\mathfrak{w} \subseteq \mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{2\alpha_2}$ *is the orthogonal complement of a subspace* $\mathfrak{w}^\perp \subseteq \mathfrak{g}_{\alpha_2}$ *of dimension* k *and constant Kähler angle* $\varphi \in [0, \frac{\pi}{2}]$ *such that:*

(1) $\varphi = 0 \Rightarrow k \in \{2, 4, \ldots, 2n\}$;

(2) $\varphi \in (0, \frac{\pi}{2}) \Rightarrow k \in \left\{2, 4, \ldots, 2\left\lfloor \frac{n}{2} \right\rfloor\right\}$;

(3) $\varphi = \frac{\pi}{2} \Rightarrow k \in \{2, 3, \ldots, n\}$.

This action has a minimal singular orbit of codimension k *and can be obtained by canonical extension of a cohomogeneity-one action on* $B_1 \simeq \mathbb{C}H^{n+1}$ *with a singular orbit.*

(g') *The actions of the connected Lie subgroups $H^{\wedge}_{1,(0,2n+2)}$ and $H^{\wedge}_{1,(\frac{\pi}{2},n+1)}$ of G with Lie algebras*

$$\mathfrak{h}^{\wedge}_{1,(0,2n+2)} = \mathfrak{k}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1 \quad \textit{and} \quad \mathfrak{h}^{\wedge}_{1,(\frac{\pi}{2},n+1)} = N_{\mathfrak{k}_1}(\mathfrak{w}_{\mathfrak{p}}) \oplus \mathfrak{w} \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1,$$

where $\mathfrak{w}$ is a $(k+1)$-dimensional totally real subspace of $\mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{2\alpha_2}$ containing $\mathfrak{a}_1$. This action has a minimal singular orbit of codimension $2n+2$ (resp., $n+1$) and can be obtained by canonical extension of a cohomogeneity-one action on $B_1 \simeq \mathbb{C}H^{n+1}$ with a single point (resp., totally geodesic $\mathbb{R}H^{n+1}$) as a singular orbit.

Proof. We consider the different cases of Theorem 5.1.7. If the orbits of H form a foliation, we get the actions in (a) and (b). As in the previous two sections, part (b)-(2) of Theorem 5.1.7 is irrelevant here.

The actions in (c), (d), and (e) are the only ones with a totally geodesic singular orbit according to [BT04]. Let us describe these three orbits in a bit more detail. Recall from Example 2.1.37 that M can be thought of as the set of 2-dimensional complex subspaces of $\mathbb{C}^{n+4}$ on which the restriction of the standard indefinite Hermitian form of signature $(n+2,2)$ is negative definite. The group $G = \mathrm{SU}(n+2,2)$ acts transitively on this set and the stabilizer of $o = \langle e_{n+3}, e_{n+4}\rangle_{\mathbb{C}}$ is precisely $K = \mathrm{S}(\mathrm{U}(n+2)\mathrm{U}(2))$.

Consider the subset S_{c} of M consisting of those 2-dimensional subspaces that lie in the complex hyperplane $\langle e_2, \dots, e_{n+4}\rangle_{\mathbb{C}}$. This subset can be described as $\mathrm{Gr}^*(2, \mathbb{C}^{n+3})$ embedded totally geodesically into M. The subgroup $\mathrm{SU}(n+1,2) \subset G$ of elements fixing the first basis vector e_1 preserves S_{c} and acts transitively on it. Its isotropy subgroup at o is $\mathrm{S}(\mathrm{U}(n+1)\mathrm{U}(2))$, whose subgroup $\mathrm{U}(2)$ acts transitively on the unit sphere in N_oS_{c}, so $\mathrm{SU}(n+1,2)$ does indeed act on M with cohomogeneity one and has S_{c} as its singular orbit. Observe that we have $S_{\mathrm{c}}^{\perp} \simeq \mathbb{C}H^2$.

Similarly, define S_{d} to be the subset of M consisting of those 2-dimensional subspaces of $\mathbb{C}^{n+4}$ that contain the last basis vector e_{n+4}. These are in bijective correspondence with the complex lines in the hyperplane $\langle e_1, \dots, e_{n+3}\rangle_{\mathbb{C}}$ on which the restriction of the standard indefinite Hermitian form of signature $(n+2,1)$ is negative definite, so S_{d} can be identified with $\mathrm{Gr}^*(1, \mathbb{C}^{n+3}) = \mathbb{C}H^{n+2}$ embedded totally geodesically into M. The subgroup $\mathrm{SU}(n+2,1) \subset G$ of elements fixing e_{n+4} preserves S_{d} and acts transitively on it. Its isotropy subgroup at o is $\mathrm{S}(\mathrm{U}(n+2)\mathrm{U}(1))$, and its slice representation at o is equivalent to the tautological representation $\mathrm{U}(n+2) \curvearrowright \mathbb{C}^{n+2}$, also known as the restricted isotropy representation of $\mathbb{C}H^{n+2}$. This is a reflection of the fact that $S_{\mathrm{d}}^{\perp} \simeq \mathbb{C}H^{n+2}$. But now observe that $(S_{\mathrm{d}}^{\perp})^{\perp} = S_{\mathrm{d}}$ is of rank 1, so, according to Subsection 5.1.1, $S_{\mathrm{d}}^{\perp}$ is also a totally geodesic singular orbit of some C1-action. We claim that this action is orbit-equivalent to the action of $\mathrm{SU}(n+2,1)$, i.e., that S_{d} and $S_{\mathrm{d}}^{\perp}$ are isometrically congruent. Indeed, S_{d} is easily seen to be a totally complex submanifold of M. Take $J \in \mathbb{S}^2_p$ such that $J(T_oS_{\mathrm{d}}) = N_oS_{\mathrm{d}}$. As follows from the discussion preceding Proposition 2.5.26, $J = dk_o$ for some $k \in K$. This k provides a congruence between S_{d} and $(S_{\mathrm{d}})^{\perp}_o$. Note that the geodesic reflection in S_{d} is an involutive holomorphic isometry. The submanifold S_{d} is called a complex form of M (see [Wol05]).

Finally, if $n = 2m$, consider the standard indefinite q-Hermitian form H on $\mathbb{H}^{m+2}$ of signature $(m+1,1)$ and identify $\mathbb{H}^{m+2}$ with $\mathbb{C}^{2m+4}$ in a standard way. Note that if we write $H = h - \omega j$, where $h, \omega \colon \mathbb{H}^{m+2} \times \mathbb{H}^{m+2} \to \mathbb{C}$, then h is precisely our indefinite

Hermitian form on $\mathbb{C}^{n+4}$. Consider the subset S_e of M consisting of quaternionic lines in $\mathbb{C}^{n+4}$, the restriction of H to which is negative definite. This subset can be described as $\mathrm{Gr}^*(1, \mathbb{H}^{m+2}) = \mathbb{H}H^{m+1}$ embedded totally geodesically into M. The group $\mathrm{Sp}(m+1, 1)$ sits naturally inside G and preserves S_e. Its isotropy subgroup at o is $\mathrm{Sp}(m+1)\mathrm{Sp}(1)$, whose slice representation at o is isomorphic to the standard representation on $\mathbb{H}^{m+1}$. Just like before, we have $S_\mathrm{e}^\perp \simeq \mathbb{H}H^{m+1} \simeq S_\mathrm{e}$, and these two submanifolds are in fact isometrically congruent. Their tangent spaces at o are mapped to each other by I_o because S_e is totally real. By Proposition 2.5.11, $I_o = dk_o$ for some $k \in K$, so k is the desired congruence. The geodesic reflection in S_e is an antiholomorphic involutive isometry, for which reason S_e is called a real form of M (see [Jaf75, Leu79b]).

Now we proceed to the canonical extension method. There are two strong orbit-equivalence classes of cohomogeneity-one actions on $B_2 \simeq \mathbb{R}H^3$ with a singular orbit, whose extensions are described in (f).

The situation with $B_1 \simeq \mathbb{C}H^{n+1}$ is much more interesting. Let us recapitulate the classification of C1-actions on complex hyperbolic spaces. As we already mentioned in the proof of Proposition 5.1.10, every C1-action on $\mathbb{C}H^{n+1}$ with a singular orbit other than $\{\mathrm{pt}\}$ and a totally geodesic $\mathbb{R}H^{n+1}$ arises via the nilpotent construction. The corresponding representation is equivalent to the tautological representation $\mathrm{U}(n) \curvearrowright \mathbb{C}^n$. We know that if two protohomogeneous subspaces of $\mathfrak{g}_{\alpha_2} \simeq \mathbb{C}^n$ are $\mathrm{U}(n)$-congruent, they produce strongly orbit-equivalent C1-actions on $\mathbb{C}H^{n+1}$. It was shown in [BT07, Th. 4.1(ii)] that the converse is true:

Proposition 5.4.2. *Suppose $M = \mathbb{C}H^{n+1}$ and let $\mathfrak{v}, \mathfrak{v}' \subseteq \mathfrak{g}_\alpha$ be two protohomogeneous subspaces. The following conditions are equivalent:*

(i) *The C1-actions on $\mathbb{C}H^{n+1}$ arising from $\mathfrak{v}$ and $\mathfrak{v}'$ are orbit-equivalent.*

(ii) *The C1-actions on $\mathbb{C}H^{n+1}$ arising from $\mathfrak{v}$ and $\mathfrak{v}'$ are strongly orbit-equivalent.*

(iii) *$\mathfrak{v}$ and $\mathfrak{v}'$ are K_0-congruent*[1].

So the classification of C1-actions on $\mathbb{C}H^{n+1}$ boils down to the classification of protohomogeneous subspaces of $\mathbb{C}^n \curvearrowleft \mathrm{U}(n)$ up to $\mathrm{U}(n)$-congruence. This was carried out by Berndt and Brück in [BB01]. They showed that a subspace of $\mathbb{C}^n$ is protohomogeneous if and only if it has constant Kähler angle, and two such subspaces are $\mathrm{U}(n)$-congruent if and only if they have the same dimension and Kähler angle. This, together with Proposition 5.1.10 and Remark 5.1.4, shows that the actions in parts (g) and (g') of Theorem 5.4.1 exhaust the list of C1-actions on M with a singular orbit arising from B_1 via canonical extension. To simplify the notation, we split off the canonical extensions of the two actions on B_1 that do not arise via the nilpotent construction into a separate group. No two actions in (g) and (g') are mutually orbit-equivalent by design: the normal spaces of their singular orbits differ in either dimension or (constant) Kähler angle. Here we use the fact that every isometry of M is either holomorphic or anti-holomorphic (Corollary 2.5.15), so it preserves the Kähler angles of tangent subspaces.

There is one action in (f) that may, in theory, have a totally geodesic singular orbit, namely the action of $H^\Lambda_{1,(0,4)}$. Its orbit $S = H^\Lambda_{1,(0,4)} \cdot o$ is a complex submanifold of M of codimension 4. But so is the singular orbit of the action of $\mathrm{SU}(n+1, 2) \subset G$ given

[1] Recall that $M_1^0 = K_1^0 = K_0$ for rank-1 spaces.

in (c). To show that these two actions are not orbit-equivalent, we prove that S is not totally geodesic. Since S is a homogeneous submanifold, we can use Proposition 2.2.43 to compute its second fundamental form. Observe that

$$T_oS = \mathfrak{w}_\mathfrak{p} \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1, \quad N_oS = \mathfrak{w}_\mathfrak{p}^\perp.$$

The subspace $\mathfrak{w}^\perp$ is contained in $\mathfrak{g}_{\alpha_2}$ if $n > 1$ and coincides with $\mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{2\alpha_2}$ if $n = 1$. Since $[\mathfrak{g}_{\alpha_1+\alpha_2}, \mathfrak{g}_{-\alpha_1}] = \mathfrak{g}_{\alpha_2}$, we can find $X \in \mathfrak{g}_{\alpha_1+\alpha_2}$ and $Y \in \mathfrak{g}_{-\alpha_1}$ such that $[X,Y] \in \mathfrak{g}_{\alpha_2} \setminus \mathfrak{w}$. By construction, $X - \theta X$ and $Y - \theta Y$ lie in T_oS. By virtue of Proposition 2.2.43, $I\!I(X - \theta X, Y - \theta Y) = \mathrm{pr}_{N_oS}([Z, Y - \theta Y])$, where $Z \in \mathfrak{k}$ is any vector such that $X - \theta X + Z \in \mathfrak{h}^\wedge_{1,(0,4)}$. We pick $Z = X + \theta X$, which gives

$$I\!I_S(X - \theta X, Y - \theta Y) = \mathrm{pr}_{N_oS}([X + \theta X, Y - \theta Y]) = \mathrm{pr}_{N_oS}([X,Y] - \theta[X,Y]) \neq 0,$$

which implies that the fundamental form of S is nonzero at o and S is not totally geodesic.

Before we proceed to the nilpotent construction, we need to prove that the actions in (f) and (g) (or (g')) are mutually nonequivalent. We are going to show that, given an action from (f) with a singular orbit S and one from (g) with a singular orbit S' of the same dimension as S, the normal spaces to S and S' have either different Kähler angles or vectors of different holomorphic sectional curvatures. Computing all such sectional curvatures by hand would be a daunting task, so we will employ a different strategy and leverage the abundance of geometric structures on M. In [Ber97], Berndt studied the complex Grassmannian of two-planes $\mathrm{Gr}(2, \mathbb{C}^{n+4})$, which is the compact dual of our symmetric space M, so we denote it by M^*. By Corollaries 2.5.10 and 2.5.27, M^* is also a Hermitian and q-Kähler symmetric space. Write I^* and $\mathcal{J}^*$ for its corresponding almost complex structure and rank-3 subbundle of $\mathfrak{so}(T(M^*))$. It turns out that these two geometric structures can be used to detect the global extrema of the holomorphic sectional curvature function of M^* (see [Ber97, Prop. 19]):

Proposition 5.4.3. *Let $M^* = \mathrm{Gr}(2, \mathbb{C}^{n+4})$, $p \in M^*$ any point, and $X \in T_pM^*$ a nonzero vector. The following conditions are equivalent:*

(i) *X is singular.*

(ii) *X is a global extremum point of the holomorphic sectional curvature function of M^*.*

(iii) *$I_p^*X \perp \mathcal{J}_p^*X$ or $I_p^*X \in \mathcal{J}_p^*X$.*

Moreover, K^*_{hol} attains its minimum (resp., maximum) value on X if and only if $I_p^*X \perp \mathcal{J}_p^*X$ (resp., $I_p^*X \in \mathcal{J}_p^*X$).

In essence, this result allows to split singular tangent vectors to M^*—and thus M—into two types. Bearing in mind that the curvatures of M and M^* are of opposite signs (Proposition 2.1.116(h)), we can introduce the following

Definition 5.4.4. Given any $p \in M = \mathrm{Gr}^*(2, \mathbb{C}^{n+4})$, a nonzero singular vector $X \in T_pM$ is said to be

- **of type A** if K_{hol} attains its maximum on $X \Leftrightarrow I_p^*X \perp \mathcal{J}_p^*X$;
- **of type B** if K_{hol} attains its minimum on $X \Leftrightarrow I_p^*X \in \mathcal{J}_p^*X$.

Recall that the set of singular vectors is preserved under isometries (Proposition 2.4.30). What makes the notion of type useful is the fact that *the type of a singular vector is preserved under isometries* as well. Indeed, any isometry of M is either holomorphic or anti-holomorphic by Corollary 2.5.15, so it preserves holomorphic sectional curvatures and hence the type of a singular vector.

In order to apply this knowledge to beat the congruence problem for C1-actions on M, we need to know explicitly what the I and $\mathcal{J}$ look like. We already know how I behaves with respect to the restricted root space decomposition (see the discussion preceding Theorem 5.4.1), so now we need to describe $\mathcal{J}$. We have $\mathfrak{g} = \mathfrak{su}(n+2,2)$, and its Cartan decomposition looks like:

$$\mathfrak{k} = \left\{ \left[\begin{array}{c|c} \mathfrak{u}(n+2) & 0 \\ \hline 0 & \mathfrak{u}(2) \end{array} \right] \middle| \ \mathrm{tr} = 0 \right\}, \quad \mathfrak{p} = \left\{ \left[\begin{array}{c|c} 0 & B \\ \hline B^* & 0 \end{array} \right] \right\},$$

where B runs through $\mathrm{Mat}((n+2)\times 2, \mathbb{C})$. The isotropy representation of the Riemannian symmetric pair $(\mathrm{SU}(n+2,2), \mathrm{S}(\mathrm{U}(n+2)\mathrm{U}(2)))$ is given by the adjoint representation of $K = \mathrm{S}(\mathrm{U}(n+2)\mathrm{U}(2))$ on $\mathfrak{p}$. Note that there is a normal subgroup $\mathrm{SU}(2) \trianglelefteq K$ isomorphic to $\mathrm{Sp}(1)$. Moreover, its intersection with $Z(\mathrm{SU}(n+2,2)) = \{\lambda E \mid \lambda^{n+4} = 1\}$ is trivial, which implies that $\mathrm{SU}(2)$ descends to an $\mathrm{Sp}(1)$-isomorphic subgroup of the isotropy group of M. In view of Proposition 2.5.26, the representation of $\mathrm{SU}(2) \subseteq K$ induces a quaternionic structure on T_oM that makes M into a q-Kähler manifold—this is how one shows that M is q-Kähler in the first place. We will, however, benefit from a more detailed description of this representation. Let us introduce the following notation:

$$\begin{aligned} \rho_n &= \text{the tautological representation of } \mathrm{U}(n) \text{ on } \mathbb{C}^n, \\ \chi_n &= \text{the standard representation of } \mathrm{Sp}(1) \text{ on } \mathbb{H}^n. \end{aligned}$$

If we identify $\mathfrak{p}$ with $\mathbb{C}^{n+2} \oplus \mathbb{C}^{n+2} \cong \mathbb{C}^{n+2} \otimes \mathbb{C}^2$ in the obvious way, the isotropy representation becomes isomorphic to the restriction of the external tensor product representation $\rho_{n+2} \otimes \rho_2$ of $\mathrm{U}(n+2) \times \mathrm{U}(2)$ to K. On the subgroup $\mathrm{SU}(2) \trianglelefteq K$, this representation is just $1 \otimes \rho_2$. Observe that we can identify $\mathrm{SU}(2)$ with $\mathrm{Sp}(1)$ and $\mathbb{C}^2$ with $\mathbb{H}$ in such a way that $\rho_2\big|_{\mathrm{SU}(2)}$ becomes the same as χ_1. Altogether, there is a normal subgroup $\mathrm{Sp}(1) \trianglelefteq K$ whose representation on $\mathfrak{p}$ is given by:

$$\begin{tikzcd} & \mathbb{C}^n \otimes_{\mathbb{C}} \mathbb{H} \arrow[d, "\wr"] \\ \mathrm{Sp}(1) \arrow[ur, "1 \otimes \chi_1"] \arrow[r, "\chi_n"'] & \mathbb{H}^n \end{tikzcd}$$

This shows explicitly how $\mathrm{SU}(2) \hookrightarrow \mathrm{SO}(T_oM)$ is equivalent to χ_n. The induced Lie algebra representation $\mathfrak{su}(2) \hookrightarrow \mathfrak{so}(T_oM)$ provides the action of imaginary quaternions on T_oM, so its image is precisely $\mathcal{J}_o$. Now we can easily identify which singular vectors in $\mathfrak{p}$ are of type A and which are of type B.

Let us take $\mathfrak{a} = \mathbb{R}(E_{n+2,n+3} + E_{n+3,n+2}) \oplus \mathbb{R}(E_{n+1,n+4} + E_{n+4,n+1})$. The restricted root space decomposition of $\mathfrak{g}$ with respect to such a choice of $\mathfrak{a}$ is explicitly described in Knapp ([Kna02, p. 371, Ex. 2]). Using that, as well as our description of the q-Kähler structure, it is straightforward to compute how $\mathcal{J}_o \subseteq \mathfrak{so}(\mathfrak{p})$ acts on different root vectors. Notice that the almost complex structure on M at o is given simply by

$$I_o\colon \quad \left[\begin{array}{c|c} 0 & B \\ \hline B^* & 0 \end{array}\right] \mapsto \left[\begin{array}{c|c} 0 & iB \\ \hline -iB^* & 0 \end{array}\right].$$

Finally, observe that $\mathfrak{su}(n+2,2)$ is a real form of $\mathfrak{sl}(n+4,\mathbb{C})$, so the Killing form B of $\mathfrak{g}$ is just the restriction of the Killing form of $\mathfrak{sl}(n+4,\mathbb{C})$, which is $(2n+8)\operatorname{tr}$. This allows to compute the inner product B_θ on $\mathfrak{g}$. The search for singular vectors of types A and B in $\mathfrak{p}$ is now a very straightforward process. For instance, one can compute that $\mathcal{J}_o$ sends $\mathfrak{p}_{2\alpha_2}$ onto $\mathbb{R}H_{\alpha_2} \oplus \mathbb{C}(E_{n+1,n+3} + E_{n+3,n+1})$, while I_o sends it onto $\mathbb{R}H_{\alpha_2}$, so we deduce that $\mathfrak{p}_{2\alpha_2}$ consists entirely (apart from zero) of singular vectors of type B. In a similar fashion, one calculates that

- $\mathfrak{p}_{\alpha_1}$ and $\mathfrak{p}_{\alpha_1+2\alpha_2}$ consist of singular vectors of type A, while
- $\mathfrak{p}_{\alpha_2}, \mathfrak{p}_{\alpha_1+\alpha_2}, \mathfrak{p}_{2\alpha_2}, \mathfrak{p}_{2\alpha_1+2\alpha_2}, \mathbb{R}H^1$, and $\mathbb{R}H_{\alpha_2}$ all consist of singular vectors of type B.

Having done all that, we can eventually tackle the congruence problem. Consider the action of $H^\Lambda_{2,1}$. The normal space of its singular orbit at o is $\mathfrak{p}_{\alpha_1}$, which is a totally real subspace of $\mathfrak{p}$. The only action in (g) or (g') whose singular orbit has totally real normal spaces of dimension 2 is the action of $H^\Lambda_{1,(\pi/2,2)}$. Whatever n is, $\mathfrak{w}^\perp$ has a nontrivial intersection with $\mathfrak{g}_{\alpha_2}$. As a consequence, $N_o(H^\Lambda_{1,(\pi/2,2)} \cdot o) = \mathfrak{w}^\perp_{\mathfrak{p}}$ contains a singular vector of type B, whereas $N_o(H^\Lambda_{2,1} \cdot o) = \mathfrak{p}_{\alpha_1}$ consists entirely of singular vectors of type A. Therefore, there cannot exist an orbit-equivalence between these two actions. Now take the action of $H^\Lambda_{2,0}$. The normal space of its singular orbit at o is $T_oB_2 = \mathbb{R}H_{\alpha_1} \oplus \mathfrak{p}_{\alpha_1}$, which is totally real. Again, the only action in (g) or (g') whose singular orbit has totally real normal spaces of dimension 3 is the action of $H^\Lambda_{1,(\pi/2,3)}$ ($n \geq 2$). Regardless of the value of n, $\dim(\mathfrak{w}^\perp \cap \mathfrak{g}_{\alpha_2}) \geq 2$, so $N_o(H^\Lambda_{1,(\pi/2,3)} \cdot o) = \mathfrak{w}^\perp_{\mathfrak{p}}$ contains a 2-dimensional subspace L of singular vectors of type B, while $N_o(H^\Lambda_{2,0} \cdot o)$ contains $\mathfrak{p}_{\alpha_1}$, a 2-dimensional subspace of singular vectors of type A. If there was an orbit-equivalence between these two actions, there would be one fixing o, which would have to send L onto a plane in $N_o(H^\Lambda_{2,0} \cdot o)$ overlapping with $\mathfrak{p}_{\alpha_1}$, which would lead to a contradiction. Consequently, no action in (f) is orbit-equivalent to an action in (g) or (g').

At last, we proceed to the nilpotent construction method. To distinguish between the Euclidean and Hermitian inner products on $\mathfrak{a} \oplus \mathfrak{n}$, we will add the letter H when talking about the latter. For example, $v \perp_H w$ means orthogonality with respect to the Hermitian inner product.

Nilpotent construction with $j = 2$. In this case we have:

$$\begin{aligned} \mathfrak{n}^1_2 &= \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \simeq \mathbb{C}^n \oplus \mathbb{C}^n \cong \mathbb{C}^n \otimes \mathbb{C}^2, \\ \mathfrak{m}_2 &= \mathfrak{z}_2 \oplus \mathfrak{g}_2 = \mathfrak{u}(n) \oplus \mathfrak{so}(3,1) \simeq \mathfrak{u}(n) \oplus \mathfrak{sl}(2,\mathbb{C}), \\ \mathfrak{k}_2 &= \mathfrak{u}(n) \oplus \mathfrak{su}(2). \end{aligned}$$

It is not hard to show that the representation of $\mathfrak{m}_2$ on $\mathfrak{n}_2^1$ is equivalent to the external tensor product representation of $\mathfrak{u}(n) \oplus \mathfrak{sl}(2,\mathbb{C})$ on $\mathbb{C}^n \otimes \mathbb{C}^2$. Consequently, the representation of $K_2^0 \simeq \mathrm{U}(n) \times \mathrm{SU}(2) \simeq \mathrm{S}(\mathrm{U}(n)\mathrm{U}(2))$ on $\mathfrak{n}_2^1$ is equivalent to the restricted isotropy representation of $\mathrm{Gr}^*(2,\mathbb{C}^{n+2}) = \mathrm{SU}(n,2)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(2))$. Pick some Hermitian-orthonormal bases $e_1,\dots,e_n \in \mathbb{C}^n$ and $f_1, f_2 \in \mathbb{C}^2$. When we think of $K_2^0 \curvearrowright \mathfrak{n}_2^1$ as the restricted isotropy representation of $\mathrm{Gr}^*(2,\mathbb{C}^{n+2})$, the two-dimensional subspace $\mathbb{R}(e_1 \otimes f_1) \oplus \mathbb{R}(e_2 \otimes f_2)$ corresponds to a maximal flat. This observation will prove useful later on. Now, let $\mathfrak{v} \subseteq \mathfrak{n}_2^1$ be an admissible and protohomogeneous subspace and pick any $v = v_1 + v_2 \in \mathfrak{v}$ with $v_1 \in \mathfrak{g}_{\alpha_2}$, $v_2 \in \mathfrak{g}_{\alpha_1+\alpha_2}$. First, mimicking the proof of [BDV15, Th. 8], we prove the following

Lemma 5.4.5. *Let $T = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathfrak{so}(2) \subseteq \mathfrak{su}(2) \subseteq \mathfrak{m}_2$. If $[T, v_1] \perp_H v_2$, then either $v_1 = 0$ or $v_2 = 0$.*

Proof of the lemma. Under the assumption $[T, v_1] \perp_H v_2$, we do not lose generality by taking $v_1 = re_1 \otimes f_1$ and $v_2 = se_2 \otimes f_2$ for some $r, s \in \mathbb{R}$. By protohomogeneity of $\mathfrak{v}$, we have:

$$\mathfrak{v} = \mathbb{R}(re_1 \otimes f_1 + se_2 \otimes f_2) \oplus^\perp N_{\mathfrak{k}_2}(\mathfrak{v})(re_1 \otimes f_1 + se_2 \otimes f_2).$$

Note that our assumption implies that the second summand here is actually Euclidean-orthogonal to both $e_1 \otimes f_1$ and $e_2 \otimes f_2$. Now let $S + A \in N_{\mathfrak{m}_2}(\mathfrak{v})$, where $S \in \mathfrak{u}(n)$, $A = \left[\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right] \in \mathfrak{sl}(2,\mathbb{C})$. We compute:

$$\begin{aligned}
&(S + A)(re_1 \otimes f_1 + se_2 \otimes f_2) = \\
&\qquad = rSe_1 \otimes f_1 + sSe_2 \otimes f_2 + re_1 \otimes (xf_1 + zf_2) + se_2 \otimes (yf_1 - xf_2) \\
&\qquad = r\mathrm{Re}(x)e_1 \otimes f_1 - s\mathrm{Re}(x)e_2 \otimes f_2 + (\text{terms in } (re_1 \otimes f_1 + se_2 \otimes f_2)^\perp).
\end{aligned}$$

We see that $r\mathrm{Re}(x)e_1 \otimes f_1 - s\mathrm{Re}(x)e_2 \otimes f_2$ must lie in $\mathbb{R}(re_1 \otimes f_1 + se_2 \otimes f_2)$, which is only possible when either r or s is zero. □

Since $\mathbb{R}(e_1 \otimes f_1) \oplus \mathbb{R}(e_2 \otimes f_2)$ corresponds to a maximal flat in $\mathrm{Gr}^*(2,\mathbb{C}^{n+2})$, it intersects the isotropy orbit of each vector, i.e., there exists some $k \in K_2^0$ such that $\mathrm{Ad}(k)v \in \mathbb{R}(e_1 \otimes f_1) \oplus \mathbb{R}(e_2 \otimes f_2)$. It then follows from Lemma 5.4.5 that $\mathrm{Ad}(k)v$ is proportional to either $e_1 \otimes f_1$ or $e_2 \otimes f_2$. Applying $T \in K_2^0$ if needed, we may assume the former and thus simply write $e_1 \otimes f_1 \in \mathfrak{v}$.

Since $\mathfrak{v}$ is protohomogeneous, we have $\mathfrak{v} \subseteq \mathrm{span}\{K_2^0 \cdot (e_1 \otimes f_1)\} = \mathfrak{g}_{\alpha_2} \oplus \mathbb{C}(e_1 \otimes f_2)$. We decompose $\mathfrak{su}(2)$ as a vector space into two pieces:

$$\ell = \left\{ \begin{bmatrix} ia & 0 \\ 0 & -ia \end{bmatrix} \,\middle|\, a \in \mathbb{R} \right\}, \quad \ell^\perp = \left\{ \begin{bmatrix} 0 & -\bar{z} \\ z & 0 \end{bmatrix} \,\middle|\, z \in \mathbb{C} \right\},$$

so $\mathfrak{su}(2) = \ell \oplus \ell^\perp$. Clearly, $\mathfrak{v} \subseteq \mathfrak{g}_{\alpha_2} \Leftrightarrow N_{\mathfrak{k}_2}(\mathfrak{v}) \subseteq \mathfrak{u}(n) \oplus \ell$. One can easily see that every subspace $\mathfrak{v}$ of $\mathfrak{g}_{\alpha_2}$ is automatically admissible. We can regard such $\mathfrak{v}$ as lying in the isometry Lie algebra $\mathfrak{g}_1 = \mathfrak{g}_1'$ of $B_1 \simeq \mathbb{C}H^{n+1}$; and $\mathfrak{v}$ is protohomogeneous in $\mathfrak{n}_2^1 \curvearrowleft M_2$ if and only if it is such in $\mathfrak{g}_{\alpha_2} \curvearrowleft \mathrm{U}(n)$ (which just means that it has constant Kähler angle). If this is the case, the canonical extension of the resulting C1-action on B_1 has the same orbits as the action obtained from $\mathfrak{v}$ by nilpotent construction performed on M. So such $\mathfrak{v}$'s produce no new actions and we may assume $\mathfrak{v}$ does not lie in $\mathfrak{g}_{\alpha_2}$ and thus $N_{\mathfrak{k}_2}(\mathfrak{v})$ has a nonzero projection to $\ell^\perp$. Let $S+A \in N_{\mathfrak{k}_2}(\mathfrak{v})$, $S=(s_{ij})_{i,j=1}^n \in \mathfrak{u}(n)$, $A=\left[\begin{smallmatrix} ia & -\bar{z} \\ z & -ia \end{smallmatrix}\right] \in \mathfrak{su}(2)$, $z \neq 0$. We compute:

$$(S+A)(e_1 \otimes f_1) = (ia+s_{11})e_1 \otimes f_1 + \sum_{i=2}^n s_{i1} e_i \otimes f_1 + z e_1 \otimes f_2,$$

$$(S+A)^2(e_1 \otimes f_1) = (\text{terms in } \mathfrak{g}_{\alpha_2} \oplus \mathbb{C}(e_1 \otimes f_2)) + \sum_{i=2}^n 2z s_{i1} e_i \otimes f_2.$$

In order for this to lie in $\mathfrak{v} \subseteq \mathfrak{g}_{\alpha_2} \oplus \mathbb{C}(e_1 \otimes f_2)$, we must have $s_{i1}=0$ for $2 \le i \le n$, which means that $(ia+s_{11})e_1 \otimes f_1 + z e_1 \otimes f_2 \in \mathfrak{v}$. But this forces $N_{\mathfrak{k}_2}(\mathfrak{v})$ to lie inside $\mathfrak{u}(1) \oplus \mathfrak{su}(2) \subseteq \mathfrak{u}(n) \oplus \mathfrak{su}(2)$, for otherwise we would have some $S+A \in N_{\mathfrak{k}_2}(\mathfrak{v}), S \in \mathfrak{u}(n) \setminus \mathfrak{u}(1)$, moving $(ia+s_{11})e_1 \otimes f_1 + z e_1 \otimes f_2$ out of $\mathfrak{g}_{\alpha_2} \oplus \mathbb{C}(e_1 \otimes f_2)$. The upshot of all this is that $\mathfrak{v} \subseteq \mathbb{C}(e_1 \otimes f_1) \oplus \mathbb{C}(e_1 \otimes f_2)$ and $N_{\mathfrak{m}_2}(\mathfrak{v}) \subseteq \mathfrak{u}(1) \oplus \mathfrak{sl}(2,\mathbb{C})$ (and we are still assuming $e_1 \otimes f_1 \in \mathfrak{v}$). So we have reduced our problem to looking for admissible and protohomogeneous subspaces of $\mathbb{C}^2$ with respect to the tautological representation of $\mathfrak{gl}(2,\mathbb{C})$. As we already know, protohomogeneity singles out precisely subspaces of constant Kähler angle φ. We consider three cases:

Case 1: $\varphi = 0$. Since we assume $\mathfrak{v} \not\subseteq \mathfrak{g}_{\alpha_2}$, we must have $\mathfrak{v} = \mathbb{C}(e_1 \otimes f_1) \oplus \mathbb{C}(e_1 \otimes f_2)$. But then we get a C1-action with a totally geodesic singular orbit isometric to $\mathrm{Gr}^*(2,\mathbb{C}^{n+3})$, see (c).

Case 2: $\varphi = \frac{\pi}{2}$. Without loss of generality, we pick $\mathfrak{v} = \mathbb{R}(e_1 \otimes f_1) \oplus \mathbb{R}(e_1 \otimes f_2)$. In this case, $N_{\mathfrak{gl}(2,\mathbb{C})}(\mathfrak{v}) = \mathfrak{gl}(2,\mathbb{R})$, so $\mathfrak{v}$ fails to be admissible.

Case 3: $\varphi \in (0, \frac{\pi}{2})$. According to [BB01, Prop. 7], we can take $\mathfrak{v}$ to be the span of $(1,0)$ and $(i\cos\varphi, i\sin\varphi)$. If we write the coordinates on $\mathbb{C}^2$ as z_1 and z_2, then $\mathfrak{v}$ is cut out by the equations $\mathrm{Re}(z_2)=0$ and $\mathrm{Im}(z_2) = \mathrm{Im}(z_1)\tan\varphi$. We need to check whether the projection of $N_{\mathfrak{sl}(2,\mathbb{C})}(\mathfrak{v})$ to the space of Hermitian traceless matrices is onto. A matrix $\left[\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right] \in \mathfrak{sl}(2,\mathbb{C})$ normalizing $\mathfrak{v}$ is subject to the following equations:

$$\begin{cases} \mathrm{Re}(z) = 0, \\ \mathrm{Im}(z) = \mathrm{Im}(x)\tan\varphi, \\ -\mathrm{Im}(z)\cos\varphi + \mathrm{Im}(x)\sin\varphi = 0, \\ \mathrm{Re}(z)\cos\varphi - \mathrm{Re}(x)\sin\varphi = (\mathrm{Re}(x)\cos\varphi + \mathrm{Re}(y)\sin\varphi)\tan\varphi. \end{cases}$$

The second and third equations are the same, so, simplifying, we are left with:

$$\begin{cases} \mathrm{Re}(z) = 0, \\ \mathrm{Im}(z) = \mathrm{Im}(x)\tan\varphi, \\ \mathrm{Re}(x) = -\mathrm{Re}(y)\dfrac{\tan\varphi}{2}. \end{cases}$$

These equations cut out a 3-dimensional subspace inside $\mathfrak{sl}(2,\mathbb{C})$. Its projection to the space of Hermitian traceless matrices is onto precisely when it does not intersect $\mathfrak{su}(2)$. But these equations have a nontrivial solution in $\mathfrak{su}(2)$, namely $\left[\begin{smallmatrix} i & -i\tan\varphi \\ i\tan\varphi & -i \end{smallmatrix}\right]$, so $\mathfrak{v}$ is not admissible. We conclude that the nilpotent construction with $j=2$ produces no new actions.

Nilpotent construction with $j = 1$. In this case, we have:

$$\begin{aligned}
\mathfrak{n}_1^1 &= \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \simeq \mathbb{C}^{n+2},\\
\mathfrak{m}_1 &= \mathfrak{g}_1 \oplus \mathfrak{z}_1 = \mathfrak{su}(n+1,1) \oplus \mathfrak{u}(1) \simeq \mathfrak{u}(n+1,1),\\
\mathfrak{k}_1 &= \mathfrak{s}(\mathfrak{u}(n+1) \oplus \mathfrak{u}(1)) \oplus \mathfrak{u}(1) \simeq \mathfrak{u}(n+1) \oplus \mathfrak{u}(1).
\end{aligned}$$

The representation of $\mathfrak{m}_1$ on $\mathfrak{n}_1^1$ can be shown to be equivalent to the tautological representation of $\mathfrak{u}(n+1,1)$. We write $\mathbb{C}^{n+2} = \mathbb{C}^{n+1} \oplus \mathbb{C}$ and notice that $\mathfrak{k}_1$ preserves the two summands. Consequently, a protohomogeneous subspace $\mathfrak{v} \subseteq \mathbb{C}^{n+2}$ intersecting nontrivially with $\mathbb{C}^{n+1}$ (which is always the case when $\dim \mathfrak{v} \geq 3$) must lie there entirely, in which case admissibility cannot be achieved. So we only need to consider 2-dimensional subspaces transversal to $\mathbb{C}^{n+1}$. Take any such $\mathfrak{v}$. We can apply the same argument to its intersection with $\mathbb{C} \subset \mathbb{C}^{n+2}$, so it must be trivial as well. In other words, the projection $\mathrm{pr}_{\mathbb{C}^{n+1}}(\mathfrak{v})$ is two-dimensional. Observe that this projection is protohomogeneous in $\mathbb{C}^{n+1}$ with respect to $\mathrm{U}(n+1)$ and thus has constant Kähler angle φ. Again, according to [BB01, Prop. 7], we may assume

$$\mathfrak{v} = \langle v_1, v_2 \rangle_{\mathbb{R}} = \langle e_1 + a e_{n+2}, i\cos\varphi e_1 + i \sin\varphi e_2 + b e_{n+2} \rangle_{\mathbb{R}}, \ a, b \neq 0.$$

Moreover, acting by $\mathrm{U}(1)$ in the last coordinate, we can take $a > 0$. Finally, since K_1^0 acts transitively on the unit circle in $\mathfrak{v}$, we must have $|b| = a$. Consequently, we can confine ourselves to working in $\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_{n+2}$. Let us denote the corresponding coordinates by x, y, and z, respectively. Note that $\mathrm{Im}(b) \neq 0$, for otherwise we would have a nontrivial intersection $\mathfrak{v} \cap \mathbb{C}^{n+1}$. We see that $\mathfrak{v}$ is cut out by the equations

$$\begin{cases}
\mathrm{Re}(y) = 0,\\
\mathrm{Im}(x)\sin\varphi = \mathrm{Im}(y)\cos\varphi,\\
\mathrm{Im}(b)\mathrm{Im}(x) = \mathrm{Im}(z)\cos\varphi,\\
a\mathrm{Re}(x) = \mathrm{Re}(z) - \dfrac{\mathrm{Re}(c)}{\mathrm{Im}(c)}\mathrm{Im}(z).
\end{cases} \tag{5.4.1}$$

The subspace $\mathfrak{v}$ is admissible if and only if for each $w \in \mathbb{C}^{n+1}$, there exists

$$S = \left[\begin{array}{c|c} A & w \\ \hline w^* & c \end{array}\right] \in \mathfrak{u}(n+1,1)$$

preserving $\mathfrak{v}$. The fact that S lies in $\mathfrak{u}(n+1,1)$ means that $A \in \mathfrak{u}(n+1)$ and $\mathrm{Re}(c) = 0$. Applying S to $(1, 0, \ldots, a)$ and $(i\cos\varphi, i\sin\varphi, 0, \ldots, b)$, we get vectors

$$\begin{bmatrix} a_{11} + a w_1 \\ a_{21} + a w_2 \\ \cdots \\ \overline{w}_1 + iac \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} i a_{11}\cos\varphi + i a_{12}\sin\varphi + w_1 b \\ i a_{21}\cos\varphi + i a_{22}\sin\varphi + w_2 b \\ \cdots \\ i\overline{w}_1\cos\varphi + i\overline{w}_2\sin\varphi + ibc \end{bmatrix}.$$

Note that $a_{12} = \overline{a}_{21}$ and $\mathrm{Re}(a_{11}) = \mathrm{Re}(a_{22}) = 0$. In order to lie in $\mathfrak{v}$, both $S(v_1)$ and

$S(v_2)$ must satisfy equations (5.4.1) (and all their other coordinates should be 0, but we do not care about that), which altogether gives a system of 8 equations, and we are left with a rather daunting and tedious task of solving it. We are not going to write all these equations here; instead, let us label them (1) through (8), where equations (1) to (4) are for $S(v_1)$ substituted into (5.4.1) and (5) to (8) are for $S(v_2)$ substituted there. Fortunately, there is no need to solve the entire system. Indeed, provided that $\varphi \neq 0$, use (1) to express $\mathrm{Re}(a_{21})$ in terms of $\mathrm{Re}(w_2)$ and plug the result into (5) to get a linear dependency on the real and imaginary parts of w_1 and w_2. Therefore, such $\mathfrak{v}$ cannot be admissible.

Now let $\varphi = 0$, which just means that $\mathrm{pr}_{\mathbb{C}^{n+1}}(\mathfrak{v})$ is a complex line. The first two equations in (5.4.1) then simply mean that $y = 0$. Assume first that $\mathrm{Re}(b) \neq 0$. Use equations (3) and (4) to express a_{11} and c in terms of w_1. Equation (7) reveals that $a\mathrm{Im}(b) = 1$. Inserting all that into (8) yields a linear dependency on $\mathrm{Re}(w_1)$ and $\mathrm{Im}(w_1)$, which implies that the coefficients in this dependency must be zero. But one can easily compute that the coefficient of $\mathrm{Im}(w_1)$ is $(a - \frac{1}{a})^2 + \mathrm{Re}(b)^2$, which cannot be zero. So we end up with the case $\mathrm{Re}(b) = 0$. Equation (4) yields $a = 1$ and thus $b = i$, so $\mathfrak{v}$ becomes the complex line spanned inside $\mathbb{C}^{n+2}$ by $e_1 + e_{n+2}$. It is not hard to verify that this subspace is admissible and protohomogeneous. Now we want to see what this $\mathfrak{v}$ looks like with respect to the restricted root space decomposition. To this end, we need to have an explicit identification between $\mathfrak{m}_1$ and $\mathfrak{u}(n+1,1)$ and also between $\mathfrak{n}_1^1$ and $\mathbb{C}^{n+2}$. We have:

$$\mathfrak{m}_1 = \left\{ \left[\begin{array}{c|ccc} A & 0 & 0 & u \\ \hline 0 & ib & & \\ 0 & & ib & \\ u^* & & & ia \end{array} \right] \middle| A \in \mathfrak{u}(n+1), u \in \mathbb{C}^{n+1}, \, a, b \in \mathbb{R}, \, \mathrm{tr} = 0 \right\},$$

where u and u^* are regarded as column and row vectors, respectively. Similarly,

$$\mathfrak{n}_1^1 = \left\{ \left[\begin{array}{c|ccc} 0 & -v & v & 0 \\ \hline v^* & 0 & 0 & -\overline{w} \\ v^* & 0 & 0 & -\overline{w} \\ 0 & -w & w & 0 \end{array} \right] \middle| v \in \mathbb{C}^{n+1}, w \in \mathbb{C} \right\},$$

where the top left zero block is of the size $(n+1) \times (n+1)$ and $\pm v$ and v^* are again regarded as column and row vectors. Now, sending the matrix in the definition of $\mathfrak{m}_1$ to

$$\left[\begin{array}{c|c} A - ibE & u \\ \hline u^* & i(a+b) \end{array} \right]$$

gives a Lie algebra isomorphism $\mathfrak{m}_1 \xrightarrow{\sim} \mathfrak{u}(n+1,1)$. Similarly, sending the matrix in the definition of $\mathfrak{n}_1^1$ to (v,w) (here v is a row vector) gives an $\mathbb{R}$-linear isomorphism $\mathfrak{n}_1^1 \xrightarrow{\sim} \mathbb{C}^{n+2}$. Moreover, under these isomorphisms, the adjoint representation of $\mathfrak{m}_1$ on $\mathfrak{n}_1^1$ becomes the tautological representation of $\mathfrak{u}(n+1,1)$. However, the isomorphism $\mathfrak{n}_1^1 \xrightarrow{\sim} \mathbb{C}^{n+2}$ is not $\mathbb{C}$-linear with respect to the complex structure on $\mathfrak{n}_1^1$ induced from $\mathfrak{a} \oplus \mathfrak{n}$! It is $\mathbb{C}$-linear in v but $\mathbb{C}$-antilinear in w. This subtlety is important if we want to transfer the subspace $\mathfrak{v}$ from $\mathbb{C}^{n+2}$ to $\mathfrak{n}_1^1$: while it is a complex line in $\mathbb{C}^{n+2}$, it will become a totally real 2-dimensional subspace in $\mathfrak{n}_1^1$. In fact, if we modify $\mathfrak{v}$ slightly by an element from $\mathrm{U}(n+1,1)$ to make it a complex line spanned by $(0,\dots,0,1,1)$, then the subspace of $\mathfrak{n}_1^1$ it corresponds to is precisely $\mathfrak{g}_{\alpha_1}$, which can be easily seen from our description of the isomorphism above. We thus have a C1-action given by the Lie subgroup $H_{1,\mathfrak{g}_{\alpha_1}}$, whose Lie algebra is

$$\begin{aligned}
\mathfrak{h}_{1,\mathfrak{g}_{\alpha_1}} &= N_{\mathfrak{m}_1}(\mathfrak{n}_1 \ominus \mathfrak{g}_{\alpha_1}) \oplus \mathfrak{a}^1 \oplus (\mathfrak{n}_1 \ominus \mathfrak{g}_{\alpha_1}) \\
&= (\mathfrak{k}_0 \oplus \mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{2\alpha_2}) \oplus \mathfrak{a}^1 \oplus (\mathfrak{n}_1 \ominus \mathfrak{g}_{\alpha_1}) \\
&= \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}_2 = \mathfrak{h}_{2,1}^{\wedge}.
\end{aligned}$$

We see that $H_{1,\mathfrak{g}_{\alpha_1}} = H_{2,1}^{\wedge}$, so this action was taken into account in (f). All in all, we conclude that the nilpotent construction does not yield any new actions for M, which completes the proof of Theorem 5.4.1. □

Chapter 6

HOMOGENEOUS COMPLEX HYPERSURFACES

In this final chapter, we make a slight change of perspective and study a natural analog of the topic of the thesis within the framework of complex geometry: homogeneous *complex* hypersurfaces in Hermitian symmetric spaces of compact and noncompact type. Such a hypersurface does not have to be an orbit of a cohomogeneity-one action, which means that we can no longer shift the study solely to such actions—although, admittedly, they will still play a big role. Homogeneous complex hypersurfaces in complex space forms and, more generally, complex flag manifolds received certain attention between the 1960s and 1980s. This was brought to a climax when Konno obtained a classification result for such hypersurfaces in complex flag manifolds with $b_2 = 1$ ([Kon88]), which includes all irreducible Hermitian symmetric spaces of compact type. We discuss Konno's findings in Section 6.2 and add a few refinements in line with the purposes of the thesis. After that, by marrying Konno's result with the classification of cohomogeneity-one actions on irreducible compact symmetric spaces ([Kol02]), we investigate whether every homogeneous complex hypersurface can be realized as a singular orbit of such an action. When it can, the other orbits of the action possess remarkable geometric properties—we discuss this in Subsections 6.2.4 and 6.2.5.

On the other hand, little to none is known about homogeneous complex hypersurfaces in Hermitian symmetric spaces of *noncompact* type. Perhaps, the only exception is the special case of totally geodesic complex hypersurfaces; those are exhausted by $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$ and $\mathrm{Gr}^*(2, \mathbb{R}^{n+1}) \subset \mathrm{Gr}^*(2, \mathbb{R}^{n+2})$, $n \geq 3$, which can be shown using the notion of index of a symmetric space. In this chapter, we make the next logical step and consider those complex hypersurfaces that fit as a leaf into a homogeneous codimension-two foliation. In Chapter 4, we saw that standard foliations are the archetype of many types of homogeneous foliations on noncompact symmetric spaces. That lets us crystallize a research goal: classify all complex hypersurfaces that appear as base leaves of standard foliations. In Subsection 6.3.2, we accomplish this goal in the irreducible case. We then proceed to show that every such hypersurface arises from one of the totally geodesic complex hypersurfaces mentioned above via canonical extension. Among other things, we will see that, in contrast to the compact case, *every* Hermitian symmetric spaces of noncompact type admits a homogeneous complex hypersurface. Here is the layout of the chapter:

- In Section 6.1, we go through some motivation for studying homogeneous complex hypersurfaces and review the original result of Smyth and Nomizu that classifies such hypersurfaces in simply connected complex spaces forms. We also formulate some objectives that will guide us throughout the chapter.

- In Section 6.2, we take a deep dive into Konno's classification result for homogeneous complex hypersurfaces in complex flag manifolds with $b_2 = 1$. We link the hypersurfaces in his classification to cohomogeneity-one actions and discuss some arising geometric phenomena.
- In Section 6.3, we study the topic of the chapter in the context of noncompact Hermitian symmetric spaces. We examine how the restricted root spaces decomposition of such a space interacts with the complex structure and obtain several structure results. We then employ that to prove Theorem 5. In the end, we propose a few ideas for how one could use Theorem 5 as a foothold for further developments in this direction.

6.1. Motivation and central questions

The starting point of the study of homogeneous complex hypersurfaces in Kähler manifolds that sparks special interest to such submanifolds is their classification in complex space forms obtained by Smyth and Nomizu ([Smy68, NS68, Smy67]):

Theorem 6.1.1. *Let M be a simply connected complex space form and $S \subseteq M$ a complete connected complex hypersurface. The following conditions are equivalent:*

(a) *S is homogeneous.*

(b) *S is intrinsically homogeneous.*

(c) *S is Einstein.*

(d) *S has parallel Ricci curvature.*

(e) *S is (strongly) isometrically congruent to one of the following complex hypersurfaces (embedded into M in a standard way):*

$$\begin{cases} \mathbb{C}^{n-1} & \textit{if } M = \mathbb{C}^n, \\ \mathbb{C}H^{n-1} & \textit{if } M = \mathbb{C}H^n, \\ \mathbb{C}P^{n-1} \textit{ or } Q^{n-1} & \textit{if } M = \mathbb{C}P^n. \end{cases}$$

Here and throughout the chapter, $Q^{n-1} \subset \mathbb{C}P^n$ stands for the standard smooth projective quadric given as the zero locus of the quadratic form $\sum_{i=0}^{n} z_i^2$. With its induced metric, it is a Hermitian symmetric space of compact type isometric to $\mathrm{Gr}^+(2, \mathbb{R}^{n+1})$. Note that a generic smooth quadric in $\mathbb{C}P^n$ is not a homogeneous submanifold. With regard to the above theorem, observe that an Einstein manifold always has parallel Ricci curvature; the converse is true provided that the manifold is irreducible, but may be false in general. Note that $Q^{n-1} \subset \mathbb{C}P^n$ is the only hypersurface in (e) that is not totally geodesic.

This theorem suggests that the property of being homogeneous is very restrictive for complex hypersurfaces in Kähler manifolds, as well as potentially related to other geometric properties like being Einstein. It is important to point out that complex space forms are very special types of Kähler manifolds, and the connection between homogeneity and

Einsteinness may not be as straightforward when the ambient space is more complex[1]. The dominant theme in this chapter is going to be the scarcity of homogeneous complex hypersurfaces, which is still present in a much wider class of ambient spaces. It is worthwhile to draw a parallel between complex and *real* hypersurfaces. First of all, as we saw in Chapters 4 and 5, homogeneous real hypersurfaces are fairly abundant: every isometric C1-action produces a 1-parameter family of those, and for many actions, no two distinct hypersurfaces in such a family are congruent. Also, in the context of real hypersurfaces, the link between being Einstein and homogeneous persists—to some degree—in spaces with higher level of complexity. For example, it was recently shown that *complete* connected real Einstein hypersurfaces in irreducible symmetric spaces are always homogeneous (see [NP23]).

Agreement. Throughout this chapter all submanifolds are assumed to be connected and properly embedded by default.

The ultimate objective in the study of homogeneous complex hypersurfaces would of course be to obtain their full classification.

Open problem 6.A. *Given a Kähler manifold M, classify all homogeneous complex hypersurfaces in M up to isometric congruence.*

Unsurprisingly, when posed in such a general form, the problem is out of reach and requires additional assumptions on the ambient space or hypersurfaces in question. If a connected Kähler manifold M admits a homogeneous complex hypersurface, the isometry group $I(M)$ acts on M with cohomogeneity at most 2. It is therefore reasonable to restrict the problem to homogeneous Kähler manifolds. One special class of such manifolds consists of compact simply connected homogeneous Kähler manifolds, also known as *complex flag manifolds*; these include Hermitian symmetric spaces of compact type and can be studied very effectively by means of Lie theory. In the noncompact case, the situation is more convoluted, and it seems sensible to restrict right away to the more well-behaved class of Hermitian symmetric spaces of noncompact type. We formulate a humbler and more down-to-earth version of the above problem.

Open problem 6.B. *Given an irreducible Hermitian symmetric space M, classify all homogeneous complex hypersurfaces in M up to isometric congruence.*

As we know from Corollary 2.5.15, every isometry of an irreducible Hermitian symmetric space is either holomorphic or anti-holomorphic. Therefore, if one reaches an answer to Open problem 6.B, it should be fairly easy to refine it and obtain a (perhaps more natural) classification of homogeneous complex hypersurfaces in M up to congruence *via a holomorphic isometry.* Notice that if two such hypersurfaces are strongly congruent, then they are of course congruent via a holomorphic isometry.

By design, a homogeneous complex hypersurface can always be realized as an orbit of an isometric action of cohomogeneity 1 or 2. As we will see below (Remark 6.2.15), for some hypersurfaces, both of these scenarios are possible. However, as of today, there is no known example of such a hypersurface that cannot be realized as an orbit of a C1-action. This leads us to the following

Conjecture 6.C. *If M is an irreducible Hermitian symmetric space and $S \subseteq M$ is a homogeneous complex hypersurface, S can be realized as a singular orbit of an isometric*

[1] Pun not intended.

C1-action on M.

In the following sections, we will address Open problem 6.B and Conjecture 6.C. As we have witnessed numerous times throughout the thesis, there is often a stark contrast between symmetric spaces of compact and noncompact type. It turns out that the case of homogeneous complex hypersurfaces is no exception. For this reason, we treat these two settings separately.

6.2. The compact case

The bulk of this section is dedicated to the classification result of Konno ([Kon88]) concerning homogeneous complex hypersurfaces in complex flag manifolds with $b_2 = 1$. After recalling the rudiments of the theory of homogeneous Kähler manifolds, we introduce the result and reformulate it a differential-geometric spirit, making minor refinements along the way. We then marry it with the classification of C1-actions on compact symmetric spaces obtained by Kollross in [Kol02]. As a result of all this, we arrive at partial answers to Open problem 6.B and Conjecture 6.C.

6.2.1. Complex flag manifolds with $b_2 = 1$

We begin with a brief introduction to the type of spaces involved in Konno's classification. For an in-depth exposition and proofs, see [Bes08, Ch. 8].

By a *homogeneous Kähler manifold* we mean a Kähler manifold that is also a Riemannian homogeneous space. Let M be a connected compact homogeneous Kähler manifold. Such a space always decomposes as a de Rham-like (recall Definition 2.1.58) product $M = M_0 \times M_1 \times \cdots \times M_k$, where each factor is homogeneous Kähler and the irreducible factors $M_1, \dots, M_k$ are simply connected. Roughly speaking, this means that the entire fundamental group of M is concentrated in the flat factor M_0, which is just a complex torus. In particular, M is simply connected if and only if it does not have a flat factor. Such spaces are intimately related to compact semisimple Lie algebras. Let G be a connected compact semisimple Lie group. Every orbit $F \subset \mathfrak{g}$ of the adjoint representation of G is simply connected and has a natural G-invariant almost complex structure that is also integrable. Moreover, F admits a unique (up to a positive constant factor) G-invariant Kähler-Einstein metric, and that metric has positive scalar curvature. Every other Kähler-Einstein metric on F is homogeneous and isometric to a G-invariant Kähler-Einstein metric via a biholomorphism[1] of F (and thus also has positive scalar curvature). What is more, every simply connected compact homogeneous Kähler manifold M is biholomorhic to an adjoint orbit of some compact connected semisimple Lie group G (we can actually take G to be $I^0(M)$). With this in mind, we introduce the following

Definition 6.2.1. A compact simply connected complex manifold M is called a **complex flag manifold** if it satisfies the following equivalent conditions:

(a) M admits a homogeneous Kähler metric.

[1] By a result of Matsushima, the group $I^0(M)$ is a compact real form of the semisimple group of inner biholomorphisms of M (see [Mat57], [Kob95, Ch. III, Th. 5.1]), and every two such compact real forms are conjugate.

(b) M is biholomorphic to an adjoint orbit of some compact connected semisimple Lie group G (equipped with its natural G-invariant complex structure).

Remark 6.2.2. This definition is not universally agreed upon. Some authors call this a *generalized complex flag manifold* and reserve the term *complex flag manifold* for principal adjoint orbits. In earlier literature on the subject, complex flag manifolds are also called *C-spaces* or *Kähler C-spaces.*

Notice that a complex flag manifold M may in general admit many nonequivalent homogeneous Kähler metrics. Among those, there is the special subclass of Kähler-Einstein metrics, which are automatically homogeneous and of positive scalar curvature, and which are all homothetic to each other via biholomorphisms of M.

Another assumption we are going to impose is a restriction on the second Betti number. Let us relinquish homogeneity for a moment, and let M be any simply connected compact Kähler manifold with $b_2 = 1$. By the Künneth formula, M has to be irreducible. Since M is simply connected, we have $H^1(M, \mathcal{O}_M) \simeq H^{0,1}(M) = 0$, where $\mathcal{O}_M$ is the structure sheaf of holomorphic functions. Since $b_2 = 1$ and M is compact Kähler, we get $h^{1,1} = 1, h^{2,0} = h^{0,2} = 0$, which implies $H^2(M, \mathcal{O}_M) \simeq H^{0,2}(M) = 0$. With this in mind, the exponential sequence for M yields

$$\cdots \longrightarrow \cancel{H^1(M, \mathcal{O}_M)} \longrightarrow \mathrm{Pic}(M) \xrightarrow[\sim]{c_1} H^2(M, \mathbb{Z}) \longrightarrow \cancel{H^2(M, \mathcal{O}_M)} \longrightarrow \cdots$$

This means that the Picard group of M is isomorphic to $\mathbb{Z}$. There are two ways to choose this isomorphism though. Observe that that $H^2(M, \mathbb{R}) = H^{1,1}(M, \mathbb{R})$ is one-dimensional. The Kähler cone $\mathcal{K}_M$ is an open ray in $H^{1,1}(M, \mathbb{R})$. We also have the integral lattice $H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{R}) = H^{1,1}(M, \mathbb{R})$. We identify $H^2(M, \mathbb{Z})$ with $\mathbb{Z}$ by sending its unique generator lying in $\mathcal{K}_M$ to 1; this lets us naturally view the first Chern class as an isomorphism $c_1\colon \mathrm{Pic}(M) \xrightarrow{\sim} \mathbb{Z}$. We call this the **degree** of a line bundle (or a divisor). Given $k \in \mathbb{Z}$, we denote the line bundle[1] over M of degree k by $\mathcal{O}_M(k)$. By construction, $\mathcal{O}_M(k)$ is positive if and only if $k > 0$. By the Kodaira embedding theorem, these are the same as ample line bundles. In particular, they exist, so M is projective. If $S \subset M$ is a (properly embedded) complex hypersurface, we can think of it as a divisor and thus talk about its **degree**. Given $k < 0$, one can show that $\mathcal{O}_M(k)$ has no nonzero sections—by using the fact that $\mathcal{O}_M(-k)$ is ample. In particular, the degree of a complex hypersurface has to be positive. If M is, in addition, a complex flag manifold, one can show that $\mathcal{O}_M(1)$ is not just ample but very ample.

One reason why imposing the restriction $b_2 = 1$ on a complex flag manifold is useful is the effect it has on the homogeneous Kähler metrics. Suppose M is a complex flag manifold with $b_2 = 1$, and suppose we have fixed an identification of M with an adjoint orbit of some compact semisimple Lie group G (G can be chosen simple due to the irreducibility of M). One can show that M admits a unique G-invariant Kähler metric up to rescaling by a positive constant (see [Bes08, 8.84]); that metric is then necessarily Einstein of positive scalar curvature.

Corollary 6.2.3. *Let M be a complex flag manifold with $b_2 = 1$.*

(a) *Any homogeneous Kähler metric on M is Einstein of positive scalar curvature.*

[1]As is the tradition, we often do not distinguish notationally between elements of the Picard group and their specific representatives.

(b) *Any two homogeneous Kähler metrics on M are homothetic via a biholomorphism of M.*

Example 6.2.4. Every Hermitian symmetric space of compact type is a complex flag manifold. It has $b_2 = 1$ if and only if it is irreducible. As complex flag manifolds, compact Hermitian symmetric spaces have the following property: every homogeneous Kähler metric on M is symmetric and differs from the existing one by some rescaling of the normalizing constants. It then makes sense to say that a complex flag manifold is *Hermitian symmetric* even if a homogeneous Kähler metric has not been specified. //

Another favorable property of the complex flag manifolds singled out by the condition $b_2 = 1$ is that they admit a neat description in terms of complex simple Lie groups. Let G be a simply connected complex simple Lie group and $\mathfrak{g}$ its Lie algebra. Pick a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a set of simple roots $\Lambda = \{\alpha_1, \dots, \alpha_r\} \subset \Delta$. Choose any $j \in \{1, \dots, r\}$ and let $\mathfrak{q}_j$ be the maximal proper parabolic subalgebra[1] of $\mathfrak{g}$ corresponding to the subset $\Phi_j = \Lambda \setminus \{\alpha_j\}$ of Λ. Write Q_j for the connected complex Lie subgroup of G corresponding to $\mathfrak{q}_j$; it is automatically a closed subgroup. With its induced complex structure, the homogeneous space $M = G/Q_j$, is a complex flag manifold with $b_2 = 1$. Indeed, for a suitable compact real form $G_0 \subset G$, one has $G_0/(G_0 \cap Q_j) \cong G/Q_j$. Here $G_0 \cap Q_j$ is the isotropy group of some $X \in \mathfrak{g}_0 = \mathrm{Lie}(G_0)$ with respect to the adjoint representation of G_0, so the quotient $G_0/(G_0 \cap Q_j)$ is the adjoint orbit $G_0 \cdot X$. One can show that the resulting identification $G_0 \cdot X \cong M$ is a biholomorphism. The space M is fully determined by the irreducible reduced root system Δ of $\mathfrak{g}$ and the removed simple root α_j, so it is common to denote this space as (Δ, α_j). Every complex flag manifold with $b_2 = 1$ arises via this construction. Note that if two simple roots $\alpha_i, \alpha_j \in \Delta$ differ by an automorphism of the Dynkin diagram, the resulting complex flag manifolds are isomorphic. But even when two pairs $(\Delta, \alpha), (\Delta', \alpha')$ have $\Delta \not\cong \Delta'$, they may give rise to isomorphic complex flag manifolds.

Example 6.2.5. Here are some complex flag manifolds with $b_2 = 1$ and pairs that represent them:

(a) $\mathbb{C}P^n$: (A_n, α_1).

(b) $\mathrm{Gr}(r, \mathbb{C}^n)$: (A_{n-1}, α_r).

(c) Q^n $(n \neq 2)$: $(\mathfrak{so}(n+2, \mathbb{C}), \alpha_1)$. Here we write the complex simple Lie algebra itself rather than its root system because the latter depends on the parity of n: it is $D_{\frac{n}{2}+1}$ if n is even, $B_{\frac{n+1}{2}}$ if $n \geq 3$ is odd, and A_1 if $n = 1$. //

Here we adopt the same root labeling as in Konno's paper [Kon88]. It is not hard to compute which of the pairs (Δ, α) give rise to Hermitian symmetric spaces, see [Kon88, Tab. 1]. The ample generator $\mathcal{O}_M(1)$ of the Picard group can be described in terms of the root system Δ and the parabolic subalgebra $\mathfrak{q}_j$. Similarly to the real semisimple case, the Lie algebra $\mathfrak{q}_j$ decomposes as a semidirect sum $\mathfrak{g}_j \niplus \mathfrak{n}_j$ of its reductive and nilpotent subalgebras. Here the summand $\mathfrak{g}_j$ contains the Cartan subalgebra $\mathfrak{h}$. From this, it is easy to see that $\mathfrak{n}_j$ must be the nilpotent nilradical of $\mathfrak{q}_j$ (see p. 149). Let P stand for the weight lattice of Δ; and let $\omega_1, \dots, \omega_r \in P$ be the fundamental weights associated with Λ. The subset $P_j^+ = \{\lambda \in P \mid \langle \lambda | \alpha_i \rangle \geq 0 \; \forall\, i \neq j\}$ is the set of highest

[1] The theory of parabolic subalgebras of complex semisimple Lie algebras is quite similar to its real counterpart (see Subsection 2.4.3), except it is arguably less complicated. See [Kna02, Sec.V.7].

weights of all irreducible representations of $\mathfrak{g}_j$. One particular element of this set is the fundamental weight ω_j. The dual of the irreducible representation of $\mathfrak{g}_j$ with highest weight ω_j can also be described as the irreducible representation with lowest weight $-\omega_j$; we are going to denote it by $(V_{-\omega_j}, \rho_{-\omega_j})$. Since $\mathfrak{n}_j$ is the nilpotent radical of $\mathfrak{q}_j$, every irreducible representation of $\mathfrak{q}_j$ vanishes on $\mathfrak{n}_j$. Essentially, this means that irreducible representations of $\mathfrak{q}_j$ are the same as those of $\mathfrak{g}_j$. We extend $\rho_{-\omega_j}$ to $\mathfrak{q}_j$ by letting it be zero on $\mathfrak{n}_j$; we keep using the same notation for the extension. One can also show that $\rho_{-\omega_j}\colon \mathfrak{q}_j \to \mathfrak{gl}(V_{-\omega_j})$ lifts (uniquely) to $\widetilde{\rho}_{-\omega_j}\colon Q_j \to \mathrm{GL}(V_{-\omega_j})$, so we obtain an irreducible holomorphic representation $(V_{-\omega_j}, \widetilde{\rho}_{-\omega_j})$ of Q_j. Since $G \twoheadrightarrow M$ is a principal Q_j-bundle, we can construct the holomorphic vector bundle $G \times_{\widetilde{\rho}_{-\omega_j}} V_{-\omega_j}$ associated to $(V_{-\omega_j}, \widetilde{\rho}_{-\omega_j})$. It turns out that this is actually a line bundle of degree 1, i.e., the ample generator $\mathcal{O}_M(1)$ of $\mathrm{Pic}(M)$. Note that G acts on $G \times_{\widetilde{\rho}_{-\omega_j}} V_{-\omega_j}$ in a way that agrees with its action on the base M. This induces a holomorphic representation of G on the space of sections $H^0(M, \mathcal{O}_M(1))$. The generalized Borel-Weil theorem (see [Kos61]) ensures that this is an irreducible representation of G with lowest weight $-\omega_j$.

6.2.2. The classification of homogeneous complex hypersurfaces

Now we are ready to discuss Konno's classification of homogeneous complex hypersurfaces. Before we do that though, let us look at the special case of totally geodesic hypersurfaces. We restrict to symmetric spaces because, in them, (complete connected) totally geodesic submanifolds are automatically homogeneous (Corollary 2.2.13(c)), but also because the study of totally geodesic submanifolds is in general more promising in symmetric spaces.

As we know from Example 2.2.15, the only symmetric spaces admitting a totally geodesic (real) hypersurface are those of constant sectional curvature. In general, having a totally geodesic submanifold of low codimension is a rather restrictive condition on a symmetric space. This is governed by the notion of *index*, first introduced and studied by Onishchik in [Oni80]. The **index** of a symmetric space M, denoted by $\boldsymbol{i(M)}$ is the lowest possible codimension of a proper totally geodesic submanifold of M. If M is irreducible, its index is bounded below by its rank: $\mathrm{rk}(M) \leq i(M)$ (see [BO18]). In particular, if M is an irreducible Hermitian symmetric space admitting a totally geodesic complex hypersurface, its rank must be 1 or 2. With the only exceptions of $G_2/\mathrm{SO}(4)$ and its dual $G_2^2/\mathrm{SO}(4)$, every irreducible symmetric space contains a reflective submanifold of codimension equal to $i(M)$. (Until resolved in [BO21], this was known as *the index conjecture*.) Together with Leung's classification of reflective submanifolds (see Subsection 2.2.1), this result allows to compute the index of every irreducible symmetric space. For low-rank spaces, it is a relatively feasible task to find all (complete connected) totally geodesic submanifolds whose codimension equals the index of the ambient space. From [BCO16, Tab. 11.1], where this is done in rank 1 and 2, we deduce:

Proposition 6.2.6. *Let M be an irreducible Hermitian symmetric space and $S \subset M$ a complete connected totally geodesic complex hypersurface. Then M is isometric to either $\mathbb{C}P^n$ or $Q^n\,(n \geq 3)$ (or their duals $\mathbb{C}H^n$, $\mathrm{Gr}^*(2, \mathbb{R}^{n+2})$), S is reflective, and*

$$S \text{ is (strongly) isometrically congruent to } \begin{cases} \mathbb{C}P^{n-1} & \text{if } M = \mathbb{C}P^n, \\ \mathbb{C}H^{n-1} & \text{if } M = \mathbb{C}H^n, \\ Q^{n-1} & \text{if } M = Q^n, \\ \mathrm{Gr}^*(2, \mathbb{R}^{n+1}) & \text{if } M = \mathrm{Gr}^*(2, \mathbb{R}^{n+2}). \end{cases}$$

Here all four hypersurfaces are embedded into the corresponding M in a standard way. Together with Theorem 6.1.1, this proposition provides three examples of homogeneous complex hypersurfaces in complex flag manifolds with $b_2 = 1$, and all of them actually occur in irreducible Hermitian symmetric spaces: $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$, $Q^{n-1} \subset \mathbb{C}P^n$, and $Q^{n-1} \subset Q^n$. In [Kon88], Konno proved that there are just two more examples—but he did so in the complex category, essentially forgetting about any Kähler metrics. This alone does not pose a serious problem thanks to Corollary 6.2.3. What is more important though, the notion of equivalence in his classification is weaker than congruence: he considers two hypersurfaces $S \subset M$ and $S' \subset M'$ equivalent if M and S are biholomorphic to M' and S', respectively; but the map $S \xrightarrow{\sim} S'$ does not have to be the restriction of $M \xrightarrow{\sim} M'$. For the sake of brevity, we shall say that *the pairs* (M, S) *and* (M', S') *are biholomorphic* and denote it by $(M, S) \simeq (M', S')$.

Theorem 6.2.7. *Let M be a complex flag manifold with $b_2 = 1$ and $S \subset M$ a connected properly embedded (nonsingular) complex hypersurface admitting a homogeneous Kähler metric (e.g., S is an intrinsically homogeneous submanifold with respect to some Kähler metric on M). Then:*

(a) *M is Hermitian symmetric.*

(b) *S is a complex flag manifold. Unless $(M, S) \simeq (\mathbb{C}P^3, Q^2)$ or (Q^3, Q^2), S has $b_2 = 1$.*

(c) *The pair (M, S) is biholomorphic to one of the following (here d is the degree of S):*

M		S		d	restrictions
$\mathbb{C}P^n$	(A_n, α_1)	$\mathbb{C}P^{n-1}$	(A_{n-1}, α_1)	1	$n \geq 1$
$\mathbb{C}P^n$	(A_n, α_1)	Q^{n-1}	[if $n \neq 3$] $(\mathfrak{so}(n+1, \mathbb{C}), \alpha_1)$	2	$n \geq 2$
Q^n	$(\mathfrak{so}(n+2, \mathbb{C}), \alpha_1)$	Q^{n-1}	[if $n \neq 3$] $(\mathfrak{so}(n+1, \mathbb{C}), \alpha_1)$	1	$n \geq 3$
$\mathrm{Gr}(2, \mathbb{C}^{2n})$	(A_{2n-1}, α_2)	$\mathrm{Sp}(n)/\mathrm{Sp}(n-2)\mathrm{U}(2)$	(C_n, α_2)	1	$n \geq 3$
$E_6/\mathrm{Spin}(10)\mathrm{U}(1)$	(E_6, α_1)	$F_4/\mathrm{Spin}(7)\mathrm{U}(1)$	(F_4, α_4)	1	

Table 6.1. Homogeneous complex hypersurfaces in complex flag manifolds with $b_2 = 1$

In this table, for both M and S (unless $S \simeq Q^2$), we list their corresponding pairs (Δ, α), where Δ is an irreducible reduced root (or a complex simple Lie algebra) and α is a simple root. Implicit in the theorem is the assertion that for each pair (M, S) as in the table, a holomorphic embedding $S \hookrightarrow M$ actually exists. For the first three rows, this is elementary. For the last two, this was shown in [Sak85] and [Kim79], respectively. In the fourth row of the table, we could have allowed $n = 2$, but then $\mathrm{Gr}(2, \mathbb{C}^4) \simeq Q^4$ and $\mathrm{Sp}(2)/\mathrm{U}(2) \simeq Q^3$, so that possibility was already accounted for in the third row. Owing to the uniqueness property of homogeneous Kähler metrics on complex flag manifolds with $b_2 = 1$ (Corollary 6.2.3(2)), Theorem 6.2.7 readily implies the following classification result, which partially answers Open problem 6.B in the compact case:

Corollary 6.2.8. *Let M be a complex flag manifold with $b_2 = 1$ endowed with a homogeneous Kähler metric, and let $S \subset M$ a connected properly embedded homogeneous complex hypersurface. Then:*

(a) *M is an irreducible Hermitian symmetric space of compact type.*

(b) *S is a complex flag manifold. Unless $(M, S) \simeq (\mathbb{C}P^3, Q^2)$ or (Q^3, Q^2), S has $b_2 = 1$.*

(c) *M and S are holomorphically isometric to one of the pairs listed in Table 6.1.*

Strictly speaking, there are two pairs (M,S) in Table 6.1 where the hypersurface has $b_2 \neq 1$, namely $(\mathbb{C}P^3, Q^2)$ and (Q^3, Q^2). In these cases, Corollary 6.2.3 is not applicable. Still, the standard metric on $Q^2 \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$ is Hermitian symmetric, hence so is every homogeneous Kähler metric on it (see Example 6.2.4). Consequently, the hypersurfaces S in these pairs will be holomorphically isometric to Q^2, possibly with nonmatching normalizing constants on its two de Rham factors. But S is actually *congruent* to the standard Q^2 inside $\mathbb{C}P^3$ or Q^3, which is Einstein and thus has equal normalizing constants thanks to Proposition 2.1.114. For $\mathbb{C}P^3$, this follows from Theorem 6.1.1, and for Q^3, we will show it in Proposition 6.2.12 below.

Let us briefly discuss the methods used in the proof of Theorem 6.2.7 given in [Kon88]. The key idea is to study cohomological invariants of the line bundles $\mathcal{O}_X(k)$, as well as the twisted vector bundles $T_X(k) = T_X \otimes \mathcal{O}_X(k)$ and $\Omega^1_X(k) = \Omega^1_X \otimes \mathcal{O}_X(k)$, of a complex flag manifold X with $b_2 = 1$ by means of the representation theory of complex simple Lie algebras and their parabolic subalgebras. One can then use the existence of an embedding $S \hookrightarrow M$ to draw certain conclusions about those cohomology groups and obtain strong restrictions on what M and S could be. We break down the proof in several steps:

(a) For an arbitrary complex flag manifold X with $b_2 = 1$, show the following:

 (1) $H^0(X, \Omega^1_X(1)) \neq 0$ unless X is (C_m, α_j), (F_4, α_4), or an irreducible Hermitian symmetric space of compact type.

 (2) Provided $X \not\simeq \mathbb{C}P^n$, $H^1(X, T_X(-d)) = 0$ for any $d \geq 1$ unless

$$X \simeq \begin{cases} Q^n \, (n \geq 3) & \text{and } d = 2, \\ (C_m, \alpha_2) \, (n \geq 3) & \text{and } d = 1, \\ (F_4, \alpha_4) & \text{and } d = 1. \end{cases}$$

(b) Let M be a complex flag manifold with $b_2 = 1$ and $S \subset M$ a (nonsingular) complex hypersurface of degree d. If $M \simeq \mathbb{C}P^n$ or Q^n (and S is intrinsically homogeneous), the assertion of Theorem 6.2.7 is a well-known fact; for this reason, *we assume $M \not\simeq \mathbb{C}P^n, Q^n$ from now on.* In particular, this ensures that $n = \dim_{\mathbb{C}}(M) \geq 5$.

(c) It follows from a sufficiently general version of the Lefschetz hyperplane theorem (see, e.g., [Fuj80]) that S is also simply connected and has $b_2 = 1$. As a result, its Picard group is isomorphic to $\mathbb{Z}$ and has a unique ample generator—according to our discussion on p. 179.

(d) Show that $H^1(S, T_S(-d)) \neq 0$. This is the crux of Konno's argument.

(e) Show that S and M are related by the following equations:

 (1) $k(M) = k(S) + d$, where k is the degree of the anticanonical bundle.

 (2) Provided that M is an irreducible Hermitian symmetric space of compact type,

$$h^0(\mathcal{O}_M(1)) = h^0(\mathcal{O}_S(1)) + \begin{cases} 0 & \text{if } d \geq 2 \\ 1 & \text{if } d = 1. \end{cases}$$

(f) Now suppose S admits a homogeneous Kähler metric—and hence is a complex flag manifold with $b_2 = 1$. Show that $H^0(M, \Omega^1_M(1))$ must vanish whenever S is (C_m, α_j), (F_4, α_4), or an irreducible Hermitian symmetric space of compact type.

(g) Combining (d) with (a)-(1), we see that there are 4 options for S:

$$S \simeq \begin{cases} \mathbb{C}P^n, & \\ Q^n\,(n \geq 3) & \text{and } d = 2, \\ (C_m, \alpha_2)\,(n \geq 3) & \text{and } d = 1, \\ (F_4, \alpha_4) & \text{and } d = 1. \end{cases}$$

(h) Combining (g), (f), and (a)-(1), we deduce that M has to be (C_m, α_j), (F_4, α_4), or an irreducible Hermitian symmetric space of compact type.

(i) Calculate the numbers $k(X)$ and $h^0(\mathcal{O}_X(1))$ for every complex flag manifold with $b_2 = 1$ (see [Kon88, Tab. 2]). Then use that in combination with (e), (g), and (h) to deduce Theorem 6.2.7.

6.2.3. The problem of congruence

As is evident from the above description, Konno's method could not possibly produce a classification of homogeneous complex hypersurfaces up to congruence. Essentially, it only tells when a complex flag manifold admits a holomorphic codimension-one embedding into another such manifold with $b_2 = 1$. Consequently, Corollary 6.2.8 does not fully resolve the classification problem for homogeneous complex hypersurfaces in irreducible compact Hermitian symmetric spaces—the way we posed it in Open problem 6.B. Here we try to address this issue.

The case of hypersurfaces in $\mathbb{C}P^n$ is taken care of by Theorem 6.1.1. It is important to point out that in their proof of the equivalence between (a) and (b) in Theorem 6.1.1, Smyth and Nomizu relied heavily on the fact that the ambient space is a complex space form. They proved that an intrinsically homogeneous complex hypersurface in a simply connected complex space form is holomorphically isometric to one of the hypersurfaces in (e), and then they invoked Calabi's rigidity theorem (see [Cal53] and [NS68, Th. 1]):

Proposition 6.2.9 (Calabi's Rigidity Theorem). *Suppose M is a simply connected complex space form and S a connected Kähler manifold with two Kähler immersions $F_{1,2}\colon S \to M$. Then there exists a holomorphic isometry g of M such that $F_2 = g \circ F_1$.*

There is little hope a similar result would hold in Hermitian symmetric spaces of nonconstant holomorphic sectional curvature, let alone arbitrary homogeneous Kähler manifolds.

The ambient space in the remaining three rows in Table 6.1 is no longer $\mathbb{C}P^n$, which makes the matters more complicated. On the other hand, the degree of the hypersurface is always one, which allows us to represent it as a hyperplane section. Let M be a complex flag manifold with $b_2 = 1$. As we mentioned earlier, the line bundle $\mathcal{O}_M(1)$ is very ample, so its complete linear system determines a holomorphic embedding $\varphi\colon M \hookrightarrow |\mathcal{O}_M(1)| = \mathbb{P}(H^0(M, \mathcal{O}_M(1))^*)$. For instance, for $M = Q^n$, this gives its standard embedding into $\mathbb{C}P^{n+1}$, whereas for $M = \mathrm{Gr}(2, \mathbb{C}^{2n})$, this is the Plücker embedding (see [Sak85]). We can represent M by a complex simple Lie algebra $\mathfrak{g}$ with a fixed simple root $\alpha_j \in \Lambda$.

Then $M = G/Q_j$, where G is the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, and $Q_j \subset G$ is the connected Lie subgroup corresponding to the parabolic subalgebra $\mathfrak{q}_j \subset \mathfrak{g}$. Recall that G acts on the total space of $\mathcal{O}_M(1)$ in a way that agrees with its action on the base. As a result, it has a representation on $V = H^0(M, \mathcal{O}_M(1))$, and that representation is irreducible with lowest weight $-\omega_j$. This is equivalent to saying that the dual representation V^* is irreducible with highest weight ω_j. This latter representation induces a holomorphic action of G on $\mathbb{P}(V^*)$. This action agrees[1] with the one on M:

Proposition 6.2.10. *The embedding* $\varphi\colon M \hookrightarrow \mathbb{P}(V^*)$ *is* G*-equivariant.*

Proof. Recall that each $p \in M$ gives rise to a linear map $\mathrm{ev}_p\colon V \twoheadrightarrow \mathcal{O}_M(1)_p, \sigma \mapsto \sigma(p)$; and $\varphi(p) = \mathrm{Ann}(\mathrm{Ker}(\mathrm{ev}_p))$, the annihilator of the hyperplane $\mathrm{Ker}(\mathrm{ev}_p) \subset V$, which is a line in V^*. The action of G on V is given by $g \cdot \sigma = g \circ \sigma \circ g^{-1}$ (where the left g acts on the total space of $\mathcal{O}_M(1)$ and the right g^{-1} acts on M). From this, we can see that the diagram

$$\begin{tikzcd} V \arrow[r, two heads, "\mathrm{ev}_p"] \arrow[d, "g", "\wr"'] & \mathcal{O}_M(1)_p \arrow[d, "g", "\wr"'] \\ V \arrow[r, two heads, "\mathrm{ev}_{g(p)}"] & \mathcal{O}_M(1)_{g(p)} \end{tikzcd}$$

is commutative for every $g \in G$. We can now compute:

$$g \cdot \varphi(p) = g(\mathrm{Ann}(\mathrm{Ker}(\mathrm{ev}_p))) = \mathrm{Ann}(g(\mathrm{Ker}(\mathrm{ev}_p))) = \mathrm{Ann}(\mathrm{Ker}(\mathrm{ev}_{g(p)})) = \varphi(g(p)),$$

which completes the proof. □

The restriction of the line bundle $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ to $M \subseteq \mathbb{P}(V^*)$ is naturally isomorphic to $\mathcal{O}_M(1)$, hence we have a linear map

$$H^0(\mathbb{P}(V^*), \mathcal{O}_{\mathbb{P}(V^*)}(1)) \to V. \tag{6.2.1}$$

But the space of sections $H^0(\mathbb{P}(V^*), \mathcal{O}_{\mathbb{P}(V^*)}(1))$ is naturally isomorphic to V, and it is easy to show that (6.2.1) is an isomorphism. What we really care about is that it is surjective[2], which implies that any (nonsingular) degree-1 hypersurface in M is a (transverse) hyperplane section of M inside $\mathbb{P}(V^*)$.

Now we are ready to deal with the third row in Table 6.1. Consider the quadric $Q^n \subset \mathbb{C}P^{n+1}, n \geq 3$. By definition, it is the projective zero locus of $q = \sum_{i=0}^{n+1} z_i^2$, and the affine cone $\widetilde{Q}^n \subset \mathbb{C}^{n+2}$ over Q^n is the affine zero locus of q. Clearly, $\widetilde{Q}^n$ (and thus Q^n) is left invariant by any linear transformation of $\mathbb{C}^{n+2}$ that preserves the symmetric bilinear form corresponding to q (we denote it by the same letter). Therefore, the subgroup $\mathrm{SU}(n+2) \cap \mathrm{O}(n+2, \mathbb{C}) = \mathrm{SO}(n+2)$ of $\mathrm{SU}(n+2)$ preserves $\widetilde{Q}^n$ and Q^n. Conversely, if $g \in \mathrm{SU}(n+2)$ leaves Q^n invariant, then the quadratic forms q and $g \cdot q = q \circ g^{-1}$ have the same zero locus and thus must be proportional: $g \cdot q = \alpha q$, where $\alpha \in \mathbb{C}^\times$. This means that $\alpha^{-1} g \in \mathrm{U}(n+2) \cap \mathrm{O}(n+2, \mathbb{C}) = \mathrm{O}(n+2)$. We deduce that the identity component of the subgroup of elements in $\mathrm{SU}(n+2)$ leaving Q^n invariant has the same image in $I^0(Q^n)$ as $\mathrm{SO}(n+2)$; this image is the whole of $I^0(Q^n)$ since

[1] Proposition 6.2.10—and this whole passage for that matter—is part of the setting for the generalized Borel-Weil theorem, see [Kos61] for more details.

[2] A projective variety satisfying this property is called *linearly normal*.

$Q^n \simeq \mathrm{Gr}^+(2,\mathbb{R}^{n+2}) = \mathrm{SO}(n+2)/\mathrm{SO}(n)\mathrm{SO}(2)$. As we know by now, every nonsingular degree-1 hypersurface in Q^n is cut out by a (unique) hyperplane $H \subset \mathbb{C}P^{n+1}$ (intersecting Q^n transversely). Let $\widetilde{H} \subset \mathbb{C}^{n+2}$ stand for the affine cone over H, which is just a hyperplane in $\mathbb{C}^{n+2}$.

Lemma 6.2.11. *A nonsingular hyperplane section $H \cap Q^n$ is a homogeneous submanifold of Q^n if and only if the orthogonal complements $\widetilde{H}^{\perp_h}$ and $\widetilde{H}^{\perp_q}$ with respect to h and q, respectively, coincide.*

Proof. It is easy to see that the intersection $H \cap Q^n$ is transverse ($\Leftrightarrow$ nonsingular) if and only if the restriction of q to $\widetilde{H}$ is nondegenerate. According to the above discussion, the hyperplane section $H \cap Q^n$ is homogeneous in Q^n if and only if the subgroup of elements of $\mathrm{SO}(n+2)$ preserving it acts transitively on it. First, assume the orthogonal complements of $\widetilde{H}$ with respect to h and q do not coincide. Then they span a 2-dimensional complex subspace of $\mathbb{C}^{n+2}$, which has to intersect $\widetilde{H}$ by some complex line ℓ. If $g \in \mathrm{SO}(n+2)$ leaves $\widetilde{H}$ invariant, then it must preserve both $\widetilde{H}^{\perp_h}$ and $\widetilde{H}^{\perp_q}$, hence it also preserves ℓ. As a result, the subgroup of elements in $\mathrm{SO}(n+2)$ preserving $\widetilde{H}$ cannot act transitively on $H \cap Q^n$. Conversely, suppose the orthogonal complements coincide. Then we can find a vector v_{n+1} in this complement such that $||v_{n+1}||^2 = q(v_{n+1}) = 1$. Take any two complex lines $L, L' \subset \widetilde{H} \cap \widetilde{Q}^n$. We can find two bases $(v_i)_{i=0}^n$ and $(v_i')_{i=0}^n$ for $\widetilde{H}$ orthonormal with respect to both h and q such that $v_0 + iv_1 \in L$ and $v_0' + iv_1' \in L'$. The linear transformation g of $\mathbb{C}^{n+2}$ sending $v_0, \ldots, v_n, v_{n+1}$ to $v_0', \ldots, v_n', v_{n+1}$ lies in $\mathrm{U}(n+2) \cap \mathrm{O}(n+2,\mathbb{C}) = \mathrm{O}(n+2)$ and preserves $\widetilde{H}$; by changing its value on v_{n+1} to $-v_{n+1}$ if necessary, we can make sure g lies in $\mathrm{SO}(n+2)$. This shows that $H \cap Q^n$ is a homogeneous submanifold of Q^n. $\square$

We can now prove the following result, which settles Open problem 6.B for $M = Q^n$.

Proposition 6.2.12. *Any connected properly embedded homogeneous complex hypersurface in Q^n is (strongly) isometrically congruent to Q^{n-1}. In particular, it is totally geodesic.*

Proof. Let S be a homogeneous complex hypersurface in Q^n. For the purposes of this proof, we may assume $n \geq 3$. In light of Corollary 6.2.8, S is a nonsingular section of Q^n by some projective hyperplane $H \subset \mathbb{C}P^{n+1}$. By Lemma 6.2.11, the affine cone $\widetilde{H} \subset \mathbb{C}^{n+2}$ satisfies $\widetilde{H}^{\perp_h} = \widetilde{H}^{\perp_q}$. This allows us to find a basis $(v_i)_{i=0}^{n+1}$ for $\mathbb{C}^{n+2}$ orthonormal with respect to both h and q such that $v_{n+1} \perp \widetilde{H}$. The linear transformation g of $\mathbb{C}^{n+2}$ sending $v_0, \ldots, v_{n+1}$ to the standard basis $e_0, \ldots, e_{n+1}$ lies in $\mathrm{U}(n+2) \cap \mathrm{O}(n+2,\mathbb{C}) = \mathrm{O}(n+2)$; by changing its value on v_{n+1} to $-e_{n+1}$ if necessary, we can make sure g lies in $\mathrm{SO}(n+2)$. The induced transformation of $\mathbb{C}P^{n+1}$ preserves Q^n and provides a (strong) isometric congruence between S and the standard subquadric $Q^{n-1} \subset Q^n$. $\square$

The situation in the last two rows of Table 6.1 is more convoluted and we do not attempt to solve the corresponding congruence problems here. It was proven in [Sak85] that every nonsingular degree-1 hypersurface in $\mathrm{Gr}(2,\mathbb{C}^{2n})$ is *holomorphically* congruent to the standard embedding $\mathrm{Sp}(n)/\mathrm{Sp}(n-2)\mathrm{U}(2) \hookrightarrow \mathrm{Gr}(2,\mathbb{C}^{2n})$, i.e., congruent via a biholomorphism $g \in \mathrm{SL}(2n,\mathbb{C})$ of the Grassmannian. (The same is clearly true for degree-2 hypersurfaces in $\mathbb{C}P^n$ and can also be shown for degree-1 hypersurfaces in Q^n.) Now, among all these hypersurfaces, there are those that are homogeneous, and we know that they are holomorphically isometric to the standard $\mathrm{Sp}(n)/\mathrm{Sp}(n-2)\mathrm{U}(2)$. The question is whether they are isometrically congruent to it. The same can be asked about degree-1 hypersurfaces in $E_6/\mathrm{Spin}(10)\mathrm{U}(1)$. We can formulate the following

Conjecture 6.2.13. (a) *Every homogeneous complex hypersurface in* $\mathrm{Gr}(2, \mathbb{C}^{2n})$, $n \geq 3$, *is isometrically congruent to the standard* $\mathrm{Sp}(n)/\mathrm{Sp}(n-2)\mathrm{U}(2)$.

(b) *Every homogeneous complex hypersurface in* $E_6/\mathrm{Spin}(10)\mathrm{U}(1)$ *is isometrically congruent to the standard* $F_4/\mathrm{Spin}(7)\mathrm{U}(1)$.

A positive answer to this conjecture would settle the compact case of Open problem 6.B. According to the discussion above, either of $\mathrm{Gr}(2, \mathbb{C}^{2n})$ and $E_6/\mathrm{Spin}(10)\mathrm{U}(1)$ embeds G-equivariantly into $\mathbb{P}(H^0(M, \mathcal{O}_M(1))^*)$, where $G = \mathrm{SL}(2n, \mathbb{C})$ or $E_6(\mathbb{C})$, respectively. With respect to this embedding, the nonsingular degree-1 hypersurfaces in M are the same as its transverse hyperplane sections. So the above conjecture boils down to showing that the action of $\mathrm{SU}(2n)$ or E_6 on $\mathbb{P}(H^0(M, \mathcal{O}_M(1))^*)$ is transitive on the set of hyperplanes intersecting M transversely along a homogeneous hypersurface. Since we know exactly what the representation of G on $H^0(M, \mathcal{O}_M(1))$ looks like, this problem might allow a solution by means of representation theory: one needs to show that two vectors in this representation giving rise to transverse homogeneous hyperplane sections lie in the same orbit of G up to a scalar multiple.

6.2.4. Homogeneous complex hypersurfaces and cohomogeneity-one actions

In this part, we look at the homogeneous complex hypersurfaces in Table 6.1 through the lens of Conjecture 6.C and try to answer which of them come from cohomogeneity-one actions. Fortunately, this is no daunting task—because C1-actions on irreducible symmetric spaces are fully classified (see [Kol02, Th. B]). As we know from Proposition 2.3.35, an isometric C1-action on a simply connected symmetric space of compact type (by a connected Lie group) must have precisely two singular orbits, and the principal orbits are the tubes of varying radii around any of the singular orbits. If the ambient space M is Hermitian, in order to figure out when a singular orbit is a complex hypersurface, one needs to compute its codimension and verify whether its tangent spaces are preserved by the complex structure of M. The former is a reasonably straightforward process—at least in case of classical symmetric spaces (see Example 6.2.17). There are only two exceptional irreducible Hermitian symmetric spaces of compact type (see Table 6.4 below); and only one of them admits an isometric C1-action (unique up to congruence), namely $F_4 \curvearrowright E_6/\mathrm{Spin}(10)\mathrm{U}(1)$. One of the singular orbits of this action is a homogeneous hypersurface $F_4/\mathrm{Spin}(7)\mathrm{U}(1)$; this was shown in [Kim79]. It turns out that the list of homogeneous complex hypersurfaces arising from isometric C1-actions coincides with the one in Table 6.1.

Proposition 6.2.14. *Let M be an irreducible Hermitian symmetric space of compact type and H a connected Lie group acting properly and isometrically on M with cohomogeneity one. If one of the singular orbits of H is a complex hypersurface, then the action of H is orbit-equivalent to one of the actions given in Table 6.2 below.*

In this table, n is subject to the same restrictions as in Table 6.1, so we do not repeat them. All five of these actions arise from a natural (local) embedding $H \hookrightarrow I^0(M)$. Note that even with Corollary 6.2.8, this proposition does not imply that every homogeneous

M	H	the complex singular orbit	the other singular orbit
$\mathbb{C}P^n$	$\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1))$	$\mathbb{C}P^{n-1}$	$\{\mathrm{pt}\}$
$\mathbb{C}P^n$	$\mathrm{SO}(n+1)$	Q^{n-1}	$\mathbb{R}P^n$
Q^n	$\mathrm{SO}(n+1)$	Q^{n-1}	$\mathbb{S}^n$
$\mathrm{Gr}(2,\mathbb{C}^{2n})$	$\mathrm{Sp}(n)$	$\mathrm{Sp}(n)/\mathrm{Sp}(n-2)\mathrm{U}(2)$	$\mathbb{H}P^{n-1}$
$E_6/\mathrm{Spin}(10)\mathrm{U}(1)$	F_4	$F_4/\mathrm{Spin}(7)\mathrm{U}(1)$	$\mathbb{O}P^2$

Table 6.2. Cohomogeneity-one actions on irreducible Hermitian symmetric spaces of compact type with a complex codimension-2 singular orbit

complex hypersurface in an irreducible Hermitian symmetric space of compact type arises as a singular orbit of an isometric C1-action; to guarantee this, we still need to show that any two such hypersurfaces lying in the same space are isometrically congruent, provided they have the same degree. Thanks to Theorem 6.1.1 and Proposition 6.2.12, this is true for $\mathbb{C}P^n$ and Q^n, but the case of $\mathrm{Gr}(2,\mathbb{C}^{2n})$ and $E_6/\mathrm{Spin}(10)\mathrm{U}(1)$ is still open. In other words, Conjecture 6.C is contingent on the answer to Conjecture 6.2.13.

Remark 6.2.15. In certain cases, a homogeneous complex hypersurface can be realized as an orbit of both a C1- and a C2-action. For example $\mathbb{C}P^{n-1}$ is a singular orbit of a C1-action $H = \mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(1)) \curvearrowright \mathbb{C}P^n$. But if we remove the U(1)-factor from H—which is responsible for a circle action in the slice representation at a point on $\mathbb{C}P^{n-1}$—then we get $\mathrm{SU}(n) \curvearrowright \mathbb{C}P^n$. This action has cohomogeneity 2 but still has $\mathbb{C}P^{n-1}$ as an orbit (this time, a nonsingular orbit).

There is a link between the actions in Table 6.2 and contact geometry. Let M be a Kähler manifold and $S \subset M$ a real hypersurface. The almost complex structure I of M allows to define a hyperplane distribution $TS \cap I(TS) = TS \ominus I(NS)$ on S. This distribution is called the *maximal holomorphic subbundle* of TS, and it may or may not determine a contact structure on S. Assume for simplicity that the normal bundle of S is trivial, and let $\zeta \in \Gamma(NM)$ be a unit normal vector field. The vector field $\xi = -I\zeta \in \mathfrak{X}(S)$ is called the *Reeb* (or sometimes *Hopf) vector field* on S. We can define a 1-form, a skew-symmetric (1,1)-tensor field, and a closed 2-form on S by the formulas

$$\eta = \xi^\flat, \quad \phi = I - \eta\otimes\zeta, \quad \omega(X,Y) = \langle \phi(X) | Y\rangle.$$

One can check ϕ does indeed take values in TS. It is called the *structure tensor field* on S; and ω is called the *fundamental 2-form* on S. We say that S is a *contact hypersurface* if there exists a non-vanishing function $f \in C^\infty(M)$ such that $d\eta = 2f\omega$ (if $\dim_{\mathbb{C}}(M) > 2$, f will automatically be a constant function). Roughly speaking, this means $d\eta$ is proportional to ω at every point, except they both have zeroes because the dimension of S is odd. If S is a contact hypersurface, the maximal holomorphic subbundle of TS determines a contact structure on S (with η as a contact 1-form). As a consequence, no complex hypersurface of M can lie in S—otherwise, it would have to be an integral submanifold of the maximal holomorphic subbundle of TS. The principal orbits of every action in Table 6.2 are contact hypersurfaces (see [BS22, Th. 3.6.6, Sect. 8.3]). Moreover, it is conjectured that these are the only complete connected contact hypersurfaces in irreducible Hermitian symmetric spaces of compact type and $\dim_{\mathbb{C}} \geq 3$ (up to isometric congruence).

Corollary 6.2.16. *Let M be an irreducible Hermitian symmetric space of compact type and $S \subset M$ a homogeneous complex hypersurface. Suppose that S can be realized as a singular orbit of an isometric C1-action on M. Then there does not exist a complex hypersurface in M (complete ot not) equidistant to S.*

Before moving on, we use the example of $\mathrm{Sp}(n) \curvearrowright \mathrm{Gr}(2, \mathbb{C}^{2n})$ to show how one can find the singular orbits of a C1-action on an irreducible symmetric space of classical type. This becomes a fairly simply task once we find a suitable geometric representation of the space.

Example 6.2.17. Let $M = \mathrm{Gr}(2, \mathbb{C}^{2n})$, $n \geq 3$. Identify $\mathbb{C}^{2n}$ with $\mathbb{H}^n$ by picking $i \in \mathbb{H}$ to be a distinguished complex structure in $\mathbb{H}^n$ and taking a $\mathbb{C}$-basis for $\mathbb{H}^n$ to be $e_1, je_1, \ldots, e_n, je_n$. This gives an orthogonal representation of $H = \mathrm{Sp}(n)$ on $\mathbb{C}^{2n}$ and thus an isometric action on M. One obvious orbit of H in M is the set of quaternionic lines in $\mathbb{H}^n$, thought of as complex 2-planes. It is easy to see that this is indeed a single orbit, and it is isometric to $\mathrm{Sp}(n)/\mathrm{Sp}(n-1)\mathrm{Sp}(1) \simeq \mathbb{H}P^{n-1}$. Its (real) codimension equals $4n - 4 \geq 8$, so it is a singular orbit. Now, consider the set S of totally complex subspaces of $\mathbb{H}^n$ of real dimension 4 that are preserved by i. In other words, S is the subset of M consisting of 2-planes L such that $jL \perp L$ ($\Rightarrow kL \perp L$). Since H acts on $\mathbb{H}^n$ orthogonally and commutes with both i and j, the subset S is easily seen to be its orbit. Consider the point $L = \mathrm{span}_{\mathbb{C}}\{e_{n-1}, e_n\} \in S$. Its isotropy subgroup H_L preserves the subspace $\mathrm{span}_{\mathbb{H}}\{e_{n-1}, e_n\}$ and its orthogonal complement $\mathbb{H}L = \mathrm{span}_{\mathbb{H}}\{e_1, \ldots, e_{n-2}\}$, so $H_L \subseteq \mathrm{Sp}(n-2)\mathrm{Sp}(2)$. But H_L also preserves the complex subspace L of $\mathbb{H}L$, which implies that the bottom right 2×2 corner of any matrix in H_L has to consist of complex numbers. From this, one can easily see that $H_L = \mathrm{Sp}(n-2)\mathrm{U}(2)$ and thus $S = \mathrm{Sp}(n)/\mathrm{Sp}(n-2)\mathrm{U}(2)$. The (real) codimension of S is 2, so it is the other singular orbit. It is not hard to see that S is in fact a complex submanifold and thus a homogeneous complex hypersurface. //

6.2.5. The complexification of projective spaces

Let us look at those actions in Table 6.2 whose complex orbit is not totally geodesic. For convenience, we put them together in a separate table:

M	H	the complex singular orbit	the other singular orbit
$\mathbb{C}P^n$	$\mathrm{SO}(n+1)$	Q^{n-1}	$\mathbb{R}P^n$
$\mathrm{Gr}(2, \mathbb{C}^{2n+2})$	$\mathrm{Sp}(n+1)$	$\mathrm{Sp}(n+1)/\mathrm{Sp}(n-1)\mathrm{U}(2)$	$\mathbb{H}P^n$
$E_6/\mathrm{Spin}(10)\mathrm{U}(1)$	F_4	$F_4/\mathrm{Spin}(7)\mathrm{U}(1)$	$\mathbb{O}P^2$

Table 6.3. Cohomogeneity-one actions on irreducible Hermitian symmetric spaces of compact type with a non-totally-geodesic complex codimension-2 singular orbit

In pursue of symmetry, we raised n in the second row by 1, so now the restriction on it in that row is $n \geq 2$. There is an obvious pattern in this table: the non-complex singular orbit is always isometric to a projective space. What is more, it is a real form of M (in the sense of [Jaf75, Leu79b]), and M is what is known as the complexification of that projective space. In this subsection we explain how to construct the complexification of a projective space and discuss some of its properties. The exposition here is based on [AB03], where this construction was carried out for projective planes. Note that every compact homogeneous space G/K can be complexified as $G(\mathbb{C})/K(\mathbb{C})$ (see, for instance, [Kul78, §5]), but the main merit of the present construction for projective spaces is that it produces a *compact complexification*.

Let $M = \mathbb{F}P^n$ be a projective space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, where $n \geq 2$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $n = 2$ for $\mathbb{F} = \mathbb{O}$. (The case of $\mathbb{O}P^2$ has to be handled separately due to the nonassociativity of the octonions, see Remark 6.2.18 below. The discussion prior to that remark applies only to the cases $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, but everything after that works for $\mathbb{F} = \mathbb{O}$, too.) Being the projectivization of the vector space $\mathbb{F}^{n+1}$, the space M admits a natural transitive action of a noncompact semisimple Lie group $\widetilde{G}_n$ with a maximal compact subgroup G_n, where

$$\widetilde{G}_n = \begin{cases} \mathrm{SL}(n+1, \mathbb{R}), & \text{if } \mathbb{F} = \mathbb{R}, \\ \mathrm{SL}(n+1, \mathbb{C}), & \text{if } \mathbb{F} = \mathbb{C}, \\ \mathrm{SL}(n+1, \mathbb{H}), & \text{if } \mathbb{F} = \mathbb{H}, \\ E_6^{-26}, & \text{if } \mathbb{F} = \mathbb{O}, \end{cases} \qquad G_n = \begin{cases} \mathrm{SO}(n+1), & \text{if } \mathbb{F} = \mathbb{R}, \\ \mathrm{SU}(n+1), & \text{if } \mathbb{F} = \mathbb{C}, \\ \mathrm{Sp}(n+1), & \text{if } \mathbb{F} = \mathbb{H}, \\ F_4, & \text{if } \mathbb{F} = \mathbb{O}. \end{cases}$$

The subgroup G_n acts on M transitively and by isometries. As we know from Example 2.1.36, up to finite covering, G_n coincides with $I^0(M)$. The key step of the complexification construction—and, coincidentally, what makes the construction so ad-hoc and hard to generalize—is to embed M into a space of matrices. Consider the (real) vector space H_{n+1} of Hermitian matrices over $\mathbb{F}$:

$$H_{n+1} = \{A \in \mathrm{Mat}(n+1, \mathbb{F}) \mid A = A^*\},$$

where $A^* = \overline{A}^t$. The trace of every matrix in H_{n+1} is real, so we can consider the affine hyperplane $H_{n+1}(1) \subset H_{n+1}$ of matrices of trace 1. The space M admits a natural embedding into H_{n+1}:

$$i\colon M \hookrightarrow H_{n+1}(1),\ \ell \mapsto \mathrm{pr}_\ell,$$

where pr_ℓ is the $\mathbb{F}$-linear operator of orthogonal projection of $\mathbb{F}^{n+1}$ onto the $\mathbb{F}$-line ℓ. Let P_n stand for the (real) projective space over H_{n+1}. In other words, P_n is the projective completion of $H_{n+1}(1)$. Since $M \subset H_{n+1}(1)$, we have $M \subset P_n$. We will denote the "infinity" $P_n \backslash H_{n+1}(1)$ via M_∞. The group $\widetilde{G}_n$ acts on H_{n+1} by $X \cdot A = XAX^*$. Under this action, G_n preserves M, $H_{n+1}(1)$, and M_∞. The action of $\widetilde{G}_n$ on H_{n+1} preserves neither M nor $H_{n+1}(1)$, but the induced action $\widetilde{G}_n \curvearrowright P_n$ does preserve $M \subset P_n$.

Remark 6.2.18. The Cayley projective plane cannot be defined as the set of "lines" in $\mathbb{O}^3$ due to the fact that $\mathbb{O}$ is not associative. Instead, it is normally defined as already sitting inside $H_3(1) \subset \mathrm{Mat}(3, \mathbb{O})$ (see, e.g., [Bae02]). The groups E_6^{-26} and F_4 (and hence their action on $\mathbb{O}P^2$) can also be defined in terms of *Hermitian* 3×3 matrices over $\mathbb{O}$ (see [Yok09]).

The actual complexification starts on the level of Lie groups and projective spaces. Let $\widetilde{G}_n(\mathbb{C})$ stand for the complexification of $\widetilde{G}_n$:

$$\widetilde{G}_n(\mathbb{C}) = \begin{cases} \mathrm{SL}(n+1, \mathbb{C}), & \text{if } \mathbb{F} = \mathbb{R}, \\ \mathrm{SL}(n+1, \mathbb{C}) \times \mathrm{SL}(n+1, \mathbb{C}), & \text{if } \mathbb{F} = \mathbb{C}, \\ \mathrm{SL}(2n+2, \mathbb{C}), & \text{if } \mathbb{F} = \mathbb{H}, \\ E_6(\mathbb{C}), & \text{if } \mathbb{F} = \mathbb{O}. \end{cases}$$

We also write $P_n(\mathbb{C})$ for the complexification of P_n, i.e., the complex projective space $\mathbb{P}(H_{n+1} \otimes \mathbb{C})$. The action $\widetilde{G}_n \curvearrowright P_n$ naturally extends to $\widetilde{G}_n(\mathbb{C}) \curvearrowright P_n(\mathbb{C})$. Since M was an orbit of $\widetilde{G}_n$ in P_n, we define $M(\mathbb{C})$ to be the corresponding orbit of $\widetilde{G}_n(\mathbb{C})$ in $P_n(\mathbb{C})$ and call it the complexification of M:

$$M(\mathbb{C}) = \widetilde{G}_n(\mathbb{C}) \cdot M = \widetilde{G}_n(\mathbb{C}) \cdot p \subset P_n(\mathbb{C}) \text{ (for any } p \in M).$$

The following can be shown essentially by computing the isotropy subgroup of $\widetilde{G}_n(\mathbb{C})$ at a suitable point of $M(\mathbb{C})$ (and we refer to [AB03] for a more detailed discussion):

Proposition 6.2.19. *The complexification $M(\mathbb{C}) \subseteq P_n(\mathbb{C})$ is a nonsingular complex projective variety and a Hermitian symmetric space. It can be described as*

$$M(\mathbb{C}) \cong \begin{cases} \mathbb{C}P^n, & \text{if } M = \mathbb{R}P^n, \\ \mathbb{C}P^n \times \mathbb{C}P^n, & \text{if } M = \mathbb{C}P^n, \\ \mathrm{Gr}(2, \mathbb{C}^{2n+2}), & \text{if } M = \mathbb{H}P^n, \\ E_6/\mathrm{Spin}(10)\mathrm{U}(1), & \text{if } M = \mathbb{O}P^2. \end{cases}$$

In particular, $M(\mathbb{C})$ is irreducible unless $M = \mathbb{C}P^n$. The submanifold M is a real form of $M(\mathbb{C})$ (the fixed point set of an antiholomorphic involutive isometry).

If we restrict the action $\widetilde{G}_n(\mathbb{C}) \curvearrowright M(\mathbb{C})$ to G_n, we obtain an isometric C1-action. One of its singular orbits is M, while the other is a homogeneous complex hypersurface. So this complex hypersurface can be described as the focal manifold of M inside $M(\mathbb{C})$.

Note that the homogeneous complex hypersurface focal to $\mathbb{C}P^n$ in its complexification $\mathbb{C}P^n \times \mathbb{C}P^n$ was not included in Table 6.1. This is simply because in Corollary 6.2.8, we restricted to complex flag manifolds with $b_2 = 1$, which are irreducible. The fact that $\mathbb{C}P^n$ is already complex itself forces its complexification to be just a product of two copies of $\mathbb{C}P^n$ (notice that $\mathbb{C}P^n$ embeds into its complexification as the diagonal). As a complex homogeneous $\mathrm{SU}(n+1)$-space, the singular orbit of $\mathrm{SU}(n+1)$ in $\mathbb{C}P^n \times \mathbb{C}P^n$ other than $\mathbb{C}P^n$ can be described as the projectivized bundle $\mathbb{P}(T_{\mathbb{C}P^n}(-1))$.

6.3. The noncompact case

In this section, we initiate a study of complex homogeneous hypersurfaces in Hermitian symmetric spaces of noncompact type. Historically, these have scarcely received any attention whatsoever—primarily because the algebraic methods that prevail in the compact case are not available for noncompact spaces. The only notable exception is the classification result Theorem 6.1.1 of Smyth and Nomizu, which implies that the totally geodesic $\mathbb{C}H^{n-1}$ is the only homogeneous complex hypersurface in $\mathbb{C}H^n$ up to isometric congruence. Notice the difference with the compact case: apart from $\mathbb{C}P^{n-1}$, there is another, non-totally-geodesic homogeneous hypersurface in $\mathbb{C}P^n$, namely the quadric Q^{n-1}.

In Proposition 6.2.6, we observed that, up to (strong) isometric congruence, there are precisely two totally geodesic complex hypersurfaces in irreducible Hermitian symmetric spaces of noncompact type: the complex hyperbolic hyperplane $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$, and the complex hyperbolic *subquadric* $\mathrm{Gr}^*(2,\mathbb{R}^{n+1}) \subset \mathrm{Gr}^*(2,\mathbb{R}^{n+2})$, , $n \geq 3$. The indisputable advantage that Hermitian symmetric spaces of noncompact type have is the canonical extension construction (first introduced in Subsection 5.1.1), which allows to extend homogeneous objects from a boundary component of M to the whole M. In Hermitian symmetric spaces of noncompact type, it can be applied to complex boundary components to produce new homogeneous complex hypersurfaces in higher-rank spaces. In particular, one can use the two totally geodesic complex hypersurfaces to generate examples of homogeneous complex hypersurfaces on *every* Hermitian symmetric space of noncompact type. We will prove a non-existence result, which puts these examples forward as candidates for *the only* homogeneous complex hypersurfaces in irreducible Hermitian symmetric spaces of noncompact type. Our main tool, which will serve as a substitute for the algebraic methods used in compact case, is going to be the restricted root space decomposition.

6.3.1. Hermitian symmetric spaces of noncompact type

In this preparatory subsection, we establish a number of results concerning Hermitian symmetric spaces of noncompact type. We begin with writing down the complete list of irreducible Hermitian symmetric spaces. In the following table, M is a space of noncompact type, DD is its Dynkin diagram, and M^* is its compact dual.

M	M^*	restrictions	rk	$\dim_{\mathbb{C}}$	DD
$\mathrm{Gr}^*(r,\mathbb{C}^{2r+n})$	$\mathrm{Gr}(r,\mathbb{C}^{2r+n})$	$r,n \geq 1$	r	$(n+r)r$	$(BC)_r\colon (2,\dots,2,(2n,1))$
$\mathrm{Gr}^*(r,\mathbb{C}^{2r})$	$\mathrm{Gr}(r,\mathbb{C}^{2r})$	$r \neq 2$	r	r^2	$C_r\colon (2,\dots,2,1)$
$\mathrm{Gr}^*(2,\mathbb{R}^{n+2})$	$\mathrm{Gr}^+(2,\mathbb{R}^{n+2})$	$n \geq 3$	2	n	$C_2\colon (n-2,1)$
$\mathrm{SO}(n,\mathbb{H})/\mathrm{U}(n)$	$\mathrm{SO}(2n)/\mathrm{U}(n)$	$n \geq 5$	$\left\lfloor \frac{n}{2} \right\rfloor$	$\frac{n(n-1)}{2}$	$[n=2k]$ $C_k\colon (4,\dots,4,1)$ $[n=2k+1]$ $(BC)_k\colon (4,\dots,4,(4,1))$
$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$	$\mathrm{Sp}(n)/\mathrm{U}(n)$	$n \geq 3$	n	$\frac{n(n-1)}{2}$	$C_n\colon (1,\dots,1,1)$
$E_6^{-14}/\mathrm{Spin}(10)\mathrm{U}(1)$	$E_6/\mathrm{Spin}(10)\mathrm{U}(1)$		2	16	$(BC)_2\colon (6,(8,1))$
$E_7^{-25}/E_6\mathrm{U}(1)$	$E_7/E_6\mathrm{U}(1)$		3	27	$C_3\colon (8,8,1)$

Table 6.4. Irreducible Hermitian symmetric spaces

Let us discuss the spaces in this table in a little more detail. First of all, the complex hyperbolic and projective spaces are all included in the first row as those Grassmannians where $r = 1$. The compact space $\mathrm{Gr}^+(2,\mathbb{R}^{n+2})$ in the third row is isometric to the smooth complex projective quadric $Q^n \subset \mathbb{C}P^{n+1}$. For this reason, we sometimes refer to its dual $\mathrm{Gr}^*(2,\mathbb{R}^{n+2})$ as the *complex hyperbolic quadric*. Its root system C_2 can also be described as B_2. In fact, it is more commonly written as B_2 because this space fits in the larger series of real Grassmannians $\mathrm{Gr}^*(r,\mathbb{R}^{n+r})$ with root system B_r—which is no longer isomorphic to C_r. There is a good reason why the root system of the complex hyperbolic quadric belongs to the C_r-family (see the discussion following Theorem 6.3.12), so we are going to stick with writing C_2, not B_2. The restrictions on r and n in the table were imposed due to the following exceptional isomorphisms in low dimensions (stated for spaces of compact type only due to duality):

- $\mathrm{Gr}(2,\mathbb{C}^4) \simeq \mathrm{Gr}^+(2,\mathbb{R}^6) = Q^4$;
- $\mathrm{Gr}^+(2,\mathbb{R}^3) \simeq \mathbb{C}P^1$, $\mathrm{Gr}^+(2,\mathbb{R}^4) \simeq \mathbb{C}P^1 \times \mathbb{C}P^1$;

- $\mathrm{SO}(4)/\mathrm{U}(2) \simeq \mathbb{C}P^1$, $\mathrm{SO}(6)/\mathrm{U}(3) \simeq \mathbb{C}P^3$, $\mathrm{SO}(8)/\mathrm{U}(4) \simeq \mathrm{Gr}^+(2,\mathbb{R}^8) = Q^6$;
- $\mathrm{Sp}(1)/\mathrm{U}(1) \simeq \mathbb{C}P^1$, $\mathrm{Sp}(2)/\mathrm{U}(2) \simeq \mathrm{Gr}^+(2,\mathbb{R}^5) = Q^3$.

The other half of exceptional isomorphisms (for the noncompact type) can be obtained by duality. As is evident from Table 6.4, *the root system of an irreducible Hermitian symmetric space of noncompact type is always isomorphic to either C_r or $(BC)_r$*. We are going to uncover the geometric reason behind this fact. Before we do that though, we need two little results that establish a deeper relation between a root system and its Weyl group.

Proposition 6.3.1. *Let (V, Δ) be an irreducible root system and $\mathrm{W}(\Delta)$ its Weyl group.*

(a) *The tautological representation of $\mathrm{W}(\Delta)$ on V is irreducible.*

(b) *$\mathrm{W}(\Delta)$ acts transitively on the set of roots of the same length*[1]*. Consequently, the orbit space $\Delta/\mathrm{W}(\Delta)$ is naturally bijective to the set of root lengths in Δ.*

In the special case of reduced root systems, these statements can be found in [Hum72] (see Lemmas B and C in Section 10.4); the general case follows easily from the reducible one. We still include a proof in the general case for the sake of greater completeness.

Proof. For part (a), suppose there is a nontrivial subrepresentation $U \subsetneq V$. By irreducibility of Δ, there has to be a root α lying in neither U nor $U^\perp$. But then the reflection s_α cannot preserve U, which leads to a contradiction.

Let us now prove part (b). For rank-2 systems, this can be shown by hand just by looking at their pictures. In general, we invoke Lemma 3.1.2: for any $\alpha, \beta \in \Delta$, there exists a chain of roots $\lambda_0, \lambda_1, \ldots, \lambda_s \in \Delta$ with $\lambda_0 = \alpha, \lambda_s = \beta$, such that $\langle \lambda_{i-1} | \lambda_i \rangle \neq 0$ for $1 \leq i \leq s$. Let us call such a sequence of roots *a chain from α to β*; we also say that s is its *length*. Take any two non-proportional roots $\alpha, \beta \in \Delta$ of the same length and suppose they lie in different orbits of the Weyl group. Let $\lambda_0, \lambda_1, \ldots, \lambda_s$ be the shortest chain from α to β (it does not have to be unique). Acting by $\mathrm{W}(\Delta)$ if necessary, we may assume α is the closest to β among members of its orbit—that is, for any $\gamma \in \mathrm{W}(\Delta) \cdot \alpha$, every chain from γ to β has length $\geq s$.

We claim that $s = 1$ or, in other words, $\alpha \not\perp \beta$. Assume for a moment that this is the case and write $U = \mathrm{span}\{\alpha, \beta\}$. Then $(U, U \cap \Delta)$ is a root system of rank 2. Moreover, by our assumption, $U \cap \Delta$ is irreducible. Consequently, there exists an element $\mathrm{W}(U \cap \Delta)$ mapping α to β. But every element of $\mathrm{W}(U \cap \Delta)$ is trivially a restriction of an element of $\mathrm{W}(\Delta)$ preserving U. We are left to prove that $s = 1$. For the sake of contradiction, assume $s > 1$. Since we picked the shortest chain, $\alpha = \lambda_0$ is orthogonal to every λ_j, $j > 1$. Denote $W = \mathrm{span}\{\alpha, \lambda_1\}$ and consider the rank-2 root system $(W, W \cap \Delta)$. By the same argument as above, it is irreducible, which implies that there exists $\alpha' \in W \cap \Delta$ of the same length as α but not proportional to it. On the one hand, we already know that α' lies in the same orbit as α. On the other hand, it can be written as $a\alpha + b\lambda_1$ with $b \neq 0$. Now, $\langle \alpha' | \lambda_2 \rangle = \langle \lambda_1 | \lambda_2 \rangle \neq 0$. Therefore, $\alpha', \lambda_2, \ldots, \lambda_s$ is a chain of length $s - 1$ from $\alpha' \in \mathrm{W}(\Delta) \cdot \alpha$ to β, which is a contradiction. □

[1] Observe that the full automorphism group $\mathrm{Aut}(\Delta)$ has the same orbits in Δ as its subgroup $\mathrm{W}(\Delta)$, as follows from Corollary 3.1.4.

Among other things, part (b) of the above proposition implies the following fact: if Δ is a root system with a fixed choice of simple roots, then every possible root length in Σ is realized by some simple root. If Σ is the restricted root system of a simple Lie algebra (necessarily irreducible by Corollary 3.2.12), we know from Section 3.2 that $\mathrm{W}(\Sigma) \subseteq \mathrm{Aut}^{\mathrm{w}}(\Sigma)$. With this in mind, Proposition 6.3.1 yields an immediate

Corollary 6.3.2. *Suppose $\mathfrak{g}$ is a real simple Lie algebra and Σ is its restricted root system. If two roots $\alpha, \beta \in \Sigma$ have the same length, then they have the same multiplicity as well.*

We are now prepared to give an intrinsic characterization to C_r and $(BC)_r$ among all irreducible root systems. Let (V, Δ) be such a system. Observe that the number of distinct root lengths in Δ is

- 1 if $\Delta \simeq A_r, D_r, E_6, E_7, E_8$;
- 2 if $\Delta \simeq B_r, C_r, F_4, G_2, (BC)_1$;
- 3 if $\Delta \simeq (BC)_r, r \geq 2$.

Definition 6.3.3. Let (V, Δ) be an irreducible root system with roots of nonequal lengths. We call a root $\alpha \in \Delta$

- **long** if it of the maximum length;
- **medium** if it is not long and 2α is not a root;
- **short** if 2α is a root.

The subsets of long, medium, and short roots are denoted by $\boldsymbol{\Delta_L}, \boldsymbol{\Delta_M}$, and $\boldsymbol{\Delta_S}$, respectively.

Trivially, a root system containing roots of nonequal lengths is reduced if and only if there are no short roots. In view of Proposition 6.3.1(b), the long roots are all congruent to each other via the Weyl group; the same holds for the medium and short roots.

Warning. In the context of reduced root systems—for example, in the theory of complex semisimple Lie algebras—if there are two roots of different lengths, it is customary to call them simply *short* and *long*. This clearly conflicts with our Definition 6.3.3: according to it, we would use the terms *medium* and *long* instead and reserve the word *short* for $(BC)_r$. This nuance should be kept in mind throughout the chapter.

Example 6.3.4 (Root lengths in $\boldsymbol{C_r}$ and $\boldsymbol{(BC)_r}$). Let $e_1, \ldots, e_r$ be the standard basis for $\mathbb{R}^r$. Recall that the root systems C_r and $(BC)_r$ are defined as:

$$\begin{aligned} C_r &= \{\pm e_i \pm e_j \mid 1 \leq i < j \leq r\} \cup \{\pm 2e_i \mid 1 \leq i \leq r\}\,, \\ (BC)_r &= \{\pm e_i \pm e_j \mid 1 \leq i < j \leq r\} \cup \{\pm e_i \mid 1 \leq i \leq r\} \cup \{\pm 2e_i \mid 1 \leq i \leq r\}\,. \end{aligned}$$

Here the medium roots are $\pm e_i \pm e_j$, the long roots are $\pm 2e_i$, and the short ones are $\pm e_i$. As a result, in these root systems one has:

$$\frac{||\text{long root}||}{||\text{medium root}||} = \frac{||\text{medium root}||}{||\text{short root}||} = \sqrt{2}.$$

//

If (V, Δ) is an irreducible root system with roots of more than one length, the long roots span V thanks to Proposition 6.3.1(a). It might happen, however, that there are "too many" long roots. For example, in $B_r\,(r \geq 3), F_4$, and G_2, the long roots are linearly dependent, even after we restrict to positive roots only. Looking at Example 6.3.4, we arrive at the following

Proposition 6.3.5 (Characterization of C_r and $(BC)_r$). *Let (V, Δ) be an irreducible root system containing roots of nonequal lengths. Then (V, Δ) is isomorphic to C_r or $(BC)_r$ if and only if its positive long roots—with respect to some ($\Leftrightarrow$ any) choice of a Weyl chamber—form a basis for V.*

Since our primary object of interest in this chapter is Hermitian symmetric spaces, we are going to confine our attention to the root systems C_r and $(BC)_r$ from now on. The first basic property of these two systems is how their roots of the same length are positioned with respect to each other.

Observation 6.3.6. Let (V, Δ) be a root system isomorphic to either C_r or $(BC)_r$. If $\alpha, \beta \in \Delta$ are two non-proportional roots of the same length, then they are orthogonal. Moreover, if α and β are long, then their sum is not a root. As a result, the positive long roots (with respect to any choice of a Weyl chamber) form an orthogonal basis of V. These claims follow from the explicit description of C_r and $(BC)_r$ in Example 6.3.4, the fact that any isomorphism between irreducible root systems is a homothety (see Proposition 3.1.3), and Proposition 6.3.5. Because of the last assertion, it is often more convenient to work with positive long roots, and not simple roots, as a set basis for V.

For future references, we recall the standard choice of positive and simple roots for these two systems:

$$
\begin{aligned}
C_r\colon \quad & \Delta^+ = \{e_i \pm e_j \mid 1 \le i < j \le r\} \cup \{2e_i \mid 1 \le i \le r\}, \\
& \Lambda = \{e_1 - e_2, \ldots, e_{r-1} - e_r, 2e_r\}, \\
(BC)_r\colon \quad & \Delta^+ = \{e_i \pm e_j \mid 1 \le i < j \le r\} \cup \{e_i \mid 1 \le i \le r\} \cup \{2e_i \mid 1 \le i \le r\}, \\
& \Lambda = \{e_1 - e_2, \ldots, e_{r-1} - e_r, e_r\}.
\end{aligned} \tag{6.3.1}
$$

Another property of these root systems that is going to be of great importance to us is that their medium roots come in pairs.

Definition 6.3.7. Let (V, Δ) be a root system isomorphic to either C_r or $(BC)_r$, and let $\Delta^+ \subseteq \Delta$ be any choice of positive roots. We are going to say that two positive medium roots $\alpha, \beta \in \Delta^+$ are **neighbors** if their sum is a long root.

A quick look at (6.3.1) reveals that every positive medium root has a unique neighbor: in either C_r or $(BC)_r$, the neighbor of $e_i \pm e_j$ is $e_i \mp e_j$. In $(BC)_r$, one could extend the notion of neighbor verbatim to positive short roots; plainly, e_i is its own unique neighbor. We will use the notation $\boldsymbol{\Delta^+_{\mathrm{L}}} = \Delta_{\mathrm{L}} \cap \Delta^+$; the same goes for $\boldsymbol{\Delta^+_{\mathrm{M}}}$ and $\boldsymbol{\Delta^+_{\mathrm{S}}}$. Suppose $\alpha, \beta \in \Delta^+_{\mathrm{M}}$ are two neighbor roots. Just like their sum, their difference is also a long root. The easiest way to see this is via an explicit description: if $\alpha = e_i \pm e_j$ and $\beta = e_i \mp e_j$, then $\alpha - \beta = \pm 2e_j \in \Delta_{\mathrm{L}}$. In particular, one of their two possible differences is a positive long root, while the other one is negative. This means that in any pair of positive medium neighbor roots, one of them can be written as the sum of the other one with a positive long root. As a result, such a root has a strictly greater height than its neighbor. It is only natural to introduce the following convenient

Definition 6.3.8. Let (V, Δ) be a root system isomorphic to either C_r or $(BC)_r$, and let $\Delta^+ \subseteq \Delta$ be any choice of positive roots. Given two neighbors $\alpha, \beta \in \Delta^+_{\mathrm{M}}$, the one of greater height is called an **upstairs neighbor**, while the other one is called a **downstairs neighbor**.

Remark 6.3.9. Up to a sign, the notion of a neighbor medium root does not depend on the choice of a Weyl chamber. But which of the neighbors is upstairs and which is downstairs does of course depend on that choice.

Now we are finally ready to proceed to Hermitian symmetric spaces. First of all, we establish a couple of restricted root space identities that will prove very useful in the sequel. Let $\mathfrak{g}$ be a real semisimple Lie algebra with a fixed restricted root space decomposition. For any $\alpha, \beta \in \Sigma$, the following holds:

$$\begin{gathered}[\mathfrak{k}_\alpha, \mathfrak{k}_\beta] \subseteq \mathfrak{k}_{\alpha+\beta} \oplus \mathfrak{k}_{\alpha-\beta}, \quad [\mathfrak{k}_\alpha, \mathfrak{p}_\beta] \subseteq \mathfrak{p}_{\alpha+\beta} \oplus \mathfrak{p}_{\alpha-\beta}, \quad [\mathfrak{p}_\alpha, \mathfrak{p}_\beta] \subseteq \mathfrak{k}_{\alpha+\beta} \oplus \mathfrak{k}_{\alpha-\beta}, \\ [\mathfrak{k}_0, \mathfrak{k}_\alpha] \subseteq \mathfrak{k}_\alpha, \quad [\mathfrak{k}_0, \mathfrak{p}_\alpha] \subseteq \mathfrak{p}_\alpha, \quad [\mathfrak{a}, \mathfrak{k}_\alpha] \subseteq \mathfrak{p}_\alpha, \quad [\mathfrak{a}, \mathfrak{p}_\alpha] \subseteq \mathfrak{k}_\alpha.\end{gathered} \tag{6.3.2}$$

Verifying these identities is fairly straightforward: for instance, take any $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta$ and compute:

$$\begin{aligned}[X + \theta X, Y + \theta Y] &= [X, Y] + [X, \theta Y] + [\theta X, Y] + [\theta X, \theta Y] \\ &= ([X, Y] + \theta[X, Y]) + ([X, \theta Y] + \theta[X, \theta Y]),\end{aligned} \tag{6.3.3}$$

which lies in $\mathfrak{k}_{\alpha+\beta} \oplus \mathfrak{k}_{\alpha-\beta}$ by Proposition 2.4.9. The rest is similar. Next, we have the following vital

Proposition 6.3.10. *Suppose M is an irreducible Hermitian symmetric space of noncompact type and rank r represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$, and let Σ be the restricted root system of $\mathfrak{g}$.*

(a) *All long roots in Σ have multiplicity 1.*

(b) *If $\alpha, \beta \in \Sigma_{\mathrm{L}}$ are such that $\alpha \neq -\beta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$.*

(c) *If $\Sigma^+ \subseteq \Sigma$ is a choice of positive roots, then the sum $\bigoplus_{\alpha \in \Sigma^+_{\mathrm{L}}} \mathfrak{g}_\alpha$, as well as its $\mathfrak{p}$-projection $\bigoplus_{\alpha \in \Sigma^+_{\mathrm{L}}} \mathfrak{p}_\alpha$, is an r-dimensional abelian subspace of $\mathfrak{g}$.*

Proof. There are various ways to prove these assertions, but most of them rely on the explicit description of C_r and $(BC)_r$ to some degree. Fix some choice of positive roots $\Sigma^+ \subseteq \Sigma$. For (a), note that the unique simple long root has multiplicity 1 according to Table 6.4. By Proposition 6.3.1, all long roots Σ are of multiplicity 1. Alternatively, one can use the fact that every root of Σ is simple for some choice of a Weyl chamber, and all the Weyl chambers are congruent by $\mathrm{W}(\Sigma) \subseteq \mathrm{Aut}^{\mathrm{w}}(\Sigma)$. To show part (b), just recall that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ (Proposition 2.4.9(d)) and $\alpha + \beta \notin \Sigma_0$ (Observation 6.3.6). Eventually, there are precisely r positive long roots due to Proposition 6.3.5, so part (c) follows from (b) and (6.3.2). Note also that part (a) can be proven via (c) without reliance on the knowledge of simple root multiplicities: the sum $\bigoplus_{\alpha \in \Sigma^+_{\mathrm{L}}} \mathfrak{p}_\alpha$ is an abelian subspace of $\mathfrak{p}$ of dimension $\geq r$, so it has to be a maximal abelian subspace and each of the r summands must be 1-dimensional. □

Corollary 6.3.11. *In the notation of Proposition 6.3.10, the subspace $\bigoplus_{\alpha \in \Sigma^+_{\mathrm{L}}} \mathfrak{p}_\alpha$ is tangent to a maximal flat in M. That flat can be realized as an orbit of the connected abelian Lie subgroup of $AN \subset G$ corresponding to the abelian subalgebra $\bigoplus_{\alpha \in \Sigma^+_{\mathrm{L}}} \mathfrak{g}_\alpha$.*

The main result of this subsection is a precise relation between the complex structure of a Hermitian symmetric space M of noncompact type and the restricted root space decomposition of its isometry Lie algebra. It turns out that root length plays a key role in this relation. In Section 5.4, we saw that the complex structure of $\mathrm{Gr}^*(2, \mathbb{C}^{n+4})$ interchanges the subspaces $\mathfrak{p}_{\alpha_1}$ and $\mathfrak{p}_{\alpha_1+2\alpha_2}$ (which correspond to the two positive medium roots in $(BC)_2$) but preserves $\mathfrak{p}_{\alpha_2}$ and $\mathfrak{p}_{\alpha_1+\alpha_2}$ (corresponding to the two positive short roots). Remarkably, the same pattern holds for all irreducible Hermitian symmetric spaces of noncompact type.

Theorem 6.3.12. *Let M be an irreducible Hermitian symmetric space of noncompact type represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. Pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and a Weyl chamber D for Σ; and write I for the almost complex structure of M at o.*

(a) *For every $\alpha \in \Sigma_{\mathrm{L}}^+$, $I(\mathbb{R}H_\alpha) = \mathfrak{p}_\alpha$ (and thus $I(\mathfrak{p}_\alpha) = \mathbb{R}H_\alpha$).*

(b) *For every $\alpha \in \Sigma_{\mathrm{M}}^+$, $I(\mathfrak{p}_\alpha) = \mathfrak{p}_\beta$, where $\beta \in \Sigma_{\mathrm{M}}^+$ is the neighbor of α.*

(c) *For every $\alpha \in \Sigma_{\mathrm{S}}^+$ (if any), $I(\mathfrak{p}_\alpha) = \mathfrak{p}_\alpha$. In other words, $\mathfrak{p}_\alpha$ is a complex subspace of $\mathfrak{p}$.*

Before embarking on the proof of this theorem, let us go through some immediate corollaries. First of all, part (a) implies that $I(\mathfrak{a}) = \bigoplus_{\alpha \in \Sigma_{\mathrm{L}}^+} \mathfrak{p}_\alpha$, which is another evidence that this sum is maximal abelian in $\mathfrak{p}$. In some sense, this observation justifies why the root system of an irreducible Hermitian symmetric space of noncompact type can only be C_r or $(BC)_r$: if there were more long roots, like in F_4, G_2, or $B_r (r \geq 3)$, $I(\mathfrak{a})$ would be a proper subspace of $\bigoplus_{\alpha \in \Sigma_{\mathrm{L}}^+} \mathfrak{p}_\alpha$. This explains why the hyperbolic quadric $\mathrm{Gr}^*(2, \mathbb{R}^{n+2})$, whose root system is $B_2 \simeq C_2$, is Hermitian, but the higher-rank noncompact Grassmann manifolds $\mathrm{Gr}^*(r, \mathbb{R}^{n+r})$, $r \geq 3$, with $\Sigma \simeq B_r \not\simeq C_r$, are not.

As we noticed in Subsection 2.4.2, given a semisimple Lie algebra $\mathfrak{g}$ and its restricted root system Σ, the decomposition $\mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{p}_\alpha$ is orthogonal with respect to $\langle - | - \rangle_B$ and thus—thanks to Corollary 2.1.110—with respect to any $\mathfrak{k}$-invariant inner product. Together with Theorem 6.3.12, Proposition 2.2.22, and Lemma 2.2.21, this implies:

Corollary 6.3.13. *Let M be an irreducible Hermitian symmetric space of noncompact type represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. Pick $\mathfrak{a} \subset \mathfrak{p}$ and $\Sigma^+ \subseteq \Sigma$.*

(a) *For each $\alpha \in \Sigma_{\mathrm{M}}^+ \cup \Sigma_{\mathrm{L}}^+$, $\mathfrak{p}_\alpha$ is a totally real subspace of $\mathfrak{p}$; the same is true for $\mathfrak{a}$.*

(b) *Any abelian subspace of $\mathfrak{p}$ is totally real.*

(c) *Any flat in M is a totally real submanifold.*

Proof of Theorem 6.3.12. We may assume $(\mathfrak{g}, \theta)$ is effective. Let (G, K) be an almost effective Riemannian symmetric pair representing M and associated with $(\mathfrak{g}, \theta)$. Pick a basis $e_1, \ldots, e_r$ for $\mathfrak{a}^*$ so that Σ and Σ^+ become as in Example 6.3.4 and (6.3.1). As per our discussion in Subsection 2.5.1, the center $\mathfrak{z}(\mathfrak{k})$ is 1-dimensional, and there exists $Z \in \mathfrak{z}(\mathfrak{k})$ such that $\mathrm{ad}(Z)\big|_{\mathfrak{p}} = I$. Moreover, the element $\exp(\frac{\pi}{2} Z) \in Z(K)^0 \simeq \mathbb{T}$ also acts on $\mathfrak{p}$ as I. In fact, the action $Z(K)^0 \curvearrowright \mathfrak{p}$ is simply the multiplication by unitary complex numbers. This action commutes with the representation of K on $\mathfrak{p}$.

Proof of part (a). As we already know that the long roots in Σ have multiplicity 1, it suffices to show that $I(H_{2e_i}) \in \mathfrak{p}_{2e_i}$ for every $1 \leq i \leq r$. In most cases, we can use a shrewd trick to impose some immediate restrictions on what I can do to the root spaces. We separate the proof into two cases accordingly.

Case 1: $\mathfrak{g}$ is not split. Recall from Proposition 2.4.16 that this is equivalent to saying that not all root multiplicities are 1. Looking at Table 6.4, we see that this case includes all spaces except for the series $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$, $n \geq 2$ (and also the hyperbolic quadric $\mathrm{Gr}^*(2,\mathbb{R}^5)$, but is isometric to $\mathrm{Sp}(4,\mathbb{R})/\mathrm{Sp}(2)$). We also see that the long roots are the only roots with multiplicity 1 in this case. Recall that the adjoint action of the compact group $K_0 = Z_K^0(\mathfrak{a})$ on $\mathfrak{g}$ is orthogonal and preserves each restricted root space and thus every $\mathfrak{p}_\alpha$. By design, it fixes each element of $\mathfrak{a}$, but it also fixes each vector in $\mathfrak{p}_{2e_i}$, $i = 1,\dots,r$, because such $\mathfrak{p}_\alpha$ is 1-dimensional. So the sum $\mathfrak{a} \oplus \bigoplus_{i=1}^r \mathfrak{p}_{2e_i}$ is contained in the subspace of invariants $\mathfrak{p}^{K_0}$. On the other hand, we know from Lemma 4.2.5 that K_0 acts with cohomogeneity 1 on each $\mathfrak{p}_\alpha$. As a consequence, its action on $\mathfrak{p}_\alpha$ is nontrivial provided α is not long. Therefore, the subspace of K_0-invariants is precisely $\mathfrak{a} \oplus \bigoplus_{i=1}^r \mathfrak{p}_{2e_i}$. But this implies that the action $\mathbb{T} \curvearrowright \mathfrak{p}$ (and, in particular, I) preserves this subspace: this action is by endomorphisms of $\mathfrak{p}$ as a K-representation and hence it must preserve $\mathfrak{p}^{K_0} = \mathfrak{a} \oplus \bigoplus_{i=1}^r \mathfrak{p}_{2e_i}$.

As a next step, we want to show that each $\lambda \in \mathbb{T}$ permutes the subspaces $\mathbb{R}H_{2e_i} \oplus \mathfrak{p}_{2e_i}$ of $\mathfrak{p}^{K_0}$. Given $X \in \mathfrak{p}^{K_0}$, let us write $Z_{\mathfrak{p}^{K_0}}(X) = \{Y \in \mathfrak{p}^{K_0} \mid [Y,X] = 0\}$. For example,

$$Z_{\mathfrak{p}^{K_0}}(H_{2e_i}) = \mathfrak{a} \oplus \bigoplus_{j \neq i} \mathfrak{p}_{2e_j}, \quad Z_{\mathfrak{p}^{K_0}}(\mathfrak{p}_{2e_i}) = (\mathfrak{a} \ominus \mathbb{R}H_{2e_i}) \oplus \bigoplus_{i=1}^r \mathfrak{p}_{2e_j},$$

because the long roots $2e_1,\dots,2e_r$ are pairwise orthogonal. More generally, given $X \in \mathfrak{p}^{K_0}$, decompose it as $X = \sum_{i=1}^r X_i$, where $X_i \in \mathbb{R}H_{2e_i} \oplus \mathfrak{p}_{2e_i}$. Then

$$Z_{\mathfrak{p}^{K_0}}(X) = \bigoplus_{\substack{1\leq i\leq r \\ X_i \neq 0}} \mathbb{R}X_i \oplus \bigoplus_{\substack{1\leq i\leq r \\ X_i = 0}} (\mathbb{R}H_{2e_i} \oplus \mathfrak{p}_{2e_i}).$$

Note that the codimension of $Z_{\mathfrak{p}^{K_0}}(X)$ in $\mathfrak{p}^{K_0}$ equals the number of nonzero X_i's. In particular, this codimension is 1 if and only if X lies in $\mathbb{R}H_{2e_i} \oplus \mathfrak{p}_{2e_i}$ for some i. Now, the representation of $Z(K)^0 \simeq \mathbb{T}$ on $\mathfrak{g}$ is by Lie algebra automorphisms. Hence, for any $\lambda \in \mathbb{T}$ and any $X \in \mathfrak{p}^{K_0}$, $\lambda Z_{\mathfrak{p}^{K_0}}(X) = Z_{\mathfrak{p}^{K_0}}(\lambda X)$. We deduce that for every $i = 1,\dots,r$, $\lambda H_{2e_i} \in \mathbb{R}H_{2e_j} \oplus \mathfrak{p}_{2e_j}$ for some $j \in \{1,\dots,r\}$, and thus λ must send

$$Z_{\mathfrak{p}^{K_0}}(H_{2e_i}) = \mathfrak{p}^{K_0} \ominus \mathfrak{p}_{2e_i} \quad \text{onto} \quad Z_{\mathfrak{p}^{K_0}}(\lambda H_{2e_i}) = \mathfrak{p}^{K_0} \ominus \ell,$$

where ℓ is the line in $\mathbb{R}H_{2e_j} \oplus \mathfrak{p}_{2e_j}$ orthogonal to λH_{2e_i}. Since the multiplication by λ is an orthogonal transformation, it must send $\mathfrak{p}_{2e_i}$ onto ℓ. We conclude that λ sends $\mathbb{R}H_{2e_i} \oplus \mathfrak{p}_{2e_i}$ onto $\mathbb{R}H_{2e_j} \oplus \mathfrak{p}_{2e_j}$.

We now know that each $\lambda \in \mathbb{T}$ permutes the subspaces $\mathbb{R}H_{2e_i} \oplus \mathfrak{p}_{2e_i}$ of $\mathfrak{p}^{K_0}$. By continuity, the whole circle group $\mathbb{T}$ must preserve each $\mathbb{R}H_{2e_i} \oplus \mathfrak{p}_{2e_i}$. Now simply notice that I is a skew-symmetric operator, so $\langle H_{2e_i} \,|\, IH_{2e_i} \rangle = 0$, which means $IH_{2e_i} \in \mathfrak{p}_{2e_i}$.

Case 2: $\mathfrak{g}$ is split. In this case, according to Proposition 2.4.16, $\mathfrak{k}_0 = \{0\}$. Then $\mathfrak{k} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{k}_\alpha$, so $Z \in \mathfrak{z}(\mathfrak{k})$ can be decomposed accordingly as $Z = \sum_{\alpha\in\Sigma^+} Z_\alpha$. We claim that $Z_\alpha = 0$ unless α is a long root. Indeed, consider any root $\alpha \in \Sigma^+$ that is not long.

Then it has to be medium because the only split space in Table 6.4 is $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$, whose root system is C_n. Let β the neighbor of α. The difference $\gamma = \beta - \alpha$ is a long root. The bracket between any two nonzero vectors in $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\gamma$, respectively, is a nonzero vector in $\mathfrak{g}_\beta$ because the pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_\gamma \twoheadrightarrow \mathfrak{g}_\beta$ is nondegenerate (Lemma 4.2.6). It then easily follows from (6.3.2) and a computation similar to (6.3.3) that the bracket between $\mathfrak{k}_\alpha$ and $\mathfrak{k}_\gamma$ is a nondegenerate pairing $\mathfrak{k}_\alpha \times \mathfrak{k}_\gamma \twoheadrightarrow \mathfrak{k}_\beta$. Take any nonzero $Y \in \mathfrak{k}_\gamma$ and consider $[Z, Y] \in \mathfrak{k} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{k}_\lambda$. The $\mathfrak{k}_\beta$-component of $[Z, Y]$ is easily seen to be $[Z_\alpha, Y]$. But $Z \in \mathfrak{z}(\mathfrak{k})$, so $[Z, Y] = 0$. Therefore, Z_α has to be zero. As α was an arbitrary medium root, we deduce that $Z \in \bigoplus_{i=1}^r \mathfrak{k}_{2e_i}$. But now simply take any $1 \le i \le r$ and compute:

$$I(H_{2e_i}) = [Z, H_{2e_i}] = \sum_{i=1}^r [Z_{2e_j}, H_{2e_i}] = -\sum_{i=1}^r \langle 2e_j, H_{2e_i}\rangle Z_{2e_j} = -4||e_i||^2 Z_{2e_i} \in \mathfrak{p}_{2e_i},$$

where the last equality follows from the fact that the positive long roots are pairwise orthogonal. This completes the proof of part (a). This is actually the bulk of the proof; and the other two parts will rely on (a).

Proof of part (b). We know by now that I preserves $\mathfrak{a} \oplus \bigoplus_{i=1}^r \mathfrak{p}_{2e_i}$. By orthogonality, it must also preserve $\bigoplus_{\alpha \in \Sigma_{\mathrm{M}}^+ \cup \Sigma_{\mathrm{S}}^+} \mathfrak{p}_\alpha$. Let us show that $I(\mathfrak{p}_{e_i - e_j}) \subseteq \mathfrak{p}_{e_i + e_j}$ for any $1 \le i < j \le r$. Since $e_i - e_j$ and $e_i + e_j$ have the same multiplicity (Corollary 6.3.2), this will do the trick. Take any $X \in \mathfrak{p}_{e_i - e_j}$. Recall that I coincides with $\mathrm{Ad}(k)\big|_{\mathfrak{p}}$, where $k = \exp(\frac{\pi}{2} Z)$. Note that $\mathrm{Ad}(k)$ is the identity on $\mathfrak{k}$. We can then compute:

$$[I(H_{2e_j}), I(X)] = \big[\mathrm{Ad}(k)(H_{2e_j}), \mathrm{Ad}(k)(X)\big] = \mathrm{Ad}(k)[H_{2e_j}, X] = [H_{2e_j}, X] \in \mathfrak{k}_{e_i - e_j},$$

On the other hand, we know that $I(H_{2e_j})$ lies in $\mathfrak{p}_{2e_j}$. If we decompose $I(X)$ as $\sum_{k<l} (I(X)_{e_k + e_l} + I(X)_{e_k - e_l}) + \sum_{k=1}^r I(X)_{e_k}$, where $I(X)_{e_k + e_l} \in \mathfrak{p}_{e_k + e_l}$, etc. (the last sum is empty if the root system is reduced), then $[I(H_{2e_j}), I(X)]$ can be written as

$$\sum_{k=1}^r \left[I(H_{2e_j}), I(X)_{e_k}\right] + \sum_{k<l} \left[I(H_{2e_j}), I(X)_{e_k + e_l}\right] + \sum_{k<l} \left[I(H_{2e_j}), I(X)_{e_k - e_l}\right].$$

Owing to (6.3.2), the only summand here that can have a nonzero component in $\mathfrak{k}_{e_i - e_j}$ is $[I(H_{2e_j}), I(X)_{e_i + e_j}]$, which means that all the other summands are zero. But since $\mathfrak{p}_{2e_j}$ is 1-dimensional, we can apply the same argument as in case 2 of part (a) to deduce that $\mathrm{ad}(I(H_{2e_j}))\colon \mathfrak{p}_\beta \xrightarrow{\sim} \mathfrak{k}_{\beta + 2e_j}$ is an isomorphism whenever β and $\beta + 2e_j$ are both roots. This implies that every component of $I(X)$ other than $I(X)_{e_i + e_j}$ is zero. In other words, $I(X) \in \mathfrak{p}_{e_i + e_j}$.

Proof of part (c). This last bit is very similar to (b). From the previous two parts, we know that I has to preserve $\bigoplus_{i=1}^r \mathfrak{p}_{e_i}$. Given $X \in \mathfrak{p}_{e_i}$, $[I(H_{2e_i}), I(X)] = [H_{2e_i}, X] \in \mathfrak{k}_{e_i}$. On the other hand, this bracket can be written as $\sum_{i=1}^r [I(H_{2e_i}), I(X)_{e_j}]$. By (6.3.2), all the components $I(X)_{e_j}$ other than $I(X)_{e_i}$ must vanish, hence $I(X) \in \mathfrak{p}_{e_i}$. This concludes the proof of Theorem 6.3.12. □

6.3.2. Classification of standard C2-foliations with a complex base leaf

In this part, armed with Theorem 6.3.12, we prove the main result of the section and make a considerable step toward a classification of homogeneous complex hypersurfaces in irreducible Hermitian symmetric spaces of noncompact type. Namely, we classify those hypersurfaces that arise as a base leaf of a standard C2-foliation.

Let $M = G/K$ be an irreducible Hermitian symmetric space of noncompact type, and let $(\mathfrak{g}, \theta)$ be the corresponding orthogonal symmetric Lie algebra. Since M is irreducible, its Riemannian metric has to be proportional to the Killing metric. To simplify the notation, we will assume that the metric is Killing, i.e., $g_o = B_\theta\big|_{\mathfrak{p}\times\mathfrak{p}} = B\big|_{\mathfrak{p}\times\mathfrak{p}}$. All the results of this subsection remain valid if the normalizing constant of M is arbitrary. As usual, we pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and a Weyl chamber D for Σ. Recall that we have a linear isomorphism $\Pi\colon \mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n} \xrightarrow{\sim} \mathfrak{p}$ given by the projection to $\mathfrak{p}$ along $\mathfrak{k}$. It is important to point out that Π is not an isometry with respect to the inner product B_θ, but it is very close to being one: it respects the orthogonal decompositions $\mathfrak{s} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and $\mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{p}_\alpha$ (being the identity on $\mathfrak{a}$ and sending $\mathfrak{g}_\alpha$ onto $\mathfrak{p}_\alpha$); it is an isometry on $\mathfrak{a}$; and it shrinks the distances by a factor of $\sqrt{2}$ on $\bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$. We pull $B_\theta\big|_{\mathfrak{p}\times\mathfrak{p}}$ back along Π and denote the resulting inner product on $\mathfrak{s}$ by $\langle - | - \rangle_\mathfrak{s}$. For reasons that will become obvious in a moment, this is going to be our default inner product on $\mathfrak{s}$. In particular, whenever we take orthogonal complements within $\mathfrak{s}$, it is with respect to $\langle - | - \rangle_\mathfrak{s}$. The decomposition $\mathfrak{s} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ is orthogonal with respect to $\langle - | - \rangle_\mathfrak{s}$; this property is going to prove crucial later on.

Notation. Since we will be working with the Lie algebra $\mathfrak{s}$ throughout this subsection, we adhere to the following notation: if $X \in \mathfrak{s}$, we denote its components with respect to the decomposition $\mathfrak{s} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ as $X_0 \in \mathfrak{a}$ and $X_\alpha \in \mathfrak{g}_\alpha$.

We pull the complex structure I on $\mathfrak{p}$ back along Π; this defines a complex structure on $\mathfrak{s}$ that we are going to denote by $\widehat{I}$. By construction, given $X \in \mathfrak{s}$, we have $(\widehat{I}X)_\mathfrak{p} = IX_\mathfrak{p}$. It follows immediately from Theorem 6.3.12 that $\widehat{I}$ interchanges $\mathbb{R}H_\alpha$ and $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma_{\mathrm{L}}^+$, interchanges $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ whenever $\alpha, \beta \in \Sigma_{\mathrm{M}}^+$ are neighbors, and preserves $\mathfrak{g}_\alpha$ if $\alpha \in \Sigma_{\mathrm{S}}^+$. By design, the complex structure $\widehat{I}$ is orthogonal with respect to $\langle - | - \rangle_\mathfrak{s}$.

A word of caution is in order. The subspace $\mathfrak{p}$ is preserved by the isotropy group K; and the complex structure I on $\mathfrak{p}$ can be given as $\operatorname{ad}(Z)\big|_\mathfrak{p}$ or $\operatorname{Ad}(k)\big|_\mathfrak{p}$, where $Z \in \mathfrak{z}(\mathfrak{k})$ and $k = \exp(\frac{\pi}{2}Z) \in Z(K)^0$. In particular, I is K-invariant. On the other hand, $\mathfrak{s}$ is preserved by neither $\operatorname{ad}(Z)$ nor $\operatorname{Ad}(k)$ (let alone K), so $\widehat{I}$ cannot be described via these operators. The single most important consequence of this fact is that the complex structure $\widehat{I}$ does not make $\mathfrak{s}$ into a complex Lie algebra. There is still some good news though: we know that there is a smaller compact subgroup $Z_K(\mathfrak{a})$ whose representation on $\mathfrak{g}$ leaves invariant each restricted root space and thus preserves $\mathfrak{s}$. As a result, the isomorphism Π is $Z_K(\mathfrak{a})$-equivariant, and hence $\widehat{I}$ is $Z_K(\mathfrak{a})$-invariant on $\mathfrak{s}$.

Suppose $\mathfrak{h} \subseteq \mathfrak{s}$ is a subalgebra and $\mathcal{F}$ its corresponding standard foliation on M. Let S denote the leaf of $\mathcal{F}$ through o. Recall that G acts on M via holomorphic isometries (Corollary 2.5.5). In particular, if T_oS is a complex subspace of T_oM, then the whole S is a homogeneous complex submanifold. Thanks to Proposition 2.2.4, and our definition of $\widehat{I}$, this is the case precisely when $\mathfrak{h}$ is a complex subspace of $\mathfrak{s}$. This means that classification of standard foliations on M with a complex base leaf boils down to classification of Lie subalgebras of $\mathfrak{s}$ that are also complex subspaces. With the next theorem, we accomplish this in complex codimension 1. It turns out that such subalgebras are very scarce, which stands in stark contrast with subalgebras of *real* codimension 1 (see Chapter 4).

Theorem 6.3.14. *Let M be an irreducible Hermitian symmetric space of noncompact type represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. Pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and a Weyl chamber D for Σ; and take a basis $e_1, \dots, e_r$ for $\mathfrak{a}^*$ so that Σ and Σ^+ become as in Example 6.3.4 and (6.3.1). Suppose $\mathfrak{h} \subset \mathfrak{s}$ is a subalgebra that is also a complex hyperplane, and write $\ell = \mathfrak{s} \ominus \mathfrak{h}$. Then there are three possibilities:*

(a) *$\Sigma \simeq C_r$ and ℓ is*

  (1) *$\mathbb{C}H_{2e_r} = \mathbb{R}H_{2e_r} \oplus \mathfrak{g}_{2e_r}$, or*

  (2) *$\mathbb{C}X$, where $X \in \mathfrak{g}_{e_{r-1}-e_r}$ is any nonzero vector.*

(b) *$\Sigma \simeq (BC)_r$ and $\ell = \mathbb{C}X$, where $X \in \mathfrak{g}_{e_r}$ is any nonzero vector.*

Conversely, for every choice of ℓ as in (a) *or* (b), *$\mathfrak{h} = \mathfrak{s} \ominus \ell$ is a subalgebra and a complex hyperplane in $\mathfrak{s}$.*

To tackle this theorem, we first prove the following technical lemma, which establishes a connection between the complex structure $\widehat{I}$ and the Lie algebra structure of $\mathfrak{s}$.

Lemma 6.3.15. *Let $X \in \mathfrak{s}$ be a nonzero vector. Then $[X, \widehat{I}X]$ is a nonzero vector in $\bigoplus_{\alpha \in \Sigma_{\mathrm{L}}^+} \mathfrak{g}_\alpha$ in each of the following cases:*

(a) *$X \in \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma_{\mathrm{L}}^+} \mathfrak{g}_\alpha$;*

(b) *$X \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$, where $\alpha, \beta \in \Sigma_{\mathrm{M}}^+$ are neighbors.*

(c) *$X \in \mathfrak{g}_\alpha$, where $\alpha \in \Sigma_{\mathrm{S}}^+$.*

The motivation for this lemma comes from the properties of the complex structure I on $\mathfrak{p}$: if $X \in \mathfrak{p}$ is any nonzero vector, then $[X, IX]$ is a nonzero vector in $\mathfrak{k}$. Indeed, otherwise $\mathbb{C}X$ would be an abelian subspace of $\mathfrak{p}$, which would go against Corollary 6.3.13.

Proof of the lemma. To simplify the notation we will write $H_i = H_{2e_i}$ and $E_i = \widehat{I}H_i \in \mathfrak{g}_{2e_i}$. Notice that $[H_i, E_i] = 4||e_i||^2 E_i$, and $[H_i, E_j] = 0$ if $i \neq j$. Part (a) is rather straightforward: decompose X as $\sum_{i=1}^r \lambda_i H_i = \sum_{i=1}^r (a_i H_i + b_i E_i)$, where $\lambda_i = a_i + ib_i$. Then $\widehat{I}X = \sum_{i=1}^r (-b_i H_i + a_i E_i)$, so we can compute:

$$\begin{aligned}
[X, \widehat{I}X] &= \left[\sum_{i=1}^r (a_i H_i + b_i E_i), \sum_{i=1}^r (-b_i H_i + a_i E_i)\right] \\
&= \sum_{i=1}^r (a_i^2 [H_i, E_i] - b_i^2 [E_i, H_i]) = \sum_{i=1}^r 4|\lambda_i|^2 ||e_i||^2 E_i \neq 0.
\end{aligned}$$

Now we proceed to part (b). Let α be upstairs and β downstairs, so we can write $\alpha = e_i + e_j, \beta = e_i - e_j$. First, assume $X \in \mathfrak{g}_{e_i+e_j}$ and hence $\widehat{I}X \in \mathfrak{g}_{e_i-e_j}$. We need to show that $[X, \widehat{I}X]$ is a nonzero element of $\mathfrak{g}_{2e_i}$. Assume the converse: $[X, \widehat{I}X] = 0$. As we mentioned above, $[X_\mathfrak{p}, (\widehat{I}X)_\mathfrak{p}] = [X_\mathfrak{p}, IX_\mathfrak{p}] \neq 0$. But

$$\begin{aligned}
[X_\mathfrak{p}, (\widehat{I}X)_\mathfrak{p}] &= \frac{1}{4}\left[X - \theta X, \widehat{I}X - \theta(\widehat{I}X)\right] \\
&= \frac{1}{4}\left([X, \widehat{I}X] + \theta[X, \widehat{I}X]\right) - \frac{1}{4}\left([X, \theta(\widehat{I}X)] + \theta[X, \theta(\widehat{I}X)]\right) \\
&= -\frac{1}{4}\left([X, \theta(\widehat{I}X)] + \theta[X, \theta(\widehat{I}X)]\right),
\end{aligned}$$

which implies that $[X, \theta(\widehat{I}X)] \neq 0$. Note that this vector lies in $\mathfrak{g}_{\alpha-\beta} = \mathfrak{g}_{2e_j}$. Consider the element of $\mathfrak{g}$ given by $[X, [\widehat{I}X, \theta(\widehat{I}X)]]$. On the one hand, we have Proposition 2.4.9(g), which implies:

$$[X, [\widehat{I}X, \theta(\widehat{I}X)]] = [X, -||\widehat{I}X||_B^2 H_{e_i-e_j}] = ||\widehat{I}X||_B^2 \langle e_i + e_j, H_{e_i-e_j}\rangle X = 0.$$

On the other hand, by our assumption and the Jacobi identity:

$$[X, [\widehat{I}X, \theta(\widehat{I}X)]] = [\widehat{I}X, [X, \theta(\widehat{I}X)]]. \tag{6.3.4}$$

Since $[X, \theta(\widehat{I}X)]$ is nonzero and the pairing $\mathfrak{g}_{e_i-e_j} \times \mathfrak{g}_{2e_j} \twoheadrightarrow \mathfrak{g}_{e_i+e_j}$ is nondegenerate (by Lemma 4.2.6), the map $\mathrm{ad}([X, \theta(\widehat{I}X)])$ provides an isomorphism between $\mathfrak{g}_{e_i-e_j}$ and $\mathfrak{g}_{e_i+e_j}$. This means that (6.3.4) cannot be zero, which is a contradiction.

We claim that the element $[X, \widehat{I}X] \in \mathfrak{g}_{2e_i}$ does not depend on the choice of a nonzero X in $\mathfrak{g}_{e_i+e_j}$ up to multiplication by a *positive* constant. Indeed, given another such $Y \in \mathfrak{g}_{e_i+e_j}$, there exists $k \in K_0$ such that $Y = c\,\mathrm{Ad}(k)X$ for some $c \neq 0$ (see Lemma 4.2.5). Since the action of K_0 on $\mathfrak{s}$ is $\mathbb{C}$-linear and preserves the restricted root spaces, we have:

$$[Y, \widehat{I}Y] = c^2\left[\mathrm{Ad}(k)X, \widehat{I}(\mathrm{Ad}(k)X)\right] = c^2\,\mathrm{Ad}(k)[X, \widehat{I}X] = c^2[X, \widehat{I}X],$$

where the last equality follows from the fact that $\mathfrak{g}_{2e_i}$ is 1-dimensional and thus the action of K_0 on it is trivial. Finally, if $Z \in \mathfrak{g}_{e_i+e_j} \oplus \mathfrak{g}_{e_i-e_j}$ is arbitrary, decompose it as $X + Y$ and observe that

$$[X + Y, \widehat{I}Y + \widehat{I}X] = [X, \widehat{I}X] + [Y, \widehat{I}Y] = [X, \widehat{I}X] + c^2[X, \widehat{I}X] = (1 + c^2)[X, \widehat{I}X] \neq 0.$$

It remains to prove part (c) (in case the root system is $(BC)_r$). Here $\widehat{I}(X)$ also lies in $\mathfrak{g}_\alpha$, so we want $[X, \widehat{I}X]$ to be a nonzero element of $\mathfrak{g}_{2\alpha}$. Here, too, we have

$$0 \neq [X_\mathfrak{p}, (\widehat{I}X)_\mathfrak{p}] = \frac{1}{4}\left([X, \widehat{I}X] + \theta[X, \widehat{I}X]\right) - \frac{1}{4}\left([X, \theta(\widehat{I}X)] + \theta[X, \theta(\widehat{I}X)]\right). \tag{6.3.5}$$

Since $\theta(\widehat{I}X) \in \mathfrak{g}_{-\alpha}$, we have $[X, \theta(\widehat{I}X)] \in \mathfrak{a}$ thanks to Proposition 2.4.9(g). This implies that the expression inside the second pair of parentheses in (6.3.5) vanishes. Consequently, $[X, \widehat{I}X]$ has to be nonzero. This completes the proof of the lemma. □

Remark 6.3.16. In many cases, stronger versions of parts (b) and (c) of the above lemma hold true: not only is $[X, \widehat{I}X]$ a nonzero element in $\mathfrak{g}_{2e_i}$ (where $2e_i$ is $\alpha + \beta$ or 2α, respectively), but in fact it is a positive multiple of E_i. This is true and easy to verify computationally in low-rank examples like[1] $M = \mathbb{C}H^n$ or $\mathrm{Gr}^*(2, \mathbb{R}^5)$. If M is arbitrary and e_i is a positive short root, the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{e_i}$ and $\mathfrak{g}_{-e_i}$ is isomorphic to $\mathfrak{i}(\mathbb{C}H^n) \simeq \mathfrak{su}(n, 1)$. Using this, one can show that $[X, \widehat{I}X]$ is a positive multiple of E_i for any nonzero $X \in \mathfrak{g}_{e_i}$. So this stronger version of (c) in fact holds for all spaces. If M is split and $e_i + e_j, e_i - e_j \in \Sigma^+_{\mathrm{M}}$ are neighbors, the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{e_i - e_j}, \mathfrak{g}_{2e_j}, \mathfrak{g}_{-e_i+e_j}$, and $\mathfrak{g}_{-2e_j}$ is isomorphic to $\mathfrak{i}(\mathrm{Gr}^*(2, \mathbb{R}^5)) \simeq \mathfrak{so}(3, 2)$. Similarly, one can use this to deduce the stronger version of (b) for M. We have all reasons to believe that it is true for all irreducible Hermitian symmetric spaces of noncompact type.

Our next step is to examine how $\mathfrak{h}$ in Theorem 6.3.14 can intersect with $\mathfrak{a}$. In theory, $\mathfrak{h} \cap \mathfrak{a}$ can have codimension 0, 1, or 2 in $\mathfrak{a}$. It turns out that the last option is not possible.

Lemma 6.3.17. *If $\mathfrak{h} \subset \mathfrak{s}$ is a subalgebra and a complex hyperplane, then $\mathfrak{h} \cap \mathfrak{a}$ is either a (real) hyperplane in $\mathfrak{a}$ or the whole $\mathfrak{a}$.*

Proof. For every pair $1 \le i < j \le r$, the intersection of $\mathfrak{h}$ with $\mathbb{C}H_i \oplus \mathbb{C}H_j$ is nontrivial for dimensional reasons. If X is any nonzero vector in that intersection, then the element $[X, \widehat{I}X] \in \mathfrak{h} \cap (\mathbb{R}E_i \oplus \mathbb{R}E_j)$ is nonzero by Lemma 6.3.15(a). Applying $\widehat{I}$ to that, we see that $\mathfrak{h}$ has a nonzero intersection with every coordinate 2-plane $\mathbb{R}H_i \oplus \mathbb{R}H_j$ in $\mathfrak{a}$. Elementary induction on $\dim(\mathfrak{a})$ shows that $\mathfrak{h} \cap \mathfrak{a}$ must then be of codimension at most 1. $\square$

The next proposition tells that $\mathfrak{h}$ has to be positioned nicely with respect to the restricted root space decomposition; and it is the crux of the proof of Theorem 6.3.14.

Proposition 6.3.18. *Let $\mathfrak{h} \subset \mathfrak{s}$ be a subalgebra and a complex hyperplane. Then*

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{a}) \oplus \bigoplus_{\alpha \in \Sigma^+} (\mathfrak{h} \cap \mathfrak{g}_\alpha).$$

Proof. We first handle the easy case when $\mathfrak{h}$ contains $\mathfrak{a}$. Notice that it then also contains $\widehat{I}(\mathfrak{a}) = \bigoplus_{\alpha \in \Sigma^+_{\mathrm{L}}} \mathfrak{g}_\alpha$. So we need to prove the following:

$$X = \sum_{i<j} (X_{e_i+e_j} + X_{e_i-e_j}) + \sum_{i=1}^{r} X_{e_i} \in \mathfrak{h} \cap \bigoplus_{\alpha \in \Sigma^+_{\mathrm{M}} \cup \Sigma^+_{\mathrm{S}}} \mathfrak{g}_\alpha \quad \Rightarrow \quad X_{e_i \pm e_j}, X_{e_i} \in \mathfrak{h}.$$

The main idea in this proof is that we can use the adjoint representation of $\mathfrak{h}$ on itself to get rid of undesired components of a vector. To make calculations less cumbersome, we will use the notation $v \sim w$ to indicate that two nonzero vectors are proportional. Observe that $[H_i, [H_j, X]] \sim X_{e_i+e_j} - X_{e_i-e_j} \in \mathfrak{h}$. By further applying $\mathrm{ad}(H_i + H_j)$, we get $X_{e_i+e_j} \in \mathfrak{h}$ and thus $X_{e_i-e_j} \in \mathfrak{h}$. This shows that $\sum_{i=1}^r X_{e_i} \in \mathfrak{h}$. By applying $\mathrm{ad}(H_i)$ to that, we have $X_{e_i} \in \mathfrak{h}$.

Now we come to the main part of the proof. Assume that $\mathfrak{h} \cap \mathfrak{a}$ is a hyperplane in $\mathfrak{a}$ and denote it by $\mathfrak{h}_0$. Note that $\widehat{I}(\mathfrak{h}_0) = \mathfrak{h} \cap \bigoplus_{\alpha \in \Sigma^+_{\mathrm{L}}} \mathfrak{g}_\alpha$ is a hyperplane in $\bigoplus_{\alpha \in \Sigma^+_{\mathrm{L}}} \mathfrak{g}_\alpha$. We will prove by induction that $\mathfrak{g}_{2e_i} \subset \mathfrak{h}$ for all $i = 1, \ldots, r-1$. Fix any i in this range and assume that $\mathfrak{g}_{2e_j} \subset \mathfrak{h}$ for every $1 \le j < i$ (if any); we will show that $\mathfrak{g}_{2e_i} \subset \mathfrak{h}$ as well. We

[1] For $\mathbb{C}H^n$, part (b) is vacuous because there are no medium roots. On the other hand, $\mathrm{Gr}^*(2, \mathbb{R}^5)$ has root system C_2, so only (b) is relevant for this space. Note that this hyperbolic quadric is the smallest split space in Table 6.4.

divide the proof into two scenarios based on whether M is split or not—the same way we did in the proof of Theorem 6.3.12.

First, assume that M is not split. This means that the medium root multiplicity is at least 2. Therefore, $\mathfrak{g}_{e_i+e_{i+1}} \oplus \mathfrak{g}_{e_i-e_{i+1}}$ is a complex subspace of $\mathfrak{s}$ of dimension ≥ 2. Its intersection with $\mathfrak{h}$ has to be nontrivial, so we can take a nonzero vector X there. In view of Lemma 6.3.15(b), $[X, \widehat{I}X] \in \mathfrak{h}$ is a nonzero vector in $\mathfrak{g}_{2e_i}$, which proves the claim.

Now, let M be split. We claim that there exists $X \in \mathfrak{h}$ with $X_{e_i-e_{i+1}} \neq 0$. Indeed, otherwise $\mathfrak{h}$ is orthogonal to $\mathfrak{g}_{e_i-e_{i+1}}$ and hence to $\mathfrak{g}_{e_i+e_{i+1}} = \widehat{I}(\mathfrak{g}_{e_i-e_{i+1}})$, which implies that $\ell = \mathfrak{s} \ominus \mathfrak{h} = \mathfrak{g}_{e_i-e_{i+1}} \oplus \mathfrak{g}_{e_i+e_{i+1}}$; in this case, the induction statement (and the statement of the proposition, for that matter) is satisfied. We fix such X. Next, the intersection $\mathfrak{h} \cap (\mathbb{R}E_i \oplus \mathbb{R}E_{i+1})$ must be nontrivial, so we take a nonzero $Y = y_i E_i + y_{i+1}E_{i+1}$ in it. We may assume $y_{i+1} \neq 0$. We first deal with a "generic" case when $\mathfrak{h}_0 \neq \mathfrak{a} \ominus \mathbb{R}(H_i - H_{i+1})$. This means that there exists $H = \sum_{i=1}^r a_i H_i \in \mathfrak{h}_0$ with $a_i \neq a_{i+1}$. If we replace X with $[H, X]$, this new element of $\mathfrak{h}$ still has a nonzero component $\mathfrak{g}_{e_i-e_{i+1}}$, but now its component in $\mathfrak{a}$ is zero. We keep denoting this vector by X. Consider the vector $Z = [X, Y] \in \mathfrak{h}$. In view of Proposition 2.4.9(d), Z can have nonzero components only in $\mathfrak{g}_{e_j+e_i}$ $(j < i)$ and $\mathfrak{g}_{e_j+e_{i+1}}$ $(j < i+1)$. Since $\operatorname{ad}(y_{i+1}E_{i+1})$ provides an isomorphism between $\mathfrak{g}_{e_i-e_{i+1}}$ and $\mathfrak{g}_{e_i+e_{i+1}}$, $Z_{e_i+e_{i+1}} \neq 0$. The element $[Z, \widehat{I}Z] \in \mathfrak{h}$ can have nonzero components only in $\mathfrak{g}_{2e_1}, \ldots, \mathfrak{g}_{2e_i}$. Moreover, its component in $\mathfrak{g}_{2e_i}$ is $[Z_{e_i+e_{i+1}}, \widehat{I}(Z_{e_i+e_{i+1}})]$, which is guaranteed to be nonzero by Lemma 6.3.15(b). By the induction hypothesis, the components of $[Z, \widehat{I}Z]$ in $\mathfrak{g}_{2e_1}, \ldots, \mathfrak{g}_{2e_{i-1}}$ lie in $\mathfrak{h}$, hence so does the one in $\mathfrak{g}_{2e_i}$.

We are left to consider the case when $\mathfrak{h}_0 = \mathfrak{a} \ominus \mathbb{R}(H_i - H_{i+1})$. In this scenario, we cannot eliminate the $\mathfrak{a}$-component of X while keeping $X_{e_i-e_{i+1}} \neq 0$. We can, however, get rid of most of the other unwanted components. To that end, first observe that y_i and y_{i+1} have to coincide, so we may as well assume $Y = E_i + E_{i+1}$. This time, the vector $Z = [X, Y] \in \mathfrak{h}$ can be written as

$$Z = \sum_{j=1}^{i-1} (Z_{e_j+e_i} + Z_{e_j+e_{i+1}}) + Z_{e_i+e_{i+1}} + Z_{2e_i} + Z_{2e_{i+1}} \tag{6.3.6}$$

with $Z_{e_i+e_{i+1}} \neq 0$. For any $j = 1, \ldots, i-1$, consider the element $H_j - H_i - H_{i+1} \in \mathfrak{h}_0$. Applying $\operatorname{ad}(H_j - H_i - H_{i+1})$ to Z eliminates the j-th summand in (6.3.6), keeps the (e_i+e_{i+1})-component nonzero, and yields a vector still lying in $\mathfrak{h}$. Therefore, by successively applying $\operatorname{ad}(H_1 - H_i - H_{i+1}), \ldots, \operatorname{ad}(H_{i-1} - H_i - H_{i+1})$ to Z, we obtain a vector

$$\mathfrak{h} \ni Z' = Z'_{e_i+e_{i+1}} + aE_i + bE_{i+1}$$

with $Z'_{e_i+e_{i+1}} \neq 0$ and $a, b \in \mathbb{R}$. By adding a suitable multiple of Y to Z', we can achieve $b = 0$; we continue to denote this new vector by Z'. Consider the following bracket in $\mathfrak{h}$:

$$\begin{aligned}\left[Z', \widehat{I}Z' + \frac{a}{2}(H_i + H_{i+1})\right] &= \left[Z'_{e_i+e_{i+1}} + aE_i, \widehat{I}(Z'_{e_i+e_{i+1}}) - \frac{a}{2}(H_i - H_{i+1})\right] \\ &= [Z'_{e_i+e_{i+1}}, \widehat{I}(Z'_{e_i+e_{i+1}})] - \frac{a^2}{2}[E_i, H_i] \\ &= [Z'_{e_i+e_{i+1}}, \widehat{I}(Z'_{e_i+e_{i+1}})] + 2a^2 \|e_i\|^2 E_i.\end{aligned}$$

This is an element of $\mathfrak{g}_{2e_i} \cap \mathfrak{h}$, and it is nonzero by Remark 6.3.16, which proves the claim.

Notice that we did not have to rely on the induction hypothesis in this case.

Now that we know that $\mathfrak{g}_{2e_i} \subset \mathfrak{h}$ for all $1 \leq i \leq r-1$, we have $\bigoplus_{i=1}^{r-1} \mathbb{R}H_i \subseteq \mathfrak{h}_0$. Since we are assuming $\mathfrak{h}_0$ is a hyperplane in $\mathfrak{a}$, it must coincide with $\bigoplus_{i=1}^{r-1} \mathbb{R}H_i$. Let us first show that this is not possible if the root system is $(BC)_r$. Indeed, for dimensional reasons, the intersection of $\mathfrak{h}$ with $\mathbb{R}H_r \oplus \mathfrak{g}_{e_r}$ has to be nontrivial. Take a nonzero vector $U = u_r H_r + U_{e_r}$ in this intersection. We can calculate:

$$\begin{aligned}
[U, \widehat{I}U] &= \left[u_r H_r + U_{e_r}, u_r E_r + \widehat{I}(U_{e_r})\right] \\
&= 4u_r^2 ||e_r||^2 E_r + 2u_r ||e_r||^2 \widehat{I}(U_{e_r}) + [U_{e_r}, \widehat{I}(U_{e_r})] \\
&= 2u_r ||e_r||^2 (u_r E_r + \widehat{I}(U_{e_r})) + 2u_r^2 ||e_r||^2 E_r + [U_{e_r}, \widehat{I}(U_{e_r})] \\
&= 2u_r ||e_r||^2 \widehat{I}U + 2u_r^2 ||e_r||^2 E_r + [U_{e_r}, \widehat{I}(U_{e_r})].
\end{aligned}$$

In the last row, the first summand lies in $\mathfrak{h}$, hence so does the sum of the other two. But this sum is a nonzero element of $\mathfrak{g}_{2e_r}$ due to Remark 6.3.16. We deduce that $\mathfrak{g}_{2e_r} \subset \mathfrak{h}$ and thus $H_r \in \mathfrak{h}$, which is a contradiction.

From now on, we assume that the root system of M is C_r. We are going to use the same approach as at the beginning of the proof. Take any $X \in \mathfrak{h}$. By subtracting vectors from $\mathfrak{h}_0 \oplus \widehat{I}(\mathfrak{h}_0)$ if necessary, we may assume $X = x_r H_r + x_r' E_r + \sum_{i<j}(X_{e_i+e_j} + X_{e_i-e_j})$. For $j < r$, we can take the consecutive brackets of X with H_i, H_j and $H_i \pm H_j$ (just like we did at the beginning) to show that $X_{e_i \pm e_j} \in \mathfrak{h}$; we assume these components are zero from now on. Next, we deal with $X_{e_i \pm e_r}$, $1 \leq i \leq r-1$. First, we commute X with H_i to get $[H_i, X] \sim X_{e_i+e_r} + X_{e_i-e_r} = V \in \mathfrak{h}$. If $\ell = \mathfrak{s} \ominus \mathfrak{h} = \mathbb{C}H_r$, the assertion of the lemma holds true, so we can rule out this case. Then, there exists $W \in \mathfrak{h}$ with W_0 a nonzero multiple of H_r. Multiplying by a suitable complex number, we can ensure $W_0 = H_r$ and $W_{2e_r} = 0$. As we know by now, the components $W_{e_j \pm e_k}$, $j < k < r$, lie in $\mathfrak{h}$, so we can subtract them and assume $W = H_r + \sum_{j=1}^{r-1}(W_{e_j+e_r} + W_{e_j-e_r}) \in \mathfrak{h}$. Let us look at $[W, V] \in \mathfrak{h}$:

$$\begin{aligned}
[W, V] &= \left[H_r + \sum_{j=1}^{r-1}(W_{e_j+e_r} + W_{e_j-e_r}), X_{e_i+e_r} + X_{e_i-e_r}\right] \\
&= 2||e_r||^2 (X_{e_i+e_r} - X_{e_i-e_r}) + [W_{e_i+e_r}, X_{e_i-e_r}] + [W_{e_i-e_r}, X_{e_i+e_r}].
\end{aligned}$$

In the second row, the last two summands lie in $\mathfrak{g}_{2e_i} \subset \mathfrak{h}$, which implies $X_{e_i+e_r} - X_{e_i-e_r} \in \mathfrak{h}$. Overall, this means that both $X_{e_i+e_r}$ and $X_{e_i+e_r}$ lie in $\mathfrak{h}$. By subtracting them from the original X for each $i = 1, \ldots, r-1$, we are only left with $x_r H_r + x_r' E_r$. If either x_r or x_r' is nonzero, we can multiply this sum by a suitable complex number to get $H_r \in \mathfrak{h}$, which is a contradiction. We conclude that $x_r = x_r' = 0$, which completes the proof of the lemma. □

With Proposition 6.3.18 under our belt, the theorem now follows rather easily from Theorem 6.3.12.

Proof of Theorem 6.3.14. Thanks to Proposition 6.3.18, $\mathfrak{h}$ is the sum of its intersections with $\mathfrak{a}$ and the restricted root spaces $\mathfrak{g}_\alpha$. The same then must be true for $\ell = \mathfrak{s} \ominus \mathfrak{h}$: $\ell = (\ell \cap \mathfrak{a}) \oplus \bigoplus_{\alpha \in \Sigma^+}(\ell \cap \mathfrak{g}_\alpha)$. Since ℓ is a complex line, there are two possibilities.

Case 1: ℓ is contained in a single $\mathfrak{g}_\alpha$ or $\mathfrak{a}$. According to Theorem 6.3.12, $\mathfrak{a}$ and

$\mathfrak{g}_\alpha$, $\alpha \in \Sigma_L^+ \cup \Sigma_M^+$, are totally real subspaces of $\mathfrak{s}$, so ℓ cannot be contained in them. The only option left is $\ell \subseteq \mathfrak{g}_\alpha$, where α is a positive short root. In particular Σ must be isomorphic to $(BC)_r$ for this to be possible. The root α must be simple. Indeed, otherwise we would have $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Sigma^+$. Since $[\mathfrak{g}_\beta, \mathfrak{g}_\gamma] = \mathfrak{g}_\alpha$ (Proposition 2.4.9(d)), $\mathfrak{h}$ cannot be a subalgebra in that case. Among the short roots $e_1, \dots, e_r$, only e_r is simple. This leads us to part (b) of the theorem.

Case 2: $\ell = \ell' \oplus \ell''$, where either of ℓ' and ℓ'' is a real line contained in a single $\mathfrak{g}_\alpha$ or $\mathfrak{a}$. It follows from Theorem 6.3.12 that there are two options: either $\ell' = \mathbb{R}H_{2e_i}$ and $\ell'' = \mathbb{R}\widehat{I}H_{2e_i} = \mathfrak{g}_{2e_i}$ (the order between ℓ' and ℓ'' obviously does not matter), or $\ell' \subseteq \mathfrak{g}_\alpha$ and $\ell'' \subseteq \mathfrak{g}_\beta$, where α and $\beta \in \Sigma_M^+$ are neighbors. In the first situation, $\mathfrak{g}_{2e_i}$ has to be a simple root space—for the same reason as in case 1. This is only possible if $i = r$ and the root system is reduced, which is exactly (a)-(1). In the second case, we may assume without loss of generality that α is upstairs and β is downstairs. If β is not simple, we end up with the same problem and $\mathfrak{h}$ cannot be a subalgebra. The only simple medium root is $e_{r-1} - e_r$, so we must have $\beta = e_{r-1} - e_r$, $\alpha = e_{r-1} + e_r$. We claim that the root system has to be reduced in this case. Indeed, otherwise we have $\mathfrak{g}_{e_{r-1}}, \mathfrak{g}_{e_r} \subseteq \mathfrak{h}$, but $[\mathfrak{g}_{e_{r-1}}, \mathfrak{g}_{e_r}] = \mathfrak{g}_{e_{r-1}+e_r} \subsetneq \mathfrak{h}$, which is a contradiction. We see that ℓ has to be as in (a)-(2), which concludes the proof. □

Note that we still need to verify that the three options for $\mathfrak{h}$ in Theorem 6.3.14 are indeed Lie subalgebras of $\mathfrak{s}$. For (a)-(1) and (b), this is trivial, because we are only meddling with $\mathfrak{a}$ and simple root spaces there. In (a)-(2), however, if we have $X \in \mathfrak{g}_{e_{r-1}-e_r}$, then $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}\widehat{I}X$, and the latter summand lies in the root space $\mathfrak{g}_{e_{r-1}+e_r}$, which is not simple. Since $e_{r-1} + e_r = e_{r-1} - e_r + 2e_r$ is the sum of two simple roots, we need to check only one thing:

$$\left[\mathfrak{g}_{2e_r}, \mathfrak{g}_{e_{r-1}-e_r} \ominus \mathbb{R}X\right] \text{ needs to be contained in } \mathfrak{g}_{e_{r-1}+e_r} \ominus \mathbb{R}\widehat{I}X. \tag{6.3.7}$$

Recall that $\operatorname{ad}(H_r)$ establishes an isomorphism $\mathfrak{g}_{e_{r-1}-e_r} \xrightarrow{\sim} \mathfrak{g}_{e_{r-1}+e_r}$. It is easy to see that (6.3.7) holds for every nonzero X in $\mathfrak{g}_{e_{r-1}-e_r}$ precisely when the restrictions of $\operatorname{ad}(H_r)$ and $\widehat{I}$ to $\mathfrak{g}_{e_{r-1}-e_r}$, thought of as operators from $\mathfrak{g}_{e_{r-1}-e_r}$ to $\mathfrak{g}_{e_{r-1}+e_r}$, are proportional. Luckily, there is no need to do this directly—it can be verified using some slick arguments involving the canonical extension procedure. We will do this in the next subsection (see Proposition 6.3.22). For now, using our discussion in Chapter 4, we reformulate the statement of Theorem 6.3.14 in the language of standard foliations. Recall that the group $Z_K(\mathfrak{a})$ acts on $\mathfrak{s}$ by unitary transformations and preserves each root space; and its action on each root space is of cohomogeneity 1. Consequently, if $X, X' \in \mathfrak{g}_{e_{r-1}-e_r}$ are any two nonzero vectors as in Theorem 6.3.14(a)-(2), then there exists an element $k \in Z_K(\mathfrak{a})$ such that $\operatorname{Ad}(k)(\mathfrak{s} \ominus \mathbb{C}X) = \mathfrak{s} \ominus \mathbb{C}X'$. The same holds for the choice of X in part (b) of the theorem. We deduce:

Corollary 6.3.19. *Let M be an irreducible Hermitian symmetric space of noncompact type represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. Pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and a Weyl chamber D for Σ; and take a basis $e_1, \dots, e_r$ for $\mathfrak{a}^*$ so that Σ and Σ^+ become as in Example 6.3.4 and (6.3.1). Every standard codimension-2 foliation on M with a complex base leaf is (strongly) isometrically congruent to the orbit foliation of the connected Lie subgroup of $AN \subset G$ with Lie algebra $\mathfrak{h} \subset \mathfrak{s}$, where $\mathfrak{h}$ is one of the following:*

(a) *If* $\Sigma \simeq C_r$*:*

  (1) $(\mathfrak{a} \ominus \mathbb{R}H_{2e_r}) \oplus (\mathfrak{n} \ominus \mathfrak{g}_{2e_r})$*, or*

  (2) $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathbb{C}X)$*, where* $X \in \mathfrak{g}_{e_{r-1}-e_r}$ *is any nonzero vector.*

(b) *If* $\Sigma \simeq (BC)_r$*:* $\mathfrak{a} \oplus (\mathfrak{n} \ominus \mathbb{C}X)$*, where* $X \in \mathfrak{g}_{e_r}$ *is any nonzero vector.*

Conversely, every choice of $\mathfrak{h}$ *as in* (a) *or* (b) *induces a standard codimension-2 foliation on* M *with a complex base leaf. Moreover, different choices of* X *in* (a)-(2) *and* (b) *lead to (strongly) isometrically congruent foliations.*

Consequently, the moduli space of standard C2-foliations on M with a complex base leaf consists of 2 points if $\Sigma \simeq C_r$ and just one point if $\Sigma \simeq (BC)_r$. Since the restricted root system of an irreducible noncompact Hermitian symmetric space is always either C_r or $(BC)_r$, we arrive at the following:

Corollary 6.3.20. *Every Hermitian symmetric space of noncompact type admits a homogeneous complex hypersurface.*

This stands in stark contrast to the the compact case: as we know from the previous section, most irreducible compact Hermitian symmetric spaces *do not* have any homogeneous complex hypersurfaces.

6.3.3. Homogeneous complex hypersurfaces in irreducible Hermitian symmetric spaces of noncompact type

In this final part of the section, we put together what is known about homogeneous complex hypersurfaces in irreducible Hermitian symmetric spaces of noncompact type, taking into account the results we have obtained earlier in the section.

First of all, let us examine how homogeneous complex hypersurfaces can be built in higher-rank spaces by means of the canonical extension. Let M be a Hermitian symmetric space of noncompact type. Suppose some boundary component B_Φ of M is a complex submanifold. Thanks to Proposition 2.5.1, B_Φ is itself a Hermitian symmetric space of noncompact type. Let $S_\Phi \subset B_\Phi$ be a homogeneous submanifold. Its canonical extension

$$S_\Phi^\wedge = S_\Phi \times A_\Phi \times N_\Phi \subset B_\Phi \times A_\Phi \times N_\Phi \cong M$$

is a homogeneous submanifold of M. Indeed, if S_Φ is an orbit of a Lie subgroup $H_\Phi \subseteq G'_\Phi$, then $S_\Phi^\wedge$ is an orbit of $H_\Phi^\wedge$.

Proposition 6.3.21. *Let* M *be a Hermitian symmetric space of noncompact type,* $B_\Phi \subseteq M$ *a complex boundary component, and* $S_\Phi \subseteq B_\Phi$ *a homogeneous submanifold.*

(a) S_Φ *is a complex submanifold of* B_Φ *if and only if* $S_\Phi^\wedge$ *is a complex submanifold of* M*. In particular,* S_Φ *is a homogeneous complex hypersurface in* B_Φ *if and only if* $S_\Phi^\wedge$ *is such in* M*.*

(b) *Suppose* S_Φ *is a homogeneous complex hypersurface in* B_Φ*, and assume it satisfies any of the following properties:*

  (1) S_Φ *is an orbit of an isometric* $C1$*-action on* B_Φ*.*

  (2) S_Φ *is an orbit of an isometric* $C2$*-action on* B_Φ*.*

(3) *S_Φ is a leaf of a homogeneous $C2$-foliation on B_Φ.*

(4) *S_Φ is the base leaf of a standard $C2$-foliation on B_Φ.*

Then S_Φ^Λ satisfies an analogous property in M.

Proof. It mostly follows from our discussion of the canonical extension procedure in Subsection 5.1.1. If $(b, a, n) \in S_\Phi^\Lambda$, then

$$N_{(b,a,n)} S_\Phi^\Lambda = N_b S_\Phi \subseteq T_b B_\Phi \subseteq T_b B_\Phi \oplus T_a A_\Phi \oplus T_n N_\Phi.$$

Plainly, S_Φ^Λ is complex if and only if this normal space is complex. But the same is true for S_Φ, which proves (1). According to our discussion on standard foliations in Chapter 4, if S_Φ is the base leaf of a standard foliation on B_Φ, we can find a subalgebra $\mathfrak{h}_\Phi$ in the solvable part $\mathfrak{a}^\Phi \oplus \bigoplus_{\alpha \in \Sigma_\Phi^+} \mathfrak{g}_\alpha$ of the Iwasawa decomposition of $\mathfrak{g}'_\Phi$ (induced from that of $\mathfrak{g}$) such that the standard foliation in question is induced by $\mathfrak{h}_\Phi$. But then $\mathfrak{h}_\Phi^\Lambda = \mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ is contained in $\mathfrak{s} = \mathfrak{a} \oplus \mathfrak{n}$, so its corresponding connected Lie subgroup H_Φ^Λ yields a standard foliation on M. The base leaf of that foliation is S_Φ^Λ. The other assertions of the proposition are trivial. ☐

Recall from Proposition 6.2.6 that the only two totally geodesic complex hypersurfaces in irreducible Hermitian symmetric spaces of noncompact type (up to isometric congruence) are the standardly embedded $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$ and $\mathrm{Gr}^*(2, \mathbb{R}^{n+1}) \subset \mathrm{Gr}^*(2, \mathbb{R}^{n+2})$, $n \geq 3$. Since the complex hyperbolic space and hyperbolic quadric have low ranks (1 and 2, respectively), they have a higher chance of appearing in other Hermitian symmetric spaces of noncompact type as boundary components. Whenever they do, we can extend their totally geodesic complex hypersurfaces to the ambient space via the canonical extension. Since any two totally geodesic complex hypersurfaces in $\mathbb{C}H^n$ (resp., $\mathrm{Gr}^*(2, \mathbb{R}^{n+2})$) are strongly isometrically congruent, the canonical extension will produce a unique homogeneous complex hypersurface up to isometric congruence. The restricted root systems of $\mathbb{C}H^n$ and $\mathrm{Gr}^*(2, \mathbb{R}^{n+2})$ are $(BC)_1$ (or $C_1 = A_1$ in case $n = 1$) and C_2, respectively. A quick glance at Table 6.4 reveals that these are the only noncompact Hermitian symmetric spaces with such root systems. So given a space M with root system Σ, we want to find $\Phi \subseteq \Lambda$ such that B_Φ is complex and Σ_Φ is isomorphic to $(BC)_1, A_1$, or C_2. The next proposition tells that this is always possible and the resulting canonically extended homogeneous complex hypersurfaces are going to be precisely those we obtained in Corollary 6.3.19.

Proposition 6.3.22. *Let M be an irreducible Hermitian symmetric space of noncompact type represented by an orthogonal symmetric Lie algebra $(\mathfrak{g}, \theta)$. Pick a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ and make a choice of positive roots $\Sigma^+ \subset \Sigma$; and take a basis $e_1, \dots, e_r$ for $\mathfrak{a}^*$ so that Σ and Σ^+ become as in Example 6.3.4 and (6.3.1).*

(a) *If $\Sigma \simeq C_r$, then:*

(1) *The only complex boundary component B_Φ isometric to $\mathbb{C}H^n$ is the one with $\Phi = \{2e_r\}$; in this case, $B_\Phi \simeq \mathbb{C}H^1$. The canonical extension of $\{\mathrm{pt}\} \subset B_\Phi$ is a homogeneous complex hypersurface and the base leaf of the standard $C2$-foliation on M induced by $\mathfrak{h} = (\mathfrak{a} \ominus \mathbb{R}H_{2e_r}) \oplus (\mathfrak{n} \ominus \mathfrak{g}_{2e_r})$.*

(2) *The only complex boundary component B_Φ isometric to $\mathrm{Gr}^*(2, \mathbb{R}^{n+2})$ is the one with $\Phi = \{e_{r-1} - e_r, 2e_r\}$. The canonical extension of a totally geodesic*

$\mathrm{Gr}^*(2,\mathbb{R}^{n+1}) \subset \mathrm{Gr}^*(2,\mathbb{R}^{n+2}) \simeq B_\Phi$ is a homogeneous complex hypersurface and the base leaf of the standard C2-foliation on M induced by $\mathfrak{h} = \mathfrak{s} \ominus \mathbb{C}X$, where $X \in \mathfrak{g}_{e_{r-1}-e_r}$ is a nonzero vector.

(b) *If $\Sigma \simeq (BC_r)$:*

(1) *The only complex boundary component B_Φ isometric to $\mathbb{C}H^n$ is the one with $\Phi = \{e_r\}$. The canonical extension of a totally geodesic $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n \simeq B_\Phi$ is a homogeneous complex hypersurface and the base leaf of the standard C2-foliation on M induced by $\mathfrak{h} = \mathfrak{s} \ominus \mathbb{C}X$, where $X \in \mathfrak{g}_{e_r}$ is a nonzero vector.*

(2) *There are no complex boundary components in M isometric to* $\mathrm{Gr}^*(2,\mathbb{R}^{n+2})$.

Each of these homogeneous complex hypersurfaces in M can be realized as a singular orbit of an isometric C1-action.

Proof. In view of Corollary 6.3.19 and Proposition 6.3.21, we only need to prove that the totally geodesic complex hypersurfaces $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$ and $\mathrm{Gr}^*(2,\mathbb{R}^{n+1}) \subset \mathrm{Gr}^*(2,\mathbb{R}^{n+2})$ arise as base leaves of standard C2-foliations as described in the proposition.

For $M = \mathbb{C}H^1 = \mathrm{SU}(1,1)/\mathrm{S}(\mathrm{U}(1)\mathrm{U}(1))$ and its (totally geodesic) submanifold $\{\mathrm{pt}\}$, the statement is trivial. Now let $M = \mathbb{C}H^n = \mathrm{SU}(n,1)/\mathrm{S}(\mathrm{U}(n)\mathrm{U}(1))$. The totally geodesic $\mathbb{C}H^{n-1}$ can be described as the orbit of $\mathrm{SU}(n-1,1)$ embedded into $\mathrm{SU}(n,1)$ in a standard way. A quick look at this embedding reveals that $\mathfrak{h} = \mathfrak{su}(n-1,1)$ is a θ-stable subalgebra of $\mathfrak{g} = \mathfrak{su}(n,1)$. We can pick $\mathfrak{a}$ to be the fixed maximal abelian subspace of $\mathfrak{h} \cap \mathfrak{p}$; in this case, the restricted root systems of $\mathfrak{h}$ and $\mathfrak{g}$ coincide. Moreover, if we write $\Sigma^+ = \{\alpha, 2\alpha\}$ according to Example 2.4.14, then $\mathfrak{h}_{2\alpha} = \mathfrak{g}_{2\alpha}$, while $\mathfrak{h}_\alpha$ is a complex hyperplane in $\mathfrak{g}_\alpha$. Note that $\mathbb{C}H^{n-1}$ is also an orbit of the solvable part of the Iwasawa decomposition of $\mathrm{SU}(n-1,1)$, whose Lie algebra is

$$\mathfrak{a} \oplus (\mathfrak{h} \cap \mathfrak{n}) = \mathfrak{a} \oplus \mathfrak{h}_\alpha \oplus \mathfrak{h}_{2\alpha} = \mathfrak{s} \ominus \ell,$$

where $\ell = \mathfrak{g}_\alpha \ominus \mathfrak{h}_\alpha$. This settles the case of $\mathbb{C}H^{n-1} \subset \mathbb{C}H^n$. The argument for $\mathrm{Gr}^*(2,\mathbb{R}^{n+1}) \subset \mathrm{Gr}^*(2,\mathbb{R}^{n+2})$ is similar. $\square$

To round off, we would like to make some remarks about potential generalizations of Theorem 6.3.14 and Proposition 6.3.22 and suggest a possible route toward a classification of homogeneous complex hypersurfaces in noncompact Hermitian symmetric spaces. Having dealt with standard foliations, the next logical step is to try to classify all homogeneous C2-foliations with a complex leaf. Let $M = G/K$ be an irreducible Hermitian symmetric space of noncompact type, and assume that $H \subset G$ is a closed connected subgroup inducing a homogeneous C2-foliation $\mathcal{F}$ on M. In a setting far more general that ours, Berndt, Tamaru, and Díaz-Ramos showed in [BDRT10, Prop. 2.2] that there exists a closed connected solvable subgroup F of H that has the same orbits. Let $\mathfrak{l}$ be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{f} = \mathrm{Lie}(F)$. It follows from the works of Mostow ([Mos61]) that $\mathfrak{l}$ has the form $\mathfrak{c} \oplus \mathfrak{n}_0$, where $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n}_0$ is a specific nilpotent subalgebra. There exists a Cartan involution θ on $\mathfrak{g}$ such that $\mathfrak{c} = \mathfrak{t}_0 \oplus \mathfrak{a}_0$, where $\mathfrak{t}_0 = \mathfrak{c} \cap \mathfrak{k}$ and $\mathfrak{a}_0 = \mathfrak{c} \cap \mathfrak{p}$ (see, e.g., [Kna02, Prop. 6.59]). In this notation, $\mathfrak{n}_0$ can be described as the sum of all positive eigenspaces of $\mathrm{ad}(H)$ for some $H \in \mathfrak{a}_0$. If $\mathcal{F}$ were hyperpolar, the Cartan subalgebra $\mathfrak{c}$ would have to be maximally noncompact, which means that $\mathfrak{a}_0$ would have to be a maximal abelian subspace of $\mathfrak{p}$. This was shown as part of the classification of homogeneous hyperpolar foliations in [BDRT10] (see Proposition 5.1 there). In our case, however, as soon as $\mathcal{F}$ has a complex leaf, it has no chance of being hyperpolar. This is because its section would also have to be a complex submanifold, which would contradict Corollary 6.3.13(c). This leaves us with two possibilities: either one can still prove that $\mathfrak{f}$ is contained in a maximally noncompact Borel subalgebra (perhaps by utilizing some results of this section like Theorem 6.3.12 or Lemma 6.3.15), or else no such Borel subalgebra exists—which could lead to new homogeneous complex hypersurfaces. Since every known example of such a hypersurface appears in Corollary 6.3.19, we formulate the following

Conjecture 6.3.23. *If $M = G/K$ is an irreducible Hermitian symmetric space of noncompact type, then every homogeneous C2-foliation on M with a complex leaf is induced by a Lie subgroup of some maximally noncompact Borel subgroup of G.*

In fact, it is not hard to show that Conjecture 6.3.23 is true for spaces of type $(BC)_r$. However, for C_r-spaces, it might happen in theory that $\mathfrak{a}_0 = \mathfrak{a} \ominus \mathbb{R}H_{2e_r}$ and $\mathfrak{n}_0 = \mathfrak{n} \ominus \mathfrak{g}_{2e_r}$. We do not know whether $\mathfrak{f}$ can have a nonzero projection to $\mathfrak{t}_0$ in that case.

A quick glance at Table 6.4 reveals that there exists precisely one family of irreducible noncompact Hermitian symmetric spaces that are split: $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$, $n \geq 1$. As we mentioned in Proposition 2.4.16, such a space has $\mathfrak{k}_0 = \{0\}$, which implies that all maximally noncompact Borel subalgebras of $\mathfrak{g}$ are conjugate to $\mathfrak{a} \oplus \mathfrak{n}$. Therefore, if Conjecture 6.3.23 holds, every homogeneous C2-foliation with a complex leaf on $\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$ will be standard. In that case, to complete the classification of such foliations for this series of spaces, one would only be left to investigate if a standard C2-foliation with a complex leaf can have its base leaf non-complex.

For other spaces, the situation is more convoluted. Assuming Conjecture 6.3.23 holds, every homogeneous C2-foliation with a complex leaf is induced by some subalgebra $\mathfrak{h}$ of $\mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{t}$ is a maximal abelian subspace of $\mathfrak{k}_0$. Again, if the foliation were hyperpolar, the projection of $\mathfrak{h}$ to $\mathfrak{s}$ along $\mathfrak{t}$ would be a subalgebra and induce the same foliation as $\mathfrak{h}$ ([BDRT10]). Since we know for a fact that $\mathcal{F}$ cannot be hyperpolar, one needs to come up with new approaches to tackle this problem.

Another logical continuation of this chapter would be to try to find all homogeneous complex hypersurfaces in M that arise as singular orbits of C1-actions. This problem should of course be investigated within the framework of Theorem 5.1.7. Using the results of this section as well as [BCO16, Tab. 11.1], one can show that no new homogeneous complex hypersurfaces in M can arise from parts (b)-(1) and (b)-(2) of Theorem 5.1.7. This means that one only needs to deal with the nilpotent construction. Take $j \in \{1, \ldots, r\}$ and consider the representation of L_j on $\mathfrak{n}_j^1$. In order to obtain a C1-action with a complex hypersurface as a singular orbit via the nilpotent construction, one has to sift through subspaces $\mathfrak{v}$ of $\mathfrak{n}_j^1$ of real dimension 2 that also happen to be complex in $\mathfrak{s}$. Plainly, no such $\mathfrak{v}$ exists if $\mathfrak{n}_j^1$ is a totally real subspace. Owing to Theorem 6.3.12, it is not hard to see that:

- If M of type C_r, $\mathfrak{n}_j^1$ is complex for $j < r$ and totally real for $j = r$.
- If M of type $(BC)_r$, $\mathfrak{n}_j^1$ is complex for every j.

If we disregard the case $j = r$ for M of type C_r, the problem can be formulated as follows:

(a) Find all (up to K_j^1) complex one-dimensional subspaces $\mathfrak{v} \subseteq \mathfrak{n}_j^1$ that are both protohomogeneous and admissible.

(b) Check which of the corresponding homogeneous complex hypersurfaces are isometrically congruent to those listed in Proposition 6.3.22.

As we can see, the nilpotent construction problem confined to this setting becomes a problem in the complex representation theory of reductive groups. As before, we do not expect to see any new homogeneous complex hypersurfaces arising in this way:

Conjecture 6.3.24. *Let M be an irreducible noncompact Hermitian symmetric space and S is homogeneous complex hypersurface. If S is an orbit of some isometric C1-action on M, then it is isometrically congruent to one of the hypersurfaces listed in Proposition 6.3.22.*

The only remaining case is homogeneous complex hypersurfaces that arise neither as singular orbits of C1-actions nor as leaves of homogeneous C2-foliations. Such a hypersurface has no choice but to be a nonsingular orbit of a C2-action with singular orbits. As we formulated in Conjecture 6.C, we do not expect that such complex hypersurfaces exist. However, with the current state of research on isometric actions on noncompact symmetric spaces, that conjecture is out of reach as it stands. As we mentioned in Example 2.3.27, every C2-action is infinitesimally polar. But even polar C2-actions on noncompact symmetric spaces are far from being fully classified. The only general result in this direction seems to be the recent classification of homogeneous polar C2-foliations by Díaz-Ramos and Lorenzo-Naveiro in [DRLN23]. As we explained above, the case of homogeneous foliations is more promising due to the existence of general structural results. Until such results are obtained for isometric actions with singular orbits and of cohomogeneity greater than 1, Open problem 6.B and Conjecture 6.C are likely to remain open.

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