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The relational complexity of linear groups acting on subspaces

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Abstract. The relational complexity of a subgroup G of $\mathrm{Sym}(\Omega)$ is a measure of the way in which the orbits of G on Ω^k for various k determine the original action of G. Very few precise values of relational complexity are known. This paper determines the exact relational complexity of all groups lying between $\mathrm{PSL}_n(\mathbb{F})$ and $\mathrm{PGL}_n(\mathbb{F})$, for an arbitrary field $\mathbb{F}$, acting on the set of 1-dimensional subspaces of $\mathbb{F}^n$. We also bound the relational complexity of all groups lying between $\mathrm{PSL}_n(q)$ and $\mathrm{P\Gamma L}_n(q)$, and generalise these results to the action on m-spaces for $m \geq 1$.

1 Introduction

The study of relational complexity began with work of Lachlan in model theory as a way of studying *homogeneous* relational structures: those in which every isomorphism between induced substructures extends to an automorphism of the whole structure. For the original definition, see, for example, [10]; an equivalent definition in terms of permutation groups was given by Cherlin [1] and, apart from a slight generalisation to group actions, is the one we now present.

Let Ω be an arbitrary set and let H be a group acting on Ω. Fix $k \in \mathbb{Z}$, and let $X := (x_1, \dots, x_k), Y := (y_1, \dots, y_k) \in \Omega^k$. For $r \leq k$, we say that X and Y are *r-equivalent* under H, denoted $X \sim_{H,r} Y$, if, for every r-subset of indices $\{i_1, \dots, i_r\} \subseteq \{1, \dots, k\}$, there exists an $h \in H$ such that

$$(x_{i_1}^h, \dots, x_{i_r}^h) = (y_{i_1}, \dots, y_{i_r}).$$

If $X \sim_{H,k} Y$, i.e. if $Y \in X^H$, then X and Y are *equivalent* under H. The *relational complexity* of H, denoted $\mathrm{RC}(H, \Omega)$, or $\mathrm{RC}(H)$ when Ω is clear, is the smallest $r \geq 1$ such that $X \sim_{H,r} Y$ implies $Y \in X^H$ for all $X, Y \in \Omega^k$ and all

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$k \geq r$. Equivalently, $\mathrm{RC}(H)$ is the smallest r such that r-equivalence of tuples implies equivalence of tuples. Note that $\mathrm{RC}(H) \geq 2$ if $H \neq 1$ and $|\Omega| > 1$, as X or Y may contain repeated entries.

Calculating the precise relational complexity of a group is often very difficult. A major obstacle is that if $K < H \leq \mathrm{Sym}(\Omega)$, then there is no uniform relationship between $\mathrm{RC}(K, \Omega)$ and $\mathrm{RC}(H, \Omega)$. For example, if $n \geq 4$, then the relational complexities of the regular action of C_n and natural actions of A_n and S_n are 2, $n-1$ and 2, respectively. In [1], Cherlin gave three families of finite primitive binary groups (groups with relational complexity two) and conjectured that this list was complete. In a dramatic recent breakthrough, this conjecture was proved by Gill, Liebeck and Spiga in [5]; this monograph also contains an extensive literature review.

In [1, 2], Cherlin determined the exact relational complexity of S_n and A_n in their actions on k-subsets of $\{1, \ldots, n\}$. The relational complexity of the remaining large-base primitive groups is considered in [4]. Looking at finite primitive groups more generally, Gill, Lodà and Spiga proved in [6] that if $H \leq \mathrm{Sym}(\Omega)$ is primitive and not large-base, then $\mathrm{RC}(H, \Omega) < 9\log|\Omega| + 1$ (our logarithms are to the base 2). This bound was tightened by the second and third author in [9] to $5\log|\Omega| + 1$. Both [6] and [9] bounded the relational complexity via base size, and the groups with the largest upper bounds are classical groups acting on subspaces of the natural module, and related product action groups. This motivated us to obtain further information about the relational complexity of these groups; this paper confirms that these bounds are tight, up to constants.

We now fix some notation for use throughout this paper. Let n be a positive integer, $\mathbb{F}$ a (not necessarily finite) field, $V = \mathbb{F}^n$, and $\Omega_m = \mathcal{PG}_m(V)$, the set of m-dimensional subspaces of V. We shall study the relational complexity of the almost simple groups $\overline{H}$ with $\mathrm{PSL}_n(\mathbb{F}) \trianglelefteq \overline{H} \leq \mathrm{P\Gamma L}_n(\mathbb{F})$, acting on Ω_m. We will generally work with the corresponding groups H with $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \Gamma\mathrm{L}_n(\mathbb{F})$, as these naturally have the same relational complexity when acting on Ω_m.

Several of our results focus on the case $\mathbb{F} = \mathbb{F}_q$. We begin with the following theorem of Cherlin.

Theorem 1.1 ([1, Example 3]). *The relational complexity of* $\mathrm{GL}_n(q)$ *acting on the nonzero vectors of* $\mathbb{F}_q^n$ *is equal to* n *when* $q = 2$, *and* $n+1$ *when* $q \geq 3$. *Hence also in the action on* 1*-spaces, we find that* $\mathrm{RC}(\mathrm{PGL}_n(2), \Omega_1) = n$.

More generally, for

$$\mathrm{PSL}_n(q) \trianglelefteq \overline{H} \leq \mathrm{PGL}_n(q),$$

Lodà [11, Corollary 5.2.7] shows that $\mathrm{RC}(\overline{H}, \Omega_1) < 2\log|\Omega_1| + 1$. Other results imply an alternative upper bound on $\mathrm{RC}(\overline{H}, \Omega_1)$. We first note that the *height* of a permutation group K on a set Ω, denoted $\mathrm{H}(K)$ or $\mathrm{H}(K,\Omega)$, is the maximum size of a subset Δ of Ω with the property that $K_{(\Gamma)} < K_{(\Delta)}$ for each $\Gamma \subsetneq \Delta$. It is easy to show (see [6, Lemma 2.1]) that $\mathrm{RC}(K) \leq \mathrm{H}(K) + 1$. By combining this with immediate generalisations of results of Hudson [8, §§5.3–5.4] and Lodà [11, Proposition 5.2.1], we obtain the following (for $|\mathbb{F}| = 2$, see Theorem 1.1; we also omit a few small exceptional cases for brevity).

Proposition 1.2. *Let* $\mathrm{PSL}_n(\mathbb{F}) \trianglelefteq \overline{H} \leq \mathrm{PGL}_n(\mathbb{F})$ *and* $|\mathbb{F}| \geq 3$.

(i) *Suppose that* $n = 2$, *with* $|\mathbb{F}| = q \geq 7$ *if* $\overline{H} \neq \mathrm{PGL}_2(\mathbb{F})$. *If* $|\mathbb{F}| \geq 4$, *then*

$$\mathrm{H}(\overline{H},\Omega_1) = 3 \quad \textit{and} \quad \mathrm{RC}(\overline{H},\Omega_1) = n + 2 = 4,$$

whilst $\mathrm{RC}(\mathrm{PGL}_2(3),\Omega_1) = n = 2$.

(ii) *If* $n \geq 3$, *then* $\mathrm{H}(\overline{H},\Omega_1) = 2n - 2$ *and* $\mathrm{RC}(\overline{H},\Omega_1) \leq 2n - 1$.

The following theorem gives the exact relational complexity of groups between $\mathrm{PSL}_n(\mathbb{F})$ and $\mathrm{PGL}_n(\mathbb{F})$ for $n \geq 3$, acting naturally on 1-spaces.

Theorem A. *Let* $n \geq 3$, *and let* $\mathbb{F}$ *be any field. Then the following hold.*

(i) *We have*

$$\mathrm{RC}(\mathrm{PGL}_n(\mathbb{F}),\Omega_1) = \begin{cases} n & \textit{if } |\mathbb{F}| \leq 3, \\ n + 2 & \textit{if } |\mathbb{F}| \geq 4. \end{cases}$$

(ii) *If* $\mathrm{PSL}_n(\mathbb{F}) \trianglelefteq \overline{H} < \mathrm{PGL}_n(\mathbb{F})$, *then*

$$\mathrm{RC}(\overline{H},\Omega_1) = \begin{cases} 2n - 1 & \textit{if } n = 3, \\ 2n - 2 & \textit{if } n \geq 4. \end{cases}$$

For most groups, we see that the relational complexity is very close to the bound in Proposition 1.2 (ii). However, the difference between the height and the relational complexity of $\mathrm{PGL}_n(\mathbb{F})$ increases with n when $|\mathbb{F}| \geq 3$. This addresses a recent question of Cherlin and Wiscons (see [5, p. 23]): there exists a family of finite primitive groups that are not large-base, where the difference between height and relational complexity can be arbitrarily large. Theorem A also provides infinitely many examples of almost simple groups $\overline{H}$ with $\mathrm{RC}(\mathrm{Soc}(\overline{H})) > \mathrm{RC}(\overline{H})$.

One way to interpret the gap between the relational complexity of $\mathrm{PGL}_n(\mathbb{F})$ and its proper almost simple subgroups with socle $\mathrm{PSL}_n(\mathbb{F})$ is to observe that preserving linear dependence and independence is a comparatively "local" phenomenon, requiring information about the images of n-tuples of subspaces but not (very much) more, whereas restricting determinants requires far richer information. This mimics the difference between the relational complexity of S_n and A_n in their natural actions, where requiring a map to be a permutation is "local", but requiring a permutation to be even is a "global" property.

We next bound the relational complexity of the remaining groups with socle $\mathrm{PSL}_n(q)$ that act on Ω_1. For $k \in \mathbb{Z}_{>0}$, the number of distinct prime divisors of k is denoted by $\omega(k)$, with $\omega(1) = 0$.

Theorem B. *Let* $\overline{H}$ *satisfy* $\mathrm{PSL}_n(q) \leq \overline{H} \leq \mathrm{P\Gamma L}_n(q)$*, and let*

$$e := |\overline{H} : \overline{H} \cap \mathrm{PGL}_n(q)|.$$

Suppose that $e > 1$ *so that* $q \geq 4$ *and* $\overline{H} \not\leq \mathrm{PGL}_n(q)$.

(i) *If* $n = 2$ *and* $q \geq 8$*, then*

$$4 + \omega(e) \geq \mathrm{RC}(\overline{H}, \Omega_1) \geq 4,$$

except that $\mathrm{RC}(\mathrm{P\Sigma L}_2(9), \Omega_1) = 3$.

(ii) *If* $n \geq 3$*, then*

$$2n - 1 + \omega(e) \geq \mathrm{RC}(\overline{H}, \Omega_1) \geq \begin{cases} n+2 & \textit{always}, \\ n+3 & \textit{if } \mathrm{PGL}_n(q) < \overline{H}, \\ 2n-2 & \textit{if } \overline{H} \leq \mathrm{P\Sigma L}_n(q) \neq \mathrm{P\Gamma L}_n(q). \end{cases}$$

In fact, the lower bound of $2n - 2$ holds for a larger family of groups; see Proposition 3.7.

Theorem C. *Let* $\overline{H}$ *satisfy* $\mathrm{PSL}_n(q) \leq \overline{H} \leq \mathrm{P\Gamma L}_n(q)$ *and let*

$$e := |\overline{H} : \overline{H} \cap \mathrm{PGL}_n(q)|.$$

Fix $m \in \{2, \ldots, \lfloor \frac{n}{2} \rfloor\}$*. Then*

$$(m+1)n - 2m + 2 + \omega(e) \geq \mathrm{RC}(\overline{H}, \Omega_m) \geq mn - m^2 + 1.$$

GAP [13] calculations using [3] yield $\mathrm{RC}(\mathrm{P\Gamma L}_2(3^5), \Omega_1) = 5 = 4 + \omega(5)$ and $\mathrm{RC}(\mathrm{P\Gamma L}_4(9), \Omega_1) = 8 = 7 + \omega(2)$, so the upper bounds of Theorem B cannot be improved in general. On the other hand, $\mathrm{RC}(\mathrm{P\Gamma L}_3(2^6), \Omega_1)$ achieves the lower bound of $6 = 3 + 3 < 7 = 5 + \omega(6)$. Additionally, $\mathrm{RC}(\mathrm{PSL}_4(2), \Omega_2)$ achieves the lower bound of 5 from Theorem C, while $\mathrm{RC}(\mathrm{PSL}_4(3), \Omega_2) = 6$ and

$$\mathrm{RC}(\mathrm{PGL}_4(3), \Omega_2) = \mathrm{RC}(\mathrm{PSL}_4(4), \Omega_2) = \mathrm{RC}(\mathrm{P\Gamma L}_4(4), \Omega_2) = 8.$$

It is straightforward to use our results to bound the relational complexity in terms of the degree. For example, $\mathrm{RC}(\mathrm{PGL}_n(q), \Omega_1) < \log(|\Omega_1|) + 3$. Many of our arguments also apply to the case where $\mathbb{F}$ is an arbitrary field; see Theorem 3.1, Lemmas 3.5 and 3.6, and Propositions 3.7 and 4.1.

This paper is structured as follows. In Section 2, we fix some more notation and prove some elementary lemmas, then prove upper bounds on the relational complexity of the relevant actions on 1-spaces. In Section 3, we shall prove corresponding lower bounds, and then prove Theorems A and B. Finally, in Section 4, we prove Theorem C.

2 Action on 1-spaces: Upper bounds

In this section, we present several preliminary lemmas, and then determine upper bounds for the relational complexity of groups H, with $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \mathrm{GL}_n(\mathbb{F})$, acting on Ω_1.

We begin with some notation that we will use throughout the remainder of the paper. Let $\{e_1, \ldots, e_n\}$ be a basis for V. For a set Γ, a tuple $X = (x_i)_{i=1}^k \in \Gamma^k$ and a permutation $\sigma \in \mathrm{S}_k$, we write X^σ to denote the k-tuple $(x_{1^{\sigma^{-1}}}, \ldots, x_{k^{\sigma^{-1}}})$. For a tuple $X \in \Omega_m^k$, we write $\langle X \rangle$ to denote the subspace of V spanned by all entries in X. For $i \in \{1, \ldots, k\}$, we shall write $(X \setminus x_i)$ to denote the subtuple of X obtained by deleting x_i.

In the remainder of this section, let $\Omega := \Omega_1 = \mathcal{PG}_1(V)$ and let H be a group such that

$$\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \mathrm{GL}_n(\mathbb{F}).$$

Recall from Theorem 1.1 that $\mathrm{RC}(\mathrm{GL}_n(\mathbb{F}), \Omega) = n$ when $|\mathbb{F}| = 2$. Thus we shall assume throughout this section that $|\mathbb{F}| \geq 3$ and $n \geq 2$.

We write D to denote the subgroup of diagonal matrices of $\mathrm{GL}_n(\mathbb{F})$ (with respect to the basis $\{e_1, \ldots, e_n\}$), and $\Delta := \{\langle e_i \rangle \mid i \in \{1, \ldots, n\}\}$. Observe that D is nontrivial since $|\mathbb{F}| > 2$, and that $D \cap H$ is the pointwise stabiliser $H_{(\Delta)}$. For a vector $v = \sum_{i=1}^n \alpha_i e_i \in V$, the *support* $\mathrm{supp}(v)$ of v is the set

$$\{i \in \{1, \ldots, n\} \mid \alpha_i \neq 0\}.$$

Additionally, the *support* $\mathrm{supp}(W)$ of a subset W of V is the set $\bigcup_{w \in W} \mathrm{supp}(w)$, and similarly for tuples. In particular, Δ is the set of subspaces of V with support of size 1, and $\mathrm{supp}(W) = \mathrm{supp}(\langle W \rangle)$ for all subsets W of V.

2.1 Preliminaries

We begin our study of the action of H on Ω with a pair of lemmas that will enable us to consider only tuples of a very restricted form.

Lemma 2.1. *Let $k \geq n$, and let $X, Y \in \Omega^k$ be such that $X \sim_{H,n} Y$. Additionally, let $a := \dim(\langle X \rangle)$. Then there exist $X' = (x'_1, \ldots, x'_k), Y' = (y'_1, \ldots, y'_k) \in \Omega^k$ such that*

(i) $x'_i = y'_i = \langle e_i \rangle$ *for* $i \in \{1, \ldots, a\}$, *and*

(ii) $X \sim_{H,r} Y$ *if and only if* $X' \sim_{H,r} Y'$ *for each* $r \in \{1, \ldots, k\}$.

Proof. Observe that there exists $\sigma \in \mathrm{S}_k$ such that $\langle X^\sigma \rangle = \langle x_{1^{\sigma^{-1}}}, \ldots, x_{a^{\sigma^{-1}}} \rangle$. Since $X \sim_{H,n} Y$ and $a \leq n$, the definition of a-equivalence yields $X^\sigma \sim_{H,a} Y^\sigma$. Hence there exists an $f \in H$ such that $x_{i^{\sigma^{-1}}}^f = y_{i^{\sigma^{-1}}}$ for all $i \in \{1, \ldots, a\}$, and so $\langle Y^\sigma \rangle = \langle y_{1^{\sigma^{-1}}}, \ldots, y_{a^{\sigma^{-1}}} \rangle$. Since $\mathrm{SL}_n(\mathbb{F})$ is transitive on n-tuples of linearly independent 1-spaces, there exists $h \in \mathrm{SL}_n(\mathbb{F}) \leq H$ such that

$$x_{i^{\sigma^{-1}}}^{fh} = y_{i^{\sigma^{-1}}}^h = \langle e_i \rangle \quad \text{for } i \in \{1, \ldots, a\}.$$

Define $X', Y' \in \Omega^k$ by

$$x'_i = x_{i^{\sigma^{-1}}}^{fh} \quad \text{and} \quad y'_i = y_{i^{\sigma^{-1}}}^h$$

so that $X' = X^{\sigma f h}$ and $Y' = Y^{\sigma h}$. Then $X' \sim_{H,r} Y'$ if and only if $X^\sigma \sim_{H,r} Y^\sigma$, which holds if and only if $X \sim_{H,r} Y$. □

Lemma 2.2. *Let $k \geq r \geq n$, and let $X, Y \in \Omega^k$ be such that $X \sim_{H,r} Y$. Additionally, let $a := \dim(\langle X \rangle)$ and assume that $a < n$. If $a = 1$, or if*

$$\mathrm{RC}(\mathrm{GL}_a(\mathbb{F}), \mathcal{PG}_1(\mathbb{F}^a)) \leq r,$$

then $Y \in X^H$.

Proof. If $a = 1$, then all entries of X are equal, so since $r \geq n \geq 2$, we see that $X \sim_{H,r} Y$ directly implies $Y \in X^H$. We will therefore suppose that $a \geq 2$ and $\mathrm{RC}(\mathrm{GL}_a(\mathbb{F}), \mathcal{PG}_1(\mathbb{F}^a)) \leq r$. By Lemma 2.1, we may assume without loss of generality that $\langle X \rangle = \langle Y \rangle = \langle e_1, \ldots, e_a \rangle$. As $X \sim_{H,r} Y$ and

$$\mathrm{RC}(\mathrm{GL}_a(\mathbb{F}), \mathcal{PG}_1(\mathbb{F}^a)) \leq r,$$

there exists an element $g \in \mathrm{GL}_a(\mathbb{F})$ mapping X to Y, considered as tuples of subspaces of $\langle e_1, \ldots, e_a \rangle$. We now let h be the diagonal matrix

$$\mathrm{diag}(\det(g^{-1}), 1, \ldots, 1) \in \mathrm{GL}_{n-a}(\mathbb{F})$$

and observe that $g \oplus h \in \mathrm{SL}_n(\mathbb{F})$ maps X to Y and lies in H, since $\mathrm{SL}_n(\mathbb{F})$ lies in H. Thus $Y \in X^H$. □

We now begin our study of some particularly nice k-tuples.

Lemma 2.3. *Let $k \geq n+1$, and let $X, Y \in \Omega^k$ be such that $x_i = y_i = \langle e_i \rangle$ for $i \in \{1, \dots, n\}$ and $X \sim_{H,n+1} Y$. Then $\operatorname{supp}(x_i) = \operatorname{supp}(y_i)$ for all $i \in \{1, \dots, k\}$.*

Proof. It is clear that $\operatorname{supp}(x_i) = \{i\} = \operatorname{supp}(y_i)$ when $i \in \{1, \dots, n\}$. Assume therefore that $i > n$. Since $X \sim_{H,n+1} Y$, there exists $g \in H$ such that

$$(\langle e_1 \rangle, \dots, \langle e_n \rangle, x_i)^g = (\langle e_1 \rangle, \dots, \langle e_n \rangle, y_i).$$

Observe that $g \in H_{(\Delta)} = D \cap H$, and so $\operatorname{supp}(x_i) = \operatorname{supp}(y_i)$. □

Our final introductory lemma collects several elementary observations regarding tuples of subspaces in $\overline{\Delta} := \Omega \setminus \Delta$, the set of 1-dimensional subspaces of support size greater than 1. For $r \geq 1$ and $A, B \in \overline{\Delta}^r$, we let $\mathbb{M}_{A,B}$ consist of all matrices in $\mathbb{M}_{n,n}(\mathbb{F})$ that fix $\langle e_i \rangle$ for $1 \leq i \leq n$ and map a_j into b_j for $1 \leq j \leq r$. Notice that all matrices in $\mathbb{M}_{A,B}$ are diagonal, and that if $g, h \in \mathbb{M}_{A,B}$, then $a_j^{g+h} = a_j^g + a_j^h \leq b_j$, so $\mathbb{M}_{A,B}$ is a subspace of $\mathbb{M}_{n,n}$. For an $n \times n$ matrix $g = (g_{ij})$ and a subset I of $\{1, \dots, n\}$, we write $g|_I$ to denote the submatrix of g consisting of the rows and columns with indices in I.

Lemma 2.4. *Let $r \geq 1$, and let $A = (a_1, \dots, a_r), B = (b_1, \dots, b_r) \in \overline{\Delta}^r$.*

(i) *Let a_i and a_j be (possibly equal) elements of A such that*

$$\operatorname{supp}(a_i) \cap \operatorname{supp}(a_j) \neq \varnothing,$$

and let $g \in \mathbb{M}_{A,A}$. Then $g|_{\operatorname{supp}(a_i, a_j)}$ is a scalar.

(ii) *Suppose $A \sim_{D,1} B$. Then, for $1 \leq i \leq r$, the space $(\mathbb{M}_{(a_i),(b_i)})|_{\operatorname{supp}(a_i)}$ is one-dimensional, so the dimension of $\mathbb{M}_{(a_i),(b_i)}$ is equal to $n + 1 - |\operatorname{supp}(a_i)|$.*

(iii) *For a subtuple A' of A, let $S := \{1, \dots, n\} \setminus \operatorname{supp}(A')$. Then*

$$\dim((\mathbb{M}_{A',A'})|_S) = |S|.$$

Proof. Part (i) is clear. For part (ii), by assumption, there is an invertible diagonal matrix mapping a_i to b_i, so $\operatorname{supp}(a_i) = \operatorname{supp}(b_i)$. Let k and ℓ be distinct elements of $\operatorname{supp}(a_i)$, which exist as $a_i \in \overline{\Delta}$. If $a_i^g \leq b_i$, then $e_k^g = \lambda e_k$ for some $\lambda \in \mathbb{F}$, and the value of λ completely determines the image μe_ℓ of e_ℓ under g. The result follows. Part (iii) holds since, for all $g \in \mathbb{M}_{A',A'}$, $s \in S$, $\lambda \in \mathbb{F}$ and $a \in A'$, the matrix obtained from g by adding λ to its s-th diagonal entry fixes a. □

2.2 Upper bounds for $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \mathrm{GL}_n(\mathbb{F})$ on 1-spaces

In this subsection, we will suppose that $n \geq 4$ and $|\mathbb{F}| \geq 3$, and let H be any group such that $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \mathrm{GL}_n(\mathbb{F})$. Our main result is Theorem 2.7, which gives upper bounds on $\mathrm{RC}(H, \Omega)$.

Lemma 2.5. *If $X \sim_{H,2n-2} Y$ implies that $X \sim_{H,2n-1} Y$ for all $X, Y \in \Omega^{2n-1}$ with $x_i = y_i = \langle e_i \rangle$ for $i \in \{1, \ldots, n\}$, then* $\mathrm{RC}(H, \Omega) \leq 2n - 2$.

Proof. Let k be at least $2n-1$, and let $A, B \in \Omega^k$ satisfy $A \sim_{H,2n-2} B$. Let A' be a subtuple of A of length $2n-1$, and B' the corresponding $(2n-1)$-subtuple of B. We shall show that $B' \in A'^H$ for all such A' and B' so that $A \sim_{H,2n-1} B$. It will then follow from Proposition 1.2 (ii) that $A \in B^H$, as required.

Let $a := \dim(\langle A' \rangle)$, and suppose first that $a < n$. We observe from Proposition 1.2 that if $a \geq 2$, then $\mathrm{RC}(\mathrm{GL}_a(\mathbb{F}), \mathcal{PG}_1(\mathbb{F}^a)) < 2n - 2$. As $A' \sim_{H,2n-2} B'$, Lemma 2.2 yields $B' \in A'^H$. If instead $a = n$, then by Lemma 2.1, there exist X and Y in Ω^{2n-1} such that $x_i = y_i = \langle e_i \rangle$ for each $i \in \{1, \ldots, n\}$, and such that, for all $r \geq 1$, the relations $A' \sim_{H,r} B'$ and $X \sim_{H,r} Y$ are equivalent. Now, $A' \sim_{H,2n-2} B'$, so $X \sim_{H,2n-2} Y$. If $X \sim_{H,2n-2} Y$ implies that $X \sim_{H,2n-1} Y$, then $B' \in A'^H$, as required. □

We shall therefore let X and Y be elements of Ω^{2n-1} with $x_i = y_i = \langle e_i \rangle$ for $i \in \{1, \ldots, n\}$ such that $X \sim_{H,2n-2} Y$. Additionally, for $i \in \{1, \ldots, 2n-1\}$ and $j \in \{1, \ldots, n\}$, define

$$\alpha_{ij}, \beta_{ij} \in \mathbb{F} \quad \text{so that} \quad x_i = \left\langle \sum_{j=1}^{n} \alpha_{ij} e_j \right\rangle \quad \text{and} \quad y_i = \left\langle \sum_{j=1}^{n} \beta_{ij} e_j \right\rangle.$$

Lemma 2.6. *With the notation above, if at least one of the following holds, then $Y \in X^H$.*

(i) *There exist $i, j \in \{n+1, \ldots, 2n-1\}$ with $i \neq j$ and* $\mathrm{supp}(x_i) \subseteq \mathrm{supp}(x_j)$.

(ii) *There exists a nonempty $R \subseteq \{n+1, \ldots, 2n-1\}$ with* $|\bigcap_{i \in R} \mathrm{supp}(x_i)| = 1$.

(iii) *There exists $i \in \{n+1, \ldots, 2n-1\}$ such that* $\mathrm{supp}(x_i) \geq 4$.

Proof. We begin by noting that Lemma 2.3 yields $\mathrm{supp}(y_i) = \mathrm{supp}(x_i)$ for all $i \in \{1, \ldots, 2n-1\}$.

(i) Since $X \sim_{H,2n-2} Y$, there exists an $h \in H$ mapping $(X \setminus x_i)$ to $(Y \setminus y_i)$, and such an h is necessarily diagonal, with fixed entries in $\mathrm{supp}(x_j)$ (up to scalar multiplication).

Now, let $\ell \in \{n+1,\dots,2n-1\} \setminus \{i,j\}$ (this is possible as $n \geq 4$). There exists an $h' \in H$ mapping $(X \setminus x_\ell)$ to $(Y \setminus y_\ell)$, and as before, each such h' is diagonal. Hence every matrix in $H \cap D$ mapping x_j to y_j maps x_i to y_i, and in particular, $x_i^h = y_i$, and so $X^h = Y$.

(ii) Let $\{\ell\} := \bigcap_{i \in R} \operatorname{supp}(x_i)$. Then $\alpha_{i\ell} \neq 0$ for all $i \in R$. Since $X \sim_{H,2n-2} Y$, there exists $h \in H$ such that $(X \setminus x_\ell)^h = (Y \setminus y_\ell)$. For all $k \in \{1,\dots,n\} \setminus \{\ell\}$, it follows that there exists $\gamma_k \in \mathbb{F}^*$ such that $e_k^h = \gamma_k e_k$. Thus, for each $i \in R$,

$$y_i = x_i^h = \Big\langle \sum_{k \in \operatorname{supp}(x_i)} \alpha_{ik} e_k^h \Big\rangle = \Big\langle \alpha_{i\ell} e_\ell^h + \sum_{k \in \operatorname{supp}(x_i) \setminus \{\ell\}} \alpha_{ik} \gamma_k e_k \Big\rangle.$$

Since $\alpha_{i\ell} \neq 0$, we deduce that $\operatorname{supp}(e_\ell^h) \subseteq \operatorname{supp}(y_i) = \operatorname{supp}(x_i)$. As this holds for all $i \in R$, we obtain $\operatorname{supp}(e_\ell^h) = \{\ell\}$. Thus $x_\ell^h = \langle e_\ell \rangle^h = \langle e_\ell \rangle = y_\ell$, so $X^h = Y$.

(iii) Permute the last $n-1$ coordinates of X and Y so that $\operatorname{supp}(x_{n+1}) \geq 4$. By (ii), we may assume that $x_i \notin \Delta$ for all $i \geq n+1$. We define

$$X_{n+1}^k := (x_{n+1},\dots,x_k) \quad \text{and} \quad Y_{n+1}^k := (y_{n+1},\dots,y_k)$$

for each $k \in \{n+1,\dots,2n-1\}$. As $\operatorname{supp}(x_i) = \operatorname{supp}(y_i)$ for all i, we see that $X_{n+1}^k \sim_{D,1} Y_{n+1}^k$, so X_{n+1}^k and Y_{n+1}^k satisfy the conditions of Lemma 2.4 (ii).

Suppose first that there exists $j \in \{n+2,\dots,2n-1\}$ such that

$$\mathbb{M}_{X_{n+1}^j, Y_{n+1}^j} = \mathbb{M}_{X_{n+1}^{j-1}, Y_{n+1}^{j-1}}.$$

As $X \sim_{H,2n-2} Y$, there exists $h \in H \cap D$ such that $(X \setminus x_j)^h = (Y \setminus y_j)$. Hence

$$h \in \mathbb{M}_{X_{n+1}^{j-1}, Y_{n+1}^{j-1}}, \quad \text{and so} \quad h \in \mathbb{M}_{X_{n+1}^j, Y_{n+1}^j}.$$

Therefore, $x_j^h = y_j$ and $X^h = Y$.

Hence we may assume instead that

$$\mathbb{M}_{X_{n+1}^j, Y_{n+1}^j} < \mathbb{M}_{X_{n+1}^{j-1}, Y_{n+1}^{j-1}} \quad \text{for all } j \in \{n+2,\dots,2n-1\}.$$

Then

$$\dim(\mathbb{M}_{X_{n+1}^j, Y_{n+1}^j}) \leq \dim(\mathbb{M}_{X_{n+1}^{j-1}, Y_{n+1}^{j-1}}) - 1.$$

Lemma 2.4 (ii) yields

$$\dim(\mathbb{M}_{X_{n+1}^{n+1}, Y_{n+1}^{n+1}}) \leq n-3,$$

and hence

$$\mathbb{M}_{X_{n+1}^{2n-2}, Y_{n+1}^{2n-2}} = \{0\} = \mathbb{M}_{X_{n+1}^{2n-1}, Y_{n+1}^{2n-2}},$$

contradicting our assumption. $\square$

We now prove the main result of this subsection.

Theorem 2.7. *Suppose that $n \geq 4$ and $|\mathbb{F}| \geq 3$, and let H be any group with $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \mathrm{GL}_n(\mathbb{F})$. Then $\mathrm{RC}(H, \Omega) \leq 2n-2$.*

Proof. Let $X, Y \in \Omega^{2n-1}$ be as defined before Lemma 2.6. By Lemma 2.5, it suffices to show that $Y \in X^H$, so assume otherwise. We may also assume that all subspaces in X are distinct so that

$$|\mathrm{supp}(x_i)| \in \{2, 3\} \quad \text{for each } i \in \{n+1, \ldots, 2n-1\}$$

by Lemma 2.6 (iii). For $k \in \{2, 3\}$, let R_k be the set of all $i \in \{n+1, \ldots, 2n-1\}$ such that $|\mathrm{supp}(x_i)| = k$. Then

$$|R_2| + |R_3| = n - 1. \tag{2.1}$$

Observe from Lemma 2.6 (i)–(ii) that if $i \in R_2$, then $\mathrm{supp}(x_i) \cap \mathrm{supp}(x_j) = \varnothing$ for each $j \in \{n+1, \ldots, 2n-1\} \setminus \{i\}$. Hence $2|R_2| \leq n$ and

$$|U| := \left|\bigcup_{j \in R_3} \mathrm{supp}(x_j)\right| \leq \left|\{1, \ldots, n\} \setminus \left(\dot{\bigcup_{i \in R_2}} \mathrm{supp}(x_i)\right)\right| = n - 2|R_2|. \tag{2.2}$$

Observe next that $|R_3| \geq 1$, else $|R_2| = n-1$ by (2.1), contradicting $2|R_2| \leq n$. We shall determine an expression for $|U|$ involving $|R_3|$, by considering the maximal subsets P of R_3 that correspond to pairwise overlapping supports. To do so, define a relation $\sim$ on R_3 by $i \sim j$ if $\mathrm{supp}(x_i) \cap \mathrm{supp}(x_j) \neq \varnothing$, let $\mathscr{P}$ be the set of equivalence classes of the transitive closure of $\sim$, and let $P \in \mathscr{P}$. We claim that

$$\left|\bigcup_{c \in P} \mathrm{supp}(x_c)\right| = 2 + |P|.$$

By Lemma 2.6 (i)–(ii), $|\mathrm{supp}(x_i) \cap \mathrm{supp}(x_j)| \in \{0, 2\}$ for all distinct $i, j \in R_3$. Thus our claim is clear if $|P| \in \{1, 2\}$.

If instead $|P| \geq 3$, then there exist distinct $c_1, c_2, c_3 \in P$ with $c_1 \sim c_2$ and $c_2 \sim c_3$. Let $I := \bigcap_{i=1}^{3} \mathrm{supp}(x_{c_i})$. We observe that $|I| \neq 0$, and so Lemma 2.6 (ii) shows that I has size two and is equal to $\mathrm{supp}(x_{c_1}) \cap \mathrm{supp}(x_{c_3})$. Hence $c_1 \sim c_3$ and

$$\bigcup_{i=1}^{3} \mathrm{supp}(x_{c_i}) = I \,\dot{\cup}\, \left(\dot{\bigcup_{i=1}^{3}} (\mathrm{supp}(x_{c_i}) \setminus I)\right).$$

If $|P| > 3$, then there exists $c_4 \in P \setminus \{c_1, c_2, c_3\}$ such that, without loss of generality, $c_4 \sim c_1$. As $c_1 \sim c_j$ for each $j \in \{2, 3\}$, the above argument shows that

$$\bigcap_{i \in \{1, j, 4\}} \mathrm{supp}(x_{c_i}) = I \quad \text{and} \quad \bigcup_{i=1}^{4} \mathrm{supp}(x_{c_i}) = I \,\dot{\cup}\, \left(\dot{\bigcup_{i=1}^{4}} (\mathrm{supp}(x_{c_i}) \setminus I)\right).$$

Repeating this argument inductively on $|P|$ shows that

$$\bigcup_{c \in P} \operatorname{supp}(x_c) = I \mathbin{\dot\cup} \Bigl(\dot{\bigcup_{c \in P}} (\operatorname{supp}(x_c) \setminus I)\Bigr),$$

which has size $2 + |P|$, as claimed.

Finally, let $r \geq 1$ be the number of parts of $\mathscr{P}$. As $|R_3| = \sum_{P \in \mathscr{P}} |P|$, it follows from our claim that $|U| = 2r + |R_3| \geq 2 + |R_3|$. Thus (2.2) yields

$$2 + |R_3| \leq n - 2|R_2|.$$

Hence $2|R_2| + |R_3| \leq n - 2 < n - 1$, which is equal to $|R_2| + |R_3|$ by (2.1), a contradiction. □

2.3 Upper bounds for $\mathrm{GL}_n(\mathbb{F})$ on 1-spaces

In this subsection, we determine a much smaller upper bound on $\mathrm{RC}(\mathrm{GL}_n(\mathbb{F}), \Omega)$ via our main result, Theorem 2.12. We shall assume throughout that n and $|\mathbb{F}|$ are at least 3, and write $G := \mathrm{GL}_n(\mathbb{F})$. Since D is the pointwise stabiliser of Δ in G, we will prove Theorem 2.12 by combining Lemmas 2.1 and 2.2 with information about the action of D on r-tuples A and B of subspaces in $\overline{\Delta} = \Omega \setminus \Delta$. If these tuples are $(r-1)$-equivalent under D, then by acting on one with a suitable element of $\overline{\Delta}$, we may assume that their first $r-1$ entries are equal. We shall denote the nonzero entries of elements g of D by just $g_1, \ldots, g_n$ rather than $g_{11}, \ldots, g_{nn}$ since g is necessarily diagonal.

Lemma 2.8. *Let $r \geq 3$, and let $A, B \in \overline{\Delta}^r$ be such that*

$$(a_1, \ldots, a_{r-1}) = (b_1, \ldots, b_{r-1}), \quad A \underset{D,r-1}{\sim} B, \quad \text{and} \quad B \notin A^D.$$

Let $C = \{a_1, \ldots, a_{r-1}\}$ and assume also that $\operatorname{supp}(C) = \{1, \ldots, n\}$. *Then (after reordering the basis for V and $(a_1, \ldots, a_{r-1})$ if necessary) the following statements hold.*

(i) *There exist integers*

$$2 \leq i_1 < i_2 < \cdots < i_{r-1} = n$$

such that, for each $t \in \{1, \ldots, r-1\}$, $\operatorname{supp}(a_1, \ldots, a_t)$ *is equal to* $\{1, \ldots, i_t\}$.

(ii) *Let $t \in \{1, \ldots, r-3\}$. Then*

$$\operatorname{supp}(a_t) \cap \operatorname{supp}(a_u) = \varnothing \quad \text{for all } u \in \{t+2, \ldots, r-1\}.$$

(iii) *The support of a_2 does not contain* 1.

(iv) *Let $t \in \{1, \ldots, r-1\}$. Then $i_t \in$* $\operatorname{supp}(a_t) \cap \operatorname{supp}(a_{t+1})$.

(v) *Each integer in* $\operatorname{supp}(a_r)$ *lies in the support of a unique subspace in C.*

Proof. We begin by fixing notation related to

$$a_r = \left\langle \sum_{\ell=1}^{n} \alpha_\ell e_\ell \right\rangle \quad \text{and} \quad b_r = \left\langle \sum_{\ell=1}^{n} \beta_\ell e_\ell \right\rangle.$$

Since $A \sim_{D,r-1} B$, there exists an element in D mapping a_r to b_r, and so it follows that $\mathrm{supp}(b_r) = \mathrm{supp}(a_r)$. On the other hand, $B \notin A^D$, and so $a_r \neq b_r$. Therefore, by scaling the basis vectors for a_r and b_r, there exist $j, k \in \{1, \dots, n\}$ such that $j < k, \alpha_j = \beta_j = 1$, and α_k and β_k are distinct and nonzero. Reordering $\{e_1, \dots, e_n\}$ if necessary, we may assume that $j = 1$. Then each element of D that maps a_r to b_r also maps $\langle e_1 + \alpha_k e_k \rangle$ to $\langle e_1 + \beta_k e_k \rangle$; we will use this fact throughout the proof.

(i) We show first that there is no partition of C into proper subsets C' and C'' such that $\mathrm{supp}(C') \cap \mathrm{supp}(C'') = \varnothing$, so suppose otherwise, for a contradiction. Then, as $|C'| < r - 1$ and $A \sim_{D,r-1} B$, there exists an $f \in D_{(C')}$ such that $a_r^f = b_r$. Multiplying f by a scalar if necessary, we may assume that $f_1 = 1$. Then $f_i = \beta_i / \alpha_i$ for each $i \in \mathrm{supp}(a_r)$. Similarly, there exists $g \in D_{(C'')}$ with the same properties. As $\mathrm{supp}(C)' \cap \mathrm{supp}(C)'' = \varnothing$, there exists an $h \in D$ such that $h|_{\mathrm{supp}(C)'} = f|_{\mathrm{supp}(C)'}$ and $h|_{\mathrm{supp}(C)''} = g|_{\mathrm{supp}(C)''}$. Since $\mathrm{supp}(C) = \{1, \dots, n\}$, we observe that $h|_{\mathrm{supp}(a_r)} = f|_{\mathrm{supp}(a_r)} = g|_{\mathrm{supp}(a_r)}$. Hence $a_r^h = b_r$. Furthermore, by construction, $h \in D_{(C')} \cap D_{(C'')} = D_C$. Thus $B \in A^D$, a contradiction.

By reordering $a_1, \dots, a_{r-1}$ if necessary, we may assume that $1 \in \mathrm{supp}(a_1)$. Then, by reordering $\{e_2, \dots, e_n\}$ if necessary, we may assume that $\mathrm{supp}(a_1)$ is equal to $\{1, 2, \dots, i_1\}$ for some $i_1 \geq 2$ since $a_1 \in \overline{\Delta}$. Thus the result holds for $t = 1$. We will use induction to prove the result in general, and to show that, for all $s \in \{2, \dots, r-1\}$,

$$\text{there exists } w \in \{1, \dots, s-1\} \text{ such that } \mathrm{supp}(a_s) \cap \mathrm{supp}(a_w) \neq \varnothing. \tag{2.3}$$

Let $t \in \{2, \dots, r-1\}$, let $U_{t-1} := \{a_1, \dots, a_{t-1}\}$, and assume inductively that

$$\mathrm{supp}(U_{t-1}) = \{1, 2, \dots, i_{t-1}\}.$$

If $t \geq 3$, assume also that (2.3) holds for all $s \in \{2, \dots, t-1\}$. Since C cannot be partitioned into two parts whose support has trivial intersection,

$$\mathrm{supp}(a_1, \dots, a_{t-1}) \cap \mathrm{supp}(a_t, \dots, a_{r-1}) \neq \varnothing,$$

so we may reorder $\{a_t, \dots, a_{r-1}\}$ so that (2.3) holds when $s = t$.

Suppose for a contradiction that $\mathrm{supp}(a_t) \subseteq \mathrm{supp}(U_{t-1})$. Then (2.3) (applied to each $s \in \{2, \dots, t-1\}$) and Lemma 2.4 imply that $D_{(C)}$ is equal to $D_{(C \setminus a_t)}$.

Since $A \sim_{D,r-1} B$, the latter stabiliser contains an element mapping a_r to b_r. Hence the same is true for $D_{(C)}$, contradicting the fact that $B \notin A^D$. Therefore, we can reorder $\{e_{i_{t-1}+1}, \ldots, e_n\}$ so that $\mathrm{supp}(a_t)$ contains $\{i_{t-1}+1, \ldots, i_t\}$ for some $i_t > i_{t-1}$, and the result and (2.3) follow by induction. Note in particular that $i_{r-1} = n$ since $\mathrm{supp}(C) = \{1, \ldots, n\}$.

(ii) Let $m \in \{1, \ldots, r-1\}$ be such that $\mathrm{supp}(a_m)$ contains the integer k from the first paragraph of this proof, and let $\mathcal{I} := \{1, \ldots, m\}$. Then, using (2.3) (for each $s \in \mathcal{I} \setminus \{1\}$) and Lemma 2.4 (i), we observe that every $g \in D_{(a_1,\ldots,a_m)}$ satisfies $g_1 = g_k$. Therefore, $a_r^g \neq b_r$ for all $g \in D_{(a_1,\ldots,a_m)}$. As $A \sim_{D,r-1} B$, we deduce that $m = r-1$. In particular, a_{r-1} is the unique subspace in C whose support contains k. Swapping e_k and e_n if necessary, we may assume that $k = n$.

Now, for a contradiction, suppose that

$$\mathrm{supp}(a_t) \cap \mathrm{supp}(a_u) \neq \varnothing$$

for some $t \in \{1, \ldots, r-3\}$ and $u \in \{t+2, \ldots, r-1\}$, and assume that u is the largest integer with this property. Then (2.3) and the maximality of u imply that $\mathrm{supp}(a_s) \cap \mathrm{supp}(a_{s-1}) \neq \varnothing$ for all $s \in \{u+1, \ldots, r-1\}$. It now follows from Lemma 2.4 (i), together with a further application of (2.3) to each $s \in \{2, \ldots, t\}$, that every $g \in E := D_{(a_1,\ldots,a_t,a_u,\ldots,a_{r-1})}$ satisfies $g_1 = g_n$. Therefore, $a_r^g \neq b_r$ for all $g \in E$. However, $|(a_1, \ldots, a_t, a_u, \ldots, a_{r-1})| < r-1$, contradicting the fact that $A \sim_{D,r-1} B$.

(iii)–(iv) As in the proof of (ii), we may assume that $k = n$. We observe from (ii) and (2.3) that $\mathrm{supp}(a_t) \cap \mathrm{supp}(a_{t+1}) \neq \varnothing$ for all $t \leq r-2$. Hence if $1 \in \mathrm{supp}(a_2)$, then Lemma 2.4 (i) shows that every $g \in D_{(a_2,\ldots,a_{r-1})}$ satisfies $g_1 = g_n$ (since $k = n$). This contradicts the fact that $A \sim_{D,r-1} B$, and so (iii) holds. Finally, since $\mathrm{supp}(a_t) \cap \mathrm{supp}(a_{t+1}) \neq \varnothing$ for each $t \leq r-2$, we obtain (iv) by defining $i_0 := 1$ and reordering the vectors in $\{e_{i_{t-1}+1}, \ldots, e_{i_t}\}$ if necessary. In particular, for $t = r-1$, the assumption that $i_{r-1} = n = k \in \mathrm{supp}(a_r)$ gives the result.

(v) Suppose for a contradiction that some $\ell \in \mathrm{supp}(a_r)$ lies in the support of more than one subspace in C. If $r = 3$, then $\ell \in \mathrm{supp}(a_1) \cap \mathrm{supp}(a_2)$ and we define $t := 2$. If instead $r > 3$, then (ii) implies that $\ell \in \mathrm{supp}(a_t)$ for at least one $t \in \{2, \ldots, r-2\}$. In either case, we deduce $\ell \neq 1$ since $1 \notin \mathrm{supp}(a_2)$ by (iii), and $1 \notin \mathrm{supp}(a_u)$ for $u \in \{3, \ldots, r-2\}$ by (i)–(ii). Furthermore, (i) shows that $\ell \neq i_{r-1} = n$.

Suppose first that $\alpha_\ell = \beta_\ell$ ($\neq 0$). Since the supports of $a_t, a_{t+1}, \ldots, a_{r-1}$ consecutively overlap, Lemma 2.4 (i) shows that $g_\ell = g_n$ for each $g \in D_{(a_t,\ldots,a_{r-1})}$. Since $\alpha_n \neq \beta_n$, no such g maps a_r to b_r, contradicting the fact that $A \sim_{D,r-1} B$. Hence $\alpha_\ell \neq \beta_\ell$. However, each $g \in D_{a_1}$ satisfies $g_1 = g_\ell$ if $r = 3$, as does each $g \in D_{(a_1,\ldots,a_t)}$ if $r > 3$. Again, no such matrix g maps a_r to b_r, a contradiction. □

Recall that G denotes $\mathrm{GL}_n(\mathbb{F})$, with $n, |\mathbb{F}| \geq 3$. Our next result is a key ingredient in the proof that $\mathrm{RC}(G, \Omega)$ is at most $n+2$.

Lemma 2.9. *Let $r \geq 2$, and let $A, B \in \overline{\Delta}^r$ be such that $A \sim_{D,r-1} B$ and $B \notin A^D$. Then there exists a subset Γ of Δ of size $n+2-r$ such that $B \notin A^{G_{(\Gamma)}}$.*

Proof. If $r = 2$, then set $\Gamma = \Delta$. Since $G_{(\Gamma)} = D$ and $B \notin A^D$, we are done. Assume therefore that $r \geq 3$. We will suppose for a contradiction that n is the smallest dimension for which the present lemma does not hold, for this value of r. Since $A \sim_{D,r-1} B$, we may also assume that $(a_1, \ldots, a_{r-1}) = (b_1, \ldots, b_{r-1})$. Let $C = \{a_1, \ldots, a_{r-1}\}$. As $B \notin A^D$, no element of $D_{(C)}$ maps a_r to b_r. Therefore, $B \notin A^{G_{(\Gamma)}}$ for a given subset Γ of Δ if and only if no element of $G_{(\Gamma \cup C)}$ maps a_r to b_r. We split the remainder of the proof into two cases, depending on whether or not $|\mathrm{supp}(C)| = n$.

Case $|\mathrm{supp}(C)| < n$. Let

$$\Delta_C := \{\langle e_j \rangle \mid j \in \mathrm{supp}(C)\},$$

let L be the subspace $\langle \Delta_C \rangle$ of V, and let a_ℓ and b_ℓ be the projections onto L of a_r and b_r, respectively. Lemma 2.4 (iii) shows that the diagonal entries corresponding to $\{1, \ldots, n\} \setminus \mathrm{supp}(C)$ of elements of $D_{(C)}$ can take any multiset of nonzero values. Since no element of $D_{(C)}$ maps a_r to b_r, it follows that there is no matrix in $D_{(C)}$ whose restriction to L maps a_ℓ to b_ℓ. By the minimality of n, there exists a subset Γ_C of Δ_C of size $|\Delta_C| + 2 - r$ such that no element of $\mathrm{GL}(L)_{(\Gamma_C \cup C)}$ maps a_ℓ to b_ℓ. Setting Γ to be $\Gamma_C \cup (\Delta \setminus \Delta_C)$ so that $|\Gamma| = n + 2 - r$, we observe that no element of $G_{(\Gamma \cup C)}$ maps a_r to b_r. This is a contradiction, and so the lemma follows in this case.

Case $|\mathrm{supp}(C)| = n$. In this case, Lemma 2.8 applies, so with the notation of that lemma, let

$$\Gamma := \Delta \setminus \{\langle e_{i_1} \rangle, \ldots, \langle e_{i_{r-2}} \rangle\}.$$

Then $|\Gamma| = n + 2 - r$ and $\langle e_1 \rangle, \langle e_n \rangle \in \Gamma$ since $i_1 \geq 2$ and $i_{r-1} = n$.

Let $g \in G_{(\Gamma \cup C)}$. To complete the proof, we will show that $a_r^g = a_r \neq b_r$, by showing that $g|_{\mathrm{supp}(a_r)}$ is scalar. We will first show that g is lower triangular. It is clear that g stabilises $\langle e_1 \rangle \in \Gamma$. Suppose inductively that g stabilises $\langle e_1, e_2, \ldots, e_s \rangle$ for some $s \in \{1, \ldots, n-1\}$. If $\langle e_{s+1} \rangle \in \Gamma$, then g stabilises

$$E_{s+1} := \langle e_1, e_2, \ldots, e_s \rangle + \langle e_{s+1} \rangle = \langle e_1, e_2, \ldots, e_{s+1} \rangle.$$

Otherwise, $s + 1 = i_t$ for some $t \in \{1, \ldots, r-2\}$, and then Lemma 2.8 (i) shows that

$$\{s+1\} \subsetneq \mathrm{supp}(a_t) \subseteq \{1, \ldots, s+1\}.$$

In this case, g again stabilises $\langle e_1, e_2, \dots, e_s\rangle + a_t = E_{s+1}$. Hence, by induction, g is lower triangular.

Now, let $\mathcal{I} := \{i_1, \dots, i_{r-1}\}$, let $\mathcal{U}$ be the set of integers that each lie in the support of a unique subspace in C, and let $\mathcal{J} := \mathcal{I} \cup \mathcal{U}$. We will show next that $g|_{\mathcal{J}}$ is diagonal, by fixing $j \in \mathcal{J}$ and proving that $g_{kj} = 0$ whenever $k > j$. First, if $\langle e_k\rangle \in \Gamma$, then it is clear that $g_{kj} = 0$, and so $g_{nj} = 0$. Hence we may also assume that $k \in \mathcal{I} \setminus \{i_{r-1}\}$.

Suppose inductively that

$$g_{i_u,j} = 0 \quad \text{for some } u \geq 2$$

(the base case here is $u = r - 1$ so that $i_u = n$). We will show that if $i_{u-1} > j$, then $g_{i_{u-1},j} = 0$. By Lemma 2.8 (iv), the indices $i_{u-1}, i_u \in \mathrm{supp}(a)_u$, and furthermore, Lemma 2.8 (i)–(ii) shows that $\mathrm{supp}(a_u) \cap \mathcal{I} = \{i_{u-1}, i_u\}$. Thus, by the previous paragraph and our inductive assumption,

$$g_{kj} = 0 \quad \text{for all } k \in \mathrm{supp}(a_u) \setminus \{j, i_{u-1}\}.$$

In fact, Lemma 2.8 (i)–(ii) shows that each integer in $\mathrm{supp}(a_u)$ less than i_{u-1} lies in $\mathrm{supp}(a_{u-1})$. As $i_{u-1} > j \in \mathcal{J}$, we deduce from the definition of $\mathcal{J}$ that $j \notin \mathrm{supp}(a_u)$. Thus $g_{kj} = 0$ for all $k \in \mathrm{supp}(a_u) \setminus \{i_{u-1}\}$. As g stabilises a_u, we deduce that $g_{i_{u-1},j} = 0$. Therefore, by induction, $g_{kj} = 0$ for all $k \neq j$, and so $g|_{\mathcal{J}}$ is diagonal.

Finally, we will show that $g|_{\mathcal{J}}$ is scalar. Let $j, k \in \mathcal{J} \cap \mathrm{supp}(a_t)$ for some $t \in \{1, \dots, r-1\}$. As g stabilises a_t, and as $g|_{\mathcal{J}}$ is diagonal, we deduce that

$$g_{jj} = g_{kk}. \tag{2.4}$$

Now, by Lemma 2.8 (iv), $i_t \in \mathrm{supp}(a_t) \cap \mathrm{supp}(a_{t+1})$ for each $t \in \{1, \dots, r-2\}$, so $i_t \in \mathcal{J} \cap \mathrm{supp}(a_t) \cap \mathrm{supp}(a_{t+1})$. Thus, starting from $t = 1$ and proceeding by induction on t, it follows from (2.4) that $g_{jj} = g_{kk}$ for all $j, k \in \mathcal{J}$, i.e. $g|_{\mathcal{J}}$ is a scalar. Since $\mathrm{supp}(a_r) \subseteq \mathcal{J}$ by Lemma 2.8 (v), we deduce that $a_r^g = a_r \neq b_r$, as required. □

The following lemma is strengthening of Lemma 2.9 in the case $|\mathbb{F}| = 3$ and $r = 2$, in which the subset Γ now has size $n + 1 - r = n - 1$.

Lemma 2.10. *Suppose $|\mathbb{F}| = 3$, and let $A, B \in \overline{\Delta}^2$. Suppose also that $A \sim_{D,1} B$ and $B \notin A^D$. Then there exists a subset Γ of Δ of size $n - 1$ such that $B \notin A^{G_{(\Gamma)}}$.*

Proof. Since $A \sim_{D,1} B$, without loss of generality, $a_1 = b_1$, and there exists an element of D mapping a_2 to b_2. Hence a_2 and b_2 have equal supports. Reordering the basis for V if necessary, we may also assume that $\operatorname{supp}(a_1) = \{1, 2, \ldots, m\}$ for some $m \geq 2$. Then, by Lemma 2.4, the upper left $m \times m$ submatrix of each matrix in D_{a_1} is a scalar, while the remaining diagonal entries can be chosen independently. As $B \notin A^D$, no matrix in D_{a_1} maps a_2 to b_2. We may therefore assume (by reordering the basis vectors in $\{e_1, \ldots, e_m\}$ and/or swapping A and B if necessary) that the projections of a_2 and b_2 onto $\langle e_1, e_2\rangle$ are $\langle e_1 + e_2\rangle$ and $\langle e_1 - e_2\rangle$, respectively.

Now, let $\Gamma := \Delta \setminus \{\langle e_2\rangle\}$, let $g \in G_{(\Gamma \cup \{a_1\})}$, and notice that g is diagonal outside of the second row. Write a_1 as

$$\left\langle \sum_{i=1}^{m} \alpha_i e_i \right\rangle,$$

with $\alpha_1 = 1$ and $\alpha_i \neq 0$ for all $i \in \{2, \ldots, m\}$. Since $a_1^g = a_1$, we deduce that, without loss of generality, the top left 2×2 submatrix of g is

$$\begin{pmatrix} 1 & 0 \\ g_{21} & 1 + \alpha_2 g_{21} \end{pmatrix}.$$

Let v be the projection of $(e_1 + e_2)^g$ onto $\langle e_1, e_2\rangle$. Recall that $\alpha_2 \neq 0$, and note that $g_{22} \neq 0$ as g is invertible. Hence if $g_{21} = 1$, then $\alpha_2 = 1$ and $v = -e_1 - e_2$; if $g_{21} = -1$, then $\alpha_2 = -1$ and $v = -e_2$; and if $g_{21} = 0$, then $v = e_1 + e_2$. Hence, in each case, v does not span $\langle e_1 - e_2\rangle = b_2|_{\langle e_1, e_2\rangle}$. Therefore, $a_2^g \neq b_2$, and hence $B \notin A^{G_{(\Gamma)}}$. □

Although the next result holds for all $\mathbb{F}$, it will only be useful in the case $|\mathbb{F}| = 3$.

Proposition 2.11. *Let $X, Y \in \Omega^{n+1}$ such that $X \sim_{G,n} Y$, and suppose $\langle X\rangle = V$. Then $Y \in X^G$.*

Proof. As $\dim(\langle X\rangle) = n$, we may assume by Lemma 2.1 that $x_i = y_i = \langle e_i\rangle$ for $i \in \{1, \ldots, n\}$. Let $S := \operatorname{supp}(x_{n+1})$ and $T := \operatorname{supp}(y_{n+1})$. We will show that $S = T$; it will then follow that there exists an element of $D = G_{(\Delta)}$ mapping x_{n+1} to y_{n+1}, and so $Y \in X^G$.

If $S = \{1, \ldots, n\} = T$, then we are done. Otherwise, exchanging X and Y if necessary (note that $\langle Y\rangle = V$), we may assume that there exists an element $t \in \{1, \ldots, n\} \setminus S$. Let $\Gamma := \Delta \setminus \{\langle e_t\rangle\}$. Then, since $X \sim_{G,n} Y$, there exists an element of $G_{(\Gamma)}$ mapping x_{n+1} to y_{n+1}. As $G_{(\Gamma)}$ stabilises each subspace $\langle e_i\rangle$ with $i \in S$, it follows that $S = T$, as required. □

We are now able to prove this section's main theorem.

Theorem 2.12. *Suppose that n and $|\mathbb{F}|$ are at least 3. Then $\mathrm{RC}(\mathrm{GL}_n(\mathbb{F}), \Omega)$ is at most $n+2$. Moreover, $\mathrm{RC}(\mathrm{GL}_n(3), \Omega) \le n$.*

Proof. Let $k \in \{n, n+1, n+2\}$, with $k = n+2$ if $|\mathbb{F}| > 3$. Additionally, let X and Y be tuples in Ω^u with $u > k$ and $X \sim_{G,k} Y$, where $G = \mathrm{GL}_n(\mathbb{F})$. It suffices to prove that $Y \in X^G$. Suppose, for a contradiction, that n is the minimal dimension for which the theorem does not hold (for a fixed $\mathbb{F}$), and that $Y \notin X^G$. Then, for each $m \in \{2, \dots, n-1\}$, using Proposition 1.2 (i) in the case $m = 2$, we obtain $\mathrm{RC}(\mathrm{GL}_m(\mathbb{F}), \mathcal{PG}_1(\mathbb{F}^m)) < k$. Since $Y \notin X^G$, Lemma 2.2 yields $\langle X \rangle = V$. Hence, by Lemma 2.1, we may assume without loss of generality[1] that

$$x_i = y_i = \langle e_i \rangle \quad \text{for } i \in \{1, \dots, n\},$$

and furthermore that all subspaces in X are distinct, so that $x_i, y_i \in \overline{\Delta}$ for each $i \ge n+1$.

We will first consider the case $k \ge n+1$. Since $X \sim_{G,n+1} Y$, Lemma 2.3 yields $\mathrm{supp}(x_i) = \mathrm{supp}(y_i)$ for all i. However, $Y \notin X^G$. Hence there exist an integer $r \ge 2$ and subtuples A of X and B of Y, with $A, B \in \overline{\Delta}^r$, such that $(x_1, \dots, x_n, a_1, \dots, a_r)$ and $(x_1, \dots, x_n, b_1, \dots, b_r)$ are $(n+r-1)$-equivalent, but not equivalent, under G. Equivalently, $A \sim_{D,r-1} B$ and $B \notin A^D$.

If $k = n+2$, then by Lemma 2.9, there exists a set $\Gamma := \{\langle e_{i_1} \rangle, \dots, \langle e_{i_{k-r}} \rangle\}$ such that $B \notin A^{G_{(\Gamma)}}$. However, this means that the subtuples

$$(x_{i_1}, \dots, x_{i_{k-r}}, a_1, \dots, a_r) \quad \text{and} \quad (x_{i_1}, \dots, x_{i_{k-r}}, b_1, \dots, b_r)$$

are not equivalent under G. This contradicts the assumption that $X \sim_{G,k} Y$. Hence, in this case, $Y \in X^G$, as required, so $\mathrm{RC}(G) \le n+2$. If $|\mathbb{F}| > 3$, then we are done.

Therefore, assume for the rest of the proof that $|\mathbb{F}| = 3$ and suppose first that $k = n+1$. By the previous paragraph, $\mathrm{RC}(G) \le n+2$. Therefore, to prove that $\mathrm{RC}(G) \le k$, it suffices to show that $X \sim_{G,n+2} Y$ whenever $X \sim_{G,k} Y$. Thus, by replacing X and Y by suitable subtuples, if necessary, we may assume that $u = n+2$. In this case, $r = 2$, and by Lemma 2.10, there exists a subset Γ of Δ of size $k - r$ such that $B \notin A^{G_{(\Gamma)}}$. Arguing as in the previous paragraph, this contradicts the assumption that $X \sim_{G,k} Y$. Thus $\mathrm{RC}(G) \le n+1$.

Finally, suppose that we have $k = n$. Since $\mathrm{RC}(G) \le n+1$, we may assume that $u = n+1$. However, since $X \sim_{G,n} Y$ and $\langle X \rangle = V$, Proposition 2.11 shows that $Y \in X^G$. Therefore, $\mathrm{RC}(G) \le n$. □

[1] If the basis vectors for V are reordered, as required by several of this section's earlier proofs, then we can reorder the subspaces in $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ in the same way to preserve this equality.

3 Action on 1-spaces: Lower bounds

In this section, we again assume that $|\mathbb{F}| \geq 3$, and write $\Omega := \Omega_1 = \mathcal{PG}_1(V)$. We drop the assumption that $n \geq 3$ and permit $n = 2$. We shall now prove lower bounds for the relational complexity of each group H satisfying

$$\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \Gamma\mathrm{L}_n(\mathbb{F}),$$

acting on Ω.

For some results in this section, we will assume that $\mathbb{F} = \mathbb{F}_q$ is finite, and when doing so, we fix a primitive element ω, and assume that $q = p^f$ for p prime. Additionally, we will write

$$\mathrm{P\Gamma L}_n(q)/\mathrm{PSL}_n(q) = \langle \delta, \phi \rangle, \quad \text{with } \mathrm{PGL}_n(q)/\mathrm{PSL}_n(q) = \langle \delta \rangle.$$

Here, the automorphism ϕ can be chosen to be induced by the automorphism of $\mathrm{GL}_n(q)$ which raises each matrix entry to its p-th power, and with a slight abuse of notation, we will also write ϕ to denote this automorphism of $\mathrm{GL}_n(q)$, and to denote a generator for $\mathrm{Aut}(\mathbb{F}_q)$. If $\mathbb{F}$ is an arbitrary field, then the group $\Gamma\mathrm{L}_n(\mathbb{F})$ is still a semi-direct product of $\mathrm{GL}_n(\mathbb{F})$ by $\mathrm{Aut}(\mathbb{F})$ (see, for example, [12, Theorem 9.36]), but of course, $\mathrm{GL}_n(\mathbb{F})/\mathrm{SL}_n(\mathbb{F})$ and $\mathrm{Aut}(\mathbb{F})$ need not be cyclic.

We let $Z := Z(\mathrm{GL}_n(\mathbb{F}))$ and will write I_n for the $n \times n$ identity matrix, and E_{ij} for the $n \times n$ matrix with 1 in the (i, j)-th position and 0 elsewhere. We write $A \oplus B$ for the block diagonal matrix with blocks A and B.

Our first result is completely general and easy to prove, although we shall later prove much tighter bounds for various special cases.

Theorem 3.1. *Let $\mathbb{F}$ be arbitrary, and let H satisfy $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \Gamma\mathrm{L}_n(\mathbb{F})$. Then $\mathrm{RC}(H, \Omega) \geq n$.*

Proof. Define $X, Y \in \Omega^n$ by $x_i = y_i = \langle e_i \rangle$ for $i \in \{1, \ldots, n-1\}$, with

$$x_n = \left\langle \sum_{i=1}^{n} e_i \right\rangle \quad \text{and} \quad y_n = \left\langle \sum_{i=1}^{n-1} e_i \right\rangle.$$

Then $\dim(\langle X \rangle) = n$ and $\dim(\langle Y \rangle) = n - 1$, so no element of $\Gamma\mathrm{L}_n(\mathbb{F})$ maps X to Y. Hence $Y \notin X^H$.

Now, let $h_\ell := I_n - E_{\ell n}$ for each $\ell \in \{1, \ldots, n-1\}$, and $h_n := I_n$. Then

$$h_\ell \in \mathrm{SL}_n(\mathbb{F}) \leq H \quad \text{and} \quad (X \setminus x_\ell)^{h_\ell} = (Y \setminus y_\ell) \quad \text{for each } \ell \in \{1, \ldots, n\}.$$

Therefore, $X \sim_{H,n-1} Y$, and so the result follows. □

Our next two results focus on the special cases $n = 2$ and $n = 3$.

Lemma 3.2. *Assume that* $q \geq 8$, *and let* H *satisfy* $\mathrm{SL}_2(q) \trianglelefteq H \leq \Gamma\mathrm{L}_2(q)$. *Then* $\mathrm{RC}(H) \geq 4$, *except that* $\mathrm{RC}(\Sigma\mathrm{L}_2(9)) = 3$.

Proof. The claim about $\Sigma\mathrm{L}_2(9)$ is an easy computation in GAP using [3], so exclude this group from now on. We divide the proof into two cases. For each, we define $X, Y \in \Omega^4$ such that $X \sim_{H,3} Y$ but $Y \notin X^H$. In both cases, we set $(X \setminus x_4) = (Y \setminus y_4) = (\langle e_1\rangle, \langle e_2\rangle, \langle e_1 + e_2\rangle)$.

Case (a): either q is even, or $H \nleq \langle Z, \Sigma\mathrm{L}_2(q)\rangle$, where $Z = Z(\mathrm{GL}_n(\mathbb{F}))$. If q is odd, then let $\alpha \in \mathbb{F}_p^* \setminus \{1\}$, and otherwise, let $\alpha = \omega^3$ so that α is not in the orbit $\omega^{\langle\phi\rangle}$. Then let $x_4 = \langle e_1 + \omega e_2\rangle$ and $y_4 = \langle e_1 + \alpha e_2\rangle$.

The stabiliser in H of $(X \setminus x_4) = (Y \setminus y_4)$ is contained in $\langle Z, \phi\rangle$. As $\alpha \notin \omega^{\langle\phi\rangle}$, no element of this stabiliser maps x_4 to y_4, and so $Y \notin X^H$. On the other hand, for each $j \in \{1, 2, 3, 4\}$, the matrix $g_j \in \mathrm{GL}_2(q)$ given below maps $(X \setminus x_j)$ to $(Y \setminus y_j)$:

$$g_1 = \begin{pmatrix} 1 & (\alpha-\omega)(1-\omega)^{-1} \\ 0 & 1-(\alpha-\omega)(1-\omega)^{-1} \end{pmatrix},$$

$$g_2 = \begin{pmatrix} 1-(\omega\alpha^{-1}-1)(\omega-1)^{-1} & 0 \\ (\omega\alpha^{-1}-1)(\omega-1)^{-1} & 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 0 \\ 0 & \alpha\omega^{-1} \end{pmatrix}, \quad g_4 = I_2.$$

If q is even, then some scalar multiple of g_j lies in H for all j, so $X \sim_{H,3} Y$, and we are done. If instead q is odd, then our assumption that $H \nleq \langle Z, \Sigma\mathrm{L}_2(q)\rangle$ implies that H contains a scalar multiple of an element $\mathrm{diag}(\omega, 1)\phi^i$ for some $i \geq 0$, as $\mathrm{diag}(\omega, 1)$ induces the automorphism δ of $\mathrm{PSL}_2(q)$. Hence, for each j, there exists $\phi^{i_j} \in \mathrm{Aut}(\mathbb{F}_q)$ such that a scalar multiple of $g_j\phi^{i_j}$ lies in H. Since $\alpha \in \mathbb{F}_p^*$, each ϕ^{i_j} fixes Y, and thus $X \sim_{H,3} Y$.

Case (b): q is odd and $H \leq \langle Z, \Sigma\mathrm{L}_2(q)\rangle$. Since $H \neq \Sigma\mathrm{L}_2(9)$ and since Proposition 1.2 (i) yields the result when $H = \mathrm{SL}_2(9)$, we may assume that $q > 9$. We generalise Hudson's [8, §5.4] proof that $\mathrm{RC}(\mathrm{SL}_2(q), \Omega) \geq 4$. First, let

$$\mathcal{S} := \mathbb{F}_q \setminus \{0, 1, -1\} \quad \text{and} \quad \mathcal{T} := \mathbb{F}_q \setminus \{0, 1\},$$

and for each $\lambda \in \mathcal{S}$, define a map $\theta_\lambda \colon \mathcal{T} \to \mathbb{F}_q$ by $\mu \mapsto (1-\lambda^2\mu)(1-\mu)^{-1}$. We will show that there exist elements $\lambda \in \mathcal{S}$ and $\tau \in \mathcal{T}$ satisfying the following conditions:

(i) $(\tau)\theta_\lambda$ is a square in $\mathbb{F}_q^*$, and

(ii) no automorphism of $\mathbb{F}_q$ maps τ to $\lambda^2\tau$.

It is easy to see that, for each $\lambda \in \mathcal{S}$, the image $\mathrm{im}(\theta_\lambda) = \mathbb{F}_q \setminus \{1, \lambda^2\}$, so the map θ_λ is injective, and the preimage of any nonzero square in $\mathrm{im}(\theta_\lambda)$ lies in $\mathcal{T}$ and satisfies condition (i). Hence, for each $\lambda \in \mathcal{S}$, there are precisely $(q-1)/2-2$ choices for $\tau \in \mathcal{T}$ satisfying condition (i).

Given $\lambda \in \mathcal{S}$, since $\lambda^2 \neq 1$, condition (ii) is equivalent to requiring that

$$\lambda^2\tau \neq \tau^{p^k} \quad \text{for all } k \in \{1, \dots, f-1\},$$

i.e. $\lambda^2 \neq \tau^{p^k-1}$ for all k. There are exactly $(q-3)/2 = (q-1)/2-1$ distinct squares of elements of $\mathcal{S}$, and precisely $(q-1)/(p-1)$ elements in $\mathbb{F}_q^*$ that are $(p-1)$-th powers. Hence if $p > 3$, then there exists $\lambda \in \mathcal{S}$ such that λ^2 is not a $(p-1)$-th power in $\mathbb{F}_q$. Observe that then λ^2 is not a (p^k-1)-th power for any k, and so this λ and any corresponding τ from the previous paragraph satisfy both conditions.

Suppose instead that $p = 3$, and fix $\lambda \in \mathcal{S}$. The number of elements $\tau \in \mathbb{F}_{3^f}^*$ not satisfying (ii), i.e. with $\lambda^2 = \tau^{3^k-1}$ for some $k \in \{1, \dots, f-1\}$, is at most

$$\begin{aligned}(3-1) + (3^2-1) + \cdots + (3^{f-1}-1)& \\ &= (3+3^2+\cdots+3^{f-1}) - (f-1).\end{aligned}$$

On the other hand, we established that the number of elements $\tau \in \mathcal{T}$ satisfying (i) is equal to

$$\begin{aligned}(3^f-1)/2-2 &= (3-1)(1+3+3^2+\cdots+3^{f-1})/2-2 \\ &= (3+3^2+\cdots+3^{f-1}) - 1.\end{aligned}$$

Since $q > 9$, and hence $f > 2$, there again exists $\tau \in \mathcal{T}$ satisfying both conditions.

Finally, fix such a $\lambda \in \mathcal{S}$ and $\tau \in \mathcal{T}$, and complete the definition of $X, Y \in \Omega^4$ by setting

$$x_4 = \langle e_1 + \tau e_2 \rangle \quad \text{and} \quad y_4 = \langle e_1 + \lambda^2 \tau e_2 \rangle.$$

The stabiliser in H of $(X \setminus x_4) = (Y \setminus y_4)$ is contained in $\langle Z, \phi \rangle$. By condition (ii), no such element maps x_4 to y_4, so $Y \notin X^H$. However, the proof of [8, Theorem 5.4.6] uses condition (i) to exhibit explicit elements of $\mathrm{SL}_2(q)$ mapping each 3-tuple of X to the corresponding 3-tuple of Y. Therefore, $X \sim_{H,3} Y$, and the result follows. □

Lemma 3.3. *Assume that* $\mathrm{PSL}_3(\mathbb{F}) \neq \mathrm{PGL}_3(\mathbb{F})$*, and let H be any group satisfying* $\mathrm{SL}_3(\mathbb{F}) \trianglelefteq H \leq \Gamma\mathrm{L}_3(\mathbb{F})$*. If $\mathbb{F}$ is finite, or if $H \leq \mathrm{GL}_3(\mathbb{F})$, then* $\mathrm{RC}(H) \geq 5$.

Proof. If $|\mathbb{F}| = 4$, then we verify the result in GAP using [3], so assume that $|\mathbb{F}| \geq 7$. If $\mathbb{F}$ is finite, then let $\lambda := \omega$, whilst if $\mathbb{F}$ is infinite, then let λ be any element of $\mathbb{F}^*$ of multiplicative order at least 3. Define $X, Y \in \Omega^5$ by

$$\begin{aligned} x_i &= y_i = \langle e_i \rangle \quad \text{for } i \in \{1, 2, 3\}, \\ x_4 &= y_4 = \langle e_1 + e_2 + e_3 \rangle, \\ x_5 &= \langle e_1 + \lambda e_2 + \lambda^2 e_3 \rangle, \\ y_5 &= \langle e_1 + \lambda^{-1} e_2 + \lambda^{-2} e_3 \rangle \end{aligned}$$

so that $x_5 \neq y_5$.

We first show that $Y \notin X^H$. The stabiliser in H of $(X \setminus x_5) = (Y \setminus y_5)$ lies in $H \cap \langle Z, \mathrm{Aut}(\mathbb{F}) \rangle$, so if $\mathbb{F}$ is infinite, then we are done. Assume therefore that $\mathbb{F} = \mathbb{F}_q$. If $x_5^{\phi^i} = y_5$, then $\lambda^{p^i} = \lambda^{-1} = \lambda^{p^f - 1}$. Since $i \in \{0, \dots, f-1\}$ and $\lambda = \omega$, we deduce that $(p, f, i) \in \{(2, 2, 1), (3, 1, 0)\}$, contradicting $q \geq 7$. Thus $Y \notin X^H$.

Next, for all $\mathbb{F}$, we show that $X \sim_{H,4} Y$. Let

$$g_1 := \begin{pmatrix} \lambda & \lambda + 1 & \lambda + \lambda^{-1} \\ 0 & -1 & 0 \\ 0 & 0 & -\lambda^{-1} \end{pmatrix}, \qquad g_2 := \begin{pmatrix} -\lambda & 0 & 0 \\ \lambda + 1 & 1 & 1 + \lambda^{-1} \\ 0 & 0 & -\lambda^{-1} \end{pmatrix},$$

$$g_3 := \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -1 & 0 \\ \lambda + \lambda^{-1} & 1 + \lambda^{-1} & \lambda^{-1} \end{pmatrix}, \quad g_4 := \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}, \quad g_5 := I_3.$$

Observe that $\det(g_\ell) = 1$ for each $\ell \in \{1, \dots, 5\}$, and so $g_\ell \in \mathrm{SL}_3(\mathbb{F}) \leq H$. It is also easy to check that $(X \setminus x_\ell)^{g_\ell} = (Y \setminus y_\ell)$ for each ℓ. Thus $X \sim_{H,4} Y$, and so $\mathrm{RC}(H) \geq 5$. □

Our remaining results hold for all sufficiently large n. The first is specific to $\mathrm{GL}_n(\mathbb{F})$.

Proposition 3.4. *If* $n \geq 3$ *and* $|\mathbb{F}| \geq 4$, *then* $\mathrm{RC}(\mathrm{GL}_n(\mathbb{F}), \Omega) \geq n + 2$.

Proof. As $|\mathbb{F}| \geq 4$, there exists an element $\lambda \in \mathbb{F}^*$ such that $\lambda \neq \lambda^{-1}$ (so $\lambda \neq -1$). Define $X, Y \in \Omega^{n+2}$ by

$$\begin{aligned} x_i &= y_i = \langle e_i \rangle \quad \text{for } i \in \{1, \dots, n\}, \\ x_{n+1} &= y_{n+1} = \left\langle \sum_{i=1}^{n} e_i \right\rangle, \\ x_{n+2} &= \langle e_1 + \lambda e_2 \rangle \quad \text{and} \quad y_{n+2} = \langle e_1 + \lambda^{-1} e_2 \rangle. \end{aligned}$$

The stabiliser in $\mathrm{GL}_n(\mathbb{F})$ of $(X \setminus x_{n+2}) = (Y \setminus y_{n+2})$ is the group of scalar matrices, so it follows that $Y \notin X^{\mathrm{GL}_n(\mathbb{F})}$. Additionally, it is easily verified that, for each $j \in \{1, \dots, n+2\}$, the matrix $g_j \in \mathrm{GL}_n(q)$ given below maps $(X \setminus x_j)$ to $(Y \setminus y_j)$:

$$\begin{aligned} g_1 &= \begin{pmatrix} \lambda & 1+\lambda \\ 0 & -1 \end{pmatrix} \oplus \lambda I_{n-2}, \\ g_2 &= \begin{pmatrix} -1 & 0 \\ 1+\lambda^{-1} & \lambda^{-1} \end{pmatrix} \oplus \lambda^{-1} I_{n-2}, \\ g_{n+1} &= \mathrm{diag}(\lambda, \lambda^{-1}, \lambda, \dots, \lambda), \\ g_j &= g_{n+1} + (\lambda - \lambda^{-1}) E_{j2} \quad \text{for } j \in \{3, \dots, n\}, \\ g_{n+2} &= I_n. \end{aligned}$$

Hence $X \sim_{\mathrm{GL}_n(\mathbb{F}),n+1} Y$, and so the result follows. □

In the light of Proposition 3.4, the next result in particular bounds the relational complexity of all remaining groups when $\mathrm{PSL}_n(\mathbb{F}) = \mathrm{PGL}_n(\mathbb{F})$.

Lemma 3.5. *Let $\mathbb{F}$ be arbitrary, assume that $n \geq 3$, and let H be any group satisfying $\mathrm{GL}_n(\mathbb{F}) \trianglelefteq H \leq \Gamma\mathrm{L}_n(\mathbb{F})$ and $H \neq \mathrm{GL}_n(\mathbb{F})$. Then $\mathrm{RC}(H) \geq n+3$.*

Proof. Since $\mathrm{GL}_n(\mathbb{F})$ is a proper subgroup of H, there exist a nontrivial

$$\psi \in H \cap \mathrm{Aut}(\mathbb{F})$$

and an element $\lambda \in \mathbb{F}^*$ with $\lambda^\psi \neq \lambda$. We define $X, Y \in \Omega^{n+3}$ by $x_i = y_i = \langle e_i \rangle$ for $i \in \{1, \dots, n\}$,

$$\begin{aligned} x_{n+1} &= y_{n+1} = \left\langle \sum_{i=1}^{n} e_i \right\rangle, \\ x_{n+2} &= y_{n+2} = \langle e_1 + e_2 + \lambda e_3 \rangle, \\ x_{n+3} &= \langle e_1 + \lambda e_2 \rangle, \quad y_{n+3} = \langle e_1 + \lambda^\psi e_2 \rangle. \end{aligned}$$

We claim that $X \sim_{H,n+2} Y$, but $Y \notin X^H$, from which the result will follow.

The stabiliser in H of

$$(x_1, \dots, x_{n+1}) = (y_1, \dots, y_{n+1})$$

is contained in $\langle Z, \mathrm{Aut}(\mathbb{F}) \rangle$. However, no element of $\langle Z, \mathrm{Aut}(\mathbb{F}) \rangle$ maps

$$(x_{n+2}, x_{n+3}) \quad \text{to} \quad (y_{n+2}, y_{n+3}),$$

so $Y \notin X^H$. The reader may verify that, for each $j \in \{1, \ldots, n+3\}$, the element $h_j \in \langle \mathrm{GL}_n(\mathbb{F}), \psi \rangle \leq H$ given below maps $(X \setminus x_j)$ to $(Y \setminus y_j)$, where we define $\tau := (\lambda - 1)^{-1}$ (notice that $\lambda \neq 1$):

$$h_1 = \begin{pmatrix} 1 & -\tau(\lambda^{\psi} - \lambda) \\ 0 & 1 + \tau(\lambda^{\psi} - \lambda) \end{pmatrix} \oplus I_{n-2},$$

$$h_2 = \begin{pmatrix} 1 - \tau(\lambda(\lambda^{-1})^{\psi} - 1) & 0 \\ \tau(\lambda(\lambda^{-1})^{\psi} - 1) & 1 \end{pmatrix} \oplus I_{n-2},$$

$$h_3 = \left(\begin{pmatrix} 1 - \tau(\lambda(\lambda^{-1})^{\psi^{-1}} - 1) & 0 & 0 \\ 0 & 1 - \tau(\lambda(\lambda^{-1})^{\psi^{-1}} - 1) & 0 \\ \tau(\lambda(\lambda^{-1})^{\psi^{-1}} - 1) & \tau(\lambda(\lambda^{-1})^{\psi^{-1}} - 1) & 1 \end{pmatrix} \oplus I_{n-3} \right) \psi,$$

$$h_j = \big(\mathrm{diag}(1, 1, \lambda^{-1}\lambda^{\psi^{-1}}, 1, \ldots, 1) + (1 - \lambda^{-1}\lambda^{\psi^{-1}}) E_{j3}\big)\psi \quad \text{for } j \in \{4, \ldots, n\},$$

$$h_{n+1} = \mathrm{diag}(1, 1, \lambda^{-1}\lambda^{\psi^{-1}}, 1, \ldots, 1)\psi, \quad h_{n+2} = \psi, \quad h_{n+3} = I_n.$$

Hence $X \sim_{H,n+2} Y$, and the result follows. □

Lemma 3.6. *Let* $\mathbb{F}$ *be arbitrary, assume that* $n \geq 4$, *and let* H *be any group satisfying* $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \Gamma\mathrm{L}_n(\mathbb{F})$ *and* $H \nleq \mathrm{GL}_n(\mathbb{F})$. *Then* $\mathrm{RC}(H) \geq n+2$.

Proof. Since $H \nleq \mathrm{GL}_n(\mathbb{F})$, there exist elements $h\psi \in H$ and $\lambda \in \mathbb{F}^*$ such that $h \in \mathrm{GL}_n(q)$, $\psi \in \mathrm{Aut}(\mathbb{F}_q)$, and $\lambda^{\psi} \neq \lambda$. Let $X, Y \in \Omega^{n+2}$ be as in the proof of Lemma 3.5, but supported only on the first $n-1$ basis vectors so that $\langle e_n \rangle$ lies in neither X nor Y, and $x_n = y_n = \langle \sum_{i=1}^{n-1} e_i \rangle$. Just as in that proof, one may check that $Y \notin X^H$, but $X \sim_{H,n+1} Y$. □

The next result applies, in particular, to all groups H such that $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H$ and either $H < \mathrm{GL}_n(\mathbb{F})$ or $H \leq \Sigma\mathrm{L}_n(\mathbb{F}) \neq \Gamma\mathrm{L}_n(\mathbb{F})$. We write $\mathbb{F}^{\times n}$ for the subgroup of $\mathbb{F}^*$ consisting of n-th powers, which is the set of possible determinants of scalar matrices in $\mathrm{GL}_n(\mathbb{F})$.

Proposition 3.7. *Assume that* $n \geq 4$ *and* $|\mathbb{F}| \geq 3$, *and let* H *be any group satisfying* $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \Gamma\mathrm{L}_n(\mathbb{F})$. *Assume also that the set*

$$\{\det(g)^{\psi}\mathbb{F}^{\times n} \mid g\psi \in H \text{ with } g \in \mathrm{GL}_n(\mathbb{F}),\ \psi \in \mathrm{Aut}(\mathbb{F})\}$$

is a proper subset of $\mathbb{F}^*/\mathbb{F}^{\times n}$. *Then* $\mathrm{RC}(H) \geq 2n-2$.

Proof. By assumption, there exists an $\alpha \in \mathbb{F}^*$ such that $\alpha \neq \det(gz)^{\psi}$ for all $g\psi \in H$ and $z \in Z$. Define $X, Y \in \Omega^{2n-2}$ as follows:

$$X := (\langle e_2\rangle, \dots, \langle e_n\rangle, \langle e_1 + e_2\rangle, \dots, \langle e_1 + e_n\rangle),$$
$$Y := (\langle e_2\rangle, \dots, \langle e_n\rangle, \langle \alpha e_1 + e_2\rangle, \dots, \langle \alpha e_1 + e_n\rangle).$$

We show first that $Y \notin X^H$. Suppose for a contradiction that there exists $g\psi \in H$, with $g \in \mathrm{GL}_n(\mathbb{F})$ and $\psi \in \mathrm{Aut}(\mathbb{F})$, such that $X^{g\psi} = Y$. As $g\psi$ fixes $\langle e_2\rangle$ and $\langle e_3\rangle$, and maps $\langle e_1 + e_2\rangle$ and $\langle e_1 + e_3\rangle$ to $\langle \alpha e_1 + e_2\rangle$ and $\langle \alpha e_1 + e_3\rangle$, respectively, we deduce that

$$e_1^{g\psi} \in \langle e_1, e_2\rangle \cap \langle e_1, e_3\rangle = \langle e_1\rangle.$$

Therefore, we see that $\langle e_i\rangle^{g\psi} = \langle e_i\rangle$ for each $i \in \{1, \dots, n\}$, and so g is diagonal. Let $\mu := \alpha^{\psi^{-1}}$. As $\langle e_1 + e_i\rangle^{g\psi} = \langle \alpha e_1 + e_i\rangle$ for each $i \in \{2, \dots, n\}$, we deduce that $g = \mathrm{diag}(\mu, 1, \dots, 1)z$ for some $z \in Z$. Hence $(\det(gz^{-1}))^{\psi} = \mu^{\psi} = \alpha$, a contradiction. Hence $Y \notin X^H$.

Now, for each $i \in \{2, \dots, n\}$, let $h_i := \mathrm{diag}(\alpha, 1, \dots, 1, \alpha^{-1}, 1, \dots, 1)$, where the α^{-1} appears in entry i. First, for $j \in \{1, \dots, n-1\}$, let $k := j + 1$ so that $x_j = y_j = \langle e_k\rangle$. It is easy to verify that $h_k + (1-\alpha)E_{k1}$ has determinant 1 and maps $(X \setminus x_j)$ to $(Y \setminus y_j)$. Finally, for $j \in \{n, \dots, 2n-2\}$, let $k := j + 2 - n$ so that $x_j = \langle e_1 + e_k\rangle$ and $y_j = \langle \alpha e_1 + e_k\rangle$. Then h_k has determinant 1 and maps $(X \setminus x_j)$ to $(Y \setminus y_j)$. Thus $X \sim_{H,2n-3} Y$, and so $\mathrm{RC}(H) \geq 2n - 2$. □

Proof of Theorem A. When $|\mathbb{F}| = 2$, this result is clear from Theorem 1.1. For the remaining fields $\mathbb{F}$, the fact that part (i) gives an upper bound on $\mathrm{RC}(\mathrm{PGL}_n(\mathbb{F}))$ is proved in Theorem 2.12, whilst we prove that it gives a lower bound in Theorem 3.1 for $|\mathbb{F}| = 3$ and Proposition 3.4 for $|\mathbb{F}| \geq 4$. That part (ii) gives upper bounds on $\mathrm{RC}(\overline{H})$ is immediate from Theorem 1.2 (ii) for $n = 3$, and from Theorem 2.7 for $n \geq 4$. Lemma 3.3 and Proposition 3.7 show that these are also lower bounds. □

Recall that $\omega(k)$ denotes the number of distinct prime divisors of the positive integer k.

Lemma 3.8 ([7, Lemma 3.1]). *Let $K \leq \mathrm{Sym}(\Gamma)$ be a finite group with normal subgroup N such that K/N is cyclic. Then* $\mathrm{H}(K, \Gamma) \leq \mathrm{H}(N, \Gamma) + \omega(|K/N|)$.

Proof of Theorem B. For the upper bound in (i), we combine Proposition 1.2 (i) with Lemma 3.8 to deduce that $\mathrm{H}(\overline{H}, \Omega_1) = 3 + \omega(e)$, so $\mathrm{RC}(\overline{H}, \Omega_1) \leq 4 + \omega(e)$. The lower bound (and the case $\overline{H} = \mathrm{P\Sigma L}_2(9)$) is Lemma 3.2.

For the upper bound in part (ii), we similarly combine Proposition 1.2 (ii) with Lemma 3.8. As for the lower bound, first let $n = 3$, and notice that, in this case,

$2n-2=4<n+2=5$. If $\overline{H}$ properly contains $\mathrm{PGL}_3(q)$, then the lower bound of 6 is proved in Lemma 3.5. Otherwise, $\mathrm{PSL}_3(q) \neq \mathrm{PGL}_3(q)$, and so the lower bound of 5 follows from Lemma 3.3. Now assume that $n \geq 4$. The general lower bound is Lemma 3.6, the bound of $n+3$ for groups properly containing $\mathrm{PGL}_n(q)$ is Lemma 3.5, and the bound of $2n-2$ is Proposition 3.7. □

4 Action on m-spaces for $m \geq 2$

In this section, we consider the action of the group H on $\Omega_m = \mathcal{PG}_m(V)$, where $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \Gamma\mathrm{L}_n(\mathbb{F})$, as before, but now $2 \leq m \leq \frac{n}{2}$. The main work is to prove a lower bound on $\mathrm{RC}(H, \Omega_m)$, as the upper bound follows from existing literature.

Proposition 4.1. *Let $\mathbb{F}$ be arbitrary, let $n \geq 2m \geq 4$, and let H be any group satisfying $\mathrm{SL}_n(\mathbb{F}) \trianglelefteq H \leq \Gamma\mathrm{L}_n(\mathbb{F})$. Then $\mathrm{RC}(H, \Omega_m) \geq mn - m^2 + 1$.*

Proof. For each $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, n-1\}$, let

$$\begin{aligned} B_i &:= \{e_1, e_2, \ldots, e_m\} \setminus \{e_i\}, \\ U_{ij} &:= \langle B_i, e_j \rangle = \langle e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_m, e_j \rangle, \\ V_i &:= \langle B_i, e_i + e_n \rangle, \quad \text{and} \quad W_i := \langle B_i, e_n \rangle \end{aligned}$$

so that $U_{ij}, V_i, W_i \in \Omega_m$. Define $X, Y \in \Omega_m^{mn-m^2+1}$ as follows:

$$\begin{aligned} x_{mn-m^2+1} &:= \left\langle e_1 + e_2, \ldots, e_1 + e_m, \sum_{i=1}^{n} e_i \right\rangle, \\ y_{mn-m^2+1} &:= \left\langle e_1 + e_2, \ldots, e_1 + e_m, -e_1 + \sum_{i=m+1}^{n} e_i \right\rangle, \\ X &:= (U_{1(m+1)}, U_{1(m+2)}, \ldots, U_{m(n-1)}, V_1, V_2, \ldots, V_m, x_{mn-m^2+1}), \\ Y &:= (U_{1(m+1)}, U_{1(m+2)}, \ldots, U_{m(n-1)}, W_1, W_2, \ldots, W_m, y_{mn-m^2+1}). \end{aligned}$$

We shall first show that $Y \notin X^{\Gamma\mathrm{L}_n(\mathbb{F})}$, so in particular $Y \notin X^H$, and then that $X \sim_{H, mn-m^2} Y$.

Assume for a contradiction that $Y \in X^{\Gamma\mathrm{L}_n(\mathbb{F})}$. Since each subspace in Y is spanned by vectors of the form

$$\sum_{i=1}^{n} \lambda_i e_i \quad \text{with } \lambda_i \in \{-1, 0, 1\},$$

it follows that there exists $g \in \mathrm{GL}_n(\mathbb{F})$ with $X^g = Y$. For each $i \in \{1, \ldots, m\}$, choose

$$k \in \{1, \ldots, m\} \setminus \{i\}.$$

Then

$$\langle e_i \rangle = \bigcap_{\ell \in \{1,\ldots,m\} \setminus \{i\}} U_{\ell(m+1)} \cap V_k = \bigcap_{\ell \in \{1,\ldots,m\} \setminus \{i\}} U_{\ell(m+1)} \cap W_k,$$

so g fixes $\langle e_i \rangle$. Similarly, g fixes

$$\langle e_j \rangle = \bigcap_{i=1}^{m} U_{ij} \quad \text{for each } j \in \{m+1, \ldots, n-1\}.$$

Therefore, there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{F}^*$ and $\mu_1, \ldots, \mu_{n-1} \in \mathbb{F}$ such that g maps e_i to $\lambda_i e_i$ for all $i \in \{1, \ldots, n-1\}$, and maps e_n to $\lambda_n e_n + \sum_{i=1}^{n-1} \mu_i e_i$. It now follows that, for each $i \in \{2, \ldots, m\}$, the element g maps $e_1 + e_i \in x_{mn-m^2+1}$ to $\lambda_1 e_1 + \lambda_i e_i$, which must lie in y_{mn-m^2+1}, and hence $\lambda_i = \lambda_1$. Similarly, $V_i^g = W_i$ for each $i \in \{1, \ldots, m\}$, and so $W_i = \langle B_i, e_n \rangle$ contains

$$(e_i + e_n)^g = \lambda_1 e_i + \lambda_n e_n + \sum_{k=1}^{n-1} \mu_k e_k.$$

Hence $\mu_i = -\lambda_1$, and $\mu_j = 0$ for all $j \in \{m+1, \ldots, n-1\}$. It now follows that g maps

$$\sum_{i=1}^{n} e_i \in x_{mn-m^2+1} \quad \text{to} \quad \sum_{i=m+1}^{n} \lambda_i e_i,$$

which is clearly not in y_{mn-m^2+1}, a contradiction. Thus $Y \notin X^H$.

We now show that $X \sim_{H,mn-m^2} Y$, by identifying an element

$$g_\ell \in \mathrm{SL}_n(\mathbb{F}) \le H$$

that maps $(X \setminus x_\ell)$ to $(Y \setminus y_\ell)$ for each $\ell \in \{1, \ldots, mn - m^2 + 1\}$. We divide the proof into three cases, which together account for all values of ℓ. To simplify our expressions, let $z := e_1 + e_2 + \cdots + e_m$, $\alpha_1 := -1$, and $\alpha_r := 1$ for all $r \in \{2, \ldots, m\}$. In each case, the element g_ℓ will be lower unitriangular and so will have determinant 1.

Case (a): $\ell \in \{1, \ldots, m(n-m-1)\}$. Let $r \in \{1, \ldots, m\}$, $s \in \{m+1, \ldots, n-1\}$ be such that $\ell = (n-m-1)(r-1) + (s-m)$ so that $x_\ell = y_\ell = U_{rs}$. Additionally, let g_ℓ fix e_i for all $i \notin \{s, n\}$, map e_s to $e_s + \alpha_r e_r$, and map e_n to $e_n - z$. Then

g_ℓ fixes U_{ij} provided $(i, j) \neq (r, s)$, and maps $e_i + e_n \in V_i$ to $e_i + e_n - z \in W_i$, and hence V_i to W_i, for all $i \in \{1, \ldots, m\}$. Finally,

$$\left(\sum_{i=1}^{n} e_i\right)^{g_\ell} = \alpha_r e_r + \sum_{i=m+1}^{n} e_i \in y_{mn-m^2+1},$$

where we have used the fact that

$$e_r + \sum_{i=m+1}^{n} e_i = (e_1 + e_r) + \left(-e_1 + \sum_{i=m+1}^{n} e_i\right)$$

when $r > 1$. Hence g_ℓ maps x_{mn-m^2+1} to y_{mn-m^2+1}, as required.

Case (b): $\ell = m(n - m - 1) + r$, where $r \in \{1, \ldots, m\}$. Here,

$$x_\ell = V_r \quad \text{and} \quad y_\ell = W_r.$$

Let g_ℓ fix e_i for each $i \in \{1, \ldots, n-1\}$ and map e_n to $\alpha_r e_r + e_n - z$. Then g_ℓ fixes U_{ij} for all i and j, and maps $e_i + e_n \in V_i$ to $e_i + \alpha_r e_r + e_n - z \in W_i$, and hence V_i to W_i, for all $i \in \{1, \ldots, m\} \setminus \{r\}$. Finally,

$$\left(\sum_{i=1}^{n} e_i\right)^{g_\ell} = \alpha_r e_r + \sum_{i=m+1}^{n} e_i \in y_{mn-m^2+1},$$

as in Case (a).

Case (c): $\ell = mn - m^2 + 1$. Let g_ℓ fix e_i for each $i \in \{1, \ldots, n-1\}$, and map e_n to $e_n - z$. Then g fixes U_{ij} for all i, j, and maps $e_i + e_n \in V_i$ to $e_i + e_n - z \in W_i$ for all i, as required. □

The *irredundant base size* $\mathrm{I}(K, \Gamma)$ of a group K acting faithfully on a set Γ is the largest size of a tuple $(\alpha_1, \ldots, \alpha_k)$ of elements of Γ such that

$$K > K_{\alpha_1} > K_{(\alpha_1,\alpha_2)} > \cdots > K_{(\alpha_1,\ldots,\alpha_k)} = 1,$$

with all inclusions strict. It is clear that $\mathrm{I}(K, \Gamma)$ is bounded below by the height $\mathrm{H}(K, \Gamma)$, which we recall (from Section 1) is bounded below by $\mathrm{RC}(K, \Gamma) - 1$.

Proof of Theorem C. In [9, Theorem 3.1], it is proved that

$$\mathrm{I}(\mathrm{PGL}_n(\mathbb{F}), \Omega_m) \leq (m + 1)n - 2m + 1.$$

Since the irredundant base size of a subgroup is at most the irredundant base size of an overgroup, and the height is at most the irredundant base size, we deduce that $\mathrm{H}(\overline{H}, \Omega_m) \leq (m+1)n - 2m + 1$ for all $\overline{H} \leq \mathrm{PGL}_n(\mathbb{F})$. From Lemma 3.8, we then see that, for all $\overline{H}$ as in the statement,

$$\mathrm{H}(\overline{H}, \Omega_m) \leq (m+1)n - 2m + 1 + \omega(e),$$

and hence the upper bound follows. The lower bound is immediate from Proposition 4.1, so the proof is complete. □

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