Mechanics Research Communications 137 (2024) 104274

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

Discontinuity waves in temperature and diffusion models

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ARTICLE INFO

Keywords:
Shock waves
Hyperbolic diffusion
Acceleration waves

ABSTRACT

We analyse shock wave behaviour in a hyperbolic diffusion system with a general forcing term which is qualitatively not dissimilar to a logistic growth term. The amplitude behaviour is interesting and depends critically on a parameter in the forcing term. We also develop a fully nonlinear acceleration wave analysis for a hyperbolic theory of diffusion coupled to temperature evolution. We consider a rigid body and we show that for three-dimensional waves there is a fast wave and a slow wave. The amplitude equation is derived exactly for a one-dimensional (plane) wave and the amplitude is found for a wave moving into a region of constant temperature and solute concentration. This analysis is generalized to allow for forcing terms of Selkov–Schnakenberg, or Al Ghoul-Eu cubic reaction type. We briefly consider a nonlinear acceleration wave in a heat conduction theory with two solutes present, resulting in a model with equations for temperature and each of two solute concentrations. Here it is shown that three waves may propagate.

1. Introduction

There has been significant interest in the propagation of heat as a wave, cf. [1–10]. However, recent studies have also focused on a hyperbolic formulation for solute transport rather than simply by diffusion, cf. [11–14]. Our definition of hyperbolicity follows that of Whitham [15, pages 113–142]. A rigorous thermodynamic derivation of a hyperbolic theory for temperature and diffusion of several constituents is given by Morro [16,17]. Structural stability of the earliest of these models was analysed by Ciarletta et al. [18]. Very notably, Al-Ghoul [12] remarks,

> The hyperbolicity of the evolution equations is a more desirable feature than the parabolicity, since disturbance characterizing waves in macroscopic systems cannot propagate at infinite speeds, …

He also remarks,

> In fact, we will show … the validity of the parabolic reaction–diffusion equations, …, becomes questionable and hyperbolic reaction–diffusion equations appear to be more suitable for describing underlying phenomena ….

It is worth drawing attention to the fact that hyperbolic theories with partial differential equations have been successfully employed in a variety of diverse areas of applied mathematics such as in diffusion with chemical reactions, [12]; in virus spread, [19]; in vegetation patterns occurring on land, [20,21]; in evolution of a gene and a culture, [22]; in social systems, [23]; and in pollution studies, [24,25].

In this article we employ the mathematical theory of waves of discontinuity in the concentration (temperature), or in the derivative of these quantities, namely, shock waves or acceleration waves. While the theory of discontinuity waves is well known it remains an exciting way to develop an *exact* analysis for a fully nonlinear theory and is still being used with great effect in many areas of continuum mechanics and even in mathematical theories pertaining to anthropological or social systems, see e.g. [1,3,6,7,23,26–32]. In particular, we encounter and solve a Bernoulli equation for the wave amplitudes. We point out that detailed analyses of such Bernoulli equations are contained in Jeffrey [33], Chen [34, page 320], Whitham [15, page 132], Boillat and Ruggeri [35], and Brini and Seccia [26], with much detail in Ruggeri and Sugiyama [27, pages 67–106].

We commence our analysis with a generalization of the work of Jordan [10] which developed shock evolution for a hyperbolic temperature model with a forcing term of logistic growth type. This has recently been further extended by Jordan and Lambers [6] and Jordan et al. [7]. Rather than employing simply a logistic term we employ a more general relation due to Richards [36]. The relation of Richards [36] allows for growth not dissimilar to logistic growth but encompasses a greater variety of possible real life scenarios. We calculate the shock speeds and the amplitude behaviour. The latter is very interesting and

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https://doi.org/10.1016/j.mechrescom.2024.104274
Received 16 October 2023; Received in revised form 2 April 2024; Accepted 5 April 2024
Available online 8 April 2024

depends critically on a parameter present in Richards [36] model. For some values of this parameter the shock amplitude may blow-up in a finite time, whereas for other values the amplitude remains bounded, regardless of the initial conditions.

We also initiate a study of nonlinear wave motion for a temperature and solute concentration in a rigid body. We note that hyperbolic diffusion with transport is also found in fluid and solid dynamics, see e.g. Peshkov and Romenski [37], and Boyaval [38]. We employ an acceleration wave analysis and fully determine the wavespeeds and the amplitude equation. The solution to the amplitude equation is derived, even when reactions are fully nonlinear such as those of Selkov-Schnakenberg, or Al-Ghoul-Eu cubic reaction, cf. [12]. The paper is completed by developing a nonlinear acceleration wave analysis for a theory with a temperature field, and for two dissolved solutes. This results in six coupled highly nonlinear partial differential equations.

2. Jordan-Cattaneo theory for hyperbolic diffusion with a Richards' term

Jordan [10] dealt with travelling waves and shock waves for a solution to a hyperbolic version of the Fisher equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + ku\left(1 - \frac{u}{u_s}\right). \tag{1}$$

He found the shock speeds via the Rankine–Hugoniot equations and found an explicit expression for the shock amplitude equation which he was able to solve exactly in analytical form. In (1), u could be temperature, concentration of a solvent, or even the density of a biological species. The constants D, k and u_s are positive, $u = u(x,t)$, D is the diffusion coefficient, u_s is a limiting value for u in the logistic term in (1), often referred to as the *carrying capacity* parameter.

If we employ a Cattaneo equation for the flux $J = J(x,t)$, then a hyperbolic form of (1) may be taken to be

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial J}{\partial x} + f(u), \\ \tau\frac{\partial J}{\partial t} + J &= -D\frac{\partial u}{\partial x}, \end{aligned} \tag{2}$$

where $\tau > 0$ is a relaxation time, and $f(u)$ is the nonlinear forcing term. For a logistic term $f(u) = ku(1 - u/u_s)$, system (2) is analysed both analytically and numerically by Jordan and Lambers [6] and Jordan et al. [7]. These writers deal with calculating shock wave behaviour in great detail.

We point out that when $\tau = 0$, system (2) reduces to one based on Fick's law and Eq. (1) is found. Shock waves in a system like (2) are referred to as Jordan-Cattaneo waves, cf. [39]. The Cattaneo version which involves Eq. $(2)_2$ avoids an infinite speed of propagation and may be thought of as introducing a delay into the physical system. Such a delay has recently been shown to be physically relevant in chemical reaction processes involving a Schnakenberg reaction, see [40]. Care must be taken with the mathematical representation of a time delay, however, as is shown by Jordan et al. [41], and Christov and Jordan [42].

Richards [36] has argued that one may employ a generalization of the logistic law for $f(u)$ to obtain an accurate representation of biological growth. In our notation Richards [36] form would involve

$$f(u) = ku\left(1 - \left\{\frac{u}{u_s}\right\}^m\right), \tag{3}$$

where $m \in \mathbb{R}$, although we restrict $m \in \mathbb{N}$. The parameter m gives one extra flexibility to fit bio-chemical behaviour but the behaviour is not qualitatively dissimilar to logistic growth.

Our goal in the first part of this article is to study the behaviour of a shock wave governed by (2) but with f given by the Richards [36] function f, namely to study the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\frac{\partial J}{\partial x} + ku\left(1 - \left\{\frac{u}{u_s}\right\}^m\right), \\ \tau\frac{\partial J}{\partial t} + J &= -D\frac{\partial u}{\partial x}. \end{aligned} \tag{4}$$

For (4) we may think of a shock wave as a plane wave in 3-D moving along the x -axis. We suppose u and J are continuous everywhere in x and t except across a surface $\mathscr{S}$ at a point x. On $\mathscr{S}$, u and J possess a finite discontinuity. This surface is called a shock wave.

Let $[f]$ be the jump of f across $\mathscr{S}$, i.e.

$$[f] = f^- - f^+$$

where $f^-(x,t) = \lim_{x\to x^-} f(x,t)$ and $f^+(x,t) = \lim_{x\to x^+} f(x,t)$ where $x \in \mathscr{S}$ and the $-$ sign indicates the limit from the left whereas the $+$ sign indicates the limit from the right.

The Rankine–Hugoniot relations for (4) yield

$$\begin{aligned} -U[u] + [J] &= 0 \\ -\tau U[J] + D[u] &= 0 \end{aligned} \tag{5}$$

where U is the shock speed, cf. Whitham [15]. From (5) we see that $U = \pm\sqrt{D/\tau}$ which shows there is a backward and a forward moving shock.

Now the Hadamard relation, see [43],

$$\frac{\delta}{\delta t}[f] = [f_t] + U[f_x], \tag{6}$$

and the product relation

$$[fg] = f^+[g] + g^+[f] + [f][g], \tag{7}$$

are necessary. In (6), $\delta/\delta t$ is the intrinsic derivative and is the rate of change as witnessed by an observer at the wavefront. Define the wave amplitudes P and Q by

$$P(t) = [u], \qquad Q(t) = [J].$$

Then take the jumps of Eqs. (4) to see that

$$\begin{aligned} &\frac{\delta P}{\delta t} - kP + \frac{k}{u_s^m}[u^{m+1}] - U[u_x] + [J_x] = 0, \\ &\tau\frac{\delta Q}{\delta t} + Q - U\tau[J_x] + D[u_x] = 0. \end{aligned} \tag{8}$$

Eliminate $[u_x]$ and $[J_x]$ from (8) using the fact that $U = \pm\sqrt{D/\tau}$ and then after use of (5) one may show that

$$\frac{\delta P}{\delta t} + \frac{1}{2}\left(\frac{1}{\tau} - k\right)P + \frac{k}{2u_s^m}[u^{m+1}] = 0. \tag{9}$$

Next, use the product rule (7) to show that

$$[u^{m+1}] = [u]^{m+1} + \sum_{k=1}^{m} {}^{m+1}C_k [u]^k (u^+)^{m+1-k}, \tag{10}$$

where ${}^{m+1}C_k$ is the combinatorial symbol. Utilizing this in (9) leads to the amplitude equation

$$\frac{\delta P}{\delta t} - \alpha P + \frac{k}{2u_s^m}\left(P^{m+1} + \sum_{k=1}^{m} {}^{m+1}C_k (u^+)^{m+1-k} P^k\right) = 0, \tag{11}$$

where $\alpha = (k - \tau^{-1})/2$. If we know u^+ ahead of the shock then Eq. (11) may be solved numerically for any m.

In the case where $u^+ = 0$ then (11) becomes

$$\frac{\delta P}{\delta t} - \alpha P + \frac{k}{2u_s^m}P^{m+1} = 0. \tag{12}$$

This is a Bernoulli equation which has exact solution

$$P^m(t) = \frac{P^m(0)e^{\alpha m t}}{1 + \frac{\hat{k}}{\alpha}(e^{\alpha m t} - 1)P^m(0)}, \tag{13}$$

where $\hat{k} = k/2u_s^m$. The solution (13) has some very interesting consequences. We here allow only $m = 1, 2, \ldots$ Eq. (13) shows that $|P(t)|$ remains bounded whenever m is even. When m is odd then it is possible for $P(0) < 0$ to have $P(t) \to -\infty$ in a finite time. Thus, the choice of parameter in Richards [36] Eq. (3) is crucial. This shows that one has to be very careful with any generalization of the logistic law to include a Richards term and it emphasizes that the modelling in such a case is very important.

3. Temperature diffusion acceleration waves

We now consider the problem of a concentration of solute or species density in a non-isothermal situation. Let $T(\mathbf{x},t)$ and $C(\mathbf{x},t)$ be the temperature and concentration of a solute in a rigid body. Let $Q_i(\mathbf{x},t)$ and $J_i(\mathbf{x},t)$ be the corresponding heat flux and flux of solute. Then the relevant hyperbolic equations may be taken to be, cf. Morro [44], [18],

$$\begin{aligned} &\rho_1 T_{,t} + Q_{i,i} = 0,\\ &C_{,t} + J_{i,i} = 0,\\ &\tau Q_{i,t} + Q_i + kT_{,i} + FC_{,i} = 0,\\ &\tau_C J_{i,t} + J_i + DC_{,i} + ST_{,i} = 0. \end{aligned} \tag{14}$$

In these equations $\rho_1 = \rho c_p$ where ρ is density and c_p is the specific heat at constant pressure, τ and τ_C are relaxation coefficients, k and D are thermal conductivity and diffusion coefficient, and F and S are the Dufour and Soret coefficients. We make the realistic assumption that k, F, D and S may depend on T and C. Standard indicial notation is employed with $,t \equiv \partial/\partial t$ and $,i \equiv \partial/\partial x_i$.

3.1. Acceleration waves and speeds

We suppose T, C, J_i and Q_i are continuous everywhere in $\mathbf{x}$ and t. However, there is a surface $\mathscr{S}$ across which the derivatives of T, C, J_i and Q_i possess a finite discontinuity. This surface $\mathscr{S}$ is called an acceleration wave.

Define the amplitudes $A, \mathscr{C}, H_i$ and B_i by

$$A = [T_{,t}], \quad \mathscr{C} = [C_{,t}], \quad H_i = [J_{i,t}], \quad B_i = [Q_{i,t}]. \tag{15}$$

We make use of the Hadamard relation, cf. [43], in the form

$$0 = \frac{\delta}{\delta t}[Q_i] = [Q_{i,t}] + U_N[Q_{i,j}n_j],$$
$$0 = \frac{\delta}{\delta t}[T] = [T_{,t}] + U_N[T_{,i}n_i],$$

where n_i is the unit outward normal to the wave surface. Next, take the jumps of Eqs. (14) and use the Hadamard and compatibility relations, see [43, eq. (176.10)], to obtain

$$\begin{aligned} &\rho_1 A - \frac{B_i n_i}{U_N} = 0,\\ &\mathscr{C} - \frac{n_i H_i}{U_N} = 0,\\ &\tau B_i - kn_i\frac{A}{U_N} - Fn_i\frac{\mathscr{C}}{U_N} = 0,\\ &\tau_C H_i - Dn_i\frac{\mathscr{C}}{U_N} - Sn_i\frac{A}{U_N} = 0. \end{aligned} \tag{16}$$

From Eqs. $(16)_{3,4}$ we observe that $B_i = Bn_i$ and $H_i = Hn_i$, i.e. the wave is longitudinal. Eqs. (16) yields a system in $(A, B, \mathscr{C}, H)$ and requiring non-zero amplitudes leads to the wavespeed equation

$$(U_N^2 - U_C^2)(U_N^2 - U_T^2) = K^2, \tag{17}$$

where $U_C^2 = D/\tau_C$, $U_T^2 = k/\rho_1\tau$ and

$$K^2 = \frac{SF}{\rho_1 \tau \tau_C}. \tag{18}$$

The quantities U_C^2 and U_T^2 are the squares of the wavespeeds for a diffusion wave in an isothermal situation and a temperature wave in the absence of a solvent.

From (18) we see that $K^2 > 0$ and then (17) leads to a fast and a slow wave with speeds U_1 and U_2 where

$$U_1^2 < \{U_C^2, U_T^2\} < U_2^2. \tag{19}$$

Furthermore, Eq. (17) yields

$$2U_N^2 = U_T^2 + U_C^2 \pm \sqrt{(U_T^2 - U_C^2)^2 + 4K^2}.$$

Rewrite the quantity under the root sign as $(U_T^2 + U_C^2)^2 + 4(K^2 - U_T^2 U_C^2)$, and this shows there are two waves provided

$$U_C^2 U_T^2 > K^2$$

or

$$Dk > FS. \tag{20}$$

We observe that system (14) can be put in the form

$$A_0(U)U_{,t} + A_i(U)U_{,i} + D(U) = 0$$

with $U = (T, Q_i, C, J_i)$, where $D(U) = (0, Q_i, 0, J_i)$ and

$$A_0(U) = \begin{pmatrix} \rho_1 & . & . & . \\ . & \tau\mathbf{I} & . & . \\ . & . & 1 & . \\ . & . & . & \tau_C\mathbf{I} \end{pmatrix},$$

$$A_i(U) = \begin{pmatrix} . & \mathbf{e}_i^T & . & . \\ k\mathbf{e}_i & . & F\mathbf{e}_i & . \\ . & . & . & \mathbf{e}_i^T \\ S\mathbf{e}_i & . & D\mathbf{e}_i & . \end{pmatrix},$$

($\mathbf{e}_i$ are the coordinate axes' unit vectors). The wavespeeds U_N are the solutions of the equation

$$\det[A_i(U)n_i - U_N A_0(U)] = 0$$

and beyond the solutions already obtained, we also get a solution $U_N = 0$ of algebraic multiplicity four. Given that the matrix $A_i(U)n_i$ has rank four, independently of the wave number $\mathbf{n}$ and the state U, we can conclude that the system is strongly hyperbolic.

3.2. Wave amplitude equation

We now proceed to calculate and solve exactly an equation for the wave amplitudes. We restrict attention to a plane wave moving along the x -axis to avoid the differential geometry involved in the three-dimensional calculation obscuring the essential physics.

It is convenient to rewrite Eqs. (14) in one-dimension and dispense with the comma notation, so e.g. $T_t \equiv T_{,t}$ and $T_x \equiv T_{,x}$. Eqs. (14) become

$$\begin{aligned} &\rho_1 T_t + Q_x = 0,\\ &C_t + J_x = 0,\\ &\tau Q_t + Q + kT_x + FC_x = 0,\\ &\tau_C J_t + J + DC_x + ST_x = 0. \end{aligned} \tag{21}$$

In the one-dimensional case it is also convenient to redefine the amplitudes in an equivalent way, namely

$$A = [T_x], \quad B = [Q_x], \quad \mathscr{C} = [C_x], \quad H = [J_x].$$

We begin by differentiating each of Eqs. (21) with respect to x and take the jumps of the result. This leads to

$$\begin{aligned} &\rho_1[T_{xt}] + [Q_{xx}] = 0,\\ &[C_{tx}] + [J_{xx}] = 0,\\ &\tau[Q_{tx}] + [Q_x] + k_T[T_x^2] + (k_C + F_T)[C_x T_x]\\ &\qquad + F_C[C_x^2] + k[T_{xx}] + F[C_{xx}] = 0,\\ &\tau_C[J_{tx}] + [J_x] + D_C[C_x^2] + (D_T + S_C)[T_x C_x]\\ &\qquad + S_T[T_x^2] + D[C_{xx}] + S[T_{xx}] = 0. \end{aligned} \tag{22}$$

We restrict attention now to the fast wave which we suppose is advancing into a region where T is constant and C is constant. Thus, $T_x^+ = 0$, $C_x^+ = 0$.

One employs the Hadamard relation to find

$$
\begin{aligned}
[T_{xt}] &= \frac{\delta A}{\delta t} - V[T_{xx}]\\
[Q_{xt}] &= \frac{\delta B}{\delta t} - V[Q_{xx}]\\
[C_{xt}] &= \frac{\delta \mathscr{C}}{\delta t} - V[C_{xx}]\\
[J_{xt}] &= \frac{\delta H}{\delta t} - V[J_{xx}]
\end{aligned}
\tag{23}
$$

where V is now the wavespeed. These relations are used in (22) and we then form the combination $\Xi \equiv (22)_1 + \lambda(22)_2 + \xi(22)_3 + \zeta(22)_4$ for constant λ, ξ, ζ at our disposal. We select $\xi = 1/\tau V$ $\lambda = (\rho_1 \tau_C / S)(V^2 - V_T^2)$ and $\zeta = (\rho_1/SV)(V^2 - V_T^2)$. In this way we employ the wavespeed equation

$$
(V^2 - V_T^2)(V^2 - V_C^2) = K^2,
$$

where $V_T \equiv U_T$ and $V_C \equiv U_C$, to remove the resulting terms in $[T_{xx}], [C_{xx}], [J_{xx}]$ and $[Q_{xx}]$.

From the jumps of (21) one has

$$
\begin{aligned}
\rho_1 V A &= B,\\
V\mathscr{C} &= H,\\
-\tau V B + kA + F\mathscr{C} &= 0,\\
-\tau_C V H + D\mathscr{C} + SA &= 0.
\end{aligned}
\tag{24}
$$

In this manner we may write B, C and H as linear functions in A, in form $(24)_1$, and

$$
\begin{aligned}
\mathscr{C} &= \frac{\rho_1 \tau}{F}(V^2 - V_T^2)A,\\
H &= \frac{V\rho_1 \tau}{F}(V^2 - V_T^2)A.
\end{aligned}
\tag{25}
$$

The equation $\Xi = 0$ may be written as

$$
\begin{aligned}
\rho_1 \frac{\delta A}{\delta t} + \lambda \frac{\delta \mathscr{C}}{\delta t} &+ \xi \left(\tau \frac{\delta B}{\delta t} + B + k_T A^2 \right.\\
&\left. + (k_C + F_T)A\mathscr{C} + F_C \mathscr{C}^2 \right)\\
&+ \zeta \left(\tau_C \frac{\delta H}{\delta t} + H + D_C \mathscr{C}^2 \right.\\
&\left. + S_T A^2 + (D_T + S_C)A\mathscr{C} \right) = 0.
\end{aligned}
\tag{26}
$$

One now uses λ, ξ and ζ together with (24) in (26) and after some calculation one may arrive at the amplitude equation

$$
\frac{\delta A}{\delta t} + \zeta_1 A + \zeta_2 A^2 = 0,
\tag{27}
$$

where

$$
\zeta_1 = \frac{\dfrac{V^2 - V_C^2}{\tau} + \dfrac{1}{\tau_C}}{2(2V^2 - V_C^2 - V_T^2)},
$$

$$
\zeta_2 = \frac{\xi_1}{\xi_2},
$$

where

$$
\begin{aligned}
\xi_1 = \frac{k_T}{\tau V} &+ (V^2 - V_T^2)\left\{ \frac{\rho_1 S_T}{SV} + \frac{\rho_1 (k_C + F_T)}{FV} \right\}\\
&+ (V^2 - V_T^2)^2 \left\{ \frac{\rho_1^2 \tau}{VSF}(D_T + S_C) + \frac{\rho_1 F_C}{VF} \right\}\\
&+ (V^2 - V_T^2)^3 \left\{ \frac{\rho_1^2 \tau D_C}{VSF} \right\},
\end{aligned}
$$

$$
\xi_2 = 2\rho_1 \left[1 + \frac{V^2 - V_T^2}{V^2 - V_C^2} \right].
$$

Once A is found from (27) then $B, \mathscr{C}$ and H follow from (24).

The solution to Eq. (27) is

$$
A(t) = \frac{1}{\dfrac{e^{\zeta_1 t}}{A(0)} + \dfrac{\zeta_2}{\zeta_1}(e^{\zeta_1 t} - 1)},
\tag{28}
$$

If $A(0) < 0$ then $T_x^- < 0$ which is a compressive wave which blows up at time

$$
T = \frac{1}{\zeta_1} \log \left[\frac{\zeta_2 A(0)}{\zeta_1 + \zeta_2 A(0)} \right].
\tag{29}
$$

This is associated with the formation of a temperature shock wave and a diffusion shock wave.

Remark. One may employ the Jordan [10] technique to investigate shock waves for system (21). A fast and a slow wave is predicted. However, when one proceeds to calculate the amplitude of the shock, one is impeded by the problem pointed out by Fu and Scott [45], where the amplitude equations continue to include terms of weaker waves, and those do not disappear naturally as in the case in Section 2.

4. Generalized temperature-diffusion waves

We now give brief details of an extension to system (14) where the right hand sides of $(14)_1$ and $(14)_2$ allow for reaction terms. While we write system (14) in terms of T and C it could easily represent two independent concentrations of chemical C_1 and C_2.

Thus, instead of (14) we consider

$$
\begin{aligned}
\rho_1 T_{,t} + Q_{i,i} &= \mathscr{E}(T, C),\\
C_{,t} + J_{i,i} &= \mathscr{F}(T, C),\\
\tau Q_{i,t} + Q_i + kT_{,i} + FC_{,i} &= 0,\\
\tau_C J_{i,t} + J_i + DC_{,i} + ST_{,i} &= 0,
\end{aligned}
\tag{30}
$$

where $\mathscr{E}$ and $\mathscr{F}$ are reaction terms. For the Selkov-Schnakenberg system the terms $\mathscr{E}$ and $\mathscr{F}$ may be taken to have form

$$
\begin{aligned}
\mathscr{E} &= \gamma(a - T + \lambda C + T^2 C),\\
\mathscr{F} &= \gamma(b - \lambda C - T^2 C),
\end{aligned}
$$

where γ, a, b, λ are positive constants, cf. Gentile and Torcicollo [46]. For the Al-Ghoul-Eu cubic reaction system one may write $\mathscr{E}$ and $\mathscr{F}$ as

$$
\begin{aligned}
\mathscr{E} &= k_2 C T^2 - k_{-2} T^3 - k_3 T + k_3 \rho_B,\\
\mathscr{F} &= k_1 \rho_A - k_{-1} C - k_2 C T^2 + k_{-2} T^3,
\end{aligned}
$$

where $k_{-1}, k_1, k_2, k_3, k_{-2}, \rho_A$ and ρ_B are positive constants, cf. Al-Ghoul [12].

An analysis analogous to that of Section 3.2 shows that the wave speeds of an acceleration wave for (30) are given by Eq. (17). Thus the wavespeeds satisfy the same restrictions as in Section 3.2.

To derive the amplitude equation for (30) one proceeds as in Section 3.2 to find that the amplitude A satisfies Eq. (27) with the same value for ζ_2 but ζ_1 must be changed to $\hat{\zeta}_1$, where

$$
\hat{\zeta}_1 = \zeta_1 - \left(\frac{\mathscr{E}_\alpha + \mathscr{F}_\alpha}{\beta} \right),
$$

where

$$
\mathscr{E}_\alpha = \frac{\partial \mathscr{E}}{\partial T} + \frac{\partial \mathscr{E}}{\partial C} \left(\frac{\tau - \rho_1 k}{\rho_1 F} \right),
$$

$$
\mathscr{F}_\alpha = \left\{ \frac{\partial \mathscr{F}}{\partial T} + \frac{\partial \mathscr{F}}{\partial C} \left(\frac{\tau - \rho_1 k}{\rho_1 F} \right) \right\} \frac{\rho_1 \tau_C}{S}(V^2 - V_T^2),
$$

and

$$
\beta = \rho_1 + \frac{1}{\rho_1 V} + \frac{2\tau_C \rho_1}{FS} \left(\frac{\tau}{\rho_1} - k \right)(V^2 - V_T^2).
$$

It should be noted that the reaction terms $\mathscr{E}$ and $\mathscr{F}$ do not alter the quadratic term in (27). However, they play a major role in amplitude behaviour since ζ_1 in (28) and (29) must be replaced by $\hat{\zeta}_1$.

5. Temperature, concentration, reaction systems

There are many real life systems where two (or more) concentrations of a chemical (or biological species) are present simultaneously, and the evolution of these is critically temperature dependent. We now consider an extension of (30) to include two concentrations C, G, and an equation for the evolution of the temperature T. In the cross diffusion terms we restrict attention to the Soret effect, rather than include cross reaction between C and G. Thus, we consider a system of equations of form

$$\begin{aligned}
&\rho_1 T_{,t} = -Q_{i,i},\\
&C_{,t} = -J_{i,i} + \mathscr{E}(C,G),\\
&G_{,t} = -K_{i,i} + \mathscr{F}(C,G),\\
&\tau Q_{i,t} + Q_i = -\kappa T_{,i} - F_1 C_{,i} - F_2 G_{,i},\\
&\tau_1 J_{i,t} + J_i = -D_1 C_{,i} - S_1 T_{,i},\\
&\tau_2 K_{i,t} + K_i = -D_2 G_{,i} - S_2 T_{,i}.
\end{aligned} \tag{31}$$

Here, $\rho_1, \tau, \tau_1, \tau_2$ are constants. The coefficients κ, F_1, F_2, D_1, D_2, S_1, S_2 may depend on C, G and T.

One may define an acceleration wave for (31) as in Sections 3 and 4. By taking the jumps of the equations in (31) and using the Hadamard and compatibility relations one may show that the wavespeed U_N of a three-dimensional acceleration wave satisfies

$$\begin{aligned}
(U_N^2 - U_\tau^2)(U_N^2 - U_1^2)(U_N^2 - U_2^2) = {}& \frac{S_1 F_1}{\rho_1 \tau \tau_1}(U_N^2 - U_2^2)\\
&+ \frac{S_2 F_2}{\rho_1 \tau \tau_2}(U_N^2 - U_1^2).
\end{aligned} \tag{32}$$

As this is a cubic equation in U_N^2 we may deduce that in general there are three forward and backward moving waves. Moreover, proceeding as for system (14), we also have a solution $U_N = 0$, in this case of algebraic multiplicity six. Again, it is easy to prove by direct calculation that the corresponding matrix has rank six, so the system is strongly hyperbolic.

One may progress to develop an amplitude equation for (31). However, we do not do this here as the calculations become very involved and we do not wish the paper to become excessively long.

6. Conclusions

We have analysed shock wave behaviour in a system of equations which allows for a growth of a species more general than logistic behaviour. This topic has been the subject of intense investigation by Jordan [10], Jordan and Lambers [6] and Jordan et al. [7]. The extension to the forcing terms of Richards [36] leads to surprising results.

We also analysed acceleration wave behaviour in a system of two equations for temperature and a species concentration or for two independent species concentrations. It is shown that, in general, two separate waves will propagate, and the amplitude of the leading wave is calculated. We additionally proposed a model for temperature dependent evolution of a set of reaction–diffusion equations for two separate species or concentrations. In this case it is shown that, in general, three separate waves may propagate.

CRediT authorship contribution statement

Michele Ciarletta: Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Brian Straughan:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **Vincenzo Tibullo:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

We are indebted to two anonymous referees who diligently read the manuscript and pointed out various points, attention to which has led to marked improvements.

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