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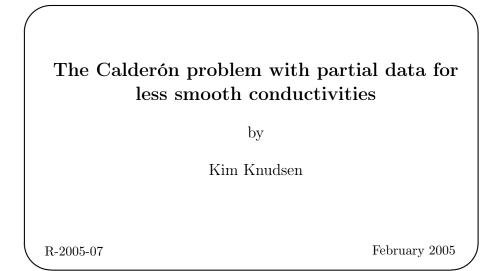
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The Calderón problem with partial data for less smooth conductivities

by

Kim Knudsen

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THE CALDERÓN PROBLEM WITH PARTIAL DATA FOR LESS SMOOTH CONDUCTIVITIES

KIM KNUDSEN

ABSTRACT. In this paper we consider the inverse conductivity problem with partial data. We prove that in dimensions $n \geq 3$ knowledge of the Dirichlet-to-Neumann map measured on particular subsets of the boundary determines uniquely a conductivity with 3/2 continuous derivatives.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a bounded domain with smooth boundary $\partial \Omega$. Let γ be a conductivity distribution in Ω that satisfies

$$\gamma \in L^\infty(\Omega),\ \gamma > 0 \text{ and } \gamma^{-1} \in L^\infty(\Omega).$$

The Dirichlet to Neumann (or voltage to current) map Λ_γ is defined by

(1) $$\Lambda_\gamma f = \gamma \partial_\nu u|_{\partial\Omega},$$

where u is the electric potential given as the unique solution to

(2) $$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \qquad u = f \text{ on } \partial\Omega$$

and ν is the outer unit normal to $\partial\Omega$.

The Calderón problem [Cal80] concerns the inversion of the map $\gamma \mapsto \Lambda_\gamma$, i.e. whether Λ_γ determines γ uniquely, and in that case how to reconstruct γ from Λ_γ. For the higher dimensional problem ($n \geq 3$) uniqueness was proved for smooth conductivities by Sylvester and Uhlmann in their foundational paper [SU87]. Generalizations to less regular conductivities has been obtained by a number of authors [NSU88, Ale88, Nac88, Cha90, Bro96, GLU03]. The sharpest uniqueness results so far seem to require essentially 3/2 derivatives of the conductivity to be continuous [PPU03] or to be in $L^p, p > 2n$ [BT03]. The reconstruction issue for $n \geq 3$ was solved in [Nac88] and [Nov88] independently. In dimension two, Nachman solved both uniqueness and reconstruction problems for conductivities $\gamma \in W^{2,p}(\Omega)$ with $p > 2$ in [Nac96]. This result was improved in [BU97, KT04] for conductivities

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having essentially one derivative. Recently Calderón's problem in dimension two was settled in the affirmative for conductivities $\gamma \in L^\infty(\Omega)$ by Astala and Päivärinta [AP03]

In recent years much attention has been devoted to the inverse problem, where only partial information about the Dirichlet to Neumann map is given. Let $\Gamma_1, \Gamma_2 \subset \partial\Omega$ be open subsets, and define the local Dirichlet to Neumann map $\tilde{\Lambda}_\gamma$ by

$$\tilde{\Lambda}_\gamma f = \Lambda_\gamma f|_{\Gamma_2}$$

on functions $f \in H^{1/2}(\partial\Omega) \cap \mathcal{E}'(\Gamma_1)$. The inverse problem then concerns the unique determination and computation of γ from $\tilde{\Lambda}_\gamma$. It is an outstanding open problem ([Isa98, Uhl98]), whether measurements taken on $\Gamma_1 = \Gamma_2 = \Gamma$ being any (possible very small) open subset of $\partial\Omega$ are sufficient for uniqueness for the inverse problem.

When γ is real-analytic, then measurements taken on any part of the boundary determine the conductivity [KV84], but beyond the real-analytic category only little is known. Bukhgeim and Uhlmann [BU02] showed that if $\Gamma_1 = \partial\Omega$ and Γ_2 is roughly speaking half of the boundary then $\tilde{\Lambda}_\gamma$ determines γ in the class $C^2(\overline{\Omega})$. Moreover, in a recent paper by Kenig, Sjöstrand and Uhlmann [KSU04] it was shown roughly speaking that taking Γ_1 as any open subset of $\partial\Omega$ and then $\Gamma_2 \subset \partial\Omega$ slightly larger than the complement also gives uniqueness in the class $C^2(\overline{\Omega})$.

The Caldéron problem arises in a number of practical problems, e.g. the medical imaging technique called electrical impedance tomography (EIT), see [CIN99]. The idea is to make current and voltage measurements through electrodes attached to a persons skin and then from these measurements compute features of the conductivity inside the body. In some applications of EIT, measurements can only taken on part of the body (see [CKK+01]), and hence the Calderón problem with partial data is of great interest also from an applied point of view.

The aim of this paper is show that partial information on the Dirichlet to Neumann map suffices for uniqueness also when γ is only in $C^{3/2}(\overline{\Omega})$. First we establish the notation. Let $\xi \in S^{n-1}$ and define the subsets

$$\partial\Omega_+(\xi) = \{x \in \Omega \colon \nu(x) \cdot \xi > 0\},$$
$$\partial\Omega_-(\xi) = \{x \in \Omega \colon \nu(x) \cdot \xi < 0\}.$$

Define further for $\epsilon > 0$ the subsets

$$\partial\Omega_{+,\epsilon}(\xi) = \{x \in \Omega \colon \nu(x) \cdot \xi > \epsilon\},$$
$$\partial\Omega_{-,\epsilon}(\xi) = \{x \in \Omega \colon \nu(x) \cdot \xi < \epsilon\}.$$

Clearly $\overline{\partial\Omega_{+,\epsilon}(\xi) \cup \partial\Omega_{-,\epsilon}(\xi)} = \partial\Omega$. The main theorem is the following:

Theorem 1.1. *Let* $\gamma_1, \gamma_2 \in C^{3/2}(\overline{\Omega}), \Omega \subset \mathbb{R}^n, n \geq 3$, *with* $\gamma_1|_{\partial\Omega_+} = \gamma_2|_{\partial\Omega_+}$ *and* $\partial_\nu\gamma_1|_{\partial\Omega_+} = \partial_\nu\gamma_2|_{\partial\Omega_+}$. *Let* $\xi \in S^{n-1}$ *be fixed and suppose*

$$(3) \qquad \Lambda_{\gamma_1}f|_{\partial\Omega_{-,\epsilon}(\xi)} = \Lambda_{\gamma_2}f|_{\partial\Omega_{-,\epsilon}(\xi)} \text{ for any } f \in H^{1/2}(\partial\Omega).$$

Then $\gamma_1 = \gamma_2$ *in* Ω.

This theorem is a generalization of [PPU03, Theorem 1.2] to the case of partial data and a generalization of [BU02, Corollary 0.2] to the case of less regular conductivities (note, however, that in [BU02, Corollary 0.2] no assumption on the normal derivative of the conductivitie at the boundary are needed). By combining our approach with that of [KSU04] it is expected that a generalization of [KSU04, Corollary 1.4] can be found.

To prove Theorem 1.1 we will combine ideas from [BU02] and [PPU03]. We will briefly outline the method of proof: suppose $\gamma \in C^1(\Omega)$ and let u be a solution to (2). Then u satisfies

$$(4) \qquad\qquad (-\Delta + A \cdot \nabla)u = 0 \quad \text{in } \Omega,$$

where

$$(5) \qquad\qquad A = -\nabla\log(\gamma).$$

The first step is to construct a suitable family of Complex Geometrical Optics (CGO) solutions to this equation. Such solutions were constructed in [PPU03] for A given by (5), however, the solutions we find here are slightly different. Also the method of proof is different. The second step is then to use an identity relating the information on the boundary to the Fourier transform of the conductivities. We will derive an identity, which seems to be new. Since we only take measurements on essentially half of the boundary, we will need to proof that the information on the remaining part of the boundary can be neglected. To do so we will derive a suitable Carleman estimate with boundary terms for a first order perturbation of the Laplacian.

The outline of the paper is the following. In Section 2 we derive Carleman estimates for first and zeroth order perturbations of the Laplacian. Then in Section 3 we prove the existence of CGO solutions to (4). Part of the argument uses the Carleman estimates. In Section 4 we derive the boundary identity, and finally in Section 5 we give the proof of Theorem 1.1.

2. Carleman estimate with first order term

In this section we will derive a Carleman estimate for the Laplacian with a zeroth order perturbation and a small first order perturbation. This estimate relies on the following proposition, which gives the estimate for the Laplacian. This estimate is sharper than the one obtained in [BU02], however a similar estimate can be found in [KSU04] in a more general setting. We will need the intermediate calculations later, so for the convenience of the reader we will give the proof.

Proposition 2.1. *Let $\xi \in S^{n-1}$ and suppose $u \in H_0^1(\Omega) \cap H^2(\Omega)$. Then there exists a constant $\tau_0 > 0$ such that for $\tau \geq \tau_0$ we have the estimate*

(6)
$$C \left(\tau^2 \|e^{-x \cdot \tau \xi} u\|_{L^2(\Omega)}^2 + \|e^{-x \cdot \tau \xi} \nabla u\|_{L^2(\Omega)}^2 \right) + C' \tau \int_{\partial \Omega} (\nu \cdot \xi) \left| e^{-x \cdot \tau \xi} \partial_\nu u \right|^2 dS$$
$$\leq \|e^{-x \cdot \tau \xi}(-\Delta)u\|_{L^2(\Omega)}^2,$$

where $C, C' > 0$ depends only on τ_0 and Ω.

Proof. For $v \in H^2(\Omega)$ we write

$$e^{-x \cdot \tau \xi}(-\Delta)(e^{x \cdot \tau \xi} v) = (-\Delta - \tau^2)v - 2\tau \partial_\xi v,$$

where $\partial_\xi = \xi \cdot \nabla$. Hence

(7) $\|e^{-x \cdot \tau \xi}(-\Delta)(e^{x \cdot \tau \xi} v)\|_{L^2(\Omega)}^2$
$$= \|(-\Delta - \tau^2)v\|_{L^2(\Omega)}^2 + \|2\tau \partial_\xi v\|_{L^2(\Omega)}^2 + 4\tau \operatorname{Re}\langle (\Delta + \tau^2)v, \partial_\xi v \rangle_{L^2(\Omega)}.$$

Note that

$$\langle (-\Delta - \tau^2)v, v \rangle_{L^2(\Omega)} = \|\nabla v\|_{L^2(\Omega)}^2 - \int_{\partial \Omega} \partial_\nu v \overline{v} dS - \tau^2 \|v\|_{L^2(\Omega)}^2$$

which together with the general estimate

$$|\langle a, b \rangle_{L^2(\Omega)}| \leq \|a\|_{L^2(\Omega)} \|b\|_{L^2(\Omega)} \leq \frac{1}{2}(\|a\|_{L^2(\Omega)}^2 + \|b\|_{L^2(\Omega)}^2)$$

implies that

$$2\|\nabla v\|_{L^2(\Omega)}^2 - 2\int_{\partial \Omega} \partial_\nu v \overline{v} dS - 2\tau^2 \|v\|_{L^2(\Omega)}^2 - \|v\|_{L^2(\Omega)}^2 \leq \|(-\Delta - \tau^2)v\|_{L^2(\Omega)}^2.$$

Furthermore the Poincaré inequality with boundary term implies that

$$C\frac{\tau^2}{d^2}\|v\|_{L^2(\Omega)}^2 - C'\tau^2 \int_{\partial \Omega} |v|^2 dS \leq \|2\tau \partial_\xi v\|_{L^2(\Omega)}^2,$$

where d is the diameter of Ω. Thus for d sufficiently small there is a $\tilde{\tau}_0 > 0$ such that for $\tau \geq \tilde{\tau}_0$ we have the estimate

$$(8) \quad C\left(\tau^2 \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2\right) - C'\tau^2 \int_{\partial\Omega} |v|^2 dS - C'' \int_{\partial\Omega} \partial_\nu v \overline{v} dS$$
$$\leq \|(-\Delta - \tau^2)v\|_{L^2(\Omega)}^2 + \|2\tau \partial_\xi v\|_{L^2(\Omega)}^2.$$

A straightforward scaling argument then shows that for a general domain of arbitrary diameter there exists a constant $\tau_0 > 0$ such that (8) holds for $\tau \geq \tau_0$.

Finally, a short calculation shows that

$$4\tau \mathrm{Re}((\Delta + \tau^2)v \partial_\xi \overline{v}) = \nabla \cdot (4\tau \mathrm{Re}(\nabla v \partial_\xi \overline{v}) - 2\tau \xi |\nabla v|^2 + 2\tau^3 \xi |v|^2),$$

which by the divergence theorem implies that

$$4\tau \mathrm{Re}\langle(\Delta + \tau^2)v, \partial_\xi v\rangle_{L^2(\Omega)}$$
$$= \int_\Omega \nabla \cdot (4\tau \mathrm{Re}(\nabla v \partial_\xi \overline{v}) - 2\tau \xi |\nabla v|^2 + 2\tau^3 \xi |v|^2)$$
$$(9) \qquad = \int_{\partial\Omega} \left(4\tau \mathrm{Re}(\partial_\nu v \partial_\xi \overline{v}) - 2\tau(\nu \cdot \xi)|\nabla v|^2 + 2\tau^3(\nu \cdot \xi)|v|^2\right) dS$$

Combining (7) with (8) and (9) gives the estimate

$$C\left(\tau^2 \|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2\right) - C'\tau^2 \int_{\partial\Omega} |v|^2 dS - C'' \int_{\partial\Omega} \partial_\nu v \overline{v} dS$$
$$(10) \qquad + \int_{\partial\Omega} \left(4\tau \mathrm{Re}(\partial_\nu v \partial_\xi \overline{v}) - 2\tau(\nu \cdot \xi)|\nabla v|^2 + 2\tau^3(\nu \cdot \xi)|v|^2\right) dS$$
$$\leq \|e^{-x \cdot \tau \xi}(-\Delta)(e^{x \cdot \tau \xi}v)\|_{L^2(\Omega)}^2$$

To obtain (6) we substitute $v = \exp(-x \cdot \tau \xi)u$ in (10). Note that

$$\|\nabla(e^{-x \cdot \tau \xi}u)\|_{L^2(\Omega)}^2 \geq \|e^{-x \cdot \tau \xi}\nabla u\|_{L^2(\Omega)}^2.$$

Then using the fact that $v|_{\partial\Omega} = u|_{\partial\Omega} = 0$ implies

$$\nabla v = \nu \partial_\nu v, \quad \partial_\xi v = (\nu \cdot \xi)\partial_\nu v$$

gives (6). $\qquad \square$

In the following proposition we obtain a Carleman estimate for the operator $-\Delta + A \cdot \nabla + q$ when the coefficient A is sufficiently small and q is bounded.

Proposition 2.2. *Suppose $v \in H^2(\Omega)$ and $q, A \in L^\infty(\Omega)$. Let C be the constant in (10) and let τ_0 be given in the previous proposition. Then*

there exists constants $A_0 > 0$ and $\tilde{\tau}_0 > 0$ satisfying

$$C_1 := C - 4A_0^2 > 0,$$
$$C_2 := C\tilde{\tau}_0^2 - 4\|q\|_{L^\infty(\Omega)}^2 - 4\|A\|_{L^\infty(\Omega)}^2 > 0,$$

such that for $\|A\|_{L^\infty(\Omega)} \leq A_0$, $\tau \geq \tilde{\tau}_0$, and $C = \min(C_1, C_2)$ we have the estimate

(11)

$$C\left(2\tau^2\|v\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(\Omega)}^2\right) - C'\tau^2 \int_{\partial\Omega} |v|^2 dS - C'' \int_{\partial\Omega} \partial_\nu v\overline{v}dS$$

$$+ \int_{\partial\Omega} \left(4\tau Re(\partial_\nu v\partial_\xi \overline{v}) - 2\tau(\nu \cdot \xi)|\nabla v|^2 + 2\tau^3(\nu \cdot \xi)|v|^2\right) dS$$

$$\leq \|e^{-x\cdot\tau\xi}(-\Delta + A \cdot \nabla + q)(e^{x\cdot\tau\xi}v)\|_{L^2(\Omega)}^2.$$

Proof. Since

$$|-\Delta e^{x\cdot\tau\xi}v|^2 = |(-\Delta + A \cdot \nabla + q)(e^{x\cdot\tau\xi}v) - A \cdot \nabla(e^{x\cdot\tau\xi}v) - q(e^{x\cdot\tau\xi}v)|^2$$

$$\leq 4|(-\Delta + A \cdot \nabla + q)(e^{x\cdot\tau\xi}v)|^2$$

$$+ 4e^{2x\cdot\tau\xi}\|A\|_{L^\infty(\Omega)}^2|\nabla v|^2 + 4e^{2x\cdot\tau\xi}(\|q\|_{L^\infty(\Omega)}^2 + \tau^2\|A\|_{L^\infty(\Omega)}^2)|v|^2,$$

the result follows from (10). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Note that if in addition to the assumptions of Proposition 2.2 v vanishes on the boundary then for $u = \exp(x \cdot \tau\xi)v$, (11) simplifies to

$$C\left(\tau^2\|e^{-x\cdot\tau\xi}u\|_{L^2(\Omega)}^2 + \|e^{-x\cdot\tau\xi}\nabla u\|_{L^2(\Omega)}^2\right) + C'\tau \int_{\partial\Omega} (\nu \cdot \xi) \left|e^{-x\cdot\tau\xi}\partial_\nu u\right|^2 dS$$

$$\leq \|e^{-x\cdot\tau\xi}(-\Delta + A \cdot \nabla + q)u\|_{L^2(\Omega)}^2,$$

which generalizes (6). We believe that this estimate will have applications in the analysis and solution of inverse boundary value problems for equations involving first order perturbations of the Laplacian.

3. Construction of complex geometrical optics solutions

In this section we show how to construct CGO solutions to (4). A general method for constructing such solutions for equations involving a first order perturbation of the Laplacian was given by Nakamura, Sun and Uhlmann [NSU95] (see also [Tol98] and [Sal04]); in case A has the particular form (5) the method simplifies considerably.

The usual method for constructing CGO solutions to (2) and (4) is to conjugate in the equation with $\gamma^{1/2}$, i.e. if u satisfies (4) then $v = \gamma^{-1/2}u$ satisfies

$$(-\Delta + q)v = 0,$$

where $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. The idea is then to construct CGO solutions to such an equation involving a zeroth order perturbation of the Laplacian, which is well-known. However, this transformation requires more smoothness than we assume here. We will proceed by introducing a smooth approximation of γ and conjugate the equation with the approximation. By doing so we will obtain an equation involving a zeroth order perturbation and a small first order perturbation of the Laplacian, which can be handled.

Let $\Phi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function with $\int_{\mathbb{R}^n} \Phi(x)dx = 1, \operatorname{supp}(\Phi) \subset B(0,1)$ and $\Phi \equiv 1$ near zero, and define the smooth approximation of the delta distribution $\Phi_\tau(x) = \tau^n\Phi(\tau x)$. Suppose $\gamma \in C^{1+s}(\overline{\Omega})$ is extended outside Ω such that $\gamma \in C^{1+s}(\mathbb{R}^n)$ with $\gamma = 1$ outside a large ball. Then for

$$\phi = \log(\gamma)$$

we define

$$\phi_\tau = \Phi_\tau * \phi,$$
$$A_\tau = \Phi_\tau * A.$$

Note that $A_\tau = \nabla\phi_\tau$. In the following proposition we collect the detailed convergence estimates for the approximations:

Proposition 3.1. *Suppose $\gamma \in C^{1+s}(\overline{\Omega}), 0 \le s \le 1$. Then*

$$\|A - A_\tau\|_{C^s(\overline{\Omega})} = o(1), \tag{12}$$
$$\|\nabla \cdot A_\tau\|_{C^s(\overline{\Omega})} = o(\tau). \tag{13}$$

Moreover,

$$\|\phi - \phi_\tau\|_{L^\infty(\Omega)} = O(\tau^{-1-s}), \tag{14}$$
$$\|A - A_\tau\|_{L^\infty(\Omega)} = O(\tau^{-s}), \tag{15}$$
$$\|\nabla \cdot A_\tau\|_{L^\infty(\Omega)} = O(\tau^{1-s}), \tag{16}$$

and

$$\|(\gamma - e^{\phi_\tau})\|_{L^\infty(\Omega)} = O(\tau^{-1-s}), \tag{17}$$
$$\|\nabla(\gamma - e^{\phi_\tau})\|_{L^\infty(\Omega)} = O(\tau^{-s}). \tag{18}$$

Proof. For $s = 0, 1$ the estimates (12)–(13) are standard (see for instance [Bur98]); for s fractional the result follow by interpolation.

For a proof of (14)–(16) we refer to [Tay97]. To prove (17) we use the series expansion of the exponential, which implies

$$\gamma - e^{\phi_\tau} = \sum_{n=1}^{\infty} \frac{1}{n!} (\log(\gamma)^n - \phi_\tau^n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} (\log(\gamma) - \phi_\tau) \sum_{k=0}^{n-1} (\log(\gamma))^k \phi_\tau^{n-1-k}.$$

Now $\|\phi_\tau\|_{L^\infty(\Omega)} \leq C\|\log(\gamma)\|_{L^\infty(\Omega)}$ and (14) implies that

$$\|\gamma - e^{\phi_\tau}\|_{L^\infty(\Omega)} \leq \|\log(\gamma) - \phi_\tau\|_{L^\infty(\Omega)} \sum_{n=0}^{\infty} \frac{n}{n!} \|\log(\gamma)\|_{L^\infty(\Omega)}^n$$

$$\leq \|\log(\gamma)\|_{L^\infty(\Omega)} e^{\|\log(\gamma)\|_{L^\infty(\Omega)}} \|\log(\gamma) - \phi_\tau\|_{L^\infty(\Omega)}$$

$$= C\|\phi - \phi_\tau\|_{L^\infty(\Omega)}$$

$$= O(\tau^{-1-s}),$$

which proves (17).

The estimate (18) follows from (15) and (17) by noting that

$$\nabla(\gamma - e^{\phi_\tau}) = \gamma \nabla(\log(\gamma) - \nabla\phi_\tau) + (\gamma - e^{\phi_\tau})\nabla\phi_\tau$$

$$= \gamma(A - A_\tau) + (\gamma - e^{\phi_\tau})\nabla\phi_\tau.$$

$\square$

The CGO solutions to (4) we will construct are of the form

(19) $$u(x, \rho) = e^{-\phi_\tau/2} e^{x \cdot \rho} (1 + \omega(x, \rho)),$$

where ω tends to zero as $\rho \to \infty$. Here the parameter $\rho \in \mathbb{C}^n \setminus \{0\}$ satisfies $\rho \cdot \rho = 0$, which implies that $\exp(x \cdot \rho)$ is harmonic. We will decompose

$$(-\Delta + A \cdot \nabla) = (-\Delta + (A - A_\tau) \cdot \nabla + A_\tau \cdot \nabla).$$

and use the fact that

$$(-\Delta + A_\tau \cdot \nabla)(e^{-\phi_\tau/2} v) = e^{-\phi_\tau/2}(-\Delta - \frac{1}{2}\nabla \cdot A_\tau + \frac{1}{4}(A_\tau)^2)v$$

since $A_\tau = \nabla\phi_\tau$. It follows that

$$(-\Delta + A \cdot \nabla)e^{-\phi_\tau/2} v = (-\Delta + (A - A_\tau) \cdot \nabla + A_\tau \cdot \nabla)e^{-\phi_\tau/2} v$$

$$= e^{-\phi_\tau/2}(-\Delta + (A - A_\tau) \cdot \nabla + q_\tau)v,$$

where

$$q_\tau = -\frac{1}{2}\nabla A_\tau - \frac{1}{4}(A_\tau)^2 - \frac{1}{2}A \cdot A_\tau.$$

Hence the equation for ω is

(20) $\qquad (-\Delta_\rho + (A - A_\tau) \cdot \nabla_\rho + q_\tau)\omega = (A - A_\tau) \cdot \rho + q_\tau,$

where

$$\Delta_\rho = e^{-x \cdot \rho} \Delta e^{x \cdot \rho} = \Delta + 2\rho \cdot \nabla,$$
$$\nabla_\rho = e^{-x \cdot \rho} \nabla e^{x \cdot \rho} = (\nabla + \rho).$$

The equation

(21) $\qquad\qquad\qquad \Delta_\rho v = f$

plays an important role in solving (20). It is a well known fact that (Δ_ρ) has a bounded inverse in certain weighted L^2-spaces on $\mathbb{R}^n$ with norm decaying like $|\rho|^{-1}$, see [SU87]. The following proposition contains results concerning the solution of (21) in a bounded domain:

Proposition 3.2. *Let $s \geq 0$ and suppose $f \in H^s(\Omega)$. Let $\xi \in S^{n-1}$ and suppose $\rho \in \mathbb{C}^n \setminus \{0\}$ with $\rho \cdot \rho = 0$ and $Re(\rho) = \tau\xi$. Then for τ sufficiently large the equation (21) has a solution $v_1 \in H^{s+1}(\Omega)$ which satisfies*

(22) $\qquad \|v_1\|_{H^{s+t}(\Omega)} = C\tau^{-1+t}\|f\|_{H^s(\Omega)}, \quad 0 \leq t \leq 1.$

(23) $\qquad v_1|_{\partial\Omega_-(\xi)} = 0.$

Furthermore, for τ sufficiently large the equation (21) has a solution $v_2 \in H^{s+2}(\Omega)$ which satisfies

(24) $\qquad \|v_2\|_{H^{s+t}(\Omega)} = C\tau^{-1+t}\|f\|_{H^s(\Omega)}, \quad 0 \leq t \leq 2.$

Proof. The existence of a solution satisfying (22) and (23) can be proved as in [BU02,] using the Carleman estimate (6) and the Hahn-Banach theorem. Furthermore, the existence of a solution satisfying (24) follows by using the inverse in the weighted L2-space composed with an extension operator to the right and a restriction operator to the left. $\qquad\square$

Note the difference between the solutions v_1 and $v_2 : v_1$ seems to be less regular than v_2, but on the other hand we have explicit knowledge of v_1 on part of the boundary. We will need both types of solutions below. We will in the sequel abuse notation and denote by $(-\Delta_\rho)^{-1}$ any of the two solution operators to for the equation (21).

We now have the necessary ingredients for proving the existence of CGO solutions to (4):

Lemma 3.3. *Suppose $A \in C^s(\overline{\Omega}), 0 \leq s \leq 1$, is of the form (5). Let $\xi \in S^{n-1}$ and suppose $\rho \in \mathbb{C}^n$ with $Re(\rho) = \tau\xi$ and $\rho \cdot \rho = 0$. Then for*

τ *sufficiently large* (4) *has a solution* $u_1 \in H^{1+s}(\Omega)$ *of the form* (19) *with*

(25)
$$\|\omega_1\|_{H^{s+t}(\Omega)} = o(\tau^t), \quad 0 \le t \le 1,$$

(26)
$$\omega|_{\partial\Omega_-(\xi)} = 0.$$

Moreover, (4) *has a solution* $u_2 \in H^{s+2}(\Omega)$ *of the form* (19), *which satisfies*

(27)
$$\|\omega_2\|_{H^{s+t}(\Omega)} = o(\tau^t), \quad 0 \le t \le 2.$$

Proof. The equation (20) for ω is equivalent to

$$(I - ((A - A_\tau) \cdot \nabla_\rho(-\Delta_\rho)^{-1}) + q_\tau(-\Delta_\rho)^{-1}))(-\Delta_\rho)\omega = (A - A_\tau) \cdot \rho + q_\tau.$$

Then $\nabla_\rho(\Delta_\rho)^{-1}$ is bounded in $H^s(\Omega)$ with norm independent of τ, and we have by (12) and (13) that

$$\|(A - A_\tau) \cdot \nabla_\rho(-\Delta_\rho)^{-1}\|_{\mathcal{B}(H^s(\Omega))} = o(1),$$
$$\|q_\tau(-\Delta_\rho)^{-1}\|_{\mathcal{B}(H^s(\Omega))} = o(1),$$

as $\tau \to \infty$. Hence by fixing τ_0 sufficiently large, the operator $(I - ((A - A_\tau) \cdot \nabla_\rho + q_\tau)(-\Delta_\rho)^{-1})$ has for $\tau \ge \tau_0$ an inverse in $H^s(\Omega)$ given by a convergent Neumann series with norm independent of τ.

Since

$$\|(A - A_\tau) \cdot \rho + q_\tau\|_{H^s(\Omega)} = o(\tau)$$

we conclude that

$$(-\Delta_\rho)\omega = ((I - (A - A_\tau) \cdot \nabla_\rho + q_\tau)(-\Delta_\rho)^{-1})^{-1}((A - A_\tau) \cdot \rho + q_\tau)$$
$$= o(\tau).$$

It follows now from Proposition 3.2 that this equation has a solution ω_1 satisfying (25) – (26) as well as a solution ω_2 satisfying (27). $\square$

4. A BOUNDARY INTEGRAL IDENTITY

In this section we derive a useful boundary integral identity.

Lemma 4.1. *Suppose* $\gamma_j \in C^1(\overline{\Omega})$, $j = 1, 2$ *and suppose* $\tilde{u}_1, u_2 \in H^1(\Omega)$ *satisfy* $\nabla \cdot \gamma_j \nabla u_j = 0$ *in* Ω *with* $\tilde{u}_1 = u_2$ *on* $\partial\Omega$. *Suppose further that* $u_1 \in H^1(\Omega)$ *satisfies* $\nabla \cdot \gamma_1 \nabla u_1 = 0$. *Then*

$$\int_\Omega \left(\gamma_1^{1/2} \nabla(\gamma_2^{1/2}) - \gamma_2^{1/2} \nabla(\gamma_1^{1/2}) \right) \cdot \nabla(u_2 u_1) dx = \int_{\partial\Omega} \gamma_1 \partial_\nu(\tilde{u}_1 - u_2) u_1 dS,$$

where the integral is understood in the sense of the dual pairing between $H^{1/2}(\partial\Omega)$ *and* $H^{-1/2}(\partial\Omega)$.

Proof. Let $h = \gamma_1^{-1/2}\gamma_2^{1/2}$ and define

$$I = \int_\Omega \nabla \cdot \gamma_1 \nabla(\tilde{u}_1 - hu_2)u_1 dx.$$

Integrating by parts twice gives

$$I = \int_\Omega (\tilde{u}_1 - hu_2)\nabla \cdot (\gamma_1 \nabla u_1)dx + \int_{\partial\Omega} \gamma_1 \partial_\nu(\tilde{u}_1 - hu_2)u_1 dS$$

$$(28) \qquad = \int_{\partial\Omega} \gamma_1 \partial_\nu(\tilde{u}_1 - u_2)u_1 dS - \int_{\partial\Omega} \gamma_1 \partial_\nu(h)u_1 u_2 dS.$$

On the other hand

$$I = -\int_\Omega \left(\nabla \cdot \gamma_1 \nabla(hu_2)u_1 - \nabla \cdot \gamma_2 \nabla(u_2)h^{-1}u_1 \right) dx$$

$$= \int_\Omega (\gamma_1 \nabla(hu_2) \cdot \nabla u_1 - \gamma_1 \nabla u_2 \cdot \nabla(h^{-1}u_1))dx$$

$$- \int_{\partial\Omega} (\gamma_1 \partial_\nu(hu_2)u_1 - \gamma_2 \partial_\nu(u_2)h^{-1}u_1)dS$$

$$= \int_\Omega \left((\gamma_1 \nabla(h) \cdot u_2 \nabla u_1 - \gamma_2 \nabla(h^{-1}) \cdot u_1 \nabla u_2) + (\gamma_1 h - \gamma_2 h^{-1})\nabla u_2 \cdot \nabla u_1 \right) dx$$

$$- \int_{\partial\Omega} \left((\gamma_1 h - \gamma_2 h^{-1})\partial_\nu(u_2)u_1 + \gamma_1 \partial_\nu(h)u_1 u_2 \right) dS.$$

Now since

$$\gamma_1 \nabla h = -\gamma_2 \nabla(h^{-1}) = \gamma_1^{1/2}\nabla(\gamma_2^{1/2}) - \gamma_2^{1/2}\nabla(\gamma_1^{1/2})$$

$$\gamma_1 h - \gamma_2 h^{-1} = 0$$

we conclude that
(29)

$$I = \int_\Omega \left(\gamma_1^{1/2}\nabla(\gamma_2^{1/2}) - \gamma_2^{1/2}\nabla(\gamma_1^{1/2}) \right) \cdot \nabla(u_1 u_2)dx - \int_{\partial\Omega} \gamma_1 \partial_\nu(h)u_2 u_1 dS.$$

The result now follows from (28) and (29). $\qquad\square$

5. The uniqueness proof

In this section we will prove the uniqueness result Theorem 1.1.

Note that from the assumption (3) it follows by [SU88] that $\gamma_1|_{\partial\Omega_-} = \gamma_2|_{\partial\Omega_-}$ and $\partial_\nu(\gamma_1)|_{\partial\Omega_-} = \partial(\gamma_2)|_{\partial\Omega_-}$. Since these boundary values also agree on the $\partial\Omega_+$ by assumption, we have $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$ and $\partial_\nu(\gamma_1)|_{\partial\Omega} = \partial(\gamma_2)|_{\partial\Omega}$.

Fix $k \in \mathbb{R}^n$ with $k \cdot \xi = 0$. Chose $l \in \mathbb{R}^n$ with $l \cdot \xi = l \cdot k = 0$ such that $\rho_2 = \tau\xi + i\frac{k+l}{2}$ satisfies $\rho_2 \cdot \rho_2 = 0$. This is possible since $n \geq 3$. Let $u_2 = e^{x \cdot \rho_2}e^{-\phi_{2\tau}/2}(1 + \omega_2)$ be a CGO solution to $\nabla \cdot \gamma_2 \nabla u_2 = 0$,

which satisfies (27), and define $\tilde{u}_1$ as the solution to $\nabla \cdot \gamma_1 \nabla \tilde{u}_1 = 0$ with $\tilde{u}_1 = u_2$ on $\partial\Omega$.

Take $\rho_1 = -\tau\xi + i\frac{k-l}{2}$ and note that $\rho_1 \cdot \rho_1 = 0$. Let $u_1 = e^{x \cdot \rho_1} e^{-\phi_1\tau/2}(1 + \omega_1)$ be a CGO solution to $\nabla \cdot \gamma_1 \nabla u = 0$ with $\omega_1 = 0$ on $\partial\Omega_+$, (cf. (25)–(26) where we have substituded $-\xi$ for ξ). Then

(30)
$$\int_\Omega (\gamma_1^{1/2} \nabla \gamma_2^{1/2} - \gamma_2^{1/2} \nabla \gamma_1^{1/2}) \cdot \nabla(u_2 u_1) dx = \int_{\partial\Omega} \gamma_1 \partial_\nu (\tilde{u}_1 - u_2) u_1 dS.$$

We will first prove that the right hand side converges to zero as $\tau \to \infty$:

Lemma 5.1. *Suppose the assumptions of Theorem 1.1 hold and define* $\tilde{u}_1, u_2$ *and* u_1 *as above. Then*

(31)
$$\int_{\partial\Omega} \gamma_1 \partial_\nu (\tilde{u}_1 - u_2) u_1 dS = o(1)$$

as $\tau \to \infty$.

Proof. The assumption $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$ and (3) implies $\partial_\nu(\tilde{u}_1 - u_2) = 0$ on $\partial\Omega_{-,\epsilon}$ and hence

$$\left| \int_{\partial\Omega} \gamma_1 \partial_\nu(\tilde{u}_1 - u_2) u_1 dS \right| \leq C \int_{\partial\Omega_{+,\epsilon}} e^{-x \cdot \tau\xi} |\partial_\nu(\tilde{u}_1 - u_2)||1 + \omega_1| dS$$

(32)
$$\leq C \int_{\partial\Omega_{+,\epsilon}} e^{-x \cdot \tau\xi} |\partial_\nu(\tilde{u}_1 - u_2)| dS.$$

To estimate (32) we will use (11), however, it seems impossible to obtain a useful estimate directly. Thus we introduce the function

$$u = e^{\phi_1\tau/2}\tilde{u}_1 - e^{\phi_2\tau/2}u_2 = u_0 + \delta u$$

where

$$u_0 = e^{\phi_1\tau/2}(\tilde{u}_1 - u_2),$$
$$\delta u = (e^{\phi_1\tau/2} - e^{\phi_2\tau/2})u_2.$$

Since $\phi_{1\tau}$ is uniformly bounded from below by a positive constant for large τ, it follows that

$$\int_{\partial\Omega_{+,\epsilon}} e^{-x\cdot\tau\xi}|\partial_\nu(\tilde{u}_1 - u_2)|dS$$

$$\leq \int_{\partial\Omega_{+,\epsilon}} e^{-x\cdot\tau\xi}|\partial_\nu(u_0)|dS$$

$$\leq C\left(\int_{\partial\Omega_{+,\epsilon}} e^{-x\cdot\tau\xi}|\partial_\nu(\delta u)|dS + \int_{\partial\Omega_{+,\epsilon}} e^{-x\cdot\tau\xi}|\partial_\nu(u)|dS\right)$$

(33)

$$\leq C\left(\int_{\partial\Omega_{+,\epsilon}} e^{-2x\cdot\tau\xi}|\partial_\nu(\delta u)|^2 dS^{1/2} + \int_{\partial\Omega_{+,\epsilon}} e^{-2x\cdot\tau\xi}|\partial_\nu(u)|^2 dS^{1/2}\right).$$

Since $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$, $\partial_\nu\gamma_1|_{\partial\Omega} = \partial_\nu\gamma_2|_{\partial\Omega}$ it follows from (17) and (18) that

$$(34) \qquad \|(e^{\phi_{1\tau}/2} - e^{\phi_{2\tau}/2})\|_{L^\infty(\partial\Omega)} \leq O(\tau^{-3/2}),$$

$$(35) \qquad \|\nabla(e^{\phi_{1\tau}/2} - e^{\phi_{2\tau}/2})\|_{L^\infty(\partial\Omega)} \leq O(\tau^{-1/2}).$$

Further using (27) we easily find the estimates

$$(36) \quad \|e^{-x\cdot\tau\xi}u_2\|_{L^2(\partial\Omega)} \leq C\|1 + \omega_2\|_{L^2(\partial\Omega)} \leq C\|1 + \omega_2\|_{H^1(\Omega)} \leq o(\tau^{1/2}),$$

(37)

$$\|e^{-x\cdot\tau\xi}\nabla u_2\|_{L^2(\partial\Omega)} \leq C(\|\omega_2\|_{H^2(\Omega)} + \tau\|1 + \omega_2\|_{H^1(\Omega)}) \leq o(\tau^{3/2}),$$

and hence

$$(38) \qquad \int_{\partial\Omega} e^{-2x\cdot\tau\xi}|\partial_\nu(\delta u)|^2 dS = o(1).$$

To estimate the second term in (33) we note that $\partial_\nu u_0 = 0$ on $\partial\Omega_{-,\epsilon}$. This and the fact that $0 < \epsilon < (\nu\cdot\xi)$ on $\partial\Omega_{+,\epsilon}$ implies that

$$\int_{\partial\Omega_{+,\epsilon}} e^{-2x\cdot\tau\xi}|\partial_\nu(u)|^2 dS$$

$$\leq C\int_{\partial\Omega_{+,\epsilon}} (\nu\cdot\xi)e^{-2x\cdot\tau\xi}|\partial_\nu(u)|^2 dS$$

$$\leq C\left(\int_{\partial\Omega} (\nu\cdot\xi)e^{-2x\cdot\tau\xi}|\partial_\nu(u)|^2 dS - \int_{\partial\Omega_{-,\epsilon}} (\nu\cdot\xi)e^{-2x\cdot\tau\xi}|\partial_\nu(\delta u)|^2 dS\right)$$

$$\leq C\int_{\partial\Omega} (\nu\cdot\xi)e^{-2x\cdot\tau\xi}|\partial_\nu(u)|^2 dS + o(1),$$

where we have used (38). To estimate the remaining term we apply the Carleman estimate (11) to the function

$$v = e^{-x \cdot \tau \xi} u_2$$

with $A = (A_1 - A_{1\tau})$ and $q = q_{1\tau}$. By (34), (35), (36) and (37) it is straightforward to establish the estimates

$$(39) \qquad \int_{\partial\Omega} |v|^2 dS = \tau^{-1} \, \mathrm{o}(1),$$

$$(40) \qquad \int_{\partial\Omega} \partial_\nu v \overline{v} dS = \int_{\partial\Omega} (\nu \cdot \xi) e^{-2x \cdot \tau \xi} |\partial_\nu u|^2 dS^{1/2} + \mathrm{o}(1),$$

$$\int_{\partial\Omega} \left(4\tau \mathrm{Re}(\partial_\nu v \partial_\xi \overline{v}) - 2\tau(\nu \cdot \xi)|\nabla v|^2 + 2\tau^3 (\nu \cdot \xi)|v|^2 \right) dS$$

$$(41) \qquad\qquad = \tau \int_{\partial\Omega} (\nu \cdot \xi) e^{-2x \cdot \tau \xi} |\partial_\nu u|^2 dS + \mathrm{o}(\tau).$$

Hence (11) and (39)–(41) implies as $\tau \to \infty$ that

$$\int_{\partial\Omega} (\nu \cdot \xi) e^{-2x \cdot \tau \xi} |\partial_\nu u|^2 dS$$

$$\leq C\tau^{-1} \int_\Omega e^{-2\tau x \cdot \xi} |(-\Delta + (A_1 - A_{1\tau}) \cdot \nabla + q_{1\tau})u|^2 dx + \mathrm{o}(1)$$

$$\leq C\tau^{-1} \int_\Omega e^{-2\tau x \cdot \xi} |(-\Delta + (A_1 - A_{1\tau}) \cdot \nabla + q_{1\tau})e^{\phi_2\tau/2} u_2|^2 dx + \mathrm{o}(1)$$

$$\leq C\tau^{-1} \int_\Omega e^{-2\tau x \cdot \xi} |(((A_2 - A_{2\tau}) - (A_1 - A_{1\tau})) \cdot \nabla)(e^{x \cdot \rho_2}(1 + \omega_2))$$

$$+ (q_{2\tau} - q_{1\tau})(e^{x \cdot \rho_2}(1 + \omega_2))|^2 dx + \mathrm{o}(1),$$

since

$$(-\Delta + (A_j - A_{j\tau}) \cdot \nabla + q_{j\tau})(e^{-\phi_{j\tau}/2} u_j) = 0, \quad j = 1, 2.$$

It follows now from the decay estimates (15)–(16) for $A_{j\tau}, q_{j\tau}$ and (27) that

$$\|e^{-\tau x \cdot \xi}(((A_2 - A_{2\tau}) - (A_1 - A_{1\tau})) \cdot \nabla + q_{2\tau} - q_{1\tau})(e^{x \cdot \rho}(1 + \omega_2))\|^2_{L^2(\Omega)} = \mathrm{o}(\tau).$$

This shows that

$$(42) \qquad \int_{\partial\Omega} (\nu \cdot \xi) e^{-2x \cdot \tau \xi} |\partial_\nu u|^2 dS = \mathrm{o}(1).$$

By combining (32), (33), (38) and (42) we obtain (31). $\qquad\qquad \square$

We remark here that since the estimates (36) and (37) are not optimal, the conclusion of the lemma remains to be valid even for less regular conductivities. We will not need the refined estimates below.

Going back to (30) we note that the decay estimate (25) and (27) shows that

$$u_2 u_1 = e^{ix \cdot k} \gamma_1^{-1/2} \gamma_2^{-1/2} + \mathrm{o}(1)$$

in $H^{1/2}(\Omega)$ and therefore

$$\nabla(u_2 u_1) = \nabla(e^{ix \cdot k} \gamma_1^{-1/2} \gamma_2^{-1/2}) + \mathrm{o}(1)$$

in $H^{-1/2}(\Omega) = (H_0^{1/2}(\Omega))^*$, where $H_0^{1/2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^{1/2}(\Omega)$. Since $\gamma_1^{1/2} \nabla \gamma_2^{1/2} - \gamma_2^{1/2} \nabla \gamma_1^{1/2} H_0^{1/2}(\Omega)$ it follows that

$$\int_\Omega (\gamma_1^{1/2} \nabla \gamma_2^{1/2} - \gamma_2^{1/2} \nabla \gamma_1^{1/2}) \cdot \nabla(u_2 u_1) dx$$
$$= \int_\Omega (\gamma_1^{1/2} \nabla \gamma_2^{1/2} - \gamma_2^{1/2} \nabla \gamma_1^{1/2}) \cdot \nabla(e^{ix \cdot k} \gamma_1^{-1/2} \gamma_2^{-1/2}) dx + \mathrm{o}(1)$$
$$= \mathrm{o}(1),$$

which by taking $\tau \to \infty$ implies

$$\int_\Omega e^{ix \cdot k} \left(-\frac{ik}{2} \cdot (\nabla \log \gamma_1 - \log \gamma_2)) + \frac{1}{4}((\nabla \log(\gamma_1))^2 - (\nabla \log(\gamma_2))^2) \right) dx = 0$$

for $k \perp \xi$. Changing ξ in a small conic neighborhood shows that

$$(\frac{1}{2}\Delta(\log \gamma_1 - \log \gamma_2) + \frac{1}{4}((\nabla \log(\gamma_1))^2 - (\nabla \log(\gamma_2))^2))\widehat{}(k) = 0$$

in an open region and by analyticity vanishes everywhere. Therefore

$$\frac{1}{2}\Delta(\log \gamma_1 - \log \gamma_2) + \frac{1}{4}\nabla(\log \gamma_1 + \log \gamma_2) \cdot \nabla(\log \gamma_1 - \log \gamma_2) = 0,$$

and since $\log \gamma_1 = \log \gamma_2$ on $\partial\Omega$, uniqueness for the boundary value problem proves that $\log \gamma_1 = \log \gamma_2$.

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