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Non-symmetric 3-class association schemes

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Abstract

There are 24 feasible parameter sets for a primitive non-symmetric association schemes with 3 classes and at most 100 vertices. Using computer search, we prove non-existence for three feasible parameter sets. Eleven cases are still open.

In the imprimitive case, we survey the known results including some constructions of infinite families of schemes. In the smallest case that has been open up to now we construct a new scheme. This scheme is equivalent to a “skew” Bush-type Hadamard matrix of order 36. We also consider directed graphs that satisfy only some of the conditions required for a non-symmetric association scheme with 3 classes.

Let X be finite set ($|X| = v$) and let $\{R_0, R_1, \ldots, R_d\}$ be a partition of $X \times X$. Then we say that $\mathcal{X} = (X, \{R_0, R_1, \ldots, R_d\})$ is an association scheme with d classes if the following conditions are satisfied

- $R_0 = \{(x, x) \mid x \in X\}$.
- for each i, $R_i^{\mathrm{t}} := \{(x, y) \mid (y, x) \in R_i\} = R_{i'}$, for some i'.
- for each triple (i, j, h), $i, j, h \in \{0, \ldots, d\}$ there exists a so-called intersection number p_{ij}^h such that for all $x, y \in X$ with $(x, y) \in R_h$ there are exactly p_{ij}^h elements $z \in X$ so that $(x, z) \in R_i$ and $(z, y) \in R_j$.

If $i = i'$ for all i then $\mathcal{X}$ is said to be symmetric, otherwise it is non-symmetric. If the graphs $R_1, \ldots, R_d$ all are connected then we say that $\mathcal{X}$ is primitive, otherwise it is imprimitive.

In this paper we consider non-symmetric association schemes with $d = 3$ classes. We will assume that the relations are enumerated so that R_1 and R_2 are non-symmetric, $R_2 = R_1^{\mathrm{t}}$, and R_3 is a symmetric relation. In this case the association scheme is determined uniquely by relation R_1.

If A denotes the adjacency matrix of the relation R_1 then the adjacency matrices of R_0, R_2 and R_3 are I, A^{t} and $J-I-A-A^{\mathrm{t}}$, respectively. The Bose-Mesner algebra of $\mathcal{X}$ is the matrix algebra $\mathcal{A}$ spanned these four matrices, see Bannai and Ito [1].

Higman [10] proved that an association scheme with $d \leq 4$ has a commutative Bose-Mesner algebra, which means that $p_{ij}^h = p_{ji}^h$, for all i, j, h.

Thus multiplication in the Bose-Mesner algebra is determined by the following equations.

$$AJ = JA = \kappa J \tag{1}$$
$$AA^{\mathrm{t}} = \kappa I + \lambda(A + A^{\mathrm{t}}) + \mu(J - I - A - A^{\mathrm{t}}) \tag{2}$$
$$A^{\mathrm{t}}A = \kappa I + \lambda(A + A^{\mathrm{t}}) + \mu(J - I - A - A^{\mathrm{t}}) \tag{3}$$
$$A^2 = \alpha A + \beta A^{\mathrm{t}} + \gamma(J - I - A - A^{\mathrm{t}}), \tag{4}$$

where $\kappa = p_{12}^0$, $\lambda = p_{12}^1 = p_{21}^1$, $\mu = p_{12}^3 = p_{21}^3$, $\alpha = p_{11}^1$, $\beta = p_{11}^2$ and $\gamma = p_{11}^3$.

We note that $\alpha = \lambda$. This is seen by counting in two ways the pairs (y, z) so that $(x, y), (x, z), (y, z) \in R_1$, for a fixed vertex x.

Since $\mathcal{A}$ is commutative and consists of normal matrices, the matrices of $\mathcal{A}$ have a common diagonalization, i.e., $\mathcal{A}$ has a basis $\{E_0, E_1, E_2, E_3\}$ of ortogonal projections.

A relation (say R_1) of a symmetric association scheme with two classes is a strongly regular graph with parameters (v, k, a, c), where $v = |X|, k = p_{11}^0, a = p_{11}^1, c = p_{11}^2$. And conversely, if R_1 is a strongly regular graph and R_2 is the complementary graph of R_1, then R_1 and R_2 form a symmetric association scheme with two classes.

A relation of a non-symmetric association scheme with two classes is called a doubly regular tournament. Reid and Brown [19] proved that there exists a doubly regular tournament with n vertices if and only if there exists a skew Hadamard matrix of order $n + 1$. Thus a necessary condition is that $n \equiv 3 \mod 4$.

Since a non-symmetric association $\mathcal{X}$ with 3 classes is commutative, the symmetrization $(X, \{R_0, R_1 \cup R_2, R_3\})$ is also an association scheme, thus R_3 is a strongly regular graph and R_1 and R_2 are orientations of a strongly regular graph. In fact $R_1 \cup R_2$ is a strongly regular graph with parameters

$$(v, k, a, c) = (v, 2p^0_{12}, p^1_{11} + p^1_{12} + p^1_{21} + p^1_{22}, p^3_{11} + p^3_{12} + p^3_{21} + p^3_{22}) \quad (5)$$
$$= (v, 2p^0_{12}, 3p^1_{12} + p^1_{22}, 2(p^3_{11} + p^3_{12})) \quad (6)$$

In [13], we prove the following.

Lemma 1 *If A is the adjacency matrix of a regular directed graph (i.e., equation (1) is satisfied), then equation (2) and equation (3) are equivalent.*

(This is also an alternative proof of the commutativity of the Bose-Mesner algebra $\mathcal{A}$.) A directed graph whose adjacency matrix satisfies these equations is called normally regular. The eigenvalues of a normally regular digraph have the following property.

Theorem 2 ([13]) *If the adjacency matrix A of a regular directed graph satisfies equation (2) then an eigenvalue $\theta \neq k$ lies on the circle in the complex plane with centre $\lambda - \mu$ and radius $\sqrt{k - \mu + (\lambda - \mu)^2}$ and $\theta + \overline{\theta}$ is an eigenvalue of $A + A^t$.*

If A satisfies all the equations (1), (2), (3), (4) then it has four eigenvalues κ, and say ρ, σ and $\overline{\sigma}$ with multiplicities 1, m_1, m_2 and m_2, respectively, and the eigenvalues of $A + A^{\mathrm{t}}$ are 2κ, 2ρ, and $\sigma + \overline{\sigma}$ with multiplicities 1, m_1, and $2m_2$.

For parameters v and p^h_{ij}, $i, j, h \in \{0, 1, 2, 3\}$ the parameters of $R_1 \cup R_2$ can be computed from equation (6). Using standard formulas, the spectrum of $R_1 \cup R_2$ can then be computed. From this it is possible to compute eigenvalues and multiplicities of R_1 (e.g. using Theorem 2). For arbitrary intersection numbers the result may be expressions for the multiplicities which are not integers.

Definition 1 *We say that v and p^h_{ij}, $i, j, h \in \{0, 1, 2, 3\}$ form a feasible parameter set for a non-symmetric association scheme with three classes if they are non-negative integers and the multiplicities of the (four) eigenvalues computed from these intersection numbers are possitive integers.*

However, Bannai and Song proved that the spectrum of A can be computed from the spectrum of $A + A^{\mathrm{t}}$. (We note that if the eigenvalues of $A + A^{\mathrm{t}}$ are $2k, r, s$ then either r or s can be split in two complex eigenvalues, if their multiplicities are even.)

Lemma 3 (Bannai and Song [2]) *If $s = \sigma + \overline{\sigma}$ if an eigenvalue of $A + A^t$ then $\sigma = \frac{1}{2}(s + i\sqrt{v\kappa/m_2})$.*

From the spectrum of A it is possible to compute the intersection numbers.

The Hadamard product of matrices $B = (b_{ij})$ and $C = c_{ij})$ is the matrix $B \circ C = (b_{ij}c_{ij})$. Since $\{I, A, A^{\mathrm{t}}, J - A - A^{\mathrm{t}} - I\}$ is a basis of $\mathcal{A}$, it follows by considering the Hadamard product of these matrices that $\mathcal{A}$ is closed under the Hadamard product. In particular, there exists numbers q_{ij}^h, for $i, j, h \in \{0, 1, 2, 3\}$, so that $E_i \circ E_j = \frac{1}{v}\sum_h q_{ij}^h E_h$. These numbers are called Krein parameters. It is known that each Krein parameter is a non-negative real number, see Bannai and Ito [1]. Since the Krein parameters can be computed from the spectrum of A, this can be used to prove non-existence for some feasible parameter sets.

Neumaier [17] found another way to exclude feasible parameter sets. Let m_i be the rank of E_i, for $i \in \{0, 1, 2, 3\}$. (Thus $m_0, \ldots, m_3$ are the multiplicities of eigenvalues.)

Theorem 4 ([17]) *The following inequalities are satisfied for a commutative association scheme.*

$$\sum_{h\,:\,q_{ii}^h > 0} m_h \leq \frac{1}{2} m_i(m_i + 1), \quad \textit{for } i = 0, \ldots, d,$$

$$\sum_{h\,:\,q_{ij}^h > 0} m_h \leq m_i m_j, \quad \textit{for } i, j = 0, \ldots, d,\ i \neq j$$

1 Primitive association schemes with three classes.

Below we give a list of feasible parameter sets for primitive association schemes with three classes and $|X| \leq 100$. For each feasible parameterset (v, k, a, c) of a strongly regular graph we investigate the feasible parameters of non-symmetric association schemes with three classes such that $R_1 \cup R_2$ has parameters (v, k, a, c). It follows from equation (6) that we need only consider parameters where k and c are even. It is also useful to know that the eigenvalues of $R_1 \cup R_2$ are integers. This follows from the next lemma.

Lemma 5 (Goldbach and Claasen [8]) *There is no non-symmetric association schemes with three classes so that* $R_1 \cup R_2$ *has parameters* $(4c+1, 2c, c-1, c)$.

In parameter sets no. 7, 11 and 21 it is known that the strongly regular graph does not exist, see Brouwer [3].

In parameter set no. 17 some of the Krein parameters are negative. Thus this case is excluded. The multiplicities of eigenvalues for parameter sets no. 16 and 22 do not satisfy Neumaier's condition.

No.	Parameters for $R_1 \cup R_2$	p_{12}^1	p_{12}^3	exists	reference
1	$(16, 10, 6, 6)$	1	2	no	Goldbach and Claasen [7]
2	$(21, 10, 3, 6)$	1	1	no	Enomoto and Mena [5]
3	$(36, 14, 4, 6)$	0	2	yes	Goldbach and Claasen [6]
4	$(36, 20, 10, 12)$	3	2	NO	Theorem 8
5	$(45, 32, 22, 24)$	6	4	NO	Theorem 7
6	$(50, 42, 35, 36)$	8	12	NO	Theorem 6
7	$(57, 42, 31, 30)$	7	9	no	Wilbrink and Brouwer [22]
8	$(64, 28, 12, 12)$	4	2	yes	Enomoto and Mena [5]
9	$(64, 36, 20, 20)$	4	6	?	
10	$(64, 42, 26, 30)$	7	6	?	
11	$(64, 42, 30, 22)$	7	6	no	absolute bound
12	$(81, 50, 31, 30)$	9	5	?	
13	$(85, 64, 48, 48)$	13	8	?	
14	$(85, 70, 57, 60)$	13	20	?	
15	$(96, 38, 10, 18)$	3	4	?	
16	$(96, 50, 22, 30)$	3	10	no	Neumaier
17	$(96, 60, 38, 36)$	11	6	no	Krein
18	$(96, 76, 60, 60)$	16	10	?	
19	$(100, 44, 18, 20)$	3	6	?	
20	$(100, 54, 28, 30)$	8	6	?	
21	$(100, 66, 39, 52)$	10	12	no	absolute bound
22	$(100, 66, 41, 48)$	8	16	no	Neumaier
23	$(100, 66, 44, 42)$	10	12	?	
24	$(100, 72, 50, 56)$	13	12	?	

For parameters no. 6, R_3 is a strongly regular graph with parameters $(50, 7, 0, 1)$, i.e., it is the Hoffman-Singleton graph. This case can be excluded, by investigating possible orientations of the complement of the Hoffmann-Singleton graph.

Theorem 6 *There is no non-symmetric association scheme with three classes where R_3 is the Hoffman-Singleton graph.*

Proof. Suppose that there exists a non-symmetric association scheme with three classes where R_3 is the Hoffman-Singleton graph. Let x be a vertex and let $x_1, \ldots, x_7$ be the neighbours of x in R_3. Let S_i be the set of neighbours of x_i other that x, for $i = 1, \ldots, 7$. Let $N^+(x)$ be the set out-neighbours of x in R_1. Then $N^+(x)$ is a set of 21 vertices in the set $N_2(x) := S_1 \cup \ldots \cup S_7$ of vertices at distance 2 from x, and $|S_i \cap N^+(x)| = p^3_{13} = 3$, for $i = 1, \ldots, 7$. The subgraph of R_3 spanned by $N^+(x)$ is regular of degree $p^1_{13} = 4$. The complement of $N^+(x)$ in $S_1 \cup \ldots \cup S_7$ is the set of in-neighbours of x in R_1 and this set also spans a 4-regular subgraph of R_3.

A computer enumeration shows that there are exactly 1140 subsets of $N_2(x)$ with the properties required for $N^+(x)$. These 1140 subsets form three orbits under the action of the subgroup of the automorphism group of the Hoffman-Singleton graph stabilizing the vertex x.

For each pair x, y of vertices, the orientation of the edges incident with x and the orientation of the edges incident with y should agree on the orientation of the edge $\{x, y\}$ if x and y are non-adjacent in R_3, and they should satisfy that for all i, j the number of vertices z so that $(x, z) \in R_i$ and $(z, y) \in R_j$ is exactly p^h_{ij} where $(x, y) \in R_h$.

A computer search shows that there are no orientations all of edges incident with x, x_1, x_2, x_3, x_4 and x_5 that satisfy these conditions. Thus the required association scheme does not exist. □

For parameters no. 5, R_3 is a strongly regular graph with parameters $(v, k, a, c) = (45, 12, 3, 3)$.

E. Spence, see [21], has shown that there are exactly 78 strongly regular graphs with these parameters. This result was verified by J. Degraer and K. Coolsaet (personal communication with Spence).

Thus the method from the previous theorem can be applied to each of these 78 graphs.

Theorem 7 *There is no primitive non-symmetric association scheme with three classes with parameterset no. 5.*

Proof. Suppose that there exists such an association scheme. Let x be a vertex and let $x_1, \ldots, x_{12}$ be the neighbours of x in R_3. Let S_i be the set of neighbours of x_i at distance 2 from x, $|S_i| = k - a - 1 = 8$, for $i = 1, \ldots, 12$. Let $N^+(x)$ be the set out-neighbours of x in R_1. Then $N^+(x)$ is a set of 16 vertices in the set $N_2(x) := S_1 \cup \ldots \cup S_{12}$, and $|S_i \cap N^+(x)| = p^3_{13} = 4$, for $i = 1, \ldots, 12$. The subgraph of R_3 spanned by $N^+(x)$ is regular of degree $p^1_{13} = 3$.

The computer search shows that if N is a set with $|S_i \cap N| = 4$, for $i = 1, \ldots, 12$, and in which every vertex has degree at most 3 then N is 3-regular and the subgraph of R_3 spanned by $N_2(x) \setminus N$ is also 3-regular.

The number of such sets N depend on the graph and the vertex x. The largest number of sets is 396, which appear in the graph with a rank 3 automorphism group.

44 of the 78 candidates for R_3 can be excluded because, for at least one vertex x, there is no such set N.

For each of the other 34 graphs we find by computer search a set W of at most 8 vertices so that there is no combination of orientations of edges in the complement of R_3 incident with w, for each $w \in W$ that satisfies the required properties. (This search took 45 minutes on a 2.4 GHz PC.)

Thus an association scheme with parameterset no. 5 does not exist. □

Using computersearch we can exclude one more case.

Theorem 8 *There is no primitive non-symmetric association scheme with three classes with parameter set no. 4.*

Proof. We use an orderly search algorithm (se Read [18]) to search for the matrix $B = 3A_3 + 2A_2 + A_1$, where A_1, A_2, A_3 are adjacency matrices of the relations R_1, R_2, R_3 of an association scheme with parameter set no. 4.

We want the vertices to be enumerated so that the matrix B is in maximal form, i.e., the sequence obtained by reading the entries of the first row followed by the entries of the second row, etc., is as large as possible (in the lexicographic order) among all enumerations of the vertices.

In turns out that with this condition (and for parameter set no. 4) it is convenient to enumerate the relations so that R_1 is symmetric and $R_2^t = R_3$.

Suppose that the first $r-1$ rows of the matrix $B = (b_{ij})$ has been filled in. We then investigate all possible ways to fill in row r with 0 on the diagonal entry, $p^0_{11'} = 15$ entries with 1's, $p^0_{22'} = 10$ entries with 2's, and $p^0_{33'} = 10$ entries with 3's in such a way that

- the first $r-1$ entries are in accordance with the entries of column r of the previous rows.
- for each $x < r$ the number of columns s, so that $b_{xs} = i$ and $b_{rs} = j'$ is exactly p_{ij}^h, where $b_{xr} = h$.
- the matrix is still in maximal form.

We find that the number of ways to fill in the first r rows is 1, 1, 100, 24161, 205671, 1116571, 52650, 39, 0, $\ldots$, 0, for $r = 1, \ldots, 36$. Thus the required association scheme does not exist. (This search took 81 minutes on a 2.4 GHz PC.) □

2 Imprimitive association schemes with three classes.

If R_3 is connected but R_1 and R_2 are disconnected then each connected component of R_1 is a doubly regular tournament on $2p_{12}^0+1$ vertices. Thus the study of these schemes reduces to the study of doubly regular tournaments.

We will thus assume that R_1 and R_2 are connected and R_3 is disconnected. Then R_3 consists of m copies of a complete graph on r vertices, for some constants m and r. We denote this graph by $m \circ K_r$. Then R_1 is an orientation of the complement $\overline{m \circ K_r}$. The vertex set of $\overline{m \circ K_r}$ is partitioned in m independent sets of size r, denoted by $V_1, \ldots, V_m$.

In [14] we introduce the following family of graphs that do not necessarily satisfy all the conditions on a relation of a non-symmetric association scheme with three classes. We say that a directed graph is a doubly regular (m, r)-team tournament if it is an orientation of $\overline{m \circ K_r}$ with adjacency matrix A satisfying equations (1) and (4).

In [14] we give a combinatorial proof of the following, i.e., we do not use eigenvalues.

Theorem 9 (Jørgensen, Jones, Klin and Song [14]) *Every doubly regular (m, r)-team tournament is of one of the following types.*

1. *For every pair i, j either all the edges between V_i and V_j are directed from V_i to V_j, or they are all directed from V_j to V_i. The graph with vertices $v_1, \ldots, v_m$ and edges $v_i \to v_j$ if edges are directed from V_i to V_j is a doubly regular tournament.*

2. *For every vertex $x \in V_i$, exactly half of the vertices in V_j ($j \neq i$) are out-neighbours of x, and $\alpha = \beta = \frac{(m-2)r}{4}$, and $\gamma = \frac{(m-1)r^2}{4(r-1)}$.*

3. *For every pair $\{i, j\}$ either V_i is partitioned in two sets V_i' and V_i'' of size $\frac{r}{2}$ so that all edges between V_i and V_j are directed from V_i' to V_j and from V_j to V_i'', or similarly with i and j interchanged. The parameters are $\alpha = \frac{(m-1)r}{4} - \frac{3r}{8}$, $\beta = \frac{(m-1)r}{4} + \frac{r}{8}$, $\gamma = \frac{(m-1)r^2}{8(r-1)}$.*

A graph of type 3 can not be a relation of an association scheme. In this case 8 divides r and $4(r-1)$ divides $m-1$. We do not know if any graph of this type exists.

Every graph of type 1 or type 2 is a relation of a non-symmetric association scheme with 3 classes. The results for these types where first proved by Goldbach and Claasen [9].

Clearly, the graph in case 1 exists if and only a doubly regular tournament of order m exists.

2.1 Type 2

We now consider graphs of type 2. We first show that a graph of this type is a relation of a non-symmetric association scheme with 3 classes. This is done by proving that equations (2) and (3) are satisfied.

Lemma 10 *Let A be the adjacency matrix of a doubly regular (m, r)-team tournament of type 2. Then A satisfies equations (2) and (3) with*

- $\lambda = \alpha = \frac{(m-2)r}{4}$ *and*
- $\mu = \frac{(m-1)r(r-2)}{4(r-1)}$.

In particular if $m = r$ then $\lambda = \mu = \frac{m(m-2)}{4}$.

Proof. Let $x \in V_i$ and $y \in V_j$, $i \neq j$, and suppose that $x \to y$. Then x has $\kappa - \frac{r}{2}$ out-neighbours outside $V_i \cup V_j$. α of these are in-neighbours of y and the remaining $\kappa - \frac{r}{2} - \alpha$ are out-neighbours of y. Thus $\lambda = \kappa - \frac{r}{2} - \alpha = \frac{(m-2)r}{4}$, since $\kappa = \frac{(m-1)r}{2}$.

Similarly, for $x, y \in V_i$, we get $\mu = \kappa - \gamma = \frac{(m-1)r(r-2)}{4(r-1)}$. Thus equation (2) is satisfied. Equation (3) can be proved in a similar way, or by applying Lemma 1. □

Since the parameters of a graph of type 2 are integers, it follows that r is even and $r-1$ divides $m-1$. Using eigenvalues, it can be shown that m is even, see [14] or Goldbach and Claasen [9].

Existence in the case $r=2$ is equivalent to existence of a doubly regular tournament of order $m-1$.

Theorem 11 ([14]) *If there exists a doubly regular $(m,2)$-team tournament Γ of type 2 then 4 divides m and the outneighbours of a vertex in Γ span a doubly regular tournament of order $m-1$.*

Conversely, for every doubly regular tournament T of order $m-1$, there exists a doubly regular $(m,2)$-team tournament Γ, such that for some vertex x in Γ the out-neighbours of x span a subgraph isomorphic to T.

No schemes with $4 \leq r < m$, where $r-1$ divides $m-1$ are known.

We will now consider the case $m=r$. We will see that such association schemes are equivalent to special cases of some well-known structures.

Definition 2 *An Hadamard matrix H of order n is an $n \times n$ matrix in which every entry is either 1 or -1 and $HH^t = nI$.*

An Hadamard matrix H of order m^2 is said to be Bush-type if H is a block matrix with $m \times m$ blocks H_{ij} of size $m \times m$ such that $H_{ii} = J_m$ and $H_{ij}J_m = J_m H_{ij} = 0$, for $i \neq j$.

Theorem 12 *An imprimitive association scheme with 3 classes of type 2 and with $r=m$ is equivalent to a Bush-type Hadamard matrix of order m^2 with the property that $H_{ij} = -H_{ji}$, for all pairs i,j with $i \neq j$.*

Proof. Let A be an adjacency matrix of relation R_1, for some imprimitive association scheme with 3 classes of type 2 and with $r=m$. We may assume that vertices are enumerated such that the vertices in V_i corresponds to coloumns/rows $mi-i+1,\ldots,mi$. Let $H = J_{m^2} - 2A$. Then H is partitioned in blocks H_{ij} of size $m \times m$ corresponding to the partition of vertices in sets $V_1,\ldots,V_m$. Clearly $H_{ii} = J_m$ and since a vertex in V_i has exactly $\frac{m}{2}$ out-neighbours and $\frac{m}{2}$ in-neighbours in V_j, $H_{ij}J_m = J_m H_{ij} = 0$.

From equations (1) and (2) we get (since $\kappa = \frac{m(m-1)}{2}$ and $\mu = \lambda = \frac{m(m-2)}{4}$)

$$HH^{\mathrm{t}} = (J_{m^2} - 2A)(J_{m^2} - 2A^{\mathrm{t}}) = (m^2 - 4\kappa)J_{m^2} + 4(\kappa I + \mu(J-I)) = m^2 I.$$

Thus H is an Hadamard matrix.

Conversely, suppose that H is a Bush-type Hadamard matrix which is skew in the sense that $H_{ij} = -H_{ji}$, for $i \neq j$.

Let $A = \frac{1}{2}(J - H)$, where $J = J_{m^2}$. Then A is a $\{0, 1\}$ matrix. Since H is Bush-type it has exactly $m + (m-1)\frac{m}{2}$ entries equal to 1 and $(m-1)\frac{m}{2}$ entries equal to -1 in each row. Thus $HJ = mJ$ and the transposed equation is $JH^{\mathrm{t}} = mJ$. Similarly $JH = mJ$. Thus $AJ = JA = \frac{m(m-1)}{2}J$ and

$$AA^{\mathrm{t}} = \frac{1}{4}(J - H)(J - H^{\mathrm{t}}) = \frac{m(m-2)}{4}J + \frac{m^2}{4}I.$$

We see that equations (1) and (2) are satisfied. Equation (3) can be proved in a similar way, or by applying Lemma 1.

Let K denote the block diagonal matrix with diagonal blocks equal to J_m. Then the Bush-type property of H implies that $HK = mK$ and the skew property of H implies that $H + H^{\mathrm{t}} = 2K$. Thus $H^2 = H(2K - H^{\mathrm{t}}) = 2mK - m^2 I$, and so

$$A^2 = \frac{1}{4}(J - H)^2 = \frac{1}{4}(m(m-2)J + 2mK - m^2 I).$$

Since $J - I - A - A^{\mathrm{t}} = K - I$, it follows that equation (4) is satisfied with $\alpha = \beta = \frac{m(m-2)}{4}$ and $\gamma = \frac{m^2}{4}$. □

Kharaghani [15] proved that if there exists an Hadamard matrix of order m then there exists a Bush-type Hadamard matrix of order m^2.

Ionin and Kharaghani [11] modified this construction and proved that if there exists an Hadamard matrix of order m then there exists a Bush-type Hadamard matrix of order m^2, which has the skew property required in Theorem 12.

Thus in many cases with $m = r$ a multiple of 4, an association scheme can be constructed.

The case with $m = r$ congruent to 2 modulo 4 seems to be more difficult and no general constructions are known. But in the special case $m = r = 6$ we may apply the computer search algorithm described in the proof of Theorem 8. However, it is estimated that a complete search would take several years. We stopped the search after a few days. At that time two association schemes were found.

Theorem 13 *There exists an imprimitive non-symmetric association scheme with 3 classes of type 2 with* $m = r = 6$.

Proof. The adjacency matrix of R_1 is listed below for one such scheme. □

A Bush-type Hadamard matrix of order 36 was first constructed Janko [12]. But a "skew" Bush-type Hadamard matrix of order 36 was not previously known. Bussemaker, Haemers and Spence [4] proved that a symmetric Bush-type Hadamard matrix of order 36 does not exist.

0 0 0 0 0 0	1 1 1 0 0 0	1 1 1 0 0 0	1 1 1 0 0 0	1 1 1 0 0 0	1 1 1 0 0 0
0 0 0 0 0 0	1 1 0 1 0 0	1 0 0 1 1 0	1 0 0 1 1 0	1 1 0 1 0 0	0 0 0 1 1 1
0 0 0 0 0 0	1 0 0 0 1 1	1 0 0 1 0 1	0 1 0 1 0 1	0 0 1 1 1 0	1 1 0 1 0 0
0 0 0 0 0 0	0 1 1 0 1 0	0 1 0 0 1 1	1 0 0 0 1 1	0 0 0 1 1 1	1 0 1 0 1 0
0 0 0 0 0 0	0 0 0 1 1 1	0 1 1 0 1 0	0 0 1 0 1 1	1 0 1 0 0 1	0 1 0 1 0 1
0 0 0 0 0 0	0 0 1 1 0 1	0 0 1 1 0 1	0 1 1 1 0 0	0 1 0 0 1 1	0 0 1 0 1 1

0 0 0 1 1 1	0 0 0 0 0 0	1 1 0 1 0 0	0 1 0 0 1 1	1 0 1 0 1 0	0 0 1 0 1 1
0 0 1 0 1 1	0 0 0 0 0 0	1 0 0 0 1 1	0 1 1 0 1 0	0 1 0 1 0 1	0 1 1 1 0 0
0 1 1 0 1 0	0 0 0 0 0 0	0 1 1 0 0 1	1 0 0 1 0 1	0 1 1 1 0 0	0 1 0 0 1 1
1 0 1 1 0 0	0 0 0 0 0 0	0 0 1 1 0 1	1 0 0 1 1 0	1 0 1 0 0 1	1 0 1 1 0 0
1 1 0 0 0 1	0 0 0 0 0 0	0 1 1 0 1 0	0 1 1 1 0 0	1 0 0 1 1 0	1 0 0 1 1 0
1 1 0 1 0 0	0 0 0 0 0 0	1 0 0 1 1 0	1 0 1 0 0 1	0 1 0 0 1 1	1 1 0 0 0 1

0 0 0 1 1 1	0 0 1 1 1 0	0 0 0 0 0 0	1 1 0 1 0 0	1 0 0 1 0 1	1 1 0 0 0 1
0 1 1 0 0 1	0 1 0 1 0 1	0 0 0 0 0 0	0 0 0 1 1 1	1 1 0 0 1 0	1 1 1 0 0 0
0 1 1 1 0 0	1 1 0 0 0 1	0 0 0 0 0 0	0 1 1 0 1 0	0 0 1 1 0 1	1 0 0 0 1 1
1 0 0 1 1 0	0 1 1 0 1 0	0 0 0 0 0 0	0 0 1 1 1 0	0 1 1 0 1 0	0 1 0 1 1 0
1 0 1 0 0 1	1 0 1 1 0 0	0 0 0 0 0 0	1 0 1 0 0 1	0 0 1 1 1 0	0 0 1 1 0 1
1 1 0 0 1 0	1 0 0 0 1 1	0 0 0 0 0 0	1 1 0 0 0 1	1 1 0 0 0 1	0 0 1 1 1 0

0 0 1 0 1 1	1 1 0 0 1 0	0 1 1 1 0 0	0 0 0 0 0 0	0 1 0 0 1 1	1 0 0 1 0 1
0 1 0 1 1 0	0 0 1 1 0 1	0 1 0 1 1 0	0 0 0 0 0 0	0 1 1 1 0 0	1 0 1 1 0 0
0 1 1 1 0 0	1 0 1 1 0 0	1 1 0 0 0 1	0 0 0 0 0 0	1 0 0 0 1 1	0 1 0 1 1 0
1 0 0 1 1 0	1 1 0 0 0 1	0 0 1 0 1 1	0 0 0 0 0 0	1 0 0 1 1 0	0 1 1 0 0 1
1 0 1 0 0 1	0 0 1 0 1 1	1 0 0 0 1 1	0 0 0 0 0 0	1 1 1 0 0 0	1 0 0 0 1 1
1 1 0 0 0 1	0 1 0 1 1 0	1 0 1 1 0 0	0 0 0 0 0 0	0 0 1 1 0 1	0 1 1 0 1 0

0 0 1 1 0 1	0 1 1 0 0 1	0 0 1 1 1 0	1 1 0 0 0 1	0 0 0 0 0 0	0 1 0 1 1 0
0 0 1 1 1 0	1 0 0 1 1 0	1 0 1 0 1 0	0 0 1 1 0 1	0 0 0 0 0 0	1 0 1 0 1 0
0 1 0 1 0 1	0 1 0 0 1 1	1 1 0 0 0 1	1 0 1 1 0 0	0 0 0 0 0 0	0 0 1 1 0 1
1 0 0 0 1 1	1 0 0 1 0 1	0 1 0 1 0 1	1 0 1 0 1 0	0 0 0 0 0 0	1 1 0 0 1 0
1 1 0 0 1 0	0 1 1 1 0 0	1 0 1 0 0 1	0 1 0 0 1 1	0 0 0 0 0 0	1 0 0 1 0 1
1 1 1 0 0 0	1 0 1 0 1 0	0 1 0 1 1 0	0 1 0 1 1 0	0 0 0 0 0 0	0 1 1 0 0 1

0 1 0 0 1 1	1 1 1 0 0 0	0 0 0 1 1 1	0 0 1 1 0 1	1 0 1 0 0 1	0 0 0 0 0 0
0 1 0 1 0 1	1 0 0 1 1 0	0 0 1 0 1 1	1 1 0 0 1 0	0 1 1 0 1 0	0 0 0 0 0 0
0 1 1 0 1 0	0 0 1 0 1 1	1 0 1 1 0 0	1 0 1 0 1 0	1 0 0 1 1 0	0 0 0 0 0 0
1 0 0 1 0 1	1 0 1 0 0 1	1 1 1 0 0 0	0 0 0 1 1 1	0 1 0 1 0 1	0 0 0 0 0 0
1 0 1 0 1 0	0 1 0 1 0 1	1 1 0 0 1 0	1 1 0 1 0 0	0 0 1 0 1 1	0 0 0 0 0 0
1 0 1 1 0 0	0 1 0 1 1 0	0 1 0 1 0 1	0 1 1 0 0 1	1 1 0 1 0 0	0 0 0 0 0 0

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