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by

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1   Abstract

There are many results dealing with the problem of decomposing a fixed graph into isomorphic subgraphs. There has also been work on characterizing graphs with the property that one can delete the edges of a number of edge disjoint copies of the subgraph and, regardless of how that is done, the graph that remains can still be decomposed (such graphs are called *randomly packable* or *randomly decomposable*). In this paper we consider the following variation. Given a fixed graph H, determine which graphs (call them *equipackable*) have the property that every *maximal* edge disjoint packing with H is *maximum*. In the case that the graph H is isomorphic to the path on 3 nodes, we characterize the equipackable graphs of girth 5 or

more.

2   Introduction

We begin with some necessary definitions. Let P_k denote a path on k vertices. A vertex with precisely one neighbour is called a *leaf* while a vertex adjacent to a leaf is called a *stem*. A stem is said to be of *odd (even) parity* if it is adjacent to an odd (even) number of leaves. Let H be a subgraph of G. An *H-packing* in G is a collection of edge disjoint copies of H, say $H_1, \ldots, H_t$ where each H_i is a subgraph of G. An H-packing in G with t isomorphic copies $H_1, \ldots, H_t$ of H is called *maximal* if $G - \cup_{i=1}^t E(H_i)$ contains no subgraph isomorphic to H. An H-packing in G with t isomorphic copies $H_1, \ldots, H_t$ of H is called *maximum* if no more than t edge disjoint copies of H can be packed into G. For example, if $G = P_5 = v_1 v_2 v_3 v_4 v_5$ and $H = P_3$, we see that $v_2 v_3 v_4$ is a maximal P_3-packing in P_5 while $v_1 v_2 v_3, v_3 v_4 v_5$ is a maximum P_3-packing. A graph G is called *H-equipackable* if every maximal packing with H in G is a maximum packing with H in G. For instance, every graph is P_2-equipackable while P_6 is P_3-equipackable. This concept is partly prompted by the study of *well-covered graphs* ([8]), those in which every maximal independent set of vertices is also a maximum as well as *equimatchable graphs* ([5, 12]), those in which every maximal matching is a maximum. Such graphs have the advantage that one only need apply a greedy algorithm to find the desired set. We also note that researchers have examined *randomly P_k-packable graphs*, that is, graphs that regardless of how one removes paths of a fixed length, one completely decomposes the edge set of the graph into paths of that fixed length. The term *randomly decomposable graphs* is also used ([1, 7, 9, 10]). We observe that these graphs are a subset of the H-equipackable ones. In particular, Beineke, Hamburger and Goddard ([1]) characterized graphs which are randomly packable with paths on k edges for $k = 4, 5, 6$ while Molina, McNally and Smith ([9]) recently extended the results to $k = 7, 8, 9$. Of course there has been much work on *packable graphs*. That is, on deciding when it is possible to decompose a graph (usually the complete graph) into isomorphic copies of a fixed subgraph ([2, 4]). In the case of the path on 3 vertices, it has been shown that any graph on an even number of edges can, in fact, be completely decomposed into edge disjoint copies of such paths. The following lemma by Caro and Schönheim ([3]) stating this fact was given an alternative proof by Ruiz ([10]) and is again established in Lemma 16 below.

Lemma 1 *A connected graph is P_3-packable if and only if it has even size.*

Observing that a connected graph G of odd size at least 3 contains an edge e such that all remaining edges are in one component of $G - e$, and hence has even size and is P_3-packable, the following corollary is clear.

Corollary 2.1 *If a connected graph is P_3-equipackable, a maximum packing either contains all edges or all but one edge of the graph.*

Since we are interested in maximal packings that are all maximum they must be of one size and since the previous result shows that for any graph there is a maximal packing missing at most one edge we have the following observation that will be useful.

Lemma 2 *If there is a maximal P_3-packing of a connected graph G which omits at least 2 edges, then the graph G is not P_3-equipackable.*

Another observation that is used (often implicitly) is the following.

Lemma 3 *If a graph G is P_k-equipackable and F is a collection of edge-disjoint P_k's, then G - F is P_k-equipackable.*

Proof: If $G - F$ had two different size maximal P_k-packings, say A and B, then $A \cup F$ and $B \cup F$ would be two different size maximal P_k-packings of G, a contradiction.

Lemma 4 *Given a connected graph G and F a collection of edge-disjoint P_3's, if $G - F$ contains at least two components with an odd number of edges, then G is not P_3-equipackable.*

Proof: If G is P_3-equipackable, then $G - F$ must also be. But any maximal P_3-packing of $G - F$ omits at least one edge from each of the odd size components. Thus, by Lemma 2, G is not P_3-equipackable.

3 The Characterization

We begin with a sequence of technical lemmas that will be useful in characterizing the graphs of girth 5 or more that are P_3-equipackable.

Lemma 5 *If a graph G has two stems that are at distance m greater than or equal to 4, then G is not P_3-equipackable.*

Proof: Assume a graph G is P_3-equipackable and has two stems, say A and B, which are at distance m $\geq$ 4. Let a shortest path joining A and B be A,$w_1, w_2, ..., w_{m-1}, B$. For every neighbour, say x, of A that is not a leaf nor w_1, select a neighbour, say y, of x and delete the edges Ax and xy. In the resulting graph repeat the process. Once the only neighbours of A are leaves (some might have been created by this procedure) and w_1 repeat the process at B.

If in the resulting graph A is of odd parity remove Aw_1, w_1w_2, otherwise A is of even parity and is adjacent to ≥ 2 leaves and we remove xA, Aw_1 where x is a leaf. In both cases we have isolated a star with center A having odd size. Similarly we isolate a star with center B and having odd size. Thus G is not P_3-equipackable by Lemma 4.

Lemma 6 *If a graph G is P_3-equipackable and has two stems that are at distance 3, then both of these stems must be of odd parity.*

Proof: Assume a graph G is P_3-equipackable and has two stems, say A and B, which are at distance 3, and at least one of the stems, say A, is of even parity, say $2r, r \geq 1$. Let a shortest path joining A and B be A, w_1, w_2, B.

(a) B is a stem of even parity:

(a)(i) B has 2s neighbours other than its leaves and w_2. In this case remove (as a P_3) Bw_2 along with an edge from B to a leaf and pair up the 2s edges to non-leaves and delete as P_3's.

(a)(ii) B has 2s + 1 neighbours other than its leaves and w_2. In this case remove (as a P_3) Bw_2 and w_1w_2. Also delete an edge from B to a leaf along with one edge from B to a non-leaf neighbour (other than w_2). Then pair up the rest of the 2s edges to non-leaves and delete as P_3's.

(b) B is a stem of odd parity:

(b)(i) B has 2s neighbours other than its leaves and w_2. Delete Bw_2 and w_1w_2 (as a P_3) and pair up and delete the 2s edges to non-leaves.

(b)(ii) B has 2s + 1 neighbours other than its leaves and w_2. Pair up Bw_2 and one of the edges to the other non-leaves and then pair up the 2s edges from B to non-leaves that remain.

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In both (a) and (b), the resulting graph contains a component that is a star with an odd number of edges and with center at B.

Now consider A. If A has $2t$ edges to non-leaves (other than w_1), then delete Aw_1 and an edge from A to a leaf and then pair up and delete the $2t$ edges to the other non-leaves.

If A has $2t + 1$ non-leaf neighbours besides w_1, then remove the P_3 consisting of Aw_1 and an edge from A to a leaf. Then for some non-leaf neighbour (not w_1) of A, say x, choose a neighbour (other than A), say y, of x. Remove the edges Ax and xy as a P_3. Then pair up the $2t$ edges to non-leaves that remain. This leaves another component (A and the remaining leaves) with an odd number of edges. Together with the component at B we now have two components with an odd number of edges and thus by Lemma 4, G is not equipackable.

Lemma 7 *If a graph G is P_3-equipackable and has two stems that are at distance 2, then these stems must be of different parity.*

Proof: Assume a graph G is P_3-equipackable and has two stems, say A and B, which are at distance 2, and the stems are of the same parity. Let AwB be a path of length two joining A and B.

First remove all but one path of length two, say AwB, joining A and B. Let A have r leaves attached and B have s leaves. Say deg(A)=(r+1)+ c and deg(B)=(s+1)+ d.

If both c and d are even, then pair up the c edges (to non-leaves other than w) and the d edges at B (to non-leaves other than w) and delete as P_3's. If exactly one of c and d, say c, is even and the other is odd, then pair up the c edges at A and at B select a neighbour, say x, that is a non-leaf but not w. Choose a neighbour, say y (other than B), of x. Then pair up Bx and xy and delete. Then pair up the remaining d - 1 edges to non-leaves at B and remove. In both these cases (exactly one or both of c and d even), if r and s are both odd, pair up Aw and wB and delete and if r and s are both even, pair up an edge from A to a leaf and Aw as well as an edge from B to a leaf and Bw and remove as P_3's. The resulting graph has at least two components with an odd number of edges and by Lemma 4 this contradicts G being equipackable.

Now consider both c and d odd. If r and s are odd, then pair up the c edges at A along with Aw and at B pair up the d edges along with Bw and delete as P_3's. If r and s are even, then first remove the edges Aw and wB as a P_3. Then pair up the c edges to non-leaves at A along with one edge

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from A to a leaf. Similarly at B pair up the d edges and an edge from B to a leaf and delete. Again we are left with two components each with an odd number of edges proving by Lemma 4 that G is not equipackable.

Lemma 8 *If a graph G is P_3-equipackable and has two stems that are adjacent, then these stems must be of the same parity.*

Proof: Assume there are two stems, say A and B, that are adjacent but of different parity. Say A is of odd parity and B of even parity. Let e denote the edge joining A and B. First observe that e must be a cut edge. Assume not. Let the length of a shortest path joining A and B that does not include the edge e be k. If k=2 then remove the P_3 consisting of e and an edge joining B to a leaf. But then the parities of A and B in the resulting graph violate Lemma 7. If k=3 and A is adjacent to ≥ 3 leaves remove BA and an edge joining A to a leaf. Then we obtain two even stems at distance 3 contradicting Lemma 6. If k=3 and A is adjacent to exactly one leaf we consider first the case deg(A)>3: Remove BA and an edge joining A to a vertex which neither is a leaf nor is on the length 3 AB-path. We obtain two stems at distance 3 of opposite parities contradicting Lemma 6. We consider next the case k=3, A adjacent to exactly one leaf and deg(A)=3: If deg(B) is even, pair off BA with the other non-leaf neighbour at A and at B pair off all remaining edges, except for one edge joining B to a leaf. We have isolated two edges and can apply Lemma 4. If deg(B) is odd, remove the first two edges on the 3-path from A to B, and remove at B all edges except one edge joining B to a leaf. This isolates two edges and we apply Lemma 4. If k is 4 or more, deleting the edge AB and an edge from B to a leaf results in a graph in which Lemma 5 is contradicted. Hence e must be a cut edge. Say A has 2a + 1 leaves and deg(A)=(2a + 1) + 1 + r and B is adjacent to 2b leaves, to A and to s non-leaf neighbours giving deg(B)=(2b) + 1 + s. If r is odd, let x be adjacent to A where x is neither a leaf nor the vertex B. Let y be adjacent to x. Remove the pair of edges Ax and xy (as a P_3) and then pair up the rest of the r edges meeting A that do not join A to leaves nor to B and remove them (as P_3's). If r is even just pair up all of these edges and remove. Repeat this procedure at the vertex B. Finally delete the P_3 consisting of the edge e and an edge from B to a leaf. But the resulting graph has two stems of odd parity as components and, by Lemma 4, we have a contradiction. Hence adjacent stems must be of the same parity.

Lemma 9 *If a graph G is P_3-equipackable and has two stems that are at distance 3, where the vertices on this shortest path are w_1 and w_2, then*

these stems must be of odd parity and have no neighbours other than leaves and w_1 or w_2. In addition w_1 and w_2 are of degree two.

Proof: Assume that G is P_3-equipackable and has two stems, say A and B, that are at distance 3, where this path is Aw_1w_2B. By Lemma 6, both A and B are of odd parity. First assume one of the stems, say A, has $s > 0$ non-leaf neighbours besides w_1. If s is odd, pair the edges to these neighbours and Aw_1 and delete as P_3's isolating A and its leaves. If s is even, for one of these neighbours, say x, select a neighbour, say y (other than A) and delete the edges Ax and xy (as a P_3). Note that this does not result in y being a (new) leaf at B else the (new) graph would have two stems of different parities at distance 3 (in violation of Lemma 6). Next pair up the remaining s - 1 edges as well as Aw_1 and remove as P_3's. Again we have isolated A and its leaves. Then isolate B and its leaves by pairing up the edges to non-leaves (if this number is even) or (if this number is odd) pair up Bw_1 and w_1w_2 and the rest of the edges as pairs. Thus neither stem can have a non-leaf neighbour other than w_1 or w_2 for otherwise we violate Lemma 4.

But if either w_1 or w_2, say w_1, has a neighbour (besides A and w_2), say x, then delete Aw_1 and wx as well as Bw_2 and w_1w_2 (as P_3's) and again two odd components are left (contradicting Lemma 4). The lemma follows.

Observe conversely that if a graph G has two stems A and B, both of odd parity, that are at distance 3, where Aw_1w_2B is a shortest path joining them and w_1 and w_2 are both of degree 2 and neither A nor B has neighbours other than its leaves and w_1 or w_2, then G is P_3-equipackable.

Lemma 10 *Say G is P_3-equipackable, has girth at least 5 and has two stems that are at distance two where w is a common neighbour of the stems. Then these two stems must be of different parity and neither stem has other neighbours than its leaves and w. Furthermore, the vertex w must be of degree two.*

Proof: Let G be a graph satisfying the hypothesis where A and B are the 2 stems sharing a common neighbour w. By Lemma 7 the stems A and B must be of different parity. Assume A is of even parity and B of odd parity. If either A or B has a neighbour other than w that is a non-leaf, remove the edge from the stem to it along with an edge from the stem to a leaf (as a P_3). But the resulting graph has two stems of the same parity at distance two which contradicts Lemma 7. Thus neither A nor B has a non-leaf neighbour other than w.

Assume w has $r > 0$ neighbours besides A and B. If r is odd, pair up the r edges to these vertices along with wB resulting in a graph with two components on an odd number of edges (violating Lemma 4). If r is even, first note that w is not a stem. If it were, then its parity would have to agree with both A and B (by Lemma 8) which is not possible. For some neigbour (neither A nor B), say x, of w, select a neighbour other than w of x, say y. Remove wx and xy as a P_3 and pair up the rest of the r edges meeting w along with wB and remove. Again this violates Lemma 4. Hence the lemma follows.

Observe that the girth restriction is sharp as the graph consisting of a 4-cycle $abcd$ with a having one leaf adjacent and c having two leaves as neighbours, is P_3-equipackable.

Say G is a graph of girth at least 5 and has two stems that are at distance two where w is the common neighbour of the stems. If it is the case that these two stems are of different parity and neither stem has other neighbours than its leaves and w and the vertex w is of degree two, then, conversely, it is easy to verify that G is P_3-equipackable.

Lemma 11 *Say G is P_3-equipackable, has girth at least 4, and has two adjacent stems. Then the stems must be of the same parity and have only each other and their leaves as neighbours.*

Proof: Say G is P_3-equipackable, has girth at least 4, and has two adjacent stems, say A and B. First we note that A and B must be of the same parity by Lemma 8. Now assume A has r non-leaf neighbours (besides B) and B has s non-leaf neighbours (besides A).

(a) Both A and B of odd parity:

(a)(i) Say exactly one of r and s, say r, is odd and the other even. Pair up the s edges from B to non-leaves other than to A and delete, also pair up the r edges at A along with AB and remove. This results in two components each with an odd number of edges which is impossible (Lemma 4).

(a)(ii) Say both r and s are odd. Pair up and delete the r edges at A along with AB. At B for some non-leaf neighbour (other than B), say x, select a neighbour (other than B), say y (possible since girth at least 4), and delete Bx and xy as a P_3. Then pair up the remaining edges at B to non-leaves and remove. This again is a contradiction (to Lemma 4).

(a)(iii) Say both r and s are even. Pair up the r edges to non-leaves at A and remove while at B for some non-leaf neighbour (other than A), say

x, select a neighbour (other than A), say y, and delete Bx and xy as a P_3. Then pair up the rest of the s edges at B along with AB and delete. A contradiction to Lemma 4 again results.

(b) Both A and B of even parity:

(b)(i) Say exactly one of r and s, say r, is odd and the other even. Pair up the r edges to non-leaves at A and an edge from A to a leaf and delete. Pair up AB and an edge from B to a leaf and remove.Then pair up the s edges and delete. This would contradict Lemma 4.

(b)(ii) Say both r and s are odd. At A remove edges as in (b)(i). At B delete the edges AB and an edge from B to leaf along with Bx and xy (where x is a non-leaf neighbour and y is a neighbour of x). Then pair up the rest of the s edges and remove. Again we have a contradiction to Lemma 4.

(b)(iii) Say both r and s are even. At A pair up the r edges to non-leaves and remove and also delete AB along with an edge from A to a leaf. At B select a non-leaf neighbour of B, say x, and call one of its neighbours y (not B). Remove Bx and xy. Also remove an edge from B to a leaf along with one of the edges from B to a non-leaf. Then pair up the remaining edges from B to non-leaves and delete. Again, by Lemma 4, the resulting graph is not possible if the original graph was P_3-equipackable. This proves the lemma.

The restriction girth ≥ 4 is sharp. Consider the 3-cycle abc having two leaves, one adjacent to a and the other to b.

Conversely, note that if G has two adjacent stems of the same parity and these stems have only each other and their leaves as neighbours, then G is P_3-equipackable.

Observe that if G is a tree and P_3-equipackable, it is either a star or is described by the previous lemmas since it would have two stems. In addition we have shown the following corollary.

Corollary 3.1 *Any connected P_3-equipackable graph which is not a tree but is of girth 5 or more can have at most one stem.*

We now complete the characterization of P_3-equipackable graphs of girth 6 or more.

Lemma 12 *Say G is a connected P_3-equipackable graph of girth 6 or more*

but is not a tree. Then the graph must be C_7.

Proof: Assume G satisfies the hypothesis of the lemma but is not C_7. First assume there is a vertex, say A, that has degree at least three, but is not a stem. Let three of its neighbours be B, C and D. Since G has at most one stem (Corollary 3.1), we can assume at least two of these neighbours, say B and C, are not stems. Consider vertex B. For each of its neighbours, say x, other than A, choose a neighbour, say y, and delete the P_3 consisting of Bx and xy. In the resulting graph B is now a leaf and hence A is a stem. Note that the resulting graph H does not contain another stem, say S. Otherwise A and S are stems in H and by Corollary 3.1 H is a tree. But outside a shortest AS-path H contains a vertex C or D, say D, which is not a leaf. That contradicts the structure described in Lemmas 6-8. So H has a unique stem, namely A.

Now consider the vertex C. For each of its neighbours, say x, other than A, choose a path $Cxyz$, and delete the P_3 consisting of xy and yz. In the graph that results both A and C are stems while D, again seen by the girth restriction, is not a leaf. But this contradicts Lemma 11.

Hence, in G, any vertex of degree more than two must be a stem. But as there can be at most one stem, say A, the other vertices must each be on a circuit with A. Select one such circuit, say $ABCDE...X$. Delete the P_3 consisting of CD and DE. But now both A and B are stems in the resulting graph and there is at least one more vertex (the vertex X adjacent to A that is neither a leaf nor a stem). This contradicts Lemma 11. Thus there are no stems in the graph. This implies that the graph is simply one circuit. It is easy to verify that the only possibility is C_7 which is P_3-equipackable. That proves Lemma 12.

Lemma 13 *Say G is a connected P_3-equipackable graph of girth 5. Then G is either C_5 or has 5+2m vertices where G consists of a circuit of length 5 along with 2m leaves attached to exactly one node on the 5-cycle.*

Proof: Let G be a connected P_3-equipackable graph of girth 5 and *abcde* be a 5-cycle in G. Assume there is a vertex, say A, that is a stem but A is different from the vertices in *abcde*. Observe that there are no other stems in G (by Corollary 3.1). Select a shortest path from A to the given 5-cycle and let a be the point on the 5-cycle to which the path joins A. Remove the edges bc and cd (as a P_3). For each neighbour, say x, of b (other than a), choose a neighbour, say y, and remove the edges bx and xy. When this process is finished b is a leaf. For the vertex d for each neighbour, say x, if

it has a neighbour, say y, remove the edges dx and xy. In the final graph d is either a stem or a leaf. Note that A is still in the same component as a by the girth restriction (no edge that was removed was in the shortest path from A to the 5-cycle). But then we have a graph with at least three stems, namely A, a and one of d,e, which means G is not P_3-equipackable (Lemmas 10 and 11). Thus no vertex not on the 5-cycle can be a stem. Next assume some vertex, say A, is not on the 5-cycle and is not adjacent to any vertex of the 5-cycle. Select a shortest path from A to the 5-cycle and for each neighbour, say x, of A not on that shortest path, choose a neighbour, say y, and delete Ax and xy. In the resulting graph A is a leaf implying there is a stem not on the 5-cycle. But this is impossible as shown in the first part of the proof. Hence any vertex not on the 5-cycle must be adjacent to some vertex of the 5-cycle. Let A be a vertex not on the 5-cycle and be adjacent to a. If A is not a leaf, let B be a neighbour (other than a) of A. By the girth restriction B is not on the 5-cycle. Since B must also be adjacent to the 5-cycle it must be adjacent to either c or d. Without loss of generality say B is adjacent to c. Delete ab and bc. But now for each vertex, say v, of the 6-cycle $aABcde$, select a neighbour, say x, not on the 6-cycle and if x has a neighbour, say y, delete vx and xy. When this operation can no longer be performed the resulting graph has a component containing a 6-cycle (namely $aABcde$) and possibly leaves attached to some of the vertices of the 6-cycle. But, by Lemma 12, such a graph is not P_3-equipackable. This shows that any vertex not on $abcde$ must be leaf. By Corollary 3.1, G has at most one stem. If this stem, say a, were of odd parity, then delete ea and ab leaving two components both with an odd number of edges which is not possible since G is P_3-equipackable. If the stem were of even parity it is easily verified that the graph is P_3-equipackable. Also C_5 itself is in the collection. This completes the proof and the characterization.

We summarize the characterization in the following theorem.

Theorem 1 *A graph G of girth 5 or more is P_3-equipackable if and only if G satisfies one of the following:*

(i) G is a tree consisting of a single star (i.e., $K_{1,n}$).

(ii) G is a tree which has two stems that are at distance 3, where the vertices on this shortest path are w_1 and w_2. Furthermore the stems are of odd parity and have no neighbours other than leaves and w_1 or w_2. In addition w_1 and w_2 are of degree two.

(iii) G is a tree which has two stems that are at distance two where w is the common neighbour of the stems. The two stems must be of different parity and neither stem has other neighbours than its leaves and w.

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Furthermore, the vertex w must be of degree two.

(iv) G is a tree which has two stems that are adjacent where these stems are of the same parity and these stems have only each other and their leaves as neighbours.

(v) G is either C_7, C_5 or has 5+2m vertices where G consists of a circuit of length 5 along with 2m leaves, all attached to the same vertex on the 5-cycle.

We have shown that if a graph has no leaves and is of girth 8 or more then it is not P_3-equipackable. This result generalizes to paths of larger order as shown by the following lemma.

Lemma 14 *Let k be an integer, $k \geq 3$. If $\delta(G) \geq 2$ and $g(G) \geq 4(k-1)$, then G is not P_k-equipackable.*

Proof: Select a circuit in the graph. Given the girth restriction and the fact that there are no leaves, it is possible to successively remove copies of P_k to isolate a P_{2k-1}. But such a resulting path is not P_k-equipackable. □

We note that the girth restriction is sharp since C_{4k-5} is P_k-equipackable, for $k \geq 2$.

Lemma 15 *Let $k \geq 2$ and G be a connected, P_k-equipackable graph with $\delta(G) \geq 2$ and $g(G) = 4k - 5$. Then G is precisely a circuit of length $4k - 5$.*

Proof: It is easy to verify that C_{4k-5} itself is P_k-equipackable. Assume G is not C_{4k-5} but is a connected, P_k-equipackable graph with $\delta(G) \geq 2$ and $g(G) = 4k-5$. Let $v_1, v_2, \ldots, v_{4k-5}$ be a cycle of length $4k-5$ in G and v_0 be a neighbour of some vertex, say v_1, on the cycle. Delete $v_0, v_1, \ldots, v_{k-1}$ and $v_{3k-3}, v_{3k-2}, \ldots, v_{4k-5}, v_1$ as P_k's. Now consider the component containing the path, say P, with $2k$-2 edges which is $v_{k-1}, v_k, \ldots, v_{3k-3}$. Observe that if any pair of vertices of P are joined by some other path, say Q, that Q must have at least $2k$-2 edges by the girth restriction. Also note that if any vertex, say x, of P is joined to v_0 by a path using only x from $v_1, v_2, \ldots, v_{4k-5}$, then such a path has more than k-1 edges (girth restriction again). Hence, since G had no leaves, we can isolate the path P by removing P_k's which have one end vertex on P but no edges in common with P. But the resulting graph, namely, a path on $2k$-2 edges, is not P_k-equipackable. □

4 Interpolation

A graph parameter $\alpha(G)$ is said to interpolate over a family $\mathcal{G}$ of graphs if for every three integers $a < b < c$ with $\alpha(G_1) = a, \alpha(G_3) = c$, $G_1, G_3 \in \mathcal{G}$, there exists $G_2 \in \mathcal{G}$ such that $\alpha(G_2) = b$.

Lemma 16 *Let G be any graph. The number of P_3's in a maximal P_3-packing interpolates over G.*

Proof: We may assume G is connected. We consider packings of G which contain a copies of P_3 and are maximal, but not maximum. Among all P_3-packings with a copies of P_3 choose A and e, f such that e, f are two edges at minimum distance from each other in G and not belonging to A. Let $e_1, \ldots, e_k$ be a shortest path in G joining an end of e to an end of f. Thus each edge $e_i, 1 \le i \le k$, must belong to A. Assume $k > 1$. Let $e = x_1 y_1$, $e_1 = x_1 x_2$. If e_1 forms a P_3 in A with $f_1 = x_1' x_1$ we form the maximal P_3-packing $A' = A + y_1 x_1 x_1' - x_2 x_1 x_1'$ with a P_3-copies and $e_1, f \notin A'$ have $d_G(e_1, f) < d_G(e, f)$, a contradiction. Hence e_1 must form a P_3 in A with $f_1 = x_2' x_2$ and we form the maximal P_3-packing $A' = A + y_1 x_1 x_2 - x_1 x_2 x_2'$ which has a P_3-copies and $x_2 x_2', f \notin A'$ and has $d_G(x_2 x_2', f) < d_G(e, f)$, again a contradiction. Thus $k = 1$ and A' constructed above along with $e_1 f$ in the first situation and $f_1 f$ in the second is a maximal P_3-packing with $a + 1$ copies of P_3. This establishes the lemma. $\square$

We observe that the proof of Lemma 16 also establishes Lemma 1 since it constructively finds a larger packing if there is more than one edge missing.

It is not obvious what the situation is for larger k and so we pose the following question.

Question. Do maximal P_4-packings interpolate over a graph? To what extent can G be decomposed into $P_4's$?

Note that maximal P_k-packings of size 1,2 and 3 exist in the graph G formed by taking P_{k-1}, P_k and P_{k+1} and identifying one end vertex of each of these paths.

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