Uncertain Buckling Load and Reliability of Columns with Uncertain Properties

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PAPER NO. 141

Submitted to "Structural Safety"

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ABSTRACT

Continuous and finite element methods are utilized to determine the buckling load of columns with material and geometrical uncertainties considering deterministic, stochastic and interval models for the bending rigidity of columns. When the bending rigidity field is assumed to be deterministic, the ordinary finite element method slightly overestimates the buckling load, and with a very few number of elements high rate of convergence to the exact results is observed. If the bending rigidity field is modelled using random fields, stochastic finite element method is utilized. The discretization is performed using weighted integrals. Then, the buckling load becomes a random variable. The sensitivity of the lower order moments of the buckling load with respect to the mesh size, the correlation length and coefficient of variation of the random field are examined. The reliability of columns designed considering safety factors are estimated by means of extensive Monte Carlo simulations. For the case, when the bending rigidity field is taken to be bounded from above and below, an integral equation formulation and optimization methods are used to determine conservative bounds for the buckling load. For structural design, the lower bound is of crucial interest. The buckling load of fixed-free, simply-supported, pinned-fixed, fixed-fixed columns and a sample frame are calculated.

1. INTRODUCTION

The paper considers the uncertainties in physical and geometrical quantities appearing in the model equations of column buckling boundary value problem. The uncertainties are due to physical imperfections, model inaccuracies and system complexities. Deterministic methods of analysis neglects these uncertainties. Modelling uncertainties as random variables or random processes suggests the use of stochastic methods; as set-theoretic variables bounded from above and below within envelope bounds requires the use of interval algebra and optimization techniques.

In this study, the problem of determining the linear buckling load of columns and frame structures with material and geometrical uncertainties under only axial static loads is considered and three different approaches, namely deterministic, stochastic and interval, are presented, deterministic one being a reference to evaluate the other two.

A. Deterministic approach:
Bernoulli-Euler beam-column equation is used to determine the deterministic buckling load.which is computed both analytically and using finite element methodology. The sensitivity of the numerical results to the mesh size of the discretization is investigated.

B. Stochastic approach:
The bending rigidity field can be modelled using random fields. Then, the buckling load becomes a random variable. To the best knowledge of the authors, there is no exact closed-form solution available for the random buckling load even for the simplest system. A finite element discretization leads to a stiffness matrix with random elements. The stochastic differential equation is discretized using the weighted integral idea of Deodatis (1990) and Takada (1990) such that the random bending rigidity field is represented by 3 random variables multiplied by deterministic shape functions. The lower order statistics of the random buckling load can be determined analytically for problems with small dimension, via simulation or perturbation techniques for problems with larger dimension. The sensitivity of the numerical results to the mesh size, correlation length and coefficient of variation of the random fields are investigated. The stability of columns and frames with random initial geometrical imperfections were studied by Boyce (1961), Elishakoff (1979), Lin and Kam (1992), Palassopoulos (1993), however uncertain material properties are considered in a few studies, Jeong (1992), Zhang and Ellingwood (1995). Zhang and Ellingwood (1995) considered a stochastic finite element formulation where random bending rigidity field is discretized using random nodal displacements and Legendre polynomials, and random eigenvalue problem is solved using a perturbation method to calculate the mean and coefficient of variation of the buckling load.

C. Interval approach:
The models of uncertainty used in the interval approach are set-theoretic. All structural uncertainties are assumed to be bounded from above and below. Analytically closed form results are given for the lower bound of the buckling load of columns. This derivation is based on integral equations and optimization techniques. Interval finite elements proposed in Köylüoğlu, Çakmak and Nielsen (1994) is also applied to the same problem, but as this formulation yields too conservative bounds, the results based on the interval finite element method are not presented in this paper. Interval algebra is thoroughly discussed in a number of books, (e.g. Alefeld and Herzberger 1983, Neumaier 1990). However, not many applications to mechanics have been reported. Introducing interval analysis for problems with uncertainty in mechanics is mentioned by Elishakoff (1991) and a book on nonprobabilistic convex modelling of uncertainty for continuous systems appeared, recently (Ben-Haim and Elishakoff 1990). A complete literature survey for continuous systems and several illustrations for the applications in mechanics are given in this book. Applications of interval algebra in connection with finite element method is given in Köylüoğlu et al. (1994), Elishakoff et al. (1995) and Qui et al. (1995).

2. DETERMINISTIC FINITE ELEMENTS

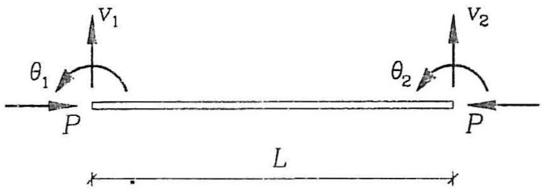

Figure 1) Bernoulli-Euler beam-column element with 4 degrees of freedom.

According to the Bernoulli-Euler beam theory, the deflection field $v(x)$ and the buckling load p_{cr} of a column of length L are related as follows :

$$\frac{d^2}{dx^2}\left((ei)(x)\frac{d^2v(x)}{dx^2}\right) - \frac{d^2}{dx^2}\left(p_{cr}v(x)\right) = 0 \quad , \quad x \in [0, L] \tag{1}$$

where $(ei)(x)$ is the deterministic bending rigidity field. Exact solutions for p_{cr} can be calculated for simple systems such as single columns, approximate solutions are needed for complicated frame problems. In what follows, finite element approach is outlined and compared to available exact solutions for fixed-free, pin-pin, pin-fixed, fixed-fixed columns and a sample frame.

The displacement field of an element $v(x)$ is approximated as a linear combination of the nodal deformations $\mathbf{v}$ with signs indicated in Figure 1, multiplied by deterministic cubic interpolation functions $\mathbf{n}(x)$ which can be compactly written as

$$v(x) = \mathbf{n}^T(x)\mathbf{v} \tag{2}$$

where

$$\mathbf{v}^T = [v_1,\ \theta_1,\ v_2,\ \theta_2] \tag{3}$$

as shown in Figure 1, and,

$$\mathbf{n}(x) = \begin{bmatrix} 1 & 0 & -\frac{3}{L^2} & \frac{2}{L^3} \\ 0 & 1 & -\frac{2}{L} & \frac{1}{L^2} \\ 0 & 0 & \frac{3}{L^2} & -\frac{2}{L^3} \\ 0 & 0 & -\frac{1}{L} & \frac{1}{L^2} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} \tag{4}$$

Following finite element methodology, e.g. Yang (1986), one can end up with the geometrical stiffness matrix $\mathbf{k}_e^G$ and the stiffness matrix $\mathbf{k}_e$ of the e^{th} beam element as

$$k^G_{e,ij} = \int_0^L \frac{dn_i(x)}{dx}\frac{dn_j(x)}{dx}dx \tag{5}$$

$$k_{e,ij} = \int_0^L (ei)(x) \frac{d^2 n_i(x)}{dx^2} \frac{d^2 n_j(x)}{dx^2} dx \qquad (6)$$

Integration in (5) can be evaluted using the listed shape functions $\mathbf{n}(x)$.

$$\mathbf{k}_e^G = \frac{1}{30} \begin{bmatrix} \frac{36}{L} & 3 & -\frac{36}{L} & 3 \\ 3 & 4L & -3 & -L \\ -\frac{36}{L} & -3 & \frac{36}{L} & -3 \\ 3 & -L & -3 & 4L \end{bmatrix} \qquad (7)$$

For the case when the bending rigidity is constant, i.e. $(ei)(x) = ei$, the element stiffness matrix becomes

$$\mathbf{k}_e = \frac{ei}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \qquad (8)$$

Next, the global matrices can be obtained from the finite element assembling and the buckling load is calculated from

$$\text{Det } |\mathbf{k} - p_{cr}\mathbf{k}^G| = 0 \qquad (9)$$

where $\mathbf{k}$ is the $n \times n$ positive-definite global stiffness matrix and $\mathbf{k}^G$ is the global geometrical stiffness matrix of elements in compression. Det $|\cdot|$ indicates the determinant of the matrix considered. The buckling load p_{cr} is the lowest eigenvalue of (9).

The buckling load p_{cr} can be calculated exactly for simple columns and for the frame shown in Figure 2.b using the differential equation (1), e.g. Simitses (1976).

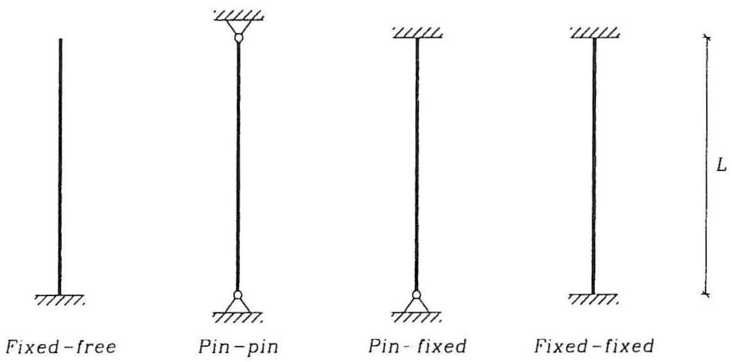

Figure 2.a) Columns.

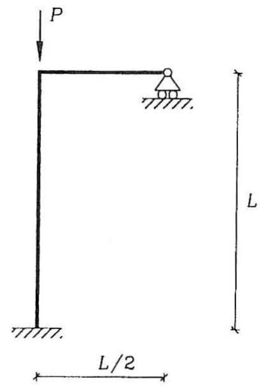

Figure 2.b) Elastic frame considered.

Table 1 is given to compare deterministic finite element results using one, two and three elements for a column with the avaliable exact results. As seen from Table 1, finite element approach overestimates the buckling load, yet gives highly accurate results using only a few elements.

Table 1. Error analysis of non-dimensional quantity $\frac{P_{cr}L^2}{\pi^2 ei}$. * Two or * three elements for the column element under compression.

Structure	Exact	Finite Element Method		
		1 element	2 elements	3 elements
Fixed-free column	0.2500	0.25188058	0.25012755	0.2500248
Pin-pin column	1.0000	1.21585420	1.00752233	1.001580
Pin-fixed column	2.0500	3.03963551	2.09824013	2.058307
Fixed-fixed column	4.0000	-	4.05284735	4.087614
Frame of Figure 2.b	0.7441	0.75433621	0.74996420*	0.7444616*

3. STOCHASTIC FINITE ELEMENTS

In what follows, the random (stochastic) quantities are denoted by capital letters. From the Bernoulli-Euler beam theory, the random deflection field $V(x)$ and the random buckling load P_{cr} of a column of length L are related as follows :

$$\frac{d^2}{dx^2}\left((EI)(x)\frac{d^2V(x)}{dx^2}\right) + \frac{d^2}{dx^2}(P_{cr}V(x)) = 0 \quad , \quad x \in [0, L] \tag{10}$$

In equation (10), the bending rigidity field $(EI)(x)$ is assumed to be a sum separable random field.

$$(EI)(x) = \overline{EI}(1 + r(x)) \qquad x \in [0, L] \tag{11}$$

where $\overline{EI}$ is the mean and $r(x)$ is the zero-mean non-dimensional stationary random field with the following mean and autocovariance functions.

$$E[r(x)] = 0 \tag{12}$$

$$E[r(x)r(x+\xi)] = A_{rr}(x, x+\xi) = A_{rr}(\xi) = \sigma^2 e^{-a^2\xi^2} \tag{13}$$

σ^2 is the variance of $r(x)$ and a is the correlation function such that $a \rightarrow 0$ indicates that the field is fully correlated and $a \rightarrow \infty$ denotes that the field is uncorrelated.

There is no exact closed-form solution for the random buckling load. In what follows, stochastic finite element formulation is utilized to determine the lower order statistics and the probability density function of P_{cr} approximately. The random displacement field of an element $V(x)$ is approximated as a linear combination of the random nodal deformations $\mathbf{V}$ multiplied by deterministic cubic interpolation functions $\mathbf{n}(x)$.

$$V(x) = \mathbf{n}^T(x)\mathbf{V} \tag{14}$$

where

$$\mathbf{V}^T = [V_1,\ \Theta_1,\ V_2,\ \Theta_2] \tag{15}$$

and $\mathbf{n}(x)$ is as given in equation (4). Following finite element methodology, one will end up with the same geometrical stiffness matrix $\mathbf{k}_e^G$ given as in equations (5) and (7). The stochastic stiffness matrix $\mathbf{K}_e$ of the beam element becomes

$$K_{e,ij} = \int_0^L (EI)(x)\frac{d^2n_i(x)}{dx^2}\frac{d^2n_j(x)}{dx^2}\,dx \tag{16}$$

Since $(EI)(x)$ is a random field, $K_{e,ij}$ is a random variable which can be written as a sum of 3 random variables

$$K_{e,ij}(\mathbf{X}^e) = K_{e,ij}^{(0)} + X_0^e\Delta K_{e,ij}^{(0)} + X_1^e\Delta K_{e,ij}^{(1)} + X_2^e\Delta K_{e,ij}^{(2)} \tag{17}$$

or in matrix form

$$\mathbf{K}_e(\mathbf{X}^e) = \mathbf{K}_e^{(0)} + X_0^e\Delta K_e^{(0)} + X_1^e\Delta K_e^{(1)} + X_2^e\Delta K_e^{(2)} \tag{18}$$

where X_i^e, $i = 0, 1, 2$ are mean-zero random variables defined as

$$X_i^e = \int_0^L x^i\ r(x)\ dx \quad , \qquad i = 0, 1, 2 \tag{19}$$

The expansion in (17)-(18) is termed the weighted integral method and has been proposed by Deodatis (1990) and Takada (1990) for the static analysis of beams with random bending rigidity. $\mathbf{K}_e^{(0)}$ is the deterministic part or the expected value which is equal to $\mathbf{k}_e$ given in equation (6) and (8) (change ei to $\overline{EI}$), and $X_0^e\mathbf{\Delta K}_e^{(0)} + X_1^e\mathbf{\Delta K}_e^{(1)} + X_2^e\mathbf{\Delta K}_e^{(2)}$ is the stochastic part of the element stiffness matrix $\mathbf{K}_e$. The integrals X_i^e defined by (19) are called weighted integrals. Given the probabilistic character of $r(x)$, joint statistical moments of arbitrary order of these random variables can be obtained from (19), e.g. if the autocovariance function $A_{rr}(x_1, x_2)$ of the random field $r(x)$ are known, the covariances $E[X_i^e X_j^e] = \kappa_{X_i^e X_j^e}$ can be obtained from (19) as

$$E[X_i^e X_j^e] = \kappa_{X_i^e X_j^e} = \int_0^L \int_0^L x_1^i x_2^j A_{rr}(x_1, x_2) dx_1 dx_2 \quad , \quad i, j = 0, 1, 2 \tag{20}$$

As seen in (20), the weighted integral method considers the correlation structure of the random field discretized. For completeness, the matrices $\mathbf{\Delta K}_e^{(0)}$, $\mathbf{\Delta K}_e^{(1)}$ and $\mathbf{\Delta K}_e^{(2)}$ of equation (18) are listed below.

$$\mathbf{\Delta K}_e^{(0)} = \frac{\overline{EI}}{L^4} \begin{bmatrix} 36 & 24L & -36 & 12L \\ 24L & 16L^2 & -24L & 8L^2 \\ -36 & -24L & 36 & -12L \\ 12L & 8L^2 & -12L & 4L^2 \end{bmatrix} \tag{21}$$

$$\mathbf{\Delta K}_e^{(1)} = \frac{\overline{EI}}{L^5} \begin{bmatrix} -144 & -84L & 144 & -60L \\ -84L & -48L^2 & 84L & -36L^2 \\ 144 & 84L & -144 & 60L \\ -60L & -36L^2 & 60L & -24L^2 \end{bmatrix} \tag{22}$$

$$\mathbf{\Delta K}_e^{(2)} = \frac{\overline{EI}}{L^6} \begin{bmatrix} 144 & 72L & -144 & 72L \\ 72L & 36L^2 & -72L & 36L^2 \\ -144 & -72L & 144 & -72L \\ 72L & 36L^2 & -72L & 36L^2 \end{bmatrix} \tag{23}$$

The stochastic global stiffness matrix can be obtained after the usual assembling operation of the finite element method. Assembling procedure consists of deterministic coordinate transformations and additions.

Then, the buckling load which is a random variable is calculated from

$$\mathrm{Det}|\mathbf{K} - P_{cr}\mathbf{k}^G| = 0 \tag{24}$$

where $\mathbf{K}$ is the $n \times n$ stochastic global stiffness matrix.

The lower order statistical moments of the random buckling load can be determined from (24) using several methods, i.e. analytically, Monte Carlo simulations, perturbations, etc.

Analytical results for the statistical moments of order m of the random buckling load can be obtained applying the total probability theorem to the conditional buckling load $P_{cr}(\mathbf{X} = \mathbf{x})$ as

$$E[P_{cr}^m] = \int_{\mathbf{x}} P_{cr}^m(\mathbf{X} = \mathbf{x}) f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad , \quad n = 1, 2, ... \tag{25}$$

This requires the conditional buckling load to be expressed as an explicit function of weighted integrals and probability density function of the weighted integrals. Such an approach can only be applied to problems where the dimension of the global stiffness matrix is low, less than 5. For an axially loaded cantilever column (fixed-free), using one finite element to represent the column, P_{cr} is related to the weighted integrals as follows :

$$P_{cr}(\mathbf{X} = \mathbf{x}) = \frac{4\overline{EI}\left(13L^3 + 18L^2X_0 - 90LX_1 + 120X_2\right)}{3L^5}$$
$$\frac{-4\overline{EI}\sqrt{(13L^3 + 18L^2X_0 - 90LX_1 + 120X_2)^2 - 45L^2\left(L^4 + 4L^3X_0 - 12L^2X_1 - 12{X_1}^2 + 12LX_2 + 12X_0X_2\right)}}{3L^5} \tag{26}$$

Other approaches consider Monte Carlo simulations where several realizations of the random variables are generated and a sample set of P_{cr} is evaluated from the solution of (24) or directly using equations similar to (26). This sample set is further used for the estimation of lower order statistics of P_{cr} and reliability of the columns. The simulation approach needs the joint probability density function (pdf) of the random variables involved to be known and one can compute the pdf of the random buckling load from the sample set available.

If only the lower order moments of the random variables are known, an approximate solution can be obtained using perturbations. The perturbation solution of the random eigenvalue problem has been given by Boyce (1968) and Scheidt and Purkert (1983), and applications to structural engineering problems are studied by Grigoriu (1991) and Pedersen (1993). Another approximate solution of the random eigenvalue problem based on crossings theory has been proposed by Grigoriu (1992).

In this paper, the bending rigidity field is assumed to be Gaussian, thus, the joint probability density function of the weighted integrals are Gaussian. In Figures 3 to 6, the mean value $E[P_{cr}]$ and the coefficient of variation $C.V.\ [P_{cr}]$ of fixed-free, pin-pin columns and the frame of Figure 2.b are plotted as a function of the correlation length a and the standard deviation σ of the random fields when three elements are used for modelling of the columns. These results are computed using 10000 independent Monte Carlo realizations of the weighted integrals that is the random stiffness matrix. In the generation of correlated random variables, an eigenvalue decomposition of the covariance matrix of the random variables is used. This is preferred to the Cholesky decomposition as numerical difficulties are experienced in the Cholesky decomposition when the dimension of the covariance matrix is increased. As seen from Figure 3 and 5 compared to the result listed in Table 1, $E[P_{cr}]$ is smaller than the buckling load of the deterministic column in all cases. The $C.V.\ [P_{cr}]$ is very sensitive to the correlation parameter a and the standard deviation of the random field σ. $C.V.\ [P_{cr}]$ grows proportional to σ and decrease with an increase in a except for the frame example. Zhang and Ellingwood (1995) considered the simply supported column and observed similar trends of behaviour in the mean and coefficient of variation of P_{cr}. It should be noted that when $a \rightarrow 0$, the random field becomes fully correlated and can be interpreted as a random variable in the limit. For this case, since the buckling problem can be solved in closed form for any realization of the bending rigidity which is a random variable, the $C.V.$ of $[P_{cr}]$ becomes equal to σ. This actually sets an upper bound for the $C.V.$ of $[P_{cr}]$ for simple columns. This upper bound can be observed in Figures 4 and 6.

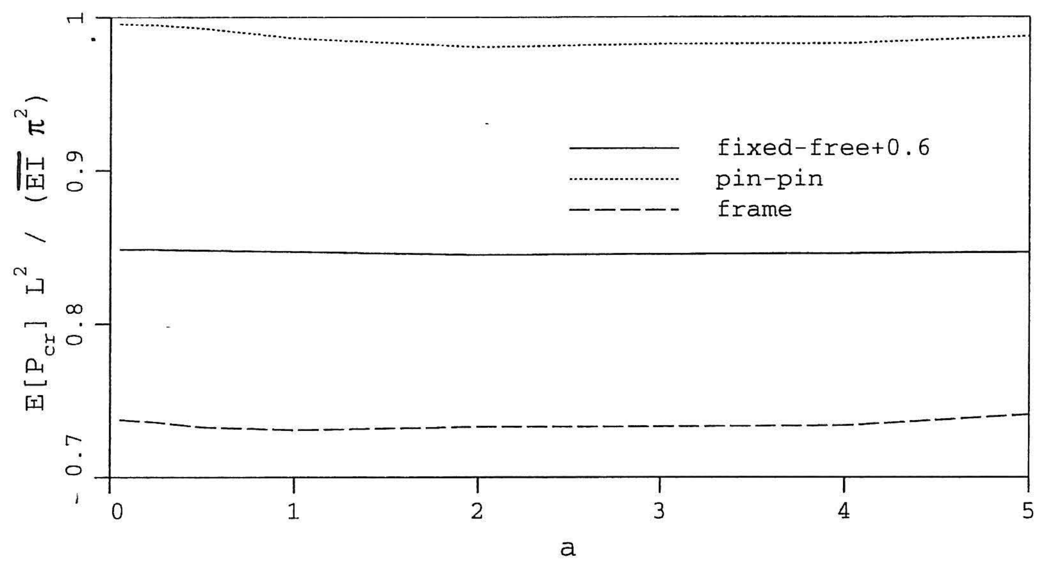

Figure 3. $\frac{E[P_{cr}]L^2}{\pi^2 \overline{EI}}$ versus the correlation parameter a of the random field. $\sigma = 0.2$. Three elements are used to model the column, and 0.6 is added to the results for the fixed-free column for presenting the results in one graph.

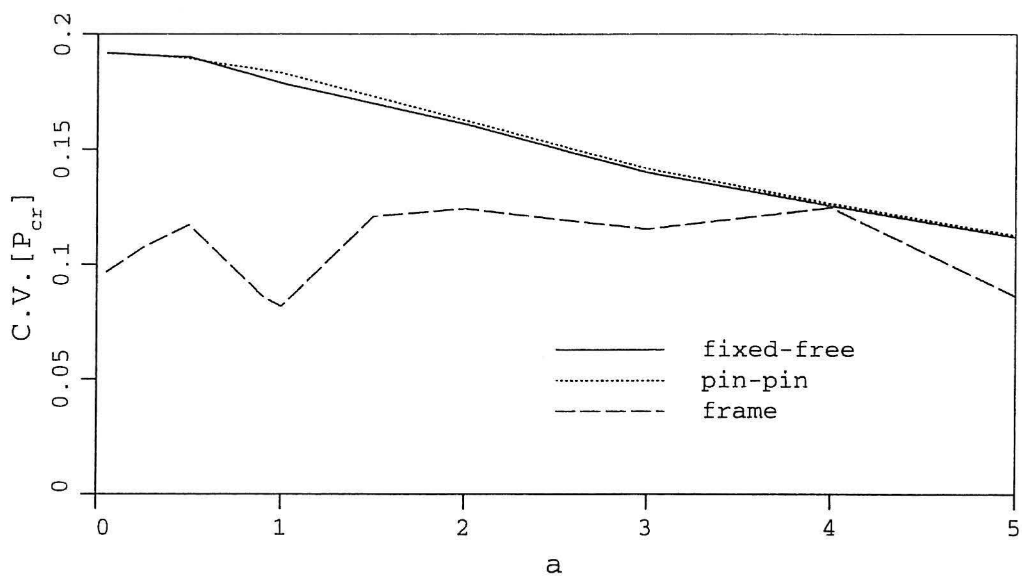

Figure 4. Coefficient of variation of P_{cr} versus a. $\sigma = 0.2$.

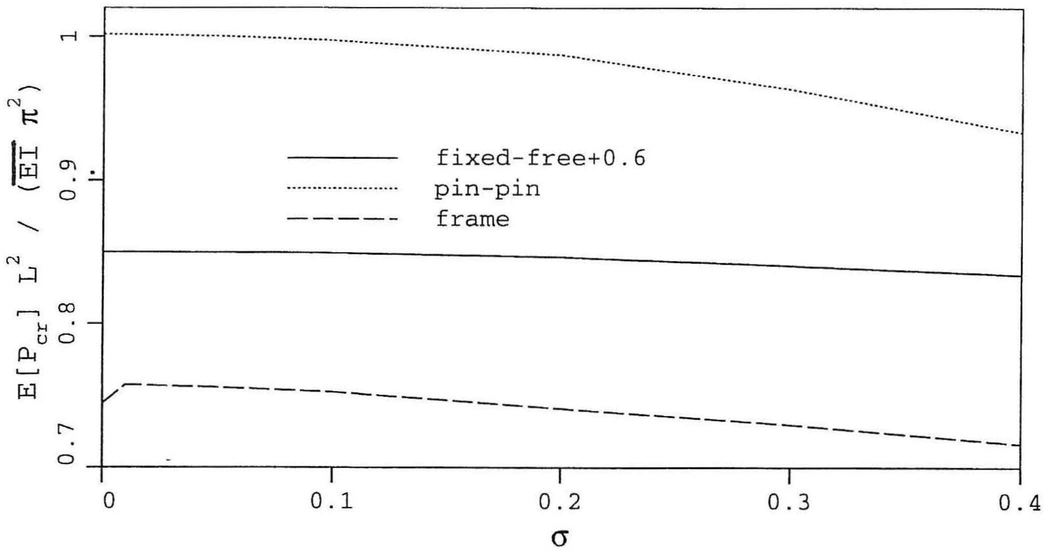

Figure 5. $\frac{E[P_{cr}]L^2}{\pi^2 \overline{EI}}$ versus the standard deviation σ of the random field. $a = 1$. Three elements are used to model the column, and 0.6 is added to the results for the fixed-free column for presenting the results in one graph.

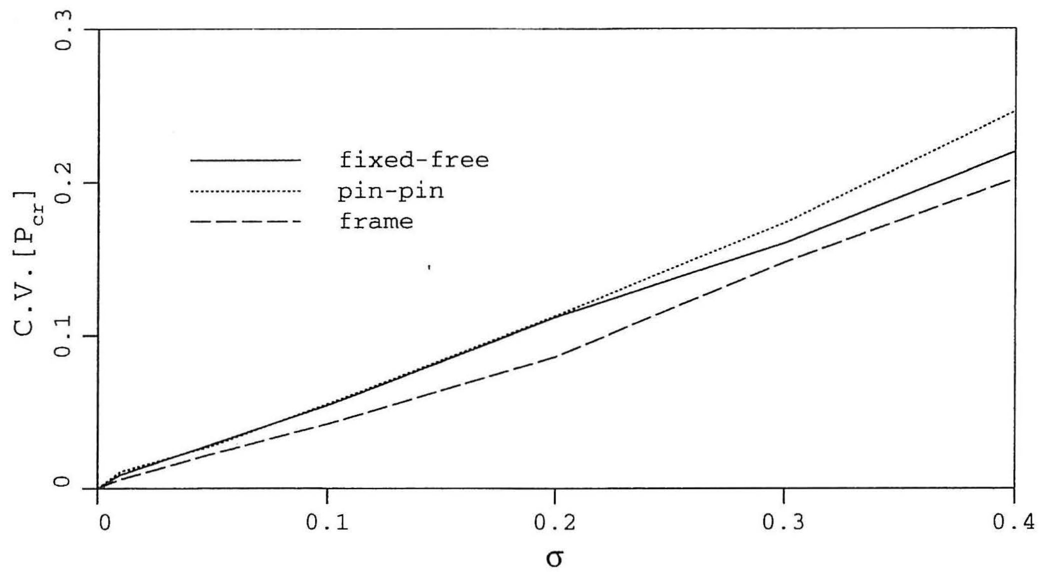

Figure 6. Coefficient of variation of P_{cr} versus σ. $a = 1$.

Tables 2 and 3 are to list the sensitivity of the mean and variance of the random buckling load P_{cr} to the mesh size for simple columns and the frame. Correlation between elements of the mesh is included in the analysis. Fast convergence is observed for the mean of P_{cr} and $C.V.$ of $[P_{cr}]$ using a few elements. Pdf of P_{cr} can be estimated using the sample set. Pdf estimations presented are evaluated using the despl and gcdf subroutines of the IMSL library. The estimated pdf of P_{cr} for the fixed-free column and the frame of Figure 2.b are given in Figures 7 and 8. It is observed that the pdf of P_{cr} resembles, but not equal to, a truncated bell (Gaussian) curve as the mesh is fined. Convergence in terms of pdf can be observed in these figures.

Table 2. Sensitivity analysis of non-dimensional quantity $\frac{E[P_{cr}]L^2}{\pi^2 EI}$. $a = 5$ and $\sigma = 0.2$.

Structure	Finite Element Method		
	1 element	2 elements	3 elements
Fixed-free column	0.247269	0.247603	0.246568
Pin-pin column	1.186187	1.001235	0.987276
Pin-fixed column	3.015973	2.077180	2.028492
Fixed-fixed column	-	4.022562	3.991981
Frame of Figure 2.b	0.748123	0.735781	0.740821

Table 3. Sensitivity analysis of non-dimensional quantity $C.V.[P_{cr}]$. $a = 5$ and $\sigma = 0.2$.

Structure	Finite Element Method		
	1 element	2 elements	3 elements
Fixed-free column	0.149308	0.113837	0.111815
Pin-pin column	0.119842	0.124792	0.112697
Pin-fixed column	0.171362	0.101680	0.112587
Fixed-fixed column	-	0.115012	0.124623
Frame of Figure 2.b	0.147260	0.086665	0.086108

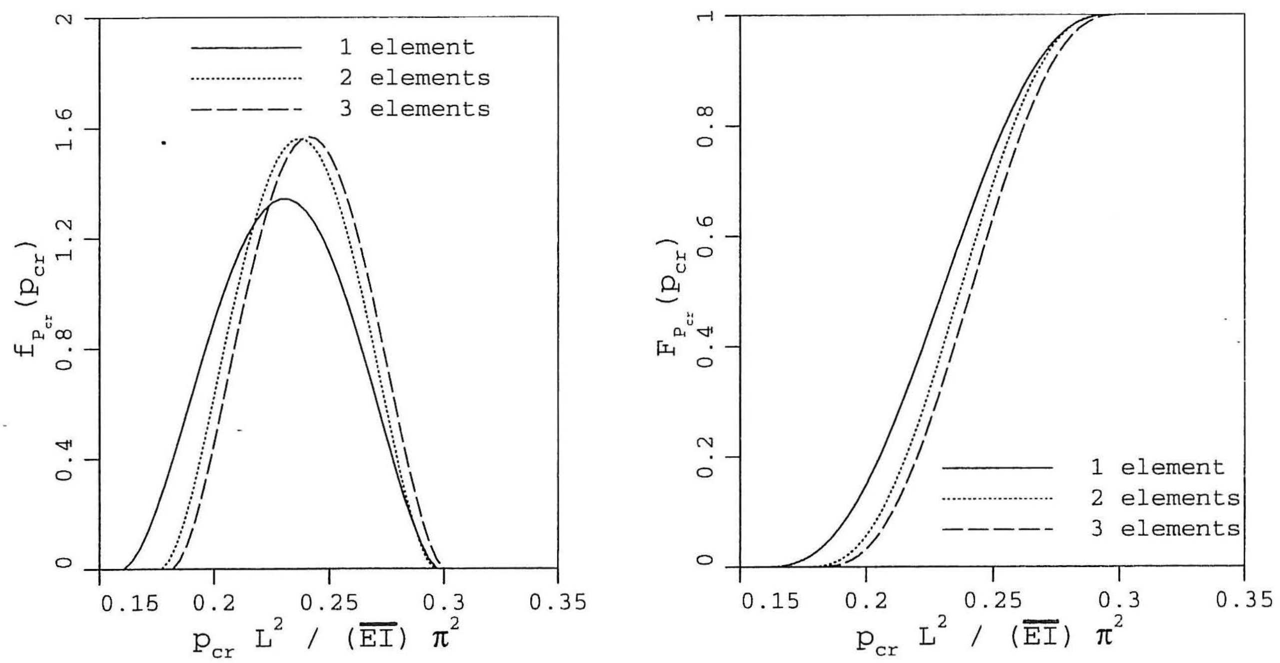

Figure 7. Pdf and probability distribution functions of P_{cr} for the fixed-free column, $\sigma = 0.2$, $a = 1$.

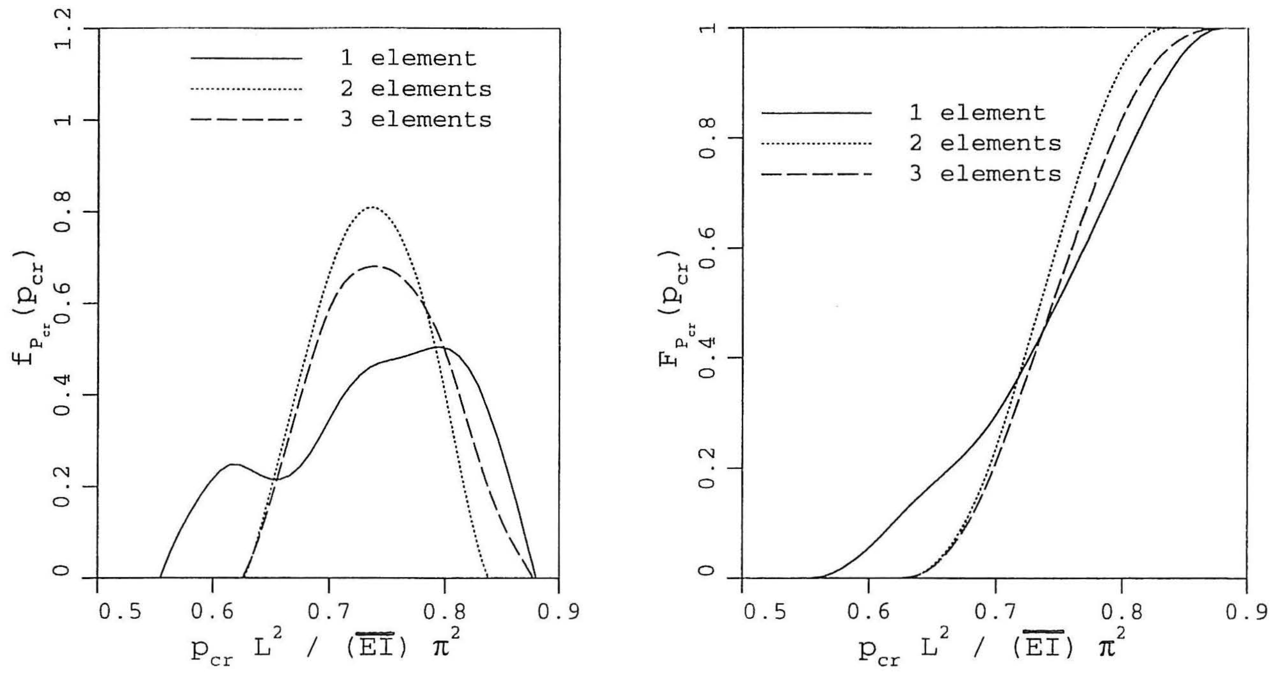

Figure 8. Pdf and probability distribution functions of P_{cr} for the frame of Figure 2.b, $\sigma = 0.2$, $a = 1$.

Reliability of the columns subject to random or determinitic loads can also be calculated. Consider buckling to be the failure event. Let the load P be a random variable, then the reliability of the column becomes $\text{Prob}(P \leq P_{cr}) = \text{Prob}(0 \leq P_{cr} - P)$.

An interesting question arises about the reliability of the already designed columns using the deterministic theory with and without the safety factors. The reliability of simple columns are estimated using 10000 simulations by simple sampling for different values of coefficient of variation of the random field σ and the correlation parameter a. The results are tabulated in Tables 4 and 5 where 3 elements are used to model the columns. Table 4 considers the columns which are deterministically designed, e.g. reliability of the fixed-free column is $\text{Prob}(\frac{0.25\pi^2 EI}{L^2} \leq P_{cr})$. A safety factor of 1.2 is introduced for the results presented in Table 5, e.g. reliability of the fixed-free column is $\text{Prob}(\frac{0.25\pi^2 EI}{1.2\ L^2} \leq P_{cr})$. From the Tables, it is seen that these columns are not very safe for the assumed uncertainties. Without the factor of safety, the reliability level is about 50 percent in the cases considered, and even a factor of safety of 1.2 is not enough to have 90 percent reliable columns when $\sigma = 0.2$ and $a = 0.05$. On the other hand, it is seen that the safety factor of 1.2 increases the reliability level about 30 percent or more in many of the cases.

Table 4. Estimations for the reliability of the structures which are designed deterministically. Note that reliability of a column with $\sigma = 0$ is 1.

Structure	$a = 5, \sigma = 0.05$	$a = 5, \sigma = 0.2$	$a = 5, \sigma = 0.3$	$a = 2, \sigma = 0.2$	$a = 0.05, \sigma = 0.2$
Fixed-free column	0.41	0.41	0.41	0.41	0.50
Pin-pin column	0.50	0.41	0.41	0.41	0.50
Pin-fixed column	0.50	0.59	0.56	0.56	0.50
Fixed-fixed column	0.69	0.53	0.47	0.50	0.53
Frame of Figure 2.b	0.75	0.50	0.50	0.50	0.50

Table 5. Estimations for the reliability of the structures which are designed with a safety factor of 1.2.

Structure	$a = 5, \sigma = 0.05$	$a = 5, \sigma = 0.2$	$a = 5, \sigma = 0.3$	$a = 2, \sigma = 0.2$	$a = 0.05, \sigma = 0.2$
Fixed-free column	1.00	0.94	0.81	0.84	0.75
Pin-pin column	1.00	0.94	0.75	0.81	0.75
Pin-fixed column	1.00	0.94	0.90	0.87	0.84
Fixed-fixed column	1.00	0.94	0.84	0.81	0.78
Frame of Figure 2.b	1.00	1.00	0.88	0.94	0.94

4. INTERVAL APPROACH

The uncertain quantities which are bounded are denoted using the caligraphic alphabet. From the Bernoulli-Euler beam theory, the deflection field $\mathcal{V}(x)$ and the buckling load $\mathcal{P}_{cr}$ of a column of length L are related as follows :

$$\frac{d^2}{dx^2}((\mathcal{E}I)(x)\frac{d^2\mathcal{V}(x)}{dx^2}) - \frac{d^2}{dx^2}(\mathcal{P}_{cr}\mathcal{V}(x)) = 0 \quad , \quad x \in [0, L] \tag{27}$$

In equation (27), the bending rigidity field $(\mathcal{E}I)(x)$ is assumed to be uncertain, yet bounded within constant envelope bounds as

$$(ei)^l \leq (\mathcal{E}I)(x) \leq (ei)^u \quad , \quad x \in [0, L] \tag{28}$$

where $(ei)^l = \inf(\mathcal{E}I)(x)$ and $(ei)^u = \sup(\mathcal{E}I)(x)$. The median bending rigidity becomes $\overline{(ei)} = \frac{(ei)^l+(ei)^u}{2}$.

Using such an interval model, a lower bound for the buckling load can be obtained by replacing (ei) of the exact solution available for deterministic model by the lower bound $(ei)^l$ for simple columns. For large frames, optimization methods similar to the one given below are needed. An analytically closed form method to show the given trivial result and calculate conservative estimates of the lower bound of the buckling load is derived below using integral equations where the trace of the kernel method is employed.

Consider the case where the both ends are pinned. Equation (27) becomes

$$(\mathcal{E}I)(x)\frac{d^2\mathcal{V}(x)}{dx^2} - \mathcal{P}_{cr}\mathcal{V}(x) = 0 \quad , \quad x \in [0, L] \tag{29}$$

The differential equation of (29) can be written as an integral equation for a realization $(ei)(x)$ of $(\mathcal{E}I)(x)$ as

$$v(x) - p_{cr}\int_0^L G(x,\xi)v(\xi)d\xi = 0 \quad , \quad x \in [0, L] \tag{30}$$

where the kernel $G(x,\xi)$ is the Green's function of the linear differential operator $\mathcal{L} = \frac{d^2}{dx^2}$ with boundary conditions $v(0) = v(L) = 0$ multiplied by $\frac{1}{ei(\xi)}$.

$$G(x,\xi) = \begin{cases} \frac{1}{(ei)(\xi)}\frac{x}{L}(\frac{\xi}{L} - 1) & 0 \leq x \leq \xi \\ \frac{1}{(ei)(\xi)}\frac{\xi}{L}(\frac{x}{L} - 1) & \xi \leq x \leq L \end{cases} \tag{31}$$

Then, a lower bound for the lowest eigenvalue of the differential equation can be calculated using the trace of the kernel, Mikhlin (1957). This yields

$$p_{cr} > \frac{1}{\sqrt[2m]{A_{2m}}} \quad , \quad m = 1, 2, \ldots \tag{32}$$

where

$$A_{2m} = \sum_{n=1}^{\infty} \lambda_n^{-2m} = \int_0^L \int_0^L K_m^2(x,\xi)dxd\xi \quad , \quad m = 1, 2, ... \tag{33}$$

λ_n denotes the n^{th} eigenvalue and λ_1 is p_{cr}. A_{2m} denotes the $2m^{th}$ trace of the kernel $G(x, \xi)$.

$$K_1(x,\xi) = G(x,\xi) \quad , \quad K_m(x,\xi) = \int_0^L K_1(x,t)K_{m-1}(t,\xi)dt \quad , \quad m = 2, 3, .. \tag{34}$$

For $m = 1$, a lower bound for any realization of $(\mathcal{E}I)(x)$ becomes

$$\mathcal{P}_{cr} > p_{cr}^l = \min\left\{ \int_0^L \int_0^L G^2(x,\xi)dxd\xi \right\}^{-\frac{1}{2}} \tag{35}$$

with the constraint in (28). Application to the column with pinned ends yields

$$p_{cr}^l = \min\left\{ \int_0^L \left(\int_0^\xi \frac{1}{(ei)(\xi)^2} \frac{x}{L}^2 (\frac{\xi}{L} - 1)^2 dx + \int_\xi^L \frac{1}{(ei)(\xi)^2} \frac{\xi}{L}^2 (\frac{x}{L} - 1)^2 dx \right) d\xi \right\}^{-\frac{1}{2}} = \sqrt{90} \frac{(ei)^l}{L^2} \tag{36}$$

Using $m > 1$ will give sharper bounds. In the limit $m \to \infty$, $p_{cr}^l = \pi^2 \frac{(ei)^l}{L^2}$.

For the fixed-free case, the lower bound is calculated similarly using the corresponding $G(x, \xi)$ and found to be

$$p_{cr}^l = \sqrt{6} \frac{(ei)^l}{L^2} \tag{37}$$

for $m = 1$ and in the limit $m \to \infty$, $p_{cr}^l = \frac{\pi^2}{4} \frac{(ei)^l}{L^2}$.

5. CONCLUSIONS

Buckling load of columns with material and geometrical uncertainties are determined considering deterministic, stochastic and interval models for the bending rigidity of columns. Stochastic finite element method based on weighted integrals is utilized when random models are assigned to represent the uncertainty. The mean of the buckling load is found to be slightly less than the buckling load of the deterministic column. Coefficient of variation of the buckling load is proportional to coefficient of variation of the random field, decreases as there is less correlation in the random field and is bounded by the coefficient of variation of the random field. In the cases considered, it is observed that the columns designed without safety factors are about 50 percent reliable and the ones designed considering a safety factor of 1.2 are 30 percent or more reliable than the ones without any safety factor. For the case, when the bending rigidity field is taken to be bounded from above and below, conservative bounds for the buckling load are determined. For structural design, the lower bound is of crucial interest.

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