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Algebraic Structures in Noncommutative Geometry: A Study of Hopf Algebras, Hopf-Galois Extensions, and Hopf Algebroids

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Preface

This thesis is the result of the work I have done during my doctoral studies at SISSA under the supervision of Prof. Dabrowski and Landi. Its main references are

- [14] L. Dabrowski, G. Landi, J.Zanchettin, ”**Hopf algebroids and twists for quantum projective spaces**”, (submitted to *Journal of Algebra*).
- [45] J.Zanchettin, ”**The Chern-Weil map for deformed Hopf-Galois extensions**”.

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Contents

Introduction

The foundation of noncommutative geometry is the duality between spaces and the algebra of functions over them. There are different incarnations of this idea, probably the most famous are: the Gelfand duality - of which one can find a detailed dissertation in [35] - between commutative C^*-algebras and Hausdorff spaces, the Serre-Swan theorem [19] that says that a module over a commutative algebra is finitely generated and projective if and only if is the module of sections of a vector bundle, and the Connes theorem [11] that relates spin manifolds with spectral triples. The leitmotiv is always that we have a geometric object on one side and a corresponding algebraic on the other

In this thesis, we deal with the following algebraic structures:

- **Hopf algebras** which are dual to groups. One can prove that every commutative Hopf algebra over a field arises from a group scheme, and conversely, the algebra of regular function on an algebraic group is a Hopf algebra [42] (the same type of result holds in a more analytical fashion considering compact Hausdorff groups and representative functions).

- **Hopf-Galois extensions** which are dual to principal bundles. Initially, a noncommutative theory of principal bundles was developed in [6] under the name of quantum principal bundles, and some years later [20] it was proven that the definition is equivalent to Hopf-Galois extensions that were originally introduced as a generalization of Galois field extensions to noncommutative rings [30].

- **Hopf algebroids**. They are dual objects to groupoids in the same spirit that Hopf algebras are dual to groups. Roughly speaking, a Hopf algebroid is a Hopf algebra where the ground field is promoted to a (noncommutative algebra). We mention that there are several definitions for Hopf algebroids and not all of them are equivalent. In this work, we mainly focus on two of them and we recall them in the second chapter.

In the first chapter of this thesis, we recall the basic definition of Hopf algebras, their comodules, Hopf-Galois extensions, and deformation theory via cocycles. Most of the material therein is already known except for the proof Theorem 1.2.13 and 1.2.15. They were both proven in [1], but without writing down explicitly the strong connection of the extensions.

The second chapter is based on the paper [14] which I wrote with my advisors. Contrary to what happens to a Hopf algebra, an antipode on a Hopf algebroid might not be unique. It is then interesting to characterize all the possible antipodes on a given bialgebroid. To do this we recall the notion of twist

[4], which is a particular type of character. After improving the proof of Theorem 2.1.4 by showing that one of the original hypotheses was not needed, we specialize in the case of the Ehresmann-Schauenburg (ES) Hopf algebroid associated with a Hopf-Galois extension. We give detailed proof that the flip is an antipode whenever the Hopf algebra of the extension is commutative and eventually, we characterize the group of twists in proposition 2.2.7. In the last part of the chapter, we apply the theory to the $U(1)$-extension $A(\mathbb{C}P_q^{n-1}) \subseteq A(S_q^{2n-1})$.

In the third chapter, we study the Chern-Weil theory for deformed Hopf-Galois extensions. We refer to the theory developed in [23] that was foreshadowed in [7]. We prove that if the cocycle deformation comes from the structure Hopf algebra of the extension, the Chern-Weil map does not change 3.2.2 while if the cocycle comes from an external symmetry we get a different map than the undeformed 3.2.3. In the last section of the chapter, we recall the concept of pullback for Hopf-Galois extensions and we prove that the noncommutative Chern-Weil map satisfies the naturality condition of its classical counterpart. This is done in proposition 3.3.4, and in the discussion that follows it, we give the result also for cocycle deformations. The material of this chapter composes a paper [45].

The last chapter is essentially an outline as it is based on an ongoing collaboration with A. Chirvasitu and M. Tobolski. We collect some partial results about a Morita theory for noncommutative Hopf algebroid. We said that Hopf algebroids are dual to groupoids, and for the latter, the Morita theory is known and has different characterizations [33]. For Hopf algebroid, so far only the Morita theory in the commutative case has been developed [16]. In general, the idea of Morita's theory is that two objects are equivalent if they have the same representation theory. In the current situation, the representation theory would be the category of comodules of the Hopf algebroids. In [16] is shown that two Hopf algebroids have (monoidal) equivalent comodules categories if and only if there exists an algebra equipped with left and right principal coaction of the algebroids (bibundle). The latter notion does not need any modification in the noncommutative case, so we use it as a temporary notion of Morita equivalence. We want to stress that this is an abuse of language since for noncommutative Hopf algebroid, admitting a bibundle does not define an equivalence relation. We prove that the principal space algebra of a Hopf-Galois extension is a bibundle of the ES algebroid and the structure Hopf algebra 4.2.2. Moreover, we prove that if a bialgebroid admits a bibudle with a Hopf algebra then is isomorphic to the ES algebroid 4.2.7. This generalizes the classical result of the Morita equivalence between the gauge groupoid of a principal bundle and the structure group [33]. We close the chapter with a conjecture, we think that we can prove the same result starting from an equivalence of the category of comodules 4.2.9.

Chapter 1

Background material

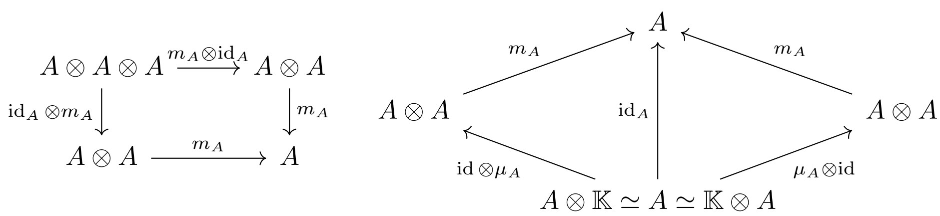
In this first chapter, we recall the concepts and known results needed in this thesis. We work over a field $\mathbb{K}$ (that can be $\mathbb{R}$ or $\mathbb{C}$), but most of the structure we introduce can be defined over a ring. Throughout the thesis, When we write vector space we mean over $\mathbb{K}$, $\mathrm{id}_V : V \longrightarrow V$ is the identity map on V, and we denote $\otimes := \otimes_{\mathbb{K}}$.

1.1 Hopf algebras

1.1.1 Bialgebras

For a more extensive treatment of the subject, we refer to the [8, 32, 34], which we also use as main sources.

Definition 1.1.1 A **unital associative algebra** is the datum of a vector space A together with the maps $m_A : A \otimes A \longrightarrow A$, $\mu_A : \mathbb{K} \longrightarrow A$ called multiplication and unit, such that the following diagrams commute



We refer to such an object just as algebra and if there is no risk of confusion we denote the multiplication by juxtaposition, i.e. $m_A(a \otimes a') := aa'$. Moreover, the unit of the algebra is denoted by $1_A := \mu_A(1_{\mathbb{K}})$. ♦

In this thesis, every algebra is taken to be unital. Thus whenever we say algebra, we mean it has the unit.

Definition 1.1.2 Given two algebras A and B, a linear map $f : A \longrightarrow B$ is said to be an **algebra morphism** if the following diagrams commute

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle m_A}\uparrow & & \uparrow{\scriptstyle m_B} \\ A\otimes A & \xrightarrow{f\otimes f} & B\otimes B \end{array} \qquad \begin{array}{ccc} \mathbb{K} & \xrightarrow{\mu_A} & A \\ & {\scriptstyle \mu_B}\searrow & \downarrow{\scriptstyle f} \\ & & B \end{array}$$

The corresponding equations read as $f(aa') = f(a)f(a')$ and $f(1_A) = 1_B$ for any $a, a' \in A$. ♦

If we take the diagrams above and reverse the direction of the arrows we get the dual notion of an algebra. In this work, we only consider unital algebra morphisms.

Definition 1.1.3 A **counital coassociative coalgebra** is the datum of a vector space C together with maps $\Delta_C : C\otimes C \longrightarrow C$, $\epsilon_C : C \longrightarrow \mathbb{K}$ called multiplication and counit, such that the following diagrams commute

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & C\otimes C \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_C\otimes \mathrm{id}_C} \\ C\otimes C & \xrightarrow{\mathrm{id}_C\otimes\Delta_C} & C\otimes C\otimes C \end{array} \qquad \begin{array}{ccccc} & {\scriptstyle \Delta_C}\swarrow & C & \searrow{\scriptstyle \Delta_C} & \\ C\otimes C & & \downarrow{\scriptstyle \mathrm{id}_C} & & C\otimes C \\ & {\scriptstyle \mathrm{id}_C\otimes\epsilon_C}\searrow & & \swarrow{\scriptstyle \epsilon_C\otimes\mathrm{id}} & \\ & & C\otimes\mathbb{K}\simeq C\simeq \mathbb{K}\otimes C & & \end{array}$$

We refer to this structure just as coalgebra. We adopt the so-called Sweedler notation for the multiplication, for all $c \in C$

$$\Delta_C(c) := c_{(1)} \otimes c_{(2)}.$$

In this way, we read the commutative diagrams as

$$c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} := c_{(1)} \otimes c_{(2)} \otimes c_{(3)}, \quad \epsilon(c_{(1)})c_{(2)} = c = c_{(1)}\epsilon(c_{(2)}).$$ ♦

The notion of morphism here is given by the following

Definition 1.1.4 Given two coalgebras C and D, a linear map $g : C \longrightarrow D$ is said to be a **coalgebra morphism** if the following diagrams commute

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_D} \\ C\otimes C & \xrightarrow{g\otimes g} & D\otimes D \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{g} & D \\ & {\scriptstyle \epsilon_C}\searrow & \downarrow{\scriptstyle \epsilon_D} \\ & & \mathbb{K} \end{array}$$

In Sweedler notation, these are given by

$$g(c)_{(1)} \otimes g(c)_{(2)} = g(c_{(1)}) \otimes g(c_{(2)})$$

$\epsilon_C = \epsilon_D \circ g$. ♦

Combining the definition of algebra and coalgebra one gets the following

Definition 1.1.5 A **bialgebra** is the datum of $(B, m_B, \mu_B, \Delta_B, \epsilon_B)$ where B is a vector space, (B, m_B, μ_B) is an algebra, $(B, \Delta_B, \epsilon_B)$ is a coalgebra, and the comultiplication and counit are algebra morphisms. A **bialgebra morphism** is a linear map $f : B \longrightarrow C$ that is both an algebra and coalgebra morphism. ⧫

Example 1.1.6 1. Consider the space $O(M_n(\mathbb{K})) := \mathbb{K}[X_{ij}|1 \leq i, j \leq n]$ of polynomial functions of $n \times n$ matrices with coefficients in $\mathbb{K}$. It is a polynomial algebra and it has a coalgebra structure given by

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}, \quad \epsilon(X_{ij}) = \delta_{ij}, \quad \forall i, j = 1, \ldots, n$$

which is compatible with the algebra structure. Then $O(M_n(\mathbb{K}))$ is a bialgebra.

2. Consider the quotient algebra $B := \mathbb{K}\langle x, y\rangle/(xy - qyx)$ with $q \in \mathbb{K}^*$. It is a bialgebra that has the coalgebra structure given by

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes 1 + x \otimes y, \quad \epsilon(x) = 1, \quad \epsilon(y) = 0.$$

This bialgebra is known in the literature as the **quantum plane**. ⧫

Given two vector spaces V and W, we denote by $\mathrm{flip}_{V.W} : V \otimes W \longrightarrow W \otimes V$ the flip map which is defined by

$$\mathrm{flip}(v \otimes w) = w \otimes v$$

For an algebra A with multiplication m_A we defined a new algebra with the same underlying vector space and multiplication

$$m_A^{op} := m_A \circ \mathrm{flip},$$

this means that $m_A^{op}(a \otimes b) = ba$ for any $a, b \in A$. This algebra is called the opposite algebra of A and is denoted by A^{op}. The unit of A^{op} is the same as A. An algebra is said to be commutative when $m_A = m_A^{op}$.

Similarly, we can associate to any coalgebra C another coalgebra C^{cop} by using the flip, i.e.

$$\Delta_C^{cop} := \mathrm{flip} \circ \Delta_C,$$

which reads $\Delta_C^{cop}(c) = c_{(2)} \otimes c_{(1)}$. This coalgebra is called the copposite coalgebra of C and has the same counit of C. The latter is said to be cocommutative if $\Delta_C^{cop} = \Delta_C$.

If we combine the two constructions we have that there is a bialgebra with flipped multiplication and multiplication associated with any bialgebra B, namely $B^{op,cop}$. The compatibility condition between m^{op} and Δ^{cop} are equivalent to the ones between m and Δ.

1.1.2 The convolution algebra of a bialgebra and the antipode

Given a bialgebra B and the linear space $\mathrm{End}_{\mathbb{K}}(B) = \mathrm{Hom}_{\mathbb{K}}(B, B)$ of endomorphism of B. We use the algebra and coalgebra structure to define a multiplication on $\mathrm{End}_{\mathbb{K}}(B)$ for any $\phi, \psi \in \mathrm{End}_{\mathbb{K}}(B)$ and $b \in B$

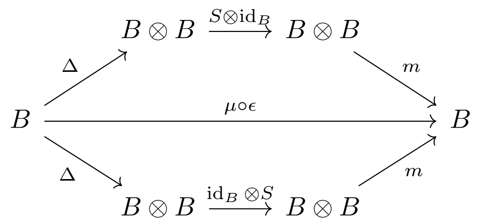
$$(\phi * \psi)(b) = \phi(b_{(1)})\psi(b_{(2)}). \qquad (1.1\text{-}1)$$

The associativity of this multiplication follows from the associativity and coassociativity of the operations on B. The composition $\mu \otimes \epsilon$ gives an element of B^* and by the definitions of unit and counit one finds that $\phi * (\mu \circ \epsilon) = \phi = (\mu \circ \epsilon) * \phi$. Then we have the following

Definition 1.1.7 The triple $(\mathrm{End}_{\mathbb{K}}(B), *, \mu \circ \epsilon)$ is called the **convolution algebra of** B. ♦

The identity morphism id_B is an element of the convolution algebra $\mathrm{End}_{\mathbb{K}}(B)$. Being the latter an unitial algebra, it makes to ask whether id_B is invertible with respect to (1.1-1). The inverse of the identity morphism goes under the name of

Definition 1.1.8 The **antipode** of a bialgebra B is the convolution inverse of the identity morphism. By definition, it is the map $S : B \longrightarrow B$ that makes the following diagram commute



A bialgebra $(B, m, \mu, \Delta, \epsilon)$ having the antipode is called a **Hopf algebra**. ♦

Remark 1.1.9
1. Some bialgebras cannot admit a Hopf algebra structure, i.e. it does not exist the antipode. For instance on the bialgebra $O(M_n(\mathbb{K}))$ of 1.3.4 there is no antipode. Therefore, being a Hopf algebra is a property of bialgebras rather than an additional structure.
2. Since S is the inverse of id_H in an algebra, it is always unique. We will see later that this is no longer the case for more general structures. ♦

In Sweedler notation the defining equations of the antipode are

$$S(b_{(1)})b_{(2)} = \epsilon(b), \quad b_{(1)}S(b_{(2)}) = \epsilon(b), \tag{1.1-2}$$

with $b \in B$. Moreover, one can prove that the antipode satisfies the following equations, i.e.

$$S(bc) = S(c)S(b), \quad S(b)_{(1)} \otimes S(b)_{(2)} = S(b_{(2)}) \otimes S(b_{(1)}) \tag{1.1-3}$$

$$S(1_B) = 1_B, \quad \epsilon \circ S = \epsilon, \tag{1.1-4}$$

$\forall b, c \in B$. Looking at these equations we can deduce that the antipode is a bialgebra morphism between B and $B^{op,cop}$.

Example 1.1.10
1. Let G be a group, the space $\mathbb{K}[G]$ generated by finite linear combination of elements of G is an algebra. It also has a coalgebra structure given by the linear extension of the maps

$$\Delta(g) = g \otimes g, \quad \epsilon(g) = 1.$$

Moreover, the linear extension of the inverse in G is an antipode

$$S(g) = g^{-1},$$

then $\mathbb{K}[G]$ is a Hopf algebra. We refer to it as the group algebra of G.

2. Let now G be an algebraic group group and $O(G)$ the space of $\mathbb{K}$ valued regular functions on G. Again this is an algebra with point-wise sum and multiplication. The coalgebra structure dualizes the group structure

$$\Delta(f)(g,h) = f(gh), \quad \epsilon(f) = f(e_G),$$

$\forall g, h \in G$ and e_G is the neutral element of G. The dualization of the inverse operation gives the antipode

$$S(f)(g) = f(g^{-1}).$$

3. As we said, the bialgebra $O(M_n(\mathbb{K}))$ of 1.1.6 is not a Hopf algebra. Nevertheless, if one considers for instance the ideal $(det(X) - 1)$, one has that $O(SL_n(\mathbb{K})) = O(M_n(\mathbb{K}))/(det(X) - 1)$ is a Hopf algebra. Other Hopf algebras can be constructed in the same way starting from $O(M_n(\mathbb{K}))$ (these are called quantum subgroups, for detail look [27]). ♦

Definition 1.1.11 Given two Hopf algebras H and K, a linear map $f : H \longrightarrow K$ is a **Hopf morphism** if it is a bialgebra morphism and $f \circ S_H = S_K \circ f$. ♦

The antipode might be (composition) invertible, in this case, we say that H is a Hopf algebra with a bijective antipode. About the invertibility of the antipode we have the following result

Proposition 1.1.12 ([34]) *H is a Hopf algebra with bijective antipode if and only if H^{cop} is a Hopf algebra with antipode $\bar{S}$. One has $\bar{S} = S^{-1}$ and moreover, if H is commutative or cocommutative $S^2 = \mathrm{id}_H$.*

Throughout the work, we consider only Hopf algebras with bijective antipode unless the contrary is stated.

1.1.3 Comodule algebras

Here we briefly recall what is the (co)representation theory for a Hopf algebra. All the definitions and results we report can be generalized to coalgebras, but for our purposes, we just deal with Hopf algebras.

Definition 1.1.13 A vector space V is said to be a right H**-comodule** if there exists a linear map $\rho : V \longrightarrow V \otimes H$ such that the following equations hold

$$\begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes H \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \Delta} \\ V \otimes H & \xrightarrow{\rho \otimes \mathrm{id}_H} & V \otimes H \otimes H \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\rho} & V \otimes H \\ & {\scriptstyle \mathrm{id}_V}\searrow & \downarrow{\scriptstyle \mathrm{id}_V \otimes \epsilon} \\ & & V \end{array}$$

The map ρ_V is called the **coaction** of H on V. ♦

For any $v \in V$, we adopt the Sweedler notation for it given by

$$\rho(v) = v_{(0)} \otimes v_{(1)}.$$

Then the defining equations read as

$$v_{(0)(0)} \otimes v_{(0)(1)} \otimes v_{(1)} = v_{(0)} \otimes v_{(1)(1)} \otimes v_{(1)(2)} := v_{(0)} \otimes v_{(1)} \otimes v_{(2)}, \quad \epsilon(v_{(1)})v_{(0)} = v.$$

For every right H-comodule one has the vector sub-space of **coaction invariant** elements

$$V^{coH} := \{v \in V | \rho(v) = v \otimes 1_H\} \tag{1.1-5}$$

Definition 1.1.14 Let (V, ρ_V) and (W, ρ_W) be right H-comodules, a linear map $f : V \to W$ is said to be **right H-colinear** or a right comodule morphism if

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ {\scriptstyle \rho_V}\downarrow & & \downarrow{\scriptstyle \rho_W} \\ V \otimes H & \xrightarrow{f \otimes \mathrm{id}_H} & W \otimes H \end{array}$$

that in Sweedler notation reads as

$$f(v)_{(0)} \otimes f(v)_{(1)} = f(v_{(0)}) \otimes v_{(1)}, \quad \forall v \in V$$ ♦

The definition of left H-comodule and left H-colinear morphism are similar, in this situation one has that coaction is a map $\lambda : V \longrightarrow H \otimes V$, for which we adopt the Sweedler notation $\lambda(v) = v_{(-1)} \otimes v_{(0)}$. Right and left H-comodules with H-colinear maps form the categories that we denote by $\mathfrak{M}^H$ and ${}^H\mathfrak{M}$. These are monoidal (or tensor) categories [17], meaning that the tensor product $V \otimes W$ with $V, W \in \mathfrak{M}^H$ is a right H-comodule. The coaction is the **diagonal** one.

$$\rho^{\otimes} : V \otimes W \longrightarrow V \otimes W \otimes H, \quad v \otimes w \longmapsto v_{(0)} \otimes w_{(0)} \otimes v_{(1)}w_{(1)}. \tag{1.1-6}$$

Example 1.1.15

1. Any Hopf algebra (H, Δ) is a right H-comodule with the coaction $\rho = \Delta$. In this case one has $H^{coH} = \mathbb{K}$: from $h_{(1)} \otimes h_{(2)} = h \otimes 1_H$ by applying $\mathrm{id}_H \otimes \epsilon$ on both sides $\mathrm{id}_H \otimes \epsilon$ we get $h = \epsilon(h)$.

2. Any Hopf algebra H can be endowed with another right H-coaction, namely the *adjoint coaction* which is given by the formula

$$\mathrm{Ad} : H \longrightarrow H \otimes H, \quad h \longmapsto h_{(2)} \otimes S(h_{(1)})h_{(3)} \tag{1.1-7}$$

We use the notation $\underline{H} = (H, \mathrm{Ad})$ to distinguished from (H, Δ) as comodule.

3. Referring to the Hopf algebra $\mathbb{K}[G]$ of 1.1.10, one can prove that the $\mathbb{K}[G]$-comodules are only vector spaces that decompose into a direct sum $V = \oplus_{g \in G} V_g$ such that $\rho_{V_g}(v_g) = v_g \otimes g$ for $v_g \in V_G$ and $g \in G$. In this case, the space of coaction invariant elements coincides with V_{e_G}. ♦

As we already said, the definition of comodule can be also given for coalgebra, in fact, neither the product nor antipode are involved in the definition.

However, thanks to the antipode one can turn a right H-comodule into a left H-comodule in the following way: if ρ is the right coaction of H on V the map

$$\lambda(v) = S^{-1}(v_{(1)}) \otimes v_{(0)}, \tag{1.1-8}$$

is a left H-coaction. Moreover, one can use S to go back to the right coaction. In this way, the antipode induces an equivalence of categories $\mathfrak{M}^H \simeq {}^H\mathfrak{M}$.

Definition 1.1.16 A right H**-comodule algebra** is the datum of an algebra A and a right coation ρ_A such that the latter is an algebra morphism, i.e.

$$\rho_A(ab) = a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}, \quad \forall a, b \in A.$$ ♦

Example 1.1.17

1. Any Hopf algebra is a right H-comodule algebra with $\rho = \Delta$.
2. The right $\mathbb{K}[G]$-comodule algebras are exactly the algebras $A = \oplus_{g\in G} A_g$ such that $A_g A_h \subseteq A_{gh}$ for every $g, h \in G$.
3. Let X be an affine algebraic variety and G an algebraic linear group. If X is a G-space, i.e. there is a regular map $X \times G \longrightarrow X, (x, g) \longmapsto xg$ satisfying $(xg)h = xgh$ and $xe_G = x$ for all $x \in X$ and $g, h \in G$, then the algebra of regular functions $O(X)$ is a right $O(G)$-comodule algebra with coaction given by

$$\rho(f)(x, g) = f(xg), \quad f \in O(X).$$

In this case one has that $O(X)^{coO(G)} \simeq O(X/G)$. ♦

Given a right H-comodule algebra A. The space of coaction invariant element A^{coH} is a subalgebra of A. We denote the category of right H-comodule algebras by $\mathcal{A}^H$. The functor induced by the antipode $\mathfrak{M}^H \longrightarrow {}^H\mathfrak{M}$ in general does not extend to comodule algebras.

Definition 1.1.18 Let H and K be Hopf algebras, vector space V is said to be a K**-**H**-bicomodule** if the left K-coaction λ_V and right H-coaction ρ_V make the following diagram commute

$$\begin{array}{ccc} V & \xrightarrow{\rho_V} & V \otimes H \\ {\scriptstyle \lambda_V}\big\downarrow & & \big\downarrow{\scriptstyle \lambda_V \otimes \mathrm{id}_H} \\ K \otimes V & \xrightarrow{\mathrm{id}_K \otimes \rho_V} & K \otimes V \otimes H \end{array}$$

Similarly, we say that A is a K-H-bicomodule algebra if it is a bicomodule and both coactions are algebra morphisms. ♦

Consider now a (K, H)-bicomodule V and a (H, L)-bicomodule W, we can define the vector space

$$V \,\square^H\, W := \{v \otimes w \in V \otimes W | \rho_V(v) \otimes w = v \otimes \lambda_W(w)\}, \tag{1.1-9}$$

where ρ_V is the right H-coaction on V and λ_W the left H-coaction on W. We refer to this space as the *cotensor product* of V and W over H.

1.1.4 Deformation via 2-cocycle

In this subsection, we review the concept of 2-cocycle which is dual to the Drinfeld twist [32] and the deformation theory of Hopf algebras and comodule algebras.

Recall that for every Hopf algebra H the tensor product $H \otimes H$ is a bialgebra with component-wise multiplication and unit, and comultiplication $\Delta^{\otimes}(h \otimes k) = h_{(1)} \otimes k_{(1)} \otimes h_{(2)} \otimes k_{(2)}$ and counit $\epsilon^{\otimes}(h \otimes k) = \epsilon(h)\epsilon(k)$.

Definition 1.1.19 Let H be a Hopf algebra, a 2**-cocycle** is linear map $\gamma : H \otimes H \longrightarrow \mathbb{K}$ such that

1. It is convolution invertible, i,e, there exists $\gamma^{-1} : H \otimes H \longrightarrow \mathbb{K}$ such that $\forall h, k \in H$

$$\gamma(h_{(1)} \otimes k_{(1)})\gamma^{-1}(h_{(2)} \otimes k_{(2)}) = \epsilon(h)\epsilon(k) = \gamma^{-1}(h_{(1)} \otimes k_{(1)})\gamma(h_{(2)} \otimes k_{(2)}) = \epsilon(h)\epsilon(k)$$

2. Satisfies the cocycle condition $\forall h, k, l \in H$

$$\gamma(h_{(1)} \otimes k_{(1)})\gamma(h_{(2)}k_{(2)} \otimes l) = \gamma(k_{(1)} \otimes l_{(1)})\gamma(h \otimes k_{(2)}l_{(2)})$$

3. It is counital $\gamma(h \otimes 1_H) = \gamma(1_H \otimes h) = \epsilon(h)$ for every $h \in H$. ⧫

With an abuse of notation, we write $\gamma(h, k)$ instead of $\gamma(h \otimes k)$.

Example 1.1.20 For the Hopf algebra $O(G)$ of regular functions on an algebraic group, the 2-cocycles are indeed the group 2-cocycles $Z^2(G)$ i.e. the functions $f : O(G) \otimes O(G) \simeq O(G \times G) \longrightarrow \mathbb{K}$ such that

$$f(g, h)f(gh, k) = f(h, k)f(g, hk), \quad f(g, e_G) = f(e_G, g) = 1,$$

for all $g, h, k \in G$. Invertibility in this case means that f is nowhere zero on $G \times G$. ⧫

Proposition 1.1.21 ([32, 1]) *Given a 2-cocycle $\gamma : H \otimes H \longrightarrow \mathbb{K}$ of a Hopf algebra H, the equation*

$$h \cdot_\gamma k := \gamma(h_{(1)}, k_{(1)})h_{(2)}k_{(2)}\gamma^{-1}(h_{(3)}, k_{(3)}), \quad h, k \in H$$

defines a new associative multiplication on H, the resulting algebra is denoted by H_γ. The latter is a Hopf algebra with the comultiplication and counit inherited from H and antipode

$$S_\gamma(h) = u_\gamma(h_{(1)})S(h_{(2)})u_\gamma^{-1}(h_{(3)}), \quad h \in H_\gamma,$$

where $u_\gamma : H \longrightarrow \mathbb{K}$ is the convolution invertible map given by $u_\gamma(h) = \gamma(h_{(1)}, S(h_{(2)}))$. Moreover, if S is bijective then also S_γ is, being its inverse given by

$$S_\gamma^{-1}(h) = v_\gamma(h_{(1)})S^{-1}(h_{(2)})v_\gamma^{-1}(h_{(3)}), \quad h \in H_\gamma$$

where $v_\gamma : H \longrightarrow \mathbb{K}$ mapping $h \longmapsto \gamma(h_{(2)}, S^{-1}(h_{(1)}))$ is convolution invertible.

Let now V be a right H-comodule. Since the deformation of H_γ of H by a 2-cocycle does not involve the coalgebra structure, we have that V is a right H_γ-comodule. The coaction is unchanged. Taking another right H-comodule W with corresponding right H_γ-comodule, we have the tensor product $V_\gamma \otimes^\gamma W_\gamma$ is the vector space $V \otimes W$ endowed with the diagonal right H_γ-coaction

$$\rho^{\otimes^\gamma} : V_\gamma \otimes^\gamma W_\gamma \longrightarrow V_\gamma \otimes^\gamma W_\gamma \otimes H_\gamma, \quad v \otimes^\gamma w \longmapsto v_{(0)} \otimes^\gamma w_{(0)} \otimes v_{(1)} \cdot_\gamma w_{(1)}. \tag{1.1-10}$$

So we have just seen that the 2-cocycle γ induces a functor from $\mathfrak{M}^H$, the category of right H-comodules, to $\mathfrak{M}^{H_\gamma}$ the category of right H_γ-comodules, and that the latter is also monoidal. Moreover, we have the following

Theorem 1.1.22 ([1]) *The functor $\mathfrak{M}^H \longrightarrow \mathfrak{M}^{H_\gamma}$ mapping $V \longmapsto V_\gamma$ is an equivalence of (monoidal) categories, being the linear map*

$$\alpha_{V,W} : V_\gamma \otimes^\gamma W_\gamma \longrightarrow (V \otimes W)_\gamma, \quad v \otimes^\gamma w \longmapsto v_{(0)} \otimes w_{(0)} \gamma^{-1}(v_{(1)}, w_{(1)})$$

a right H_γ-comodule isomorphism with inverse

$$\alpha^{-1}_{V,W} : (V \otimes W)_\gamma \longrightarrow V_\gamma \otimes^\gamma W_\gamma, \quad v \otimes w \longmapsto v_{(0)} \otimes^\gamma w_{(0)} \gamma(v_{(1)}, w_{(1)})$$

If A is rather a right H-comodule algebra, to get a right H_γ-comodule algebra we have to deform the product of A using γ and its properties. This is done by defining

$$a \cdot_\gamma \tilde{a} := a_{(0)} \tilde{a}_{(0)} \gamma^{-1}(a_{(1)}, \tilde{a}_{(1)}). \tag{1.1-11}$$

we denote by A_γ the resulting algebra. By applying the same procedure to A_γ using γ^{-1} we get back the algebra A, i.e. $(A_\gamma)_{\gamma^{-1}} \simeq A$.

Deformation of left K-comodules by a 2-cocycle $\sigma : K \otimes K \longrightarrow \mathbb{K}$ works in the same way. Also, in this case, one has the equivalence of categories of left K-comodules $({}^K\mathfrak{M}, \otimes)$ and left K_σ-comodules $({}^{K_\sigma}\mathfrak{M}, {}^\sigma\otimes)$, now the isomorphism between tensor products is given by

$$\phi_{V,W} : {}_\sigma V \; {}^\sigma\!\otimes {}_\sigma W \longrightarrow {}_\sigma(V \otimes W), \quad v \; {}^\sigma\!\otimes w \longmapsto \sigma(v_{(-1)}, w_{(-1)}) v_{(0)} \otimes w_{(0)}. \tag{1.1-12}$$

For a left K-comodule algebra A and a 2-cocycle σ of K. we deform the product on A via

$$a \bullet_\sigma \tilde{a} := \sigma(a_{(-1)}, \tilde{a}_{(-1)}) a_{(0)} \tilde{a}_{(0)}. \tag{1.1-13}$$

The resulting left K_σ-comodule algebra is denoted by ${}_\sigma A$ and as before if we deform ${}_\sigma A$ with σ^{-1} we get back A.

1.2 Hopf-Galois extensions

Now that we have reviewed the theory of comodule algebras, we can recall what a Hopf-Galois extension is. Initially, a similar structure was called a quantum principal bundle [6] and, sometime later, in [20], it was proved that quantum principal bundles (with the universal calculus) are equivalent to Hopf-Galois extensions.

In this section, we give the basic definitions, several examples that we need later, and the deformation theory via 2-cocycle.

1.2.1 The canonical map and its properties

Let (A, ρ) be a right H-comodule algebra with $B := A^{coH}$. The *canonical map* is defined as

$$\begin{aligned} \mathrm{can} := (\mathrm{m}_A \otimes \mathrm{id}_H) \circ (\mathrm{id}_A \otimes_B \rho) : A \otimes_B A &\longrightarrow A \otimes H \\ a \otimes_B \tilde{a} &\longmapsto a\tilde{a}_{(0)} \otimes \tilde{a}_{(1)}. \end{aligned}$$

The domain of can is the balanced tensor product $A \otimes_B A$, the quotient of $A \otimes A$ by the ideal generated by elements of the form $ab \otimes \tilde{a} - a \otimes b\tilde{a}$ with $a, \tilde{a} \in A$ and $b \in B$. Moreover, can is a left A-linear and right H-colinear morphism. The comodule structure of H is the one given by the right adjoint coaction (1.1-7).

Definition 1.2.1 A **H-Hopf-Galois extension** is an algebra extension $B \subseteq A$, where A i aright H-comodule algebra, $B = A^{coH}$ and can is invertible. ⧫

Remark 1.2.2 One can associate a different canonical map to $B \subseteq A$, namely

$$\mathrm{can}' : A \otimes_B A \longrightarrow A \otimes H, \quad a \otimes_B \tilde{a} \longmapsto a_{(0)}\tilde{a} \otimes a_{(1)}.$$

If the antipode of H is bijective the map $\varphi : A \otimes H \to A \otimes H$, mapping $a \otimes h \to a_{(0)} \otimes S^{-1}(h)a_{(1)}$ is bijective too and moreover $\mathrm{can} = \varphi \circ \mathrm{can}'$. Thus the invertibility of can is equivalent to the invertibility of can'. ⧫

Using the bijectivity of the canonical map associated with a H-Hopf-Galois extension, one defines the *translation map*

$$\tau := \mathrm{can}^{-1}|_{1_A \otimes H} : H \longrightarrow A \otimes_B A, \tag{1.2-1}$$

that satisfies the identity $\mathrm{can}(\tau(h)) = 1_A \otimes h$. In the following, we use a Sweedler-like convention for this map

$$\tau(h) = h^{\langle 1 \rangle} \otimes_B h^{\langle 2 \rangle}, \quad h \in H.$$

Thus, by definition,

$$h^{\langle 1 \rangle} {h^{\langle 2 \rangle}}_{(0)} \otimes {h^{\langle 2 \rangle}}_{(1)} = 1_A \otimes h, \quad \forall h \in H \tag{1.2-2}$$

The translation map fulfills a series of useful properties that we list below [41, 9]

Proposition 1.2.3 *Let H be a Hopf algebra with counit ϵ and antipode S and A a right H-comodule algebra such that $B \subseteq A$ is a H-Hopf-Galois extension. Then the translation map τ satisfies, for any $h, k \in H$ and $a \in A$*

$$h^{\langle 1 \rangle} \otimes_B {h^{\langle 2 \rangle}}_{(0)} \otimes {h^{\langle 2 \rangle}}_{(1)} = {h_{(1)}}^{\langle 1 \rangle} \otimes_B {h_{(1)}}^{\langle 2 \rangle} \otimes h_{(2)}, \tag{1.2-3}$$

$${h^{\langle 1 \rangle}}_{(0)} \otimes_B h^{\langle 2 \rangle} \otimes {h^{\langle 1 \rangle}}_{(1)} = {h_{(2)}}^{\langle 1 \rangle} \otimes_B {h_{(2)}}^{\langle 2 \rangle} \otimes S(h_{(1)}), \tag{1.2-4}$$

$$h^{\langle 1 \rangle} h^{\langle 2 \rangle} = \epsilon(h) 1_A, \tag{1.2-5}$$

$$(hk)^{\langle 1 \rangle} \otimes_B (hk)^{\langle 2 \rangle} = k^{\langle 1 \rangle} h^{\langle 1 \rangle} \otimes_B h^{\langle 2 \rangle} k^{\langle 2 \rangle}, \tag{1.2-6}$$

$$a_{(0)} {a_{(1)}}^{\langle 1 \rangle} \otimes_B {a_{(1)}}^{\langle 2 \rangle} = 1_A \otimes_B a. \tag{1.2-7}$$

Example 1.2.4 1. One can define the canonical map also for a bialgebra since its definition does not involve the antipode. The latter actually is related to the invertibility of can. Consider a bialgebra H and the extension $\mathbb{K} \subseteq H$. If the canonical map is invertible, then the map

$$S : H \longrightarrow H, \quad h \longmapsto h^{\langle 1 \rangle} \epsilon(h^{\langle 2 \rangle}),$$

is the antipode of H, for any $h \in H$

$$S(h_{(1)})h_{(2)} = h^{\langle 1 \rangle} \epsilon(h^{\langle 2 \rangle}{}_{(1)}) h^{\langle 2 \rangle}{}_{(2)} = h^{\langle 1 \rangle} h^{\langle 2 \rangle} = \epsilon(h)$$
$$h_{(1)} S(h_{(2)}) = h_{(1)} h_{(2)}{}^{\langle 1 \rangle} \epsilon(h_{(2)}{}^{\langle 2 \rangle}) = \epsilon(h).$$

We used the definition of the counit and (1.2-7) respectively. Vice versa, if H is a Hopf algebra with antipode S then the map

$$\mathrm{can}^{-1}(h \otimes k) = hS(k_{(1)}) \otimes k_{(2)}.$$

is the inverse of can

$$(\mathrm{can} \circ \mathrm{can}^{-1})(h \otimes k) = hS(k_{(1)})k_{(2)} \otimes k_{(3)} = h \otimes k$$
$$(\mathrm{can}^{-1} \circ \mathrm{can})(h \otimes k) = hk_{(1)}S(k_{(2)}) \otimes k_{(3)} = h \otimes k$$

for any $h \otimes k \in H \otimes H$.

2. Consider the a G-space X as in example 1.1.17. The canonical map of the extension $O(X/G) \subseteq O(X)$ is the pullback of the map

$$\alpha : X \times G \longrightarrow X \times_{X/G} X, \quad (x, g) \longmapsto (x, xg),$$

where $X \times_{X/G} X$ is the fibered product of X with itself over X/G. It is easy to see that the injectivity of α is equivalent to the freeness of the action of G on X and its surjectivity is equivalent to the transitivity along the fibers of the projection $X \twoheadrightarrow X/G$. In other words, for the $O(G)$-extension $O(X/G) \subseteq O(X)$ is Hopf-Galois if and only if is $X \twoheadrightarrow X/G$ is a principal G-bundle [26].

3. For the group algebras $\mathbb{K}[G]$ we have that every comodule algebra has the form $A = \oplus_{g \in G} A_g$ with $A_g A_h \subseteq A_{gh}$. The extension $A_{e_G} \subseteq A$ is $\mathbb{K}[G]$-Hopf-Galois if and only if $A_g A_h = A_{gh}$ for all $g, h \in G$ (strongly graded) [44, 34]. ♦

1.2.2 Principal comodule algebras

In most situations, it is easy to check the surjectivity of the canonical map. On the other hand, it might be very difficult to prove that can is injective. The next result is of great importance since allows us to prove the bijectivity of the canonical map without dealing directly with injectivity

Theorem 1.2.5 (Schneider's [40]) *Let H a Hopf algebra with bijective antipode and A a right H-comodule algebra with $A^{coH} = B$. Then the following are equivalent*

1. *A is an injective right H-comodule and* can *is surjective;*
2. *The functor $\mathfrak{M}_B \longrightarrow \mathfrak{M}_A^H, M \longmapsto M \otimes_B A$, is an equivalence;*
3. *The functor ${}_B\mathfrak{M} \longrightarrow {}_A\mathfrak{M}^H, M \longmapsto A \otimes_B M$, is an equivalence;*
4. *A is a faithfully flat left B-module and* can *is invertible;*
5. *A is a faithfully flat right B-module and* can *is invertible.*

In the statement of this theorem, there are several concepts we have not dealt with in this thesis. So, in the following remark, we briefly recall some definitions

Remark 1.2.6 1. Any right comodule $V \in \mathfrak{M}^H$ induces a functor $\mathrm{Hom}^H(-, V) : \mathfrak{M}^H \longrightarrow \mathrm{Vect}_{\mathbb{K}}$. If the latter is exact V is said to be an injective comodule. If the Hopf algebra H is cosemisimple then any right H-comodule is injective.

2. A left B-module $M \in {}_B\mathfrak{M}$ is said to be faithfully flat if the functor $- \otimes_B M : \mathfrak{M}_B \longrightarrow \mathrm{Vect}_{\mathbb{K}}$ preserves and reflects exact sequences. Similarly for the right modules.

3. The adjoint functor of $- \otimes_B A$ is given by $(-)^{coH} : \mathfrak{M}_A^H \longrightarrow \mathfrak{M}_B$. So, for Hopf-Galois extensions such that A is a faithfully flat B-module one has

$$(M \otimes_B A)^{coH} \simeq M, \quad V^{coH} \otimes_B A \simeq V,$$

for every $M \in \mathfrak{M}_B$ and $V \in \mathfrak{M}_A^H$. And the same functor is the adjoint of $A \otimes_B -$. ♦

Definition 1.2.7 A H-Hopf-Galois extension such that A is a right (or left) faithfully flat B-module is said to be a **principal comodule algebra**. ♦

So for cosemisimple Hopf algebra with bijective antipode it is enough to check the surjectivity of the canonical map to prove its invertibility and faithfully flatness. In this thesis, we only deal with cosemisimple Hopf algebras.

Principal comodule algebras are characterized by another important property that deals with connections. First of all, one has the following

Definition 1.2.8 Let $B \subseteq A$ be a H-Hopf-Galois extension and $m_B : B \otimes A \longrightarrow A$ the multiplication map $m_B(b \otimes a) = ba$. A **strong connection** on $B \subseteq A$ is a unital left B-linear and right H colinear map $s : A \longrightarrow B \otimes A$ such that $m_B \circ s = \mathrm{id}_A$. ♦

This notion of connection was introduced initially in [20] and further studied in [13]. From the computational point of view, the next result is extremely important

Theorem 1.2.9 *Let H be a Hopf algebra with bijective antipode and $B \subseteq A$ a H-Hopf-Galois extension, then strong connections are in one-to-one correspondence with unital linear maps $l : H \longrightarrow A \otimes A$ satisfying*

$$(\mathrm{id}_A \otimes \rho) \circ l = (l \otimes \mathrm{id}_H) \circ \Delta, \tag{1.2-8}$$

$$(\lambda \otimes \mathrm{id}_A) \circ l = (\mathrm{id}_H \otimes l) \circ \Delta, \tag{1.2-9}$$

$$\pi_B \circ l = \tau, \tag{1.2-10}$$

where λ is the left H-coaction (1.1-8), $\pi_B : A \otimes A \longrightarrow A \otimes_B A$ is the canonical projection, and τ the translation map. The correspondence is given by

$$s = (m_A \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes l) \circ \rho, \quad l = m \circ (\mathrm{id}_A \otimes_B s) \circ \tau$$

We also refer to l as the strong connection of the Hopf-Galois extension. In other words, a strong connection is a lift of the translation map.

For principal comodule algebras, a strong connection always exists

Theorem 1.2.10 ([39]) *Let H be a Hopf algebra with bijective antipode and A a right H-comodule algebra with $A^{coH} = B$, then the following are equivalent*

1. *There exists a strong connection $l : H \longrightarrow A \otimes A$;*
2. *$B \subseteq A$ is a principal comodule algebra.*

In the following we use the Sweedler notation for strong connection $l(h) = h^{\langle 1 \rangle} \otimes h^{\langle 2 \rangle}$ so that we can use equations the lift of equations 1.2.3.

1.2.3 Deformations and external symmetries

Consider a H principal comodule algebra $B \subseteq A$ and another Hopf algebra K. If A is a K-H-bicomodule algebra, we say that K is an external symmetry. Denoting by $\rho_A(a) = a_{(0)} \otimes a_{(1)}$, $\lambda_A(a) = a_{(-1)} \otimes a_{(0)}$, we have the equation

$$a_{(0)(-1)} \otimes a_{(0)(0)} \otimes a_{(1)} = a_{(-1)} \otimes a_{(0)(0)} \otimes a_{(0)(1)}.$$

In general, the subalgebra of coaction invariant elements A^{coH} is different from the coaction invariant elements ${}^{coK}A$.

Example 1.2.11 Let G a, say compact, Lie group and $P \longrightarrow X$ be a principal G-bundle. If L is another Lie group acting on both P and X so that it commutes with the G-action, we say that L is an external symmetry of the bundle. At the algebraic level, we have that $O(X) \subseteq O(P)$ is a $O(G)$-Hopf-Galois extension and both $O(P)$ and $O(X)$ are $(O(L), O(G))$-bicomodule algebras. ♦

Consider now a $\gamma : H \otimes H \longrightarrow \mathbb{K}$ a 2-cocycle of H and deform H into H_γ and A into A_γ according to 1.1.21 and (1.1-11) respectively. It is easy to see that the subalgebra B does not change under the deformation, for any pair b, b' in the subalgebra of coaction invariant elements B one has $b \cdot_\gamma b' = bb'$.

If we indicate the functor $\Gamma : \mathfrak{M}^H \longrightarrow \mathfrak{M}^{H_\gamma}, V \longmapsto V_\gamma$, induced by the 2-cocycle γ, we have the following

Theorem 1.2.12 ([1]) *The algebra extension $B \subseteq A$ is H-Hopf-Galois if and only if $B \subseteq A_\gamma$ is H_γ-Hopf-Galois.*

This result is a consequence of the commutativity of the following diagram

$$\begin{array}{ccc} A_\gamma \otimes_B A_\gamma & \xrightarrow{\mathrm{can}_\gamma} & A_\gamma \otimes H_\gamma \\ \simeq \downarrow & & \downarrow \simeq \\ (A \otimes_B A)_\gamma & \xrightarrow[\Gamma(\mathrm{can})]{} & (A \otimes H)_\gamma, \end{array} \quad (1.2\text{-}11)$$

where the vertical arrows are the corresponding isomorphism of Theorem 1.1.22 for $A \otimes_B A$ and $A \otimes H$. Here H and H_γ are respectively right H and H_γ-comodule endowed with the adjoint coaction. Then has that can_γ is invertible if and only if $\Gamma(\mathrm{can})$.

The same result holds for principal comodule algebras, here we prove it by showing the explicit formula for a strong connection

Theorem 1.2.13 *The algebra extension $B \subseteq A$ is a principal H-comodule algebra if and only if $B \subseteq A_\gamma$ is a principal H_γ-comodule algebra.*

Proof To write down the right diagram, we need to introduce the H_γ-comodule isomorphism [1]

$$\mathfrak{f} : \underline{H_\gamma} \longrightarrow \underline{H}_\gamma, \quad h \longmapsto h_{(3)} u_\gamma(h_{(1)}) \gamma^{-1}(S(h_{(2)}), h_{(4)})$$

whose inverse is given by

$$\mathfrak{f}^{-1} : \underline{H}_\gamma \longrightarrow \underline{H_\gamma}, \quad h \longmapsto h_{(3)} u_\gamma^{-1}(h_{(2)}) \gamma(S(h_{(1)}), h_{(4)}).$$

We recall that $\underline{H}$ indicates H endowed with the right adjoint coaction (1.1-7). At this point, defines l_γ the map making the following diagram commute

$$\begin{array}{ccc} \underline{H}_\gamma & \xrightarrow{\Gamma(l)} & (A \otimes A)_\gamma \\ {\scriptstyle \mathfrak{f}^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha_{A,A}} \\ \underline{H_\gamma} & \xrightarrow{l_\gamma} & A_\gamma \otimes^\gamma A_\gamma \end{array}$$

Explicitly, we have

$$l_\gamma := \alpha_{A,A} \circ \Gamma(l) \circ \mathfrak{f} : \underline{H_\gamma} \longrightarrow A_\gamma \otimes^\gamma A_\gamma$$

mapping $h \longmapsto {h_{(2)}}^{\langle 1 \rangle} \otimes^\gamma {h_{(2)}}^{\langle 2 \rangle} u_\gamma(h_{(1)})$. We now prove that it is a strong connection for the extension $B \subseteq A_\gamma$. For this, we need to check the defining equations for any $h \in H_\gamma$

$$\begin{aligned} [(\mathrm{id}_A \otimes \rho^\gamma) \circ l_\gamma](h) &= {h_{(2)}}^{\langle 1 \rangle} \otimes^\gamma {{h_{(2)}}^{\langle 2 \rangle}}_{(0)} \otimes {{h_{(2)}}^{\langle 2 \rangle}}_{(1)} u_\gamma(h_{(1)}) \\ &= {h_{(2)}}^{\langle 1 \rangle} \otimes^\gamma {h_{(2)}}^{\langle 2 \rangle} \otimes h_{(3)} u_\gamma(h_{(1)}) \\ &= [(\mathrm{id}_H \otimes l_\gamma) \circ \Delta](h) \end{aligned}$$

$$\begin{aligned} [(\lambda^\gamma \otimes \mathrm{id}_A) \circ l_\gamma](h) &= S_\gamma^{-1}({{h_{(2)}}^{\langle 1 \rangle}}_{(1)}) \otimes {{h_{(2)}}^{\langle 1 \rangle}}_{(0)} \otimes {h_{(2)}}^{\langle 2 \rangle} u_\gamma(h_{(1)}) \\ &= S_\gamma^{-1}(S(h_{(2)})) \otimes {h_{(3)}}^{\langle 1 \rangle} \otimes^\gamma {h_{(3)}}^{\langle 1 \rangle} u_\gamma(h_{(1)}) \\ &= S_\gamma^{-1}(S_\gamma(h_{(3)}) \otimes {h_{(5)}}^{\langle 1 \rangle} \otimes^\gamma {h_{(5)}}^{\langle 2 \rangle} u_\gamma(h_{(1)}) u_\gamma^{-1}(h_{(2)}) u_\gamma(h_{(4)}) \\ &= h_{(1)} \otimes {h_{(3)}}^{\langle 1 \rangle} \otimes^\gamma {h_{(3)}}^{\langle 2 \rangle} u_\gamma(h_{(2)}) = [(\mathrm{id}_H \otimes l_\gamma) \circ \Delta]\,(h) \end{aligned}$$

$$\begin{aligned} (\mathrm{can}_\gamma \circ \pi_B \circ l_\gamma)(h) &= {h_{(2)}}^{\langle 1 \rangle} \cdot_\gamma {{h_{(2)}}^{\langle 2 \rangle}}_{(0)} \otimes {{h_{(2)}}^{\langle 2 \rangle}}_{(1)} u_\gamma(h_{(1)}) \\ &= {h_{(2)}}^{\langle 1 \rangle} \cdot {h_{(2)}}^{\langle 2 \rangle} \otimes h_{(3)} u_\gamma(h_{(1)}) \\ &= 1_A \otimes h_{(5)} \gamma(h_{(1)}, S(h_{(2)})) \gamma^{-1}(S(h_{(3)}), h_{(4)}) \\ &= 1_A \otimes h_{(2)} (u_\gamma * u_\gamma^{-1})(h_{(1)}) = 1_A \otimes h \end{aligned}$$

We used (1.2-8), (1.2-9) and (1.2-5) respectively. Being $\mathfrak{f}$ and $\alpha_{A,A}$ isomorphism, also the opposite is true. ■

Let now A be a (K,H)-bicomodule algebra and put $B := A^{coH}$. If a 2-cocycle σ of K is given, we denote by $\Sigma : {}^K\mathfrak{M} \longrightarrow {}^{K_\sigma}\mathfrak{M}$ the corresponding monoidal tensor realizing the equivalence, so $\Sigma(V) = {}_\sigma V$ for any left K-comodule. The Hopf algebra H is a left K-comodule algebra with the trivial coaction $h \longmapsto 1_K \otimes h$, so one has ${}_\sigma H \simeq H$.

Theorem 1.2.14 *The algebra extension $B \subseteq A$ is H-Hopf-Galois if and only if ${}_\sigma B \subseteq {}_\sigma A$ is H-Hopf-Galois.*

As for the case where the Hopf algebra H is deformed, the proof of this result follows from the commutativity of a diagram, namely

$$\begin{array}{ccc} {}_\sigma A \;{}^\sigma\!\otimes_{{}_\sigma B}\; {}_\sigma A & \xrightarrow{\mathrm{can}_\sigma} & {}_\sigma A \;{}^\sigma\!\otimes H \\ \simeq\downarrow & & \downarrow\simeq \\ {}_\sigma(A \otimes_B A) & \xrightarrow[\Sigma(\mathrm{can})]{} & {}_\sigma(A \otimes H), \end{array} \qquad (1.2\text{-}12)$$

where the vertical arrows are the corresponding isomorphism (1.1-12).

Again the same result holds for principal comodule algebras

Theorem 1.2.15 *The algebra extension $B \subseteq A$ with external symmetry K is a principal H-comodule algebra if and only if ${}_\sigma B \subseteq {}_\sigma A$ is a principal H-comodule algebra with external symmetry ${}_\sigma K$.*

Proof Recall that the inverse isomorphism of left K-comodule (1.1-12) for $A \otimes A$ is given by

$$\phi^{-1}_{A,A}(a \otimes \tilde{a}) = \sigma^{-1}\left(a_{(-1)}, \tilde{a}_{(-1)}\right) a_{(0)} \;{}^\sigma\!\otimes \tilde{a}_{(0)}.$$

We define the map $l_\sigma := \phi^{-1}_{A,A} \circ \Sigma(l) : H \longrightarrow {}_\sigma A \;{}^\sigma\!\otimes {}_\sigma A$ sending $h \longmapsto \sigma^{-1}\left(h^{\langle 1\rangle}{}_{(-1)}, h^{\langle 2\rangle}{}_{(-1)}\right) h^{\langle 1\rangle}{}_{(0)} \;{}^\sigma\!\otimes h^{\langle 2\rangle}{}_{(0)}$ that makes the diagram

$$\begin{array}{ccc} H & \xrightarrow{{}_\sigma l} & {}_\sigma A \;{}^\sigma\!\otimes {}_\sigma A \\ & {}_{\Sigma(l)}\searrow & \downarrow \phi_{A,A} \\ & & {}_\sigma(A \otimes A) \end{array}$$

Assume that l is a strong connection dor $B \subseteq A$, then for ${}_\sigma l$ we find that for any $h \in H$

$$\begin{aligned} \left[(\mathrm{id}_{{}_\sigma A} \otimes \rho) \circ {}_\sigma l\right](h) &= \sigma^{-1}\left(h^{\langle 1\rangle}{}_{(-1)}, h^{\langle 2\rangle}{}_{(-1)}\right) h^{\langle 1\rangle}{}_{(0)} \;{}^\sigma\!\otimes h^{\langle 2\rangle}{}_{(0)(0)} \otimes h^{\langle 2\rangle}{}_{(0)(1)} \\ &= \sigma^{-1}\left(h^{\langle 1\rangle}{}_{(0)}, h^{\langle 2\rangle}{}_{(0)(-1)}\right) h^{\langle 1\rangle}{}_{(0)} \;{}^\sigma\!\otimes h^{\langle 2\rangle}{}_{(0)(0)} \otimes h^{\langle 2\rangle}{}_{(1)} \\ &= \sigma^{-1}\left(h_{(1)}{}^{\langle 1\rangle}{}_{(0)}, h_{(1)}{}^{\langle 2\rangle}{}_{(-1)}\right) h_{(1)}{}^{\langle 1\rangle}{}_{(0)} \otimes h_{(1)}{}^{\langle 2\rangle}{}_{(0)} \;{}^\sigma\!\otimes h_{(2)}{}^{\langle 2\rangle} \\ &= \left[({}_\sigma l \otimes \mathrm{id}_H) \circ \Delta\right](h), \end{aligned}$$

$$\begin{aligned}
\left[(\lambda \otimes \mathrm{id}_{{}_\sigma A}) \circ {}_\sigma l\right](h) &= \sigma^{-1}\left({h^{\langle 1\rangle}}_{(-1)}, {h^{\langle 2\rangle}}_{(-1)}\right) S^{-1}\left({h^{\langle 1\rangle}}_{(0)(1)}\right) \otimes {h^{\langle 1\rangle}}_{(0)(0)} \,{}^{\sigma}\!\otimes {h^{\langle 2\rangle}}_{(0)} \\
&= \sigma^{-1}\left({h^{\langle 1\rangle}}_{(0)(-1)}, {h^{\langle 2\rangle}}_{(-1)}\right) S^{-1}\left({h^{\langle 1\rangle}}_{(1)}\right) \otimes {h^{\langle 1\rangle}}_{(0)(0)} \,{}^{\sigma}\!\otimes {h^{\langle 2\rangle}}_{(0)} \\
&= S^{-1}\left(S(h_{(1)})\right) \otimes \sigma^{-1}\left({h_{(2)}}^{\langle 1\rangle}{}_{(-1)}, {h_{(2)}}^{\langle 2\rangle}{}_{(-1)}\right) {h_{(2)}}^{\langle 1\rangle}{}_{(0)} \,{}^{\sigma}\!\otimes {h_{(2)}}^{\langle 2\rangle}{}_{(0)} \\
&= \left[(\mathrm{id}_H \otimes {}_\sigma l) \circ \Delta\right](h),
\end{aligned}$$

$$\begin{aligned}
({}_\sigma\mathrm{can} \circ \pi_{{}_\sigma B} \circ {}_\sigma l)(h) &= \sigma^{-1}\left({h^{\langle 1\rangle}}_{(-1)}, {h^{\langle 2\rangle}}_{(-1)}\right) {h^{\langle 1\rangle}}_{(0)} \bullet_\sigma {h^{\langle 2\rangle}}_{(0)(0)} \otimes {h^{\langle 2\rangle}}_{(1)} \\
&= \sigma\left({h_{(1)}}^{\langle 1\rangle}{}_{(0)(-1)}, {h_{(1)}}^{\langle 2\rangle}{}_{(0)(-1)}\right) \sigma^{-1}\left({h_{(1)}}^{\langle 1\rangle}{}_{(-1)}, {h_{(1)}}^{\langle 2\rangle}{}_{(-1)}\right) {h_{(1)}}^{\langle 1\rangle}{}_{(0)} {h_{(1)}}^{\langle 2\rangle}{}_{(0)} \otimes h_{(2)} \\
&= (\sigma * \sigma^{-1})\left({h_{(1)}}^{\langle 1\rangle}{}_{(-1)}, {h_{(1)}}^{\langle 2\rangle}{}_{(-1)}\right) {h_{(1)}}^{\langle 1\rangle}{}_{(0)} {h_{(1)}}^{\langle 2\rangle}{}_{(0)} \otimes h_{(2)} \\
&= {h_{(1)}}^{\langle 1\rangle} {h_{(1)}}^{\langle 2\rangle} \otimes h_{(2)} = 1_A \otimes h,
\end{aligned}$$

proving that ${}_\sigma l$ is a strong connection for ${}_\sigma B \subseteq {}_\sigma A$. We used the equations (1.2-3)-(1.2-5) and that ${}_\sigma A$ is a $({}_\sigma K, H)$-bicomodule. We have also dropped the symbol Σ for the right and left H-coactions in the equations to simplify the expression. Since $\phi_{A,A}$ is an isomorphism we have that also the opposite is true. ■

1.3 Hopf algebroids

In this section, we review the general theory of bialgebroids and Hopf algebroids both in the sense of Schauenburg [38] and Böhm-Szlachanyi [5, 3]. For the part that regards rings, coring, and bialgebroids we mostly follow [8].

1.3.1 Bialgebroids

As their name suggests, bialgebroids generalize bialgebras. The latter are algebras over a field that are equipped with a compatible coalgebra structure. Roughly speaking, a bialgebroid is a bialgebra over a (noncommutative) algebra rather than a field. We introduce bialgebroids in the same spirit of bialgebras, i.e. giving the definition of an algebra, a coalgebra and then the compatibility conditions.

In this setting, the algebra role is played by:

Definition 1.3.1 Let B be an algebra and denote by $B^e := B \otimes B^{op}$ its enveloping algebra. A B^e**-ring** is the datum of (U, t, s) where

- U is algebra;
- $s : B \to U$ and $t : B^{op} \to U$ are algebra morphisms;
- the ranges of s and t commute in U, i.e. $s(b)t(b) = t(b)s(b)\ \forall b \in B$.

We refer to s and t as *source* and *target* map respectively. ♦

A B-bimodule structure is then defined on the algebra U by

$$bub' := s(b)t(b')u, \quad u \in U, \quad b, b' \in B. \tag{1.3-1}$$

The balanced tensor product over the algebra B is obtained as the quotient of $U \otimes U$ by the ideal generated by elements of the form $t(b)u \otimes u' - u \otimes s(b)u'$ with $u, u' \in U$ and $b \in B$. For later use we write explicitly this tensor product structure:

$$U \otimes_B U := U \otimes U/(t(b)u \otimes u' - u \otimes s(b)u'). \tag{1.3-2}$$

With the source and target maps we can endow U also with a B^{op}-bimodule structure getting in this way different balanced tensor products. If we denote by ${}^{op} : B \longrightarrow B^{op}$ the linear anti-multiplicative isomorphism, then we have for any $u \in U$ and $b, b' \in B$

$$b^{op} \cdot u \cdot b'^{op} := t(b)ut(b'), \tag{1.3-3}$$

$$b^{op} * u * b'^{op} := s(b')us(b). \tag{1.3-4}$$

Accordingly, we define

$$U \odot_{B^{op}} U := U \otimes U/(ut(b) \otimes u' - u \otimes t(b)u'), \tag{1.3-5}$$

$$U \circledast_{B^{op}} U := U \otimes U/(s(b)u \otimes u' - u \otimes u's(b)). \tag{1.3-6}$$

The notion of coalgebra is extended by

Definition 1.3.2 Let B be an algebra, a B**-coring** is the datum of $(\mathcal{C}, \underline{\Delta}, \underline{\epsilon})$ such that

- $\mathcal{C}$ is a B-bimodule;
- $\underline{\Delta} : \mathcal{C} \longrightarrow \mathcal{C} \otimes_B \mathcal{C}$ is a B-bimodule morphism satisfying

$$(\underline{\Delta} \otimes_B \mathrm{id}_{\mathcal{C}}) \circ \underline{\Delta} = (\mathrm{id}_{\mathcal{C}} \otimes_B \underline{\Delta}) \circ \underline{\Delta}; \tag{1.3-7}$$

- $\underline{\epsilon}$ is a B-bimodule morphism with the property

$$(\underline{\epsilon} \otimes_B \mathrm{id}_{\mathcal{C}}) \circ \underline{\Delta} = \mathrm{id}_{\mathcal{C}} = (\mathrm{id}_{\mathcal{C}} \otimes_B \underline{\epsilon}) \circ \underline{\Delta}\,. \tag{1.3-8}$$

We call $\underline{\Delta}$ the *comultiplication* and $\underline{\epsilon}$ the *counit* as in the case of bialgebras. ♦

Then we put these two structures together to get

Definition 1.3.3 Let B be an algebra, the datum of $(\mathcal{H}, \underline{\Delta}, \underline{\epsilon}, s, t)$ is a (left) B-**bialgebroid** when:

- The triple $(\mathcal{H}, s, t)$ is a B^e-ring;
- The triple $(\mathcal{H}, \underline{\Delta}, \underline{\epsilon})$ is a B-coring;
- Given the *Takeuchi product* [43]

$$\mathcal{H} \times_B \mathcal{H} := \{h \otimes_B h' \in \mathcal{H} \otimes_B \mathcal{H} | ht(b) \otimes_B h' = h \otimes_B h's(b), \forall b \in B\}, \tag{1.3-9}$$

  one has that $\mathrm{Im}(\underline{\Delta}) \subseteq \mathcal{H} \times_B \mathcal{H}$ and it is an algebra morphism if corestricted $\mathcal{H} \times_B \mathcal{H}$.

- The counit is unital $\underline{\epsilon}(1_{\mathcal{H}}) = 1_A$ and satisfies

$$\underline{\epsilon}(hh') = \underline{\epsilon}(hs(\underline{\epsilon}(h'))) = \underline{\epsilon}(ht(\underline{\epsilon}(h'))), \quad \forall h, h' \in \mathcal{H}\,. \tag{1.3-10}$$

♦

In the above equation (1.3-9), we have the Takeuchi product [43] that is an algebra with component-wise multiplication and unit $1_{\mathcal{H}} \otimes 1_{\mathcal{H}}$. This condition is important since in general $\mathcal{H} \otimes_B \mathcal{H}$ does not have an algebra structure, unless $\mathcal{H}$ is a symmetric B-bimodule.

We use the Sweedler summation notation for the comultiplication $\underline{\Delta}(h) = h_{(1)} \otimes_B h_{(2)}$ and by recalling the B-bimodule structure (1.3-1) one has for equation (1.3-8)

$$\underline{\epsilon}(h_{(1)})h_{(2)} = s(\underline{\epsilon}(h_{(1)}))h_{(2)} = h = t(\underline{\epsilon}(h_{(2)}))h_{(1)} = h_{(1)}\underline{\epsilon}(h_{(2)}). \tag{1.3-11}$$

For a left bialgebroid $\mathcal{H}$ over an algebra B, the category of left $\mathcal{H}$-module ${}_{\mathcal{H}}\mathfrak{M}$ is a monoidal category with respect to the tensor product $\otimes_B$. In contrast, the category of right $\mathcal{H}$-module $\mathfrak{M}_{\mathcal{H}}$ is not monoidal in general. Using a left-right symmetry argument one can define a right bialgebroid over B [8]. Given a left (right) bialgebroid it is possible that a right (left) bialgebroid structure cannot be defined. We return to this point later in the text.

Example 1.3.4 Let A be an algebra and B a bialgebra. We have that the space $\mathcal{H} := A \otimes B \otimes A^{op}$ a bialgebroid over A if endowed with the following maps

$$s(a) = a \otimes 1_B \otimes 1_A, \quad t(a) = 1_A \otimes 1_B \otimes a^{op}$$
$$\underline{\Delta}(a \otimes b \otimes a'^{op}) = a \otimes b_{(1)} \otimes 1_A \otimes 1_A \otimes b_{(2)} \otimes a'^{op}, \quad \underline{\epsilon}(a \otimes b \otimes a'^{op}) = \epsilon(b)aa'$$ ♦

1.3.2 Canonical maps, antipodes, and comodule algebras

We saw in example 1.2.4 that the existence of the antipode for a bialgebra H is equivalent to the Hopf-Galois condition for $\mathbb{K} \subseteq H$. In [38] the author was guided by this feature of Hopf algebras to give the following

Definition 1.3.5 A B-bialgebroid $\mathcal{H}$ is a (left) **weak Hopf algebroid** if the canonical map

$$\beta : \mathcal{H} \odot_{B^{op}} \mathcal{H} \longrightarrow \mathcal{H} \otimes_B \mathcal{H}, \quad h \odot_{B^{op}} h' \longmapsto h_{(1)} \otimes_B h_{(2)}h' \tag{1.3-12}$$

is bijective. Notice that the map is well-defined over the tensor product (1.3-5) since $\underline{\Delta}(t(b)) = 1 \otimes_B t(b)$ for every $b \in B$. ♦

If $B = \mathbb{K}$ we retrieve the Hopf algebra case. In the same work, the invertibility of β is proved to be equivalent to the inner-hom functor preserving property of the forgetful functor ${}_{\mathcal{H}}\mathfrak{M} \to {}_B\mathfrak{M}_B$ from the category of left $\mathcal{H}$-modules to the one of B-bimodules.

There is another canonical map associated with a bialgebroid, namely

$$\lambda : \mathcal{H} \circledast_{B^{op}} \mathcal{H} \longrightarrow \mathcal{H} \otimes_B \mathcal{H}, \quad h \circledast_{B^{op}} h' \longmapsto h'_{(1)}h \otimes_B h'_{(2)}, \tag{1.3-13}$$

using the tensor product (1.3-6), which is well-defined due to $\underline{\Delta}(s(b)) = s(b) \otimes_B 1_{\mathcal{H}}$.

An alternative way to define a Hopf algebroid was proposed in [5], which mimics the Hopf algebra case

Definition 1.3.6 Let $(\mathcal{H}, \underline{\Delta}, \underline{\epsilon}, s, t)$ be a bialgebroid over an algebra B, then an **antipode** on it is the datum of a bijective anti-algebra morphism $\underline{S} : \mathcal{H} \to \mathcal{H}$, with inverse $\underline{S}^{-1} : \mathcal{H} \to \mathcal{H}$, such that,

$$\underline{S} \circ t = s, \tag{1.3-14}$$

and, for all $h \in \mathcal{H}$,

$$\underline{S}(h_{(1)})_{(1')} h_{(2)} \otimes_B \underline{S}(h_{(1)})_{(2')} = 1_{\mathcal{H}} \otimes_B \underline{S}(h), \tag{1.3-15}$$

$$\underline{S}^{-1}(h_{(2)})_{(1')} \otimes_B \underline{S}^{-1}(h_{(2)})_{(2')} h_{(1)} = \underline{S}^{-1}(h) \otimes_B 1_{\mathcal{H}}. \tag{1.3-16}$$

A B-bialgebroid equipped with an antipode is called a **full Hopf algebroid** over B and here it is denoted by $(\mathcal{H}, \underline{S})$. ♦

Remark 1.3.7 One notices that eq. (1.3-14) implies that for the inverse it holds that

$$\underline{S}^{-1} \circ s = t. \tag{1.3-17}$$

Moreover eqs.(1.3-15) and (1.3-16) imply that the following equations hold true for all $h \in \mathcal{H}$

$$\underline{S}(h_{(1)}) h_{(2)} = (t \circ \underline{\epsilon} \circ \underline{S})(h), \tag{1.3-18}$$

$$\underline{S}^{-1}(h_{(2)}) h_{(1)} = (s \circ \underline{\epsilon} \circ \underline{S}^{-1})(h). \tag{1.3-19}$$

♦

Remark 1.3.8 The names weak and full Hopf algebroids are not established in the literature, mostly because there is no wide consensus on the right definition of Hopf algebroid yet. Despite this, we decided to adopt these names because of the following result ♦

Proposition 1.3.9 ([25]) *Let $(\mathcal{H}, \underline{S})$ be a full Hopf algebroid over B then $\mathcal{H}$ is a Hopf algebroid in the sense of Definition 1.3.5.*

Remark 1.3.10 One can prove that if an invertible antipode exists then the map (1.3-13) is bijective too, with its inverse given by

$$\lambda^{-1}(h \otimes_B h') = \underline{S}^{-1}(h')_{(2)} h \circledast_{B^{op}} \underline{S}(\underline{S}^{-1}(h')_{(1)}). \tag{1.3-20}$$

Furthermore the simultaneous bijectivity of β and λ is equivalent to the existence of an invertible antipode for a left bialgebroid with a compatible right bialgebroid structure [5]. As mentioned before this is not always the case. Indeed there are situations where an antipode does not exist at all. We refer to [29] for an example. ♦

Having an invertible antipode one can give a B^{op}-bimodule structure to $\mathcal{H}$ that differs from the ones in (1.3-3)-(1.3-4). This is given by

$$b^{op} h b'^{op} = h \underline{S}^{-1}(t(b)) t(b'); \tag{1.3-21}$$

with associated balanced tensor product over B^{op}

$$\mathcal{H} \otimes_{B^{op}} \mathcal{H} := \mathcal{H} \otimes \mathcal{H} / (h t(b) \otimes h' - h \otimes \underline{S}^{-1}(t(b)) h'). \tag{1.3-22}$$

Lemma 1.3.11 *Let $(\mathcal{H}, \underline{S})$ be a full Hopf algebroid over B, and define the map from $\mathcal{H}$ to $\mathcal{H} \otimes_{B^{op}} \mathcal{H}$ given by the formula*

$$h \longmapsto h^{[1]} \otimes_{B^{op}} h^{[2]} := \underline{S}(\underline{S}^{-1}(h)_{(2)}) \otimes_{B^{op}} \underline{S}(\underline{S}^{-1}(h)_{(1)}). \tag{1.3-23}$$

It satisfies the following identities

$$h^{[1]}\underline{S}(h^{[2]}) = s(\underline{\epsilon}(h)), \tag{1.3-24}$$

$$\lambda^{-1}(h \otimes_B h') = \underline{S}^{-1}(h'^{[1]})h \circledast_{B^{op}} h'^{[2]}, \tag{1.3-25}$$

$$h^{[1]} \otimes_{B^{op}} {h^{[2]}}_{(1)} \otimes_B {h^{[2]}}_{(2)} = {h_{(1)}}^{[1]} \otimes_{B^{op}} {h_{(1)}}^{[2]} \otimes_B h_{(2)}, \tag{1.3-26}$$

$$\underline{S}(h)^{[1]} \otimes_{B^{op}} \underline{S}(h)^{[2]} = \underline{S}(h_{(2)}) \otimes_{B^{op}} \underline{S}(h_{(1)}). \tag{1.3-27}$$

for all $h, h' \in \mathcal{H}$, in (1.3-25) we have the inverse of the canonical map (1.3-13).

Proof For the first equation, we use the anti-multiplicative property of the antipode $\underline{S}$ and (1.3-18). For the second one just rewrite (1.3-20) using the definition (1.3-23). The third equation is proved by applying on both sides the isomorphism $\lambda \otimes_B \mathrm{id}_{\mathcal{H}}$ and showing that one gets the same result. The last equation comes from evaluating (1.3-23) at $k = \underline{S}(h)$ with $h \in \mathcal{H}$. ■

We conclude this subsection by discussing the notion of comodule algebra for a bialgebroid/Hopf algebroid. Since we are dealing now with B-modules rather than vector spaces we have to take care of the B-linearity of the maps

Definition 1.3.12 A B-bimodule M is said to be a right $\mathcal{H}$-comodule if there exists a B-bilinear map $\underline{\rho}_M : M \longrightarrow M \otimes_B \mathcal{H}$ that satisfies

$$(\underline{\rho}_M \otimes_B \mathrm{id}_{\mathcal{H}}) \circ \underline{\rho}_M = (\mathrm{id}_M \otimes_B \underline{\Delta}) \circ \underline{\rho}_M, \quad (\mathrm{id}_M \otimes_B \underline{\epsilon}) \circ \underline{\rho}_M = \mathrm{id}_M\,. \qquad \blacklozenge$$

Chapter 2

Hopf algebroids and twists for quantum projective spaces

2.1 Twists

In this section, we introduce the notion of a twist for a bialgebroid. Since we are working on a bimodule over an algebra we have to specify which module structure we are referring to. We exclusively work with the right B-module structure of (1.3-1) given by the action of the target map.

2.1.1 General aspects

Definition 2.1.1 Let $\mathcal{H}$ be a B-bialgebroid, the (right) **generalized characters** are given by the set

$$\mathcal{H}_* := \{\phi_* : \mathcal{H} \longrightarrow B | \phi_*(t(b)h) = \phi_*(h)b, \quad \forall h \in \mathcal{H}, \forall b \in B\}.$$

In other words, a generalized character is a map from $\mathcal{H}$ to the base algebra B that is a right B-module morphism. ♦

Using the source s and target map t of $\mathcal{H}$ one has other types of generalized characters. For instance maps such that $\phi(s(b)h) = b\phi(h)$ (left B-module morphisms) are denoted by ${}_*\mathcal{H}$. It is easy to define all the other possible generalized characters. In the case of Hopf algebras, $B\mathbb{K}$, the generalized characters are linear maps from the Hopf algebra into the ground field. The choice we made about the name might cause some trouble because characters for a Hopf algebra are not just linear maps from the algebra into the ground field, but also algebra morphisms. Since in the current thesis, there is no risk of confusion we can still use the name generalized character for $\mathcal{H}_*$.

We state and prove several properties of generalized characters in the next lemma (cf. [4]).

Lemma 2.1.2 *Let B be an algebra and $\mathcal{H}$ a B-algebroid then one has:*

1. *The set of generalized characters $\mathcal{H}_*$ is a unital algebra if endowed with product*
$$\phi_*\psi_*(h) := \psi_*\left(s(\phi_*(h_{(1)}))h_{(2)}\right); \tag{2.1-1}$$
where $s : B \longrightarrow \mathcal{H}$ is the source map. The unit element of $\mathcal{H}_*$ is the counit $\underline{\epsilon} : \mathcal{H} \longrightarrow B$;

2. *The space $\mathcal{H}$ is a right $\mathcal{H}_*$-module if endowed with*
$$h \triangleleft \phi_* := s(\phi_*(h_{(1)}))h_{(2)}; \tag{2.1-2}$$
for $\phi_* \in \mathcal{H}_*$ and $h \in \mathcal{H}$. The counit $\underline{\epsilon}$ acts trivially on $\mathcal{H}$ and for all $b \in B$ one has $t(b) \triangleleft \phi_* = t(b)$. As a consequence of the last property, we have $1_{\mathcal{H}} \triangleleft \phi_* = 1_{\mathcal{H}}$ for all $\phi_* \in \mathcal{H}_*$.

Proof For point (1).

Let $b \in B$, $h \in H$ and recall that the comultiplication $\underline{\Delta}$ on $\mathcal{H}$ is a B-bimodule morphism
$$\underline{\Delta}(t(b)h) = h_{(1)} \otimes_B t(b)h_{(2)}.$$
By taking any two elements $\phi_*, \psi_* \in \mathcal{H}_*$ one gets from (2.1-1)
$$\begin{aligned} \phi_*\psi_*(t(b)h) &= \psi_*(s(\phi_*(h_{(1)}))t(b)h_{(2)}) \\ &= \psi_*(t(b)s(\phi_*(h_{(1)}))h_{(2)}) \\ &= \psi_*(s(\phi_*(h_{(1)}))h_{(2)})b = \phi_*\psi_*(h)b, \end{aligned}$$
where in the second line we used the ranges of the source and target commute in $\mathcal{H}$. Thus, we conclude that $\phi_*\psi_* \in \mathcal{H}_*$.

Being $\underline{\epsilon}$ a B-bimodule morphism it is in particular a right B-module morphism, so is an element of $\mathcal{H}_*$. Moreover, for any $\phi_* \in \mathcal{H}_*$ and $h \in \mathcal{H}$ one has
$$\begin{aligned} \phi_*\underline{\epsilon}(h) &= \underline{\epsilon}(s(\phi_*(h_{(1)}))h_{(2)}) \\ &= \underline{\epsilon}(s(\phi_*(h_{(1)}))s(\underline{\epsilon}(h_{(2)}))) \\ &= \underline{\epsilon}(s(\phi_*(h_{(1)})\underline{\epsilon}(h_{(2)}))) \\ &= \underline{\epsilon}(s(\phi_*(t(\underline{\epsilon}(h_{(2)})h_{(1)})))) \\ &= (\underline{\epsilon} \circ s \circ \phi_*)(h) = \phi_*(h). \end{aligned}$$
Here in the second line we used the multiplicative property $\underline{\epsilon}(hk) = \underline{\epsilon}(hs(\underline{\epsilon}(k)))$, in the third one that the source map s is an algebra morphism, in the fourth one the Definition 2.1.1, and in the last one the equation $t(\underline{\epsilon}(h_{(2)}))h_{(1)} = h$ and that the source map is a section of the counit $\underline{\epsilon} \circ s = \mathrm{id}_B$. On the other, from the equation $s(\underline{\epsilon}(h_{(1)}))h_{(2)} = h$ it easily follows that
$$\underline{\epsilon}\phi_*(h) = \phi_*(s(\underline{\epsilon}(h_{(1)}))h_{(2)}) = \phi_*(h).$$
We conclude that $\phi_*\underline{\epsilon} = \underline{\epsilon}\phi_* = \phi_*$.

We only need to prove that $h \triangleleft (\phi_*\psi_*) = (h \triangleleft \phi_*) \triangleleft \psi_*$. Given $\phi_*, \psi_* \in \mathcal{H}_*$ and $h \in \mathcal{H}$ we have

$$\begin{aligned}(h \triangleleft \phi_*) \triangleleft \psi_* &= (s(\phi_*(h_{(1)})h_{(2)})) \triangleleft \psi_* \\ &= s(\psi_*(s(\phi_*(h_{(1)})h_{(2)})))h_{(3)} \\ &= s(\phi_*\psi_*(h_{(1)}))h_{(2)} = h \triangleleft (\phi_*\psi_*).\end{aligned}$$

where we just used the definition of the product (2.1-1) and the coassociativity os the comultiplication (1.3-7). ■

Inside the space of generalized characters, we find a group of its units, i.e. the invertible elements for the multiplication (2.1-1). In what follows we focus on a particular subgroup of this group, the one of twists.

Definition 2.1.3 Let $\mathcal{H}$ be a B-bialgebroid, we say that an element $\phi_* \in \mathcal{H}_*$ is a **twists** if it is invertible with respect to (2.1-1) and moreover

$$(h \triangleleft \phi_*)(h' \triangleleft \phi_*) = hh' \triangleleft \phi_*, \quad \forall h, h' \in \mathcal{H}. \tag{2.1-3}$$

♦

The counit of $\mathcal{H}$ is the neutral element of this group. For the rest of the paper, we denote the group of twists by $\mathcal{T}_*$.

2.1.2 Twists and antipodes

In the original definition of twists [4], it is required an additional property

$$\underline{S}(h_{(1)}) \triangleleft \phi_* \otimes'_B h_{(2)} = \underline{S}(h_{(1)}) \otimes'_B h_{(2)} \triangleleft \phi_*^{-1};$$

that involves the balanced tensor product $\otimes'_B$ obtained from $\mathcal{H} \otimes \mathcal{H}$ over the ideal generated by $hs(b) \otimes h' - h \otimes s \circ \phi_*^{-1} \circ s(b)h'$, for $b \in B$ and $h, h' \in \mathcal{H}$. For us, a twist does not need to fulfill this property, and in particular we have that the notion of twist makes sense even for bialgebroids.

Theorem 2.1.4 *([4]) Let $(\mathcal{H}, \underline{S})$ be a full Hopf algebroid over an algebra B, then $(\mathcal{H}, \underline{S}')$ is a Hopf algebroid with the same underlying B-bialgebroid structure if and only if there exists a twist $\phi_* \in \mathcal{T}_*$ such that*

$$\underline{S}'(h) = \underline{S}(h \triangleleft \phi_*) \quad h \in \mathcal{H}.$$

Proof Our proof differs slightly from [4], thus, we report it.

Let $\phi_* \in \mathcal{T}_*$ and $\underline{S}$ be an antipode on $\mathcal{H}$. Then the map $\underline{S}'$ defined above is invertible with inverse

$$\underline{S}'^{-1}(h) := \underline{S}(h) \triangleleft \phi^{-1}.$$

One checks easily that $\underline{S}'$ is an anti-algebra morphism because $\underline{S}$ is such and property (2.1-3). From 2.1.2 we have $t(b) \triangleleft \phi_* = b$ for every $b \in B$ so $\underline{S}' \circ t = s$. Now to prove (1.3-15) and (1.3-16) we first notice that for any $\psi_* \in \mathcal{H}_*$ one has

$$(h \triangleleft \psi_*)_{(1)} \otimes_B (h \triangleleft \psi_*)_{(2)} = \underline{\Delta}(h \triangleleft \psi_*) = h_{(1)} \triangleleft \psi_* \otimes_B h_{(2)},$$

since the comultiplication is a left B-module morphism. Thus for any $h \in \mathcal{H}$ we compute,

$$\begin{aligned}\underline{S}'(h_{(1)})_{(1)'}h_{(2)} \otimes_B \underline{S}'(h_{(1)})_{(2)'} &= \underline{S}(h_{(1)} \triangleleft \phi_*)_{(1)'}h_{(2)} \otimes_B \underline{S}(h_{(1)} \triangleleft \phi_*)_{(2)'} \\ &= \underline{S}((h \triangleleft \phi_*)_{(1)})_{(1)'}(h \triangleleft \phi_*)_{(2)} \otimes_B \underline{S}((h \triangleleft \phi_*)_{(1)})_{(2)'} \\ &= 1_{\mathcal{H}} \otimes_B \underline{S}(h \triangleleft \phi*) = 1_{\mathcal{H}} \otimes_B \underline{S}'(h);\end{aligned}$$

$$\begin{aligned}\underline{S}'^{-1}(h_{(2)})_{(1)'} \otimes_B \underline{S}'^{-1}(h_{(2)})_{(2)'}h_{(1)} &= (\underline{S}^{-1}(h_{(2)}) \triangleleft \phi_*)_{(1)'} \otimes (\underline{S}^{-1}(h_{(2)}) \triangleleft \phi_*)_{(2)'}h_{(1)} \\ &= \underline{S}^{-1}(h_{(2)})_{(1)'} \triangleleft \phi_* \otimes_B \underline{S}^{-1}(h_{(2)})_{(2)'}h_{(1)} \\ &= \underline{S}^{-1}(h) \triangleleft \phi_* \otimes_B 1_{\mathcal{H}} = \underline{S}'^{-1}(h) \otimes_B 1_{\mathcal{H}}.\end{aligned}$$

We conclude that $\underline{S}'$ is an invertible antipode on $\mathcal{H}$.

On the other hand consider $\underline{S}$ and $\underline{S}'$ two antipodes on $\mathcal{H}$. The map $\underline{\epsilon} \circ \underline{S}'^{-1} \circ \underline{S}$ from $\mathcal{H} \to B$ lies in $\mathcal{T}_*$. Right B-linearity is easy to check. The map $\underline{\epsilon} \circ \underline{S}^{-1} \circ \underline{S}'$ is the inverse: indeed we have

$$\begin{aligned}(\underline{\epsilon} \circ \underline{S}'^{-1} \circ \underline{S})(\underline{\epsilon} \circ \underline{S}^{-1} \circ \underline{S}')(h) &= \underline{\epsilon} \circ \underline{S}^{-1} \circ \underline{S}' \left[s(\underline{\epsilon} \circ \underline{S}'^{-1} \circ \underline{S}(h_{(1)}))h_{(2)}\right] \\ &= \underline{\epsilon} \circ \underline{S}^{-1} \circ \underline{S}' \left[s \circ \underline{\epsilon} \circ \underline{S}'^{-1}(\underline{S}(h_{(1)}))h_{(2)}\right] \\ &= \underline{\epsilon} \circ \underline{S}^{-1} \circ \underline{S}' \left[s \circ \underline{\epsilon} \circ \underline{S}'^{-1}(\underline{S}(h)^{[2]})\underline{S}^{-1}(\underline{S}(h)^{[1]})\right] \\ &= \underline{\epsilon} \circ \underline{S}^{-1} \circ \underline{S}' \left[s \circ \underline{\epsilon} \circ \underline{S}'^{-1}(\underline{S}(h)^{[2]'})\underline{S}'^{-1}(\underline{S}(h)^{[1]'})\right] \\ &= \underline{\epsilon} \circ \underline{S}^{-1} \left[\underline{S}(h)^{[1]'}\underline{S}' \circ s \circ \underline{\epsilon} \circ \underline{S}'^{-1}(\underline{S}(h)^{[2]'})\right] \\ &= \underline{\epsilon} \circ \underline{S}^{-1} \left[\underline{S}(h)^{[1]'}\underline{S}'(\underline{S}(h)^{[2]'}{}_{(1)})\underline{S}(h)^{[2]'}{}_{(2)}\right] \\ &= \underline{\epsilon} \circ \underline{S}^{-1} \left[\underline{S}(h)_{(1)}{}^{[1]'}\underline{S}'(\underline{S}(h)_{(1)}{}^{[2]'})\underline{S}(h)_{(2)}\right] \\ &= \underline{\epsilon} \circ \underline{S}^{-1} \left[s(\underline{\epsilon}(\underline{S}(h)_{(1)}))\underline{S}(h)_{(2)}\right] \\ &= \underline{\epsilon} \circ \underline{S}^{-1} \circ \underline{S}(h) = \underline{\epsilon}(h).\end{aligned}$$

where in order we used eqs.(1.3-24)-(1.3-27) of Lemma 1.3.11.

To conclude the proof we have to show that the action of $\underline{\epsilon} \circ \underline{S}'^{-1} \circ \underline{S}$ is unital and multiplicative. The first property comes from the fact that both maps are a composition of unital maps, for the second one for any $h, h' \in \mathcal{H}$ we compute

$$\begin{aligned}(hh') \triangleleft \underline{\epsilon} \circ \underline{S}'^{-1} \circ \underline{S} &= s\left[\epsilon \circ \underline{S}'^{-1} \circ \underline{S}(h_{(1)}h'_{(1)})\right] h_{(2)}h'_{(2)} \\ &= s \circ \underline{\epsilon}\left[\underline{S}'^{-1} \circ \underline{S}(h_{(1)})\underline{S}'^{-1} \circ \underline{S}(h'_{(1)})\right] h_{(2)}h'_{(2)} \\ &= s \circ \underline{\epsilon}\left[\underline{S}'^{-1} \circ \underline{S}(h_{(1)})\underline{S}'^{-1} \circ \underline{S}(h'_{(1)})\right] h_{(2)}h'_{(2)} \\ &= s \circ \underline{\epsilon}\left[\underline{S}'^{-1} \circ \underline{S}(h_{(1)})t \circ \underline{\epsilon} \circ \underline{S}'^{-1} \circ \underline{S}(h'_{(1)})\right] h_{(2)}h'_{(2)} \\ &= s \circ \underline{\epsilon}\left[\underline{S}'^{-1} \circ \underline{S}(h_{(1)}t \circ \underline{\epsilon} \circ \underline{S}'^{-1} \circ \underline{S}(h'_{(1)}))\right] h_{(2)}h'_{(2)} \\ &= s \circ \underline{\epsilon}\left[\underline{S}'^{-1} \circ \underline{S}(h_{(1)})\right] h_{(2)}s \circ \underline{\epsilon}\left[\circ\underline{S}'^{-1} \circ \underline{S}(h'_{(1)})\right] h'_{(2)} \\ &= (h \triangleleft \underline{\epsilon} \circ \underline{S}'^{-1} \circ \underline{S})(h' \triangleleft \underline{\epsilon} \circ \underline{S}'^{-1} \circ \underline{S}).\end{aligned}$$

Here in the third line, we have used that $\underline{S'}^{-1} \circ \underline{S}$ is an algebra morphism, in the fourth equation (1.3-10), in the fifth the combination of eqs.(1.3-18) and (1.3-19) and in the last one the Takeuchi product (1.3-9). ∎

This is a stronger version of Theorem 4.1 in the reference since we showed that the requirement in the remark before 2.1.4 is not needed.

2.2 The Ehresmann–Schauenburg Hopf algebroid

We turn to the study of the theory of twists introduced in the previous section, in the special case of the Erhesmann-Schauenburg (ES) bialgebroid associated with a principal comodule algebra $B \subseteq A$ extension.

We first give the following lemma that gathers some results about the equivalence of the space of coaction invariant elements $(A \otimes A)^{coH}$ of the diagonal coaction (1.1-6) on $A \otimes A$ with other spaces in the case of Hopf-Galois extensions.

Lemma 2.2.1 *Let $B \subseteq A$ be a right H-Hopf-Galois extension and define*

$$\mathcal{L}_A := \{a \otimes \tilde{a} \in A \otimes A | a \otimes \tilde{a}_{(0)} \otimes \tilde{a}_{(1)} = a_{(0)} \otimes \tilde{a} \otimes S(a_{(1)})\},$$
$$(A \otimes A)^{coH} := \{a \otimes \tilde{a} \in A \otimes A | a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)} = a \otimes \tilde{a} \otimes 1_H\},$$
$$\mathcal{C}(A, H) := \{a \otimes \tilde{a} \in A \otimes A | a_{(0)} \otimes {a_{(1)}}^{\langle 1 \rangle} \otimes_B {a_{(1)}}^{\langle 2 \rangle}\tilde{a} = a \otimes \tilde{a} \otimes_B 1_A\}.$$

Then one has $\mathcal{L}_A = (A \otimes A)^{coH} = \mathcal{C}(A, H)$. Moreover, if the antipode of H is bijective, the above spaces are also identified with $A \,\square^H A$.

Proof We first prove the inclusion $\mathcal{L}_A \subseteq (A \otimes A)^{coH}$ by taking $a \otimes \tilde{a} \in \mathcal{L}_A$, and applying on both sides of the defining equation the map $\rho \otimes \mathrm{id}_H$ we get

$$a_{(0)} \otimes a_{(1)} \otimes \tilde{a}_{(0)} \otimes \tilde{a}_{(1)} = a_{(0)} \otimes a_{(1)} \otimes \tilde{a} \otimes S(a_{(2)}).$$

By applying again on both sides the map $(\mathrm{id}_{A \otimes A} \otimes \mathrm{m}_H) \circ (\mathrm{id}_A \otimes \mathrm{flip}_{HA} \otimes \mathrm{id}_H)$:

$$a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)} = a_{(0)} \otimes \tilde{a} \otimes a_{(1)}S(a_{(2)}) = a \otimes \tilde{a} \otimes 1_H,$$

where we used $a_{(1)}S(a_{(2)}) = \epsilon(a_{(1)})1_H$ and $a_{(0)}\epsilon(a_{(1)}) = a$, which proves the inclusion. We next prove $(A \otimes A)^{coH} \subseteq \mathcal{L}_A$. Take $a \otimes \tilde{a} \in (A \otimes A)^{coH}$ and by applying $(\mathrm{id}_A \otimes S \circ \rho) \otimes \mathrm{id}_{A \otimes H}$ on both sides of the defying equation we get

$$a_{(0)} \otimes S(a_{(1)}) \otimes \tilde{a}_{(0)} \otimes a_{(2)}\tilde{a}_{(1)} = a_{(0)} \otimes S(a_{(1)}) \otimes \tilde{a} \otimes 1_H.$$

By applying again $(\mathrm{id}_{A \otimes A} \otimes \mathrm{m}_H) \circ (\mathrm{id}_A \otimes \mathrm{flip}_{HA} \otimes \mathrm{id}_H)$ we have the equation

$$a_{(0)} \otimes \tilde{a}_{(0)} \otimes S(a_{(1)})a_{(2)}\tilde{a}_{(1)} = a_{(0)} \otimes \tilde{a} \otimes S(a_{(1)}).$$

Lhe LHS of the latter reduces to $a \otimes \tilde{a}_{(0)} \otimes \tilde{a}_{(1)}$ once we use $a_{(1)}S(a_{(2)}) = \epsilon(a_{(1)})1_H$ and $a_{(0)}\epsilon(a_{(1)}) = a$. We conclude that $(A \otimes A)^{coH} = \mathcal{L}_A$.

To prove the other equivalence we use bijectivity of the canonical map can. Take $a \otimes \tilde{a} \in \mathcal{C}(A, H)$ and apply $\mathrm{id}_A \otimes \mathrm{can}$ on both sides of the defying equation to get

$$a_{(0)} \otimes {a_{(1)}}^{\langle 1 \rangle} {{a_{(1)}}^{\langle 2 \rangle}}_{(0)} \tilde{a}_{(0)} \otimes {{a_{(1)}}^{\langle 2 \rangle}}_{(1)} \tilde{a}_{(1)} = a \otimes \tilde{a} \otimes 1_H.$$

Using (1.2-3), the LHS reduces to $a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)}$, thus $\mathcal{C}(A, H) \subseteq (A \otimes A)^{coH}$. Conversely, take $a \otimes \tilde{a} \in (A \otimes A)^{coH}$ and apply $\mathrm{id}_A \otimes \mathrm{can}^{-1}$ to both sides to the coinvariance equation. The left A-linearity of can^{-1} yields

$$a_{(0)} \otimes \tilde{a}_{(0)}(a_{(1)}\tilde{a}_{(1)})^{\langle 1 \rangle} \otimes_B (a_{(1)}\tilde{a}_{(1)})^{\langle 2 \rangle} = a \otimes \tilde{a} \otimes_B 1_A.$$

Using equations (1.2-6) and (1.2-7) the LHS can be rewritten as $a_{(0)} \otimes \tau(a_{(1)})\tilde{a}$, showing the converse inclusion. So we have $(A \otimes A)^{coH} = \mathcal{C}(A, H)$.

If the antipode of H is bijective, we can endow A with the H-coaction (1.1-8) and construct the cotensor product (1.1-9). The last statement follows easily from the definition of $\mathcal{L}_A$. ■

In the rest of the chapter, we use $\mathcal{C}(A, H)$. The latter can be endowed with a B-bialgebroid structure [37, 8] and it is named the Ehresmann-Schauenburg (ES) bialgebroid. The B-coring structure on $\mathcal{C}(A, H)$ is given by the following comultiplication and counit:

$$\underline{\Delta} : \mathcal{C}(A, H) \longrightarrow \mathcal{C}(A, H) \otimes_B \mathcal{C}(A, H), \quad a \otimes \tilde{a} \longmapsto a_{(0)} \otimes a_{(1)}{}^{\langle 1 \rangle} \otimes_B a_{(1)}{}^{\langle 2 \rangle} \otimes \tilde{a} \tag{2.2-1}$$

$$\underline{\epsilon} : \mathcal{C}(A, H) \longrightarrow B, \quad a \otimes \tilde{a} \longmapsto a\tilde{a}\,. \tag{2.2-2}$$

The space $\mathcal{C}(A, H)$ is a subalgebra of $A \otimes A^{op}$, i.e. the product in $\mathcal{C}(A, H)$ is

$$(a \otimes \tilde{a})(c \otimes \tilde{c}) = ac \otimes \tilde{c}\tilde{a}, \tag{2.2-3}$$

with $a \otimes \tilde{a}$ and $c \otimes \tilde{c} \in \mathcal{C}(A, H)$, and the B^e-ring structure is given by the source and target maps

$$s : B \longrightarrow \mathcal{C}(A, H), \quad b \longmapsto b \otimes 1_A, \tag{2.2-4}$$

$$t : B \longrightarrow \mathcal{C}(A, H), \quad b \longmapsto 1_A \otimes b. \tag{2.2-5}$$

All the compatibility conditions between the coring and ring structure are ensured in this situation, so $\mathcal{C}(A, H)$ is a (left) B-bialgebroid.

Remark 2.2.2 (The classical case) Let G be a compact Lie group, and $\pi : P \longrightarrow M$ be a compact principal G-bundle with translation function $\tau : P \times_M P \longrightarrow G$. Recall that the latter is defined by the equation

$$p\tau(p, q) = q, \quad (p, q) \in P \times_M P \tag{2.2-6}$$

In other words, $\tau(p, q)$ is the unique element in G connecting two points in the same fiber. We need the following identities, for all $(p, q) \in P \times_M P$ and $g \in G$ one has

$$\tau(pg, q) = g^{-1}\tau(p, q), \tag{2.2-7}$$

$$\tau(p, qg) = \tau(p, q)g, \tag{2.2-8}$$

$$\tau(p, q)^{-1} = \tau(q, p) \tag{2.2-9}$$

The gauge groupoid [33] over the base manifold M is given by the space $\Omega := (P \times P)/G$ of the orbits of the diagonal action, with source and target maps

$$s : \Omega \longrightarrow M, \quad [p, q] \longmapsto \pi(q) \tag{2.2-10}$$

$$t : \Omega \longrightarrow M, \quad [p, q] \longmapsto \pi(p) \tag{2.2-11}$$

and space of composable pairs

$$\Omega^{(2)} := \{(\omega, \omega') \in \Omega \times \Omega | s(\omega) = t(\omega')\}. \tag{2.2-12}$$

For which it is defined the composition rule

$$[p, q]\,[p', q'] := [p\tau(q, p'), q']\,, \quad ([p, q]\,, [p', q']) \in \Omega^{(2)} \tag{2.2-13}$$

♦

Remark 2.2.3 Faithfully flatness of A as a left B-module is sufficient to prove that the above set of data gives a bialgebroid over B. For the details see section 34 in [8]. An example for which faithful flatness is not needed is in section 3 in [24]. ♦

Remark 2.2.4 A *Galois object* is a H-Hopf–Galois extension such that the subalgebra of coaction invariant elements is the ground field $B = \mathbb{K}$. In this case, the ES bialgebroid is proved to be a Hopf algebra [36]. For a Hopf algebra, twists are convolution invertible characters and their action on the antipode provides an antipode in the sense of Hopf algebroid. In other words, starting from a Hopf algebra, using twists we get a full Hopf algebroid, a phenomenon which is central in [12]. ♦

Remark 2.2.5 Since in the first part of the proof we do not use that A is an algebra nor that $B \subseteq A$ is a Hopf-Galois extension, the equation $\mathcal{L}_V = (V \otimes V)^{coH}$ holds for any right H-comodule V. ♦

2.2.1 The flip

We give a sufficient condition for the map $\mathrm{flip}_A : A \otimes A \to A \otimes A$ to be an antipode for the ES bialgebroid of a Hopf–Galois extension $B \subseteq A$. Clearly flip_A is an anti-algebra morphism and moreover it satisfies for any $b \in B$

$$\mathrm{flip}_A(t(b)) = \mathrm{flip}(1_A \otimes b) = b \otimes 1_A = s(b).$$

Recall that $\mathcal{C}(A, H)$ is a sub-algebra of $A \otimes A^{op}$. To prove that flip_A is an antipode it only remains to check equations (1.3-15) and (1.3-16).

Proposition 2.2.6 *Let $B \subseteq A$ be a principal H-comodule algebra and let $\mathcal{C}(A, H)$ be the associated ES bialgebroid. If flip_A is a right H-comodule endomorphism of $(A \otimes A, \rho^{\otimes})$ then is an antipode for $\mathcal{C}(A, H)$.*

Proof The coinvariance hypothesis on flip_A reads

$$\rho^{\otimes} \circ \mathrm{flip}_A = (\mathrm{flip}_A \otimes \mathrm{id}_H) \circ \rho^{\otimes}.$$

Then if $a \otimes \tilde{a} \in \mathcal{C}(A, H) = (A \otimes A)^{coH}$ so does $\mathrm{flip}_A(a \otimes \tilde{a}) = \tilde{a} \otimes a$, i.e. the flip maps $\mathcal{C}(A, H)$ into itself. By the lemma 2.2.1 we also have $\tilde{a} \otimes a \in \mathcal{L}_A$. For the LHS of equation (1.3-15) we need

$$(a \otimes \tilde{a})_{(1)} \otimes_B (a \otimes \tilde{a})_{(2)} = a_{(0)} \otimes {a_{(1)}}^{\langle 1\rangle} \otimes_B {a_{(1)}}^{\langle 2\rangle} \otimes \tilde{a},$$

$$(\mathrm{flip}_A(a \otimes \tilde{a})_{(1)})_{(1')} \otimes_B (\mathrm{flip}_A(a \otimes \tilde{a})_{(1)})_{(2')} = {{a_{(1)}}^{\langle 1\rangle}}_{(0)} \otimes {{{a_{(1)}}^{\langle 1\rangle}}_{(1)}}^{\langle 1\rangle} \otimes_B {{{a_{(1)}}^{\langle 1\rangle}}_{(1)}}^{\langle 2\rangle} \otimes a_{(0)}.$$

So we get for LHS

$$\begin{aligned}
&{a_{(1)}}^{\langle 1\rangle}{}_{(0)}{a_{(1)}}^{\langle 2\rangle}\otimes\tilde{a}{a_{(1)}}^{\langle 1\rangle}{}_{(1)}{}^{\langle 1\rangle}\otimes_B{a_{(1)}}^{\langle 1\rangle}{}_{(1)}{}^{\langle 2\rangle}\otimes a_{(0)}=\\
&={a_{(2)}}^{\langle 1\rangle}{a_{(2)}}^{\langle 2\rangle}\otimes\tilde{a}S(a_{(1)})^{\langle 1\rangle}\otimes_B S(a_{(1)})^{\langle 2\rangle}\otimes a_{(0)}\\
&=1_A\otimes\tilde{a}S(a_{(1)})^{\langle 1\rangle}\otimes_B S(a_{(1)})^{\langle 2\rangle}\otimes a_{(0)}\\
&=1_A\otimes\tilde{a}\tau(S(a_{(1)}))\otimes a_{(0)}\\
&=1_A\otimes\tilde{a}_{(0)}\tau(\tilde{a}_{(1)})\otimes a\\
&=1_A\otimes 1_A\otimes_B\tilde{a}\otimes a=1_{\mathcal{C}(A,H)}\otimes\mathrm{flip}_A(a\otimes\tilde{a})
\end{aligned}$$

in the first line we used (1.2-4), in the second one the equation (1.2-5) and identity $a_{(1)}\epsilon(a_{(2)})=a_{(1)}$, in the third one just the definition of the translation map, in the fourth one the fact that $\tilde{a}\otimes a\in\mathcal{L}_A$ and finally in the fifth line equation (1.2-7).

Notice that $\mathrm{flip}_A^{-1}=\mathrm{flip}_A$, then to write down the LHS of equation (1.3-16) in this case we need

$$(\mathrm{flip}_A(a\otimes\tilde{a})_{(2)})_{(1')}\otimes_B(\mathrm{flip}_A(a\otimes\tilde{a})_{(2)})_{(2')}=\tilde{a}_{(0)}\otimes{\tilde{a}_{(1)}}^{\langle 1\rangle}\otimes_B{\tilde{a}_{(1)}}^{\langle 2\rangle}\otimes{a_{(1)}}^{\langle 2\rangle}$$

so it becomes

$$\begin{aligned}
\tilde{a}_{(0)}\otimes{\tilde{a}_{(1)}}^{\langle 1\rangle}\otimes_B{\tilde{a}_{(1)}}^{\langle 2\rangle}a_{(0)}\otimes{a_{(1)}}^{\langle 1\rangle}{a_{(1)}}^{\langle 2\rangle}&=\tilde{a}_{(0)}\otimes{\tilde{a}_{(1)}}^{\langle 1\rangle}\otimes_B{\tilde{a}_{(1)}}^{\langle 2\rangle}a_{(0)}\otimes\epsilon(a_{(1)})\\
&=\tilde{a}_{(0)}\otimes\tau(\tilde{a}_{(1)})a\otimes 1_A\\
&=\tilde{a}\otimes a\otimes_B 1_A\otimes 1_A=\mathrm{flip}_A(a\otimes\tilde{a})\otimes 1_{\mathcal{C}(A,H)}.
\end{aligned}$$

We used again (1.2-5), the identity $a_{(0)}\epsilon(a_{(1)})=a$, and that $\tilde{a}\otimes a\in\mathcal{C}(A,H)$. ■

Using this result we have the following Let $B\subseteq A$ a principal H-comodule algebra extension such that H is commutative, then flip_A is an antipode for $\mathcal{C}(A,H)$.

Proof Because of the previous proposition, it is enough to show that flip_A is a right H-comodule endomorphism of $(A\otimes A,\rho^{\otimes})$. This is easily checked since for any $a\otimes\tilde{a}\in A\otimes A$ one finds

$$\begin{aligned}
\rho^{\otimes}(\mathrm{flip}_A(a\otimes\tilde{a}))&=\tilde{a}_{(0)}\otimes a_{(0)}\otimes\tilde{a}_{(1)}a_{(1)}\\
&=\tilde{a}_{(0)}\otimes a_{(0)}\otimes a_{(1)}\tilde{a}_{(1)}\\
&=(\mathrm{flip}_A\otimes\mathrm{id}_H)(a_{(0)}\otimes\tilde{a}_{(0)}\otimes a_{(1)}\tilde{a}_{(1)})\\
&=(\mathrm{flip}_A\otimes\mathrm{id}_H)(\rho^{\otimes}(a\otimes\tilde{a})).
\end{aligned}$$

This concludes the proof. ■

2.2.2 Twists for the ES bialgebroid

For the ES bialgebroid associated to a Hopf–Galois extension $\mathcal{C}(A,H)$, Definitions 2.1.1 and 2.1.3 read

$$a\otimes\tilde{a}\triangleleft\phi_*=\phi_*(a_{(0)}\otimes{a_{(1)}}^{\langle 1\rangle}){a_{(1)}}^{\langle 2\rangle}\otimes\tilde{a},\tag{2.2-14}$$

$$\phi_*(1_A\otimes 1_A)=1_A,\tag{2.2-15}$$

and

$$\phi_*(a_{(0)}c_{(0)} \otimes c_{(1)}{}^{\langle 1\rangle}a_{(1)}{}^{\langle 1\rangle})a_{(1)}{}^{\langle 2\rangle}c_{(1)}{}^{\langle 2\rangle} \otimes \tilde{c}\tilde{a}$$
$$= \phi_*(a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle})a_{(1)}{}^{\langle 2\rangle}\phi_*(c_{(0)} \otimes c_{(1)}{}^{\langle 1\rangle})c_{(1)}{}^{\langle 2\rangle} \otimes \tilde{c}\tilde{a}, \quad (2.2\text{-}16)$$

for $a \otimes \tilde{a}, c \otimes \tilde{c} \in \mathcal{C}(A, H)$.

We now describe the group of twists for the ES bialgebroid. Let $\mathrm{Alg}^H(A)$ be the group of unital right H-comodule algebra automorphisms of A, with product

$$F \cdot G = G \circ F. \quad (2.2\text{-}17)$$

Proposition 2.2.7 *Let $B \subseteq A$ be a principal H-comodule algebra, the formulas*

$$\phi_*^F(a \otimes \tilde{a}) := F(a)\tilde{a}, \quad a \otimes \tilde{a} \in \mathcal{C}(A, H), F \in \mathrm{Alg}^H(A), \quad (2.2\text{-}18)$$
$$F_{\phi_*}(a) := \phi_*(a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle})a_{(1)}{}^{\langle 2\rangle}, \quad a \in A, \phi_* \in \mathcal{T}_*, \quad (2.2\text{-}19)$$

provide a group isomorphism between the group of twits $\mathcal{T}_*$ of the bialgebroid $\mathcal{C}(A, H)$ and the group $\mathrm{Alg}^H(A)$.

Proof Let $F \in \mathrm{Alg}^H(A)$ and consider ϕ_*^F as in (2.2-18). Its image lies in B since F is a H-comodule map.

$$\rho(F(a)\tilde{a}) = F(a_{(0)})\tilde{a}_{(0)} \otimes a_{(1)}\tilde{a}_{(1)} = F(a)\tilde{a} \otimes 1_H,$$

for any $a \otimes \tilde{a} \in \mathcal{C}(A, H)$. Moreover, one easily checks that it is a right B-module morphism and it is unital too. The right action (2.1-2) here reads

$$a \otimes \tilde{a} \triangleleft \phi_*^F = F(a) \otimes \tilde{a}$$

and with this we have

$$\begin{aligned} ac \otimes \tilde{c}\tilde{a} \triangleleft \phi_*^F &= F(ac) \otimes \tilde{c}\tilde{a} \\ &= (F(a) \otimes \tilde{a})(F(c) \otimes \tilde{c}) \\ &= (a \otimes \tilde{a} \triangleleft \phi_*^F)(c \otimes \tilde{c} \triangleleft \phi_*^F) \end{aligned}$$

for any $a \otimes \tilde{a}$ and $c \otimes \tilde{c} \in \mathcal{C}(A, H)$. Now with a second $G \in \mathrm{Alg}^H(A)$, for any $a \otimes \tilde{a} \in \mathcal{C}(A, H)$, one finds

$$\phi_*^{F \cdot G}(a \otimes \tilde{a}) = F \cdot G(a)\tilde{a} = G(F(a))\tilde{a} = \phi_*^G(F(a) \otimes \tilde{a}) = \phi_*^F\phi_*^G(a \otimes \tilde{a})$$

and finally $\phi_*^{id_A} = \underline{\epsilon}$. All of this shows that $F \to \phi_*^F$ is a group morphism with $(\phi_*^F)^{-1} = \phi_*^{F^{-1}}$.

Conversely, if $\phi_* \in \mathcal{T}_*$ is a twist the expression (2.2-19) is well-defined due to the right B-linearity of ϕ_* and $a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle} \otimes a_{(1)}{}^{\langle 2\rangle} \in \mathcal{C}(A, H) \otimes A$. One can easily check that F_{ϕ_*} is unital and from (2.2-16) for any $a, c \in A$ one has

$$\rho \circ F_{\phi_*}(a) = \phi_*(a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle})a_{(1)}{}^{\langle 1\rangle} \otimes a_{(2)} = (F_{\phi_*} \otimes \mathrm{id}_H) \circ \rho(a)$$

$$\begin{aligned} F_{\phi_*}(ac) &= \phi_*\left(a_{(0)}c_{(0)} \otimes c_{(1)}{}^{\langle 1\rangle}a_{(1)}{}^{\langle 1\rangle}\right)a_{(1)}{}^{\langle 2\rangle}c_{(1)}{}^{\langle 2\rangle} \\ &= \phi_*\left((a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle})(c_{(0)} \otimes c_{(1)}{}^{\langle 1\rangle})\right)a_{(1)}{}^{\langle 2\rangle}c_{(1)}{}^{\langle 2\rangle} \\ &= \phi_*(a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle})a_{(1)}{}^{\langle 2\rangle}\phi_*(c_{(0)} \otimes c_{(1)}{}^{\langle 1\rangle})c_{(1)}{}^{\langle 2\rangle} \\ &= F_{\phi_*}(a)F_{\phi_*}(c). \end{aligned}$$

Thus the map F_{ϕ_*} is a right H-comodule algebra morphism. To prove these equations, we used properties of the translation map in Proposition 1.2.3. Furthermore, it is easy to check that $\phi_* \to F_{\phi_*}$ is a group morphism

$$\begin{aligned} F_{\phi_*\psi_*}(a) &= \phi_*\psi_*(a_{(0)} \otimes {a_{(1)}}^{\langle 1\rangle})\, {a_{(1)}}^{\langle 2\rangle} \\ &= \psi_*\left(\phi_*(a_{(0)} \otimes {a_{(1)}}^{\langle 1\rangle}){a_{(1)}}^{\langle 2\rangle} \otimes {a_{(2)}}^{\langle 1\rangle}\right) {a_{(2)}}^{\langle 2\rangle} \\ &= F_{\psi_*}(\phi_*(a_{(0)} \otimes {a_{(1)}}^{\langle 1\rangle}){a_{(1)}}^{\langle 2\rangle}) \\ &= F_{\psi_*}(F_{\phi_*}(a)) = F_{\phi_*} \cdot F_{\psi_*}(a) \end{aligned}$$

for any $a \in A$ and $F_{\underline{\epsilon}} = \mathrm{id}_A$, thus $(F_{\phi_*})^{-1} = F_{\phi_*^{-1}}$. Finally one has

$$\phi_*^{F_{\phi_*}} = \phi_* \qquad F_{\phi_*^F} = F,$$

thus finishing the proof. ■

Remark 2.2.8 The equations (2.2-18) and (2.2-19) realizing the isomorphism between $\mathcal{T}_*$ and $\mathrm{Alg}^H(A)$ are the same that give the isomorphism between the group of bisections of $\mathcal{B}(\mathcal{C}(A,H))$ and unital H-comodule algebra maps that preserve the base B (vertical gauge transformations) ${}_B\mathrm{Alg}^H(A)$ as proved in [24] in the context of Hopf-Galois extensions and [10] in the context of quantum groups. The invertibility of a H-comodule algebra map which is not vertical needs to be assumed in general. ♦

2.3 The $U(1)$-extension $A(\mathbb{C}P_q^{n-1}) \subseteq A(S_q^{2n-1})$

In this final section we study a special case of Hopf–Galois extension and associated Ehresmann–Schauenburg bialgebroid, the (non-commutative) principal $U(1)$-bundle $S_q^{2n-1} \to \mathbb{C}P_q^{n-1}$ over quantum projective spaces.

2.3.1 Hopf–Galois structure

Let $q \in (0,1)$ and consider the $*$-algebra generated by elements $\{z_i, z_i^*\}_{i=1,\dots,n}$ subjected to relations:

$$z_i z_j = q z_j z_i \quad \forall i < j, \quad z_i^* z_j = q z_j z_i^* \qquad \forall i \neq j \tag{2.3-1}$$

$$[z_1^*, z_1] = 0, \quad [z_k^*, z_k] = (1-q^2)\sum_{j=1}^{k-1} z_j z_j^* \qquad \forall 1 < k \leq n \tag{2.3-2}$$

$$\sum_{j=1}^{n} z_j z_j^* = 1, \tag{2.3-3}$$

with $[\cdot,\cdot]$ denoting the commutator. This algebra is called the *quantum* $(2n-1)$-*dimensional sphere* denoted $A(S_q^{2n-1})$. The equations (2.3-2) can be used to rewrite the sphere relation (2.3-3) as

$$\sum_{j=1}^{n} q^{2(n-j)} z_j^* z_j = 1\,. \tag{2.3-4}$$

The entries of the projection $P_{ij} := z_i^* z_j$ form a $*$-algebra which is the q-deformation of the coordinate algebra of the complex projective space that is denoted by $A(\mathbb{C}P_q^{n-1})$. The commutation relations among the generators p_{ij} come from those in (2.3-1)-(2.3-3).

Let us denote by $O(U(1)) = \mathbb{C}[t, t^{-1}]$ (Laurent polynomials) the Hopf $*$-algebra generated by t and its inverse with involution $t^* = t^{-1}$ and with comultiplication, counit and antipode given by

$$\Delta(t^{\pm}) = t^{\pm} \otimes t^{\pm}, \quad \epsilon(t^{\pm}) = 1, \quad S(t^{\pm}) = t^{\mp} \tag{2.3-5}$$

The algebra $A(S_q^{2n-1})$ is a right $\mathbb{C}[t, t^{-1}]$-comodule $*$-algebra if endowed with

$$\rho : A(S_q^{2n-1}) \longrightarrow A(S_q^{2n-1}) \otimes \mathbb{C}[t, t^{-1}], \quad z_i \longmapsto z_i \otimes t, \quad z_i^* \longmapsto z_i^* \otimes t^{-1} \tag{2.3-6}$$

One checks that the subalgebra of coaction invariants in $A(S_q^{2n-1})$ is $A(\mathbb{C}P_q^{n-1})$.

One has that $A(\mathbb{C}P_q^{n-1}) \subseteq A(S_q^{2n-1})$ is a faithfully flat $\mathbb{C}[t, t^{-1}]$-Hopf–Galois extension. This is shown by observing that the canonical map:

$$\text{can} : A(S_q^{2n-1}) \otimes_{A(\mathbb{C}P_q^{n-1})} A(S_q^{2n-1}) \longrightarrow A(S_q^{2n-1}) \otimes \mathbb{C}[t, t^{-1}]$$

is surjective. One has the translation map:

$$\tau : \mathbb{C}[t, t^{-1}] \longrightarrow A(S_q^{2n-1}) \otimes_{A(\mathbb{C}P_q^{n-1})} A(S_q^{2n-1})$$

$$\tau(t) = \sum_{j=1}^{n} q^{2(n-j)} z_j^* \otimes_{A(\mathbb{C}P_q^{n-1})} z_j, \quad \tau(t^{-1}) = \sum_{j=1}^{n} z_j \otimes_{A(\mathbb{C}P_q^{n-1})} z_j^* \tag{2.3-7}$$

for which one verifies, using (2.3-3) and (2.3-4) that $\text{can}(\tau(t^{\pm})) = 1 \otimes t^{\pm}$ (which is indeed sufficient for surjectivity). Moreover, the Hopf algebra $\mathbb{C}[t, t^{-1}]$ is cosemisimple and has a bijective antipode, then from Theorem 1.2.5 and the following remark 1.2.6 one has that $A(\mathbb{C}P_q^{n-1}) \subseteq A(S_q^{2n-1})$ is a principal $\mathbb{C}[t, t^{-1}]$-comodule algebra.

2.3.2 K-theory of the base and the bialgebroid

Equation (2.3-7) can be written more compactly using the elements in the free module $A(S_q^{2n-1})^n \simeq A(S_q^{2n-1}) \otimes \mathbb{C}^n$,

$$v = \begin{pmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_n^* \end{pmatrix}, \quad w = \begin{pmatrix} q^{(n-1)} z_1 \\ q^{(n-2)} z_2 \\ \vdots \\ z_n \end{pmatrix} \tag{2.3-8}$$

which satisfy

$$\tau(t) = w^{\dagger} \dot{\otimes}_{A(\mathbb{C}P_q^{n-1})} w, \quad \tau(t^{-1}) = v^{\dagger} \dot{\otimes}_{A(\mathbb{C}P_q^{n-1})} v\,. \tag{2.3-9}$$

Here the symbol $\dot{\otimes}$ stands for the matrix multiplication composed with the tensor product. From (2.3-3) and (2.3-4) both v and w are partial isometries of norm 1 with respect to the natural $A(S_q^{2n-1})$ hermitian product in $A(S_q^{2n-1})^n$:

$$(\xi, \eta) := \sum_{j=1}^{n} \xi_j^* \eta_j$$

for $\xi = (\xi_j)$ and $\eta = (\eta_j) \in A(S_q^{2n-1})^n$. As a consequence the two $n \times n$ matrices $P = vv^\dagger$ and $Q = ww^\dagger$ are projections with entries in the algebra $A(\mathbb{C}P_q^{n-1})$. They define two inequivalent classes [21, 15] in the K-theory $K_0(A(\mathbb{C}P_q^{n-1}))$.

2.3.3 Full Hopf algebroid structure

Referring to the above section, set $A = A(S_q^{2n-1})$, $B = A(\mathbb{C}P_q^{n-1})$ and $H = \mathbb{C}[t, t^{-1}]$. The two column vectors (2.3-8) give the generators of $\mathcal{C}(A, H)$ in the form of matrices $V := v\dot{\otimes}v^\dagger$ and $W := w\dot{\otimes}w^\dagger$ whose entries are

$$V_{ij} = z_i^* \otimes z_j, \qquad W_{ij} = q^{(2n-i-j)} z_i \otimes z_j^* . \tag{2.3-10}$$

It is easy to check that they are coaction invariant in $A \otimes A$.

Any coaction invariant element in $A \otimes A$ can be written as a combination of V_{ij}'s and W_{ij}'s. This result follows from a simple argument. In our convention (2.3-6), z_i^* are of weight -1 and z_i of weight 1, then any element of weight 0 in $A \otimes A$ is a combination of elements of the form $z_i^* \otimes z_j$ and $z_i \otimes z_j^*$. The latter are proportional to V_{ij} and W_{ij}.

The commutations relations of the components of the matrices V and W can be derived from eqs.(2.3-1) and (2.3-2). For the first we have

$$V_{ik}V_{jk} = q^{-1}V_{jk}V_{ik}, \quad V_{ki}V_{kj} = q^{-1}V_{kj}V_{ki}, \quad \forall i < j \tag{2.3-11}$$

$$V_{ik}V_{jl} = V_{jl}V_{ik}, \quad V_{il}V_{jk} = q^{-2}V_{jk}V_{il}, \quad \forall i < j, l < k \tag{2.3-12}$$

while for the entries of W one finds

$$W_{ik}W_{jk} = qW_{jk}W_{ik}, \quad W_{ki}W_{kj} = qW_{kj}W_{ki}, \quad \forall i < j \tag{2.3-13}$$

$$W_{ik}W_{jl} = W_{jl}W_{ik}, \quad W_{il}W_{jk} = q^2W_{jk}W_{il}, \quad \forall i < j, l < k\,. \tag{2.3-14}$$

Following (2.2-4) and (2.2-5) we have that the source and target maps are given by

$$s : P_{ij} \longmapsto \sum_{k=1}^{n} q^{(j-k)} V_{ik}W_{jk}, \quad Q_{ij} \longmapsto \sum_{k=1}^{n} q^{(k-j)} W_{ik}V_{jk} \tag{2.3-15}$$

$$t : P_{ij} \longmapsto \sum_{k=1}^{n} q^{(i-k)} V_{kj}W_{ki}, \quad Q_{ij} \longmapsto \sum_{k=1}^{n} q^{(k-i)} W_{kj}V_{ki}\,. \tag{2.3-16}$$

From these we see that the two embeddings of B into $\mathcal{C}(A, H)$ are given by combination of V_{ij} and W_{ij}.

For the comultiplication and counit we find a similar formula as before

$$\underline{\Delta} : V_{ij} \longmapsto \sum_{k=1}^{n} V_{ik} \otimes_B V_{kj}, \quad W_{ij} \longmapsto \sum_{k=1}^{n} W_{ik} \otimes_B W_{kj} \tag{2.3-17}$$

$$\underline{\epsilon} : V_{ij} \longmapsto P_{ij}, \quad W_{ij} \longmapsto Q_{ij}\,. \tag{2.3-18}$$

Having set the notation for the bialgebroid we are ready for the next result.

Proposition 2.3.1 *Define on the generators of $\mathcal{C}(A, H)$ the map*

$$\underline{S} : V_{ij} \longmapsto q^{(j-i)} W_{ji}, \quad W_{ij} \longmapsto q^{(i-j)} V_{ji}\,. \tag{2.3-19}$$

This is an antipode of $\mathcal{C}(A, H)$ with inverse given by

$$\underline{S}^{-1} : V_{ij} \longmapsto q^{(i-j)} W_{ji}, \quad W_{ij} \longmapsto q^{(j-i)} V_{ji} \tag{2.3-20}$$

Proof We now check all the properties listed before. First we have that $\underline{S}$ is an anti-algebra morphism; for any $i < j$ one computes

$$\begin{aligned}\underline{S}(V_{jk})\underline{S}(V_{ik}) &= q^{(k-j)}q^{(k-i)}W_{kj}W_{ki} \\ &= q^{-1}(q^{(j-k)}W_{jk})(q^{(i-k)}W_{ik}) \\ &= q^{-1}\underline{S}(V_{ik})\underline{S}(V_{jk})\end{aligned}$$

when we used (2.3-11). The same goes for the other relations.

Looking at (2.3-15) and the definition of $\underline{S}$ in this case, for (1.3-14) one has

$$\begin{aligned}\underline{S}(t(P_{ij})) &= \sum_{k=1}^{n} q^{(i-k)}\underline{S}(W_{ki})\underline{S}(V_{kj}) \\ &= \sum_{k}^{n} q^{(i-k)}q^{(k-i)}V_{ik}q^{(j-k)}W_{jk} \\ &= \sum_{k=1}^{n} q^{(j-k)}V_{ik}W_{jk} = s(P_{ij})\end{aligned}$$

and with similar computations one finds the same for Q_{ij}.

For the last property, (1.3-15), firstly we have

$$\underline{\Delta}(V_{ij}) = \sum_{k=1}^{n} V_{ik} \otimes_B V_{kj}, \quad \underline{\Delta}(\underline{S}(V_{ik})) = q^{(k-i)}\sum_{l=1}^{n} W_{kl} \otimes_B W_{li}.$$

Then (1.3-15) becomes

$$\begin{aligned}\sum_{k,l=1}^{n} q^{(k-i)}W_{kl}V_{kj} \otimes_B W_{li} &= \sum_{k=1}^{n}\left(\sum_{l=1}^{n} q^{(k-j)}W_{kl}V_{kj}\right) \otimes_B q^{(j-i)}W_{li} \\ &= \sum_{l=1}^{n} t(Q_{jl}) \otimes_B q^{(j-i)}W_{li} \\ &= 1_{\mathcal{C}(A,H)} \otimes_B q^{(j-i)}\sum_{l=1}^{n} s(Q_{jl})W_{li} \\ &= 1_{\mathcal{C}(A,H)} \otimes_B q^{(j-i)}W_{ji} = 1_{\mathcal{C}(A,H)} \otimes_B \underline{S}(V_{ij})\end{aligned}$$

where in second line we used the tensor product over B in (1.3-2), while in the third one the relation

$$\sum_{l=1}^{n} s(Q_{jl})W_{li} = W_{ji}$$

this equation is (1.3-11). The same goes for the generators W_{ij}. We prove (1.3-16) now taking the W_{ij}. Firstly,

$$\underline{\Delta}(W_{ij}) = \sum_{k=1}^{n} W_{ik} \otimes_B W_{kj}, \quad \underline{\Delta}(\underline{S}^{-1}(W_{kj})) = q^{(j-k)}\sum_{l=1}^{n} V_{jl} \otimes_B V_{lk}.$$

Thus, one gets

$$\begin{aligned}\sum_{k,l=1}^{n} q^{(j-k)} V_{jl} \otimes_B V_{lk} W_{ik} &= q^{(j-i)} \sum_{l=1}^{n} V_{jl} \otimes_B \sum_{k=1}^{n} q^{(i-k)} V_{lk} W_{ik} \\ &= q^{(j-i)} \sum_{l=1}^{n} V_{jl} \otimes_B s(P_{li}) \\ &= q^{(j-i)} \sum_{l=1}^{n} t(P_{li}) V_{jl} \otimes_B 1_{\mathcal{C}(A,H)} \\ &= q^{(j-i)} V_{ji} \otimes_B 1_{\mathcal{C}(A,H)} = \underline{S}^{-1}(W_{ij}) \otimes_B 1_{\mathcal{C}(A,H)}.\end{aligned}$$

Between the second and third line, we used again the identification (1.3-2) and

$$\sum_{l=1}^{n} t(P_{li}) V_{jl} = V_{ji},$$

is again (1.3-11). Similar computations follow if one considers the generators V_{ij}.

Being V_{ij}, W_{ij} generators of $\mathcal{C}(A, H)$ we can conclude that the map $\underline{S}$ of (2.3-19) is the invertible antipode of the Ehresmann–Schauenburg bialgebroid associated to the $\mathbb{C}[t, t^{-1}]$-Hopf–Galois extension $A(\mathbb{C}P_q^{n-1}) \subseteq A(S_q^{2n-1})$, bialgebroid that is indeed a Hopf algebroid. ■

We are now in the situation of having two antipodes on the bialgebroid associated with the extension $A(\mathbb{C}P_q^{n-1}) \subseteq A(S_q^{2n-1})$. Indeed, being the Hopf algebra $\mathbb{C}[t, t^{-1}]$ commutative, we saw in Proposition 2.2.6 and 2.2.1 that the map $\mathrm{flip}_{A(S_q^{2n-1})}$ is an antipode, its action on the generators is given by

$$\mathrm{flip}_{A(S_q^{2n-1})}(V_{ij}) = q^{(i+j-2n)} W_{ji}, \quad \mathrm{flip}_{A(S_q^{2n-1})}(W_{ij}) = q^{(2n-i-j)} V_{ji}.$$

We have proved also that the map (2.3-19) is. Using the theory developed in subsection 2.2, we find that the two are related by the twist

$$\psi_*(V_{ij}) = q^{2(i-n)} P_{ij}, \quad \psi_*(W_{ij}) = q^{2(n-j)} Q_{ij}, \tag{2.3-21}$$

and thus by 2.1.4 we get

$$\mathrm{flip}_{A(S_q^{2n-1})}(\cdot) = \underline{S}(\cdot \triangleleft \psi_*). \tag{2.3-22}$$

We retrieve the whole group of twists from Proposition 2.2.7 For the $U(1)$-Hopf–Galois extension $A(\mathbb{C}P_q^{n-1}) \subseteq A(S_q^{2n-1})$ one has $\mathcal{T}_* \simeq (\mathbb{C}^*)^n$.

Proof We first find the group $\mathrm{Alg}^{O(U(1))}(A(S_q^{2n-1}))$ and then use the isomorphism of Proposition 2.2.7 as mentioned before. One can check easily that any right H-comodule endomorphism of A has the form

$$F(z_i) = X_i z_i, \qquad F(z_i^*) = Y_i z_i^*,$$

where $X_i, Y_i \in A(\mathbb{C}P_q^{n-1})$. The invertible ones are such that $X_i, Y_i \in \mathbb{C}^*$ (non-zero complex numbers) since they are all the invertible elements in $A(\mathbb{C}P_q^{n.1})$.

Moreover, we require $F(1_A) = 1_A$. The above expression is an algebra morphism if $Y_i = X_i^{-1}$ (so that it preserves the sphere relations). Using (2.2-18) the action of twists on the generators of $\mathcal{C}(A(S_q^{2n-1}), O(U(1)))$ is

$$\phi_*^F(V_{ij}) = X_i^{-1}P_{ij} \qquad \phi_*^F(W_{ij}) = X_iQ_{ij}. \tag{2.3-23}$$

This finishes the proof. ■

Chapter 3

Chern-Weil theory for deformed Hopf-Galois extensions

In this chapter we first review the Chern-Weil theory developed in [23] for coalgebras extensions.

The classical Chern-Weil map [28] allows one to find characteristic classes of principal and vector bundles. Characteristic classes are global invariants of the base manifold M of the bundle and allow one to prove the non-triviality of the bundle: if some class is non-trivial, then also the bundle is. The Chern-Weil map computes characteristic classes in terms of curvature form of connections in the de Rham cohomology of the base manifold $H^*_{dR}(M)$. Given a principal G-bundle on M (G is a Lie group), the curvature form of a connection ∇ is a Lie algebra-valued $2n$-form Ω_∇ where n is the dimension of M. If we denote by $\mathbb{K}[\mathfrak{g}]^G$ ring of adjoint-invariant $\mathbb{K}$-valued polynomials of the Lie algebra $\mathfrak{g}$, the Chern-Weil map is given by

$$\mathrm{Chw} : \mathbb{K}[\mathfrak{g}]^G \longrightarrow H^{2n}_{dR}(M), \quad f \longmapsto f(\Omega_\nabla).$$

Examples of classes obtained in this way are the Chern classes, the Chern character, and the Pontrjagin classes.

In noncommutative geometry, and more specifically in Hopf-Galois theory, the Chern character was first found [7]. While the general theory has been recently developed in [23]. In this setting, the domain of the Chern-Weil map is the spaces of *cotraces* of the coalgebra and is valued in the cyclic homology of the base algebra.

3.1 The general construction

In this section, we recall some definitions of the cyclic homology of algebras that we need to define the Chern-Weil map properly, Our main reference is [31]. This homology theory is the noncommutative version of the de Rham cohomology theory.

3.1.1 Hochschild and cyclic homology

In its full generality, Hochschild homology for any bimodule M over any algebra B. For us, the case $M = B$ is enough. The motivation for this homology theory lies in the Hochschild-Kostant-Rosenberg theorem [18] that characterizes Hochschild homology for commutative algebras.

Given an algebra B, define the complex $C_n(B) := B^{\otimes(n+1)}$. Moreover, introduce the face operators

Lemma 3.1.1 *Given the maps $d_i : C_n(B) \longrightarrow C_{n-1}(B)$*

$$\begin{aligned} d_0(b_0 \otimes \cdots \otimes b_n) &:= b_0 b_1 \otimes \cdots \otimes b_n, \\ d_i(b_0 \otimes \cdots \otimes b_n) &:= b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n, \quad 1 \le i \le n \\ d_n(b_0 \otimes \cdots \otimes b_n) &:= b_n b_0 \otimes \cdots \otimes b_{n-1}. \end{aligned}$$

One has that $d := \sum_{i=0}^{n} (-1)^i d_i$ satisfies $d \circ d = 0$. The same holds for the truncated operator $d' := \sum_{i=0}^{n-1} (-1)^i d_i$.

Having this, we can give the following

Definition 3.1.2 For any algebra B the **Hochschild complex** is the datum of $(C_*(B), d)$, as above for which one has the sequence

$$\ldots \xrightarrow{d} C_n(B) \xrightarrow{d} C_{n-1}(B) \xrightarrow{d} C_{n-2}(B) \xrightarrow{d} \ldots \xrightarrow{d} C_0(B)$$

The $n-th$ **Hochschild homology group** of B is the quotient

$$HH_n(B) := \frac{\mathrm{Ker}(d : C_n(B) \longrightarrow C_{n-1}(B))}{\mathrm{Im}(d : C_{n+1}(B) \longrightarrow C_n(B))}$$

We denote by $HH_*(B) = \oplus_{n\ge 0} HH_n(B)$. ♦

There is a natural action of the cyclic group $\mathbb{Z}_{n+1}$ on $C_n(B)$ given by

$$t_n(b_0 \otimes \cdots \otimes b_n) = (-1)^n (b_n \otimes b_0 \cdots \otimes b_{n-1}). \tag{3.1-1}$$

for $t_n \in \mathbb{Z}_{n+1}$ the generator. It is straightforward to check that $t^{n+1} = \mathrm{id}$. The $\mathbb{Z}_{n+1}$-invariant elements in $C_n(B)$ are called *cyclic tensor*, and we denote them by $C_n^\delta(B) := C_n(B)/\mathrm{Ker}(\delta)$ where $\delta := \mathrm{id} - t_n$. Thanks to this result

Lemma 3.1.3 *Let $N := 1 + t + \cdots + t^n : C_n(B) \longrightarrow C_n(B)$ be the norm operator, then the following equations hold*

$$(\mathrm{id} - t) d' = d(\mathrm{id} - t), \quad d' N = N d,$$

where d and d' are the operators in 3.1.1.

we can define a new complex

Definition 3.1.4 The **Connes complex** is the datum of $(C_*^\delta(B), d)$ for which one has the sequence

$$\ldots \xrightarrow{d} C_n^\delta(B) \xrightarrow{d} C_{n-1}^\delta(B) \xrightarrow{d} C_{n-2}^\delta(B) \xrightarrow{d} \ldots \xrightarrow{d} C_0^\delta(B)$$

The $n-th$ **cyclic homology group** of B is the quotient

$$HC_n(B) := \frac{\mathrm{Ker}(d : C_n^\delta(B) \longrightarrow C_{n-1}^\delta(B))}{\mathrm{Im}(d : C_{n+1}^\delta(B) \longrightarrow C_n^\delta(B))}$$ ♦

For commutative algebras, one finds that the cyclic homology is expressed by the de Rham cohomology of the space associated with the algebras [11].

Hochschild and cyclic homology are related via a long exact sequence discovered by Connes who introduced the so-called *periodicity operator* $P : HC_n(B) \longrightarrow HC_{n-2}(B)$, To write down explicitly P, we need to introduce some tools. First of all, if $x \in C_n(B)$ we denote by $\bar{x}$ the corresponding element in $C_n^\delta(B)$. Moreover, we define the map

$$d^{[2]} := \sum_{i,j=0}^{n} (-1)^{i+j} d_i d_j : C_n(B) \longrightarrow C_{n-2}(B).$$

At this point, the periodicity operator has the following expression

$$P([\bar{x}]) := -\frac{1}{n(n-1)}[\overline{d^{[2]}(x)}], \tag{3.1-2}$$

where $[\bar{x}] \in HC_n(B)$ denotes the homology class of $\bar{x}$.

3.1.2 The space of cotraces and the Chern-Weil map

Given a Hopf algebra H we have the following vector space

$$H^{tr} := \{h \in H | h_{(1)} \otimes h_{(2)} = h_{(2)} \otimes h_{(1)}\} \tag{3.1-3}$$

that we call the space of *cotraces*. The name is justified by the fact that any element of H^{tr} defines a trace in the space $\mathrm{Hom}(H, \mathbb{K})$, in fact for $h \in H^{tr}$ one has

$$\tau_h : \mathrm{Hom}(H, \mathbb{K}) \longrightarrow \mathbb{K}, \quad f \longmapsto f(h) \tag{3.1-4}$$

for which one finds that $\tau_h(f * g) = \tau_h(g * f)$ for all $f, g \in \mathrm{Hom}(H, \mathbb{K})$. For the Hopf algebra $O(G)$ of example 1.1.10 we have that

$$\begin{aligned} O(G)^{tr} &= \{f \in O(G) | f(gh) = f(hg), \quad \forall g, h \in G\} \\ &= \{f \in O(G) | f(hgh^{-1}) = f(g), \forall g, h \in G\} \\ &= O(\mathrm{Ad}(G))^G, \end{aligned}$$

which is the algebra of adjoint invariant elements. Via a filtration using the ideal $\ker(\epsilon)$ (as shown in [23]), this algebra gives the algebra $\mathbb{K}[\mathfrak{g}]^G$ of adjoint-invariant polynomials on the Lie algebra $\mathfrak{g}$, which we saw being the domain of the classical Chern-Weil map.

In general, the space of cotraces is characterized by the following property

Lemma 3.1.5 ([23]) *For any $n \in \mathbb{N}$ the multiplication of H induces a linear isomorphism*

$$H^{tr} \simeq H \,\square^{H \otimes H^{op}} (\underbrace{H \,\square^H \cdots \square^H H}_{n+1})$$

Applying the counit to the leftmost factor of $H \,\square^{H \otimes H^{op}} (\underbrace{H \,\square^H \cdots \square^H H}_{n+1})$ we have a circular cotensor product.

Consider now a principal H-comodule algebra $B \subseteq A$ and the space $A \,\square^H A$ which is identified with $(A \otimes A)^{coH}$ via Lemma 2.2.1. For the rest of the section, we use the short notation $M := A \,\square^H\, A$. Recall that the latter has a B-coring structure with counit given by equation (2.2-2). Because this map is left B-linear we can define a multiplication in the following way

$$m \cdot m' := \underline{\epsilon}(m)m' \tag{3.1-5}$$

Remark 3.1.6 We saw in the previous chapter that M is a subalgebra of $A \otimes sA^{op}$. In general, this algebra structure and the one given by (3.1-5) are different. ♦

With this algebra structure, we consider the cyclic homology $HC_*(M)$ and we collect a series of lemmas in the following

Proposition 3.1.7 ([23]) *A strong connection $l : H \longrightarrow A \otimes A$ induces a map*

$$c_n(l) : H^{tr} \longrightarrow M^{\otimes(n+1)}, \quad h \longmapsto l(h_{(1)}) \otimes \cdots \otimes l(h_{(n+1)})$$

for any $n \in \mathbb{N}$, where $M^{\otimes(n+1)}$ is thought as a circular tensor product. Moreover this element

$$c_n(l)(h) := \left(h_{(n+1)}{}^{\langle 2\rangle} \otimes h_{(1)}{}^{\langle 1\rangle}\right) \otimes \cdots \otimes \left(h_{(n)}{}^{\langle 2\rangle} \otimes h_{(n+1)}{}^{\langle 1\rangle}\right)$$

is cyclic-symmetric in $M^{\otimes(n+1)}$ and for any face operator d_i with $i = 0, \ldots, n$ one has

$$d_i c_n(l)(h) = c_{n-1}(l)(h), \quad \forall h \in H^{tr}$$

This result allows one allows us to define a $2n$-cycle in the cyclic homology of $\mathrm{HC}_{2n}(M)$, which form is given by

$$\widetilde{\mathrm{chw}}_n(l)(h) := \sum_{i=0}^{2n} (-1)^{\lfloor \frac{i}{2} \rfloor} \frac{i!}{\lfloor \frac{i}{2} \rfloor !} c_i(l)(h), \quad h \in H^{tr}. \tag{3.1-6}$$

The fact that $\widetilde{\mathrm{chw}}_n(l)(h)$ defines a $2n$-cycle follows from the identities

$$d(2c_2n(l)(h)) = (1-t)c_{2n-1}(l)(h) \tag{3.1-7}$$

$$d'(nc_{2n-1}(l)(h)) = Nc_{2n-2}(l)(h) \tag{3.1-8}$$

for all $h \in H^{tr}$. Moreover, it is stable under the periodicity operator (3.1-2).

We can now use the counit (2.2-2) to induce a map from $M^{\otimes(n+1)}$ into $B^{\otimes(n+1)}$, this is done by applying it to every factor of the tensor product. For any $n \in \mathbb{N}$ one has the formula for any $h \in H^{tr}$

$$x_n(l,h) := \left(\underline{\epsilon}^{\otimes(n+1)} \circ c_n(l)\right)(h) = h_{(n+1)}{}^{\langle 2\rangle} h_{(1)}{}^{\langle 1\rangle} \otimes \cdots \otimes h_{(n)}{}^{\langle 2\rangle} h_{(n+1)}{}^{\langle 1\rangle} \in B^{\otimes(n+1)}. \tag{3.1-9}$$

Thus, we have the following

Definition 3.1.8 For any principal H-comodule algebra $B \subseteq A$ with strong connection l, the **Chern-Weil map** is given by

$$\mathrm{chw}_n(l) : H^{tr} \longrightarrow \mathrm{HC}_{2n}(B), \quad h \longmapsto \sum_{i=0}^{2n} (-1)^{\lfloor \frac{i}{2} \rfloor} \frac{i!}{\lfloor \frac{i}{2} \rfloor !} x_i(l,h)$$ ♦

3.2 Chern-Weil map for deformed extensions

We now study how the Chern-Weil map behaves under a 2-cocycle deformation of a principal comodule algebra. First, we deform the structure Hopf algebra H and the algebra A, then we consider extensions with an external symmetry K and deform both A and B.

3.2.1 Deformation of the structure Hopf algebra

Let now $B \subseteq A$ be a principal H-comodule algebra and $\gamma : H \otimes H \longrightarrow \mathbb{K}$ an invertible 2-cocycle of H. Since the deformation H_γ involves only the multiplication and not the multiplication, we have

$$H_\gamma^{tr} \simeq H^{tr} \tag{3.2-1}$$

Moreover, we have the following result

Lemma 3.2.1 *The spaces $M = A \,\square^H\, A$ and $M_\gamma = A_\gamma \,\square^{H_\gamma}\, A_\gamma$ are isomorphic as vector space.*

Proof From lemma 2.2.1 we have that $M \simeq (A \otimes A)^{coH}$ and $M_\gamma \simeq (A_\gamma \otimes A_\gamma)^{coH_\gamma}$. The statement follows from the isomorphism of Theorem 1.1.22. ■

Despite having the same underlying linear structure M and M_γ have different algebra structures. The multiplication on M_γ is given by the formula

$$m \cdot_\gamma m' := \underline{\epsilon}_\gamma(m)m' \tag{3.2-2}$$

Explicitly, given $a \otimes^\gamma \widetilde{a}, a' \otimes^\gamma \widetilde{a}' \in M_\gamma$ one has

$$(a \otimes^\gamma \widetilde{a}) \cdot_\gamma (a' \otimes^\gamma \widetilde{a}') = (a \cdot_\gamma \widetilde{a}) \cdot_\gamma a' \otimes^\gamma \widetilde{a}'$$

where now $\cdot_\gamma$ is the multiplication (1.1-11).

We saw in the first chapter that the subalgebra B does not change under the deformation via γ, i.e. $B_\gamma = B$. Thus, the cyclic homology is the same $HC_*(B_\gamma) = HC_*(B)$ and we have the diagram

$$\begin{array}{ccc} H_\gamma & \xrightarrow{\mathrm{chw}_n(l_\gamma)} & HC_{2n}(B_\gamma) \\ \simeq\downarrow & & \downarrow\simeq \\ H & \xrightarrow{\mathrm{chw}_n(l)} & HC_{2n}(B), \end{array}$$

so that $\mathrm{chw}_n(l_\gamma) = \mathrm{chw}_n(l)$, where l_γ is the strong connection of 1.2.13.

We now prove this equivalence explicitly by showing how there are cancellations of the 2-cocycle in the formulas.

Proposition 3.2.2 *For any $n \in \mathbb{N}$ the map*

$$c_n(l_\gamma) : H_\gamma^{tr} \longrightarrow M_\gamma^{\otimes(n+1)}$$

$$h \longmapsto {h_{(2n+2)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2)}}^{\langle 1\rangle} \otimes^\gamma \cdots \otimes^\gamma {h_{(2n)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2n+2)}}^{\langle 1\rangle} u_\gamma(h_{(1)}) u_\gamma(h_{(3)}) \ldots u_\gamma(h_{(2n+1)})$$

is well-defined and its image lies in the cyclic-symmetric part of $M_\gamma^{\otimes(n+1)}$. Moreover for any face operator d_i with $i = 0, \ldots, n$ one has

$$d_i c_n(l_\gamma) = c_{n-1}(l_\gamma)$$

Proof We are looking at $M_\gamma^{\otimes(n+1)}$ as a circular tensor product, so that for $n=1$ we have that

$$c_1(l_\gamma)(h) = {h_{(4)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2)}}^{\langle 1\rangle} \otimes^\gamma {h_{(2)}}^{\langle 2\rangle} \otimes^\gamma {h_{(4)}}^{\langle 1\rangle} u_\gamma(h_{(1)})u_\gamma(h_{(3)})$$

To check that this element lies in $M_\gamma^{\otimes 2}$ we apply first $\rho^{\otimes^\gamma} \otimes \mathrm{id}$ and $\mathrm{id} \otimes \rho^{\otimes^\gamma}$

$$\begin{aligned}
&(\rho^{\otimes^\gamma} \otimes^\gamma \mathrm{id})(c_1(l_\gamma)(h)) = \\
&= {{h_{(4)}}^{\langle 2\rangle}}_{(0)} \otimes^\gamma {{h_{(2)}}^{\langle 1\rangle}}_{(0)} \otimes {{h_{(4)}}^{\langle 2\rangle}}_{(1)} \cdot_\gamma {{h_{(2)}}^{\langle 1\rangle}}_{(1)} \otimes {h_{(2)}}^{\langle 2\rangle} \otimes^\gamma {h_{(4)}}^{\langle 1\rangle} u_\gamma(h_{(1)})u_\gamma(h_{(3)}) \\
&= {h_{(5)}}^{\langle 2\rangle} \otimes^\gamma {h_{(3)}}^{\langle 1\rangle} \otimes h_{(6)} \cdot_\gamma S(h_{(2)}) \otimes {h_{(3)}}^{\langle 2\rangle} \otimes^\gamma {h_{(5)}}^{\langle 1\rangle} u_\gamma(h_{(1)})u_\gamma(h_{(4)}) \\
&= {h_{(5)}}^{\langle 2\rangle} \otimes^\gamma {h_{(3)}}^{\langle 1\rangle} \otimes h_{(6)} \cdot_\gamma S_\gamma(h_{(3)}) \otimes {h_{(3)}}^{\langle 2\rangle} \otimes^\gamma {h_{(5)}}^{\langle 1\rangle} u_\gamma(h_{(1)})u_\gamma^{-1}(h_{(2)})u_\gamma(h_{(4)})u_\gamma(u_{(5)}) \\
&= {h_{(5)}}^{\langle 2\rangle} \otimes^\gamma {h_{(3)}}^{\langle 1\rangle} \otimes h_{(6)} \cdot_\gamma S_\gamma(h_{(1)}) \otimes {h_{(3)}}^{\langle 2\rangle} \otimes^\gamma {h_{(5)}}^{\langle 1\rangle} u_\gamma(h_{(2)})u_\gamma(h_{(4)}) \\
&= {h_{(4)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2)}}^{\langle 1\rangle} \otimes 1_{H_\gamma} \otimes {h_{(2)}}^{\langle 2\rangle} \otimes^\gamma {h_{(4)}}^{\langle 1\rangle} u_\gamma(h_{(1)})u_\gamma(h_{(3)})
\end{aligned}$$

where in the second equality we used the properties of the translation map, in the third one the definition of S_γ, then the fact that u_γ is convolution invertible, in the second last equality we used the identification $H_\gamma^{tr} \simeq H^{tr}$ and Lemma something for H^{tr} and finally the definition of the antipode.

$$\begin{aligned}
&(\mathrm{id} \otimes^\gamma \rho^{\otimes^\gamma})(c_1(l_\gamma)(h)) = \\
&= {h_{(4)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2)}}^{\langle 1\rangle} \otimes^\gamma {{h_{(2)}}^{\langle 2\rangle}}_{(0)} \otimes^\gamma {{h_{(4)}}^{\langle 1\rangle}}_{(0)} \otimes {{h_{(2)}}^{\langle 2\rangle}}_{(1)} \cdot_\gamma {{h_{(4)}}^{\langle 1\rangle}}_{(1)} u_\gamma(h_{(1)})u_\gamma(h_{(3)}) \\
&= {h_{(6)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2)}}^{\langle 1\rangle} \otimes^\gamma {h_{(2)}}^{\langle 2\rangle} \otimes^\gamma {h_{(6)}}^{\langle 1\rangle} \otimes h_{(3)} \cdot_\gamma S(h_{(5)})u_\gamma(h_{(1)})u_\gamma(h_{(4)}) \\
&= {h_{(8)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2)}}^{\langle 1\rangle} \otimes^\gamma {h_{(2)}}^{\langle 2\rangle} \otimes^\gamma {h_{(8)}}^{\langle 1\rangle} \otimes h_{(3)} \cdot_\gamma S_\gamma(h_{(6)})u_\gamma(h_{(1)})u_\gamma(h_{(4)})u_\gamma^{-1}(h_{(5)})u_\gamma(h_{(7)}) \\
&= {h_{(6)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2)}}^{\langle 1\rangle} \otimes^\gamma {h_{(2)}}^{\langle 2\rangle} \otimes^\gamma {h_{(6)}}^{\langle 1\rangle} \otimes h_{(3)} \cdot_\gamma S_\gamma(h_{(4)})u_\gamma(h_{(1)})u_\gamma(h_{(5)}) \\
&= {h_{(4)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2)}}^{\langle 1\rangle} \otimes^\gamma {h_{(2)}}^{\langle 2\rangle} \otimes^\gamma {h_{(4)}}^{\langle 1\rangle} \otimes 1_{H_\gamma} u_\gamma(h_{(1)})u_\gamma(h_{(3)})
\end{aligned}$$

where, in order, we once more used the properties of the translation map, the definition of S_γ, and the invertibility of u_γ.

For the second part of the statement, we just need to check that the equality holds for $i=0$, and then it follows from induction that it is true for any other $i>0$. For any $h \in H_\gamma$, one has

$$\begin{aligned}
&d_0(c_n(l_\gamma)(h)) = \\
&= ({h_{(2n+2)}}^{\langle 2\rangle} \cdot_\gamma {h_{(2)}}^{\langle 1\rangle}) \cdot_\gamma {h_{(2)}}^{\langle 2\rangle} \otimes^\gamma \cdots \otimes^\gamma {h_{(2n)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2n+2)}}^{\langle 1\rangle} u_\gamma(h_{(1)}) \dots u_\gamma(h_{(2n+1)}) \\
&= {h_{(2n)}}^{\langle 2\rangle} \cdot_\gamma ({h_{(2)}}^{\langle 1\rangle} \cdot_\gamma {h_{(2)}}^{\langle 2\rangle} u_\gamma(h_{(1)})) \otimes^\gamma \cdots \otimes^\gamma {h_{(2n)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2n+2)}}^{\langle 1\rangle} u_\gamma(h_{(3)}) \dots u_\gamma(h_{(2n+1)}) \\
&= {h_{(2n)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2)}}^{\langle 1\rangle} \otimes^\gamma \cdots \otimes^\gamma {h_{(2n-2)}}^{\langle 2\rangle} \otimes^\gamma {h_{(2n)}}^{\langle 1\rangle} u_\gamma(h_{(1)}) \dots u_\gamma(h_{(2n-1)}) = c_{n-1}(l_\gamma)(h)
\end{aligned}$$

we used the definition of the counit and 1.2.13. ■

This lemma allows us to define a map $\widetilde{\mathrm{chw_n}}(l_\gamma) : H_\gamma^{tr} \longrightarrow \mathrm{HC_{2n}}(M_\gamma)$ in the same way as the non-deformed case, equation 3.1-6. Applying the counit

$\underline{\epsilon}_{\gamma}^{\otimes(n+1)}$ after the map $c_n(l_\gamma)$ we have the element $(\underline{\epsilon}_{\gamma}^{\otimes^\gamma(n+1)} \circ c_n(l_\gamma))(h)$ with $h \in H_\gamma^{tr}$

$$\begin{aligned}
&{h_{(2n+2)}}^{\langle 2\rangle} \cdot_\gamma {h_{(2n+2)}}^{\langle 2\rangle} \otimes^\gamma \dots \otimes^\gamma {h_{(2n)}}^{\langle 2\rangle} \cdot_\gamma {h_{(2n+2)}}^{\langle 1\rangle} u_\gamma(h_{(1)}) \dots u_\gamma(h_{(2n+1)})\\
&= {h_{(5n+4)}}^{\langle 2\rangle} {h_{(4)}}^{\langle 1\rangle} \otimes^\gamma \dots \otimes^\gamma {h_{(5n-1)}}^{\langle 2\rangle} {h_{(5n+4)}}^{\langle 1\rangle}\\
&\gamma^{-1}(h_{(5n+5)}, S(h_{(3)})) \dots \gamma^{-1}(h_{(5n)}, S(h_{(5n+3)}))\gamma(h_{(1)}, S(h_{(2)})) \dots \gamma(h_{(5n+1)}, S(h_{(5n+2)}))\\
&= {h_{(n+4)}}^{\langle 2\rangle} {h_{(4)}}^{\langle 1\rangle} \otimes^\gamma \dots \otimes^\gamma {h_{(n+3)}}^{\langle 2\rangle} {h_{(n+4)}}^{\langle 1\rangle} \gamma^{-1}(h_{(n+5)}, S(h_{(3)}))\gamma(h_{(1)}, S(h_{(2)}))\\
&= {h_{(n+3)}}^{\langle 2\rangle} {h_{(3)}}^{\langle 1\rangle} \otimes^\gamma \dots \otimes^\gamma {h_{(n+2)}}^{\langle 2\rangle} {h_{(n+3)}}^{\langle 1\rangle} (\gamma^{-1} * \gamma)(h_{(1)}, S(h_{(2)}))\\
&= {h_{(n+1)}}^{\langle 2\rangle} {h_{(1)}}^{\langle 1\rangle} \otimes^\gamma \dots \otimes^\gamma {h_{(n)}}^{\langle 2\rangle} {h_{(n+1)}}^{\langle 1\rangle}
\end{aligned}$$

that belongs to the algebra $B^{\otimes^\gamma(n+1)}$ and when we put it in the formula of the Chern-Weil map we get

$$\mathrm{chw}_n(l) = \mathrm{chw}_n(l_\gamma) \tag{3.2-3}$$

In other words, the cyclic homology Chern-Weil map a is deformation invariant.

3.2.2 Deformation of an external symmetry

In this case, we consider a (K, H)-bicomodule algebra such that $B \subset A$ is a principal H-comodule algebra and we take a 2-cocycle of the external symmetry $\sigma : K \otimes K \longrightarrow \mathbb{K}$. We saw that both A and B are deformed into ${}_\sigma A$ and ${}_\sigma B$, while the structure Hopf algebra H remains the same.

Like in the preview case, we have the linear isomorphism $M = A \,\square^H\, A \simeq {}_\sigma M = {}_\sigma A \,\square^H\, {}_\sigma A$. The reason is the same as in Lemma 3.2.1. The counit on ${}_\sigma M$ is

$${}_\sigma\underline{\epsilon} : {}_\sigma M \longrightarrow {}_\sigma B, \quad a \;{}^\sigma\!\otimes \tilde{a} \longmapsto \sigma\left(a_{(-1)}, \tilde{a}_{(-1)}\right) a_{(0)}\tilde{a}_{(0)}, \tag{3.2-4}$$

thus the algebra structure is given by the multiplication

$$m \bullet_\sigma m' := {}_\sigma\underline{\epsilon}(m)m'. \tag{3.2-5}$$

We remark that in the latter formula, the product (1.1-13) is used, so explicitly we have

$$\mu_{{}_\sigma M}(a \;{}^\sigma\!\otimes \tilde{a} \otimes a' \;{}^\sigma\!\otimes \tilde{a}') = {}_\sigma\underline{\epsilon}(a \;{}^\sigma\!\otimes \tilde{a}) \bullet_\sigma a' \;{}^\sigma\!\otimes \tilde{a}'$$

where now $\bullet_\sigma$ is the multiplication in ${}_\sigma A$.

For a deformation via a 2-cocycle σ of an external Hopf algebra K we have the following result

Proposition 3.2.3 *For any $n \in \mathbb{N}$ the map*

$$\begin{gathered}
c_n({}_\sigma l) : H^{tr} \longrightarrow {}_\sigma M^{\otimes(n+1)}\\
h \longmapsto \sigma^{-1}({h_{(1)}}^{\langle 1\rangle}{}_{(-1)}, {h_{(1)}}^{\langle 2\rangle}{}_{(-1)}) \dots \sigma^{-1}({h_{(n+1)}}^{\langle 1\rangle}{}_{(-1)}, {h_{(n+1)}}^{\langle 2\rangle}{}_{(-1)})\\
{h_{(n+1)}}^{\langle 2\rangle}{}_{(0)} \;{}^\sigma\!\otimes {h_{(1)}}^{\langle 1\rangle}{}_{(0)} \;{}^\sigma\!\otimes \dots \;{}^\sigma\!\otimes {h_{(n)}}^{\langle 2\rangle}{}_{(0)} \;{}^\sigma\!\otimes {h_{(n+1)}}^{\langle 1\rangle}{}_{(0)}
\end{gathered}$$

is well-defined and its image lies in the cyclic-symmetric part of ${}_\sigma M^{\otimes(n+1)}$. Moreover for any face operator d_i with $i = 0, \dots, n$ one has

$$d_i c_n({}_\sigma l) = c_{n-1}({}_\sigma l)$$

Proof The proof goes in the same way as in 3.2.2. Let $\rho^{\otimes}$ be the diagonal coaction of H on ${}_{\sigma}A$, then

$$
\begin{aligned}
&(\rho^{\otimes}\ {}^{\sigma}\!\otimes \mathrm{id})(c_n({}_{\sigma}l)(h)) = \\
&= \sigma^{-1}\left(h_{(1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(1)}{}^{\langle 2\rangle}{}_{(-1)}\right)\dots\sigma^{-1}\left(h_{(n+1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(n+1)}{}^{\langle 2\rangle}{}_{(-1)}\right) \\
&h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)(0)}\ {}^{\sigma}\!\otimes h_{(1)}{}^{\langle 1\rangle}{}_{(0)(0)} \otimes h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)(1)} h_{(1)}{}^{\langle 1\rangle}{}_{(0)(1)}\ {}^{\sigma}\!\otimes \dots {}^{\sigma}\!\otimes h_{(n)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(n+1)}{}^{\langle 1\rangle}{}_{(0)} \\
&= \sigma^{-1}\left(h_{(1)}{}^{\langle 1\rangle}{}_{(0)(-1)}, h_{(1)}{}^{\langle 2\rangle}{}_{(-1)}\right)\dots\sigma^{-1}\left(h_{(n+1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)(-1)}\right) \\
&h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)(0)}\ {}^{\sigma}\!\otimes h_{(1)}{}^{\langle 1\rangle}{}_{(0)(0)} \otimes h_{(n+1)}{}^{\langle 2\rangle}{}_{(1)} h_{(1)}{}^{\langle 1\rangle}{}_{(1)} \otimes \dots {}^{\sigma}\!\otimes h_{(n)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(n+1)}{}^{\langle 1\rangle}{}_{(0)} \\
&= \sigma^{-1}\left(h_{(2)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(2)}{}^{\langle 2\rangle}{}_{(-1)}\right)\dots\sigma^{-1}\left(h_{(n+2)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(n+2)}{}^{\langle 2\rangle}{}_{(-1)}\right) \\
&h_{(n+2)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(2)}{}^{\langle 1\rangle}{}_{(0)} \otimes h_{(n+3)} S(h_{(1)}) \otimes \dots {}^{\sigma}\!\otimes h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(n+2)}{}^{\langle 1\rangle}{}_{(0)} \\
&= \sigma^{-1}\left(h_{(1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(1)}{}^{\langle 2\rangle}{}_{(-1)}\right)\dots\sigma^{-1}\left(h_{(n+1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(n+1)}{}^{\langle 2\rangle}{}_{(-1)}\right) \\
&h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(1)}{}^{\langle 1\rangle}{}_{(0)} \otimes h_{(n+2)} S(h_{(n+3)}) \otimes \dots {}^{\sigma}\!\otimes h_{(n)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(n+1)}{}^{\langle 1\rangle}{}_{(0)} \\
&= \sigma^{-1}\left(h_{(1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(1)}{}^{\langle 2\rangle}{}_{(-1)}\right)\dots\sigma^{-1}\left(h_{(n+1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(n+1)}{}^{\langle 2\rangle}{}_{(-1)}\right) \\
&h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(1)}{}^{\langle 1\rangle}{}_{(0)} \otimes 1_H \otimes \dots {}^{\sigma}\!\otimes h_{(n)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(n+1)}{}^{\langle 1\rangle}{}_{(0)}
\end{aligned}
$$

where we used the fact that ${}_{\sigma}A$ is a K-H-bicomodule, the properties of the strong connection and the cyclic property in H^{tr}.

Now, if we apply the face operator d_0, what we get is

$$
\begin{aligned}
&d_0(c_n({}_{\sigma}l)(h)) = \\
&= \sigma^{-1}\left(h_{(1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(1)}{}^{\langle 2\rangle}{}_{(-1)}\right)\dots\sigma^{-1}\left(h_{(n+1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(n+1)}{}^{\langle 2\rangle}{}_{(-1)}\right) \\
&(h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)} \bullet_{\sigma} h_{(1)}{}^{\langle 1\rangle}{}_{(0)}) \bullet_{\sigma} h_{(2)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes \dots {}^{\sigma}\!\otimes h_{(n)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(n+1)}{}^{\langle 1\rangle}{}_{(0)} \\
&= \sigma^{-1}\left(h_{(2)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(2)}{}^{\langle 2\rangle}{}_{(-1)}\right)\dots\sigma^{-1}\left(h_{(n+1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(n+1)}{}^{\langle 2\rangle}{}_{(-1)}\right) \\
&h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)} \bullet_{\sigma} \left(\sigma^{-1}\left(h_{(1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(1)}{}^{\langle 2\rangle}{}_{(-1)}\right) h_{(1)}{}^{\langle 1\rangle} \bullet_{\sigma} h_{(1)}{}^{\langle 2\rangle}\right)\ {}^{\sigma}\!\otimes \dots {}^{\sigma}\!\otimes h_{(n)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(n+1)}{}^{\langle 1\rangle}{}_{(0)} \\
&= \sigma^{-1}\left(h_{(1)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(1)}{}^{\langle 2\rangle}{}_{(-1)}\right)\dots\sigma^{-1}\left(h_{(n)}{}^{\langle 1\rangle}{}_{(-1)}, h_{(n)}{}^{\langle 2\rangle}{}_{(-1)}\right) \\
&h_{(n)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(1)}{}^{\langle 1\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes \dots {}^{\sigma}\!\otimes h_{(n-1)}{}^{\langle 2\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes h_{(n)}{}^{\langle 1\rangle}{}_{(0)} = c_{n-1}({}_{\sigma}l)(h)
\end{aligned}
$$

where we used the associativity of the product $\bullet_{\sigma}$ and 1.2.15 ■

The counit ${}_{\sigma}\underline{\epsilon}$, together with map $\widetilde{\mathrm{chw}}_n({}_{\sigma}l) : H^{tr} \longrightarrow \mathrm{HC}_{2n}({}_{\sigma}M)$, induces a map $\mathrm{chw}_n({}_{\sigma}l) : H^{tr} \longrightarrow \mathrm{HC}_{2n}({}_{\sigma}B)$ valued in the cyclic homology of the base algebra ${}_{\sigma}B$ that is defined starting with $x_n({}_{\sigma}l, h) := ({}_{\sigma}\underline{\epsilon}^{\sigma\otimes(n+1)} \circ c_n({}_{\sigma}l))(h)$

$$
\begin{aligned}
x_n({}_{\sigma}l, h) =& \sigma^{-1}\left(h_{(1)}{}^{\langle 1\rangle}{}_{(-2)}, h_{(1)}{}^{\langle 2\rangle}{}_{(-2)}\right)\dots\sigma^{-1}\left(h_{(n+1)}{}^{\langle 1\rangle}{}_{(-2)}, h_{(n+1)}{}^{\langle 2\rangle}{}_{(-2)}\right) \\
&\sigma\left(h_{(n+1)}{}^{\langle 2\rangle}{}_{(-1)}, h_{(1)}{}^{\langle 1\rangle}{}_{(-1)}\right)\dots\sigma\left(h_{(n)}{}^{\langle 2\rangle}{}_{(-1)}, h_{(n+1)}{}^{\langle 1\rangle}{}_{(-1)}\right) \\
&h_{(n+1)}{}^{\langle 2\rangle}{}_{(0)} h_{(1)}{}^{\langle 1\rangle}{}_{(0)}\ {}^{\sigma}\!\otimes \dots {}^{\sigma}\!\otimes h_{(n)}{}^{\langle 2\rangle}{}_{(0)} h_{(n+1)}{}^{\langle 1\rangle}{}_{(0)}.
\end{aligned}
\tag{3.2-6}
$$

So we have the formula

$$\mathrm{chw}_n({}_\sigma l)(h) = \sum_{i=0}^{2n}(-1)^{\lfloor \frac{i}{2} \rfloor}\frac{i!}{\lfloor \frac{i}{2} \rfloor !}x_i({}_\sigma l, h), \quad h \in H^{tr} \tag{3.2-7}$$

We notice in this case that there is no cancellation of the cocycle σ in the final formula. Then under a deformation of this type, we have the Chern-Weil map changes.

3.2.3 Deformations combined

If we now take 2-cocycles γ of H and σ of K for a (K, H)-bicomoule algebra A such that $B \subseteq A$ is a H-comodule algebra, we consider the deformations ${}_\sigma K$, H_γ, ${}_\sigma A_\gamma$ and ${}_\sigma B$ we saw that we get a principal H_γ-comodule algebra ${}_\sigma B_\gamma \subseteq {}_\sigma A_\gamma$ with strong connection ${}_\sigma l_\gamma$.

The Chern-Weil map in this case takes the form of the equation 3.2-6 since the deformations commute and we proved that $\mathrm{chw}_n(l) = \mathrm{chw}_n(l_\gamma)$.

3.3 Pushforward property of the Chern-Weil map

In this section that closes the chapter, we review the theory of the noncommutative Chern character for Hopf-Galois extensions, which is known as the Chern-Galois character. Under suitable assumptions on the structure Hopf algebra H this object allows us to prove the functoriality of the Chern-Weil map.

3.3.1 The Chern-Galois character

Let $B \subseteq A$ be a principal H-comodule algebra and (V, φ) a finite-dimensional corepresentation of H. By this we mean a left H-comodule V with coaction $\varphi : V \longrightarrow H \otimes V$. For any basis $\{e_i\}_{i=1,\ldots,n:=\dim(V)}$ of V, we have

$$\varphi(e_i) = \sum_{j=1}^{n} c_{ij} \otimes e_j. \tag{3.3-1}$$

Define for any corepresentation the element

$$c_\varphi := \sum_{i=1}^{n} c_{ii} \in H. \tag{3.3-2}$$

Following [7] one can construct matrix with coefficients valued in B using (3.3-2) and any unital left B-linear map $\varpi \in {}_B\mathrm{Hom}(A, B)$. Given a basis $\{a_I\}$ of A and its dual $\{\alpha_I\}$ the coefficient of the matrix are given by

$$E_{(I,i)(J,j)} := \varpi(\alpha_I(c_{ij})a_J). \tag{3.3-3}$$

Denoting the matrix with coefficients (3.3-3) by E, one can prove that

$$E^2 = E, \quad B^N E \simeq A \,\Box^H\, V, \tag{3.3-4}$$

for some $N \in \mathbb{N}$ and the isomorphism is in the category of left B-modules. Thus, the left B-module $A \,\square^H\, V$ is finitely generated and projective.

The set of isomorphism classes of finite-dimensional corepresentations of H is a semi-group with respect to the direct sum. Its Grothendieck group [2] is denoted by $\mathrm{Corep}_f(H)$. The above construction then defines a group morphism

$$\mathrm{Corep}_f(H) \longrightarrow K_0(B), \quad [(\varphi, V)] \longmapsto [A \,\square^H\, V]. \tag{3.3-5}$$

If one applies now the *Chern character* [31, 11] which is a map

$$\mathrm{ch}_n : K_0(B) \longrightarrow HC_{2n}(B), \tag{3.3-6}$$

to the class $[A \,\square^H\, V]$ obtains a map

$$\mathrm{chg}_n : \mathrm{Corep}_f(H) \longrightarrow HC_{2n}(B)$$
$$[(\varphi, V)] \longmapsto \sum_{j=0}^{2n} (-1)^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{\lfloor \frac{j}{2} \rfloor !} [\sum_{i_1,\ldots,i_{j+1}} {c_{i_1 i_2}}^{\langle 2 \rangle} {c_{i_2 i_3}}^{\langle 1 \rangle} \otimes \cdots \otimes {c_{i_{j+1} i_1}}^{\langle 2 \rangle} {c_{i_1 i_2}}^{\langle 1 \rangle}], \tag{3.3-7}$$

which is called the **Chern-Galois character**. The latter does not depend on the choice of the strong connection $l(h) = h^{\langle 1 \rangle} \otimes h^{\langle 2 \rangle}$ because different connections yield an idempotent representing the same class in $K_0(B)$ and the Chern character ch_n does depend on the choice of the idempotent.

Recalling the equation

$$\Delta(c_{ij}) = \sum_{k=1}^{n} c_{ik} \otimes c_{kj},$$

we have that the element (3.3-1) is a cotrace. Thus, we can define $\chi : \mathrm{Corep}_f(H) \longrightarrow H^{tr}$ sending $(\varphi, V) \longmapsto c_\varphi$. This is a **character** of the coreprensentation (φ, V) and does not depend on the basis chosen. Applying the Chern-Weil map yields another map

$$\mathrm{Corep}_f(H) \longrightarrow HC_{2n}(B),$$

that makes the following diagram commute

$$\begin{array}{ccc} \mathrm{Corep}_f(H) & \xrightarrow{[A \square^H -]} & K_0(B) \\ \chi \downarrow & & \downarrow \mathrm{ch}_n \\ H^{tr} & \xrightarrow{\mathrm{chw}_n(l)} & HC_{2n}(B). \end{array} \tag{3.3-8}$$

One has that the diagonal of the diagram is the Chern-Galois character (3.3-7) and because the latter does not depend on the choice of the strong connection, so does not chw_n. In this particular situation, we retrieve the independence of the Chern-Weil map on the connection of the bundle (Hopf-Galios extension). In general, one has the following

Theorem 3.3.1 ([23]) *Let $B \subseteq A$ be a principal H-comodule algebra. Suppose the space of cotraces is linearly generated by the characters of the finite-dimensional corepresentations of H. In that case, the Chern-Weil map does not depend on the choice of a strong connection.*

Considering deformations via 2-cocycles, we have that the same result holds. For a deformation of the structure Hopf algebra H and the algebra A, we have that the Chern-Galois character does not change. In proposition 3.2.2 and following discussion we saw that also the Chern-Weil map does not change, so theorem 3.3.1 applies. If we deform the algebras A and B with a 2-cocycle of an external symmetry K, one can show with the same type of computations performed in the proof of 3.2.3 that the Chern-Galois character make the following diagram commute

$$\begin{array}{ccc} \mathrm{Corep}_f(H) & \xrightarrow{[{}_\sigma A\square^H -]} & K_0({}_\sigma B) \\ \chi\downarrow & & \downarrow \mathrm{ch}_n \\ H^{tr} & \xrightarrow{\mathrm{chw}_n({}_\sigma l)} & HC_{2n}({}_\sigma B), \end{array}$$

and under the hypothesis of 3.3.1, the same result holds.

3.3.2 Pushforward

In this subsection, that closes the chapter, we study how the Chern-Weil map behaves with respect to a pullback of bundles. We start by stating and proving the following

Proposition 3.3.2 *Let $B \subseteq A$ be a principal H-comodule algebra, $\bar{A}$ a right H-comodule algebra with coaction invariant elements $\bar{B}$, and $f : A \longrightarrow \bar{A}$ a unital right H-comodule algebra morphism. Then the have*

1. *The restriction of f to B and corestricts to $\bar{B}$,*
2. *$\bar{B} \subseteq \bar{A}$ is a principal H-comodule algebra,*
3. *The is a left $\bar{B}$-linear and right H-colinear isomorphism $\bar{A} \simeq \bar{B} \otimes_B A$.*

Proof The first statement is trivial. For the second we prove that the map

$$\bar{l} := (f \otimes f) \circ l : H \longrightarrow \bar{A} \otimes \bar{A}$$

is a strong connection for $\bar{B} \subseteq \bar{A}$ by checking the defining equations (1.2-8)-(1.2-10). Let h be any element in H:

$$\begin{aligned}[(\mathrm{id}_{\bar{A}} \otimes \bar{\rho}) \circ \bar{l}](h) &= f(h^{\langle 1\rangle}) \otimes f(h^{\langle 2\rangle})_{(0)} \otimes f(h^{\langle 2\rangle})_{(1)} \\ &= f(h^{\langle 1\rangle}) \otimes f({h^{\langle 2\rangle}}_{(0)}) \otimes {h^{\langle 2\rangle}}_{(1)} \\ &= f({h_{(1)}}^{\langle 1\rangle}) \otimes f({h_{(1)}}^{\langle 2\rangle}) \otimes h_{(2)} = [(\bar{l} \otimes \mathrm{id}_H) \circ \Delta](h),\end{aligned}$$

where in the second line we used that f is a right H-comodule morphism and in the third one equation (1.2-3). Moving on, we find

$$\begin{aligned}[(\lambda_{\bar{A}} \otimes \mathrm{id}_{\bar{A}}) \circ \bar{l}](h) &= S^{-1}(f(h^{\langle 1\rangle})_{(1)}) \otimes f(h^{\langle 1\rangle})_{(0)} \otimes f(h^{\langle 2\rangle}) \\ &= S^{-1}(S(h_{(1)})) \otimes f({h_{(2)}}^{\langle 1\rangle}) \otimes f({h_{(2)}}^{\langle 2\rangle}) \\ &= h_{(1)} \otimes f({h_{(2)}}^{\langle 1\rangle}) \otimes f({h_{(2)}}^{\langle 2\rangle}) = [(\mathrm{id}_H \otimes \bar{l}) \circ \Delta](h),\end{aligned}$$

we used again that f is a right H-comodule morphism and in the third line equation (1.2-4). Finally, we have

$$\begin{aligned}(\mathrm{can}_{\bar{A}} \circ \pi_B \circ \bar{l})(h) &= f(h^{\langle 1\rangle})f(h^{\langle 2\rangle})_{(0)} \otimes f(h^{\langle 2\rangle})_{(1)} \\ &= f(h^{\langle 1\rangle})f({h^{\langle 2\rangle}}_{(0)}) \otimes {h^{\langle 2\rangle}}_{(1)} \\ &= f(h^{\langle 1\rangle}{h^{\langle 2\rangle}}_{(0)}) \otimes {h^{\langle 2\rangle}}_{(1)} = 1_{\bar{A}} \otimes h.\end{aligned}$$

We used that f is a H-colinear algebra morphism and the definition of translation map (1.2-2).

To prove the third statement, observe that the algebra $\bar{B}$ is a B-bimodule via the map $f|_B$. Then $\bar{B} \otimes_B A$ is well-defined and inside this space, we have the identification

$$\bar{b}f(b) \otimes_B a = \bar{b} \otimes_B ba$$

for any $\bar{b} \in \bar{B}$, $b \in B$ and $a \in A$. Define the linear maps

$$\begin{aligned}&\phi : \bar{A} \longrightarrow \bar{B} \otimes_B A, \quad \bar{a} \longmapsto \bar{a}_{(0)} f({\bar{a}_{(1)}}^{\langle 1\rangle}) \otimes_B {\bar{a}_{(1)}}^{\langle 2\rangle} \\ &\psi : \bar{B} \otimes_B A \longrightarrow \bar{A}, \quad \bar{b} \otimes_B a \longmapsto \bar{b}f(a)\end{aligned}$$

it is easy to check that they are left $\bar{B}$-module and right H-comodule morphism. Moreover ϕ lands into $\bar{B} \otimes_B A$ since

$$\begin{aligned}(\bar{\rho} \otimes_B \mathrm{id}_A)(\phi(\bar{a})) &= \bar{a}_{(0)} f({{\bar{a}_{(2)}}^{\langle 1\rangle}}_{(0)}) \otimes \bar{a}_{(1)} {{\bar{a}_{(2)}}^{\langle 2\rangle}}_{(1)} \otimes_B {\bar{a}_{(2)}}^{\langle 2\rangle} \\ &= \bar{a}_{(0)} f({\bar{a}_{(2)(2)}}^{\langle 1\rangle}) \otimes \bar{a}_{(1)} S(\bar{a}_{(2)(1)}) \otimes_B {\bar{a}_{(2)(2)}}^{\langle 2\rangle} \\ &= \bar{a}_{(0)} f({\bar{a}_{(3)}}^{\langle 1\rangle}) \otimes \bar{a}_{(1)} S(\bar{a}_{(2)}) \otimes_B {\bar{a}_{(3)}}^{\langle 2\rangle} \\ &= \bar{a}_{(0)} f({\bar{a}_{(1)}}^{\langle 1\rangle}) \otimes 1_H \otimes_B {\bar{a}_{(1)}}^{\langle 2\rangle}\end{aligned}$$

in order we used the fact that f is a right H-comodule morphism, the coassociativity of the coproduct, equation (1.2-4) and the antipode equation $h_{(1)}S(h_{(2)}) = \epsilon(h)$ for all $h \in H$.

We are now ready to prove that they are the inverse of each other:

$$(\phi \circ \psi)(\bar{b} \otimes_B a) = \bar{b}f(a_{(0)})f({a_{(1)}}^{\langle 1\rangle}) \otimes_B {a_{(1)}}^{\langle 2\rangle} = \bar{b}f(a_{(0)}{a_{(1)}}^{\langle 1\rangle}) \otimes_B {a_{(1)}}^{\langle 2\rangle} = \bar{b} \otimes_B a$$

$$(\psi \circ \phi)(\bar{a}) = \bar{a}_{(0)} f({\bar{a}_{(1)}}^{\langle 1\rangle}) f({\bar{a}_{(1)}}^{\langle 2\rangle}) = \bar{a}_{(0)} f({\bar{a}_{(1)}}^{\langle 1\rangle}{\bar{a}_{(1)}}^{\langle 2\rangle}) = \bar{a}_{(0)}\varepsilon(\bar{a}_{(1)}) = \bar{a}$$

■

Because we are assuming that the antipode of H is bijective, the right H-colinear morphism f is also left H-colinear. Thus, the above result also holds in the left case, namely

$$\bar{A} \simeq A \otimes_B \bar{B}, \tag{3.3-9}$$

where $\bar{B}$ is a left B-module with the action given by f. This is an isomorphism of right $\bar{B}$-modules and left H-comodules.

Remark 3.3.3 This result is the noncommutative analog of the pullback principal bundle. If $P \longrightarrow P/G$ is a topological principal G-bundle and $F : P' \longrightarrow P$ is a G-equivariant continuous map, then P' is a G-space such that the action is free and transitive on the fibers of $P' \longrightarrow P'/G$. Moreover, one has the homeomorphism $P' \simeq P'/G \times_{P/G} P$ [22]. ♦

If we denote by $HCf : HC_*(B) \longrightarrow HC_*(\bar{B})$ the induced cyclic homology map by f, we have the following

Proposition 3.3.4 *The Chern-Weil map associated with the principal H-comodule algebra satisfies for any $n \in \mathbb{N}$*

$$\mathrm{chw}_n(\bar{l}) = HCf \circ \mathrm{chw}_n(l).$$

Proof Consider the strong connection of proposition 3.3.2 and the element that defines the homology class $x_n(\bar{l}, h)$ for $h \in H^{tr}$, then

$$\begin{aligned} x_n(\bar{l}, h) &= f({h_{(n+1)}}^{\langle 2 \rangle} {h_{(1)}}^{\langle 1 \rangle}) \otimes \cdots \otimes f({h_{(n)}}^{\langle 2 \rangle} {h_{(n+1)}}^{\langle 1 \rangle}) \\ &= f({h_{(n+1)}}^{\langle 2 \rangle}) f({h_{(1)}}^{\langle 1 \rangle}) \otimes \cdots \otimes f({h_{(n)}}^{\langle 2 \rangle}) f({h_{(n+1)}}^{\langle 1 \rangle}) = HCf(x_n(l, h)). \end{aligned}$$

this concludes the proof. ■

If we consider a deformation via a 2-cocycle of the structure of algebra H, we have that for any unital right H-comodule algebra morphism $f : A \longrightarrow \bar{A}$ is also a unital right H_γ-comodule algebra $f : A_\gamma \longrightarrow \bar{A}_\gamma$

$$\begin{aligned} f(a) \cdot_\gamma f(a') &= f(a)_{(0)} f(a')_{(0)} \gamma^{-1}(f(a)_{(1)}, f(a')_{(1)}) \\ &= f(a_{(0)}) f(a'_{(0)}) \gamma^{-1}(a_{(1)}, a'_{(1)}) \\ &= f(a_{(0)} a'_{(0)}) \gamma^{-1}(a_{(1)}, a'_{(1)})) = f(a \cdot_\gamma a'). \end{aligned} \tag{3.3-10}$$

Thus, we can prove the proposition 3.3.2 in the same way to get the strong connection $\bar{l}_\gamma$ making $\bar{B} \subseteq \bar{A}_\gamma$ a principal H_γ-comodule algebra. After this consideration, we have also that proposition 3.3.4 is valid.

In case we consider a deformation via a 2-cocycle σ of an external symmetry K, to have the same type of result, it is sufficient to ask that map $f : A \longrightarrow \bar{A}$ is a (K, H)-bicomodule algebra morphism. In this way, we have an induced map from ${}_\sigma A$ to ${}_\sigma \bar{A}$ because of (3.3-10). Thus, the results 3.3.2 and 3.3.4 hold also in this case.

Chapter 4

Morita equivalence for the Ehresmann-Schauenburg algebroid

In this final chapter, we collect some partial results about the Morita theory of noncommutative Hopf algebroid. The commutative case has been studied in detail in [16] and is a dualization of the corresponding theory for groupoids. We define in this chapter what a bibundle is for two bialgebroids. Then we prove the noncommutative version of the classical result of the Morita equivalence between the gauge groupoid of a principal G-bundle and the structure group.

In the following when we say Hopf algebroid, we refer to definition of Schauenburg 1.3.5.

4.1 Bibundles

We briefly review the theory of bibundle for Lie groupoids and then give the corresponding definition for Hopf algebroid. We point out that in the latter case, the notion of bibundle does not define an equivalence relation in general.

4.1.1 The case of groupoids

Let $\Omega \rightrightarrows \Omega_0$ be Lie groupoid and P another manifold

Definition 4.1.1 An **action** of $\Omega \rightrightarrows \Omega_0$ on P is the datum of a smooth map $f : P \longrightarrow \Omega_0$ and a smooth map

$$\Omega \times_{\Omega_0} P \longrightarrow P, \quad (\omega, p) \longrightarrow \omega \triangleright p,$$

such that

$$(\omega \circ \omega') \triangleright p = \omega \triangleright (\omega' \triangleright p), \quad \mathrm{id}_{s(\omega_0)} \triangleright p = p,$$

for all $(\omega, \omega') \in \Omega_2$, $p \in P$ and $\omega_0 \in \Omega_0$.

We say that a manifold with a groupoid action is a Ω-**space**. ♦

In the above definition, we have the fiber product of Ω and P given by

$$\Omega \times_{\Omega_0} P := \{(\omega, p) \in \Omega \times P | s(\omega) = f(p)\}.$$

As in the case of group action, we are interested in characterizing the freeness and transitivity of groupoid spaces. Our guiding principle is the canonical map of example 1.2.4. Let P be a Ω-space

Definition 4.1.2 A **Ω-bundle** is the datum of a Ω-space (P,f) and a smooth map $\varpi : P \longrightarrow N$ that satisfying

$$\varpi(p) = \varpi(\omega \triangleright p),$$

for all $(\omega, p) \in \Omega \times_{\Omega_0} P$. We say that $(\Omega, P, f, N, \varpi)$ is a principal Ω **principal** when π is a submersion and the smooth map

$$\Omega \times_{\Omega_0} P \longrightarrow P \times_N P, \quad (\omega, p) \longmapsto (\omega \triangleright p, p),$$

is bijective. ♦

In the case of a Lie group, $\Omega_0 = \{*\}$, we retrieve the notion of a principal bundle over the quotient manifold.

We defined groupoid space with action from the left, the definition with the action from the right is similar. We denote by $p \triangleleft \omega$ the right action.

Definition 4.1.3 A manifold P with a right groupoid action of $\Omega \rightrightarrows \Omega_0$ and a left action $\Omega' \rightrightarrows \Omega'_0$ such that

$$(\omega \triangleright p) \triangleleft \omega' = \omega \triangleright (p \triangleleft \omega'),$$

is said to be a $\Omega - \Omega'$**-space**. Moreover, if both actions are principal we say that P is a $\Omega - \Omega'$**-bibundle**. ♦

We remark that, for groupoids, a bibundle exists if and only if they are Morita equivalent [16, 33].

Remark 4.1.4 (The classical case) Given a principal G-bundle $\pi : P \longrightarrow M$, there is a natural left action of the gauge groupoid Ω 2.2.2 on the total space P. We recall that in this case $\Omega_0 = M$, and the map f of the Definition 4.1.1 is the projection π, the action map is given by

$$\triangleright : \Omega \times_M P \longrightarrow P, \quad [p, q] \triangleright r := p\tau(q, r). \tag{4.1-1}$$

It is well-defined since

$$[pg, qg] \triangleright r = pg\tau(qg, r) = pgg^{-1}\tau(q, r) = p\tau(q, r) = [p, q] \triangleright r, \quad \forall g \in G \tag{4.1-2}$$

where we used (2.2-7). We can easily see that is a left action

$$\begin{aligned} [p, q] \triangleright ([p', q'] \triangleright r) &= [p, q] \triangleright p'\tau(q', r) \\ &= p\tau(q, p'\tau(q', r)) \\ &= p\tau(q, p')\tau(q', r) \\ &= p\tau(q'\tau(p', q), r) \\ &= [p, q'\tau(p', q)] \triangleright r = [p, q]\,[p', q'] \triangleright r, \end{aligned}$$

where in order we used (2.2-8), (2.2-7) and (2.2-9). This action commute with the right G-action on P , in fact from (2.2-8) we have

$$[p,q] \triangleright (rg) = p\tau(q,rg) = p\tau(q,r)g = ([p,q] \triangleright r)g.$$

Finally, we have that the Ω-action is principal, being the canonical map

$$\Omega \times_M P \longrightarrow P \times P, \quad ([p,q]\,,r) \longmapsto (p\tau(q,r),r),$$

a diffeomorphism with inverse given by

$$P \times P \longrightarrow \Omega \times_M P, \quad (p,q) \longrightarrow ([p,q]\,,q).$$ ♦

4.1.2 The case of Hopf algebroids

In the first chapter of this thesis, we saw that for a bialgebroid the notion of comodule algebra is the same as for bialgebra, and in addition, we asked the coaction to be a B-bilinear morphism.

Before defining bibundles for Hopf algebroid, we need the notion of principality of action in the algebraic sense

Definition 4.1.5 Let $(\mathcal{H}, B)$ be a Hopf algebroid and A a left $\mathcal{H}$-comodule algebra with coaction $\underline{\lambda} : A \longrightarrow \mathcal{H} \otimes_B A$ sending $a \longmapsto a_{(-1)} \otimes a_{(0)}$. We say that the coaction is **principal** if the canonical map

$$\mathrm{can}_{\mathcal{H}} : A \otimes_{{}^{co\mathcal{H}}A} A \longrightarrow \mathcal{H} \otimes_B A, \quad a \otimes \tilde{a} \longmapsto a_{(-1)} \otimes_B a_{(0)}\tilde{a}$$

is bijective. ♦

Consider now two Hopf algebroids $(\mathcal{H}, B)$ and $(\mathcal{H}', B')$ then we have the following

Definition 4.1.6 An algebra A is said to be a $(\mathcal{H}, \mathcal{H}')$**-bicomodule algebra** if it is a B and B'-bimodule, a left $\mathcal{H}$-comodule algebra with a right B'-linear coaction $\underline{\lambda} : A \longrightarrow \mathcal{H} \otimes_B A$, and a right $\mathcal{H}'$-comodule algebra with a left B-linear coaction $\underline{\rho} : A \longrightarrow A \otimes_{B'} \mathcal{H}'$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\quad \underline{\rho} \quad} & A \otimes_{B'} \mathcal{H}' \\ {\scriptstyle \underline{\lambda}}\downarrow & & \downarrow{\scriptstyle \underline{\lambda} \otimes_{B'} \mathrm{id}_{\mathcal{H}'}} \\ H \otimes_B A & \xrightarrow{id_{\mathcal{H}} \otimes_B \underline{\rho}} & \mathcal{H} \otimes_B A \otimes_{B'} \mathcal{H}' \end{array}$$ ♦

We remark that the definitions can be given just for bialgebroids since they do not involve antipode or the canonical map of 1.3.5. We now give the central definition of this section

Definition 4.1.7 A $(\mathcal{H}, \mathcal{H}')$-bicomodule algebra A is said to be a **bibundle** if ${}^{co\mathcal{H}}A = B'$ and $A^{co\mathcal{H}'} = B$, A is a faithfully flat B and B' module, and the canonical maps the canonical maps

$$\mathrm{can}_{\mathcal{H}} : A \otimes_{B'} A \longrightarrow \mathcal{H} \otimes_B A$$
$$\mathrm{can}_{\mathcal{H}'} : A \otimes_B A \longrightarrow A \otimes_{B'} \mathcal{H}'.$$ ♦

Remark 4.1.8 The requirement of faithful flatness of the algebra A seems to be too strong in the noncommutative case. So, this is not a definitive notion of bibundle in this setting, but for our purpose is good enough, and that is because for the results we prove later in 4.2.2 and 4.2.7 such a requirement is fulfilled.♦

Given two commutative Holf algebroids, if they admit a bibundle then their categories of comodules are equivalent, and also the converse is true [16]. In representation theory, being Morita equivalent means that two objects have the same (co)representation theory. So the result can be stated as two Hopf algebroids are Morita equivalent if and only if they admit a bibundle.

We also mention that in the case of Hopf algebras, i.e. $B = \mathbb{K} = B'$, a bibundle reduces to a bi-Galois object [36]. In the same paper is also proved that the comodule categories of two Hopf algebras admitting a bi-Galois object are equivalent.

4.2 A Morita theory result

In this section, we prove that, given a Hopf-Galois extension $B \subseteq A$, the algebra A is a $(\mathcal{C}(A,H),H)$-bibundle. Moreover, we prove that any bialgebroid $\mathcal{L}$ admitting a $(\mathcal{L},H)$-bibundle A with a Hopf algebra H is isomorphic to the ES algebroid; generalizing the classical result in 2.2.2 to the noncommutative algebraic setting.

4.2.1 Coactions and canonical maps

For the rest of the subsection, let $B \subseteq A$ be a right principal H-comodule algebra, so that the ES algebroid $\mathcal{C}(A,H)$ is well-defined. There is a natural map

$$\underline{\lambda} : A \to \mathcal{C}(A,H) \otimes_B A \tag{4.2-1}$$

that sends

$$\underline{\lambda}(a) := a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle} \otimes_B a_{(1)}{}^{\langle 2\rangle}. \tag{4.2-2}$$

It is easy to see that

Lemma 4.2.1 *The pair $(A,\underline{\lambda})$ is a left $\mathcal{C}(A,H)$-comodule algebra, and moreover ${}^{co\mathcal{C}(A,H)}A = \mathbb{K}$*

Proof One easily checks that $\underline{\lambda}$ is both left and right B-linear. Let $\rho^{\otimes}$ be the diagonal coaction of H on $A \otimes A$. By applying the map $\rho^{\otimes} \otimes_B \mathrm{id}_A$ to $\underline{\lambda}(a)$ and using (1.2-4) we get

$$\begin{aligned} a_{(0)(0)} \otimes a_{(1)}{}^{\langle 1\rangle}{}_{(0)} \otimes a_{(0)(1)} a_{(1)}{}^{\langle 1\rangle}{}_{(1)} \otimes_B a_{(1)}{}^{\langle 2\rangle} &= a_{(0)} \otimes a_{(2)(2)}{}^{\langle 1\rangle} \otimes a_{(1)} S(a_{(2)(1)}) \otimes_B a_{(2)(2)}{}^{\langle 2\rangle} \\ &= a_{(0)} \otimes a_{(3)}{}^{\langle 1\rangle} \otimes a_{(1)} S(a_{(2)}) \otimes_B a_{(3)}{}^{\langle 2\rangle} \\ &= a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle} \otimes 1_H \otimes_B a_{(1)}{}^{\langle 2\rangle} \end{aligned}$$

This proves that $\mathrm{Im}(\underline{\lambda})(A) \subseteq (A \otimes A)^{coH} \otimes_B A = \mathcal{C}(A,H) \otimes_B A$. Moreover, we have

$$a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle} \otimes_B a_{(1)}{}^{\langle 2\rangle} b = a_{(0)} \otimes b a_{(1)}{}^{\langle 1\rangle} \otimes_B a_{(1)}{}^{\langle 2\rangle}$$

and

$$\begin{aligned}\underline{\lambda}(a\tilde{a}) &= a_{(0)}\tilde{a}_{(0)} \otimes {\tilde{a}_{(1)}}^{\langle 1\rangle}{a_{(1)}}^{\langle 1\rangle} \otimes_B {a_{(1)}}^{\langle 2\rangle}{\tilde{a}_{(1)}}^{\langle 2\rangle} \\ &= (a_{(0)} \otimes {a_{(1)}}^{\langle 1\rangle}) \cdot_{\mathcal{C}(A,H)} (\tilde{a}_{(0)} \otimes {\tilde{a}_{(1)}}^{\langle 1\rangle}) \otimes_B {a_{(1)}}^{\langle 2\rangle}{\tilde{a}_{(1)}}^{\langle 2\rangle} \\ &= \underline{\lambda}(a)\underline{\lambda}(\tilde{a})\end{aligned}$$

here we used (1.2-6). Thus, $\underline{\lambda}$ is an algebra morphism if corestricted to $\mathcal{C}(A,H)\times_B A$.

Denote by (Δ, ϵ) the B-coring structure on $\mathcal{C}(A,H)$, then one has for any $a \in A$

$$\begin{aligned}[(\mathrm{id}_{\mathcal{C}(A,H)} \otimes_B \underline{\lambda}) \circ \underline{\lambda}](a) &= a_{(0)} \otimes {a_{(1)}}^{\langle 1\rangle} \otimes_B {{a_{(1)}}^{\langle 2\rangle}}_{(0)} \otimes {{{a_{(1)}}^{\langle 2\rangle}}_{(1)}}^{\langle 1\rangle} \otimes_B {{{a_{(1)}}^{\langle 2\rangle}}_{(1)}}^{\langle 2\rangle} \\ &= a_{(0)} \otimes {a_{(1)(1)}}^{\langle 1\rangle} \otimes_B {a_{(1)(1)}}^{\langle 2\rangle} \otimes {a_{(1)(2)}}^{\langle 1\rangle} \otimes_B {a_{(1)(2)}}^{\langle 2\rangle} \\ &= a_{(0)} \otimes {a_{(1)}}^{\langle 1\rangle} \otimes_B {a_{(1)}}^{\langle 2\rangle} \otimes {a_{(2)}}^{\langle 1\rangle} \otimes_B {a_{(2)}}^{\langle 2\rangle} \\ &= (\Delta \otimes_B \mathrm{id}_A)\left(a_{(0)} \otimes {a_{(1)}}^{\langle 1\rangle} \otimes_B {a_{(1)}}^{\langle 2\rangle}\right) \\ &= [(\Delta \otimes_B \mathrm{id}_A) \circ \underline{\lambda}](a)\end{aligned}$$

where in the first line we used (1.2-3). For the counit we have

$$[(\epsilon \otimes_B \mathrm{id}_A) \circ \underline{\lambda}](a) = a_{(0)}{a_{(1)}}^{\langle 1\rangle} \otimes_B {a_{(1)}}^{\langle 2\rangle} = 1_A \otimes_B a$$

where we recognize (1.2-7). This ends the proof. ■

We can now return to the original goal of exhibiting a "Morita equivalence" between the original Hopf algebra H and $\mathcal{C}(A,H)$:

Theorem 4.2.2 *For any right principal H-comodule algebra $B \subseteq B$, the algebra A is a principal $(\mathcal{C}(A,H), H)$-bibundle when equipped with its corresponding left coaction (4.2-1) of $\mathcal{C}(A,H)$ and right coaction ρ of H.*

Proof Write $\mathcal{H} := \mathcal{C}(A,H)$ for brevity. We already know from Lemma 4.2.1 that A is a left $\mathcal{H}$-comodule algebra in the sense of [8, §31.23]. There are a few other items to check.

(I) **The H- and $\mathcal{H}$-coactions commute.** for any $a \in A$ we have one one side

$$[(\underline{\lambda} \otimes \mathrm{id}_H) \circ \rho](a) = a_{(0)} \otimes {a_{(1)}}^{\langle 1\rangle} \otimes_B {a_{(1)}}^{\langle 2\rangle} \otimes a_{(2)},$$

and on the other

$$[(\mathrm{id}_{\mathcal{H}} \otimes_B \rho) \circ \underline{\lambda}](a) = a_{(0)} \otimes {{a_{(1)}}^{\langle 1\rangle}}_{(0)} \otimes_B {{a_{(1)}}^{\langle 2\rangle}}_{(1)} \otimes a_{(2)}.$$

Using (1.2-3) the two expressions are equal, then the diagram

$$\begin{array}{ccc} A & \longrightarrow & A \otimes H \\ \downarrow & & \downarrow \\ \mathcal{H} \otimes_B A & \longrightarrow & \mathcal{H} \otimes_B A \otimes H, \end{array}$$

commutes.

(II) **Right principality.** Since we are considering a principal comodule algebra, we have that the canonical map $\mathrm{can}_H : A\otimes_B \longrightarrow A \otimes H$ is bijective.

(III) **Left principality.** This means that the other, left-hand canonical map

$$\begin{aligned}&\mathrm{can}_{\mathcal{H}} : A \otimes A \longrightarrow \mathcal{H} \otimes_B A\\&\mathrm{can}_{\mathcal{H}}(a \otimes \widetilde{a}) = a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle} \otimes_B a_{(1)}{}^{\langle 2\rangle}\widetilde{a}\end{aligned} \tag{4.2-3}$$

is bijective. It is very easy to write down the inverse explicitly:

$$a \otimes \widetilde{a} \otimes_B a' \longmapsto a \otimes \widetilde{a}a' \in A \otimes A. \tag{4.2-4}$$

(4.2-3) followed by (4.2-4) is

$$A \otimes A \ni a \otimes \widetilde{a} \longmapsto a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle} a_{(1)}{}^{\langle 2\rangle}\widetilde{a} \in A \otimes A,$$

and the right-hand side equals $a \otimes \widetilde{a}$. On the other hand, applying (4.2-4) followed by (4.2-3) to

$$a \otimes \widetilde{a} \otimes_B 1 \in \mathcal{H} \otimes_B A \tag{4.2-5}$$

produces

$$a_{(0)} \otimes a_{(1)}{}^{\langle 1\rangle} \otimes_B a_{(1)}{}^{\langle 2\rangle}\widetilde{a}.$$

The fact that this is nothing but the original element (4.2-5) follows from one of the characterizations of $\mathcal{H}$ 2.2.1..

This concludes the proof. ■

In the case of a Galois object ($B = \mathbb{K}$), a universal property of the ES algebroid - that in that case is proven to be a Hopf algebra - is shown in lemma 3.2 therein. We now prove that an analogous result holds in general; that the statement is fairly expansive, illustrating the various interactions between the multiple left/right module structures characteristic of the present situation (of working over a non-commutative base ring B).

Recall the Takeuchi product 1.3-9. We have the following

Theorem 4.2.3 *Let $B \subseteq A$ be a right principal H-comodule algebra, then*

(1) We have a natural isomorphism

$$\mathfrak{M}_B\left(\mathcal{C}(A,H),\ -\right) \cong \mathfrak{M}^H\left(A,\ - \otimes_B A\right) \tag{4.2-6}$$

of functors $\mathfrak{M}_B \longrightarrow {}_B\mathfrak{M}_B$, where the B-bimodule structures on the two sides of (4.2-8) are as follows:

- *on the right, transported over from that on the domain A of the maps $A \to V \otimes_B A$ for $V \in \mathfrak{M}_B$;*
- *on the left, the 'B' subscript is induced by the B-module structure on the second tensorand of*

$$A \otimes A \supseteq \mathcal{C}(A,H) \in {}_B\mathfrak{M}_B, \tag{4.2-7}$$

and the leftover B-bimodule structure on $\mathfrak{M}_B\left(\mathcal{C}(A,H),\ -\right)$ is transported from that on the first tensorand in (4.2-7).

(2) We have a natural isomorphism

$$ {}_B\mathfrak{M}_B\left(\mathcal{C}(A,H),\ -\right) \cong \mathfrak{M}^H\left(A,\ - \times_B A\right) \tag{4.2-8}$$

of functors ${}_B\mathfrak{M}_B \longrightarrow {}_B\mathfrak{M}_B$, where the B-bimodule structures on the two sides of (4.2-8) are as before, with the two 'B' subscripts in ${}_B\mathfrak{M}_B\left(\mathcal{C}(A,H),\ -\right)$ induced by the B-bimodule structure on the right-hand tensorand in (4.2-7).

(3) Similarly, we have an isomorphism

$$ {}_{B^e}\mathfrak{M}_{B^e}\left(\mathcal{C}(A,H),\ -\right) \cong {}_B\mathfrak{M}^H_B\left(A,\ - \times_B A\right) \tag{4.2-9}$$

of functors ${}_{B^e}\mathfrak{M}_{B^e} \longrightarrow$ VECT, where for $V \in {}_{B^e}\mathfrak{M}_{B^e}$

- *$V \otimes_B A$ and $V \times_B A$ are built using the B-bimodule structure on $V \in {}_{B^e}\mathfrak{M}_{B^e}$ induced by the two (left and right) B^{op}-actions;*
- *and the B-bimodule structure on $V \times_B A$ then results from the two (left and right) B-actions on V.*

(4) There is an isomorphism

$$ {}_B\text{CORNG}\left(\mathcal{C}(A,H),\ -\right) \cong \text{COMOD}^H\left(A,\ - \otimes_B A\right) \tag{4.2-10}$$

of functors ${}_B$CORNG $\longrightarrow$ SET defined on the category of B-corings, where the right-hand side of (4.2-10) denotes the set of H-comodule morphisms that are also left comodule structures on A for the B-coring '–'.

Proof We will denote by V the generic object of any of the various categories under consideration, filling in the '–' blank in (4.2-8) and analogues.

(1) This follows very much as in [36]:

$$\begin{aligned}
\mathfrak{M}^H\left(A,\ V \otimes_B A\right) &\cong \mathfrak{M}^H_A\left(A \otimes A,\ V \otimes_B A\right) \quad \text{(hom-tensor adjunction internal to } \mathfrak{M}^H\text{)}\\
&\cong \mathfrak{M}_B\left((A \otimes A)^{coH},\ V\right) \quad 1.2.5 \qquad (4.2\text{-}11)\\
&= \mathfrak{M}_B\left(\mathcal{C}(A,H),\ V\right) \quad \text{(definition of } \mathcal{C}(A,H)\text{)}.
\end{aligned}$$

To conclude, note that the B-bimodule structure on the domain A of the maps $A \to V \otimes_B A$ (and hence, later, on the left-hand tensorand of $A \otimes A$) run undisturbed through the isomorphisms.

The other items are consequences of part (1): as we will outline, the requisite isomorphisms are all (co)restrictions of (4.2-8).

(2) Running through the isomorphism chain (4.2-11), note that an H-comodule morphism $A \to V \otimes_B A$ takes values in the smaller space

$$V \times_B A \subseteq V \otimes_B A$$

precisely when, recast as a map $A \otimes A \longrightarrow V \otimes_B A$ as in the first line of (4.2-11), it intertwines left B-multiplication on the right-hand A tensorand and left B-multiplication on V. This, in turn, is equivalent to the map's avatar through (4.2-11) lying in

$$ {}_B\mathfrak{M}_B\left(\mathcal{C}(A,H),\ V\right) \subseteq \mathfrak{M}_B\left(\mathcal{C}(A,H),\ V\right).$$

(3) The argument is very similar to the preceding one: having made the identification

$$\mathfrak{M}^H(A,\ V \times_B A) \simeq {}_B\mathfrak{M}_B\left(\mathcal{C}(A,H),\ V\right)$$

in part (2), note that a map in the left-hand space respects the B-bimodule structures on the two sides precisely when it does so on the right for

- the B-bimodule structure on $\mathcal{C}(A,H)$ inherited from its left-hand tensorand A in
$$\mathcal{C}(A,H) \subseteq A \otimes A;$$
- the B-bimodule structure on V resulting from the two (left and right) B-actions given by $V \in {}_{B^e}\mathfrak{M}_{B^e}$.

(4) This too is a fairly quick consequence of part (1). For a B-coring

$$(V,\ \Delta,\ \varepsilon) \in {}_B\text{CoRng},$$

recall first [8] that V-comodule structures

$$\underline{\lambda} : A \longrightarrow \mathcal{L} \otimes_B A$$

are required to be left B-module morphisms. Transporting that requirement along (4.2-11), the counterpart of $\underline{\lambda}$ is (at least) a B-*bi*module morphism $\psi : \mathcal{C}(A,H) \longrightarrow \mathcal{L}$.

As for preserving the coring structure, consider, say, the comultiplication. The composition

$$\begin{array}{ccc} A & \xrightarrow{\quad\underline{\lambda}\quad} & A \otimes_B \mathcal{L} \\ {\scriptstyle\underline{\lambda}}\downarrow & & \downarrow{\scriptstyle(\mathrm{id}_{\mathcal{L}} \otimes_B \underline{\lambda})\circ\underline{\lambda}} \\ \mathcal{L} \otimes_B A & \xrightarrow[(\underline{\Delta}_{\mathcal{L}} \otimes_B \mathrm{id}_A)\circ\underline{\lambda}]{} & \mathcal{L} \otimes_B \mathcal{L} \otimes_B A \end{array} \tag{4.2-12}$$

corresponds via (4.2-8)

- to
$$\mathcal{C}(A,H) \xrightarrow{\quad\psi\quad} \mathcal{L} \xrightarrow{\quad\Delta_{\mathcal{L}}\quad} \mathcal{L} \otimes_B \mathcal{L}$$
for a unique B-bimodule morphism θ, if we follow the lower path in (4.2-12);
- and to
$$\mathcal{C}(A,H) \xrightarrow{\quad\underline{\Delta}\quad} \mathcal{C}(A,H) \otimes \mathcal{C}(A,H) \xrightarrow{\quad\psi\otimes\psi\quad} \mathcal{L} \otimes_B \mathcal{L}$$
is we follow the upper path instead.

The two must be equal, by the commutativity of (4.2-12) (and (4.2-8) again). The same type of argument goes through for counit preservation.

This concludes the proof. ■

Recall from [8] that the category ${}_{B^e}\text{Rng}$ of B^e-rings is monoidal under the Takeuchi product '$\times_A$'. In fact, more is true: since construction of $R \times_A S$ in the reference and the proof therein only make use of the B^{op}-ring structure on R and the B-ring structure on S, the two factors need not both be full B^e-rings. We spell out the various possibilities (without proof, since [8] handles the matter):

Proposition 4.2.4 *The bifunctor*

$$ {}_{B^{op}}\mathrm{R}\small{\mathrm{NG}} \times {}_{B}\mathrm{R}\small{\mathrm{NG}} \xrightarrow{\times_B} \mathrm{R}\small{\mathrm{NG}} $$

extends

(1) to a functor

$$ {}_{B^e}\mathrm{R}\small{\mathrm{NG}} \times {}_{B}\mathrm{R}\small{\mathrm{NG}} \xrightarrow{\times_B} {}_{B}\mathrm{R}\small{\mathrm{NG}}; $$

(2) as well as a functor

$$ {}_{B^{op}}\mathrm{R}\small{\mathrm{NG}} \times {}_{B^e}\mathrm{R}\small{\mathrm{NG}} \xrightarrow{\times_B} {}_{B^{op}}\mathrm{R}\small{\mathrm{NG}}. $$

■

Remark 4.2.5 The functors of Proposition 4.2.4 almost look like they would make ${}_{B}\mathrm{RNG}$ and ${}_{B^{op}}\mathrm{RNG}$ into *module categories* [17] over ${}_{B^e}\mathrm{RNG}$, save for the fact that the latter is not, in general, a monoidal category: [43] and [38] both note that '$\times_A$' need not be associative. ♦

There is a multiplicative counterpart to Theorem 4.2.3 (whose part (4), for instance, is concerned with comultiplicative structure):

Theorem 4.2.6 *Let $B \subseteq A$ be a right principal H-comodule algebra, then:*

(1) There is a natural isomorphism

$$ {}_{B^{op}}\mathrm{RNG}\,(\mathcal{C}(A,H),\ -) \cong \mathrm{RNG}\,(A,\ - \times_B A) $$

of functors ${}_{B^{op}}\mathrm{RNG} \to \mathrm{SET}$.

(2) Similarly, there is a natural isomorphism

$$ {}_{B^e}\mathrm{RNG}\,(\mathcal{C}(A,H),\ -) \cong {}_{B}\mathrm{RNG}\,(A,\ - \times_B A) $$

of functors ${}_{B^e}\mathrm{RNG} \to \mathrm{SET}$.

Proof The proof of [36], showing that for $B = \mathbb{K}$ multiplicativity on one side of (4.2-11) entails multiplicativity on the other, goes through verbatim in the present setting: one would instead be working with the identifications (4.2-8) and (4.2-9) instead. ■

We now have the following characterization of ES bialgebroid, analogous to (a portion of) [33].

Theorem 4.2.7 *Let H be a Hopf algebra with bijective antipode.*
A left bialgebroid $\mathcal{L}$ admits a principal $(\mathcal{L}, H)$-bibundle if and only if it is the ES bialgebroid attached to a right principal H-comodule algebra.

Proof Theorem 4.2.2 settles the '$\Longleftarrow$' implication, so we handle the converse. Assume, to that end, that A is a principal $(\mathcal{L}, H)$-bibundle. The universality properties of $\mathcal{C}(A,H)$ in Theorem 4.2.3 (4) and Theorem 4.2.6 (2) provide a unique B-bialgebroid morphism

$$ \psi : \mathcal{C}(A,H) \longrightarrow \mathcal{L} $$

inducing the $\mathcal{L}$-comodule structure

$$\underline{\lambda}_{\mathcal{L}} : A \longrightarrow \mathcal{L} \otimes_B A$$

on A via

$$\begin{array}{ccc} A & \xrightarrow{\underline{\lambda}} & \mathcal{C}(A,H) \\ & \underset{\underline{\lambda}_{\mathcal{L}}}{\searrow} & \Big\downarrow {\scriptstyle \psi\otimes_B \mathrm{id}_A} \\ & & \mathcal{L} \end{array}$$

for the left $\mathcal{C}(A,H)$-comodule algebra structure $\underline{\lambda}$ of (4.2-2).

As a straightforward consequence, the following diagram commutes

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{can}_{\mathcal{C}(A,H)}} & \mathcal{C}(A,H) \otimes_B A \\ & \underset{\mathrm{can}_{\mathcal{L}}}{\searrow} & \Big\downarrow {\scriptstyle \psi\otimes_B \mathrm{id}_A} \\ & & \mathcal{L} \otimes_B A \end{array} \qquad \blacksquare$$

Since A is both a $(\mathcal{L}, H)$-bibundle and $(\mathcal{C}(A,H), A)$-bibundle the canonical maps $\mathrm{can}_{\mathcal{L}}$ and $\mathrm{can}_{\mathcal{C}(A,H)}$ are bijective. Then, also $\psi \otimes_B \mathrm{id}_A$ is and from the faithful flatness of A we have that ψ is bijective.

4.2.2 Category theory point of view

Let now B be an algebra and $\mathcal{L}$ a B-bialgebroid, we denote by ${}^{\mathcal{L}}\mathfrak{M}$ the category of *strict* left $\mathcal{L}$-comodules [8]. The objects M are B-bimodules endowed with a left $\mathcal{L}$-coaction $\lambda : M \longrightarrow \mathcal{L} \otimes_B M$ such that $\lambda(M) \subseteq \mathcal{L} \times_B M$ and the morphisms are B-bimodule maps preserving the $\mathcal{L}$-coaction. It is a monoidal category with respect to the tensor product $\otimes_B$ as proved in [37, Proposition 5.6].

For the case $\mathcal{L} = \mathcal{C}(A,H)$ the bialgebroid associated with a principal H-comodule algebra $B \subseteq A$ we have the following result [37, Proposition 5.16]

Proposition 4.2.8 *There is an equivalence of monoidal categories* ${}^{\mathcal{C}(A,H)}\mathfrak{M} \simeq {}^{H}\mathfrak{M}$ *given by the functor*

$${}^{H}\mathfrak{M} \ni V \longmapsto A \,\Box^H\, V \in {}^{\mathcal{C}(A,H)}\mathfrak{M}$$

This result is the algebraic version of the fact that the structure Lie group G of a principal bundle and the gauge Lie groupoid Ω are Morita equivalent. In this classical setting also the converse holds true, if a Lie groupoid is Morita equivalent to a Lie Group then it is isomorphic to a gauge Lie groupoid of some principal bundle [33, Proposition 5.14]. In the realm of Hopf-Galois extensions, we have the following

Proposition 4.2.9 (Conjecture) *Let H be a Hopf algebra with bijective antipode and $\mathcal{L}$ a B-bialgebroid. If ${}^{\mathcal{L}}\mathfrak{M} \simeq {}^{H}\mathfrak{M}$ as monoidal categories then there exists a principal comodule algebra $B \subseteq A$ such that $\mathcal{L} \simeq \mathcal{C}(A,H)$.*

Bibliography

[1] Paolo Aschieri, Pierre Bieliavsky, Chiara Pagani, and Alexander Schenkel. Noncommutative principal bundles through twist deformation. *Communications in Mathematical Physics*, 352(1):287–344, November 2016.

[2] Michael Atiyah. *K-theory*. CRC press, 2018.

[3] Gabriella Böhm. Hopf algebroids. In *Handbook of algebra*, volume 6, pages 173–235. Elsevier, 2009.

[4] Gabriella Böhm. An alternative notion of hopf algebroid. In *Hopf algebras in noncommutative geometry and physics*, pages 31–54. CRC Press, 2019.

[5] Gabriella Böhm and Kornél Szlachányi. Hopf algebroids with bijective antipodes: axioms, integrals, and duals. *Journal of Algebra*, 274(2):708–750, 2004.

[6] Thomasz Brzeziński and Shahn Majid. Quantum group gauge theory on quantum spaces. *Communications in Mathematical Physics*, 157:591–638, 1993.

[7] Tomasz Brzeziński and Piotr M Hajac. The chern–galois character. *Comptes rendus. Mathématique*, 338(2):113–116, 2004.

[8] Tomasz Brzezinski and Robert Wisbauer. *Corings and comodules*, volume 309 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.

[9] Tomasz Brzeziński. Translation map in quantum principal bundles. *Journal of Geometry and Physics*, 20(4):349–370, November 1996.

[10] Alexandru Chirvasitu. Translations in quantum groups. *arXiv preprint arXiv:1810.00370*, 2018.

[11] Alain Connes. *Noncommutative geometry*. Springer, 1994.

[12] Alain Connes and Henri Moscovici. Hopf algebras, cyclic cohomology and the transverse index theorem. *Communications in Mathematical Physics*, 198(1):199–246, 1998.

[13] Ludwik Dabrowski, Harald Grosse, and M Hajac. Strong connections and chern–connes pairing¶ in the hopf–galois theory. *Communications in Mathematical Physics*, 220:301–331, 2001.

[14] Ludwik Dabrowski, Giovanni Landi, and Jacopo Zanchettin. Hopf algebroids and twists for quantum projective spaces, 2023. http://arxiv.org/abs/2302.12073v1.

[15] Francesco D'Andrea and Giovanni Landi. Anti-selfdual connections on the quantum projective plane: instantons. *Communications in Mathematical Physics*, 333:505–540, 2015.

[16] Laiachi El Kaoutit and Niels Kowalzig. Morita theory for Hopf algebroids, principal bibundles, and weak equivalences. *Doc. Math.*, 22:551–609, 2017.

[17] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[18] Victor Ginzburg. Lectures on noncommutative geometry. *arXiv preprint math/0506603*, 2005.

[19] José M Gracia-Bondía, Joseph C Várilly, and Héctor Figueroa. *Elements of noncommutative geometry*. Springer Science & Business Media, 2013.

[20] Piotr M Hajac. Strong connections on quantum principal bundles. *Communications in mathematical physics*, 182:579–617, 1996.

[21] Piotr M Hajac and Shahn Majid. Projective module description of the q-monopole. *Communications in mathematical physics*, 206:247–264, 1999.

[22] Piotr M Hajac and Tomasz Maszczyk. Pullbacks and nontriviality of associated noncommutative vector bundles. *Journal of noncommutative geometry*, 12(4):1341–1358, 2018.

[23] Piotr M Hajac and Tomasz Maszczyk. Cyclic-homology chern–weil theory for families of principal coactions. *Communications in Mathematical Physics*, 381(2):707–734, 2021.

[24] Xiao Han and Giovanni Landi. Gauge groups and bialgebroids. *Letters in Mathematical Physics*, 111:1–43, 2021.

[25] Xiao Han and Shahn Majid. Hopf-galois extensions and twisted hopf algebroids. *Journal of Algebra*, 2023.

[26] Dale Husemöller. *Fibre bundles*, volume GTM 20. Springer, 1966.

[27] Anatoli Klimyk and Konrad Schmüdgen. *Quantum groups and their representations*. Springer Science & Business Media, 2012.

[28] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry, Volume 2*, volume 61. John Wiley & Sons, 1996.

[29] Ulrich Krähmer and Ana Rovi. A lie–rinehart algebra with no antipode. *Communications in Algebra*, 43(10):4049–4053, 2015.

[30] Herbert F Kreimer and Masaru Takeuchi. Hopf algebras and galois extensions of an algebra. *Indiana University mathematics journal*, 30(5):675–692, 1981.

[31] Jean-Louis Loday. *Cyclic homology*, volume 301. Springer Science & Business Media, 2013.

[32] Shahn Majid. *Foundations of Quantum Group Theory*. Cambridge University Press, 1995.

[33] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003.

[34] Susan Montgomery. Hopf algebras and their actions on rings. 1993.

[35] Gerald J Murphy. *C*-algebras and operator theory*. Academic press, 2014.

[36] Peter Schauenburg. Hopf bi-Galois extensions. *Comm. Algebra*, 24(12):3797–3825, 1996.

[37] Peter Schauenburg. Bialgebras over noncommutative rings and a structure theorem for hopf bimodules. *Applied Categorical Structures*, 6:193–222, 1998.

[38] Peter Schauenburg. Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras). In *New trends in Hopf algebra theory (La Falda, 1999)*, volume 267 of *Contemp. Math.*, pages 273–299. Amer. Math. Soc., Providence, RI, 2000.

[39] Peter Schauenburg and Hans-Jürgen Schneider. On generalized hopf galois extensions. *Journal of Pure and Applied Algebra*, 202(1-3):168–194, 2005.

[40] Hans-Jürgen Schneider. Principal homogeneous spaces for arbitrary hopf algebras. *Israel Journal of Mathematics*, 72:167–195, 1990.

[41] Hans Jürgen Schneider. Representation theory of hopf galois extensions. *Israel Journal of Mathematics*, 72:196–231, 1990.

[42] Hans-Jürgen Schneider. Lectures on hopf algebras. *Trabajos de Matemática*, 31:95, 1995.

[43] Mitsuhiro Takeuchi. Groups of algebras over $A \otimes \overline{A}$. *J. Math. Soc. Japan*, 29(3):459–492, 1977.

[44] K H Von Ulbrich. Vollgraduierte algebren. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 51, pages 136–148. Springer, 1981.

[45] Jacopo Zanchettin. The chern-weil homomorphism for deformed hopf-galois extensions, 2023. https://arxiv.org/abs/2312.12927.