Work statistics, quantum signatures, and enhanced work extraction in quadratic fermionic models

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In quadratic fermionic models, we determine a quantum correction to the work statistics after both a sudden quench and a time-dependent driving. Such a correction lies in the noncommutativity of the initial quantum state and the time-dependent Hamiltonian, and is revealed via the Kirkwood-Dirac quasiprobability (KDQ) approach to two-times correlators. Thanks to the latter, one can assess the onset of nonclassical signatures in the KDQ distribution of work, in the form of negative and complex values that no classical theory can reveal. By applying these concepts on the one-dimensional transverse-field Ising model, we relate nonclassical behaviors of the KDQ statistics of work in correspondence of the critical points of the model. Finally, we also prove the enhancement of the extracted work in nonclassical regimes where the noncommutativity takes a role.

I. INTRODUCTION

A fundamental quantity in nonequilibrium thermodynamics is the statistics of the work done on a quantum system by an external coherent source varying its Hamiltonian over time [1–3]. In the quantum regime, the task of determining energy-change fluctuations is still debated in the current literature, especially when considering many-body quantum systems [4–10] and addressing cases where the initial density matrix ρ_0 and the system Hamiltonian H_t are noncommuting operators [11–17].

It is known that there is no quantum observable that allow us to directly measure differences of energy values in different realizations of the system dynamics [18]. Hence, different protocols for their evaluation have been proposed in the last few decades [14,15,19–26]. In this regard, a celebrated protocol is the two-point measurement (TPM) scheme [27–30] that well reproduces the quantum work statistics when ρ_0 and H_t commute. However, in the case of noncommutativity, the results provided by the TPM scheme suffer from the uncertainty due to quantum measurement back action. On the other hand, in agreement with the no-go theorems in Refs. [15,31,32], it is also known that there is not a unique measurement scheme to characterize—quantum mechanically—the work statistics or, more in general, functions of measurement outcomes defined at two times.

In this paper, we compute the characteristic function of the work distribution for a quadratic fermionic many-body

system [33], using the Kirkwood-Dirac quasiprobability (KDQ) approach [34–37] applied to quantum thermodynamics [15,16,38]. In fact, quadratic models are of paramount importance since they allow us to investigate equilibrium and nonequilibrium properties of probability distribution functions exactly [39–43]. They give us the opportunity to rigorously inspect at quantum many-particle phenomena, thus going beyond the few-particle results but still keeping manageable the complexity of the computation.

Thanks to KDQs, we are able to determine a quantum correction to the distribution of work by thus amending the result obtained from applying the TPM scheme, which rids us off any quantum coherence of ρ_0 in the initial Hamiltonian basis. In this way, we recover the unperturbed expression of the average work, i.e., $\text{Tr}\{H_t\rho_t\} - \text{Tr}\{H_0\rho_0\}$, which exhibits a natural classical-quantum correspondence [44]. Another key property of the KDQ approach is that, by evaluating the characteristic function, one can determine whether the work statistics is originated by a quasiprobability distribution, thus with negative real terms or even complex ones. The latter cannot be reproduced by any corresponding classical theories, and for this reason their presence represents a *signature of nonclassicality* [45]. A work distribution can be nonclassical at a given time if the initial state and the system Hamiltonian do *not* commute. In such a case, a quantum correction to the work statistics needs to be applied. Here, these concepts are derived for a generic many-body fermionic system and directly linked to the critical behavior of the one-dimensional (1D) transverse field Ising model [46] across its magnetic phases. In this way, we determine that the KDQ distribution of the stochastic work becomes nonclassical across the critical point, where the noncommutativity of ρ_0 and H_t, if present, is made evident. We also find a work extraction enhancement in those regions where noncommutativity takes a role. As an

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important remark, we stress that the results of our analysis hold both in the case the initial Hamiltonian H_{t_1} of the work protocol is powered by a sudden quench and in the more realistic scenario where the Hamiltonian change $H_{t_1} \to H_{t_2}$ is enabled by a sufficiently fast ramp driving.

II. PROTOCOL

Let us take into account a quantum system starting from a state ρ_0 subjected to a time-dependent driving of its Hamiltonian parameters from time t_1 to t_2. By expressing the Hamiltonian H_t in spectral decomposition, i.e., $H_t = \sum_n E_t^{(n)} \Pi_t^{(n)}$ with $\Pi_t^{(n)} = |E_t^{(n)}\rangle\langle E_t^{(n)}|$, the probability to do/extract a given amount of stochastic work $W_{[t_1,t_2]} = E_{t_2} - E_{t_1}$ on/from the system obeys the distribution

$$P(W_{[t_1,t_2]}) = \sum_{m,n} p_{m,n} \delta\big(W - \big(E_{t_2}^{(m)} - E_{t_1}^{(n)}\big)\big), \qquad (1)$$

where $\delta(\cdot)$ is the Dirac delta, and $E_{t_1}^{(n)}$ and $E_{t_2}^{(m)}$ are, respectively, the internal energies of the system at the beginning and at the end of the driving. In fact, since the energy of the system is assumed to change according to a time-dependent driving of its Hamiltonian, the internal energy variations can be ascribed as work. Moreover, with $p_{m,n}$ we denote the joint probability of the energy at times t_1 and t_2. As argued in the Introduction, if $[\rho_0, H_{t_1}] \neq 0$ and/or $[H_{t_1}, H_{t_2}] \neq 0$, there is not an unique way of defining multitime joint probabilities in the quantum regime. In fact, quantum mechanically, one cannot obtain the statistics of outcomes originating from noncompatible quantum observable without losing information [15,31,32]. Therefore, we are going to analyze the following two possibilities:

$$p_{m,n} = \begin{cases} \text{Tr}\big\{U_{[t_1,t_2]}\Pi_{t_1}^{(n)}\rho_0\,\Pi_{t_1}^{(n)}U_{[t_1,t_2]}^{\dagger}\Pi_{t_2}^{(m)}\big\} & \text{TPM} \\ \text{Tr}\big\{U_{[t_1,t_2]}\Pi_{t_1}^{(n)}\rho_0\,U_{[t_1,t_2]}^{\dagger}\Pi_{t_2}^{(m)}\big\} & \text{KDQ}, \end{cases} \qquad (2)$$

where $U_{[t_1,t_2]} = \mathcal{T}\exp(-i\int_{t_1}^{t_2} H_t dt)$ is the time-ordered exponential of the Hamiltonian. The expressions in Eq. (2) are the joint probabilities returned by the TPM and KDQ approach, respectively, to the two-times work statistics. It is worth noting that, if $[\rho_0, H_{t_1}] = 0$, then the two schemes are equivalent. Otherwise, the noncommutativity of ρ_0 and H_{t_1} can entail that some $p_{m,n}$ are negative real numbers or even complex. In this context, the characteristic function $G(u) = \langle e^{iuW_{[t_1,t_2]}}\rangle$ of the work distribution is thus defined as

$$G(u) = \begin{cases} \text{Tr}\big\{\Delta_1(\rho_0)e^{-iuH_{t_1}}e^{iuH_{t_2}^{\text{H}}}\big\} & \text{TPM} \\ \text{Tr}\big\{\rho_0\,e^{-iuH_{t_1}}e^{iuH_{t_2}^{\text{H}}}\big\} & \text{KDQ}, \end{cases} \qquad (3)$$

where $u \in \mathbb{C}$, $H_{t_2}^{\text{H}} = U_{[t_1,t_2]}^{\dagger}H_{t_2}U_{[t_1,t_2]}$ denotes the evolution of H_{t_2} expressed in the Heisenberg representation and $\Delta_1(\rho_0) = \sum_n \Pi_{t_1}^{(n)}\rho_0\Pi_{t_1}^{(n)}$ is the diagonal part of the initial density matrix ρ_0 in the basis of H_{t_1}. Note that the characteristic function of the KDQ work distribution is the *quantum correlation function* of the operators $e^{-iuH_{t_1}}$ and $e^{iuH_{t_2}^{\text{H}}}$ that, in the general case, do not commute among them and with ρ_0, the arbitrary initial quantum state.

III. MODEL

We consider quadratic fermionic models with the Hamiltonian

$$H_t = -\sum_{k,j}(T_{kj}c_k^{\dagger}c_j + \Delta_{kj}c_k^{\dagger}c_j^{\dagger} + \text{H.c.}) - h(t)\sum_k(2n_k - 1), \qquad (4)$$

where $c_k^{\dagger}$ and c_k are the fermionic creation and annihilation operators, such that $n_k = c_k^{\dagger}c_k$ and $\{c_k, c_j^{\dagger}\} = \delta_{kj}$. $\{\cdot, \cdot\}$ denotes the anticommutator, while δ_{kj} is the Kronecker delta. Moreover, T_{kj} and Δ_{kj} are, respectively, the hopping (or tunneling) and pairing amplitudes, while $h(t)$ is the time-dependent strength of an external field. Under the assumption that the Hamiltonian H_t is translational invariant, i.e., $T_{kj} = T(|k-j|)$ and $\Delta_{kj} = \Delta(|k-j|)$, H_t admits the quadratic form

$$H_t = \sum_{p>0} \Psi_p^{\dagger}\mathbb{H}_p(t)\Psi_p \qquad (5)$$

in the momentum component p, with $\Psi_p^{\dagger} = (\tilde{c}_p^{\dagger}, \tilde{c}_{-p})$ where

$$\tilde{c}_p = \frac{e^{i\frac{\pi}{4}}}{\sqrt{L}}\sum_k e^{-ipk}c_k,$$

$$\mathbb{H}_p(t) = \begin{pmatrix} h(t) - \tilde{T}_p & \tilde{\Delta}_p \\ \tilde{\Delta}_p & \tilde{T}_p - h(t) \end{pmatrix}$$

in terms of the Fourier transform $\tilde{T}_p = \sum_r T(r)\cos(pr)$ and $\tilde{\Delta}_p = \sum_r \Delta(r)\sin(pr)$ of the hopping and pairing amplitudes, respectively. Quadratic fermionic models are quite versatile, since they can be directly mapped to quantum spin systems via the Jordan-Wigner transformation [47]. In this regard, both the transverse field Ising model and the XY model can be recovered and then analytically solved [48].

The 2×2 Hamiltonian $\mathbb{H}_p(t)$ becomes diagonal after a proper SU(2) rotation around the y axis. Formally, $\mathbb{H}_p(t) = R_y^{\dagger}(\phi_p(t))\mathbb{D}_p(t)R_y(\phi_p(t))$, with $\mathbb{D}_p(t) = \omega_p(t)\sigma^z$ and $R_y^{\dagger}(\phi_p(t)) = \exp(-i\phi_p(t)\sigma^y/2)$, where $\phi_p(t)$ denotes the Bogoliubov angle and σ^{α} ($\alpha \in \{x,y,z\}$) are Pauli matrices. We thus get the instantaneous Bogoliubov fermions $\Gamma_p(t) = (\gamma_p, \gamma_{-p}^{\dagger})^T = R_y^{\dagger}(\phi_p(t))\Psi_p$ such that, for any time t,

$$H_t = \sum_{p>0} \Gamma_p^{\dagger}(t)\mathbb{D}_p(t)\Gamma_p(t). \qquad (6)$$

During the work protocol, we vary the external field over time according to a specific time-dependent function $h(t)$. In the characteristic function $G(u)$, the full time dependence enters via the Heisenberg representation of the final Hamiltonian $H_{t_2}^{\text{H}}$, whose closed-form expression reads

$$H_{t_2}^{\text{H}} = \sum_{p>0} \Psi_p^{\dagger}\mathbb{H}_p^{[2]\text{H}}\Psi_p, \qquad (7)$$

with

$$\mathbb{H}_p^{[2]\text{H}} = \mathbb{U}_{p,t_1:t_2}^{\dagger}R_y^{\dagger}\big(\phi_p^{[2]}\big)\omega_p^{[2]}\sigma^z R_y\big(\phi_p^{[2]}\big)\mathbb{U}_{p,t_1:t_2}, \qquad (8)$$

where $\mathbb{U}_{p,t_1:t_2} = \mathcal{T}\exp(-i\int_{t_1}^{t_2}\mathbb{H}_p(s)\,ds)$ and we used the shortcut notation $\omega_p^{[j]} = \omega_p(t_j)$ and $\phi_p^{[j]} = \phi_p(t_j)$ for the sake of presentation. Note that the time-ordered exponential entering the matrix $\mathbb{U}_{p,t_1:t_2}$ is carried out from solving the

Heisenberg differential equation

$$\dot{\Psi}_p = i[H_t, \Psi_p] = -i\mathbb{H}_p(t)\Psi_p, \tag{9}$$

with $\hbar$ set to 1 and such that $\mathbb{U}_{p,t_1:t}\Psi_p = \Psi_{p,t}^{\mathrm{H}}$.

IV. KDQ CHARACTERISTIC FUNCTION OF WORK

Let us demonstrate how to analytically compute the characteristic function of the KDQ distribution of work, as well as its derivatives, for a generic quadratic fermionic model. We thus recall that $G(u)$ depends on the initial density matrix that is taken equal to $\rho_0 = \exp\{-\beta H_{t_0}\}/Z$, with $Z = \mathrm{Tr}\{e^{-\beta H_{t_0}}\}$ such that $[H_{t_0}, H_{t_1}] \neq 0$. In this regard, note that some efficient variational quantum algorithms are now available for the preparation of thermal states, such as ρ_0, on a quantum device [49–54]. It is thus shown that also the preparation of a desired thermal state does not come without a cost, as auxiliary systems (ancillae) are needed. However, high-fidelity preparation can be achieved by means of algorithmic procedures whose complexity scales polynomially with the product of the size of the quantum system one aims to initialize and the number of ancillae. Thus, by preventing the use of algorithms scaling exponentially with the system's size, an efficient initial states preparation is, in principle, feasible.

By substituting ρ_0, H_{t_1}, and $H_{t_2}^H$ in the characteristic function, one gets

$$G(u) = \prod_p \frac{g_p(u)}{g_p(0)}, \tag{10}$$

where

$$g_p(u) = \mathrm{Tr}\left\{e^{-\beta\Psi_p^\dagger\mathbb{H}_p^{[0]}\Psi_p}e^{-iu\Psi_p^\dagger\mathbb{H}_p^{[1]}\Psi_p}e^{iu\Psi_p^\dagger\mathbb{H}_p^{[2]H}\Psi_p}\right\}. \tag{11}$$

We note that the trace in Eq. (11) is evaluated in the p-momentum Fock subspace $\{|\emptyset\rangle, c_p^\dagger|\emptyset\rangle, c_{-p}^\dagger|\emptyset\rangle, c_p^\dagger c_{-p}^\dagger|\emptyset\rangle\}$, where $|\emptyset\rangle$ denotes the vacuum state such that $c_{\pm p}|\emptyset\rangle = 0$.

By repeatedly applying the group composition law of SU(2) matrices to Eq. (11), one gets

$$e^{-\beta\Psi_p^\dagger\mathbb{H}_p^{[0]}\Psi_p}e^{-iu\Psi_p^\dagger\mathbb{H}_p^{[1]}\Psi_p}e^{iu\Psi_p^\dagger\mathbb{H}_p^{[2]H}\Psi_p} = e^{\Psi_p^\dagger\mathbb{B}_p\Psi_p}. \tag{12}$$

The spectral decomposition of $\mathbb{B}_p$ provides us eigenvalues $b_p, -b_p$ such that

$$g_p(u) = \mathrm{Tr}\left\{e^{\Psi_p^\dagger\mathbb{B}_p\Psi_p}\right\} = 2(1 + \cosh b_p). \tag{13}$$

V. QUANTUM SIGNATURES IN WORK STATISTICS

The KDQ distribution of work $P(W_{[t_1,t_2]})$ can exhibit nonclassical properties, which no classical model can reproduce. Specifically, the nonclassicality of $P(W_{[t_1,t_2]})$ means that $\mathrm{Re}\,p_{m,n} < 0$ and/or $\mathrm{Im}\,p_{m,n} \neq 0$ for some indices (m, n). To witness $\mathrm{Im}\,p_{m,n} \neq 0$, one can use the following statement:

$$G^*(u) = \int dW_{[t_1,t_2]}e^{-iuW_{[t_1,t_2]}}P^*\big(W_{[t_1,t_2]}\big) = G(-u) \tag{14}$$

if and only if $\mathrm{Im}\,p_{m,n} = 0$ for any (m, n). Hence, a violation of the identity $G^*(u) = G(-u)$ is directly linked to the presence of complex values in the KDQ distribution of work. In the previous paragraph, we have determined that $G(u) =$

$\prod_p g_p(u)/g_p(0)$. As a result,

$$G^*(u) = G(-u) \quad \text{if} \quad g_p(u) = g_p^*(-u) \tag{15}$$

for any p, u. In this regard, we can determine that

$$\begin{aligned}
g_p(u) - g_p^*(-u) = &-4\sin\left(\phi_p^{[0]} - \phi_p^{[1]}\right)\sin\left(\phi_p^{[2]}\right)\sinh\left(\beta\omega_p^{[0]}\right) \\
&\times \left(\mathrm{Im}s_p^2 + \mathrm{Im}z_p^2\right)\sin\left(u\omega_p^{[1]}\right)\sin\left(u\omega_p^{[2]}\right),
\end{aligned} \tag{16}$$

where z_p and s_p are the independent variables that define the unitary 2×2 matrix

$$\mathbb{U}_{p,t_1:t_2} = \begin{pmatrix} z_p & -s_p^* \\ s_p & z_p^* \end{pmatrix} \tag{17}$$

associated to the p mode. The right-hand side (r.h.s.) of Eq. (16), witness of non-classicality, is equal to zero if $[\rho_0, H_{t_1}] = 0$, i.e., if $\phi_p^{[0]} = \phi_p^{[1]}\ \forall p$.

In the sudden quench limit, namely, when $H_{t_2}^H = H_{t_2}$, $z_p = 1$ and $s_p = 0$. Therefore, the KDQ distribution of work $P(W_{[t_1,t_2]})$ is a real-valued function of real variable. Nonetheless, we can find quantum signatures due to negative quasiprobabilities. This is shown in detail in Appendix F, where the sign of the fourth central moment $\langle(W_{[t_1,t_2]} - \langle W_{[t_1,t_2]}\rangle)^4\rangle$ is investigated for paradigmatic case studies.

Albeit nonclassical, the statistical moments of the KDQ distribution of work can still be computed from making the derivatives of the characteristic with respect to u. Specifically, for the average work, one gets

$$\langle W \rangle = L\int_0^\pi \frac{dp}{2\pi}\tanh\left(\frac{\beta\omega_p^{[0]}}{2}\right)\left(\omega_p^{[1]}Q_p^{[0,1]} - \omega_p^{[2]}Q_p^{[0,2]}\right), \tag{18}$$

where $Q_p^{[0,1]} = 2P_p^{[0,1]} - 1 = \cos\left(\phi_p^{[0]} - \phi_p^{[1]}\right)$ and $Q_p^{[0,2]} = 2P_p^{[0,2]} - 1$, with $1 - P_p^{[\ell,j]}$ denoting the *transition probability* from the energy eigenstates at time t_ℓ towards the ones at time t_j in the momentum domain (see Appendix B for the derivation). Notice that in Eq. (18), $\langle W_{[t_1,t_2]}\rangle/L =: \langle w\rangle$ becomes the *work density* in the thermodynamic limit of $L \to \infty$, with L size of the fermionic system. This result generalizes the findings of Refs. [8,10], where the work protocol operates from an initial state that commutes with the initial Hamiltonian, and the average work is computed using the TPM scheme. In Eq. (18), the TPM result is retrieved when $Q_p^{[0,1]} = 1\ \forall p$.

Equation (18) can be applied to any initial state of the form $\rho_0 = \exp\{e^{\beta H_{t_0}}\}/Z$ that does not commute, in general, with the Hamiltonian H_{t_1} at the beginning of the work protocol. As a consequence, quantum coherences in the initial energy basis also start playing a relevant role in energy fluctuations. Their effect on the average work is encoded in the parameters $Q_p^{[0,1]}$ and $Q_p^{[0,2]}$. In particular, $Q_p^{[0,1]}$ is related to the overlap between the eigenbases of the initial Hamiltonian and the initial state, respectively. Interestingly, the average work extracted by the external driving, $-\langle W_{[t_1,t_2]}\rangle$, increases as $Q_p^{[0,1]}$ approaches the minimum value -1 for any p. $Q_p^{[0,1]} = -1$ corresponds to $\phi_p^{[0]} - \phi_p^{[1]} = \pi$, meaning that the eigenbases of $\mathbb{H}_p^{[0]}$ and $\mathbb{H}_p^{[1]}$ are orthogonal. In other terms, concerning $Q_p^{[0,1]}$, the work extraction is optimized when the operators ρ_0 and H_{t_1} are maximally noncommuting. On the other hand, the

parameters $Q_p^{[0,2]}$ are associated to the transition probabilities among the instantaneous eigenstates of the (time-dependent) Hamiltonian H_t from time t_0 to t_2 as an effect of the external driving. If the transitions operated by the driving field occur in a *nonadiabatic* fashion, then part of the internal energy variation is converted in *irreversible work* [9,55–57]. Accordingly, in the general case, the $Q_p^{[0,2]}$s describe, on average, how the presence of quantum coherence affects nonadiabatic irreversible work in quadratic fermionic models.

To analyze the dynamical contribution of $Q_p^{[0,2]}$ to the average extracted work, we compare the r.h.s. of Eq. (18) with the average work obtained by applying the TPM scheme that considers the completely dephased initial state $\Delta_1(\rho_0) = \sum_n \Pi_{t_1}^{(n)} \rho_0 \Pi_{t_1}^{(n)}$. The dephasing operator does not modify the initial average energy, but the absence of initial quantum coherence in the eigenbasis of H_{t_1} unavoidably alters the final energy probabilities. Specifically, one has that

$$\langle w[\rho_0] \rangle - \langle w[\Delta_1(\rho_0)] \rangle$$
$$= \frac{1}{L} \mathrm{Tr}\left\{ [\rho_0 - \Delta_1(\rho_0)] H_{t_2}^H \right\}$$
$$= \int_0^\pi \frac{dk}{2\pi} \tanh\left(\frac{\beta \omega_p^{[0]}}{2} \right) \omega_p^{[2]} (Q_p^{[0,1]} Q_p^{[1,2]} - Q_p^{[0,2]}). \quad (19)$$

Hence, enhanced energy extraction can be obtained in a finite and connected region of parameters. A significant advantage, with respect to what is returned by the TPM scheme, is always achieved when $Q_p^{[0,1]} = 0$ and $Q_p^{[0,2]} > 0$. Remarkably, as shown in the next paragraph, these conditions are originated by the interplay between quantum coherences and quantum critical points.

VI. ENHANCED ENERGY EXTRACTION IN THE QUANTUM ISING MODEL

To conclude the analysis, we show analytical and numerical results, concerning the enhancement of work extraction aided by nonclassicality, in the concrete example of the quantum transverse field Ising model, which Hamiltonian reads

$$H = -\sum_{j=1}^L \sigma_j^x \sigma_{j+1}^x + h \sum_{j=1}^L \sigma_j^z, \quad (20)$$

where σ_j^α are the local Pauli matrices, with $\alpha = x, y, z$, $j = 1, \ldots, L$, and L the size of the spin system. The diagonalization of this model and its mapping to a fermionic model [47,58] are detailed in Appendix A. Following the protocol introduced above, we initialize the system in $\rho_0 = \exp\{-\beta H_{t_0}\}/Z$. ρ_0 is identified by the magnetic field $h = h_0$, while the Hamiltonian H_{t_1} at the beginning of the work protocol is with $h = h_1$. Afterwards, we quench the external field from $h = h_1$ to $h = h_2$. Figure 1(a) shows the average work density exchanged during this driving process as a function of h_0 and h_1 with fixed $h_2 = 0.5$ and $\beta = 15$. Energy extraction corresponds to negative values of the average work density. This has to be compared with Fig. 1(b), where we plot $\overline{Q_p^{[0,1]}} = \int_0^\pi Q_p^{[0,1]} dp/\pi$ as an estimate of the average overlap between the eigenbases of H_{t_0} and H_{t_1}. In agreement with our general analysis, the amount of energy
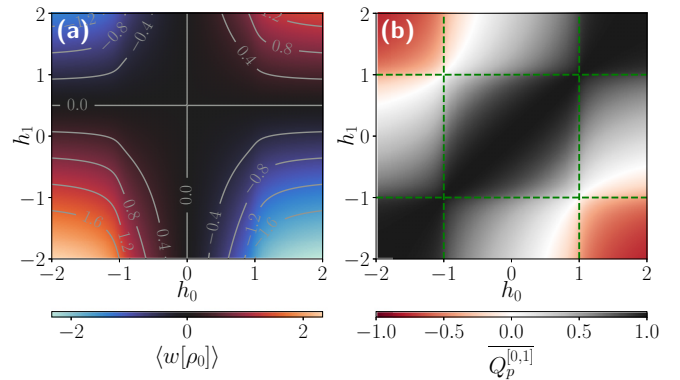

FIG. 1. (a) Average work density originated by quenching the quantum Ising Hamiltonian from h_1 to $h_2 = 0.5$ for different values of h_0. The inverse temperature of the initial state $\rho_0 = \exp\{-\beta H_{t_0}\}/Z$ is taken equal to $\beta = 15$. (b) Average overlaps between the eigenbases of H_{t_1} and H_{t_2}.

extraction increases in the parameters region where $\overline{Q_p^{[0,1]}}$ is negative, i.e., the quantum coherences of ρ_0 in the basis of H_{t_1} significantly change the work density. Interestingly, the optimal region for energy extraction corresponds to $h_0 > 1$ in the ferromagnetic phase and $h_1 < -1$ in the antiferromagnetic phase or vice versa. In Fig. 2(a), we plot the r.h.s. of Eq. (19) as a function of h_0 and h_1, with $h_2 = 0.5$ and $\beta = 15$. It represents the quantum correction to the average work density, provided by an extra contribution (than the TPM scheme) due to noncommutativity. In this regard, in Fig. 2(a), we also highlight a region (green shaded area) where the sign of the fourth central moment of the work distribution is negative. This signals the presence of a nonclassical region, corresponding to the fact that the KDQ distribution of work has negative values. Accordingly, in such a nonclassical
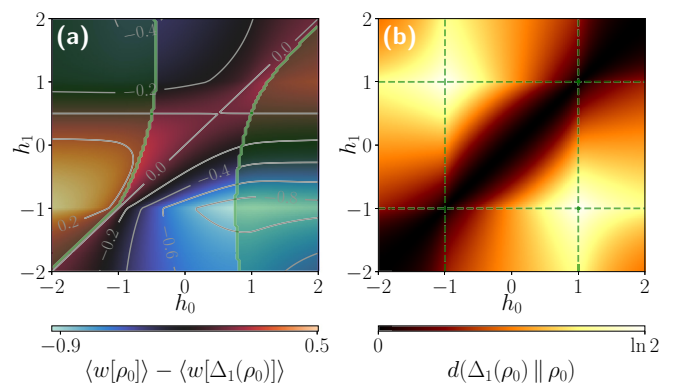

FIG. 2. (a) Quantum correction to the average work density in the quantum Ising model, using the same Hamiltonian as in Fig. 1(a), and enhancement of the amount of extracted work due to noncommutativity of the Hamiltonian operators at different times of the work protocol. The shaded green areas mark the nonclassical region of parameters where the fourth central moment of the KDQ distribution of work is negative. (b) Relative entropy of quantum coherence density, $D(\rho_0 || \Delta_1(\rho_0))/L$, as a function of h_0 and h_1. The inverse temperature of the initial state is taken equal to $\beta = 15$.

region, $\langle w[\rho_0] \rangle < 0$ witnesses an enhancement of energy extraction that is boosted by quantum coherence of the initial state ρ_0, expressed in the basis of H_{t_1}.

For quadratic fermionic models, we have already argued that $Q_p^{[0,1]} = 0$ and $Q_p^{[0,2]} > 0$ always allow for energy extraction enhancement. In Fig. 1(b) [details are in Appendix C], we specialize our analysis to the quantum Ising model where the maximum enhancement in energy extraction Eq. (19) is exactly satisfied by choosing $h_0 = \pm 1$ and $h_1 = \mp 1$, i.e., in parameter regions where the initial state and the initial Hamiltonian sit at the two opposite quantum critical points of the model. Remarkably, these points also correspond to the maximum of the *relative entropy of quantum coherences* [59,60] that is defined as

$$D(\rho_0 || \Delta_1(\rho_0)) = \mathrm{Tr}\{\rho_0(\ln \rho_0 - \ln \Delta_1(\rho_0))\}. \quad (21)$$

Its density $d(\rho_0 || \Delta_1(\rho_0)) = D(\rho_0 || \Delta_1(\rho_0))/L$ is thus plotted in Fig. 2(b) as a function of h_0 and h_1. Consequently, the correspondence between the maximum value of $D(\rho_0 || \Delta_1(\rho_0))$ and the enhancement of the work extraction benchmarks that this advantage originates from the noncommutativity of the initial state ρ_0 with the Hamiltonian H_{t_1} at the beginning of the work protocol. At the critical points of the quantum Ising model, it becomes a quite useful thermodynamic resource.

VII. LANDAU-ZENER-STÜCKELBERG-MAJORANA (LZSM) DYNAMICS

The above results are obtained by assuming a sudden change of the initial Hamiltonian with a quench dynamics. However, the validity of such findings can be also confirmed by changing the Hamiltonian of the work protocol via a sufficiently fast, but time-finite, ramp drive. Specifically, the magnetic field $h(t)$ varies over time as a linear ramp at velocity δ, i.e.,

$$h(t) = h_1 + (t - t_1)\delta, \quad (22)$$

with $t \in [t_1, t_2]$ and $\delta = (h_2 - h_1)/(t_2 - t_1)$. In this case, the system evolution is still exactly solvable since it can be decomposed in the dynamics of L independent Landau-Zener-Stückelberg-Majorana models [61,62], one for each Fourier mode.

To properly take into account changes of the Hamiltonian at finite velocity, we have to solve the following time-dependent equation for

$$\mathbb{U}_{p,t_1:t} = \begin{pmatrix} z_p(t) & -s_p^*(t) \\ s_p(t) & z_p^*(t) \end{pmatrix} \quad (23)$$

namely,

$$i\partial_t \begin{pmatrix} c_p(t) \\ c_{-p}^\dagger(t) \end{pmatrix} = [(h(t) - \cos p)\sigma^z - (\sin p)\sigma^x] \begin{pmatrix} c_p(t) \\ c_{-p}^\dagger(t) \end{pmatrix}. \quad (24)$$

The solution of Eq. (24) can be determined by introducing the ansatz

$$c_p(t) = z_p(t)c_p - s_p^*(t)c_{-p}^\dagger$$
$$c_{-p}^\dagger(t) = s_p(t)c_p + z_p^*(t)c_{-p}^\dagger, \quad (25)$$

where the time dependence of the fermionic operators is encoded in the parameters $z_p(t)$ and $s_p(t)$. Hence, the equation for $c_p(t)$ can be mapped into an equation for $z_p(t)$ and $s_p(t)$. In fact, from

$$i\dot{z}_p(t)c_p - i\dot{s}_p^*(t)c_{-p}^\dagger = [h(t) - \cos p](z_p(t)c_p - s_p^*(t)c_{-p}^\dagger)$$
$$- \sin p(s_p(t)c_p + z_p^*(t)c_{-p}^\dagger), \quad (26)$$

we end up with the following equation in matrix form:

$$i\begin{pmatrix} \dot{z}_p(t) \\ \dot{s}_p(t) \end{pmatrix} = \begin{pmatrix} \Omega_p(t) & \Delta_p \\ \Delta_p & -\Omega_p(t) \end{pmatrix}\begin{pmatrix} z_p(t) \\ s_p(t) \end{pmatrix} = \mathbb{H}_p(t)\begin{pmatrix} z_p(t) \\ s_p(t) \end{pmatrix}, \quad (27)$$

where $\Omega_p(t) = h(t) - \cos p$ and $\Delta_p = -\sin p$, with initial conditions $z_p(t_1) = 1$ and $s_p(t_1) = 0$. Being dependent on each other, the first-order equations in Eq. (27) can be thus recast into a second-order equation for $z_p(t)$, i.e.,

$$i\ddot{z}_p = \dot{\Omega}_p z_p + \Omega_p \dot{z}_p + \Delta_p \dot{s}_p$$
$$= \delta z_p - i\Omega_p[\Omega_p z_p + \Delta_p s_p] - i\Delta_p[\Delta_p z_p - \Omega_p s_p], \quad (28)$$

that reduces to

$$\ddot{z}_p + [\Omega_p^2(t) + \Delta_p^2 + i\delta]z_p = 0. \quad (29)$$

This equation has to be solved with initial conditions $z_p(t_1) = 1$ and $\dot{z}_p = -i\Omega_p(t_1)$, while

$$s_p(t) = \frac{1}{\Delta_p}[i\partial_t - \Omega_p(t)]z_p(t). \quad (30)$$

The general solution for $\mathbb{U}_{p,t_1:t_2}$ is readily available in terms of the Weber D functions [61]. From the latter, we can compute the exact transition probabilities as well as $Q_p^{[0,2]}$. In this way, from Eq. (18), we can also determine the explicit average work that is originated by a linear driving.

In Fig. 3, we plot the difference of the average work densities obtained by initializing the system in the quantum state ρ_0 and in its dephased counterpart $\Delta_1(\rho_0)$, respectively. In the figure, both the fast ($\delta = 4$) and slow ($\delta = 0.5$) drivings are considered. It is worth noting that the average work density $\langle w[\rho_0] \rangle$ witnesses an enhancement of the work extraction, which is boosted by the noncommutativity of ρ_0 with H_{t_1}. The latter makes available quantum coherence of the initial quantum state ρ_0 that is *robust* against finite but sufficiently fast velocity of the driving. In the opposite limit of an infinitely slow driving, the difference $\langle w[\rho_0] \rangle - \langle w[\Delta_1(\rho_0)] \rangle$ approaches zero whenever the driving ramp does not cross a quantum critical point, while it takes a finite value around the criticality [see Fig. 3(f)]. This may be ascribed to the fact that, in case the adiabatic hypothesis is satisfied, $Q_p^{[1,2]} \to 1$ and $Q_p^{[0,2]} \to Q_p^{[0,1]}$ as $\delta \to 0$, thus leading to a zero average work difference. However, the adiabatic hypothesis is violated whenever a gapless quantum critical point is crossed during a
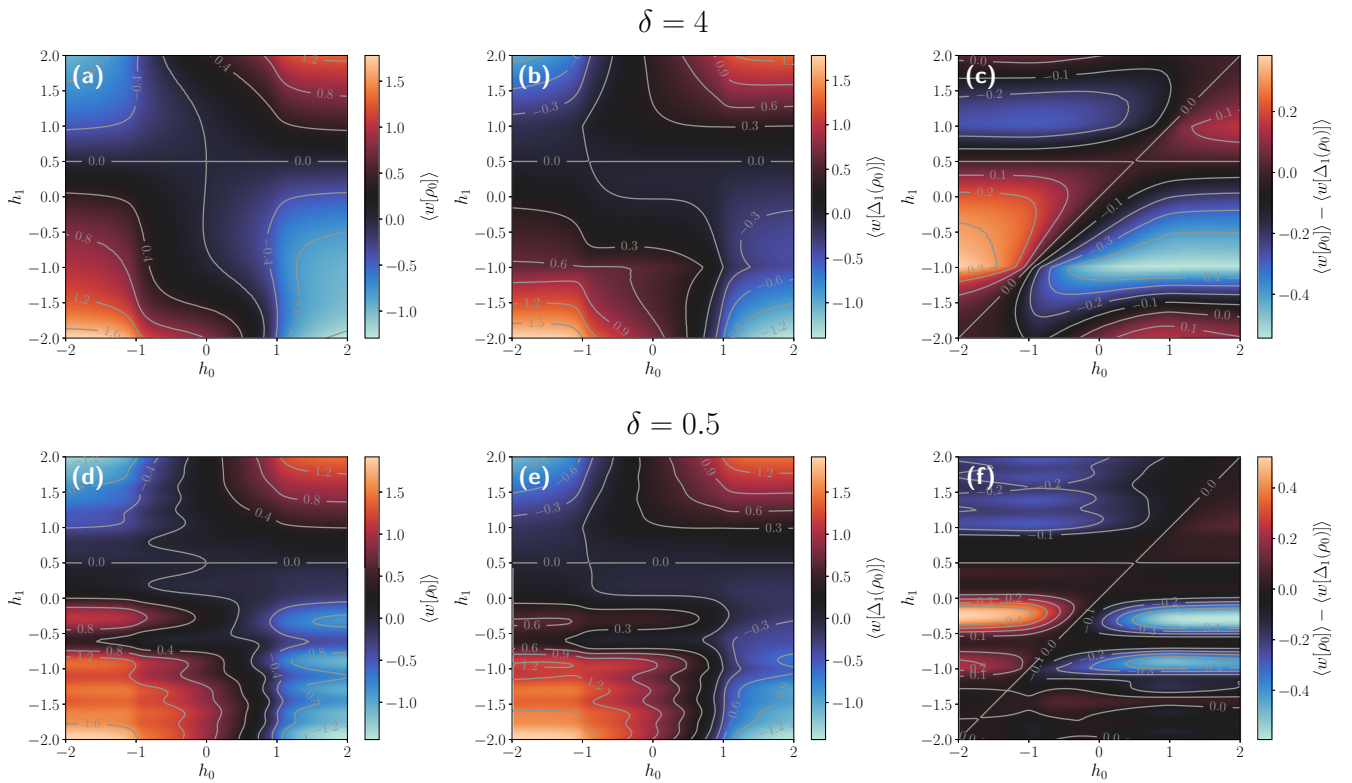
FIG. 3. Average work density of the quantum Ising model obtained by initializing the system in the quantum state ρ_0 with magnetic field h_0 [(a), (d)] or in the dephased state $\Delta_1(\rho_0)$ [(b), (e)]. In all cases, the Hamiltonian changes are due to a linear driving of the magnetic field $h_1 \to h_2$. (c), (f) Enhancement of work extraction. A faster driving allows for the extraction of a larger amount of work. (a)–(c) Fast driving with velocity $\delta = 4$; (d)–(f) slow driving with velocity $\delta = 0.5$.

quantum dynamics. Such violation allows for a finite enhancement of extracted work that also persists for slow drivings.

VIII. CONCLUSIONS

We have analyzed a fundamental aspect of quantum thermodynamics when applied to many-body systems, namely, the understanding of how the noncommutativity between a generic initial quantum state and the Hamiltonian of the system, evaluated at least over two times, modifies work fluctuations. To answer this question, we make use of the KDQ approach, introduced for two-time quantum correlators, by obtaining analytical results for a generic quadratic fermionic model, among which the transverse field Ising model.

In summary, beyond determining a correction to the work statistics that strictly depends on noncommutativity, the main outcome of our analysis is to show a connection between the following concepts: (i) nonclassical signatures in the KDQ distribution of work, (ii) enhanced work extraction under nonadiabatic coherent driving and noncommutativity of ρ_0 and H_t during the work protocol, and (iii) critical behaviors in quantum fermionic systems across different matter phases.

We believe that our study can open a research line for determining nonclassical fluctuations of thermodynamic quantities in a generic quantum many-body system thanks to the use of KDQ distributions. In this regard, we propose the application

of this quasiprobability approach to shed light, with a unique perspective, at the quantum nature of the phase transitions occurring in spin and fermionic many-body systems. Finally, in line with Refs. [57,63], our results about the enhancement of work extraction would be worth further investigation for the design of quantum heat engines [64–67] and quantum batteries [68,69] that are powered by genuinely quantum features and show a clear quantum advantage.

Note added. Recently, Ref. [70] appeared, dealing with topics related to the ones discussed by us.

APPENDIX A: FROM QUANTUM ISING MODEL TO QUADRATIC FERMIONIC HAMILTONIANS

The Hamiltonian operator of the transverse field quantum Ising model reads

$$H_t = -J \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x + h(t) \sum_{j=1}^{L} \sigma_j^z, \quad \text{(A1)}$$

where σ_j^α are the local Pauli matrices, with $\alpha = x, y, z$ and $j = 1, \ldots, L$. It thus holds that $[\sigma_i^\alpha, \sigma_j^\beta] = 2i\delta_{ij}\epsilon^{\alpha\beta\gamma}\sigma_j^\gamma$, where δ denotes the Kronecker delta and ϵ is the Levi-Civita symbol. We set $J = 1$ to fix the energy scale of the system. At zero temperature, the model has a ferro-/paramagnetic phase transition for $h = \pm 1$. As is customary, we transform the Hamiltonian Eq. (A1) by means of the Jordan-Wigner transformation [47,58]

$$\sigma_k^x = \prod_{j=1}^{k-1}(1 - 2n_j)(c_k^\dagger + c_k),$$

$$\sigma_k^y = i\prod_{j=1}^{k-1}(1 - 2n_j)(c_k^\dagger - c_k), \qquad \text{(A2)}$$

$$\sigma_k^z = 1 - 2n_k,$$

where $\{c_i, c_j^\dagger\} = \delta_{ij}$ and $n_j = c_j^\dagger c_j$. After the Jordan-Wigner transformation, the Hamiltonian takes the following form:

$$H_t = -\sum_{j=1}^{L}(c_j^\dagger c_{j+1} + c_j^\dagger c_{j+1}^\dagger + \text{H.c.}) - h(t)\sum_{j=1}^{L}(2c_j^\dagger c_j - 1).$$

$$\text{(A3)}$$

Since the model is translationally invariant, the Hamiltonian H_t can be diagonalized by means of the discrete Fourier transform

$$c_j = \frac{e^{-i\pi/4}}{\sqrt{L}}\sum_p e^{ipj}\tilde{c}_p, \quad \tilde{c}_p = \frac{e^{i\pi/4}}{\sqrt{L}}\sum_{j=1}^{L} e^{-ipj}c_j, \quad \text{(A4)}$$

with $p = 2\pi m/L$ and $m = -L/2 + 1, \ldots, L/2$. Moreover, the invariance under the inversion symmetry $p \to -p$ allows us to restrict the computations to positive momenta by defining $\Psi_p = (\tilde{c}_p, \tilde{c}_{-p}^\dagger)^T$. Therefore,

$$H_t = \sum_{p>0}\Psi_p^\dagger \mathbb{H}_p(t)\Psi_p, \qquad \text{(A5)}$$

where

$$\mathbb{H}_p(t) = \begin{pmatrix} -2\cos p + 2h(t) & -2\sin p \\ -2\sin p & 2\cos p - 2h(t) \end{pmatrix}. \quad \text{(A6)}$$

The $R_y(\phi_p(t)) = \exp(i\phi_p\sigma^y/2)$ rotation applied to the new fermions $\Gamma_p(t) = (\gamma_p, \gamma_{-p}^\dagger)^T$ diagonalizes the problem. Formally, one has

$$\Psi_p = \exp\left(i\frac{\phi_p(t)}{2}\sigma^y\right)\Gamma_p(t) \equiv R_y(\phi_p(t))\Gamma_p(t), \quad \text{(A7)}$$

where the eigenvectors of the rotation are given by

$$v_{+,p} = (\cos\phi_p(t)/2 \quad \sin\phi_p(t)/2)^T,$$

$$v_{-,p} = (-\sin\phi_p(t)/2 \quad \cos\phi_p(t)/2)^T. \quad \text{(A8)}$$

The rotation angles ϕ_p are implicitly defined by the conditions $\cos\phi_p(t) = 2(h(t) - \cos p)/\omega_p(t)$ and $\sin\phi_p(t) = 2\sin p/\omega_p(t)$; note that $\phi_p(t) = -\phi_{-p}(t)$. The Hamiltonian, written in terms of the new fermionic operators, then reads

$$H_t = \sum_{p>0}\Gamma_p^\dagger(t)\mathbb{D}_p(t)\Gamma_p(t)$$

$$= \sum_{p>0}\Psi_p^\dagger R_y^\dagger(\phi_p(t))\omega_p(t)\sigma^z R_y(\phi_p(t))\Psi_p, \quad \text{(A9)}$$

where $\mathbb{D}_p(t) = \omega_p(t)\sigma^z$ and the energies of each mode are given by $\omega_p(t) = 2\sqrt{(\cos p - h(t))^2 + \sin^2 p}$, with $\omega_p(t) = \omega_{-p}(t)$.

APPENDIX B: DERIVATION OF THE AVERAGE WORK

In this Appendix, we show an alternative derivation of Eq. (18) in the main text by directly computing

$$\langle W_{[t_1,t_2]}\rangle = \text{Tr}\{U\rho_0 U^\dagger H_{t_2}\} - \text{Tr}\{\rho_0 H_{t_1}\}. \quad \text{(B1)}$$

First, we consider the Hamiltonian H_t in diagonalized form, i.e.,

$$H_{t_j} = \sum_{p>0}\Psi_p^\dagger \mathbb{H}_p^{[j]}\Psi_p = \sum_{p>0}\omega_p^{[j]}\left(\gamma_p^{[j]\dagger}\gamma_p^{[j]} - \gamma_{-p}^{[j]}\gamma_{-p}^{[j]\dagger}\right).$$

$$\text{(B2)}$$

Then, from Eq. (B2), the initial state can be written as

$$\rho_0 = \frac{e^{-\beta H_{t_0}}}{Z^{[0]}},$$

$$= \prod_{p>0}\frac{\exp\left\{-\beta\omega_p^{[0]}\left(n_p^{[0]} + n_{-p}^{[0]} - 1\right)\right\}}{Z_p^{[0]}} \equiv \prod_{p>0}\rho_p^{[0]}, \quad \text{(B3)}$$

where we have defined the number operator in the p-momentum subspace $n_{\pm p}^{[0]} = \gamma_{\pm p}^{[0]\dagger}\gamma_{\pm p}^{[0]}$, and $Z^{[0]} = \text{Tr}e^{-\beta H^{[0]}} = \prod_{p>0}Z_p^{[0]} = \prod_{p>0}2(1 + \cosh(\beta\omega_p^{[0]}))$. To compute the average work, we have to consider

$$\text{Tr}\rho_0 U^\dagger H^{[2]}U = \prod_{q>0}\text{Tr}\rho_q^{[0]}\sum_{p>0}H_p^{[2]\text{H}} = \sum_{p>0}\text{Tr}\rho_p^{[0]}\Psi_p^\dagger \mathbb{U}_{p,t_1:t_2}^\dagger R_y^\dagger(\phi_p^{[2]})\omega_p^{[2]}\sigma^z R_y(\phi_p^{[2]})\mathbb{U}_{p,t_1:t_2}\Psi_p$$

$$= \sum_{p>0}\text{Tr}\rho_p^{[0]}\Gamma_p^{[0]\dagger}R_y^\dagger(\phi_p^{[0]})\mathbb{U}_{p,t_1:t_2}^\dagger R_y^\dagger(\phi_p^{[2]})\omega_p^{[2]}\sigma^z R_y(\phi_p^{[2]})\mathbb{U}_{p,t_1:t_2}R_y(\phi_p^{[0]})\Gamma_p^{[0]} \equiv \sum_{p>0}\text{Tr}\rho_p^{[0]}\Gamma_p^{[0]\dagger}\mathbb{M}\Gamma_p^{[0]}, \quad \text{(B4)}$$

with $\Psi_p = R_y(\phi_p^{[0]})\Gamma_p^{[0]}$ and

$$\mathbb{M} = R_y^\dagger(\phi_p^{[0]})\mathbb{U}_{p,t_1:t_2}^\dagger R_y^\dagger(\phi_p^{[2]})\omega_p^{[2]}\sigma^z R_y(\phi_p^{[2]})\mathbb{U}_{p,t_1:t_2}R_y(\phi_p^{[0]}). \tag{B5}$$

Then, by computing the trace over the Fock space $\{|\emptyset\rangle, \gamma_p^\dagger|\emptyset\rangle, \gamma_{-p}^\dagger|\emptyset\rangle, \gamma_p^\dagger\gamma_{-p}^\dagger|\emptyset\rangle\}$, we can find that

$$\begin{aligned}
&\mathrm{Tr}\{\rho_p^{[0]}\Gamma_p^{[0]\dagger}\mathbb{M}\Gamma_p^{[0]}\} \\
&\quad = \mathbb{M}_{11}\mathrm{Tr}\{\rho_p^{[0]}n_p^{[0]}\} + \mathbb{M}_{22}\mathrm{Tr}\{\rho_p^{[0]}(1-n_{-p}^{[0]})\} \\
&\quad = \mathbb{M}_{11}\frac{1+e^{-\beta\omega_p^{[0]}}}{Z_p^{[0]}} + \mathbb{M}_{22}\frac{e^{\beta\omega_p^{[0]}}+1}{Z_p^{[0]}}.
\end{aligned} \tag{B6}$$

From the cyclic property of the trace, it is easy to check that $\mathrm{tr}\{\mathbb{M}\} = 0$, and therefore $\mathbb{M}_{11} = -\mathbb{M}_{22}$, whereby

$$\begin{aligned}
\mathrm{Tr}\{\rho_p^{[0]}\Gamma_p^{[0]\dagger}\mathbb{M}\Gamma_p^{[0]}\} &= \mathbb{M}_{22}\frac{e^{\beta\omega_p^{[0]}}-e^{-\beta\omega_p^{[0]}}}{Z_p^{[0]}} \\
&= \frac{\mathbb{M}_{22}}{2}\tanh\frac{\beta\omega_p^{[0]}}{2}.
\end{aligned} \tag{B7}$$

If we define $\hat{e}_+ = (1 \quad 0)^T$ and $\hat{e}_- = (0 \quad 1)^T$, $\mathbb{M}_{22}$ can be expressed as

$$\begin{aligned}
\mathbb{M}_{22} = \omega_p^{[2]}\big[\hat{e}^T R_y^\dagger(\phi_p^{[0]})\mathbb{U}_p^\dagger(t)R_y^\dagger(\phi_p^{[2]})(\hat{e}_+ \cdot \hat{e}_+^T \\
- \hat{e}_- \cdot \hat{e}_-^T)R_y(\phi_p^{[2]})\mathbb{U}_p(t)R_y(\phi_p^{[0]})\hat{e}_-\big].
\end{aligned} \tag{B8}$$

From the explicit expression of the eigenvectors in Eqs. (A8), $R(\phi_p^{[j]}) = (v_{+,p}^{[j]}|v_{-,p}^{[j]})$, so $R(\phi_p^{[j]})\hat{e}_i = v_{i,p}^{[j]}$ with $i = \pm$. Therefore,

$$\begin{aligned}
\mathbb{M}_{22} &= \omega_p^{[2]}\big[|v_{+,p}^{[2]\dagger}\mathbb{U}_{p,t_1:t_2}v_{-,p}^{[0]}|^2 - |v_{-,p}^{[2]\dagger}\mathbb{U}_{p,t_1:t_2}v_{-,p}^{[0]}|^2\big] \\
&= -\omega_p^{[2]}\big[P_p^{[0,2]} - (1-P_p^{[0,2]})\big] = -\omega_p^{[2]}(2P_p^{[0,2]}-1),
\end{aligned} \tag{B9}$$

where $P_p^{[0,2]}$ denotes the probability of not transitioning between the instantaneous eigenstates of $H_p^{[2]}$. It is worth pointing out that the transition matrix with elements $\mathcal{P}_{i,j}^{[0,2]} = |v_{i,p}^{[2]\dagger}\mathbb{U}_{p,t_1:t_2}v_{j,p}^{[0]}|^2$ is bistochastic, which implies that $\sum_i \mathcal{P}_{i,j}^{[0,2]} = \sum_j \mathcal{P}_{i,j}^{[0,2]} = 1$. As a result, we can define

$$\begin{aligned}
|v_{+,p}^{[2]\dagger}\mathbb{U}_{p,t_1:t_2}v_{+,p}^{[0]}|^2 &= |v_{-,p}^{[2]\dagger}\mathbb{U}_{p,t_1:t_2}v_{-,p}^{[0]}|^2 = P_p^{[0,2]} \\
|v_{+,p}^{[2]\dagger}\mathbb{U}_{p,t_1:t_2}v_{-,p}^{[0]}|^2 &= |v_{-,p}^{[2]\dagger}\mathbb{U}_{p,t_1:t_2}v_{+,p}^{[0]}|^2 = 1 - P_p^{[0,2]}.
\end{aligned} \tag{B10}$$

Moreover, $\mathrm{Tr}\{\rho_0 H^{[1]}\}$ is obtained setting $\mathbb{U}_{p,t_1:t_2} = \mathbb{I}_p$. Hence, substituting $2 \to 1$ concludes our derivation of the average work that leads to

$$\begin{aligned}
\langle W_{[t_1,t_2]}\rangle = \frac{L}{2\pi}\int_0^\pi dp \tanh\left(\frac{\beta\omega_p^{[0]}}{2}\right)\big[\omega_p^{[1]}(2P_p^{[0,1]}-1) \\
- \omega_p^{[2]}(2P_p^{[0,2]}-1)\big].
\end{aligned} \tag{B11}$$

The formula above reduces to the expression for the average work shown in the main text upon introducing the parameters $Q_p^{[i,j]} = 2P_p^{[i,j]} - 1$. More details on the properties of such parameters as functions of the Fourier modes and for different

values of the chemical potential h are provided in the next Appendix.

APPENDIX C: TRANSITION PROBABILITIES AND BOGOLIUBOV ANGLES

We here provide a detailed analysis of the properties of both the transition probabilities $P_p^{[i,j]}$ and the parameters $Q_p^{[i,j]} = (2P_p^{[i,j]} - 1)$ entering the expression for the average work in the main text. Let us start from $Q_p^{[0,1]}$; this quantity describes the overlap between the eigenbasis of the initial Hamiltonian H_{t_1}, with chemical potential h_1, and the eigenbasis of the initial state ρ_0. The latter is chosen of the form $\rho_0 = e^{-\beta H_{t_0}}/Z$, where H_{t_0} is a quadratic fermionic Hamiltonian with chemical potential h_0. Accordingly, $Q_p^{[0,1]}$ can be written in terms of the Bogoliubov angles $\phi_p^{[0]}$, $\phi_p^{[1]}$ associated to the rotations that diagonalize the Hamiltonians H_{t_0} and H_{t_1}, i.e.,

$$Q_p^{[0,1]} = \cos\left(\phi_p^{[0]} - \phi_p^{[1]}\right). \tag{C1}$$

It thus follows that the properties of $Q_p^{[0,1]}$ derive directly from the properties of the Bogoliubov angles. Figure 4 shows the Bogoliubov angles as a function of the Fourier modes p in the different phases of the model, i.e., for different values of the control parameter h. One can note that the different phases of the model can be identified by behavior of the Bogoliubov angles at $p = 0, \pi$. In fact, for $p \to 0$ we have that

$$\lim_{p\to 0}\phi_p = \begin{cases} 0 & \text{if } h > 1 \\ \pi/2 & \text{if } h = 1 \\ \pi & \text{if } h < 1, \end{cases} \tag{C2}$$

with a jump discontinuity at the critical point $h = 1$. Similarly, for $p \to \pi$ we find

$$\lim_{p\to\pi}\phi_p = \begin{cases} 0 & \text{if } h > -1 \\ \pi/2 & \text{if } h = -1 \\ \pi & \text{if } h < -1, \end{cases} \tag{C3}$$

with another discontinuity in correspondence with the second critical point $h = -1$. Moreover, note that for $h = 0, \pm 1$ the Bogoliubov angle ϕ_p takes the simple form

$$\phi_p = \begin{cases} \pi/2 - p/2 & \text{if } h = 1 \\ \pi - p & \text{if } h = 0 \\ -p/2 & \text{if } h = -1. \end{cases} \tag{C4}$$

These properties of the Bogoliubov angles translate into properties of $Q_p^{[0,1]}$, which are estimated in the main text through their average over the Fourier modes: $\overline{Q_p^{[i,j]}} = \frac{1}{\pi}\int_0^\pi Q_p^{[i,j]}dp$.

APPENDIX D: SYMMETRIES OF THE AVERAGE WORK

In this Appendix, we provide a proof that the average work $\langle W_{[t_1,t_2]}\rangle$ (from here on simply denoted with $\langle W(h)\rangle$ unless specified) is an *odd function* of h_0, i.e.,

$$\langle W(h_0)\rangle = -\langle W(-h_0)\rangle. \tag{D1}$$

For the sake of presentation, we are going to explicitly write the dependence on $h(t)$, namely, $h_j = h(t_j)$.

Let us thus start from a summary of properties of both the single-particle spectrum and the Bogoliubov angles, which are
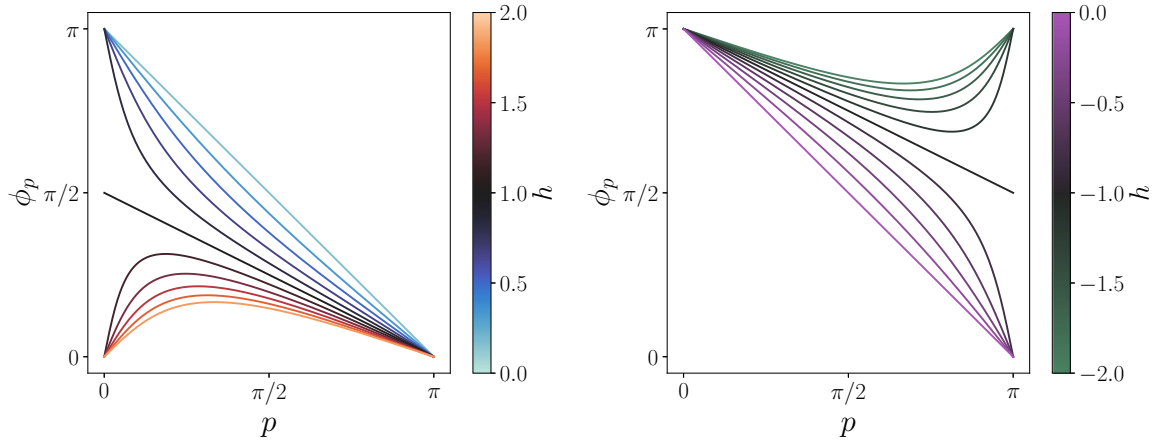
FIG. 4. Rotation angles for different values of h as a function of p; left panel, ferromagnetic phase; right panel, paramagnetic phase.

necessary to carry out the proof. In particular, directly from their definitions in the main text, we find that

$$\omega_p(h) = \omega_{-p}(h) = \omega_{\pi-p}(-h), \tag{D2}$$

while for the Bogoliubov angles it holds that

$$\phi_p(h) = -\phi_{-p}(h) = \pi - \phi_{\pi-p}(-h). \tag{D3}$$

With the above properties in mind, we consider the average work that can be expressed as

$$\langle W(h_0) \rangle = \frac{L}{2\pi} \int_0^\pi dp \tanh\left(\frac{\beta \omega_p(h_0)}{2}\right)$$
$$\times [\omega_p(h_1) \cos(\phi_p(h_0) - \phi_p(h_1))$$
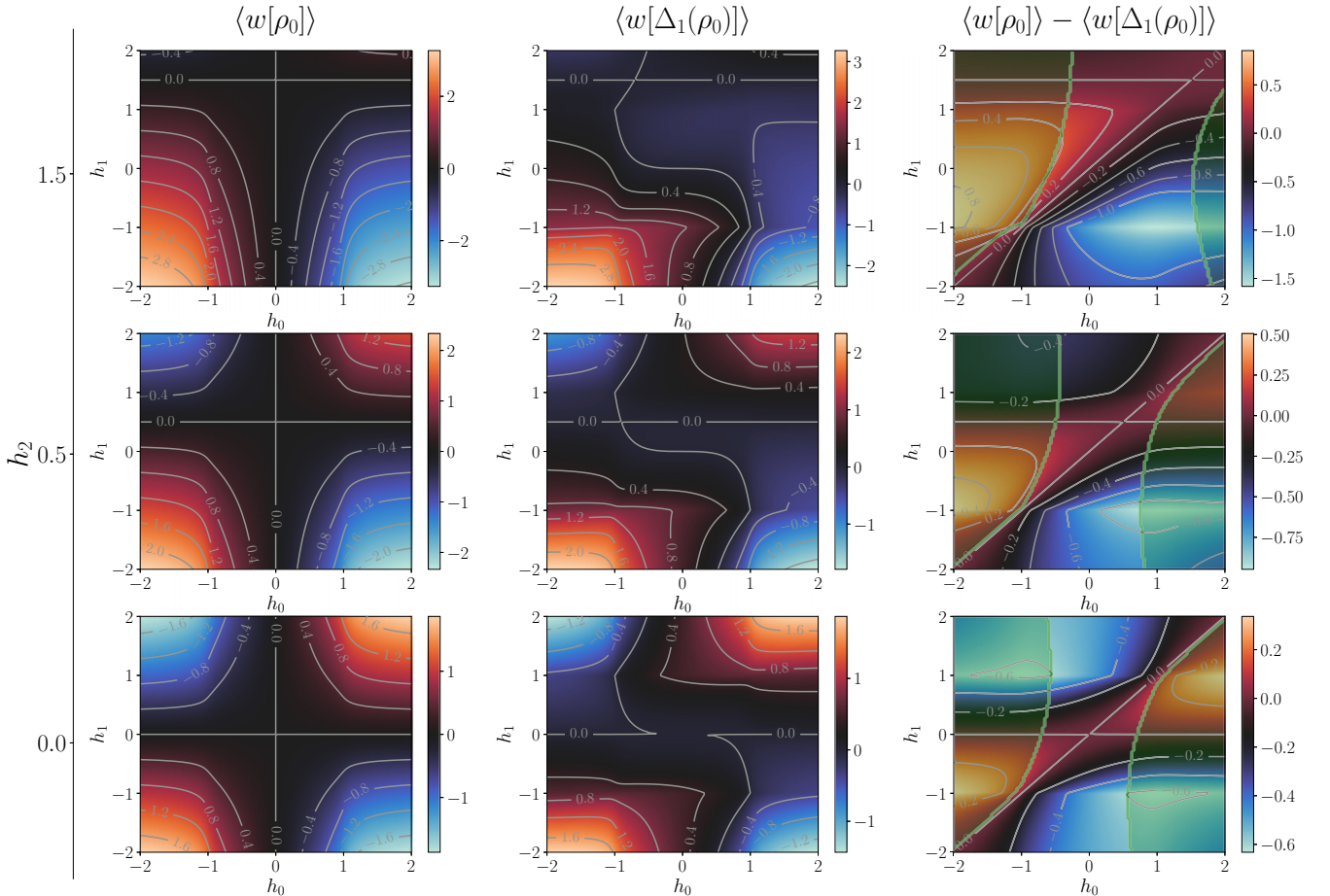$$- \omega_p(h_2) \cos(\phi_p(h_0) - \phi_p(h_2))], \tag{D4}$$

FIG. 5. Average work density after a sudden quench from h_1 to h_2 for ρ_0, its dephased counterpart $\Delta_1(\rho_0)$, and the difference of the two. In all plots, $\beta = 15$.

where we have used the fact that $Q_p^{[i,j]} = \cos(\phi_p(h_i) - \phi_p(h_j))$ for quench dynamics. To proceed with our proof, it is convenient to translate the integration variable by $\pi/2$. Thus, we introduce the change of variables $k = p + \pi/2$, so

$$
\langle W(h_0) \rangle = \frac{L}{2\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} dp \tanh\left(\frac{\beta \omega_{k-\frac{\pi}{2}}(h_0)}{2}\right)
$$
$$
\times \left[\omega_{k-\frac{\pi}{2}}(h_1) \cos(\phi_{k-\frac{\pi}{2}}(h_0) - \phi_{k-\frac{\pi}{2}}(h_1)) \right.
$$
$$
\left. - \omega_{k-\frac{\pi}{2}}(h_2) \cos(\phi_{k-\frac{\pi}{2}}(h_0) - \phi_{k-\frac{\pi}{2}}(h_2)) \right]. \quad \text{(D5)}
$$

Then, exploiting that $\omega_{k-\frac{\pi}{2}}(h_0) = \omega_{\frac{\pi}{2}-k}(-h_0) = \omega_{k-\frac{\pi}{2}}(-h_0)$ and $\phi_{k-\frac{\pi}{2}}(h_0) = \pi - \phi_{\frac{\pi}{2}-k}(-h_0) = \pi + \phi_{k-\frac{\pi}{2}}(-h_0)$, we determine that

$$
\langle W(h_0) \rangle = \frac{L}{2\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} dp \tanh\left(\frac{\beta \omega_{k-\frac{\pi}{2}}(-h_0)}{2}\right)
$$
$$
\times \left[\omega_{k-\frac{\pi}{2}}(h_1) \cos(\pi + \phi_{k-\frac{\pi}{2}}(-h_0) - \phi_{k-\frac{\pi}{2}}(h_1)) \right.
$$
$$
\left. - \omega_{k-\frac{\pi}{2}}(h_2) \cos(\pi + \phi_{k-\frac{\pi}{2}}(-h_0) - \phi_{k-\frac{\pi}{2}}(h_2)) \right].
$$
$$
\text{(D6)}
$$

Finally, using the trigonometric identity $\cos(\pi + x) = -\cos(x)$ and translating back the integration variable to $p = k - \pi/2$, we obtain

$$
\langle W(h_0) \rangle = -\frac{L}{2\pi} \int_0^\pi dp \tanh\left(\frac{\beta \omega_p(-h_0)}{2}\right)
$$
$$
\times \left[\omega_p(h_1) \cos(\phi_p(-h_0) - \phi_p(h_1)) \right.
$$
$$
\left. - \omega_p(h_2) \cos(\phi_p(-h_0) - \phi_p(h_2)) \right] = -\langle W(-h_0) \rangle.
$$
$$
\text{(D7)}
$$

This concludes our proof.

APPENDIX E: AVERAGE WORK WITH INITIAL DEPHASED QUANTUM STATE

In this Appendix, we aim to extend the above discussion to the case of a dephased quantum state $\Delta_1(\rho_0) \equiv$ $\sum_\alpha \Pi_\alpha^{[1]} \rho_0 \Pi_\alpha^{[1]}$, where $\Pi_\alpha^{[1]}$ are projectors in the $H^{[1]}$ basis. In this case, we find that the corresponding average work, identified by $\langle W[\Delta_1(\rho_0)] \rangle$, equals

$$
\langle W[\Delta_1(\rho_0)] \rangle = \text{Tr}\{\Delta_1(\rho_0) U^\dagger H_{t_2} U\} - \text{Tr}\{\Delta_1(\rho_0) H_{t_1}\}
$$
$$
= \text{Tr}\{\Delta_1(\rho_0) U^\dagger H_{t_2} U\} - \text{Tr}\{\rho_0 H_{t_1}\}. \quad \text{(E1)}
$$

To find an explicit expression for the average work originated by a work protocol starting from $\Delta_1(\rho_0)$, we need to project the state ρ_0 onto the basis that decomposes the initial Hamiltonian H_{t_1}. For this purpose, we consider the state with

$$
Z_p^{[0]} \rho_p^{[0]} = e^{\beta \omega_p^{[0]}} \left\{ \left[1 + n_p^{[0]}(e^{-\beta \omega_p^{[0]}} - 1) \right] \right.
$$
$$
\left. \times \left[1 + n_{-p}^{[0]}(e^{-\beta \omega_p^{[0]}} - 1) \right] \right\}. \quad \text{(E2)}
$$

Then, we have to rewrite such a state in terms of the fermionic operators that diagonalize H_{t_1}. The latter are related to the fermionic operators diagonalizing ρ_0 through the condition

$$
\gamma_p^{[0]} = C_p \gamma_p^{[1]} + S_p \gamma_{-p}^{[1]\dagger}, \quad \text{(E3)}
$$

where

$$
C_p = \cos\left(\frac{\phi_p^{[1]} - \phi_p^{[0]}}{2}\right) \quad \text{and} \quad S_p = \sin\left(\frac{\phi_p^{[1]} - \phi_p^{[0]}}{2}\right). \quad \text{(E4)}
$$

Therefore, the number of zero fermions in the p mode can be expressed in terms of $\gamma_p^{[1]}$, i.e.,

$$
n_p^{[0]} = \left[C_p^2 n_p^{[1]} + S_p^2 (1 - n_{-p}^{[1]}) \right] + C_p S_p \left[\eta_p^{[1]\dagger} + \eta_p^{[1]} \right], \quad \text{(E5)}
$$

where we have introduced the bosonic operator $\eta_p = \gamma_p \gamma_{-p}$. Thus, using the relations above, we can write ρ_0 in the basis of H_{t_1}:

$$
Z_p^{[0]} \rho_p^{[0]} = e^{\beta \omega_p^{[0]}} \left\{ \left[C_p^2 \left(1 + n_p^{[1]}(e^{-\beta \omega_p^{[0]}} - 1) \right) + S_p^2 \left(1 + (1 - n_{-p}^{[1]})(e^{-\beta \omega_p^{[0]}} - 1) \right) + C_p S_p \left(\eta_p^{[1]\dagger} + \eta_p^{[1]} \right)(e^{-\beta \omega_p^{[0]}} - 1) \right] [(p \to -p)] \right\},
$$
$$
\text{(E6)}
$$

with $S_{-p} = -S_p$. In the following, to extract only the diagonal contribution, we are going to discard the terms that are not diagonal in $n_p^{[1]}, n_{-p}^{[1]}$. Accordingly, by defining $\rho_p^{[1|0]} = \sum_\alpha \Pi_\alpha^{[1]} \rho_p^{[0]} \Pi_\alpha^{[1]}$, we find that

$$
Z_p^{[0]} \rho_p^{[1|0]} = \left[C_p^4 e^{-\beta \omega_p^{[0]}(n_p^{[1]} + n_{-p}^{[1]} - 1)} + S_p^4 e^{\beta \omega_p^{[0]}(n_p^{[1]} + n_{-p}^{[1]} - 1)} + 2 S_p^2 C_p^2 - (Z_p^{[0]} - 4) S_p^2 C_p^2 \left(\eta_p^{[1]\dagger} + \eta_p^{[1]} \right) \left(\eta_{-p}^{[1]\dagger} + \eta_{-p}^{[1]} \right) \right]
$$
$$
= \left[C_p^4 e^{-\beta \omega_p^{[0]}(n_p^{[1]} + n_{-p}^{[1]} - 1)} + S_p^4 e^{\beta \omega_p^{[0]}(n_p^{[1]} + n_{-p}^{[1]} - 1)} + 2 S_p^2 C_p^2 - (Z_p^{[0]} - 4) S_p^2 C_p^2 (n_p + n_{-p} - 1 - 2 n_p n_{-p}) \right], \quad \text{(E7)}
$$

whereby, as expected, $\text{Tr}\{\rho_p^{[1|0]}\} = 1$.

We are now ready to compute the explicit expression of the average work $\langle W[\Delta_1(\rho_0)]\rangle$. This is achieved by following the same steps of the previous section and simply replacing $\rho_p^{[0]}$ with its diagonal part in the $\rho_p^{[1|0]}$ basis of H_{t_1}. Accordingly,

$$\mathrm{Tr}\big\{\rho_p^{[1|0]}\Gamma_p^{[1]\dagger}\mathbb{M}\Gamma_p^{[1]}\big\} = \mathbb{M}_{11}\mathrm{Tr}\big\{\rho_p^{[1|0]}n_p^{[1]}\big\} + \mathbb{M}_{22}\mathrm{Tr}\big\{\rho_p^{[1|0]}\big(1 - n_{-p}^{[1]}\big)\big\} = \frac{\mathbb{M}_{22}}{2}\tanh\left(\frac{\beta\omega_p^{[0]}}{2}\right)\big(C_p^4 - S_p^4\big)$$

$$= \frac{\mathbb{M}_{22}}{2}\tanh\left(\frac{\beta\omega_p^{[0]}}{2}\right)\cos\big(\phi_p^{[1]} - \phi_p^{[0]}\big), \qquad \text{(E8)}$$

where

$$\mathbb{M} = R_y^\dagger\big(\phi_p^{[1]}\big)\mathbb{U}_{p,t_1:t_2}^\dagger R_y^\dagger\big(\phi_p^{[2]}\big)\omega_p^{[2]}\sigma^z R_y\big(\phi_p^{[2]}\big)\mathbb{U}_{p,t_1:t_2} R_y\big(\phi_p^{[1]}\big). \tag{E9}$$

As a result, given that $\mathbb{M}_{22} = -\omega_p^{[2]}(2P_p^{[1,2]} - 1)$ and $\cos(\phi_p^{[1]} - \phi_p^{[0]}) = 2P_p^{[0,1]} - 1$, we finally get that

$$\langle W[\Delta_1(\rho_0)]\rangle = \frac{L}{2\pi}\int_0^\pi dp\tanh\left(\frac{\beta\omega_p^{[0]}}{2}\right)$$
$$\times\big[\omega_p^{[1]}\big(2P_p^{[0,1]} - 1\big) - \omega_p^{[2]}\big(2P_p^{[1,2]} - 1\big)$$
$$\times\big(2P_p^{[0,1]} - 1\big)\big]. \tag{E10}$$

The formula above reduces to the expression for the average work shown in the main text upon substituting the parameters $Q_p^{[i,j]} = 2P_p^{[i,j]} - 1$.

APPENDIX F: AVERAGE WORK DENSITY AND NONCLASSICAL REGIONS FOR DIFFERENT VALUES OF THE MODEL PARAMETERS

For completeness, in Fig. 5 we show the value of the average work density for ρ_0, its dephased counterpart $\Delta_1(\rho_0)$ and the difference of the two, with different choices of h_2. They are plotted by considering a sudden quench $h_1 \rightarrow h_2$ and an inverse temperature of the initial quantum state equal to

$\beta = 15$. As in the main text, the shaded green areas mark the nonclassicality regions, for which the fourth central moment of the KDQ distribution of work is negative.

Moreover, in Fig. 6 we plot the regions of nonclassicality, where the fourth central moment of the KDQ distribution work is negative, against h_0 and h_1 for a sudden quench with $h_2 = 0.5$. Different values of the initial inverse temperature β are also considered. From the figure, it is worth noting that the nonclassical region is reduced by increasing the temperature of the initial state. This signals that nonclassical effects are reduced in the limit of high temperature.

APPENDIX G: RELATIVE ENTROPY OF QUANTUM COHERENCES

Let us consider the quantum relative entropy between state ρ_0 and its projection on the basis of H_{t_1}. It is defined as

$$D(\Delta_1(\rho_0)\,\|\,\rho_0) = \mathrm{Tr}\rho_0(\ln\rho_0 - \ln\Delta_1(\rho_0)). \tag{G1}$$

It can be shown that

$$D(\Delta_1(\rho_0)\,\|\,\rho_0) = S[\Delta_1(\rho_0)] - S[\rho_0], \tag{G2}$$

where $S[\rho] = -\mathrm{Tr}\rho\ln\rho$ is the von Neumann entropy of a quantum state ρ. Remember that $\rho_0 = \prod_p \rho_p^{[0]}$ and $\Delta_1(\rho_0) = \prod_p \rho_p^{[1|0]}$. Therefore, the quantum relative entropy among these two states can be also expressed by the following sum:

$$D(\Delta_1(\rho_0)\,\|\,\rho_0) = \sum_p \mathrm{Tr}\rho_p^{[0]}\ln\rho_p^{[0]} - \rho_p^{[1|0]}\ln\rho_p^{[1|0]}. \tag{G3}$$
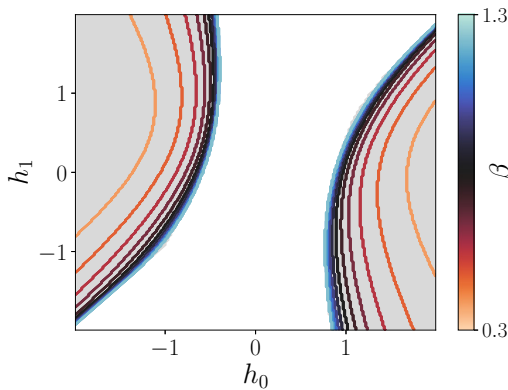

FIG. 6. Quantum signatures of the KDQ distribution of work in the form of nonclassical regions where the fourth central moment of the corresponding work distribution is negative. Such regions (in grey) are obtained by varying the temperature of the initial state ρ_0, with a fixed value ($h_2 = 0.5$) of the final magnetic field. The borders of the nonclassical regions are marked with different colors, corresponding to different values of temperature.
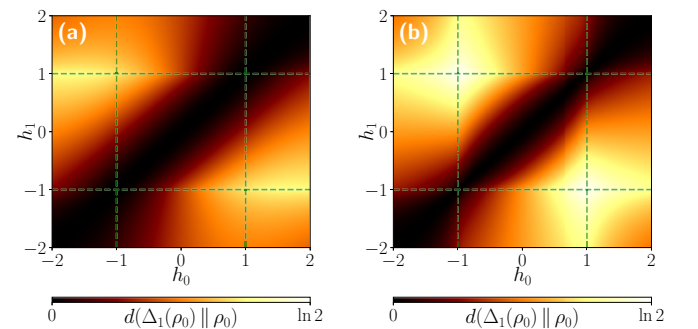

FIG. 7. Relative entropy of the coherence density for the quantum Ising model between a thermal state $\rho_0 = \exp\{-\beta H_{t_0}\}/Z$ with magnetic field $h = h_0$, and the dephased state $\Delta_1(\rho_0)$ with respect to the Hamiltonian H_{t_1} with magnetic field $h = h_1$. (a) $\beta = 1$, (b) $\beta = 15$.

Moreover, after some algebra, we also arrive at an explicit expression for the relative entropy of the coherence density:

$$d(\Delta_1(\rho_0) \| \rho_0) = \frac{D(\Delta_1(\rho_0) \| \rho_0)}{L} = \frac{1}{\pi} \int_0^\pi dp \, \beta\omega_p^{[0]} \tanh\frac{\beta\omega_p^{[0]}}{2} + - \frac{1}{Z_p^{[0]}} [(P_p^{[1,0]} e^{\beta\omega_p^{[0]}} + (1 - P_p^{[1,0]}) e^{-\beta\omega_p^{[0]}})$$

$$\times \ln (P_p^{[1,0]} e^{\beta\omega_p^{[0]}} + (1 - P_p^{[1,0]}) e^{-\beta\omega_p^{[0]}}) + (P_p^{[1,0]} e^{-\beta\omega_p^{[0]}} + (1 - P_p^{[1,0]}) e^{\beta\omega_p^{[0]}})$$

$$\times \ln (P_p^{[1,0]} e^{-\beta\omega_p^{[0]}} + (1 - P_p^{[1,0]}) e^{\beta\omega_p^{[0]}})]. \tag{G4}$$

For the quantum Ising model, Fig. 7 shows $d(\Delta_1(\rho_0) \| \rho_0)$ as a function of h_0 and h_1 for two different values of the initial inverse temperature $\beta = 1$ [Fig. 7(a) and $\beta = 15$ [Fig. 7(b)]. It is worth noting that $d(\Delta_1(\rho_0) \| \rho_0)$ has two maxima corresponding to $h_0 = \pm 1$ and $h_1 = \mp 1$, i.e., when h_0 and h_1 sit at the two quantum critical points of the model. At these points, $P_p^{[0,1]} = 1/2$ ($Q_p^{[0,1]} = 0$) $\forall p$, thus leading to the maximum value

$$\max_{h_0, h_1} d(\Delta_1(\rho_0) \| \rho_0) = \frac{1}{\pi} \int_0^\pi dp \left[\beta\omega_p^{[0]} \tanh\left(\frac{\beta\omega_p^{[0]}}{2}\right) - \frac{\cosh\left(\beta\omega_p^{[0]}\right)}{1 + \cosh\left(\beta\omega_p^{[0]}\right)} \ln\left(\cosh\left(\beta\omega_p^{[0]}\right)\right) \right]. \tag{G5}$$

Interestingly,

$$\lim_{\beta \to 0} \max_{h_0, h_1} d(\Delta_1(\rho_0) \| \rho_0) = 0, \tag{G6}$$

which again signals the fact that quantum coherences in ρ_0 (with respect to the basis of H_{t_1}) does not entail relevant effects to work statistics in the high-temperature limit. On the other hand, at low temperatures the relative entropy of quantum coherences increases, until saturating the maximum value allowed by the Hilbert space dimension in the zero-temperature (infinite β) limit:

$$\lim_{\beta \to \infty} \max_{h_0, h_1} d(\Delta_1(\rho_0) \| \rho_0) = \ln 2. \tag{G7}$$

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