NUMERICAL COMPUTATION OF EGG AND DOUBLE-EGG CURVES WITH CLOTHOIDS

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ABSTRACT

The clothoid, also known as Cornu spiral or Euler spiral, is a curve widely used as a transition curve when designing the layout of railway tracks and roads because of a key feature: its curvature is proportional to its length. The classical method to compute a clothoid is based on the use of Taylor expansions of sine and cosine functions, usually starting with zero curvature at the initial point. In this paper the clothoid is presented as the only curve with a constant rate of change of curvature,which parametrization can be obtained by solving an initial value problem. In this initial value problem the curvature at the starting point can be chosen, being able to develop simple, efficient and accurate algorithms to connect two oriented circumferences by means of clothoids. These algorithms are presented as an useful tool for designing egg and double-egg curves in highway connections and interchanges.

INTRODUCTION

The use of transition curves in the design of the layout of railway tracks and roads is a key to reach a gradual change of the centrifugal force experimented by a vehicle, which not only increases the comfort of the passengers but also improves the visual perception of the road by the driver and considerably decreases the risk of accident.

Those kind of curves have been used principally for horizontal alignment, but transition curves for highway vertical alignments have been recently introduced (Easa

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and Hassan 2000a; Easa and Hassan 2000b; Kobryń 2016a). In the recent Kobryń (2017) different types of curves are presented as transition curves, among others: Bloss curves, Grabowski curves, sinusoidal and co-sinusoidal, parabolic, polynomial (Baykal et al. 1997; Tari and Baykal 2005; Kobryń 2011; Bosurgi and D'Andrea 2012; Kobryń 2016b), and general transition curves (Kobryń 2011; Kobryń 2014), but the transition curve that has been most widely used in design of roads is the clothoid (Baass 1984; Kobryń 1993; Dong et al. 2007).

Originally, chlotoids were manually represented by draughtsmen, which while being a simple process, is also usually quite laborious, and some trial and error can not be avoided. As the clothoid is a spiral defined parametrically in terms of Fresnel integrals, it seems that a clear step forward in computing the curve can come hand in hand with improving efficiency in computing those integrals. As such, some results about approximations to the Fresnel integrals by terms of direct evaluation of the standard Maclaurin and asymptotic series (see Heald (1985) and the references therein) will be most useful.

Different approaches in computing the clothoid can be found in Wang et al. (2001), where the clothoid is approximated by a high degree Bézier polynomial, in Sánchez-Reyes and Chacón (2003), where the approximation is given by an s-power series (the two point version of Taylor series), or in Meek and Walton (2004a), where the spiral is approximated by an arc spline. In Press et al. (2002) some approximations based in power series and continued fractions are given, although in this case, as in the case of the rational approximation by Maclaurin series, the methods are not specifically designed to compute clothoids.

In a previous paper (Vázquez-Méndez and Casal 2016), authors have proposed an alternative method for computing clothoids and have shown its usefulness for connecting two oriented straight segments and also an oriented straight segment with an arc of an oriented circumference. These connections are basic in highway alignments, and recently the method has been successfully used in road design and reconstruction (Casal et al. 2017; Vázquez-Méndez et al. 2018). For highway connections and interchanges, shorter links are highly desirable, and it is frequent to have to connect two oriented circumference arcs with a transition curve, avoiding straight segments between them. Different transition curves between circumferences, forming egg and double-egg curves, have been proposed in the literature (Bosurgi and D'Andrea 2012; Koç et al. 2015). There are also oval-shaped transition curves that can be used to linking two circumferences with the same orientation (Kobryń 2011; Kobryń 2016b). The use of clothoid arcs (partial spirals) for linking circumferences has also been studied from a mathematical point of view (Meek and Walton 1989; Meek and Walton 2004b; Stoer 1982). In road design, these partial spirals are usually obtained as an arch of the *standard clothoid* (clothoid starting at $(0,0)$ with zero curvature and tangent to OX^+) suitably rotated and translated (Lovell et al. 2001). In this paper, the alternative method studied in Vázquez-Méndez and Casal (2016) for calculating clothoids is extended to partial spirals (Algorithm 1), and this method is used for linking any two given circumferences. Then, simple, efficient and accurate algorithms for computing egg curves (Algorithm 2) and double-egg curves (Algorithm 3) are detailed.

In order to do it, the next section is devoted to characterize the clothoid as the only curve with a constant rate of change of curvature. Then, a partial spiral is presented as the solution of an initial value problem, where the length of curve, the rate of change of curvature, and the azimuth and the oriented curvature at initial point should be given as input data.

In the second section we analyze the case of linking two circumferences with only one clothoid arc. First, based on the numerical solution of the initial value problem given the partial spiral, we propose an alternative method for computing it. This algorithm is simple, efficient and accurate (even when the classical procedure fails) and then it is used for designing an egg curve linking two any interior circumferences.

In order to be linked by a partial spiral, the circumference with smaller radius has to be completely inside the larger one, without intersecting and being concentric. And even in this case, the clothoid arc that links them may not be useful. These situations are analyzed in the penultimate section, where an algorithm for computing the double egg curve is detailed and used for linking two any (interiors, exteriors and secants) circumferences.

Finally, the last section presents some brief and interesting conclusions.

CURVES WITH A CONSTANT RATE OF CHANGE OF CURVATURE: THE CLOTHOID

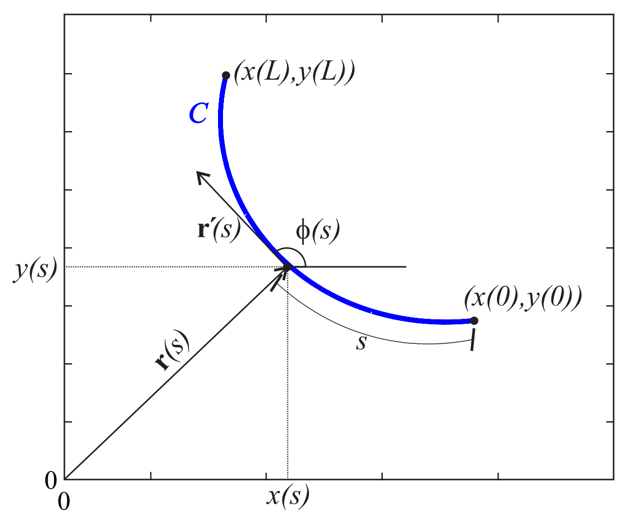

Fig. 1. Arc length parametrization

Let $C \subset \mathbb{R}^2$ be a smooth plane curve of length L and let $\boldsymbol{r}(s) = (x(s), y(s))$ be a parametrization of C where $s \in [0, L]$ is the *arc length* parameter. If $\Phi(s) \in [0, 2\pi)$ denotes the angle between the tangent vector $\boldsymbol{r}'(s)$ at the point $\boldsymbol{r}(s)$ and the positive abscissa axis OX^+ (see Fig. 1), then $\Phi' = d\Phi/ds$ represents the rate of change of Φ with respect to the arc length s, that is, the *oriented curvature* of C,

$$\Phi'(s) = \lambda(s)|\Phi'(s)|,$$

where $|\Phi'(s)|$ is usually known as curvature of C, and $\lambda(s) = \pm 1$ indicates the way in which the unit tangent vector $r'(s)$ rotates as a function of the arc length parameter along the curve (if $\lambda = -1$, the angle is decreasing and the unit tangent vector rotates clockwise, while if $\lambda = 1$, the angle is increasing and the unit tangent vector rotates counterclockwise).

In road and railway horizontal alignment, in order to improve the transition connecting straight line segments and circular arcs, or linking two circular arcs, it is common to use curves $C \in \mathbb{R}^2$ with constant rate of change of curvature. It is not difficult to see that a curve with such a property is uniquely defined by the system,

$$\left\{ \begin{aligned} \frac{d\Phi'}{ds} = \Phi''(s) = v_c, \quad & s \in (0, L), \\ \Phi(0) = \Phi_0, & \\ \Phi'(0) = \Phi'_0, & \end{aligned} \right. \tag{1}$$

where, following the previous notation:

1. $\Phi_0 \in [0, 2\pi)$ is the angle between the tangent vector at the initial point of the curve C and the positive abscissa axis OX^+.
2. $\Phi'_0 = \lambda|\Phi'_0|$ is the orientated curvature at the initial point.
3. $v_c = v|v_c|$, where $v = \pm 1$ and $|v_c| \geq 0$ gives the rate of change of the curvature. Whenever $v_c \neq 0$, the case $\lambda v = 1$ indicates that the curvature is increasing (the curve is "closing"), while the case $\lambda v = -1$ indicates that the curvature is decreasing (the curve is "opening").

So it is not difficult to obtain that

$$\Phi(s) = \frac{1}{2} v_c s^2 + \Phi'_0 s + \Phi_0. \tag{2}$$

Remark 1 *The case $v_c = 0$ in system (1) corresponds either to a circumference or to a straight line, depending on the value of the curvature $|\Phi'_0|$. Whenever $|\Phi'_0| > 0$ system (1) models a circumference of radius $1/|\Phi'_0|$, while the case $|\Phi'_0| = 0$ corresponds to a straight line.*

In both of cases, of course, the curvature is a constant value in all the points of the curve, what follows from the rate of change of the curvature, $|v_c|$, is equal to zero.

By the other hand, as $\boldsymbol{r}(s)$ is the arc length parametrization of C it follows that $||\boldsymbol{r}'(s)|| = 1$ and, therefore (see Fig. 1),

$$\left\{ \begin{aligned} x'(s) = \cos(\Phi(s)),\ s \in (0, L), \\ y'(s) = \sin(\Phi(s)),\ s \in (0, L). \end{aligned} \right. \tag{3}$$

Thus, if the initial point of the curve C is (x_0, y_0), then the arc length parametrization

$\boldsymbol{r}(s)$ of C is the solution of the following initial value problem:

$$\begin{cases} x'(s) = \cos\left(\frac{1}{2}v_c s^2 + \Phi_0' s + \Phi_0\right), & s \in (0, L), \\ x(0) = x_0, \\ y'(s) = \sin\left(\frac{1}{2}v_c s^2 + \Phi_0' s + \Phi_0\right), & s \in (0, L), \\ y(0) = y_0. \end{cases} \tag{4}$$

Now, with a suitable change of variable, problem (4) would become

$$\begin{cases} x'(s) = \cos\left(\frac{1}{2}v_c s^2 + \tilde{\Phi}_0\right), & s \in (s_0, s_0 + L), \\ x(s_0) = x_0, \\ y'(s) = \sin\left(\frac{1}{2}v_c s^2 + \tilde{\Phi}_0\right), & s \in (s_0, s_0 + L), \\ y(s_0) = y_0, \end{cases} \tag{5}$$

where

$$s_0 = \frac{\Phi_0'}{v_c}, \qquad \tilde{\Phi}_0 = \Phi_0 - \frac{(\Phi_0')^2}{2v_c}.$$

In fact, (5) is the equation of an arc of the clothoid with parameter $A = \sqrt{1/|v_c|}$, that is traversed as indicates v for $s > 0$ and satisfies that the angle between the unit vector tangent at the initial point and the axis OX^+ takes the value $\tilde{\Phi}_0$ (see Vázquez-Méndez and Casal (2016)).

It has just presented that the only curve with a constant rate of change of curvature is the clothoid. From (2) it is easy to see that this characterization is, of course, equivalent to the usual characterization of the clothoids as the curves in what their curvature is a linear function of the arc of length (see, for instance Stoer (1982)).

In Vázquez-Méndez and Casal (2016) it is shown how to solve the problem (5) by using a numerical method, and how much useful that algorithm is in order to compute the transition curves (clothoids) in a horizontal road alignment made up of straight lines and circular arcs. Moreover, it was shown how to connect two fixed points of two oriented circumferences with clothoid-line-clothoid, whenever their center points are enough far apart (see Figures 7 and 8 in Vázquez-Méndez and Casal (2016)).

In that mentioned paper, the authors started with system (5) in order to obtain the arc length parametrization of the clothoid, but a zero curvature at the initial point of the clothoid was imperative. Due to this, the classical egg curve (circular arc-clothoide-circular arc) and double-egg curve (circular arc-clothoid- circular arc-clothoid-circular arc) have not been considered.

In this paper, system (4) will be used to obtain the arc length parametrization of arc of clothoids. In this model the curvature at the initial point of the curve can be chosen and, as will be pointed out below, it provides a simple method to compute partial spirals linking circular arcs.

In the sequel, for $i \in \{1, 2, 3\}$ let $C_i \subset \mathbb{R}^2$ be a circumference with center $c_i = (x_i, y_i)$ and radius $R_i > 0$. The distance between the centers c_i and c_j will be denoted by $d_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$, and the minimum distance between circumferences C_i and C_j, by D_{ij} (see Fig. 2). We say that two circumferences are *interior* if $0 < d_{ij} < |R_i - R_j|$ (see, for instance, Fig. 2, 3, 6 and 7). On the contrary, we say that they are *secant* (Fig. 9) or *exterior* (Fig. 8) whenever $|R_i - R_j| < d_{ij} < R_i + R_j$ or $d_{ij} > R_i + R_j$ respectively. It is convenient to note that concentric or tangent circumferences are not considered.

LINKING TWO CIRCUMFERENCES WITH AN ARC OF CLOTHOID. THE EGG CURVE

The objective is to find an arc of clothoid C linking two circumferences C_1 and C_2, that is: the curve C has to meet the circumferences C_1 and C_2 at points P_1 and P_2 respectively and, at such points, the curvature and the slope have to be the same both in the clothoid and the corresponding circumference. Notice that, in order to be linked correctly, the two circumferences has to be traversed with the same orientation, given by the λ parameter, and they have to be interior (Stoer 1982).

If L denotes the length of the arc of clothoid between the points P_1 and P_2, it has to be fulfilled that $\Phi'_0 = \lambda/R_1$ and $\Phi'(L) = \lambda/R_2$ and, consequently,

$$v_c = \lambda \frac{\frac{1}{R_2} - \frac{1}{R_1}}{L} = \frac{\lambda(R_1 - R_2)}{LR_1R_2}. \tag{6}$$

Remark 2 *From* (6) *it is not difficult to see that if $R_1 > R_2$ the signs of v_c and λ match and the clothoid has to be closing, while if $R_1 < R_2$ then v_c and λ have opposing signs and the clothoid has to be opening. In the limiting case, when $R_1 = R_2$ holds, it has that $v_c = 0$, there is an only circumference $C_1 = C_2$, and the points P_1 and P_2 are connected by a circular arc, so it is not necessary to calculate the arc of clothoid C.*

From (2) it follows that

$$\Phi(L) = \frac{R_1 + R_2}{2R_1R_2}\lambda L + \Phi_0. \tag{7}$$

By the other hand, if $\Delta\Phi > 0$ denotes the increase in the azimuth between the initial and the end points of the arc of clothoid, then necessarily

$$\Phi(L) = \Phi_0 + \lambda\Delta\Phi, \tag{8}$$

and, therefore

$$L = \Delta\Phi \frac{2R_1R_2}{R_1 + R_2}. \tag{9}$$

Next, two different cases will be studied.

Changing the Curvature and the Azimuth with a Fixed Initial Point

Let us suppose that the circumference C_1 (with radius R_1) is traversed with the orientation determined by the parameter λ and, in a given point $P_1=(x_0, y_0) \in C_1$, it is wanted to link with an arc of clothoid C in order to reach a radius R_2 changing the azimuth $\Delta\Phi$ radians. Such an arc is the solution of (4), where v_c is given in (6), L is given in (9), $\Phi'_0 = \lambda/R_1$, and Φ_0 is the angle between the tangent vector to the circumference C_1 at (x_0, y_0) and OX^+, that is:

$$\Phi_0 = \begin{cases} \arccos\left(\dfrac{\lambda(y_1 - y_0)}{R_1}\right), & \text{if } x_0 \geq x_1, \\ 2\pi - \arccos\left(\dfrac{\lambda(y_1 - y_0)}{R_1}\right), & \text{if } x_0 < x_1. \end{cases} \tag{10}$$

There are many suitable numerical methods to solve such a problem (4) (see Atkinson et al. (2009)). For example, the classical Euler method (used previously in Vázquez-Méndez and Casal (2016)) leads to the following algorithm to compute the arc of clothoid C.

Algorithm 1

1. Choose a positive natural number $N \in \mathbb{N}$.
2. Take $\Delta s = L/N$, $x^0 = x_0$, $y^0 = y_0$, and, for each $n \in \{0, \ldots, N\}$, set $s^n = n\Delta s$.
3. For each $n \in \{0, 1, \ldots, N-1\}$, compute

$$x^{n+1} = x^n + \Delta s \cos\left(\frac{1}{2}v_c(s^n)^2 + \Phi'_0 s^n + \Phi_0\right), \tag{11}$$

$$y^{n+1} = y^n + \Delta s \sin\left(\frac{1}{2}v_c(s^n)^2 + \Phi'_0 s^n + \Phi_0\right), \tag{12}$$

4. For $n \in \{0, 1, \ldots, N-1\}$ take the approximations $(x(s^{n+1}), y(s^{n+1})) \approx (x^{n+1}, y^{n+1})$ as the N equispaced points of the arc of clothoid C.

The Fig. 2 presents an example of arc of clothoid obtained with the Algorithm 1. It shows a spiral linking a circumference with center $c_1 = (200, 300)$ and radius $R_1 = 500$ m to a circumference with radius $R_2 = 300$ m, starting at the point $(x_0, y_0) = (-155, -55)$, with azimuth of $7\pi/4$, and ending at a point with azimuth of $5\pi/12$, so that $\Delta\Phi = 2\pi/3$.

Once obtained the final point $(x(L), y(L))$ of the arc of clothoid, the center of the circumference C_2 is given by

$$c_2 = \big(x(L), y(L)\big) + R_2\lambda\big(-\sin(\Phi(L)), \cos(\Phi(L))\big). \tag{13}$$

In the literature, this kind of transition curve (partial spiral) is obtained as an arch of the *standard clothoid* suitably rotated and translated (Lovell et al. 2001). The *standard clothoid*, which starts at $(0, 0)$ with zero curvature and is is tangent to OX^+ at this

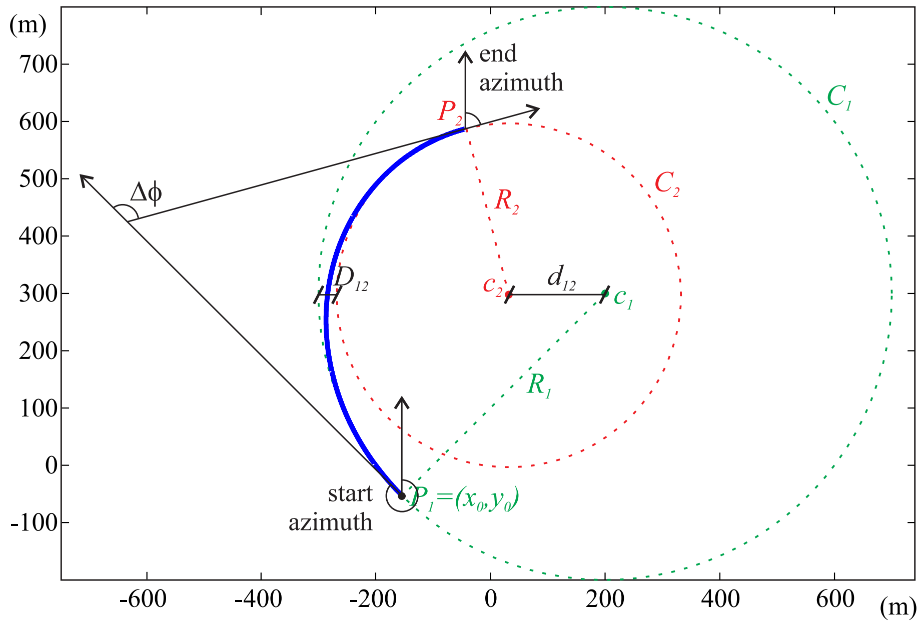

Fig. 2. Egg curve starting at a fixed initial point

point, is traditionally obtained by using Taylor series expansions of the cosine and sine functions (e.g. Lovell (1999)). This type of approximations is sharp whenever the spiral angle is small (as usually in simple applications of surveying engineering) but, as it is pointed out in Vázquez-Méndez and Casal (2016), can deviate from the clothoid as the angle grows, what can happen when the clothoid is used to connect two circular arcs in the design of egg curves in link roads.

Fig. 3 shows, as an example, the transition curve of length $L = 169.56$ m connecting two interior circumferences of radii $R_1 = 130$ m and $R_2 = 100$ m. The solid curve (in blue) is the partial spiral obtained using the Algorithm 1, while the dashed one (in magenta) is the corresponding *standard clothoid* computed with Taylor series of sixth degree for cosine function and seventh degree for sine, as it is proposed in Lovell et al. (2001). Both methods give the same curve in the beginning, but the approximation obtained by the Taylor series gives a end point P_2' outside C_2, and the partial spiral obtained with this method does not link both circumferences. Cartesian coordinates of some characteristic points of the egg curve can be seen in Table 1. The end point of the clothoid arc obtained with Algorithm 1 (P_2) is very good (it is on C_2, with appropriate curvature and azimuth), but the computed with Taylor series (P_2') is not a good approximation.

Connecting Two Fixed Interior Circumferences

It will be now considered the case of two given fixed interior circumferences C_1 and C_2, which are traversed with the same orientation determined by λ. Assuming that $R_1 > R_2$, an arc of clothoid $\bar{C}$ with initial point $(x_1, y_1 - R_1)$ will link C_1 to a new circumference $\bar{C}_2$, with radius R_2 and center $\bar{c}_2$, such that de distance between c_1 and $\bar{c}_2$ is equal to d_{12}. Once this arc has been calculated, the final arc of clothoid C linking

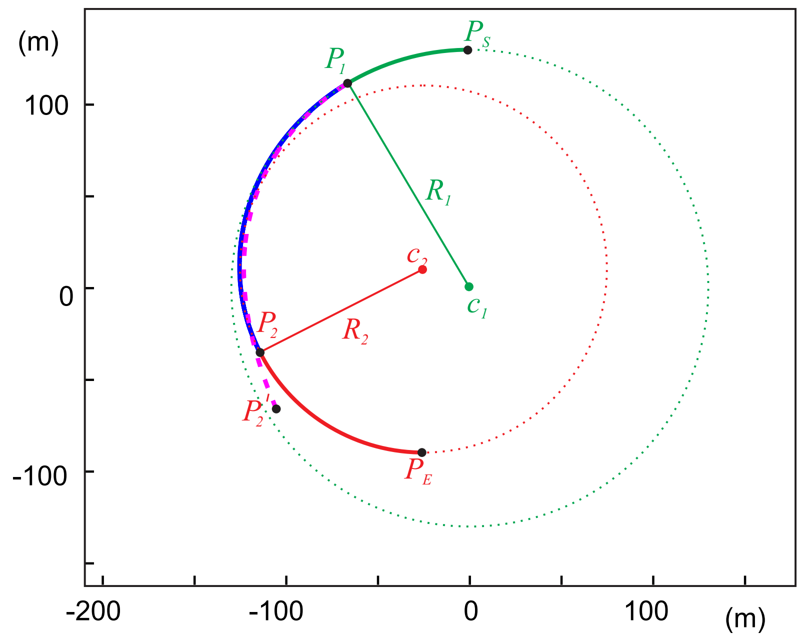

Fig. 3. Egg curve: arc of clothoid linking two circular arcs of radii $R_1 = 130$ m and $R_2 = 100$ m. The solid line (blue) is the arc of clothoid computed with Algorithm 1 and the dashed line (magenta) is the standard clothoid computed with the classical method (Taylor series).

TABLE 1. Coordinates of some characteristic points of the egg curve depicted in Fig. 3, corresponding with radii $R_1 = 130$ m and $R_2 = 100$ m

Arc length s (meters)	Cartesian coordinates $P_i = (x_i, y_i)$ (meters)	
0.00	P_S=(0,130)	
69.20	P_1=(-65.98,112.01)	
238.77	P_2=(-114.80,-34.16)	P_2'=(-105.35,-66.11)
349.68	P_E=(-25.27,-89.61)	
—	C_1=(0,0)	
—	C_2=(-25.27,10.39)	

C_1 and C_2 is obtained rotating $\bar{C}$ (and $\bar{C}_2$) around c_1, until $\bar{C}_2$ and C_2 match.

As it has been explained previously, an arc of clothoid, with initial point $(x_0, y_0) = (x_1, y_1 - R_1)$, linking C_1 to a circumference of radius R_2 is uniquely defined by the increase in the azimuth $\Delta\Phi$. Moreover, once $\Delta\Phi > 0$ is known, the center of $\bar{C}_2$ is given by (13) and, therefore, the distance between the centers c_1 and $\bar{c}_2$ is

$$d(\Delta\Phi) = \sqrt{(x(L) - R_2 \sin(\Phi(L)) - x_1)^2 + (y(L) + R_2 \cos(\Phi(L)) - y_1)^2}. \qquad (14)$$

It is further required that such a distance has to be equal to the distance between the centers c_1 and c_2, that is to say,

$$d(\Delta\Phi) = d_{12}. \tag{15}$$

Remark 3 *It is not difficult to see that equation* (15) *has solution if and only if there exists an arc of clothoid linking* C_1 *to* C_2 *under the conditions presented at the beginning of this subsection. Theorem* (2.4) *in Stoer (1982) guarantees that such a spiral exists in this case and, consequently, equation* (15) *has solution.*

Thus, the arc of clothoid C joining C_1 and C_2 can be computed with the following algorithm:

Algorithm 2

1. Take $(x_0, y_0) = (x_1, y_1 - R_1)$ and consider $d(\Delta\Phi)$ given by (14), where $x(L)$ and $y(L)$ are computed by Algorithm 1, $\Phi(L)$ is given by (8), L is given by (9) and Φ_0 is given by (10), taking into account that, in this case, $\Phi_0 = \arccos(\lambda)$.
2. Get $\Delta\Phi$ by solving equation (15).
3. From the value $\Delta\Phi$ calculated in the previous step, use Algorithm 1 to compute the arc of clothoid $\bar{C}$ linking C_1 and $\bar{C}_2$, that is, N points $(\bar{x^n}, \bar{y^n})$ for $n \in \{1, \ldots, N\}$ setting $\bar{C}$.
4. Rotate the obtained arc of clothoid $\bar{C}$ until $\bar{C}_2$ and C_2 match. That is, compute points (x^n, y^n) setting the arc of clothoid C, following the next steps:

   a) Compute $\bar{c}_2 = (\bar{x}_2, \bar{y}_2)$, the center of $\bar{C}_2$, as

   $$\bar{x}_2 = x^N - \lambda R_2 \sin(\Phi(L)), \tag{16}$$

   $$\bar{y}_2 = y^N + \lambda R_2 \cos(\Phi(L)). \tag{17}$$

   b) Compute the rotate angle as following:

   - Take

   $$\alpha = \arccos\left(\frac{(c_2 - c_1)\cdot(\bar{c}_2 - c_1)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right). \tag{18}$$

   - Define $\tilde{c_2} = (\tilde{x_2}, \tilde{y_2})$ as

   $$\tilde{x_2} = x_1 + (\bar{x}_2 - x_1)\cos(\alpha) - (\bar{y}_2 - y_1)\sin(\alpha), \tag{19}$$

   $$\tilde{y_2} = y_1 + (\bar{x}_2 - x_1)\sin(\alpha) + (\bar{y}_2 - y_1)\cos(\alpha). \tag{20}$$

   - If $c_2 \neq \tilde{c_2}$, take

   $$\alpha = 2\pi - \arccos\left(\frac{(c_2 - c_1)\cdot(\bar{c}_2 - c_1)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right).$$

   c) Rotate points on the arc of clothoid, that is, for $n \in \{1, \ldots, N\}$, compute

   $$x^n = x_1 + (\bar{x^n} - x_1)\cos(\alpha) - (\bar{y^n} - y_1)\sin(\alpha), \tag{21}$$

   $$y^n = y_1 + (\bar{x^n} - x_1)\sin(\alpha) + (\bar{y^n} - y_1)\cos(\alpha). \tag{22}$$

There exist diverse numerical methods that can be employed in order to solve equation (15) (step 2). In this work, the authors have used the *trust-region-reflective method* (see Powell (1970)), that is also implemented in the `fsolve` command of MATLAB R2012a.

To show the usefulness of Algorithm 2 and its accuracy for obtaining arcs of clothoids linking circular arcs, we consider the example given in Koç et al. (2015): we take circumferences C_1, C_2 and C_3 of radii $R_1 = 200$ m, $R_2 = 150$ m and $R_3 = 100$ m and centers c_1=(6736.338,4146.877), c_2=(6736.461,4196.1287) and c_3=(6687.231,4198.388). We use Algorithm 2 for linking C_1 with C_2 and C_2 with C_3, and the final curve is showed in Fig. 4. In Table 2, the coordinates of characteristic points obtained with our method are compared with points given in Koç et al. (2015). As we can see, our method is accuracy and very useful for obtaining the parametrization of the final curve.

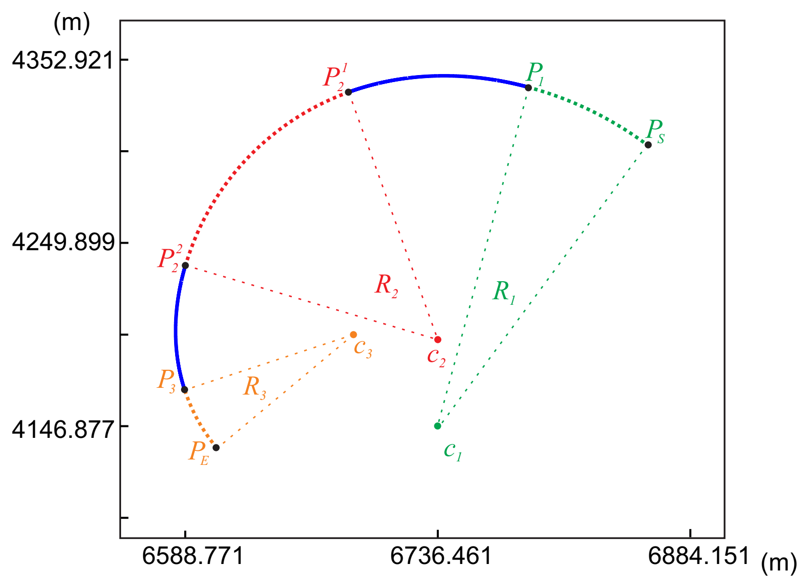

Fig. 4. Arcs of clothoid (solid line) obtained by applying Algorithm 2 to link three circumferences given in the literature. Centers and radii of these circumferences are, respectively, c_1=(6736.338,4146.877), c_2=(6736.461,4196.1287), c_3=(6687.231,4198.388) and $R_1 = 200$ m, $R_2 = 150$ m, $R_3 = 100$ m. The cartesian coordinates of characteristic points P_i are given in Table 2.

Nevertheless, it can not be excluded that some solutions of equation (15) could be not suitable since, depending on the considered problem, the increase in the azimuth can be bounded above. In fact, arcs of clothoid rotating more than once are often out of interest and, as a general rule, only solutions in which $\Delta\Phi$ is smaller than a given threshold will be useful.

For each pair of values of R_1 and R_2 will be very helpful to analyze the graph of the function $d(\Delta\Phi)$. For example, in Fig. 5 the graph of $d(\Delta\Phi)$ for $R_1 = 500$ m and $R_2 = 200$ m is showed. In this case it can be observed that the circumferences will only be linked with an arc of clothoid rotating less than one time, if the value of d_{12} is greater

TABLE 2. Coordinates of some characteristic points of the double-egg curve depicted in Fig. 4, obtained by linking the circumferences of centers c_1=(6736.338,4146.877), c_2=(6736.461,4196.1287), c_3=(6687.231,4198.388) and radii $R_1 = 200$ m, $R_2 = 150$ m, $R_3 = 100$ m. Results obtained with Algorithm 2 are compared with results given in the literature.

Characteristic Point in Fig. 4	Cartesian coordinates (meters) in Koç et al. (2015)	Cartesian coordinates (meters) obtained with Algorithm 2
P_S	(6856.861,4306.484)	——
P_1	(6788.271,4340.017)	(6788.376,4339.989)
P_2^1	(6685.739,4337.290)	(6685.705,4337.273)
P_2^2	(6592.540,4238.393)	(6592.560,4238.466)
P_3	(6592.139,4167.443)	(6592.167,4167.364)
P_E	(6615.414,4128.800)	——

than $d_{min} = 124$ m, i.e., if the distance $D_{12} = R_1 - R_2 - d_{12}$ between the circumferences is smaller than $D_{max} = 176$ m.

So, there are cases where no acceptable clothoid arc can be found to form an egg curve. In Fig. 6 two different arcs of clothoid built using Algorithm 2 are presented: one of them (Fig. 6(a)) connects a circumference C_1 with $c_1 = (0, 0)$ and $R_1 = 500$ m, to a circumference C_2 with $c_2 = (100, 0)$ and $R_2 = 200$ m, while the other (Fig. 6(b)) connects the same C_1 with a circumference C_2 with $c_2 = (200, 0)$ and $R_2 = 200$ m. In the first case the distance between the circumferences is $D_{12} = 200 > D_{max}$, so that the arc of clothoid rotates more than once, while in the other case it happens that $D_{12} = 100 < D_{max}$ and the arc of clothoid is much shorter.

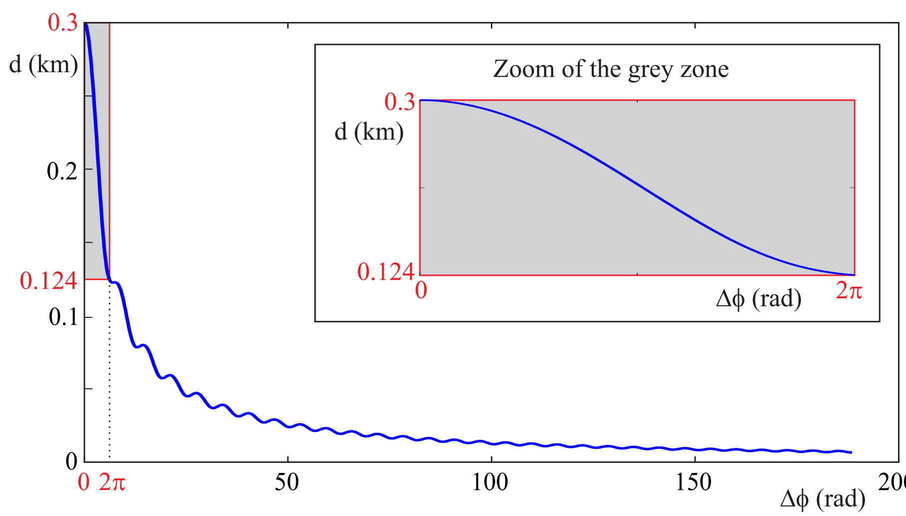

Fig. 5. Graphic of function $d(\Delta\Phi)$ for radii $R_1 = 500$ m and $R_2 = 200$ m

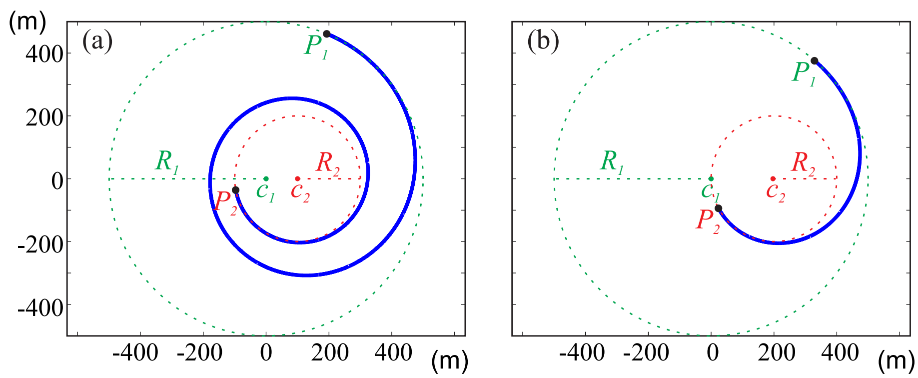

Fig. 6. Egg curves joining two interior circumferences

LINKING WITH CLOTHOID-CIRCUMFERENCE-CLOTHOID. THE DOUBLE-EGG CURVE

As explained in the previous section, whenever the increasing of the azimuth is greater than a given threshold, the egg curve linking the two interior circumferences is not useful for the design of roads. In such cases there exists an alternative transition curve connecting the circumferences known as *double-egg* curve: it is needed to introduce a new circumference C_3, and the double-egg curve will consist in an arc of clothoid joining C_1 and C_3, an arc of C_3 and an arc of clothoid joining C_3 and C_2. In Fig. 7 can be seen how to connect with a double-egg curve the circumferences that could not be linked with a suitable arc of clothoid in Fig. 6(a) .

Connecting Two Given Interior Circumferences

A double-egg curve linking two given interior circumferences C_1 and C_2 can be computed with the next Algorithm 3. Notice that whenever C_1 and C_2 are interior, $R_2 < R_3 < R_1$ and $D_{12} \geq D_{13} + D_{23}$ must be satisfied.

Algorithm 3

1. In order to set the auxiliary circumference C_3, take $R_3 > 0$, $D_{13} > 0$ and $D_{23} > 0$.
2. Calculate $d_{13} = |R_1 - R_3| - D_{13}$ and $d_{23} = |R_3 - R_2| - D_{23}$.
3. Calculate c_3 to completely determine C_3, in the following way:

   a) Compute

$$\mathbf{v} = \frac{1}{d_{12}}((x_2 - x_1, y_2 - y_1), \tag{23}$$

$$\mathbf{v}_\perp = \frac{\lambda}{d_{12}}(y_2 - y_1, x_1 - x_2). \tag{24}$$

   b) Compute

$$\alpha = \arccos\left(\frac{d_{13}^2 + d_{12}^2 - d_{23}^2}{2d_{13}d_{12}}\right). \tag{25}$$

c) Compute

$$c_3 = c_1 + d_{13}\cos(\alpha)\mathbf{v} + d_{13}\sin(\alpha)\mathbf{v}_\perp. \tag{26}$$

4. Use Algorithm 2 to calculate the clothoids joining C_1 to C_3 and C_3 to C_2.

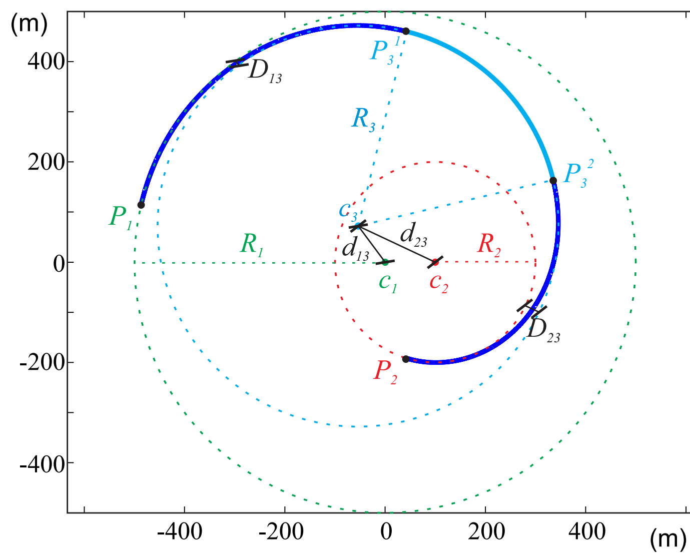

Fig. 7. Double-egg curve joining two interior circumferences: C_1 (dashed green) with center $c_1 = (0, 0)$ and radius $R_1 = 500$ m, and C_2 (dashed red) with center $c_2 = (100, 0)$ and radius $R_2 = 200$ m. In dashed light blue, an auxiliary circumference C_3; in solid dark blue, arcs of clothoid linking C_1 to C_3 and C_3 to C_2; in solid light blue, a circular arc in C_3.

Connecting Two Circumferences Either Exterior or Secant

Whenever the circumferences C_1 and C_2 are not interior and are traversed with the same orientation, it is also required an auxiliary circumference C_3, in order to join them, and both pairs C_1C_3 and C_2C_3 of circumferences must be interior (see Stoer (1982)).

In the case of C_1 and C_2 are exterior, both circumferences must be inside the auxiliary circumference C_3^{out}, that is, $R_3^{out} > R_1$ and $R_3^{out} > R_2$ must hold. By the other hand, if C_1 and C_2 are secant there are two options: a case analogous to the previous, with C_1 and C_2 inside C_3^{out}, or an auxiliary circumference C_3^{in} can be chosen such that is both inside C_1 and C_2, that is, $R_3^{in} < R_1$ and $R_3^{in} < R_2$ must hold.

In any case, the auxiliary circumference C_3 has to fulfill

$$D_{i3} < |R_3 - R_i|, \text{ for } i \in \{1, 2\}, \tag{27}$$

$$D_{13} + D_{23} < |R_3 - R_1| + |R_3 - R_2| - d_{12}, \tag{28}$$

and additionally, depending on the case,

$$R_3^{out} > \frac{1}{2}(R_1 + R_2 + d_{12}), \tag{29}$$

$$R_3^{in} < \frac{1}{2}(R_1 + R_2 - d_{12}). \tag{30}$$

The former Algorithm 3 provide the way to compute the double-egg curve corresponding to each chosen auxiliary circumference C_3. For example, in Fig. 8 is presented a transition curve linking two exterior circumferences, while in Fig. 9 two different curves joining two secant circumferences are shown: one of them built from an auxiliary circumference C_3^{in} that is inside both C_1 and C_2, and the other built with the help of an auxiliary circumference C_3^{out}.

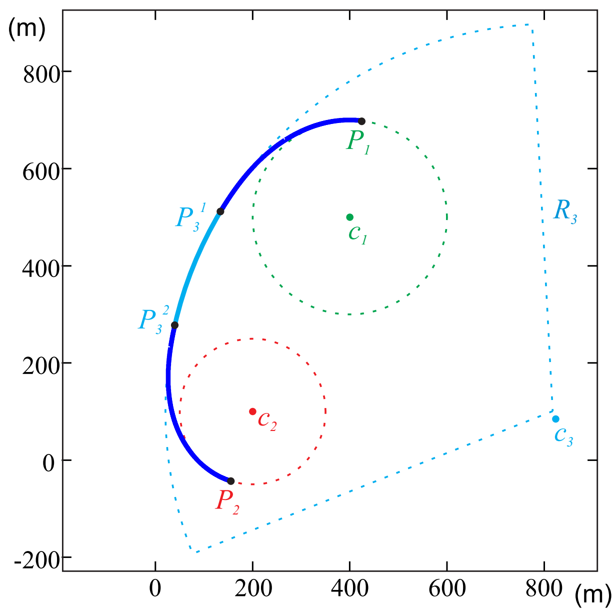

Fig. 8. Double-egg curve joining two exterior circumferences: C_1 (dashed green) with center $c_1 = (400, 500)$ and radius $R_1 = 200$ m, and C_2 (dashed red) with center $c_2 = (200, 100)$ and radius $R_2 = 150$ m. In dashed light blue, an auxiliary circumference C_3; in solid dark blue, arcs of clothoid linking C_1 to C_3 and C_3 to C_2; in solid light blue, a circular arc in C_3.

CONCLUSIONS

In this paper, the method presented in Vázquez-Méndez and Casal (2016) for calculating clothoids is extended to compute partial spirals for linking two circumferences in a suitable way. Partial spirals are very useful for designing transition curves between two circular curves in highway connections and intersections. The classical method to compute clothoids is not accurate for partial spirals and, in some practical applications, it can give wrong results. On the contrary, the alternative method proposed in this paper

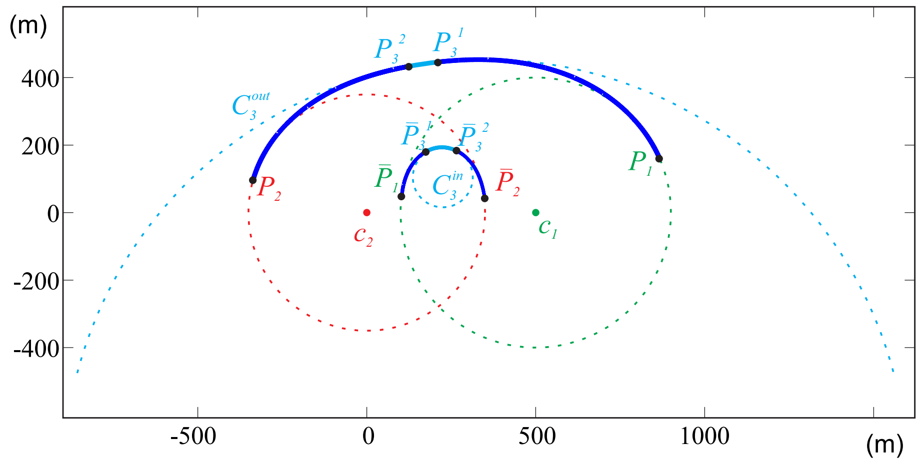

Fig. 9. Two different double-egg curves joining two secant circumferences: C_1 (dashed green) with center $c_1 = (500, 0)$ and radius $R_1 = 400$ m, and C_2 (dashed red) with center $c_2 = (0, 0)$ and radius $R_2 = 300$ m. In dashed light blue, both auxiliary circumferences C_3^{in} and C_3^{out}; in solid dark blue, arcs of clothoid linking C_1 and C_2 to the auxiliary circumference; in solid light blue, a circular arc in the auxiliary circumference.

results very useful and it has allowed us to develop two algorithms for designing the two classical transition curves between circular curves: egg and double-egg curves. These algorithms are very simple, efficient and accurate, and they can be easily included in any model (computer application) for horizontal road design. Particularly, they are very useful tools for designing highway connections and intersections.

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