WITTEN'S PERTURBATION ON STRATA WITH GENERAL ADAPTED METRICS

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Abstract. Let M be a stratum of a compact stratified space A. It is equipped with a general adapted metric g, which is slightly more general than the adapted metrics of Nagase and Brasselet-Hector-Saralegi. In particular, g has a general type, which is an extension of the type of an adapted metric. A restriction on this general type is assumed, and then g is called good. We consider the maximum/minimum ideal boundary condition, $d_{\text{max/min}}$, of the compactly supported de Rham complex on M, in the sense of Brüning-Lesch. Let $H^*_{\text{max/min}}(M)$ and $\Delta_{\text{max/min}}$ denote the cohomology and Laplacian of $d_{\text{max/min}}$. The first main theorem states that $\Delta_{\text{max/min}}$ has a discrete spectrum satisfying a weak form of the Weyl's asymptotic formula. The second main theorem is a version of Morse inequalities using $H^*_{\text{max/min}}(M)$ and what we call rel-Morse functions. An ingredient of the proofs of both theorems is a version for $d_{\text{max/min}}$ of the Witten's perturbation of the de Rham complex. Another ingredient is certain perturbation of the Dunkl harmonic oscillator previously studied by the authors using classical perturbation theory.

The condition on g to be good is general enough in the following sense. Assume that A is a stratified pseudomanifold, and consider its intersection homology $I^{\bar p}H_*(A)$ with perversity $\bar p$; in particular, the lower and upper middle perversities are denoted by $\bar m$ and $\bar n$, respectively. Then, for any perversity $\bar p \le \bar m$, there is an associated good adapted metric on M satisfying the Nagase isomorphism $H^r_{\text{max}}(M) \cong I^{\bar p}H_r(A)^*$ ($r \in \mathbb{N}$). If M is oriented and $\bar p \ge \bar n$, we also get $H^r_{\text{min}}(M) \cong I^{\bar p}H_r(A)$. Thus our version of the Morse inequalities can be described in terms of $I^{\bar p}H_*(A)$.

Contents

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1. Introduction

1.1. **Ideal boundary conditions of the de Rham complex.** The following usual notation is used for a densely defined linear operator T in a Hilbert space. Its domain and range are denoted by $\mathsf{D}(T)$ and $\mathsf{R}(T)$. If T is essentially self-adjoint, its closure is denoted by $\overline{T}$. If T is self-adjoint, its *smooth core* is $\mathsf{D}^\infty(T) := \bigcap_{m=1}^\infty \mathsf{D}(T^m)$, and its spectrum is denoted by $\sigma(T)$.

A *Hilbert complex* $(\mathsf{D}, \mathbf{d})$ is a differential complex of finite length given by a densely defined closed operator $\mathbf{d}$ in a graded separable Hilbert space $\mathfrak{H}$ [9]. Then the operator $\mathbf{D} = \mathbf{d} + \mathbf{d}^*$, with $\mathsf{D}(\mathbf{D}) = \mathsf{D}(\mathbf{d}) \cap \mathsf{D}(\mathbf{d}^*)$, is self-adjoint in $\mathfrak{H}$, and therefore the *Laplacian* $\mathbf{\Delta} = \mathbf{D}^2 = \mathbf{d}\mathbf{d}^* + \mathbf{d}^*\mathbf{d}$ is also self-adjoint. Moreover $\mathsf{D}^\infty(\mathbf{\Delta})$ is a subcomplex of $(\mathsf{D}, \mathbf{d})$ with the same homology [9, Theorem 2.12]; it may be also said that $\mathsf{D}^\infty(\mathbf{\Delta})$ is the *smooth core* of $\mathbf{d}$.

The above notion is applied here in the following case. For a Riemannian manifold M, let $\Omega_0(M)$ be the space of compactly supported differential forms, and $L^2\Omega(M)$ the graded Hilbert space of square integrable differential forms. Let d and δ be the de Rham derivative and coderivative acting on $\Omega_0(M)$, and let $D = d + \delta$ and $\Delta = D^2 = d\delta + \delta d$ (the Laplacian). Every Hilbert complex extension $\mathbf{d}$ of d in $L^2\Omega(M)$ is called an *ideal boundary condition* (*i.b.c.*) [9], giving rise to self-adjoint extensions $\mathbf{D}$ and $\mathbf{\Delta}$ of D and Δ in $L^2\Omega(M)$. There exists a minimum/maximum i.b.c., $d_{\min} = \overline{d}$ and $d_{\max} = \delta^*$, inducing self-adjoint extensions $D_{\max/\min}$ and $\Delta_{\max/\min}$ of D and Δ. If M is oriented, then $\Delta_{\max}$ corresponds to $\Delta_{\min}$ by the Hodge star operator. The corresponding cohomologies, $H_{\max/\min}(M)$, are quasi-isometric invariants of M; for instance, $H_{\max}(M)$ is the usual L^2 cohomology $H_{(2)}(M)$ [12]. They give rise to versions of Betti numbers and Euler characteristic, $\beta^r_{\max/\min} = \beta^r_{\max/\min}(M)$ and $\chi_{\max/\min} = \chi_{\max/\min}(M)$. These concepts can indeed be defined for arbitrary elliptic complexes [9]. It is well known that $d_{\min} = d_{\max}$ if M is complete. Thus considering an i.b.c. becomes interesting when M is not complete. For example, if M is the interior of a compact Riemannian manifold N with with $\partial N \neq \emptyset$, then $d_{\max/\min}$ is defined by taking absolute/relative boundary conditions. With more generality, we will assume that M is a stratum of a compact stratified space A [41, 31, 32, 42], equipped with a generalization of the adapted metrics considered in [33, 34, 8]. As we will see, we can assume $\overline{M} = A$ if desired (it can be said that M is the *regular strum* in this case).

1.2. **Stratified spaces.** Roughly speaking, a (*Thom-Mather*) *stratified space* (or *stratification*) is a Hausdorff, locally compact and second countable space A equipped with a partition into C^∞ manifolds (the *strata*), satisfying certain conditions [41, 31]. In particular, an order relation on the family of strata is defined by declaring $X \leq Y$ when $X \subset \overline{Y}$. With respect to this ordering, the maximum length of chains of strata less or equal than a stratum X is called the *depth* of X. The supremum of the strata depth is called the *depth* of A, denoted $\operatorname{depth} A$. The precise definition and needed preliminaries were collected in [4, Section 3], where we have mainly followed [42]. Instead of recalling it, let us describe how the strata of A fit together, describing also *morphisms/isomorphisms* of stratifications, and, in particular, the group of *automorphisms*, $\operatorname{Aut}(A)$. We proceed by induction on its depth. If $\operatorname{depth} A = 0$, then A is just a C^∞ manifold, and $\operatorname{Aut}(A)$ consists of its diffeomorphisms.

Now, given any $k \in \mathbb{Z}_+$, assume that any stratified space L is described if $\operatorname{depth} L < k$, as well as $\operatorname{Aut}(L)$. If L is compact, the *cone* with *link* L is $c(L) = (L \times [0,\infty))/(L \times \{0\})$, whose *vertex* is the point $* = L \times \{0\} \in c(L)$. Let L' be another compact stratification of depth $< k$, and $\phi : L \to L'$ a morphism. Then let $c(\phi) : c(L) \to c(L')$ be the map induced by $\phi \times \mathrm{id} : L \times [0,\infty) \to L' \times [0,\infty)$; in particular, we get the group $c(\operatorname{Aut}(L)) = \{\, c(\phi) \mid \phi \in \operatorname{Aut}(L)\,\}$. It is also declared that $c(\emptyset) = \{*\}$, for the empty stratification, and $c(\emptyset) = \mathrm{id}$, for the empty map. The cone $c(L)$ is used as a model stratified space of depth k if L is of depth $k-1$, whose strata are $\{*\}$ and the manifolds $Y \times \mathbb{R}_+$ for strata Y of L. The second factor projection $L \times [0,\infty) \to [0,\infty)$ defines a $c(\operatorname{Aut}(L))$-invariant function $\rho : c(L) \to [0,\infty)$, called the *radial function*. The restrictions of ρ to the strata are C^∞. A *conic bundle* is a fiber bundle T over a manifold X with typical fiber $c(L)$ and structural group $c(\operatorname{Aut}(L))$. Then ρ induces a *radial function* on T, also denoted by ρ, and the vertex of $c(L)$ defines the *vertex section* of T, whose image is identified with X. Moreover the stratified structure defined on $c(L)$ can be used to define a stratified structure on T, where X becomes the *vertex stratum*.

For any stratification A of depth k, every stratum X has an open neighborhood (a *tube* representative) that is isomorphic to an open neighborhood of X in some conic bundle T_X over X (with the obvious restrictions of stratified structures to open subsets). The typical fiber of T_X is of the form $c(L_X)$ for some compact stratification L_X (the *link* of X) with $\operatorname{depth} L_X < \operatorname{depth} A$. The vertex and radial function of $c(L_X)$ are denoted by $*_X$ and ρ_X. Two such neighborhoods of X represent the same *tube* if their structure is equal on some smaller neighborhood of X. Note that X is open in A if and only if $L_X = \emptyset$.

Finally, a *morphism* between two stratifications is a continuous map sending every stratum to another stratum, whose restrictions to the strata are C^∞, and whose restrictions to small enough tube representatives are restrictions of conic bundle morphisms. Then *isomorphisms* and *automorphisms* of stratifications have the obvious meaning. This completes the description because the depth is locally finite by the local compactness.

The (topological) dimension of a stratification A equals the supremum of the dimensions of its strata. It may be infinite, but it is locally finite. The *codimension* of every stratum X is $\dim A - \dim X$. Our main results will assume that the stratification is compact, but non-compact stratifications will be also used in the proofs. In any case, we will only consider stratifications of finite dimension. If the above description of A is modified by requiring that, at every inductive step, only stratifications with no strata of codimension 1 are used, then A is called a *stratified pseudomanifold*.

A locally closed subset $B \subset A$ is called a *substratification* of A if the restrictions of the strata and tubes of A to B define a stratified structure on B. For instance, A can be restricted to any open subset, to any locally closed union of strata, and to the closure of any stratum. If moreover there are tube representatives of A whose restrictions to B have the same fibers over points of B, then B is called *saturated*.

Let x be a point of a stratum X of dimension m_X in a stratification A. A local trivialization of T_X on some open neighborhood U of x defines a *chart* $O \equiv O'$ of A for some open $O' \subset \mathbb{R}^{m_X} \times c(L_X)$. We can assume $O' = U' \times c_\epsilon(L_X)$, where U' is some open neighborhood of 0 in $\mathbb{R}^{m_X}$ and $c_\epsilon(L_X)$ is the subset of $c(L_X)$ defined by the condition $\rho_X < \epsilon$, for some $\epsilon > 0$. This chart is said to be *centered* at x if $x \equiv (0, *_X) \in O'$. The corresponding concept of *atlas* has the obvious meaning. These concepts can be generalized as follows. Any finite product of stratifications has a non-canonical stratified structure [4, Section 3.1.2]; in particular, any finite product of cones is isomorphic to a cone [4, Lemma 3.8]. Moreover $\operatorname{Aut}(P) \times \operatorname{Aut}(Q)$ is canonically injected in $\operatorname{Aut}(P \times Q)$ for stratifications P and Q. Thus it makes sense to consider a decomposition $c(L_X) \cong \prod_{i=1}^{a_X} c(L_{X,i})$ ($a_X \in \mathbb{N}$), for compact stratifications $L_{X,i}$. The vertex and radial function of every $c(L_{X,i})$ are denoted by $*_{X,i}$ and $\rho_{X,i}$. Then we can also consider *general tube* representatives given by bundles T_X with typical fibers $\prod_{i=1}^{a_X} c(L_{X,i})$ and structural groups $\prod_{i=1}^{a_X} c(\operatorname{Aut}(L_{X,i}))$. This gives rise to a *general chart* $O \equiv O'$ around x for some open $O' \subset \mathbb{R}^{m_X} \times \prod_{i=1}^{a_X} c(L_{X,i})$, which is *centered* at x if $x \equiv (0, *_{X,1}, \dots, *_{X,a_X}) \in O'$. As above, we can assume $O' = U' \times \prod_{i=1}^{a_X} c_\epsilon(L_{X,i})$ for some $\epsilon > 0$. Let $\rho_{X,0}$ denote the norm function on $\mathbb{R}^{m_X}$. The function $\rho = (\rho_{X,0}^2 + \dots + \rho_{X,a_X}^2)^{1/2}$ is called the *radial* function of $\mathbb{R}^{m_X} \times \prod_{i=1}^{a_X} c(L_{X,i})$, even though, when $m_X = 0$, ρ is not the radial function of any cone structure on $\prod_{i=1}^{a_X} c(L_{X,i})$ [4, Example 3.6 and Proof of Lemma 3.8]. A collection of general charts covering A is called a *general atlas*.

We can suppose that the strata of A are connected [4, Remark 1 (v)]. Fix a stratum M of dimension n in A. Since the stratified structure of A can be restricted to $\overline{M}$ [4, Section 3.1.1], we can also assume without loss of generality that $\overline{M} = A$ (any other stratum is $< M$); in particular, $\operatorname{depth} A = \operatorname{depth} M$ and $\dim A = n$. With the above notation, for a chart $O \equiv O'$ centered at x, we get $M \cap O \equiv M' \cap O'$, where $M' = \mathbb{R}^{m_X} \times N \times \mathbb{R}_+$ for some dense stratum N on L_X. In the case of a general chart $O \equiv O'$ centered at x, we have $M \cap O \equiv M' \cap O'$ for $M' = \mathbb{R}^{m_X} \times \prod_{i=1}^{a_X} (N_i \times \mathbb{R}_+)$, where every N_i is some dense stratum of $L_{X,i}$. We will use the notation $k_{X,i} = \dim N_i + 1$.

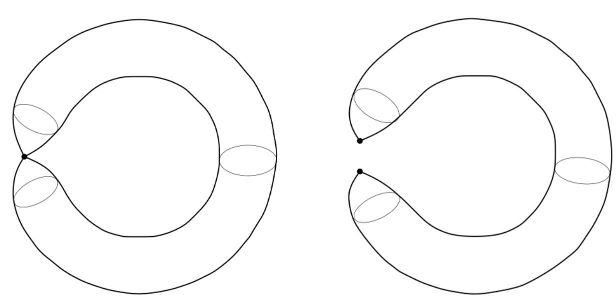
1.3. **General adapted metrics.** A *general adapted metric* g on M is defined by induction on the depth of M. It is any (Riemannian) metric if $\operatorname{depth} M = 0$. Now, assume that $\operatorname{depth} M > 0$ and general adapted metrics are defined for lower depth. Given any general chart $O \equiv O'$ as above, take any general adapted metric $\tilde{g}_i$ on every N_i ($\operatorname{depth} N_i < \operatorname{depth} M$), and let $g_i = \rho_{X,i}^{2u_{X,i}} \tilde{g}_i + (d\rho_{X,i})^2$ on $N_i \times \mathbb{R}_+$ for some $u_{X,i} > 0$. Let also g_0 be the Euclidean metric on $\mathbb{R}^{m_X}$. Then g is a *general adapted metric* if, via any such general chart, $g|_O$ is quasi-isometric to $(\sum_{i=0}^{a_X} g_i)|_{O'}$. In this case, the mapping $X \mapsto u_X := (u_{X,1}, \dots, u_{X,a_X}) \in \mathbb{R}_+^{a_X}$ ($X < M$) is called the *general type* of g. Such a general chart is called *compatible* with g, or with its general type.

Let us point out that a general metric does not completely determine its general type. For instance, suppose $u_{X,i} = u_{X,j} = 1$ for indices $i \neq j$. Write $c(L_{X,i}) \times c(L_{X,j}) \equiv c(L)$, with radial function ρ, for some stratification L. Then $N_i \times \mathbb{R}_+ \times N_j \times \mathbb{R}_+ \equiv N \times \mathbb{R}_+$ for some dense stratum N of L. Moreover there is a general adapted metric $\tilde{g}$ on N such that $g_i + g_j$ is quasi-isometric to $\rho^2 \tilde{g} + (d\rho)^2$ via the above identity. Therefore we can omit $u_{X,i}$ or $u_{X,j}$ in u_X, obtaining a different type of g. This cannot be done if $u_{X,i} = u_{X,j} \neq 1$ (Proposition 2.1).

If the above definition of general adapted metric is modified by requiring that, at every inductive step, the general type satisfies $u_{X,i} \leq 1$ for all $X < M$ and $i = 1, \dots, a_X$, then the general adapted metric is called *good* for the scope of this paper. On the other hand, if the definition is modified by requiring at every inductive step that $a_X = 1$ and u_X depends only on $k := k_{X,1} = \operatorname{codim} X$ for all $X < M$, then we get the *adapted metrics* considered in [33, 34, 8]. In this case, the general charts compatible with the general type are indeed charts. Writing $u_k = u_X \equiv u_{X,1} \in \mathbb{R}_+$, the condition on an adapted metric g to be good becomes $u_k \leq 1$ for all k, at every inductive step of its definition. In [33, 34, 8], it is assumed that A is a stratified pseudomanifold, and then $\hat{u} = (u_2, \dots, u_n)$ stands for the *type* of g. This $\hat{u}$ is determined by g. In particular, if the definition is modified by taking $u_k = 1$ for all k at every inductive step, we get the adapted metrics of *conic type* considered in [12, 13, 14]. Be alerted about the three slightly different terms used for the scope of this paper: adapted metrics of conic type, adapted metrics and general adapted metrics. The class of (good) general adapted metrics is preserved by products, as well as the class of adapted metrics of conic type, but the class of adapted metrics does not have this property. The existence of general adapted metrics with any possible general type can be shown like in the case of adapted metrics [33, Lemma 4.3], [8, Appendix].

Like in [4], the term "relative(ly)" (or simply "rel-") usually means that some condition is required in the intersection of M with small neighborhoods of the points in $\overline{M}$, or that some concept can be described using those intersections.

Let M be equipped with a general adapted metric g, with a general type $X \mapsto u_X$ as above. The *rel-local metric completion* $\widehat{M}$ of M consists of the points in the metric completion represented by Cauchy sequences that converge in $\overline{M}$ ($\widehat{M}$ is the metric completion of M if $\overline{M}$ is compact). Figure 1 illustrates this concept. The limits of Cauchy sequences define a continuous map $\lim : \widehat{M} \to \overline{M}$. The following properties can be proved like in the case of conic metrics [4, Proposition 3.20 (i),(ii)]. $\widehat{M}$ has a unique stratified structure with connected strata so that $\lim : \widehat{M} \to \overline{M}$ is a morphism whose restrictions to the strata are local diffeomorphisms. Moreover g is also a general adapted metric with respect to $\widehat{M}$.



(a) $\overline{M}$ (b) $\widehat{M}$

FIGURE 1. The stratified space $\widehat{M}$.

1.4. **Relatively Morse functions.** A smooth function f on M is called *rel-admissible* when the functions f, $|df|$ and $|\operatorname{Hess} f|$ are rel-bounded. In this case,

f may not have any continuous extension to $\overline{M}$, but it has a continuous extension to $\widehat{M}$. So it makes sense to say that $x \in \widehat{M}$ is a *rel-critical point* of f when $\liminf |df(y)| = 0$ as $y \to x$ in $\widehat{M}$ with $y \in M$. The set of rel-critical points of f is denoted by $\mathrm{Crit}_{\mathrm{rel}}(f)$. It is said that f is a *rel-Morse function* if it is rel-admissible and has the following description around every $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$:

- there is a general chart $O \equiv O'$ of $\widehat{M}$, centered at x and compatible with g, such that $M \cap O \equiv M' \cap O'$ for $M' = \mathbb{R}^{m_X} \times \prod_{i=1}^{a_X}(N_i \times \mathbb{R}_+)$, where X is the stratum of $\widehat{M}$ containing x; and
- $f|_{M\cap O} \equiv f(x) + \frac{1}{2}(\rho_+^2 - \rho_-^2)|_{M'\cap O'}$, where $\rho_\pm$ is the radial function of $\mathbb{R}^{m_\pm} \times \prod_{i\in I_\pm} c(L_{X,i})$ for some expression $m_X = m_+ + m_-$ ($m_\pm \in \mathbb{N}$) and some partition of $\{1, \dots, a_X\}$ into sets $I_\pm$.

This local condition is used instead of requiring that $\mathrm{Hess}\, f$ is "rel-non-degenerate" at the rel-critical points because a "rel-Morse lemma" is missing. Moreover, for every $r \in \{0, \dots, n\}$, let

$$\nu^r_{x,\max/\min} = \sum_{(r_1,\dots,r_{a_X})} \prod_{i=1}^{a_X} \beta^{r_i}_{\max/\min}(N_i)\ , \tag{1}$$

where $(r_1, \dots, r_{a_X})$ runs in the subset of $\mathbb{N}^{a_X}$ determined by

$$\left.\begin{array}{c} r = m_- + \sum_{i=1}^{a_X} r_i + |I_-|\ , \\ \left.\begin{array}{ll} r_i < \frac{k_{X,i}-1}{2} + \frac{1}{2u_{X,i}} & \text{if } i \in I_+ \\ r_i \ge \frac{k_{X,i}-1}{2} + \frac{1}{2u_{X,i}} & \text{if } i \in I_- \end{array}\right\} \text{ for } \nu^r_{x,\max}\ , \\ \left.\begin{array}{ll} r_i \le \frac{k_{X,i}-1}{2} - \frac{1}{2u_{X,i}} & \text{if } i \in I_+ \\ r_i > \frac{k_{X,i}-1}{2} - \frac{1}{2u_{X,i}} & \text{if } i \in I_- \end{array}\right\} \text{ for } \nu^r_{x,\min}\ . \end{array}\right\} \tag{2}$$

When $a_X = 0$ in (1), the singleton $\mathbb{N}^0$ consists of the empty sequence, obtaining[1] $\nu^r_{x,\max/\min} = \delta_{r,m_-}$ with the convention that the value of empty products is 1. Finally, let $\nu^r_{\max/\min} = \sum_x \nu^r_{x,\max/\min}$ with x running in $\mathrm{Crit}_{\mathrm{rel}}(f)$. The notation $\nu^r_{x,\max/\min}(f)$ and $\nu^r_{\max/\min}(f)$ may be used if necessary. The existence of rel-Morse functions for general adapted metrics holds like in the case of adapted metrics [4, Proposition 4.9].

1.5. **Main theorems.** The following is our first main theorem, where property (ii) is a weak version of the Weyl's asymptotic formula.

Theorem 1.1. *The following properties hold on any stratum of a compact stratification with a good general adapted metric:*

(*i*) $\Delta_{\max/\min}$ *has a discrete spectrum,* $0 \le \lambda_{\max/\min,0} \le \lambda_{\max/\min,1} \le \cdots$, *where every eigenvalue is repeated according to its multiplicity.*
(*ii*) $\liminf_k \lambda_{\max/\min,k}\, k^{-\theta} > 0$ *for some* $\theta > 0$.

Our second main result is the following version of Morse inequalities for rel-Morse functions.

[1] Kronecker's delta symbol is used.

Theorem 1.2. *For any rel-Morse function on a stratum of dimension n of a compact stratification, equipped with a good general adapted metric, we have*

$$\sum_{r=0}^{k}(-1)^{k-r}\,\beta^r_{\mathrm{max/min}} \le \sum_{r=0}^{k}(-1)^{k-r}\,\nu^r_{\mathrm{max/min}} \quad (0 \le k < n)\;,$$

$$\chi_{\mathrm{max/min}} = \sum_{r=0}^{n}(-1)^r\,\nu^r_{\mathrm{max/min}}\;.$$

In the case of adapted metrics of conic type, Theorem 1.1 (i) is essentially due to Cheeger [12, 13] (see also [1, 2, 4]), Theorem 1.1–(ii) was proved by the authors [4], and Theorem 1.2 was proved by the authors [4] and Ludwig [30] (with more restrictive conditions but stronger consequences). Other developments of elliptic theory on strata were made in [10, 25, 23, 39, 16, 2, 1], all of them using adapted metrics of conic type. The main novelty of our paper is the extension of the elliptic theory on strata to the wider class of good general adapted metrics, including good adapted metrics.

1.6. **Applications to intersection homology.** Consider now the case where A is a stratified pseudomanifold, and therefore M is its regular stratum. Let $I^{\bar p}H_*(A)$ denote its intersection homology with perversity $\bar p$ [19, 20], taking real coefficients. Let $\beta^{\bar p}_r = \beta^{\bar p}_r(A)$ and $\chi^{\bar p} = \chi^{\bar p}(A)$ denote the versions of Betti numbers and Euler characteristic for $I^{\bar p}H_*(A)$. Every perversity can be considered as a sequence $\bar p = (p_2, p_3, \dots)$ in $\mathbb{N}$ satisfying $p_2 = 0$ and $p_k \le p_{k+1} \le p_k + 1$. For example, the zero perversity is $\bar 0 = (0, 0, \dots)$, the top perversity is $\bar t = (0, 1, 2, \dots)$ ($t_k = k - 2$), the lower middle perversity is $\bar m = (0, 0, 1, 1, 2, 2, 3, \dots)$ ($m_k = \lfloor \frac{k}{2} \rfloor - 1$), and the upper middle perversity is $\bar n = (0, 1, 1, 2, 2, 3, 3, \dots)$ ($n_k = \lceil \frac{k}{2} \rceil - 1$). Recall also that two perversities $\bar p$ and $\bar q$ are called complementary if $\bar p + \bar q = \bar t$. Write $\bar p \le \bar q$ if $p_k \le q_k$ for all k. Let g be an adapted metric on M of type $\hat u = (u_2, \dots, u_n)$. If $\hat u$ is *associated* with a perversity $\bar p \le \bar m$ in the sense

$$\left.\begin{array}{lll} \frac{1}{k-1-2p_k} \le u_k < \frac{1}{k-3-2p_k} & \text{if} & 2p_k \le k-3\;, \\ 1 \le u_k < \infty & \text{if} & 2p_k = k-2\;, \end{array}\right\} \tag{3}$$

then $H^r_{(2)}(M) \cong I^{\bar p}H_r(A)^*$ [33, 34, 8], and therefore $\beta^{\bar p}_r = \beta^r_{\max}$. In particular, $H^r_{(2)}(M) \cong I^{\bar m}H_r(A)^*$ if g is an adapted metric of conic type [14]. Thus the incompatibility of adapted metrics with products is related to the subtleties of the versions of the Künneth theorem for intersection homology [15, 17]. For instance, the isomorphism $I^{\bar p}H_*(P \times Q) \cong I^{\bar p}H_*(P) \otimes I^{\bar p}H_*(Q)$, for arbitrary pseudomanifolds P and Q, only holds with some special perversities $\bar p$, including $\bar p = \bar m$. According to (3), there exist good adapted metrics on M whose type is associated with any given perversity $\le \bar m$.

In (3), only the choices $2p_k = k-2, k-4, \dots$ are possible if k is even, and only the choices $2p_k = k-3, k-5, \dots$ are possible if k is odd. Thus, for every k, (3) establishes a bijection between the possibilities for p_k and a partition of $[\frac{1}{k-1}, \infty)$ into semi-open intervals, where u_k is taken.

Let f be a rel-Morse function on M, let $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$, let X be the stratum of $\widehat M$ containing x, and let $k = \operatorname{codim} X$. With the above notation for a chart $O \equiv O'$ of $\widehat M$ centered at x, there is an adapted metric $\tilde g$ on N so that, via the chart, $g|_O$ is quasi-isometric to the restriction of $g_0 + \rho_X^{2u_k}\tilde g + (d\rho_X)^2$ to $M' \cap O'$.

Then the type of $\tilde g$ is also associated with $\bar p$. Moreover there is some expression, $m_X = m_+ + m_-$ ($m_\pm \in \mathbb{N}$), and some decomposition, $c(L_X) \equiv c(L_+) \times c(L_-)$, so that $M' \equiv \mathbb{R}^{m_+} \times N_+ \times \mathbb{R}_+ \times \mathbb{R}^{m_-} \times N_- \times \mathbb{R}_+$ for dense strata $N_\pm$ of $L_\pm$, and $f|_O \equiv f(x) + \frac{1}{2}(\rho_+^2 - \rho_-^2)|_{O'}$, where $\rho_\pm$ is the radial function of $\mathbb{R}^{m_\pm} \times c(L_\pm)$. Let $k_\pm = \dim N_\pm + 1$; thus $k = k_+ + k_-$. Here, some of the stratifications $L_\pm$ may be empty; in fact, $L_+ \neq \emptyset \neq L_-$ only can happen if $u_k = 1$ (Section 1.3). From (1) and (2), it follows that the numbers $\nu^r_{x,\max}$ are independent of the choice of $\hat u$ associated with $\bar p$, and therefore the notation $\nu^{\bar p}_{x,r} = \nu^{\bar p}_{x,r}(f)$ will be used. Precisely, they have the following expressions:

- If $L_+ \neq \emptyset \neq L_-$ (only if $u_k = 1$), then
$$\nu^{\bar p}_{x,r} = \sum_{(r_+,r_-)} \beta^{\bar p}_{r_+}(L_+)\,\beta^{\bar p}_{r_-}(L_-)\;,$$
where (r_+, r_-) runs in the subset of $\mathbb{N}^2$ determined by the conditions
$$r = m_- + r_+ + r_- + 1\;,\quad r_+ < \tfrac{k_+}{2}\;,\quad r_- \geq \tfrac{k_-}{2}\;.$$
- If $L_X = L_+ \neq \emptyset$ ($L_- = \emptyset$), then
$$\nu^{\bar p}_{x,r} = \sum_{r_+} \beta^{\bar p}_{r_+}(L_X)\;,$$
where r_+ runs in the subset of $\mathbb{N}$ determined by the conditions
$$r = m_- + r_+\;,\quad r_+ < \begin{cases} k - 1 - p_k & \text{if } u_k < 1 \\ \frac{k}{2} & \text{if } u_k = 1\;. \end{cases}$$
- If $L_X = L_- \neq \emptyset$ ($L_+ = \emptyset$), then
$$\nu^{\bar p}_{x,r} = \sum_{r_-} \beta^{\bar p}_{r_-}(L_X)\;,$$
where r_- runs in the subset of $\mathbb{N}$ determined by the conditions
$$r = m_- + r_- + 1\;,\quad r_- \geq \begin{cases} k - 1 - p_k & \text{if } u_k < 1 \\ \frac{k}{2} & \text{if } u_k = 1\;. \end{cases}$$
- If $L_X = \emptyset$, then $\nu^{\bar p}_{x,r} = \delta_{r,m_-}$.

Finally, let $\nu^{\bar p}_r = \nu^{\bar p}_r(f) = \sum_x \nu^{\bar p}_{x,r}$ ($x \in \mathrm{Crit}_{\mathrm{rel}}(f)$), which equals $\nu^r_{\max}$.

Suppose now that A is oriented (M is oriented) and compact. We have $\beta^r_{\min} = \beta^{n-r}_{\max}$ for all r because $\Delta_{\min}$ corresponds to $\Delta_{\max}$ by the Hodge star operator. On the other hand, for any perversity $\bar q \geq \bar n$, if $\bar p \leq \bar m$ is complementary of $\bar q$, then $I^{\bar q}H_r(A) \cong I^{\bar p}H_{n-r}(A)^*$ [19, 20], and therefore $\beta^{\bar q}_r = \beta^{\bar p}_{n-r}$, obtaining $\beta^{\bar q}_r = \beta^r_{\min}$. As before, it follows from (1) and (2) that the numbers $\nu^r_{x,\min}$ are independent of the choice of $\hat u$ associated with $\bar p$. Precisely, with the notation $\nu^{\bar q}_{x,r} = \nu^{\bar q}_{x,r}(f) = \nu^r_{x,\min}$, they have the following expressions:

- If $L_+ \neq \emptyset \neq L_-$ (only if $u_k = 1$), then
$$\nu^{\bar q}_{x,r} = \sum_{(r_+,r_-)} \beta^{\bar q}_{r_+}(L_+)\,\beta^{\bar q}_{r_-}(L_-)\;,$$
where (r_+, r_-) runs in the subset of $\mathbb{N}^2$ determined by the conditions
$$r = m_- + r_+ + r_- + 1\;,\quad r_+ \leq \tfrac{k_+}{2} - 1\;,\quad r_- > \tfrac{k_-}{2} - 1\;.$$

- If $L_X = L_+ \neq \emptyset$ ($L_- = \emptyset$), then
$$\nu^{\bar q}_{x,r} = \sum_{r_+} \beta^{\bar q}_{r_+}(L_X)\ ,$$
where r_+ runs in the subset of $\mathbb{N}$ determined by the conditions
$$r = m_- + r_+\ ,\quad r_+ \leq \begin{cases} k-2-q_k & \text{if } u_k < 1 \\ \frac{k}{2} - 1 & \text{if } u_k = 1\ . \end{cases}$$
- If $L_X = L_- \neq \emptyset$ ($L_+ = \emptyset$), then
$$\nu^{\bar q}_{x,r} = \sum_{r_-} \beta^{\bar q}_{r_-}(L_X)\ ,$$
where r_- runs in the subset of $\mathbb{N}$ determined by the conditions
$$r = m_- + r_- + 1\ ,\quad r_- > \begin{cases} k-2-q_k & \text{if } u_k < 1 \\ \frac{k}{2} - 1 & \text{if } u_k = 1\ . \end{cases}$$
- If $L_X = \emptyset$, then $\nu^{\bar q}_{x,r} = \delta_{r,m_-}$.

Like $\nu^{\bar p}_r$, we also define $\nu^{\bar q}_r = \nu^{\bar q}_r(f) = \sum_x \nu^{\bar q}_{x,r}$ ($x \in \mathrm{Crit}_{\mathrm{rel}}(f)$), which equals $\nu^r_{\min}$.

Theorem 1.2 has the following direct consequence.

Corollary 1.3. *Let A be a compact pseudomanifold of dimension n, let M be its regular stratum, and let $\bar p$ be a perversity. If $\bar p \leq \bar m$, or if A is oriented and $\bar p \geq \bar n$, then, for any rel-Morse function on M (with respect to any good adapted metric), we have*
$$\sum_{r=0}^{k} (-1)^{k-r}\, \beta^{\bar p}_r \leq \sum_{r=0}^{k} (-1)^{k-r}\, \nu^{\bar p}_r \quad (0 \leq k < n)\ ,$$
$$\chi^{\bar p} = \sum_{r=0}^{n} (-1)^r\, \nu^{\bar p}_r\ .$$

Stratified Morse theory was introduced by Goresky and MacPherson [21], and has a great wealth of applications. In particular, Goresky and MacPherson have proved Morse inequalities on complex analytic varieties with Whitney stratifications, involving the intersection homology with perversity $\bar m$ [21, Chapter 6, Section 6.12]. Ludwig also gave an analytic interpretation of Morse theory in the spirit of Goresky and MacPherson for conformally conic manifolds [26, 27, 28, 29]. Our version of Morse functions, critical points and associated numbers is different from those used in [21], even in the case of perversity $\bar m$. To the authors' knowledge, Corollary 1.3 is the first version of Morse inequalities for intersection homology with perversity $\neq \bar m$.

1.7. **Ideas of the proofs.** In the proofs of Theorems 1.1 and 1.2, several steps are like in the case of adapted metrics of conic type [4]. Only brief indications of those steps are given in this paper, whereas the parts with new ideas are explained with detail. We adapt the well-known analytic method of Witten [43]; specially, as described in [36, Chapters 9 and 14]. Thus, given a rel-Morse function f on M, we consider the Witten's perturbation $d_s = e^{-sf} d e^{sf} = d + s\, df\wedge$ on $\Omega_0(M)$ ($s > 0$). Let $d_{s,\max/\min}$ denote its maximum/minimum i.b.c., with corresponding Laplacian $\Delta_{s,\max/\min}$. Since $\Delta_{s,\max/\min} - \Delta_{\max/\min}$ is bounded, it is enough to prove the properties of Theorem 1.1 for $\Delta_{s,\max/\min}$. Moreover, using a globalization procedure [4, Propositions 14.2 and 14.3] and a version of the Künneth theorem [9, Corollary 2.15], [4, Lemma 5.1], it is enough to consider the case of a stratum $M = N \times \mathbb{R}_+$ of a cone $c(L)$ (a non-compact stratification), with a good general adapted metric of the form $g = \rho^{2u}\tilde{g} + d\rho^2$, and the rel-Morse function $\pm\frac{1}{2}\rho^2$, where ρ is the radial function and L a compact stratification of smaller depth. A tilde is added to the notation of concepts considered for N. By induction on the depth, it is assumed that $\widetilde{\Delta}_{\max/\min}$ satisfies the properties of Theorem 1.1. Then its eigenforms are used like in [4] to split $d_{s,\max/\min}$ into a direct sum of Hilbert complexes of length one and two, which can be described as the maximum/minimum i.b.c. of certain elliptic complexes on $\mathbb{R}_+$. The elliptic complexes of length one are of the same kind as in [4], so that the Laplacian of their maximum/minimum i.b.c. is induced by the Dunkl harmonic oscillator on $\mathbb{R}$ [3], whose spectrum is well known. However, the Laplacian of the elliptic complexes of length two is a perturbation of the Dunkl harmonic oscillator containing new terms of the form ρ^{-2u} and ρ^{-2u-1}. A different analytic tool is used here, which was developed by the authors [5]. Precisely, classical perturbation methods were used in [5] to determine self-adjoint operators with discrete spectra defined by this perturbation of the Dunkl harmonic oscillator, giving also upper and lower estimates of its eigenvalues. The application of this analytic tool is what requires g to be good. The information obtained for this perturbation is weaker than for the Dunkl harmonic oscillator. For instance, such self-adjoint operators are only known to exist in some cases, and only a core of their square root is known. Thus more work is needed here than in [4] to describe the Laplacians of the maximum/minimum i.b.c. of the simple elliptic complexes of length two, using those self-adjoint operators. The proof of Theorem 1.1 can be completed with such information like in [4]. On the other hand, only eigenvalue estimates of those self-adjoint operators are known, which makes it more difficult to determine the "cohomological contribution" of the rel-critical points. This is the key idea to complete the proof of Theorem 1.2 like in [4].

1.8. **Some open problems.** We do not know whether the condition on g to be good could be deleted. It depends on whether the result used from [5] holds with weaker hypothesis.

The applications would increase by extending our version of Morse inequalities to "rel-Morse-Bott functions." Their rel-critical point set would be a finite union of substratifications.

There should be an extension of the isomorphism $H^r_{(2)}(M) \cong I^{\bar{p}}H_r(A)^*$ to the case of general adapted metrics and general perversities [18]. In that direction, an extension of the de Rham theorem with general perversities was proved in [37, 38]. The case with classical perversities was previously considered in [11, 7].

It is also natural to continue with the following program, already achieved on closed manifolds. First, it should be shown that there is a spectral gap of the form $\sigma(\Delta_{s,\max/\min}) \cap (C_1 e^{-C_2 s}, C_3 s) = \emptyset$, for some $C_1, C_2, C_3 > 0$. This would define a finite-dimensional complex $(\mathcal{S}_{s,\max/\min}, d_s)$ generated by the eigenforms corresponding to eigenvalues in $[0, C_1 e^{-C_2 s}]$ ("small eigenvalues"). Second, it should be proved that $(\mathcal{S}_{s,\max/\min}, d_s)$ "converges" to the "rel-Morse-Thom-Smale complex," assuming that the function satisfies the "rel-Morse-Smale transversality condition." It seems that the existence of the above spectral gap would follow easily by adapting the arguments of [4, Propositions 14.2 and 14.3]. The comparison of $(\mathcal{S}_{s,\max/\min}, d_s)$

with the "rel-Morse-Thom-Smale complex" would require additional techniques, according to the case of closed manifolds [22], [6, Section 6]. This program was developed by Ludwig in a special case [30].

2. Preliminaries

2.1. **Products of cones.** Let L and L' be compact stratifications, and let $*$ and ρ, and $*'$ and ρ' be the vertices and radial functions of $c(L)$ and $c(L')$. Any morphism $\psi : c(L) \to c(L')$ is of the form $c(\phi)$ around $*$ for some morphism $\phi : L \to L'$. In particular, $\psi(*) = *'$, and $\psi^*\rho' = \rho$ around $*$.

The product of two stratifications, $A \times A'$, has a stratification structure whose strata are the products of strata of A and A'. However the tubes in $A \times A'$ depend on the choice of a function $h : [0,\infty)^2 \to [0,\infty)$ that is continuous, homogeneous of degree one, smooth on $\mathbb{R}^2_+$, with $h^{-1}(0) = \{(0,0)\}$, and such that, for some $C > 1$, we have $h(r,r') = \max\{r,r'\}$ if $C \min\{r,r'\} < \max\{r,r'\}$ [4, Section 3.1.2]. Thus the stratification structure of $A \times A'$ is not unique.

In the case of two cones, $c(L) \times c(L')$ can be described as another cone in the following way [4, Lemma 3.8]. The function $h(\rho\times\rho') : c(L)\times c(L') \to [0,\infty)$ satisfies that $L'' = (h(\rho \times \rho'))^{-1}(1)$ is a compact saturated substratification of $c(L) \times c(L')$. Then the map

$$\phi : c(L'') \to c(L) \times c(L') \ , \quad [([x,r],[x',r']),s] \mapsto ([x,rs],[x',r's]) \ ,$$

is an isomorphism of stratifications. The vertex of $c(L'')$ is $*'' = \phi^{-1}(*,*')$, and its radial function is $\rho'' = \phi^*(h(\rho \times \rho'))$. Thus the radial function of $c(L) \times c(L')$, $(\rho^2 + \rho'^2)^{1/2}$, does not correspond to ρ'' via ϕ if $L \neq \emptyset \neq L'$.

Assume that $L \neq \emptyset \neq L'$. Let N and N' be strata of L and L', and let $M = N \times \mathbb{R}_+$ and $M' = N' \times \mathbb{R}_+$ be the corresponding strata of $c(L)$ and $c(L')$. Take general adapted metrics $\tilde g$ and $\tilde g'$ on N and N', and fix any $u > 0$. We get general adapted metrics $g = \rho^{2u}\tilde g + (d\rho)^2$ and $g' = \rho'^{2u}\tilde g' + (d\rho')^2$ on M and M'. On the other hand, with the above notation, we have $\phi^{-1}(M \times M') = N'' \times \mathbb{R}_+ =: M''$, where $N'' = (M \times M') \cap L''$ (a stratum of L''). Let $\tilde g''$ be any general adapted metric on N'' so that $N'' \hookrightarrow M \times M'$ is quasi-isometric; for instance, we may take $\tilde g'' = (g + g')|_{N''}$. We get the general adapted metric $g'' = \rho''^{2u}\tilde g'' + (d\rho'')^2$ on M''. Equip $M \times M'$ with $g + g'$ and M'' with g''.

Proposition 2.1. *(i) If $u = 1$, then $\phi : M'' \to M \times M'$ is a quasi-isometry.*

(ii) If $u < 1$, then $\phi : M'' \cap O \to (M \times M') \cap \phi(O)$ is not quasi-isometric for any neighborhood O of $*''$ in $c(L'')$.

Proof. Without lost of generality, we can assume $\tilde g'' = (g + g')|_{N''}$. We have

$$M'' = N'' \times \mathbb{R}_+ \subset M \times M' \times \mathbb{R}_+ = N \times \mathbb{R}_+ \times N' \times \mathbb{R}_+ \times \mathbb{R}_+ \ .$$

According to this expression, an arbitrary point $p \in M''$ can be written as $p = (x,r,x',r',r'') \equiv (\bar p, r'')$, obtaining

$$\phi(p) = (x, rr'', x', r'r'') \in M \times M' = N \times \mathbb{R}_+ \times N' \times \mathbb{R}_+ \ .$$

Thus we can canonically consider

$$\begin{aligned} T_{\bar p}N'' &\subset T_xN \oplus \mathbb{R} \oplus T_{x'}N' \oplus \mathbb{R} \ , \\ T_pM'' &\subset T_xN \oplus \mathbb{R} \oplus T_{x'}N' \oplus \mathbb{R} \oplus \mathbb{R} \ , \\ T_{\phi(p)}(M \times M') &= T_xN \oplus \mathbb{R} \oplus T_{x'}N' \oplus \mathbb{R} \ . \end{aligned}$$

We easily get

$$\phi_*(\partial_{\rho''}(p)) = (0, r\partial_\rho(rr''), 0, r'\partial_{\rho'}(r'r'')) ,$$
$$\phi_*(X,0) = (Y, cr''\partial_\rho(rr''), Y', c'r''\partial_{\rho'}(r'r'')) ,$$

for $X = (Y, c\partial_\rho(r), Y', c'\partial_{\rho'}(r')) \in T_{\bar p}N''$. Hence

$$\|\partial_{\rho''}(p)\|^2_{g''} = 1 , \tag{4}$$
$$\|\phi_*(\partial_{\rho''}(p))\|^2_{g+g'} = r^2 + {r'}^2 , \tag{5}$$
$$\begin{aligned}\|(X,0)\|^2_{g''} &= {r''}^{2u}\,\|X\|^2_{\tilde g+\tilde g'}\\ &= {r''}^{2u}\left(\|Y\|^2_{\tilde g} + c^2 + \|Y'\|^2_{\tilde g'} + {c'}^2\right) ,\end{aligned} \tag{6}$$
$$\begin{aligned}\|\phi_*(X,0)\|^2_{g+g'} &= \left({r''}^{2u}\,\|Y\|^2_{\tilde g} + c^2{r''}^2 + {r''}^{2u}\,\|Y'\|^2_{\tilde g'} + {c'}^2{r''}^2\right)\\ &= {r''}^{2u}\left(\|Y\|^2_{\tilde g} + c^2{r''}^{2(1-u)} + \|Y'\|^2_{\tilde g'} + {c'}^2{r''}^{2(1-u)}\right) ,\end{aligned} \tag{7}$$

where every metric is added as subindex of the corresponding norm.

Observe that $C_0 := \min_{N''}(\rho^2 + {\rho'}^2) > 0$ and $C_1 := \max_{N''}(\rho^2 + {\rho'}^2) < \infty$ by the properties of h. So, by (4) and (5),

$$C_0\,\|\partial_{\rho''}(p)\|^2_{g''} \le \|\phi_*(\partial_{\rho''}(p))\|^2_{g+g'} \le C_1\,\|\partial_{\rho''}(p)\|^2_{g''} .$$

Moreover, if $u = 1$, then $\|\phi_*(X,0)\|^2_{g+g'} = \|(X,0)\|^2_{g''}$ by (6) and (7), obtaining (i).

Now, suppose that $u < 1$. With the above notation, by the conditions satisfied by h, we can take $\bar p = (x, r, x', 1) \in N''$ and $X = (0, \partial_\rho(r), 0, 0) \in T_{\bar p}N''$ for all r small enough. By (6) and (7), it follows that

$$\frac{\|\phi_*(X,0)\|^2_{g+g'}}{\|(X,0)\|^2_{g''}} = {r''}^{2(1-u)} \to 0$$

as $r'' \to 0$, giving (ii). □

Similar observations apply to the product of any finite number of cones.

2.2. **General adapted metrics.** Consider the notation of Section 1.3.

Remark 1. For every $m \in \mathbb{Z}_+$, there is a canonical homeomorphism $c(\mathbb{S}^{m-1}) \approx \mathbb{R}^m$, $[x,\rho] \mapsto \rho x$, so that the radial function ρ corresponds to the norm on $\mathbb{R}^m$ [4, Example 3.7]. This is not an isomorphism of stratifications: $c(\mathbb{S}^{m-1})$ has two strata and $\mathbb{R}^m$ only one; the stratum $\mathbb{S}^{m-1}\times\mathbb{R}_+$ of $c(\mathbb{S}^{m-1})$ corresponds to $\mathbb{R}^m \smallsetminus \{0\}$. If $\tilde g$ denotes the standard metric on $\mathbb{S}^{m-1}$, then $\rho^2\tilde g + (d\rho)^2$ on $\mathbb{S}^{m-1}\times\mathbb{R}_+$ corresponds to the Euclidean metric on $\mathbb{R}^m \smallsetminus \{0\}$. Thus, with the notation of Section 1, the factors $\mathbb{R}^{m_X}$ or $\mathbb{R}^{m_\pm}$ could be also described as cones, or as strata of cones after removing one point.

Remark 2. By taking charts and using induction on the depth, we get the following (cf. [4, Remark 7]):

(i) If two general adapted metrics on M have the same type with respect to the same general tubes, then they are rel-locally quasi-isometric. In particular, they are quasi-isometric if $\overline{M}$ is compact.

(ii) Any point in $\overline{M}$ has a countable base $\{ O_m \mid m \in \mathbb{N} \}$ of open neighborhoods such that, with respect to any general adapted metric, $\operatorname{vol}(M \cap O_m) \to 0$ and $\max\{ \operatorname{diam} P \mid P \in \pi_0(M \cap O_m) \} \to 0$ as $m \to \infty$. Thus, if $\overline{M}$ is compact, then $\operatorname{vol} M < \infty$ and $\operatorname{diam} P < \infty$ for all $P \in \pi_0(M)$.

Remark 3. The argument of [8, Appendix] also shows the following. Let $\{O_a\}$ be a locally finite open covering of $\overline{M}$, let $\{\lambda_a\}$ be a smooth partition of unity of M subordinated to the open covering $\{M \cap O_a\}$, and let g_a be a general adapted metric on every $M \cap O_a$. Suppose that the metrics g_a have the same general type with respect to restrictions to the sets O_a of the same general tubes. Then the metric $\sum_a \lambda_a g_a$ is general adapted on M and has the same general type with respect to those general tubes.

When M is not connected, $\widehat{M}$ is defined as the disjoint union of the rel-local completion of the connected components of M (Section 1.3), using [4, Remark 1 (v)].

Remark 4. (i) By Remark 2 (i), $\widehat{M}$ is independent of the choice of the general adapted metric of a given general type. In fact, by Remark 2 (ii) and [4, Example 3.19], $\widehat{M}$ is also independent of the general type.

(ii) For any open $O \subset A$, we have $\widehat{M \cap O} \equiv \lim^{-1}(\overline{M} \cap O) \subset \widehat{M}$.

Remark 5. The following is a direct consequence of Remark 4 (i) and [4, Remark 9 (i),(ii) and Proposition 3.20 (iii)]:

(i) $\lim : \widehat{M} \to \overline{M}$ is surjective with finite fibers.

(ii) M is rel-locally connected with respect to $\widehat{M}$.

(iii) Let M' be a connected stratum of another stratification A' equipped with a general adapted metric, and let $\phi : A \to A'$ be a morphism with $\phi(M) \subset M'$. Then the restriction $\phi : M \to M'$ extends to a morphism $\hat{\phi} : \widehat{M} \to \widehat{M'}$. Moreover $\hat{\phi}$ is an isomorphism if ϕ is an isomorphism.

2.3. **Relatively Morse functions.** Consider the notation of Section 1.4. Besides the observations given in that section, the following holds like in the case of adapted metrics of conic type [4, Section 4].

Remark 6. (i) The rel-local boundedness of $|df|$ is invariant by rel-local quasi-isometries, and therefore it depends only on the general type of g. Similarly, the definition of rel-critical point depends only on the general type of g. But the rel-local boundedness of $|\operatorname{Hess} f|$ depends on the choice of g. However it follows from (iv) and (v) below that the existence of g so that f is rel-admissible with respect to g is a rel-local property.

(ii) If $\operatorname{depth} M = 0$, then any smooth function is admissible, and its rel-critical points are its critical points.

(iii) With the notation of Section 2.1, let $h \in C^\infty(\mathbb{R}_+)$ with $h' \in C^\infty_0(\mathbb{R}_+)$. Then the function $h(\rho)$ is rel-admissible on the stratum M of $c(L)$ with respect to any general adapted metric.

(iv) Let $\{ O_a \mid a \in \mathcal{A} \}$ be a locally finite covering of $\overline{M}$ by open subsets of A. Then there is a C^∞ partition of unity $\{\lambda_a\}$ on M subordinated to $\{M \cap O_a\}$ such that $|d\lambda_a|$ is rel-locally bounded for all general adapted metrics on M of any fixed general type.

(v) Suppose that $\{\lambda_a\}$ and $\{g_a\}$ satisfy the conditions of Remark 3 and (iv). Let $f \in C^\infty(M)$ such that every $f|_{M \cap O_a}$ is rel-admissible with respect to g_a. Then

f is rel-admissible with respect to the general adapted metric $g = \sum_a \lambda_a g_a$ on M.

(vi) Let $\mathcal{F} \subset C^\infty(M)$ denote the subset of functions with continuous extensions to $\overline{M}$ that restrict to rel-Morse functions with respect to all general adapted metrics of all possible general types on all strata $\leq M$. Then $\mathcal{F}$ is dense in $C^\infty(M)$ with the weak C^∞ topology.

2.4. **Hilbert and elliptic complexes.** Consider the notation of Section 1.1.

2.4.1. *Hilbert complexes with a discrete positive spectrum.* Let $(\mathsf{D}, \mathbf{d})$ be a Hilbert complex in a graded separable Hilbert space $\mathfrak{H}$, defining self-adjoint operators $\mathbf{D}$ and $\boldsymbol{\Delta}$ according to Section 1.1. The direct sum of homogeneous subspaces of even/odd degree are denoted with the subindex "ev/odd". The same subindex is used to denote the restriction of homogeneous operators to such subspaces.

Lemma 2.2. *The positive spectrum of $\boldsymbol{\Delta}_{\mathrm{ev}}$ is discrete*[2] *and bounded away from zero if and only if the positive spectrum of $\boldsymbol{\Delta}_{\mathrm{odd}}$ is discrete and bounded away from zero. In this case, both operators have the same positive eigenvalues, with the same multiplicity.*

Proof. For instance, suppose that the positive spectrum of $\boldsymbol{\Delta}_{\mathrm{ev}}$ is discrete and bounded away from zero. It follows from the spectral theorem that

$$\mathsf{D}^\infty(\boldsymbol{\Delta}_{\mathrm{ev/odd}}) = \ker \boldsymbol{\Delta}_{\mathrm{ev/odd}} \oplus \boldsymbol{\Delta}(\mathsf{D}^\infty(\boldsymbol{\Delta}_{\mathrm{ev/odd}}))\;,$$

and

$$\mathbf{D}_{\mathrm{ev}} : \boldsymbol{\Delta}(\mathsf{D}^\infty(\boldsymbol{\Delta}_{\mathrm{ev}})) \to \boldsymbol{\Delta}(\mathsf{D}^\infty(\boldsymbol{\Delta}_{\mathrm{odd}}))$$

is a linear isomorphism satisfying $\mathbf{D}_{\mathrm{ev}}\boldsymbol{\Delta}_{\mathrm{ev}} = \boldsymbol{\Delta}_{\mathrm{odd}}\mathbf{D}_{\mathrm{ev}}$. □

2.4.2. *Elliptic complexes with a term that is a direct sum.* Let $E = \bigoplus_r E_r$ be a graded Riemannian or Hermitian vector bundle over a Riemannian manifold M. The space of its smooth sections is denoted by $C^\infty(E)$, its subspace of compactly supported smooth sections is denoted by $C_0^\infty(E)$, and the Hilbert space of square integrable sections of E is denoted by $L^2(E)$. All of these are graded spaces. Consider differential operators of the same order, $d_r : C^\infty(E_r) \to C^\infty(E_{r+1})$, such that $(C^\infty(E), d = \bigoplus_r d_r)$ is an elliptic[3] complex. The simpler notation (E, d) (or even d) will be preferred. Elliptic complexes with nonzero terms of negative degrees or homogeneous differential operators of degree -1 may be also considered without any essential change. For instance, we have the formal adjoint elliptic complex (E, δ).

Suppose that there is an orthogonal decomposition $E_{r+1} = E_{r+1,1} \oplus E_{r+1,2}$ for some degree $r+1$. Thus

$$\begin{aligned} C^\infty(E_{r+1}) &\equiv C^\infty(E_{r+1,1}) \oplus C^\infty(E_{r+1,2})\;,\\ C_0^\infty(E_{r+1}) &\equiv C_0^\infty(E_{r+1,1}) \oplus C_0^\infty(E_{r+1,2})\;,\\ L^2(E_{r+1}) &\equiv L^2(E_{r+1,1}) \oplus L^2(E_{r+1,2})\;, \end{aligned}$$

[2]Recall that a complex number is in the discrete spectrum of a normal operator in a Hilbert space when it is an eigenvalue of finite multiplicity.

[3]Recall that ellipticity means that the sequence of principal symbols of the operators d_r is exact over every nonzero cotangent vector.

and we can write

$$d_r = \begin{pmatrix} d_{r,1} \\ d_{r,2} \end{pmatrix} , \qquad \delta_r = \begin{pmatrix} \delta_{r,1} & \delta_{r,2} \end{pmatrix} ,$$

$$d_{r+1} = \begin{pmatrix} d_{r+1,1} & d_{r+1,2} \end{pmatrix} , \quad \delta_{r+1} = \begin{pmatrix} \delta_{r+1,1} \\ \delta_{r+1,2} \end{pmatrix} .$$

The operators $d_{r,i}$ and $\delta_{r,i}$ can be also considered as elliptic complexes of length one, and therefore they have a maximum/minimum i.b.c., $d_{r,i,\max/\min}$ and $\delta_{r,i,\max/\min}$.

Lemma 2.3 ([4, Lemma 8.2])**.** *We have:*

$$\mathsf{D}(d_{\max,r}) = \mathsf{D}(d_{r,1,\max}) \cap \mathsf{D}(d_{r,2,\max}) , \quad d_{\max,r} = \begin{pmatrix} d_{r,1,\max}|_{\mathsf{D}(d_{\max,r})} \\ d_{r,2,\max}|_{\mathsf{D}(d_{\max,r})} \end{pmatrix} .$$

Lemma 2.4. *We have:*

$$\mathsf{D}(d_{r+1,1,\max/\min}) \oplus \mathsf{D}(d_{r+1,2,\max/\min}) \subset \mathsf{D}(d_{\max/\min,r+1}) . \tag{8}$$

Proof. Take any $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathsf{D}(d_{r+1,1,\min}) \oplus \mathsf{D}(d_{r+1,2,\min})$, and let $u' = d_{r+1,1,\min}u$ and $v' = d_{r+1,2,\min}v$. This means that there are sequences, u_i in $C^\infty_0(E_{r+1,1})$ and v_i in $C^\infty_0(E_{r+1,2})$, such that $u_i \to u$ in $L^2(E_{r+1,1})$, $v_i \to v$ in $L^2(E_{r+1,2})$, $d_{r+1,1}u_i \to u'$ and $d_{r+1,2}v_i \to v'$ in $L^2(E_{r+2})$. So $\left(\begin{smallmatrix} u_i \\ v_i \end{smallmatrix}\right) \in C^\infty_0(E_{r+1,1}) \oplus C^\infty_0(E_{r+1,2}) \equiv C^\infty_0(E_{r+1})$, $\left(\begin{smallmatrix} u_i \\ v_i \end{smallmatrix}\right) \to \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ in $L^2(E_{r+1})$ and $d_{r+1}\left(\begin{smallmatrix} u_i \\ v_i \end{smallmatrix}\right) \to u' + v'$ in $L^2(E_{r+2})$, obtaining $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathsf{D}(d_{\min,r+1})$.

Now, take any $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathsf{D}(d_{r+1,1,\max}) \oplus \mathsf{D}(d_{r+1,2,\max})$, and let $u' = d_{r+1,1,\max}u$ and $v' = d_{r+1,2,\max}v$. This means that $\langle u, \delta_{r+1,1}w\rangle = \langle u', w\rangle$ and $\langle v, \delta_{r+1,2}w\rangle = \langle v', w\rangle$ for all $w \in C^\infty_0(E_{r+2})$. Thus $\langle \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), \delta_{r+1}w\rangle = \langle u' + v', w\rangle$ for all $w \in C^\infty_0(E_{r+2})$, obtaining that $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \mathsf{D}(d_{\max,r+1})$. □

3. A perturbation of the Dunkl harmonic oscillator

This section is devoted to recall the study of self-adjoint operators on $\mathbb{R}_+$ induced by the Dunkl harmonic oscillator on $\mathbb{R}$ [3], and also by certain perturbation of the Dunkl harmonic oscillator on $\mathbb{R}$ [5]. This is the main analytic tool of the paper.

Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the real-/complex-valued Schwartz space on $\mathbb{R}$, with its Fréchet topology. It decomposes as direct sum of subspaces of even and odd functions, $\mathcal{S} = \mathcal{S}_{\mathrm{ev}} \oplus \mathcal{S}_{\mathrm{odd}}$. For $\sigma > -\frac{1}{2}$, the sequence of generalized Hermite polynomials, $p_k = p_{s,\sigma,k}(x)$, consists of the orthogonal polynomials associated with the measure $e^{-sx^2}|x|^{2\sigma}\,dx$ on $\mathbb{R}$ [40, p. 380, Problem 25]. It is assumed that every p_k is normalized and has positive leading coefficient. They give rise to the general Hermite functions $\phi_k = \phi_{s,\sigma,k}(x) = p_k e^{-sx^2/2} \in \mathcal{S}$. If k is odd, then $p_{s,\tau,k}$ and $\phi_{s,\tau,k}$ also make sense for $\tau > -\frac{3}{2}$.

Now, let ρ denote the canonical coordinate of $\mathbb{R}_+$. Consider the spaces of real-/complex-valued functions, $C^\infty = C^\infty(\mathbb{R})$, $C^\infty_+ = C^\infty(\mathbb{R}_+)$ and $C^\infty_{+,0} = C^\infty_0(\mathbb{R}_+)$, where the subindex 0 is used for compactly supported functions or sections. For every $a \in \mathbb{R}$, the operator of multiplication by the function ρ^a on C^∞_+ will be also denoted by ρ^a. We have

$$[\tfrac{d}{d\rho}, \rho^a] = a\rho^{a-1} , \quad [\tfrac{d^2}{d\rho^2}, \rho^a] = 2a\rho^{a-1}\,\tfrac{d}{d\rho} + a(a-1)\rho^{a-2} . \tag{9}$$

For every $\phi \in C^\infty$, let $\phi_+ = \phi|_{\mathbb{R}_+}$, and let $\mathcal{S}_{\mathrm{ev/odd},+} = \{\, \phi_+ \mid \phi \in \mathcal{S}_{\mathrm{ev/odd}} \,\}$. For $c, d > -\frac{1}{2}$, let $L^2_{c,+} = L^2(\mathbb{R}_+, \rho^{2c}\,d\rho)$ and $L^2_{c,d,+} = L^2_{c,+} \oplus L^2_{d,+}$, whose scalar products are denoted by $\langle\ ,\ \rangle_c$ and $\langle\ ,\ \rangle_{c,d}$, and the corresponding norms by $\|\ \|_c$

and $\| \ \|_{c,d}$, respectively. The simpler notation L^2_+, $\langle\ ,\ \rangle$ and $\|\ \|$ is used when $c = 0$. Recall that the harmonic oscillator on C^∞_+ is the operator $H = -\frac{d^2}{d\rho^2} + s^2\rho^2$ ($s > 0$). For $c_1, c_2, d_1, d_2 \in \mathbb{R}$, let

$$P_0 = H - 2c_1\rho^{-1}\frac{d}{d\rho} + c_2\rho^{-2}\ , \quad Q_0 = H - 2d_1\frac{d}{d\rho}\,\rho^{-1} + d_2\rho^{-2}\ . \tag{10}$$

Proposition 3.1 ([3, Theorem 1.4])**.** *If $a \in \mathbb{R}$ satisfies*

$$a^2 + (2c_1 - 1)a - c_2 = 0\ , \tag{11}$$

$$\sigma := a + c_1 > -\tfrac{1}{2}\ , \tag{12}$$

then the following holds:

(*i*) *P_0, with $\mathsf{D}(P_0) = \rho^a\mathcal{S}_{\mathrm{ev},+}$, is essentially self-adjoint in $L^2_{c_1,+}$.*

(*ii*) *The spectrum of $\mathcal{P}_0 := \overline{P_0}$ consists of the eigenvalues*

$$\lambda_k = (2k + 1 + 2\sigma)s\ , \tag{13}$$

for $k \in 2\mathbb{N}$, with multiplicity one and corresponding normalized eigenfunctions $\chi_k = \chi_{s,\sigma,a,k} := \sqrt{2}\,\rho^a\phi_{s,\sigma,k,+}$.

(*iii*) $\mathsf{D}^\infty(\mathcal{P}_0) = \rho^a\mathcal{S}_{\mathrm{ev},+}$.

Proposition 3.2 (See [3, Section 5])**.** *If $b \in \mathbb{R}$ satisfies*

$$b^2 + (2d_1 + 1)b - d_2 = 0\ , \tag{14}$$

$$\tau := b + d_1 > -\tfrac{3}{2}\ , \tag{15}$$

then the following holds:

(*i*) *Q_0, with $\mathsf{D}(Q_0) = \rho^b\mathcal{S}_{\mathrm{odd},+}$, is essentially self-adjoint in $L^2_{d_1,+}$.*

(*ii*) *The spectrum of $\mathcal{Q}_0 := \overline{Q_0}$ consists of the eigenvalues given by the expression* (13)*, for $k \in 2\mathbb{N}+1$ and using τ instead of σ, with multiplicity one and corresponding normalized eigenfunctions $\chi_k = \chi_{s,\tau,b,k} := \sqrt{2}\,\rho^b\phi_{s,\tau,k,+}$.*

(*iii*) $\mathsf{D}^\infty(\mathcal{Q}_0) = \rho^b\mathcal{S}_{\mathrm{odd},+}$.

Proposition 3.3 ([5, Corollary 8.1])**.** *Let $\xi > 0$ and*

$$0 < u < 1\ . \tag{16}$$

If $a \in \mathbb{R}$ satisfies (11) *and*

$$\sigma := a + c_1 > u - \tfrac{1}{2}\ , \tag{17}$$

then there is a positive self-adjoint operator $\mathcal{P}$ in $L^2_{c_1,+}$ satisfying the following:

(*i*) *$\rho^a\mathcal{S}_{\mathrm{ev},+}$ is a core of $\mathcal{P}^{1/2}$ and, for all $\phi, \psi \in \rho^a\mathcal{S}_{\mathrm{ev},+}$,*

$$\langle\mathcal{P}^{1/2}\phi, \mathcal{P}^{1/2}\psi\rangle_{c_1} = \langle P_0\phi, \psi\rangle_{c_1} + \xi\langle\rho^{-u}\phi, \rho^{-u}\psi\rangle_{c_1}\ . \tag{18}$$

(*ii*) *$\mathcal{P}$ has a discrete spectrum. Let $\lambda_0 \le \lambda_2 \le \cdots$ be its eigenvalues, repeated according to their multiplicity. There is some $D = D(\sigma, u) > 0$ and, for any $\epsilon > 0$, there is some $C = C(\epsilon, \sigma, u) > 0$ so that, for all $k \in 2\mathbb{N}$,*

$$\lambda_k \ge (2k + 1 + 2\sigma)s + \xi D s^u (k+1)^{-u}\ , \tag{19}$$

$$\lambda_k \le (2k + 1 + 2\sigma)(s + \xi\epsilon s^u) + \xi C s^u\ . \tag{20}$$

Proposition 3.4 ([5, Corollary 8.2])**.** *For ξ and u like in Proposition 3.3, if $b \in \mathbb{R}$ satisfies* (14) *and*

$$\tau := b + d_1 > u - \tfrac{3}{2}\ , \tag{21}$$

then there is a positive self-adjoint operator $\mathcal{Q}$ in $L^2_{d_1,+}$ satisfying the following:

(*i*) $\rho^b\mathcal{S}_{\mathrm{odd},+}$ *is a core of* $\mathcal{Q}^{1/2}$ *and, for all* $\phi,\psi\in\rho^b\mathcal{S}_{\mathrm{odd},+}$,

$$\langle\mathcal{Q}^{1/2}\phi,\mathcal{Q}^{1/2}\psi\rangle_{d_1}=\langle Q_0\phi,\psi\rangle_{d_1}+\xi\langle\rho^{-u}\phi,\rho^{-u}\psi\rangle_{d_1}\ . \tag{22}$$

(*ii*) $\mathcal{Q}$ *has a discrete spectrum. Let* $\lambda_1\le\lambda_3\le\cdots$ *be its eigenvalues, repeated according to their multiplicity. There is some* $D=D(\tau,u)>0$ *and, for any* $\epsilon>0$*, there is some* $C=C(\epsilon,\tau,u)>0$ *so that* (19) *and* (20) *are satisfied, for* $k\in 2\mathbb{N}+1$ *and with* τ *instead of* σ.

Proposition 3.5 ([5, Corollary 8.3])**.** *Consider the notation and conditions of Propositions 3.3 and 3.4. Fix also some* $\eta\in\mathbb{R}$*, and let*

$$\theta>-\tfrac{1}{2}\ . \tag{23}$$

Moreover suppose that the following properties hold:

(*a*) *If* $\sigma=\theta\ne\tau$ *and* $\tau-\sigma\notin-\mathbb{N}$*, then*

$$\sigma-1<\tau<\sigma+1,2\sigma+\tfrac{1}{2}\ . \tag{24}$$

(*b*) *If* $\sigma\ne\theta=\tau$ *and* $\sigma-\tau\notin-\mathbb{N}$*, then*

$$-\tau,\tau-1<\sigma<3\tau+1,11\tau+2,\tau+1\ . \tag{25}$$

(*c*) *If* $\sigma\ne\theta=\tau+1$ *and* $\sigma-\tau-1\notin-\mathbb{N}$*, then*

$$\tau+1<\sigma<\tau+3,2\tau+\tfrac{7}{2}\ . \tag{26}$$

(*d*) *If* $\sigma\ne\theta\ne\tau$ *and* $\sigma-\theta,\tau-\theta\notin-\mathbb{N}$*, then*

$$\left.\begin{aligned}\tfrac{\sigma-\tau}{2}-1,\tfrac{\tau-\sigma}{2},\tfrac{\sigma+\tau-1}{4},\tfrac{\sigma+3\tau-2}{14},\tfrac{3\sigma+\tau-4}{14},\tfrac{\sigma+\tau-1}{2}<\theta<\tfrac{\sigma+\tau+1}{2}\ ,\\ \tau-1<\sigma<\tau+3\ .\end{aligned}\right\} \tag{27}$$

Then there is a positive self-adjoint operator $\mathcal{W}$ *in* $L^2_{c_1,d_1,+}$ *satisfying the following:*

(*i*) $\rho^a\mathcal{S}_{\mathrm{ev},+}\oplus\rho^b\mathcal{S}_{\mathrm{odd},+}$ *is a core of* $\mathcal{W}^{1/2}$*, and, for* $\phi=(\phi_1,\phi_2)$ *and* $\psi=(\psi_1,\psi_2)$ *in* $\rho^a\mathcal{S}_{\mathrm{ev},+}\oplus\rho^b\mathcal{S}_{\mathrm{odd},+}$,

$$\begin{aligned}\langle\mathcal{W}^{1/2}\phi,\mathcal{W}^{1/2}\psi\rangle_{c_1,d_1}&=\langle(P_0\oplus Q_0)\phi,\psi\rangle_{c_1,d_1}+\xi\langle\rho^{-u}\phi,\rho^{-u}\psi\rangle_{c_1,d_1}\\&\quad+\eta\left(\langle\rho^{-a-b-1}\phi_2,\psi_1\rangle_\theta+\langle\phi_1,\rho^{-a-b-1}\psi_2\rangle_\theta\right)\ .\end{aligned} \tag{28}$$

(*ii*) $\mathcal{W}$ *has a discrete spectrum. Its eigenvalues form two groups,* $\lambda_0\le\lambda_2\le\cdots$ *and* $\lambda_1\le\lambda_3\le\cdots$*, repeated according to their multiplicity, such that there is some* $D=D(\sigma,\tau,u)>0$ *and, for every* $\epsilon>0$*, there are some* $C=C(\epsilon,\sigma,\tau,u)>0$ *and* $E=E(\epsilon,\sigma,\tau,\theta)>0$ *so that, for all* $k\in\mathbb{N}$,

$$\lambda_k\ge(2k+1+2\varsigma_k)\big(s-2|\eta|\epsilon s^{\frac{v+1}{2}}\big)+\xi Ds^u(k+1)^{-u}-2|\eta|Es^{\frac{v+1}{2}}\ , \tag{29}$$

$$\lambda_k\le(2k+1+2\varsigma_k)\big(s+\epsilon\big(\xi s^u+2|\eta|s^{\frac{v+1}{2}}\big)\big)+\xi Cs^u+2|\eta|Es^{\frac{v+1}{2}}\ , \tag{30}$$

where $v=\sigma+\tau-2\theta$*,* $\varsigma_k=\sigma$ *if* k *is even, and* $\varsigma_k=\tau$ *if* k *is odd.*

(*iii*) *Let* $\tilde u\in\mathbb{R}$ *such that*

$$0,v,\tau-2\theta+\tfrac{1}{2},\sigma-2\theta-\tfrac{1}{2}<\tilde u<1,v+1,\sigma+\tfrac{1}{2},\tau+\tfrac{3}{2}\ , \tag{31}$$

and let $\hat u=\max\{\tilde u,v+1-\tilde u\}$*. There is some* $D=D(\sigma,\tau,u)>0$ *and, for any* $\epsilon>0$*, there is some* $\widetilde C=\widetilde C(\epsilon,\sigma,\tau,u)>0$ *so that, for all* $k\in\mathbb{N}$,

$$\lambda_k\ge(2k+1+2\varsigma_k)\big(s-|\eta|\epsilon s^{\hat u}\big)+\xi Ds^u(k+1)^{-u}-|\eta|\widetilde Cs^{\hat u}\ . \tag{32}$$

(*iv*) *If* $u = \frac{v+1}{2}$ *and* $\xi \geq |\eta|$, *then there is some* $\widetilde{D} = \widetilde{D}(\sigma, \tau, u) > 0$ *so that, for all* $k \in \mathbb{N}$,

$$\lambda_k \geq (2k + 1 + 2\varsigma_k)s + (\xi - |\eta|)\widetilde{D}s^u(k+1)^{-u} . \tag{33}$$

(*v*) *If we add the term* $\xi'\langle\phi_1, \psi_1\rangle_{c_1} + \xi''\langle\phi_2, \psi_2\rangle_{d_1}$ *to the right-hand side of* (28), *for some* $\xi', \xi'' \in \mathbb{R}$, *then the result holds as well with the additional term* $\max\{\xi', \xi''\}$ *in the right-hand side of* (30), *and the additional term,* ξ' *for* $k \in 2\mathbb{N}$ *and* ξ'' *for* $k \in 2\mathbb{N}+1$, *in the right-hand sides of* (29), (32) *and* (33).

Remark 7. (i) If h is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho) \to 1$ as $\rho \to 0$, then $\langle h\chi_0, \chi_0\rangle_{c_1} \to 1$ as $s \to \infty$ [4, Lemma 7.3].

(ii) The existence of $a \in \mathbb{R}$ satisfying (11) is characterized by the condition $(2c_1 - 1)^2 + 4c_2 \geq 0$, which holds if $c_2 \geq \min\{0, 2c_1\}$. If $c_2 = 0$, then (11) means that $a \in \{0, 1 - 2c_1\}$. If $c_2 = 2c_1$, then (11) means that $a \in \{1, -2c_1\}$.

(iii) The existence of $b \in \mathbb{R}$ satisfying (14) is characterized by the condition $(2d_1 + 1)^2 + 4d_2 \geq 0$, which holds if $d_2 \geq \min\{0, -2d_1\}$. If $d_2 = 0$, then (14) means that $b \in \{0, -1 - 2d_1\}$. If $d_2 = -2d_1$, then (14) means that $b \in \{-1, -2d_1\}$.

(iv) Propositions 3.1 and 3.2 are indeed equivalent, as well as Propositions 3.3 and 3.4, because, if $c_1 = d_1 + 1$ and $c_2 = d_2$, then $Q_0 = \rho P_0 \rho^{-1}$ by (9), and $\rho : L^2_{c_1,+} \to L^2_{d_1,+}$ is a unitary isomorphism.

(v) We have $\mathcal{P} = \overline{P}$, $\mathcal{Q} = \overline{Q}$ and $\mathcal{W} = \overline{W}$, where

$$P = P_0 + \xi\rho^{-2u} , \quad Q = Q_0 + \xi\rho^{-2u} , \tag{34}$$

$$W = \begin{pmatrix} P & \eta\rho^{2(\theta-c_1)-a-b-1} \\ \eta\rho^{2(\theta-d_1)-a-b-1} & Q \end{pmatrix} , \tag{35}$$

with $\mathsf{D}(P) = \mathsf{D}^\infty(\mathcal{P})$, $\mathsf{D}(Q) = \mathsf{D}^\infty(\mathcal{Q})$ and $\mathsf{D}(W) = \mathsf{D}^\infty(\mathcal{W})$ [5, Remark 1.4 (i) and Section 8].

(vi) We have

$$\mathsf{D}(\mathcal{P}^{1/2}) = \mathsf{D}(\mathcal{P}_0^{1/2}) , \quad \mathsf{D}(\mathcal{Q}^{1/2}) = \mathsf{D}(\mathcal{Q}_0^{1/2}) , \quad \mathsf{D}(\mathcal{W}^{1/2}) = \mathsf{D}((\mathcal{P}_0 \oplus \mathcal{Q}_0)^{1/2}) .$$

Thus the expressions (18), (22) and (28) can be extended to ϕ and ψ in $\mathsf{D}(\mathcal{P}^{1/2})$, $\mathsf{D}(\mathcal{Q}^{1/2})$ and $\mathsf{D}(\mathcal{W}^{1/2})$, respectively, using

$$\langle\mathcal{P}_0^{1/2}\phi, \mathcal{P}_0^{1/2}\psi\rangle_{c_1} , \quad \langle\mathcal{Q}_0^{1/2}\phi, \mathcal{Q}_0^{1/2}\psi\rangle_{d_1} , \quad \langle(\mathcal{P}_0 \oplus \mathcal{Q}_0)^{1/2}\phi, (\mathcal{P}_0 \oplus \mathcal{Q}_0)^{1/2}\psi\rangle_{c_1,d_1}$$

instead of

$$\langle P_0\phi, \psi\rangle_{c_1} , \quad \langle Q_0\phi, \psi\rangle_{d_1} , \quad \langle(P_0 \oplus Q_0)\phi, \psi\rangle_{c_1,d_1} ,$$

respectively [5, Remark 3.21 and Section 8].

(vii) In Proposition 3.5 (iii), the condition (31) means that (16), (17) and (21) also hold with $\tilde{u}$ and $v + 1 - \tilde{u}$ instead of u. There exists $\tilde{u}$ satisfying (31) just when

$$0, v, \tau - 2\theta + \tfrac{1}{2}, \sigma - 2\theta - \tfrac{1}{2} < 1, v + 1, \sigma + \tfrac{1}{2}, \tau + \tfrac{3}{2} . \tag{36}$$

This property is satisfied in the cases (b) and (d) by (16), (17), (21), (23), (25) and (27); in particular, we can take $\tilde{u} = \frac{v+1}{2}$. By (16), (17), (21), (23) and (24) (respectively, (26)), in the case (a) (respectively, in the case (c)), we have (36) if and only if $\tau < 3\sigma$ (respectively, $\sigma < 3\tau + 4$).

Consider the conditions and notation of Proposition 3.3, and the notation of Proposition 3.1. Take a complete orthonormal system $\{\, \hat{\chi}_k = \hat{\chi}_{\mathcal{P},k} \mid k \in 2\mathbb{N} \,\}$ of $L^2_{c_1,+}$ so that every $\hat{\chi}_k$ is a λ_k-eigenfunction of $\mathcal{P}$. Let $\hat{\chi}'_k = \hat{\chi}'_{\mathcal{P},k}$ and $\hat{\chi}''_k = \hat{\chi}''_{\mathcal{P},k}$

denote the orthogonal projections of every $\hat\chi_k$ to the subspaces spanned by χ_k and $\{\,\chi_i \mid k > i \in 2\mathbb{N}\,\}$, respectively; in particular, $\hat\chi_0'' = 0$. Let also $\hat\chi_k''' = \hat\chi_{\mathcal{P},k}''' = \hat\chi_k - \hat\chi_k' - \hat\chi_k''$.

Lemma 3.6. $\|\hat\chi_{\mathcal{P},k}'\|_{c_1} \to 1$ *as* $s \to \infty$ *for every* $k \in 2\mathbb{N}$.

Proof. We proceed by induction on k. For $k = 0$, take some $\epsilon > 0$ and $C > 0$ satisfying (20). By Propositions 3.1 (ii) and 3.3 (ii), and Remark 7 (vi),

$$\begin{aligned}(1+2\sigma)(s+\xi\epsilon s^u)+\xi C s^u \geq \lambda_0 &= \langle \mathcal{P}^{1/2}\hat\chi_0, \mathcal{P}^{1/2}\hat\chi_0\rangle_{c_1} > \langle \mathcal{P}_0^{1/2}\hat\chi_0, \mathcal{P}_0^{1/2}\hat\chi_0\rangle_{c_1}\\ &= \langle \mathcal{P}_0^{1/2}\hat\chi_0', \mathcal{P}_0^{1/2}\hat\chi_0'\rangle_{c_1} + \langle \mathcal{P}_0^{1/2}\hat\chi_0''', \mathcal{P}_0^{1/2}\hat\chi_0'''\rangle_{c_1}\\ &\geq (1+2\sigma)s\,\|\hat\chi_0'\|_{c_1}^2 + (5+2\sigma)s\,\|\hat\chi_0'''\|_{c_1}^2 = (1+2\sigma)s + 4s\,\|\hat\chi_0'''\|_{c_1}^2\ ,\end{aligned}$$

giving

$$\|\hat\chi_0'''\|_{c_1}^2 < \frac{((1+2\sigma)\epsilon + C)\xi}{4s^{1-u}} \to 0$$

as $s \to \infty$, and therefore $\|\hat\chi_0'\|_{c_1}^2 \to 1$.

Now, take any even integer $k > 0$ and suppose that the result holds for all even indices $< k$. This yields $\|\hat\chi_k''\|_{c_1} \to 0$ as $s \to \infty$. Thus, given any $\delta > 0$, we have $\|\hat\chi_k''\|_{c_1}^2 < \delta/k$ for s large enough. Take some $\epsilon > 0$ and $C > 0$ satisfying (20). By Propositions 3.1 (ii) and 3.3 (ii), and Remark 7 (vi),

$$\begin{aligned}(2k+1+2\sigma)(s+\xi\epsilon s^u)+\xi C s^u \geq \lambda_k &= \langle \mathcal{P}^{1/2}\hat\chi_k, \mathcal{P}^{1/2}\hat\chi_k\rangle_{c_1} > \langle \mathcal{P}_0^{1/2}\hat\chi_k, \mathcal{P}_0^{1/2}\hat\chi_k\rangle_{c_1}\\ &= \langle \mathcal{P}_0^{1/2}\hat\chi_k', \mathcal{P}_0^{1/2}\hat\chi_k'\rangle_{c_1} + \langle \mathcal{P}_0^{1/2}\hat\chi_k'', \mathcal{P}_0^{1/2}\hat\chi_k''\rangle_{c_1} + \langle \mathcal{P}_0^{1/2}\hat\chi_k''', \mathcal{P}_0^{1/2}\hat\chi_k'''\rangle_{c_1}\\ &\geq (2k+1+2\sigma)s\,\|\hat\chi_k'\|_{c_1}^2 + (1+2\sigma)s\,\|\hat\chi_k''\|_{c_1}^2 + (2k+5+2\sigma)s\,\|\hat\chi_k'''\|_{c_1}^2\\ &= (1+2\sigma)s + 2ks(\|\hat\chi_k'\|_{c_1}^2 + \|\hat\chi_k'''\|_{c_1}^2) + 4s\,\|\hat\chi_k'''\|_{c_1}^2\\ &> (1+2\sigma)s + 2ks(1-\delta/k) + 4s\,\|\hat\chi_k'''\|_{c_1}^2\ ,\end{aligned}$$

giving

$$\|\hat\chi_k'''\|_{c_1}^2 < \frac{((2k+1+2\sigma)\epsilon + C)\xi}{4s^{1-u}} + \frac{\delta}{2} < \delta$$

for s large enough. Thus $\|\hat\chi_k'''\|_{c_1}^2 \to 0$ as $s \to \infty$, and the result follows. □

Corollary 3.7. *If* h *is a bounded measurable function on* $\mathbb{R}_+$ *such that* $h(\rho) \to 1$ *as* $\rho \to 0$, *then* $\langle h\hat\chi_{\mathcal{P},0}, \hat\chi_{\mathcal{P},0}\rangle_{c_1} \to 1$ *as* $s \to \infty$.

Proof. This follows from Lemma 3.6 and Remark 7 (i). □

Similar results hold for $\mathcal{Q}$ and $\mathcal{W}$, but they are omitted because they are not used.

4. Two simple types of elliptic complexes

Here, we study two simple elliptic complexes on $\mathbb{R}_+$, which will show up in a direct sum splitting of the rel-local model of Witten's perturbation (Section 6).

4.1. **An elliptic complex of length one.** Consider the standard metric on $\mathbb{R}_+$. Let E be the graded Riemannian/Hermitian vector bundle over $\mathbb{R}_+$ whose nonzero terms are E_0 and E_1, which are real/complex trivial line bundles equipped with the standard Riemannian/Hemitian metrics. Thus

$$C^\infty(E_0) \equiv C^\infty_+ \equiv C^\infty(E_1)\;, \quad L^2(E_0) \equiv L^2_+ \equiv L^2(E_1)\;,$$

where real-/complex-valued functions are considered in C^∞_+ and L^2_+. For any fixed $s > 0$ and $\kappa \in \mathbb{R}$, let

$$C^\infty(E_0) \underset{\delta}{\overset{d}{\rightleftarrows}} C^\infty(E_1)$$

be the differential operators defined by

$$d = \tfrac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\;, \quad \delta = -\tfrac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\;.$$

It is easy to check that (E, d) is an elliptic complex, and that[4] $\delta = d^\dagger$.

4.1.1. *Self-adjoint operators defined by the Laplacian.* By (9), the homogeneous components of Δ (or $\Delta^\pm$) are:

$$\Delta_0 = H + \kappa(\kappa - 1)\rho^{-2} \mp s(1 + 2\kappa)\;, \tag{37}$$

$$\Delta_1 = H + \kappa(\kappa + 1)\rho^{-2} \pm s(1 - 2\kappa)\;, \tag{38}$$

where H is the harmonic oscillator on C^∞_+ defined with the constant s. Then Δ_0 and Δ_1 are like P_0 and Q_0 in (10), with $c_1 = 0 = d_1$, plus a constant. Then, by Propositions 3.1 and 3.2, Δ_0 and Δ_1 define the self-adjoint operators $\mathcal{A}_i$ and $\mathcal{B}_i$ in L^2_+ indicated in Table 1, where the conditions come from (12) and (15). The notation $\mathcal{A}_i^\pm$ and $\mathcal{B}_i^\pm$ may be used as well to specify that these operators are defined by $\Delta_0^\pm$ and $\Delta_1^\pm$. In these cases, we have $c_1 = d_1 = 0$, and therefore $\sigma = a$ and $\tau = b$, which are given by (11) and (14).

		σ	τ	Condition
Δ_0	$\mathcal{A}_1$	κ		$\kappa > -\frac{1}{2}$
	$\mathcal{A}_2$	$1-\kappa$		$\kappa < \frac{3}{2}$
Δ_1	$\mathcal{B}_1$		κ	$\kappa > -\frac{3}{2}$
	$\mathcal{B}_2$		$-1-\kappa$	$\kappa < \frac{1}{2}$

TABLE 1. Self-adjoint operators defined by Δ_0 and Δ_1

There are the following overlaps in Table 1:

- Both $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined if $-\frac{1}{2} < \kappa < \frac{3}{2}$, and they are equal just when $\kappa = \frac{1}{2}$.
- Both $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined if $-\frac{3}{2} < \kappa < \frac{1}{2}$, and they are equal just when $\kappa = -\frac{1}{2}$.

The cores of $\mathcal{A}_i$ and $\mathcal{B}_i$, given by Propositions 3.1 and 3.2, will be denoted by $\mathcal{E}_i^0$ and $\mathcal{E}_i^1$, respectively. Note that the graded subspace $\mathcal{E}_i = \mathcal{E}_i^0 \oplus \mathcal{E}_i^1$ of $C^\infty(E) \cap L^2(E)$, whenever defined, is preserved by $D = d + \delta$. Propositions 3.1 and 3.2 also describe the spectra of $\mathcal{A}_i$ and $\mathcal{B}_i$:

[4]The superindex $\dagger$ is used to denote the formal adjoint.

- The spectrum of $\mathcal{A}_1$ consists of the eigenvalues
$$(2k + (1 \mp 1)(1 + 2\kappa))s \quad (k \in 2\mathbb{N}) \tag{39}$$
of multiplicity one.
- The spectrum of $\mathcal{A}_2$ consists of the eigenvalues
$$(2k + 4 - (1 \pm 1)(1 + 2\kappa))s \quad (k \in 2\mathbb{N}) \tag{40}$$
of multiplicity one.
- The spectrum of $\mathcal{B}_1$ consists of the eigenvalues
$$(2k + 2 + (1 \mp 1)(-1 + 2\kappa))s \quad (k \in 2\mathbb{N} + 1) \tag{41}$$
of multiplicity one.
- The spectrum of $\mathcal{B}_2$ consists of the eigenvalues
$$(2k - 2 - (1 \pm 1)(-1 + 2\kappa))s \quad (k \in 2\mathbb{N} + 1) \tag{42}$$
of multiplicity one.

These eigenvalues have normalized eigenfunctions χ_k, defined for the corresponding values of $a = \sigma$ and $b = \tau$. For $\mathcal{A}_1^+$, (39) becomes $2ks$. For $\mathcal{A}_1^-$, (39) is $2(k+1+2\kappa)s$. For $\mathcal{A}_2^+$, (40) becomes $2(k + 1 - 2\kappa)s$. For $\mathcal{A}_2^-$, (40) is $2(k + 2)s$. For $\mathcal{B}_1^+$, (41) is $2(k + 1)s$. For $\mathcal{B}_1^-$, (41) becomes $2(k + 2\kappa)s$. For $\mathcal{B}_2^+$, (42) is $2(k - 2\kappa)s$. For $\mathcal{B}_2^-$, (42) becomes $2(k - 1)s$. Using this, we get the information about the sign of the eigenvalues of $\mathcal{A}_i$ and $\mathcal{B}_i$ given in Table 2. In the tables, grey color is used for cases that will be disregarded later (for instance, if there may exist some negative eigenvalue), and a question mark is used for unknown information.

		Sign of eigenvalues
$\mathcal{A}_1^+$		0 if $k = 0$
		$+$ if $k \geq 2$ even
$\mathcal{A}_1^-$		$+$ $\forall k \in 2\mathbb{N}$
$\mathcal{A}_2^+$	$\kappa > \frac{1}{2}$	$-$ if $k < 2\kappa - 1$
		0 if $k = 2\kappa - 1$
		$+$ if $k > 2\kappa - 1$
	$\kappa = \frac{1}{2}$	0 if $k = 0$
		$+$ if $k \geq 2$ even
	$\kappa < \frac{1}{2}$	$+$ $\forall k \in 2\mathbb{N}$
$\mathcal{A}_2^-$		$+$ $\forall k \in 2\mathbb{N}$

		Sign of eigenvalues
$\mathcal{B}_1^+$		$+$ $\forall k \in 2\mathbb{N} + 1$
$\mathcal{B}_1^-$	$\kappa > -\frac{1}{2}$	$+$ $\forall k \in 2\mathbb{N} + 1$
	$\kappa = -\frac{1}{2}$	0 if $k = 1$
		$+$ if $k \geq 3$ odd
	$\kappa < -\frac{1}{2}$	$-$ if $k < -2\kappa$
		0 if $k = -2\kappa$
		$+$ if $k > -2\kappa$
$\mathcal{B}_2^+$		$+$ $\forall k \in 2\mathbb{N} + 1$
$\mathcal{B}_2^-$		0 if $k = 1$
		$+$ if $k \geq 3$ odd

TABLE 2. Sign of the eigenvalues of $\mathcal{A}_i$ and $\mathcal{B}_i$

4.1.2. *Laplacians of the maximum/minimum i.b.c.*

Proposition 4.1 ([4, Proposition 8.4]). *Table 3 describes* $\Delta_{\max/\min}$.

	$\Delta_{\max,0}$	$\Delta_{\min,0}$	$\Delta_{\max,1}$	$\Delta_{\min,1}$
$\kappa \geq \frac{1}{2}$	$\mathcal{A}_1$		$\mathcal{B}_1$	
$\lvert\kappa\rvert < \frac{1}{2}$	$\mathcal{A}_1$	$\mathcal{A}_2$	$\mathcal{B}_1$	$\mathcal{B}_2$
$\kappa \leq -\frac{1}{2}$	$\mathcal{A}_2$		$\mathcal{B}_2$	

TABLE 3. Description of $\Delta_{\max/\min}$

Remark 8. (i) In [4], the proof of Proposition 4.1 uses the following property [4, Lemma 8.5]. Suppose that either $\theta > \frac{1}{2}$, or $\theta = \frac{1}{2} = \kappa$ (respectively, $\theta = \frac{1}{2} = -\kappa$). Then, for every $\xi \in \rho^\theta \mathcal{S}_{\mathrm{ev},+}$, considered as subspace of $C^\infty(E_0)$ (respectively, $C^\infty(E_1)$), there is a sequence (ξ_n) in $C_0^\infty(E_0)$ (respectively, $C_0^\infty(E_1)$), independent of κ, such that $\lim_n \xi_n = \xi$ in $L^2(E_0)$ (respectively, $L^2(E_1)$) and $\lim_n d\xi_n = d\xi$ in $L^2(E_1)$ (respectively, $\lim_n \delta\xi_n = \delta\xi$ in $L^2(E_0)$). In particular, $\rho^\theta \mathcal{S}_{\mathrm{ev},+}$ is contained in $\mathsf{D}(d_{\min})$ (respectively, $\mathsf{D}(\delta_{\min})$). Moreover, according to the proof of [4, Lemma 8.5], given $0 < a < b$, we can take $\xi_n = \alpha_n \xi$ for some $\alpha_n \in C_+^\infty$ satisfying $\chi_{[\frac{b}{n}, na]} \leq \alpha_n \leq \chi_{[\frac{a}{n}, nb]}$, where χ_S denotes the characteristic function of every subset $S \subset \mathbb{R}_+$.

(ii) $\mathcal{E}_i^0$ (respectively, $\mathcal{E}_i^1$) is also a core of $d_{\max/\min}$ (respectively, $\delta_{\min/\max}$) when $\Delta_{\max/\min,0} = \mathcal{A}_i$ (respectively, $\Delta_{\max/\min,1} = \mathcal{B}_i$).

4.2. **An elliptic complex of length two.** Consider again the standard metric on $\mathbb{R}_+$. Let F be the graded Riemannian/Hermitian vector bundle over $\mathbb{R}_+$ whose nonzero terms are F_0, F_1 and F_2, which are trivial real/complex vector bundles of ranks 1, 2 and 1, respectively, equipped with the standard Riemannian/Hermitian metrics. Thus

$$C^\infty(F_0) \equiv C_+^\infty \equiv C^\infty(F_2)\ , \quad C^\infty(F_1) \equiv C_+^\infty \oplus C_+^\infty\ ,$$
$$L^2(F_0) \equiv L_+^2 \equiv L^2(F_2)\ , \quad L^2(F_1) \equiv L_+^2 \oplus L_+^2\ ,$$

where real-/complex-valued functions are considered in C_+^∞ and L_+^2. Fix $s, \mu > 0$, $0 < u < 1$ and $\kappa \in \mathbb{R}$. Let

$$C^\infty(F_0) \underset{\delta_0 \equiv \begin{pmatrix}\delta_{0,1} & \delta_{0,2}\end{pmatrix}}{\overset{d_0 \equiv \begin{pmatrix}d_{0,1} \\ d_{0,2}\end{pmatrix}}{\rightleftarrows}} C^\infty(F_1) \underset{\delta_1 \equiv \begin{pmatrix}\delta_{1,1} \\ \delta_{1,2}\end{pmatrix}}{\overset{d_1 \equiv \begin{pmatrix}d_{1,1} & d_{1,2}\end{pmatrix}}{\rightleftarrows}} C^\infty(F_2)$$

be the differential operators defined by

$$\begin{aligned} d_{0,1} &= \mu\rho^{-u}\ , & d_{0,2} &= \tfrac{d}{d\rho} - (\kappa+u)\rho^{-1} \pm s\rho\ , \\ d_{1,1} &= \tfrac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\ , & d_{1,2} &= -\mu\rho^{-u}\ , \\ \delta_{0,1} &= \mu\rho^{-u}\ , & \delta_{0,2} &= -\tfrac{d}{d\rho} - (\kappa+u)\rho^{-1} \pm s\rho\ , \\ \delta_{1,1} &= -\tfrac{d}{d\rho} - \kappa\rho^{-1} \pm s\rho\ , & \delta_{1,2} &= -\mu\rho^{-u}\ . \end{aligned}$$

Observe that $\delta_0 = d_0^\dagger$ and $\delta_1 = d_1^\dagger$. We may also use the more explicit notation $d_r^\pm$, $\delta_r^\pm$, $d_{r,i}^\pm$ and $\delta_{r,i}^\pm$. A direct computation shows that d_0 and d_1 define an elliptic complex (F, d) of length two. Note that, by (9),

$$d_{1,1} = \rho^{-u}\, d_{0,2}\, \rho^u\ , \quad \delta_{0,2} = \rho^{-u}\, \delta_{1,1}\, \rho^u\ . \tag{43}$$

4.2.1. *Self-adjoint operators defined by the Laplacian.* By (9), the homogeneous components of the corresponding Laplacian Δ (or $\Delta^{\pm}$) are given by

$$\begin{aligned}
\Delta_0 &= H + (\kappa+u)(\kappa+u-1)\rho^{-2} + \mu^2\rho^{-2u} \mp s(1+2(\kappa+u)) \;, \\
\Delta_2 &= H + \kappa(\kappa+1)\rho^{-2} + \mu^2\rho^{-2u} \pm s(1-2\kappa) \;, \\
\Delta_1 &= \begin{pmatrix} \Delta_{1,1} & -2\mu u\rho^{-u-1} \\ -2\mu u\rho^{-u-1} & \Delta_{1,2} \end{pmatrix} \;, \\
\Delta_{1,1} &= H + \kappa(\kappa-1)\rho^{-2} + \mu^2\rho^{-2u} \mp s(1+2\kappa) \;, \\
\Delta_{1,2} &= H + (\kappa+u)(\kappa+u+1)\rho^{-2} + \mu^2\rho^{-2u} \pm s(1-2(\kappa+u)) \;.
\end{aligned}$$

(We may also use (37) and (38) to compute easily some parts of the above components of Δ.) The operators Δ_0, Δ_2, $\Delta_{1,1}$ and $\Delta_{1,2}$ are like P and Q in (34), with $c_1 = 0 = d_1$, plus a constant term. Write $\Delta_1 = U \mp sV$, where

$$V = \begin{pmatrix} 1+2\kappa & 0 \\ 0 & -1+2(\kappa+u) \end{pmatrix} \;. \tag{44}$$

Then, by Propositions 3.3, 3.4 and 3.5, and Remark 7 (v), Δ_0, Δ_2 and Δ_1 define the self-adjoint operators $\mathcal{P}_i$ and $\mathcal{Q}_j$ in L^2_+, and $\mathcal{W}_{i,j}$ in $L^2_+ \oplus L^2_+$, indicated in Table 4, where the conditions come from (17), (21), (23), (24), (25), (26) and (27). The notation $\mathcal{P}_i^{\pm}$, $\mathcal{Q}_j^{\pm}$ and $\mathcal{W}_{i,j}^{\pm}$ may be used as well to specify that these operators are defined by $\Delta_0^{\pm}$, $\Delta_2^{\pm}$ and $\Delta_1^{\pm}$. Note that $v = u$ for all $\mathcal{W}_{i,j}$. The cores of $\mathcal{P}_i^{1/2}$, $\mathcal{Q}_j^{1/2}$ and $\mathcal{W}_{i,j}^{1/2}$, given by Propositions 3.3, 3.4 and 3.5, will be denoted by $\mathcal{F}_i^0$, $\mathcal{F}_j^2$ and $\mathcal{F}_{i,j}^1 = \mathcal{F}_i^{1,1} \oplus \mathcal{F}_j^{1,2}$, respectively.

Remark 9. In contrast to $\mathcal{E}_i$ in Section 4.1.1, note that the graded subspace $\mathcal{F}_i^0 \oplus \mathcal{F}_{i,j}^1 \oplus \mathcal{F}_j^2$ of $C^\infty(F) \cap L^2(F)$, whenever defined, is not preserved by $D = d + \delta$. For instance, it is preserved by d but not by δ when $i = j = 1$, and it is preserved by δ but not by d when $i = j = 2$.

		σ	τ	θ	Condition
Δ_0	$\mathcal{P}_1$	$\kappa+u$			$\kappa > -\frac{1}{2}$
	$\mathcal{P}_2$	$1-\kappa-u$			$\kappa < \frac{3}{2} - 2u$
Δ_2	$\mathcal{Q}_1$		κ		$\kappa > u - \frac{3}{2}$
	$\mathcal{Q}_2$		$-1-\kappa$		$\kappa < \frac{1}{2} - u$
Δ_1	$\mathcal{W}_{1,1}$	κ	$\kappa+u$	κ	$\kappa > u - \frac{1}{2}$
	$\mathcal{W}_{2,2}$	$1-\kappa$	$-1-\kappa-u$	$-\kappa-u$	$\kappa < \frac{1}{2} - 2u$
	$\nexists\ \mathcal{W}_{1,2}$	κ	$-1-\kappa-u$	$-\frac{1}{2}-u$	Impossible
	$\mathcal{W}_{2,1}$	$1-\kappa$	$\kappa+u$	$\frac{1}{2}$	$-1-\frac{u}{2} < \kappa < 1 - \frac{u}{2}$

TABLE 4. Self-adjoint operators defined by Δ_0, Δ_2 and Δ_1

Let us explain the contents of Table 4. Since $c_1 = d_1 = 0$, we have $\sigma = a$ and $\tau = b$, which are given by (11) and (14). Moreover σ, τ and u determine θ in Table 4 so that U is of the form (35) because $2\theta - \sigma - \tau = -u$. Let us check the conditions written in this table, which are given by the hypothesis of Propositions 3.3–3.5. For

$\mathcal{P}_i$ and $\mathcal{Q}_j$, only (17) and (21) are required. For $\mathcal{W}_{i,j}$, we also require (23), and the hypothesis (a)–(d) of Proposition 3.5, obtaining the following:

- For $\mathcal{W}_{1,1}$, we have $\sigma = \theta \neq \tau$ and $\tau - \sigma = u \notin -\mathbb{N}$. Thus (a) applies in this case. Note that (17), (21) and (23) mean $\kappa > u - \frac{1}{2}$. Then (24) holds because $0 < u < 1$ and $\kappa > u - \frac{1}{2}$. So (a) is satisfied.
- For $\mathcal{W}_{2,2}$, we have $\sigma \neq \theta = \tau + 1$ and $\sigma - \tau - 1 = 1 + u \notin -\mathbb{N}$. Thus (c) applies in this case. Now, (17), (21) and (23) mean $\kappa < \frac{1}{2} - 2u$. Then (26) holds because $0 < u < 1$ and $\kappa < \frac{1}{2} - 2u$. So (c) is satisfied.
- There is no $\mathcal{W}_{1,2}$ because $\theta < -\frac{1}{2}$ in that case.
- For $\mathcal{W}_{2,1}$, (17), (21) and (23) mean $-\frac{3}{2} < \kappa < \frac{3}{2} - u$, and we have the following possibilities:
  - The case $\sigma = \theta = \tau$ is not possible because $u \neq 0$.
  - The case $\sigma = \theta \neq \tau$ happens when $\kappa = \frac{1}{2}$. Then $\sigma = \frac{1}{2}$ and $\tau = \frac{1}{2} + u$, obtaining $\tau - \sigma = u \notin -\mathbb{N}$. Thus (a) applies in this case. Moreover (24) holds because $0 < u < 1$. So (a) is satisfied.
  - The case $\sigma \neq \theta = \tau$ happens when $\kappa = \frac{1}{2} - u$. Then $\sigma = \frac{1}{2} + u$ and $\tau = \frac{1}{2}$, obtaining $\sigma - \tau = u \notin -\mathbb{N}$. Thus (b) applies in this case. Moreover (25) holds because $0 < u < 1$. Hence (b) is satisfied.
  - The case $\sigma \neq \theta = \tau + 1$ happens when $\kappa = -\frac{1}{2} - u$. Then $\sigma = \frac{3}{2} + u$ and $\tau = -\frac{1}{2}$, obtaining $\sigma - \tau - 1 = 1 + u \notin -\mathbb{N}$. Thus (c) applies in this case. Moreover (26) holds because $0 < u < 1$. Hence (c) is satisfied.
  - Finally, assume that $\sigma \neq \theta \neq \tau$. The condition $\sigma - \theta, \tau - \theta \notin -\mathbb{N}$ means that $\kappa \notin (\frac{1}{2} + \mathbb{N}) \cup (\frac{1}{2} - u - \mathbb{N})$, which in turn means that $\kappa \neq \frac{1}{2}, \frac{1}{2} - u, -\frac{1}{2} - u$ because $-\frac{3}{2} < \kappa < \frac{3}{2} - u$. But $\sigma = \theta$ if $\kappa = \frac{1}{2}$, $\tau = \theta$ if $\kappa = \frac{1}{2} - u$, and $\theta = \tau + 1$ if $\kappa = -\frac{1}{2} - u$, as we have seen in the previous cases. So $\sigma - \theta, \tau - \theta \notin -\mathbb{N}$, and (d) applies in this case. Moreover, since $0 < u < 1$, (27) holds just when $-1 - \frac{u}{2} < \kappa < 1 - \frac{u}{2}$. Thus (d) is satisfied assuming the stated conditions on κ.

  Therefore $\mathcal{W}_{2,1}$ is defined in one of the above ways if $-1 - \frac{u}{2} < \kappa < 1 - \frac{u}{2}$.

There are the following overlaps of the conditions in Table 4:

- Both $\mathcal{P}_1$ and $\mathcal{P}_2$ are defined for $-\frac{1}{2} < \kappa < \frac{3}{2} - 2u$, and $\mathcal{P}_1 = \mathcal{P}_2$ just when $\kappa = \frac{1}{2} - u$.
- Both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined for $u - \frac{3}{2} < \kappa < \frac{1}{2} - u$, and $\mathcal{Q}_1 = \mathcal{Q}_2$ just when $\kappa = -\frac{1}{2}$.
- Both $\mathcal{W}_{1,1}$ and $\mathcal{W}_{2,2}$ are defined for $u - \frac{1}{2} < \kappa < \frac{1}{2} - 2u$ (if $u < \frac{1}{3}$), but $\mathcal{W}_{1,1} \neq \mathcal{W}_{2,2}$ for all such κ.
- Both $\mathcal{W}_{1,1}$ and $\mathcal{W}_{2,1}$ are defined for $u - \frac{1}{2} < \kappa < 1 - \frac{u}{2}$, and $\mathcal{W}_{1,1} = \mathcal{W}_{2,1}$ just when $\kappa = \frac{1}{2}$.
- Both $\mathcal{W}_{2,2}$ and $\mathcal{W}_{2,1}$ are defined for $-1 - \frac{u}{2} < \kappa < \frac{1}{2} - 2u$, and $\mathcal{W}_{2,2} = \mathcal{W}_{2,1}$ just when $\kappa = -\frac{1}{2} - u$.

Propositions 3.3, 3.4 and 3.5 also give the following spectral estimates, for all $\epsilon > 0$:

- The spectrum of $\mathcal{P}_1$ consists of eigenvalues $\lambda_0 \leq \lambda_2 \leq \cdots$, taking multiplicity into account, such that there are some $D = D(\kappa, u) > 0$ and

$C = C(\epsilon, \kappa, u) > 0$ so that, for all $k \in 2\mathbb{N}$,

$$\lambda_k \geq (2k + (1 \mp 1)(1 + 2(\kappa + u)))s + \mu^2 D s^u (k+1)^{-u} \ , \tag{45}$$

$$\begin{aligned}\lambda_k \leq\ & (2k + (1 \mp 1)(1 + 2(\kappa + u)))s \\ & + (2k + 1 + 2(\kappa + u))\mu^2 \epsilon s^u + \mu^2 C s^u \ .\end{aligned} \tag{46}$$

The first term of the right-hand side of (45) and (46) for $\mathcal{P}_1^+$ and $\mathcal{P}_1^-$ is $2ks$ and $2(k + 1 + 2(\kappa + u))s$, respectively.

- The spectrum of $\mathcal{P}_2$ consists of eigenvalues $\lambda_0 \leq \lambda_2 \leq \cdots$, taking multiplicity into account, such that there are some $D = D(\kappa, u) > 0$ and $C = C(\epsilon, \kappa, u) > 0$ so that, for all $k \in 2\mathbb{N}$,

$$\lambda_k \geq (2k + 4 - (1 \pm 1)(1 + 2(\kappa + u)))s + \mu^2 D s^u (k+1)^{-u} \ , \tag{47}$$

$$\begin{aligned}\lambda_k \leq\ & (2k + 4 - (1 \pm 1)(1 + 2(\kappa + u)))s \\ & + (2k + 3 - 2(\kappa + u))\mu^2 \epsilon s^u + \mu^2 C s^u \ .\end{aligned} \tag{48}$$

The first term of the right-hand side of (47) and (48) for $\mathcal{P}_2^+$ and $\mathcal{P}_2^-$ becomes $2(k + 1 - 2(\kappa + u))s$ and $2(k + 2)s$, respectively.

- The spectrum of $\mathcal{Q}_1$ consists of eigenvalues $\lambda_1 \leq \lambda_3 \leq \cdots$, taking multiplicity into account, such that there are some $D = D(\kappa, u) > 0$ and $C = C(\epsilon, \kappa, u) > 0$ so that, for all $k \in 2\mathbb{N} + 1$,

$$\lambda_k \geq (2k + 2 - (1 \mp 1)(1 - 2\kappa))s + \mu^2 D s^u (k+1)^{-u} \ , \tag{49}$$

$$\lambda_k \leq (2k + 2 - (1 \mp 1)(1 - 2\kappa))s + (2k + 1 + 2\kappa)\mu^2 \epsilon s^u + \mu^2 C s^u \ . \tag{50}$$

The first term of the right-hand side of (49) and (50) for $\mathcal{Q}_1^+$ and $\mathcal{Q}_1^-$ is $2(k + 1)s$ and $2(k + 2\kappa)s$, respectively.

- The spectrum of $\mathcal{Q}_2$ consists of eigenvalues $\lambda_1 \leq \lambda_3 \leq \cdots$, taking multiplicity into account, such that there are some $D = D(\kappa, u) > 0$ and $C = C(\epsilon, \kappa, u) > 0$ so that, for all $k \in 2\mathbb{N} + 1$,

$$\lambda_k \geq (2k - 2 + (1 \pm 1)(1 - 2\kappa))s + \mu^2 D s^u (k+1)^{-u} \ , \tag{51}$$

$$\lambda_k \leq (2k - 2 + (1 \pm 1)(1 - 2\kappa))s + (2k - 1 - 2\kappa)\mu^2 \epsilon s^u + \mu^2 C s^u \ . \tag{52}$$

The first term of the right-hand side of (51) and (52) for $\mathcal{Q}_2^+$ and $\mathcal{Q}_2^-$ is $2(k - 2\kappa)s$ and $2(k - 1)s$, respectively.

- For $\mathcal{W}_{2,1}$, we can take $\tilde{u} = \frac{u+1}{2}$ satisfying (31). Moreover the maximum eigenvalue of $\mp sV$ is $s(1 \mp (2\kappa + u) - u)$. Thus the spectrum of $\mathcal{W}_{2,1}$ consists of two groups of eigenvalues, $\lambda_0 \leq \lambda_2 \leq \cdots$ and $\lambda_1 \leq \lambda_3 \leq \cdots$, repeated according to multiplicity, such that there are some $D = D(\kappa, u) > 0$, $C = C(\epsilon, \kappa, u) > 0$, $\widetilde{C} = \widetilde{C}(\epsilon, \kappa, u) > 0$ and $E = E(\epsilon, \kappa) > 0$ so that, for all $k \in 2\mathbb{N}$,

$$\begin{aligned}\lambda_k \geq\ & \big(1 - 2\mu u \epsilon s^{\frac{u-1}{2}}\big)(2k + 3 - 2\kappa)s \\ & + \mu^2 D s^u (k+1)^{-u} - 2\mu u \widetilde{C} s^{\frac{u+1}{2}} \mp (1 + 2\kappa)s \ ,\end{aligned} \tag{53}$$

$$\begin{aligned}\lambda_k \leq\ & (2k + 4 - (1 \pm 1)(2\kappa + u))s \\ & + (2k + 3 - 2\kappa)\epsilon(\mu^2 s^u + 4\mu u s^{\frac{u+1}{2}}) + \mu^2 C s^u + 4\mu u E s^{\frac{u+1}{2}} \ ,\end{aligned} \tag{54}$$

and, for all $k \in 2\mathbb{N}+1$,

$$\begin{aligned}\lambda_k &\geq \left(1 - 2\mu u \epsilon s^{\frac{u-1}{2}}\right)(2k+1+2(\kappa+u))s \\ &\quad + \mu^2 D s^u (k+1)^{-u} - 2\mu u \widetilde{C} s^{\frac{u+1}{2}} \pm (1+2(\kappa+u))s \ , \qquad (55)\\ \lambda_k &\leq (2k+2+(1\mp 1)(2\kappa+u))s \\ &\quad + (2k+1+2(\kappa+u))\epsilon(\mu^2 s^u + 4\mu u s^{\frac{u+1}{2}}) + \mu^2 C s^u + 4\mu u E s^{\frac{u+1}{2}} \ . \qquad (56)\end{aligned}$$

- $\mathcal{W}_{1,1}$ and $\mathcal{W}_{2,2}$ also have a discrete spectrum, which has the lower bound given by (29) and Proposition 3.5 (v). We omit its explicit expression because it will not be used. The lower estimate of Proposition 3.5 (iii) may not be possible for $\mathcal{W}_{1,1}$ and $\mathcal{W}_{2,2}$ in general. In fact, according to Remark 7 (vii), the existence of $\tilde{u}$ for $\mathcal{W}_{1,1}$ (respectively, $\mathcal{W}_{2,2}$) is characterized by the additional condition $2\kappa > u$ (respectively, $2\kappa < -3u$), which is an additional restriction.

Table 5 contains the information about the sign of the eigenvalues of $\mathcal{P}_i$, $\mathcal{Q}_j$ and $\mathcal{W}_{i,j}$ given by the above spectral estimates.

		Sign of eigenvalues
$\mathcal{P}_1$		$+$ $\forall k \in 2\mathbb{N}$
$\mathcal{P}_2^+$	$\kappa > \frac{1}{2} - u$	$?$ if $k < 2(\kappa+u)-1$ even
		$+$ if $k \geq 2(\kappa+u)-1$ even
	$\kappa \leq \frac{1}{2} - u$	$+$ $\forall k \in 2\mathbb{N}$
$\mathcal{P}_2^-$		$+$ $\forall k \in 2\mathbb{N}$
$\mathcal{Q}_1^+$		$+$ $\forall k \in 2\mathbb{N}+1$
$\mathcal{Q}_1^-$	$\kappa \geq -\frac{1}{2}$	$+$ $\forall k \in 2\mathbb{N}+1$
	$\kappa < -\frac{1}{2}$	$?$ if $k < -2\kappa$ odd
		$+$ if $k \geq -2\kappa$ odd
$\mathcal{Q}_2$		$+$ $\forall k \in 2\mathbb{N}+1$
$\mathcal{W}_{i,j}$		$+$ if $k \gg 0$

TABLE 5. Sign of the eigenvalues of $\mathcal{P}_i$, $\mathcal{Q}_i$ and $\mathcal{W}_{i,j}$

4.2.2. *Laplacians of the maximum/minimum i.b.c.*

Proposition 4.2. *Tables 6, 7 and 8 describe* $\Delta_{\max/\min}$ *for the stated values of* κ.

	$\Delta_{\max,0}$
$\kappa > -\frac{1}{2}$	$\mathcal{P}_1$
$-\frac{1}{2} - u < \kappa \leq -\frac{1}{2}$	?
$\kappa \leq -\frac{1}{2} - u$	$\mathcal{P}_2$

	$\Delta_{\min,0}$
$\kappa \geq \frac{1}{2} - u$	$\mathcal{P}_1$
$\kappa < \frac{1}{2} - u$	$\mathcal{P}_2$

TABLE 6. Description of $\Delta_{\max/\min,0}$

	$\Delta_{\max,2}$
$\kappa > -\frac{1}{2}$	$\mathcal{Q}_1$
$\kappa \leq -\frac{1}{2}$	$\mathcal{Q}_2$

	$\Delta_{\min,2}$
$\kappa \geq \frac{1}{2}$	$\mathcal{Q}_1$
$\frac{1}{2} - u \leq \kappa < \frac{1}{2}$	?
$\kappa < \frac{1}{2} - u$	$\mathcal{Q}_2$

TABLE 7. Description of $\Delta_{\max/\min,2}$

	$\Delta_{\max,1}$
$\kappa > u - \frac{1}{2}$	$\mathcal{W}_{1,1}$
$-\frac{1}{2} < \kappa \leq u - \frac{1}{2}$	?
$-\frac{1}{2} - u < \kappa \leq -\frac{1}{2}$	$\mathcal{W}_{2,1}$
$\kappa \leq -\frac{1}{2} - u$	$\mathcal{W}_{2,2}$

	$\Delta_{\min,1}$
$\kappa \geq \frac{1}{2}$	$\mathcal{W}_{1,1}$
$\frac{1}{2} - u \leq \kappa < \frac{1}{2}$	$\mathcal{W}_{2,1}$
$\frac{1}{2} - 2u \leq \kappa < \frac{1}{2} - u$	?
$\kappa < \frac{1}{2} - 2u$	$\mathcal{W}_{2,2}$

TABLE 8. Description of $\Delta_{\max/\min,1}$

Proof. The operators $d_{0,2}$, $\delta_{0,2}$, $d_{1,1}$ and $\delta_{1,1}$ are like d and δ in Section 4.1. So Proposition 4.1 and Remark 8 (ii) give the following:

$$\mathsf{D}(d_{0,2,\max}) \supset \begin{cases} \mathcal{F}_1^0 & \text{if } \kappa > -\frac{1}{2} - u \\ \mathcal{F}_2^0 & \text{if } \kappa \leq -\frac{1}{2} - u \,, \end{cases} \tag{57}$$

$$\mathsf{D}(d_{0,2,\min}) \supset \begin{cases} \mathcal{F}_1^0 & \text{if } \kappa \geq \frac{1}{2} - u \\ \mathcal{F}_2^0 & \text{if } \kappa < \frac{1}{2} - u \,, \end{cases} \tag{58}$$

$$\mathsf{D}(\delta_{0,2,\max}) \supset \begin{cases} \mathcal{F}_1^{1,2} & \text{if } \kappa \geq \frac{1}{2} - u \\ \mathcal{F}_2^{1,2} & \text{if } \kappa < \frac{1}{2} - u \,, \end{cases} \tag{59}$$

$$\mathsf{D}(\delta_{0,2,\min}) \supset \begin{cases} \mathcal{F}_1^{1,2} & \text{if } \kappa > -\frac{1}{2} - u \\ \mathcal{F}_2^{1,2} & \text{if } \kappa \leq -\frac{1}{2} - u \,, \end{cases} \tag{60}$$

$$\mathsf{D}(d_{1,1,\max}) \supset \begin{cases} \mathcal{F}_1^{1,1} & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_2^{1,1} & \text{if } \kappa \leq -\frac{1}{2} \,, \end{cases} \tag{61}$$

$$\mathsf{D}(d_{1,1,\min}) \supset \begin{cases} \mathcal{F}_1^{1,1} & \text{if } \kappa \geq \frac{1}{2} \\ \mathcal{F}_2^{1,1} & \text{if } \kappa < \frac{1}{2} \,, \end{cases} \tag{62}$$

$$\mathsf{D}(\delta_{1,1,\max}) \supset \begin{cases} \mathcal{F}_1^2 & \text{if } \kappa \geq \frac{1}{2} \\ \mathcal{F}_2^2 & \text{if } \kappa < \frac{1}{2} \,, \end{cases} \tag{63}$$

$$\mathsf{D}(\delta_{1,1,\min}) \supset \begin{cases} \mathcal{F}_1^2 & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_2^2 & \text{if } \kappa \leq -\frac{1}{2} \,, \end{cases} \tag{64}$$

$$d_{0,2,\max} = d_{0,2,\min}\ , \quad \delta_{0,2,\max} = \delta_{0,2,\min} \quad \text{if} \quad |\kappa+u| \geq \tfrac{1}{2}\ ,$$
$$d_{1,1,\max} = d_{1,1,\min}\ , \quad \delta_{1,1,\max} = \delta_{1,1,\min} \quad \text{if} \quad |\kappa| \geq \tfrac{1}{2}\ .$$

On the other hand, since $d_{0,1}$, $\delta_{0,1}$, $d_{1,2}$ and $\delta_{1,2}$ are multiplication operators, we have

$$d_{0,1,\max} = d_{0,1,\min}\ , \quad \delta_{0,1,\max} = \delta_{0,1,\min}\ ,$$
$$d_{1,2,\max} = d_{1,2,\min}\ , \quad \delta_{1,2,\max} = \delta_{1,2,\min}\ .$$

These are maximal multiplication operators [24, Examples III-2.2 and V-3.22]. They satisfy the following:

$$\mathsf{D}(d_{0,1,\max/\min}) \supset \begin{cases} \mathcal{F}_1^0 & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_2^0 & \text{if } \kappa < \frac{3}{2} - 2u\ , \end{cases} \tag{65}$$

$$\mathsf{D}(\delta_{0,1,\max/\min}) \supset \begin{cases} \mathcal{F}_1^{1,1} & \text{if } \kappa > u - \frac{1}{2} \\ \mathcal{F}_2^{1,1} & \text{if } \kappa < \frac{3}{2} - u\ , \end{cases} \tag{66}$$

$$\mathsf{D}(d_{1,2,\max/\min}) \supset \begin{cases} \mathcal{F}_1^{1,2} & \text{if } \kappa > -\frac{3}{2} \\ \mathcal{F}_2^{1,2} & \text{if } \kappa < \frac{1}{2} - 2u\ , \end{cases} \tag{67}$$

$$\mathsf{D}(\delta_{1,2,\max/\min}) \supset \begin{cases} \mathcal{F}_1^2 & \text{if } \kappa > u - \frac{3}{2} \\ \mathcal{F}_2^2 & \text{if } \kappa < \frac{1}{2} - u\ . \end{cases} \tag{68}$$

By Remark 8 (i), we also get

$$\mathsf{D}(d_{\min,0}) = \mathsf{D}(d_{0,1,\min}) \cap \mathsf{D}(d_{0,2,\min})\ , \quad d_{\min,0} = \begin{pmatrix} d_{0,1,\min}|_{\mathsf{D}(d_{\min,0})} \\ d_{0,2,\min}|_{\mathsf{D}(d_{\min,0})} \end{pmatrix}\ , \tag{69}$$

$$\mathsf{D}(\delta_{\min,1}) = \mathsf{D}(\delta_{1,1,\min}) \cap \mathsf{D}(\delta_{1,2,\min})\ , \quad \delta_{\min,1} = \begin{pmatrix} \delta_{1,1,\min}|_{\mathsf{D}(\delta_{\min,1})} \\ \delta_{1,2,\min}|_{\mathsf{D}(\delta_{\min,1})} \end{pmatrix}\ , \tag{70}$$

complementing Lemma 2.3 in this case.

From (57)–(70), Lemmas 2.3 and 2.4, and [44, Chapter XI-12, p. 338, Eq. (1)], it follows that

$$\mathsf{D}(\Delta_{\max,0}^{1/2}) = \mathsf{D}(d_{\max,0}) = \mathsf{D}(d_{0,1,\max}) \cap \mathsf{D}(d_{0,2,\max}) \supset \begin{cases} \mathcal{F}_1^0 & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_2^0 & \text{if } \kappa \leq -\frac{1}{2} - u\ , \end{cases}$$

$$\mathsf{D}(\Delta_{\min,0}^{1/2}) = \mathsf{D}(d_{\min,0}) = \mathsf{D}(d_{0,1,\min}) \cap \mathsf{D}(d_{0,2,\min}) \supset \begin{cases} \mathcal{F}_1^0 & \text{if } \kappa \geq \frac{1}{2} - u \\ \mathcal{F}_2^0 & \text{if } \kappa < \frac{1}{2} - u\ , \end{cases}$$

$$\mathsf{D}(\Delta_{\max,2}^{1/2}) = \mathsf{D}(\delta_{\min,1}) = \mathsf{D}(\delta_{1,1,\min}) \cap \mathsf{D}(\delta_{1,2,\min}) \supset \begin{cases} \mathcal{F}_1^2 & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_2^2 & \text{if } \kappa \leq -\frac{1}{2}\ , \end{cases}$$

$$\mathsf{D}(\Delta_{\min,2}^{1/2}) = \mathsf{D}(\delta_{\max,1}) = \mathsf{D}(\delta_{1,1,\max}) \cap \mathsf{D}(\delta_{1,2,\max}) \supset \begin{cases} \mathcal{F}_1^2 & \text{if } \kappa \geq \frac{1}{2} \\ \mathcal{F}_2^2 & \text{if } \kappa < \frac{1}{2} - u\ , \end{cases}$$

$$\begin{aligned}
\mathsf{D}(\Delta^{1/2}_{\max,1}) &= \mathsf{D}(\delta_{\min,0}+d_{\max,1}) = \mathsf{D}(\delta_{\min,0})\cap\mathsf{D}(d_{\max,1})\\
&\supset (\mathsf{D}(\delta_{0,1,\min})\oplus\mathsf{D}(\delta_{0,2,\min}))\cap(\mathsf{D}(d_{1,1,\max}\oplus\mathsf{D}(d_{1,2,\max}))\\
&\supset\begin{cases}\mathcal{F}^1_{1,1} & \text{if } \kappa>u-\frac12\\ \mathcal{F}^1_{2,1} & \text{if } -\frac12-u<\kappa\le-\frac12\\ \mathcal{F}^1_{2,2} & \text{if } \kappa\le-\frac12-u\,,\end{cases}\\
\mathsf{D}(\Delta^{1/2}_{\min,1}) &= \mathsf{D}(\delta_{\max,0}+d_{\min,1}) = \mathsf{D}(\delta_{\max,0})\cap\mathsf{D}(d_{\min,1})\\
&\supset (\mathsf{D}(\delta_{0,1,\max})\oplus\mathsf{D}(\delta_{0,2,\max}))\cap(\mathsf{D}(d_{1,1,\min}\oplus\mathsf{D}(d_{1,2,\min}))\\
&\supset\begin{cases}\mathcal{F}^1_{1,1} & \text{if } \kappa\ge\frac12\\ \mathcal{F}^1_{2,1} & \text{if } \frac12-u\le\kappa<\frac12\\ \mathcal{F}^1_{2,2} & \text{if } \kappa<\frac12-2u\,.\end{cases}
\end{aligned}$$

Since $\mathcal{F}^0_i$, $\mathcal{F}^2_j$ and $\mathcal{F}^1_{i,j}$ are cores of $\mathcal{P}^{1/2}_i$, $\mathcal{Q}^{1/2}_j$ and $\mathcal{W}^{1/2}_{i,j}$, respectively, and taking into account Table 4, it follows that

$$\Delta^{1/2}_{\max,0}\supset\begin{cases}\mathcal{P}^{1/2}_1 & \text{if } \kappa>-\frac12\\ \mathcal{P}^{1/2}_2 & \text{if } \kappa\le-\frac12-u\,,\end{cases}\qquad \Delta^{1/2}_{\min,0}\supset\begin{cases}\mathcal{P}^{1/2}_1 & \text{if } \kappa\ge\frac12-u\\ \mathcal{P}^{1/2}_2 & \text{if } \kappa<\frac12-u\,,\end{cases}$$

$$\Delta^{1/2}_{\max,2}\supset\begin{cases}\mathcal{Q}^{1/2}_1 & \text{if } \kappa>-\frac12\\ \mathcal{Q}^{1/2}_2 & \text{if } \kappa\le-\frac12\,,\end{cases}\qquad \Delta^{1/2}_{\min,2}\supset\begin{cases}\mathcal{Q}^{1/2}_1 & \text{if } \kappa\ge\frac12\\ \mathcal{Q}^{1/2}_2 & \text{if } \kappa<\frac12-u\,,\end{cases}$$

$$\Delta^{1/2}_{\max,1}\supset\begin{cases}\mathcal{W}^{1/2}_{1,1} & \text{if } \kappa>u-\frac12\\ \mathcal{W}^{1/2}_{2,1} & \text{if } -\frac12-u<\kappa\le-\frac12\\ \mathcal{W}^{1/2}_{2,2} & \text{if } \kappa\le-\frac12-u\,,\end{cases}$$

$$\Delta^{1/2}_{\min,1}\supset\begin{cases}\mathcal{W}^{1/2}_{1,1} & \text{if } \kappa\ge\frac12\\ \mathcal{W}^{1/2}_{2,1} & \text{if } \frac12-u\le\kappa<\frac12\\ \mathcal{W}^{1/2}_{2,2} & \text{if } \kappa<\frac12-2u\,.\end{cases}$$

But these inclusions are equalities because they involve self-adjoint operators. □

Proposition 4.3. *We have* $\ker\Delta_{\max/\min}=0$.

Proof. We have $\ker\Delta_{\max/\min,\mathrm{ev}}=0$ because $\ker d_{\max/\min,0}=0$ and $\ker\delta_{\max/\min,1}=0$ by Lemma 2.3, (69) and (70), since $d_{0,1,\max/\min}$ and $\delta_{1,2,\max/\min}$ are maximal multiplication operators in L^2_+ by continuous non-vanishing functions.[5]

Since $\sigma(\Delta_{\max/\min,\mathrm{ev}})$ is bounded away from 0, we get $\mathsf{R}(\Delta_{\max/\min,0})=L^2_+=\mathsf{R}(\Delta_{\max/\min,2})$ by the spectral theorem. The maximal multiplication operator by $\rho^{\pm u}$ in L^2_+ will be also denoted by $\rho^{\pm u}$. Let $\phi\in\mathsf{D}(\Delta_{\max/\min,0})$ such that $\Delta_{\max/\min,0}\phi\in\mathsf{D}(\rho^u)$. By (43),

$$\begin{aligned}
\psi:=\frac{1}{\mu}\rho^u d_{0,2,\max/\min}\phi &\in \mathsf{D}(\delta_{0,2,\max/\min}\,\rho^{-u})\cap\mathsf{D}(\rho^u\,\delta_{0,2,\max/\min}\,\rho^{-u})\\
&=\mathsf{D}(\rho^{-u}\,\delta_{1,1,\max/\min})\cap\mathsf{D}(\delta_{1,1,\max/\min})\,.
\end{aligned}$$

Then $\psi\in\mathsf{D}(\delta_{\max/\min,1})$ by (70) since $\rho^{-u}\psi\in L^2_+$ and $\delta_{1,2,\max/\min}$ is the maximal multiplication operator by $-\mu\rho^{-u}$. In the following, for the sake of simplicity, the

[5]We may also use Table 5 and Proposition 4.2 for some values of κ (Tables 6 and 7).

notation $d_{0,2}$, $\delta_{1,1}$, $\delta_{0,2}$ and Δ_0 is used for $d_{0,2,\max/\min}$, $\delta_{1,1,\max/\min}$, $\delta_{0,2,\max/\min}$ and $\Delta_{\max/\min,0}$, respectively. It also follows from (43) that

$$\begin{aligned} d_{\max/\min,0}(\phi) + \delta_{\max/\min,1}(\psi) &= \begin{pmatrix} \mu\rho^{-u}\phi + \delta_{1,1}\psi \\ d_{0,2}\phi - \mu\rho^{-u}\psi \end{pmatrix} \\ &= \begin{pmatrix} \mu\rho^{-u}\phi + \frac{1}{\mu}\delta_{1,1}\rho^u d_{0,2}\phi \\ 0 \end{pmatrix} = \begin{pmatrix} \mu\rho^{-u}\phi + \frac{1}{\mu}\rho^u \delta_{0,2} d_{0,2}\phi \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\mu}\rho^u \Delta_0\phi \\ 0 \end{pmatrix} . \end{aligned}$$

Since $\mathsf{R}(\Delta_{\max/\min,0}) = L^2_+$, we get

$$\mathsf{R}(\rho^u) \oplus 0 \subset \mathsf{R}(d_{\max/\min,0}) + \mathsf{R}(\delta_{\max/\min,1}) .$$

With an analogous argument, using Lemma 2.3 instead of (70), we get

$$0 \oplus \mathsf{R}(\rho^u) \subset \mathsf{R}(d_{\max/\min,0}) + \mathsf{R}(\delta_{\max/\min,1}) .$$

Therefore

$$\mathsf{R}(\rho^u) \oplus \mathsf{R}(\rho^u) \subset \mathsf{R}(d_{\max/\min,0}) + \mathsf{R}(\delta_{\max/\min,1}) ,$$

obtaining that $\mathsf{R}(d_{\max/\min,0}) + \mathsf{R}(\delta_{\max/\min,1})$ is dense in $L^2_+ \oplus L^2_+$ because $\mathsf{R}(\rho^u)$ is dense in L^2_+. Thus $\ker \Delta_{\max/\min,1} = 0$ [9, Lemma 2.1]. □

Corollary 4.4. $\Delta_{\max/\min,\mathrm{ev}}$ *and* $\Delta_{\max/\min,1}$ *have the same eigenvalues, with the same multiplicity.*

Proof. This is a direct consequence of Proposition 4.3 and Lemma 2.2. □

Remark 10. Some generalities about this complex of length two hold for all $u > 0$, like (57)–(70), Proposition 4.3 and Corollary 4.4. But the main results require $0 < u < 1$.

Concerning the spectrum, the following corollary fills the gaps in Tables 6–8.

Corollary 4.5. *Tables 9 and 10 describe the spectra of* $\Delta_{\max/\min,\mathrm{ev}}$ *and* $\Delta_{\max/\min,1}$ *in terms of the spectra of* $\mathcal{P}_i$, $\mathcal{Q}_j$ *and* $\mathcal{W}_{i,j}$ *for the stated values of* κ.

	$\sigma(\Delta_{\max,\mathrm{ev}})$
$\kappa > -\frac{1}{2}$	$\sigma(\mathcal{P}_1 \oplus \mathcal{Q}_1)$
$-\frac{1}{2} - u < \kappa \le -\frac{1}{2}$	$\sigma(\mathcal{W}_{2,1})$
$\kappa \le -\frac{1}{2} - u$	$\sigma(\mathcal{P}_2 \oplus \mathcal{Q}_2)$

	$\sigma(\Delta_{\min,\mathrm{ev}})$
$\kappa \ge \frac{1}{2}$	$\sigma(\mathcal{P}_1 \oplus \mathcal{Q}_1)$
$\frac{1}{2} - u \le \kappa < \frac{1}{2}$	$\sigma(\mathcal{W}_{2,1})$
$\kappa < \frac{1}{2} - u$	$\sigma(\mathcal{P}_2 \oplus \mathcal{Q}_2)$

TABLE 9. Spectrum of $\Delta_{\max/\min,\mathrm{ev}}$

	$\sigma(\Delta_{\max,1})$
$\kappa > u - \frac{1}{2}$	$\sigma(\mathcal{W}_{1,1})$
$-\frac{1}{2} < \kappa \le u - \frac{1}{2}$	$\sigma(\mathcal{P}_1 \oplus \mathcal{Q}_1)$
$-\frac{1}{2} - u < \kappa \le -\frac{1}{2}$	$\sigma(\mathcal{W}_{2,1})$
$\kappa \le -\frac{1}{2} - u$	$\sigma(\mathcal{W}_{2,2})$

	$\sigma(\Delta_{\min,1})$
$\kappa \ge \frac{1}{2}$	$\sigma(\mathcal{W}_{1,1})$
$\frac{1}{2} - u \le \kappa < \frac{1}{2}$	$\sigma(\mathcal{W}_{2,1})$
$\frac{1}{2} - 2u \le \kappa < \frac{1}{2} - u$	$\sigma(\mathcal{P}_2 \oplus \mathcal{Q}_2)$
$\kappa < \frac{1}{2} - 2u$	$\sigma(\mathcal{W}_{2,2})$

TABLE 10. Spectrum of $\Delta_{\max/\min,1}$

Proof. This is a direct consequence of Proposition 4.2 and Corollary 4.4. □

4.3. **The wave operator.** For the Hermitian bundle versions of E and F, consider the wave operator $\exp(itD_{\max/\min})$ ($i=\sqrt{-1}$) on $L^2(E)$ or $L^2(F)$, which is bounded.

Proposition 4.6. *For ϕ in $L^2(E)$ or $L^2(F)$, let $\phi_t=\exp(itD_{\max/\min})\phi$. If $\operatorname{supp}\phi\subset(0,a]$ for some $a>0$, then $\operatorname{supp}\phi_t\subset(0,a+|t|]$ for all $t\in\mathbb{R}$.*

Proof. The case of E is given by [4, Proposition 8.7 (ii)]. Then consider the case of F, where the proof needs a slight change because the needed description of $\mathsf{D}^\infty(\Delta_{\max/\min})$ is not available. Since $\exp(itD_{\max/\min})$ is bounded, we can assume that $\phi\in\mathsf{D}^\infty(\Delta_{\max/\min})$. Write $\phi_t=\phi_{t,0}+\phi_{t,1}+\phi_{t,2}$ with $\phi_{t,r}\in C^\infty(F_r)\equiv C^\infty_+$ ($r=0,2$), and $\phi_{t,1}\equiv\left(\begin{smallmatrix}\phi_{t,1,1}\\ \phi_{t,1,2}\end{smallmatrix}\right)\in C^\infty(F_1)\equiv C^\infty_+\oplus C^\infty_+$. Suppose that $t\ge0$, the other case being analogous. For any $c>a$,

$$
\begin{aligned}
\frac{d}{dt}\int_{a+t}^c|\phi_t(\rho)|^2\,d\rho&=\int_{a+t}^c((iD\phi_t,\phi_t)+(\phi_t,iD\phi_t))(\rho)\,d\rho-|\phi_t(a+t)|^2\\
&=i\int_{a+t}^c((D\phi_t,\phi_t)-(\phi_t,D\phi_t))(\rho)\,d\rho-|\phi_t(a+t)|^2\;.
\end{aligned}
$$

Now, $d_{0,1}\equiv\delta_{0,1}$ and $d_{1,2}\equiv\delta_{1,2}$ are multiplication operators by real valued functions. Moreover $d_{0,2}$ and $\delta_{0,2}$ are equal to $\frac{d}{d\rho}$ and $-\frac{d}{d\rho}$, respectively, up to the sum of multiplication operators by the same real valued functions, and the same is true for $d_{1,1}$ and $\delta_{1,1}$. Thus

$$
\begin{aligned}
&(D\phi_t,\phi_t)-(\phi_t,D\phi_t)\\
&\qquad=(\delta_{0,1}\phi_{t,1,1}+\delta_{0,2}\phi_{t,1,2},\phi_{t,0})+(d_{1,1}\phi_{t,1,1}+d_{1,2}\phi_{t,1,2},\phi_{t,2})\\
&\qquad\quad+(d_{0,1}\phi_{t,0}+\delta_{1,1}\phi_{t,2},\phi_{t,1,1})+(d_{0,2}\phi_{t,0}+\delta_{1,2}\phi_{t,2},\phi_{t,1,2})\\
&\qquad\quad-(\phi_{t,0},\delta_{0,1}\phi_{t,1,1}+\delta_{0,2}\phi_{t,1,2})-(\phi_{t,2},d_{1,1}\phi_{t,1,1}+d_{1,2}\phi_{t,1,2})\\
&\qquad\quad-(\phi_{t,1,1},d_{0,1}\phi_{t,0}+\delta_{1,1}\phi_{t,2})-(\phi_{t,1,2},d_{0,2}\phi_{t,0}+\delta_{1,2}\phi_{t,2})\\
&\qquad=-\phi'_{t,1,2}\overline{\phi_{t,0}}+\phi'_{t,1,1}\overline{\phi_{t,2}}-\phi'_{t,2}\overline{\phi_{t,1,1}}+\phi'_{t,0}\overline{\phi_{t,1,2}}\\
&\qquad\quad+\phi_{t,0}\overline{\phi'_{t,1,2}}-\phi_{t,2}\overline{\phi'_{t,1,1}}+\phi_{t,1,1}\overline{\phi'_{t,2}}-\phi_{t,1,2}\overline{\phi'_{t,0}}\\
&\qquad=2i\,\Im(\phi_{t,0}\overline{\phi'_{t,1,2}}+\phi'_{t,1,1}\overline{\phi_{t,2}}+\phi_{t,1,1}\overline{\phi'_{t,2}}+\phi'_{t,0}\overline{\phi_{t,1,2}})\\
&\qquad=2i\,\Im(\phi_{t,1,1}\overline{\phi_{t,2}}+\phi_{t,0}\overline{\phi_{t,1,2}})'\;.
\end{aligned}
$$

Therefore

$$
i\int_{a+t}^c((D\phi_t,\phi_t)-(\phi_t,D\phi_t))(\rho)\,d\rho\in\mathbb{R}\;,
$$

and

$$
\begin{aligned}
&\left|\int_{a+t}^c((D\phi_t,\phi_t)-(\phi_t,D\phi_t))(\rho)\,d\rho\right|\\
&\qquad\le2|(\phi_{t,1,1}\overline{\phi_{t,2}}+\phi_{t,0}\overline{\phi_{t,1,2}})(c)-(\phi_{t,1,1}\overline{\phi_{t,2}}+\phi_{t,0}\overline{\phi_{t,1,2}})(a+t)|\\
&\qquad\le|\phi_{t,1,1}(c)|^2+|\phi_{t,2}(c)|^2+|\phi_{t,1,2}(c)|^2+|\phi_{t,0}(c)|^2\\
&\qquad\quad+|\phi_{t,1,1}(a+t)|^2+|\phi_{t,2}(a+t)|^2+|\phi_{t,1,2}(a+t)|^2+|\phi_{t,0}(a+t)|^2\\
&\qquad=|\phi_t(c)|^2+|\phi_t(a+t)|^2\;.
\end{aligned}
$$

Since $t \mapsto \phi_t$ defines a differentiable map with values in $L^2(F)$, it follows that there is a sequence $a < c_i \uparrow \infty$ such that $\phi_t(c_i) \to 0$, and

$$\frac{d}{dt}\int_{a+t}^{\infty} |\phi_t(\rho)|^2\, d\rho = \lim_i \frac{d}{dt}\int_{a+t}^{c_i} |\phi_t(\rho)|^2\, d\rho \le \lim_i |\phi_t(c_i)|^2 = 0\ .$$

So

$$\int_{a+t}^{\infty} |\phi_t(\rho)|^2\, d\rho \le \int_{a}^{\infty} |\phi_0(\rho)|^2\, d\rho = \int_{a}^{\infty} |\phi(\rho)|^2\, d\rho = 0\ . \quad \square$$

5. Witten's perturbation on a cone

For rel-Morse functions, the rel-local analysis of the Witten's perturbed Laplacian will be reduced to the case of the functions $\pm\frac{1}{2}\rho^2$ on a stratum of a cone with a model adapted metric, where ρ denotes the radial function. This kind of rel-local analysis begins in this section.

5.1. **Witten's perturbation.** To begin with, recall the following generalities about the Witten's perturbation. Let $M \equiv (M, g)$ be a Riemannian n-manifold. For all $x \in M$ and $\alpha \in T_xM^*$, let

$$\alpha\lrcorner = (-1)^{nr+n+1} \star \alpha\wedge \star = -\iota_{\alpha^\sharp} \quad \text{on} \quad \bigwedge^r T_xM^*\ ,$$

involving the Hodge star operator $\star$ on $\bigwedge T_xM^*$ defined by any choice of orientation of T_xM. For any $f \in C^\infty(M)$, E. Witten [43] has introduced the following perturbations of d, δ, D and Δ, depending on $s \ge 0$:

$$d_s = e^{-sf}\, d\, e^{sf} = d + s\, df\wedge\ , \tag{71}$$

$$\delta_s = e^{sf}\, \delta\, e^{-sf} = \delta - s\, df\lrcorner\ , \tag{72}$$

$$D_s = d_s + \delta_s = D + sR\ ,$$

$$\Delta_s = D_s^2 = d_s\delta_s + \delta_s d_s = \Delta + s(RD + DR) + s^2R^2\ , \tag{73}$$

where $R = df\wedge - df\lrcorner$. Notice that $\delta_s = d_s^\dagger$; thus D_s and Δ_s are formally self-adjoint. By analyzing the terms $RD + DR$ and R^2, the expression (73) becomes

$$\Delta_s = \Delta + s\,\mathbf{Hess} f + s^2\,|df|^2\ , \tag{74}$$

where $\mathbf{Hess} f$ is an endomorphism defined by $\operatorname{Hess} f$ [36, Lemma 9.17], satisfying $|\mathbf{Hess} f| = |\operatorname{Hess} f|$ [4, Section 9].

5.2. **De Rham operators on a cone.** Let L be a non-empty compact stratification. Consider a stratum N of L, and the corresponding stratum $M = N \times \mathbb{R}_+$ of $c(L)$. We use the notation $\tilde n = \dim N$ and $n = \dim M = \tilde n + 1$. Let $\pi : M \to N$ be the first factor projection, and ρ the radial function on $c(L)$. From $\bigwedge TM^* = \bigwedge TN^* \boxtimes \bigwedge T\mathbb{R}_+^*$, we get a canonical identity

$$\bigwedge^r TM^* \equiv \pi^* \bigwedge^r TN^* \oplus d\rho\wedge\pi^* \bigwedge^{r-1} TN^* \equiv \pi^* \bigwedge^r TN^* \oplus \pi^* \bigwedge^{r-1} TN^* \tag{75}$$

for every degree r. So

$$\Omega^r(M) \equiv C^\infty(\mathbb{R}_+, \Omega^r(N)) \oplus d\rho\wedge C^\infty(\mathbb{R}_+, \Omega^{r-1}(N)) \tag{76}$$

$$\equiv C^\infty(\mathbb{R}_+, \Omega^r(N)) \oplus C^\infty(\mathbb{R}_+, \Omega^{r-1}(N))\ . \tag{77}$$

Here, smooth functions $\mathbb{R}_+ \to \Omega(N)$ are defined by considering $\Omega(N)$ as Fréchet space with the weak C^∞ topology. In this section, all matrix expressions of vector bundle homomorphisms on $\bigwedge^r TM^*$ or differential operators on $\Omega^r(M)$ will be considered with respect to the decompositions (75) and (77).

Let d and $\tilde d$ denote the exterior derivatives on $\Omega(M)$ and $\Omega(N)$, respectively. We have [4, Lemma 10.1]

$$d \equiv \begin{pmatrix} \tilde d & 0 \\ \frac{d}{d\rho} & -\tilde d \end{pmatrix} . \tag{78}$$

Fix a general adapted metric $\tilde g$ on N. For $u > 0$, the metric $g = \rho^{2u}\tilde g + d\rho^2$ is a general adapted metric on M. The induced metrics on $\bigwedge TM^*$ and $\bigwedge TN^*$ are also denoted by g and $\tilde g$, respectively. Fix some degree $r \in \{0, 1, \dots, n\}$, and, to simplify the expressions, let

$$\kappa = (n - 2r - 1)\tfrac{u}{2} . \tag{79}$$

According to (75),

$$g \equiv \rho^{-2ru}\, \tilde g \oplus \rho^{-2(r-1)u}\, \tilde g \tag{80}$$

on $\bigwedge^r TM^*$. Choose an orientation on an open subset $W \subset N$, and let $\tilde\omega$ denote the corresponding $\tilde g$-volume form on W. Consider the orientation on $W \times \mathbb{R}_+ \subset M$ so that the corresponding g-volume form is

$$\omega = \rho^{(n-1)u}\, d\rho \wedge \tilde\omega . \tag{81}$$

The corresponding Hodge star operators on $\bigwedge T(W \times \mathbb{R}_+)^*$ and $\bigwedge TW^*$ will be denoted by $\star$ and $\tilde\star$, respectively. Like in [4, Lemma 10.2], from (80) and (81), it follows that

$$\star \equiv \begin{pmatrix} 0 & \rho^{2(\kappa+u)}\, \tilde\star \\ (-1)^r \rho^{2\kappa}\, \tilde\star & 0 \end{pmatrix} \tag{82}$$

on $\bigwedge^r T(W \times \mathbb{R}_+)^*$. Let $L^2\Omega^r(M) = L^2\Omega^r(M, g)$ and $L^2\Omega^r(N) = L^2\Omega^r(N, \tilde g)$. From (80) and (81), we also get that (77) induces the identity of Hilbert spaces[6]

$$L^2\Omega^r(M) \equiv \left(L^2_{\kappa,+} \mathbin{\widehat\otimes} L^2\Omega^r(N)\right) \oplus \left(L^2_{\kappa+u,+} \mathbin{\widehat\otimes} L^2\Omega^{r-1}(N)\right) . \tag{83}$$

Let δ and $\tilde\delta$ denote the exterior coderivatives on $\Omega(M)$ and $\Omega(N)$, respectively. Like in [4, Lemma 10.3], using (78), (82) and (9), we get

$$\delta \equiv \begin{pmatrix} \rho^{-2u}\, \tilde\delta & -\frac{d}{d\rho} - 2(\kappa + u)\rho^{-1} \\ 0 & -\rho^{-2u}\, \tilde\delta \end{pmatrix} \tag{84}$$

on $\Omega^r(M)$. Let Δ and $\widetilde\Delta$ denote the Laplacians on $\Omega(M)$ and $\Omega(N)$, respectively. Like in [4, Corollary 10.4], from (78), (84) and (9), it follows that

$$\Delta \equiv \begin{pmatrix} P & -2u\rho^{-1}\, \tilde d \\ -2u\rho^{-2u-1}\, \tilde\delta & Q \end{pmatrix} \tag{85}$$

on $\Omega^r(M)$, where

$$P = \rho^{-2u}\, \widetilde\Delta - \tfrac{d^2}{d\rho^2} - 2\kappa\rho^{-1}\, \tfrac{d}{d\rho} , \tag{86}$$

$$Q = \rho^{-2u}\, \widetilde\Delta - \tfrac{d^2}{d\rho^2} - 2(\kappa + u)\tfrac{d}{d\rho}\, \rho^{-1} . \tag{87}$$

---

[6] Recall that, for Hilbert spaces $\mathfrak{H}'$ and $\mathfrak{H}''$, with scalar products $\langle\ ,\ \rangle'$ and $\langle\ ,\ \rangle''$, the notation $\mathfrak{H}' \mathbin{\widehat\otimes} \mathfrak{H}''$ is used for the Hilbert space tensor product. This is the Hilbert space completion of the algebraic tensor product $\mathfrak{H}' \otimes \mathfrak{H}''$ with respect to the scalar product defined by $\langle u' \otimes u'', v' \otimes v'' \rangle = \langle u', v' \rangle' \, \langle u'', v'' \rangle''$.

5.3. **Witten's perturbation on a cone.** Let d_s, δ_s, D_s and Δ_s ($s \ge 0$) denote the Witten's perturbations of d, δ, D and Δ induced by the function $f = \pm\frac{1}{2}\rho^2$ on M. The more explicit notation $d_s^\pm$, $\delta_s^\pm$, $D_s^\pm$ and $\Delta_s^\pm$ may be used if needed. In this case, $df = \pm\rho\, d\rho$. According to (77),

$$\rho\, d\rho\wedge \equiv \begin{pmatrix} 0 & 0 \\ \rho & 0 \end{pmatrix} , \quad -\rho\, d\rho\lrcorner \equiv \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix} .$$

So, by (78), (84), (71) and (72),

$$d_s \equiv \begin{pmatrix} \tilde d & 0 \\ \frac{d}{d\rho} \pm s\rho & -\tilde d \end{pmatrix} , \tag{88}$$

$$\delta_s \equiv \begin{pmatrix} \rho^{-2u}\,\tilde\delta & -\frac{d}{d\rho} - 2(\kappa+u)\rho^{-1} \pm s\rho \\ 0 & -\rho^{-2u}\,\tilde\delta \end{pmatrix} , \tag{89}$$

on $\Omega^r(M)$. Now,

$$R = \pm\rho(d\rho\wedge - d\rho\lrcorner) \equiv \pm\begin{pmatrix} 0 & \rho \\ \rho & 0 \end{pmatrix} ,$$

and therefore

$$R^2 \equiv \begin{pmatrix} \rho^2 & 0 \\ 0 & \rho^2 \end{pmatrix} \equiv \rho^2 . \tag{90}$$

Like in [4, Lemma 10.6], we get

$$RD + DR = \mp V \tag{91}$$

on $\Omega^r(M)$, where V is given by (44). As a consequence of (73), (85) and (91), we obtain

$$\Delta_s \equiv \begin{pmatrix} P_s & -2u\rho^{-1}\tilde d \\ -2u\rho^{-2u-1}\tilde\delta & Q_s \end{pmatrix} \tag{92}$$

on $\Omega^r(M)$, where

$$P_s = \rho^{-2u}\widetilde\Delta + H - 2\kappa\rho^{-1}\,\frac{d}{d\rho} \mp s(1+2\kappa) , \tag{93}$$

$$Q_s = \rho^{-2u}\widetilde\Delta + H - 2(\kappa+u)\frac{d}{d\rho}\,\rho^{-1} \mp s(-1+2(\kappa+u)) . \tag{94}$$

6. Splitting of the Witten's complex on a cone

6.1. **Spectral decomposition on the link of the cone.** Theorem 1.1 is proved by induction on the depth. Thus, with the notation of Section 5, suppose that $\tilde g$ is good, and $\widetilde\Delta_{\max/\min}$ satisfies the statement of Theorem 1.1. Moreover suppose that g is also good; that is, $u \le 1$.

Let $\widetilde{\mathcal H}_{\max/\min} = \ker\widetilde D_{\max/\min} = \ker\widetilde\Delta_{\max/\min}$, which is a graded subspace of $\Omega(N) \cap L^2\Omega(N)$. For every degree r, let $\widetilde{\mathsf R}_{\max/\min,r-1}, \widetilde{\mathsf R}^*_{\max/\min,r} \subset L^2\Omega^r(N)$ be the images of $\tilde d_{\max/\min,r-1}$ and $\tilde\delta_{\max/\min,r}$, respectively, which are closed subspaces. By restriction, $\widetilde\Delta_{\max/\min}$ defines self-adjoint operators in $\widetilde{\mathsf R}_{\max/\min,r-1}$ and $\widetilde{\mathsf R}^*_{\max/\min,r-1}$, with the same eigenvalues [4, Section 5.1]. For any eigenvalue $\tilde\lambda$ of the restriction of $\widetilde{\Delta}_{\max/\min}$ to $\widetilde{\mathsf{R}}_{\max/\min,r-1}$, let $\widetilde{\mathsf{R}}_{\max/\min,r-1,\tilde\lambda}$ and $\widetilde{\mathsf{R}}^*_{\max/\min,r-1,\tilde\lambda}$ denote the corresponding $\tilde\lambda$-eigenspaces. We have[7]

$$L^2\Omega^r(N)=\widetilde{\mathcal{H}}^r_{\max/\min}\oplus\widehat{\bigoplus_{\tilde\lambda,\tilde\lambda'}}\left(\widetilde{\mathsf{R}}_{\max/\min,r-1,\tilde\lambda}\oplus\widetilde{\mathsf{R}}^*_{\max/\min,r,\tilde\lambda'}\right)\ , \tag{95}$$

where $\tilde\lambda$ and $\tilde\lambda'$ run in the spectrum of the restrictions of $\widetilde{\Delta}_{\max/\min}$ to $\widetilde{\mathsf{R}}_{\max/\min,r-1}$ and $\widetilde{\mathsf{R}}^*_{\max/\min,r}$, respectively.

6.2. **Subcomplexes of length one.** Given $0\neq\gamma\in\widetilde{\mathcal{H}}^r_{\max/\min}$, consider the canonical identities

$$C^\infty_+\equiv C^\infty_+\,\gamma\subset\Omega^r(M)\ ,\quad C^\infty_+\equiv C^\infty_+\,d\rho\wedge\gamma\subset\Omega^{r+1}(M)\ . \tag{96}$$

The following result follows from (88) and (89).

Lemma 6.1. *For $s\geq0$, d_s and δ_s define maps*

$$0\ \underset{\delta_{s,r-1}}{\overset{d_{s,r-1}}{\rightleftarrows}}\ C^\infty_+\,\gamma\ \underset{\delta_{s,r}}{\overset{d_{s,r}}{\rightleftarrows}}\ C^\infty_+\,d\rho\wedge\gamma\ \underset{\delta_{s,r+1}}{\overset{d_{s,r+1}}{\rightleftarrows}}\ 0\ .$$

Moreover, using (96),

$$d_{s,r}=\tfrac{d}{d\rho}\pm s\rho\ ,\quad \delta_{s,r}=-\tfrac{d}{d\rho}-2\kappa\rho^{-1}\pm s\rho\ .$$

Let $\mathcal{E}_{\gamma,0}$ denote the subcomplex of length one of $(\Omega(M),d_s)$ defined by

$$\mathcal{E}^r_{\gamma,0}=C^\infty_{+,0}\,\gamma\equiv C^\infty_{+,0}\ ,\quad \mathcal{E}^{r+1}_{\gamma,0}=C^\infty_{+,0}\,d\rho\wedge\gamma\equiv C^\infty_{+,0}\ .$$

The closure of $\mathcal{E}_{\gamma,0}$ in $L^2\Omega(M)$ is denoted by $L^2\mathcal{E}_\gamma$. By (83),

$$L^2\mathcal{E}^r_\gamma=L^2_{\kappa,+}\,\gamma\equiv L^2_{\kappa,+}\ ,\quad L^2\mathcal{E}^{r+1}_\gamma=L^2_{\kappa,+}\,d\rho\wedge\gamma\equiv L^2_{\kappa,+}\ .$$

Assume now that $s>0$. With the notation of Section 4.1, consider the real version of the elliptic complex (E,d) determined by s and κ (given by (79)). Using Lemma 6.1 and (9), like in [4, Proposition 12.3], we get the following.

Proposition 6.2. *The operator $\rho^\kappa:L^2_{\kappa,+}\to L^2_+$ defines a unitary isomorphism $L^2\mathcal{E}_\gamma\to L^2(E)$, which restricts to an isomorphism of complexes, $(\mathcal{E}_{\gamma,0},d_s)\to(C^\infty_0(E),d)$, up to a shift of degree.*

By Proposition 6.2, $(\mathcal{E}_{\gamma,0},d_s)$ has a maximum/minimum Hilbert complex extension in $L^2\mathcal{E}_\gamma$. Let $(\mathsf{D}_\gamma,\mathbf{d}_{s,\gamma})$ be the maximum/minimum Hilbert complex extension of $(\mathcal{E}_{\gamma,0},d_s)$ if $\gamma\in\widetilde{\mathcal{H}}^r_{\max/\min}$, and $\mathbf{\Delta}_{s,\gamma}$ the corresponding Laplacian. Let $\mathcal{H}_{s,\gamma}=\mathcal{H}^r_{s,\gamma}\oplus\mathcal{H}^{r+1}_{s,\gamma}=\ker\mathbf{\Delta}_{s,\gamma}$, with the induced grading. The more explicit notation $\mathbf{d}^\pm_{s,\gamma}$, $\mathbf{\Delta}^\pm_{s,\gamma}$ and $\mathcal{H}^\pm_{s,\gamma}=\mathcal{H}^{\pm,r}_{s,\gamma}\oplus\mathcal{H}^{\pm,r+1}_{s,\gamma}$ may be also used.

Corollary 6.3. (*i*) *$\mathbf{\Delta}_{s,\gamma}$ has a discrete spectrum.*

(*ii*) *The dimensions of $\mathcal{H}^{\pm,r}_{s,\gamma}$ and $\mathcal{H}^{\pm,r+1}_{s,\gamma}$ are given in Table 11.*

(*iii*) *If $e_s\in\mathcal{H}_{s,\gamma}$ with norm one for every s, and h is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho)\to1$ as $\rho\to0$, then $\langle he_s,e_s\rangle\to1$ as $s\to\infty$.*

(*iv*) *All nonzero eigenvalues of $\mathbf{\Delta}_{s,\gamma}$ are positive and in $O(s)$ as $s\to\infty$.*

<table>
<tr><td></td><td colspan="4">$\gamma \in \widetilde{\mathcal{H}}^r_{\text{max}}$</td><td colspan="4">$\gamma \in \widetilde{\mathcal{H}}^r_{\text{min}}$</td></tr>
<tr><td></td><td>$\mathcal{H}^{+,r}_{s,\gamma}$</td><td>$\mathcal{H}^{+,r+1}_{s,\gamma}$</td><td>$\mathcal{H}^{-,r}_{s,\gamma}$</td><td>$\mathcal{H}^{-,r+1}_{s,\gamma}$</td><td>$\mathcal{H}^{+,r}_{s,\gamma}$</td><td>$\mathcal{H}^{+,r+1}_{s,\gamma}$</td><td>$\mathcal{H}^{-,r}_{s,\gamma}$</td><td>$\mathcal{H}^{-,r+1}_{s,\gamma}$</td></tr>
<tr><td>$\kappa \geq \frac{1}{2}$</td><td rowspan="2">1</td><td rowspan="3" colspan="2">0</td><td rowspan="2">0</td><td>1</td><td rowspan="3" colspan="2">0</td><td>0</td></tr>
<tr><td>$\lvert\kappa\rvert < \frac{1}{2}$</td><td rowspan="2">0</td><td rowspan="2">1</td></tr>
<tr><td>$\kappa \leq -\frac{1}{2}$</td><td>0</td><td>1</td></tr>
</table>

TABLE 11. Dimensions of $\mathcal{H}^{\pm,r}_{s,\gamma}$ and $\mathcal{H}^{\pm,r+1}_{s,\gamma}$

Proof. This follows from Propositions 6.2 and 4.1, Corollary 3.7, Section 4.1.1, and the choice made to define $\mathbf{d}_{s,\gamma}$. □

6.3. **Subomplexes of length two.** Let $\mu = \sqrt{\tilde\lambda}$ for an eigenvalue $\tilde\lambda$ of the restriction of $\widetilde{\Delta}_{\text{max/min}}$ to $\widetilde{\mathsf{R}}_{\text{max/min},r-1}$. According to [4, Section 5.1], there are nonzero differential forms,

$$\alpha \in \widetilde{\mathsf{R}}_{\text{max/min},r-1,\tilde\lambda} \subset \Omega^r(N)\ , \quad \beta \in \widetilde{\mathsf{R}}^*_{\text{max/min},r-1,\tilde\lambda} \subset \Omega^{r-1}(N)\ ,$$

such that $\tilde d\beta = \mu\alpha$ and $\tilde\delta\alpha = \mu\beta$. Consider the canonical identities

$$C^\infty_+ \equiv C^\infty_+\,\beta \subset \Omega^{r-1}(M)\ , \quad C^\infty_+ \equiv C^\infty_+\,d\rho\wedge\alpha \subset \Omega^{r+1}(M)\ , \tag{97}$$

$$C^\infty_+ \oplus C^\infty_+ \equiv C^\infty_+\,\alpha + C^\infty_+\,d\rho\wedge\beta \subset \Omega^r(M)\ . \tag{98}$$

The following result follows from (88) and (89).

Lemma 6.4. *For $s \geq 0$, d_s and δ_s define maps*

$$0 \underset{\delta_{s,r-2}}{\overset{d_{s,r-2}}{\rightleftarrows}} C^\infty_+\,\beta \underset{\delta_{s,r-1}}{\overset{d_{s,r-1}}{\rightleftarrows}} C^\infty_+\,\alpha + C^\infty_+\,d\rho\wedge\beta$$

$$\underset{\delta_{s,r}}{\overset{d_{s,r}}{\rightleftarrows}} C^\infty_+\,d\rho\wedge\alpha \underset{\delta_{s,r+1}}{\overset{d_{s,r+1}}{\rightleftarrows}} 0\ .$$

Moreover, according to (97) *and* (98),

$$\begin{aligned}
d_{s,r-1} &= \begin{pmatrix} \mu \\ \frac{d}{d\rho} \pm s\rho \end{pmatrix}\ ,\\
\delta_{s,r-1} &= \begin{pmatrix} \mu\rho^{-2u} & -\frac{d}{d\rho} - 2(\kappa+u)\rho^{-1} \pm s\rho \end{pmatrix}\ ,\\
d_{s,r} &= \begin{pmatrix} \frac{d}{d\rho} \pm s\rho & -\mu \end{pmatrix}\ ,\\
\delta_{s,r} &= \begin{pmatrix} -\frac{d}{d\rho} - 2\kappa\rho^{-1} \pm s\rho \\ -\mu\rho^{-2u} \end{pmatrix}\ .
\end{aligned}$$

[7]Consider a family of Hilbert spaces, $\mathfrak{H}_a$ with scalar product $\langle\ ,\ \rangle_a$. Recall that the Hilbert space direct sum, $\widehat{\bigoplus}_a \mathfrak{H}^a$, is the Hilbert space completion of the algebraic direct sum, $\bigoplus_a \mathfrak{H}^a$, with respect to the scalar product $\langle\langle (u^a), (v^a) \rangle\rangle = \sum_a \langle u^a, v^a \rangle_a$. Thus $\widehat{\bigoplus}_a \mathfrak{H}^a = \bigoplus_a \mathfrak{H}^a$ if and only if the family is finite.

Let $\mathcal{F}_{\alpha,\beta,0} = \mathcal{F}^{r-1}_{\alpha,\beta,0} \oplus \mathcal{F}^{r}_{\alpha,\beta,0} \oplus \mathcal{F}^{r+1}_{\alpha,\beta,0}$ denote the subcomplex of length two of $(\Omega(M), d_s)$ defined by

$$\mathcal{F}^{r-1}_{\alpha,\beta,0} = C^\infty_{+,0}\,\beta \equiv C^\infty_{+,0}\ , \quad \mathcal{F}^{r+1}_{\alpha,\beta,0} = C^\infty_{+,0}\,d\rho\wedge\alpha \equiv C^\infty_{+,0}\ ,$$
$$\mathcal{F}^{r}_{\alpha,\beta,0} = C^\infty_{+,0}\,\alpha + C^\infty_{+,0}\,d\rho\wedge\beta \equiv C^\infty_{+,0}\oplus C^\infty_{+,0}\ .$$

The closure of $\mathcal{F}_{\alpha,\beta,0}$ in $L^2\Omega(M)$ is denoted by $L^2\mathcal{F}_{\alpha,\beta}$. By (83),

$$L^2\mathcal{F}^{r-1}_{\alpha,\beta} = L^2_{\kappa+u,+}\,\beta \equiv L^2_{\kappa+u,+}\ , \quad L^2\mathcal{F}^{r+1}_{\alpha,\beta} = L^2_{\kappa,+}\,d\rho\wedge\alpha \equiv L^2_{\kappa,+}\ ,$$
$$L^2\mathcal{F}^{r}_{\alpha,\beta} = L^2_{\kappa,+}\,\alpha + L^2_{\kappa+u,+}\,d\rho\wedge\beta \equiv L^2_{\kappa,+}\oplus L^2_{\kappa+u,+}\ .$$

Assume now that $s>0$. With the notation of Section 4.2, consider the real version of the elliptic complex (F,d) determined by s and κ (given by (79)). Using Lemma 6.4 and (9), we get the following (cf. [4, Proposition 12.9]).

Proposition 6.5. *If $u<1$, then $\rho^\kappa : L^2_{\kappa,+}\to L^2_+$ and $\rho^{\kappa+u} : L^2_{\kappa+u,+}\to L^2_+$ define a unitary isomorphism $L^2\mathcal{F}_{\alpha,\beta}\to L^2(F)$, which restricts to an isomorphism of complexes, $(\mathcal{F}_{\alpha,\beta,0}, d_s)\to (C^\infty_0(F), d)$, up to a shift of degree.*

By Proposition 6.5, $(\mathcal{F}_{\alpha,\beta,0}, d_s)$ has a maximum/minimum Hilbert complex extension in $L^2\mathcal{F}_{\alpha,\beta}$. Let $(\mathsf{D}_{\alpha,\beta}, \mathbf{d}_{s,\alpha,\beta})$ be the maximum/minimum Hilbert complex extension of $(\mathcal{F}_{\alpha,\beta,0}, d_s)$ if $\alpha\in\widetilde{\mathsf{R}}_{\max/\min,r-1,\tilde\lambda}$ and $\beta\in\widetilde{\mathsf{R}}^*_{\max/\min,r-1,\tilde\lambda}$. Let $\boldsymbol{\Delta}_{s,\alpha,\beta}$ denote the corresponding Laplacian. The more explicit notation $\mathbf{d}^\pm_{s,\alpha,\beta}$ and $\boldsymbol{\Delta}^\pm_{s,\alpha,\beta}$ may be used.

Corollary 6.6. *(i) $\boldsymbol{\Delta}_{s,\alpha,\beta}$ has a discrete spectrum.*
(ii) The eigenvalues of $\boldsymbol{\Delta}_{s,\alpha,\beta}$ are positive and in $O(s)$ as $s\to\infty$.

Proof. In the case $u<1$, this follows from Proposition 6.5 and Corollary 4.5. In the case $u=1$, this is the content of [4, Proposition 12.11]. □

Remark 11. According to (91)–(94), we have

$$\begin{aligned}
\Delta_s &\equiv H - 2\kappa\rho^{-1}\tfrac{d}{d\rho} \mp s(1+2\kappa) && \text{on } C^\infty_+\equiv C^\infty_+\,\gamma\ ,\\
\Delta_s &\equiv H - 2\kappa\tfrac{d}{d\rho}\,\rho^{-1} \mp s(-1+2\kappa) && \text{on } C^\infty_+\equiv C^\infty_+\,d\rho\wedge\gamma\ ,\\
\Delta_s &\equiv H - 2(\kappa+u)\rho^{-1}\tfrac{d}{d\rho} + \mu^2\rho^{-2u} \mp s(1+2(\kappa+u)) && \text{on } C^\infty_+\equiv C^\infty_+\,\beta\ ,\\
\Delta_s &\equiv H - 2\kappa\tfrac{d}{d\rho}\,\rho^{-1} + \mu^2\rho^{-2u} \mp s(-1+2\kappa) && \text{on } C^\infty_+\equiv C^\infty_+\,d\rho\wedge\alpha\ ,
\end{aligned}$$

and

$$\Delta_s \equiv \begin{pmatrix} P_{\mu,s} & -2\mu u\rho^{-1}\\ -2\mu u\rho^{-2u-1} & Q_{\mu,s}\end{pmatrix}$$

on $C^\infty_+\oplus C^\infty_+\equiv C^\infty_+\,\alpha + C^\infty_+\,d\rho\wedge\beta$, where

$$P_{\mu,s} = H - 2\kappa\rho^{-1}\tfrac{d}{d\rho} + \mu^2\rho^{-2u} \mp s(1+2\kappa)\ ,$$
$$Q_{\mu,s} = H - 2(\kappa+u)\tfrac{d}{d\rho}\,\rho^{-1} + \mu^2\rho^{-2u} \mp s(-1+2(\kappa+u))\ .$$

So the results of Section 3 could be applied to these expressions. We opted for analyzing first the complexes of Section 4 for the sake of simplicity because we have $a=b=0$, L^2_+ is used instead of $L^2_{\kappa,+}$ or $L^2_{\kappa+u,+}$, and Remark 8 is directly applied.

6.4. **Splitting into subcomplexes.** Let $\mathcal{C}_{\max/\min,0}$ denote an orthonormal frame of $\widetilde{\mathcal{H}}_{\max/\min}$ consisting of homogeneous differential forms. For every positive eigenvalue μ of $\widetilde{D}_{\max/\min}$, let $\mathcal{C}_{\max/\min,\mu}$ be an orthonormal frame of the μ-eigenspace of $\widetilde{D}_{\max/\min}$ consisting of differential forms $\alpha+\beta$ like in Section 6.3. Then let

$$\mathbf{d}_{s,\max/\min} = \bigoplus_{\gamma} \mathbf{d}_{s,\gamma} \oplus \widehat{\bigoplus_{\mu}} \bigoplus_{\alpha+\beta} \mathbf{d}_{s,\alpha,\beta} \ ,$$

where γ runs in $\mathcal{C}_{\max/\min,0}$, μ runs in the positive spectrum of $\widetilde{D}_{\max/\min}$, and $\alpha+\beta$ runs in $\mathcal{C}_{\max/\min,\mu}$. The notation $\mathbf{d}^{\pm}_{s,\max/\min}$ may be also used when $\mathbf{d}^{\pm}_{s,\gamma}$ and $\mathbf{d}^{\pm}_{s,\alpha,\beta}$ are considered.

Proposition 6.7. *We have* $d_{s,\max/\min} = \mathbf{d}_{s,\max/\min}$.

Proof. This follows like [4, Proposition 12.12], using [4, Lemma 5.2], [9, Lemma 3.6 and (2.38b)], (76) and (95). □

Let $\mathcal{H}_{s,\max/\min} = \bigoplus_r \mathcal{H}^r_{s,\max/\min} = \ker \Delta_{s,\max/\min}$, with the induced grading. The superindex "±" may be added to this notation to indicate that we are referring to $\Delta^{\pm}_{s,\max/\min}$.

Corollary 6.8. (*i*) $\Delta_{s,\max/\min}$ *has a discrete spectrum.*

(*ii*) *Table 12 describes the isomorphism class of* $\mathcal{H}^{\pm,*}_{s,\max/\min}$.

(*iii*) *If* $e_s \in \mathcal{H}_{s,\max/\min}$ *has norm one for every* s, *and* h *is a bounded measurable function on* $\mathbb{R}_+$ *with* $h(\rho) \to 1$ *as* $\rho \to 0$, *then* $\langle he_s, e_s\rangle \to 1$ *as* $s \to \infty$.

(*iv*) *Let* $0 \le \lambda_{s,\max/\min,0} \le \lambda_{s,\max/\min,1} \le \cdots$ *be the eigenvalues of* $\Delta_{s,\max/\min}$, *repeated according to their multiplicities. Given* $k \in \mathbb{N}$, *if* $\lambda_{s,\max/\min,k} > 0$ *for some* s, *then* $\lambda_{s,\max/\min,k} > 0$ *for all* s, *and* $\lambda_{s,\max/\min,k} \in O(s)$ *as* $s \to \infty$.

(*v*) *There is some* $\theta > 0$ *such that* $\liminf_k \lambda_{s,\max/\min,k} k^{-\theta} > 0$.

	$\mathcal{H}^{+,r}_{s,\max}$	$\mathcal{H}^{-,r+1}_{s,\max}$	$\mathcal{H}^{+,r}_{s,\min}$	$\mathcal{H}^{-,r+1}_{s,\min}$
$\kappa \ge \frac{1}{2}$	$H^r_{\max}(N)$	0	$H^r_{\min}(N)$	0
$\lvert\kappa\rvert < \frac{1}{2}$	$H^r_{\max}(N)$	0	0	$H^r_{\min}(N)$
$\kappa \le -\frac{1}{2}$	0	$H^r_{\max}(N)$	0	$H^r_{\min}(N)$

TABLE 12. Spaces isomorphic to $\mathcal{H}^{\pm,*}_{s,\max/\min}$

Proof. In the case $u = 1$, this result was already shown in [4, Corollary 12.13]. So we consider only the case $0 < u < 1$. For all γ, μ and $\alpha+\beta$ as above, $\mathbf{\Delta}_{s,\gamma}$ and $\mathbf{\Delta}_{s,\alpha,\beta}$ have a discrete spectrum by Corollaries 6.3 (i) and 6.6 (i). Moreover the union of their spectra has no accumulation points according to Section 4 and since $\widetilde{\Delta}_{\max/\min}$ is discrete. Then (i) follows by Proposition 6.7.

Now, properties (ii)–(iv) follow directly from Corollaries 6.3 and 6.6, and Proposition 6.7.

To prove (v), let $0 \le \tilde{\lambda}_{\max/\min,0} \le \tilde{\lambda}_{\max/\min,1} \le \cdots$ denote the eigenvalues of $\widetilde{\Delta}_{\max/\min}$, repeated according to their multiplicities. Since N satisfies Theorem 1.1 (ii) with $\tilde{g}$, there is some $C_0, \theta_0 > 0$ such that

$$\tilde{\lambda}_{\max/\min,\ell} \ge C_0 \ell^{\theta_0} \tag{99}$$

for all ℓ large enough. Consider the counting function

$$\mathfrak{N}^{\pm}_{s,\max/\min}(\lambda) = \#\left\{ k \in \mathbb{N} \mid \lambda^{\pm}_{s,\max/\min,k} < \lambda \right\} \quad (\lambda > 0)\ .$$

From Proposition 4.3, Corollary 4.5, (39)–(42), (45), (47), (49), (51), (53), (55) and (99), and the choices made to define $\mathbf{d}_\gamma$ and $\mathbf{d}_{\alpha,\beta}$ (Sections 6.2 and 6.3), it follows that there are some $C_1, C_2 > 0$ and $C_3, C_3' \in \mathbb{R}$ such that

$$\begin{aligned}
&\mathfrak{N}^{\pm}_{s,\max/\min}(\lambda)\\
&\quad\le \#\left\{ (k,\ell) \in \mathbb{N}^2 \mid C_1 k + C_2\, \tilde{\lambda}_{\max/\min,\ell}(k+1)^{-u} + C_3' \le \lambda \right\}\\
&\quad\le \#\{ (k,\ell) \in \mathbb{N}^2 \mid C_1 k + C_2 C_0 \ell^{\theta_0}(k+1)^{-u} + C_3 \le \lambda \}\\
&\quad\le \#\left\{ (k,\ell) \in \mathbb{N}^2 \,\middle|\, 0 \le \frac{\lambda - C_3}{C_1},\ \ell \le \left(\frac{\lambda - C_3 - C_1 k}{C_2 C_0}\right)^{\frac{1}{\theta_0}} (k+1)^{\frac{u}{\theta_0}} \right\}\ .
\end{aligned}$$

Consider the function

$$f : \left[-1, a := \frac{\lambda - C_3}{C_1}\right] \to [0,\infty)\ , \quad f(x) = \left(\frac{\lambda - C_3 - C_1 x}{C_2 C_0}\right)^{\frac{1}{\theta_0}} (x+1)^{\frac{u}{\theta_0}}\ .$$

Elementary calculus shows that f vanishes at $x = -1, a$, it reaches its maximum at

$$x = b := \frac{\lambda u - C_3 u - C_1}{C_1(1+u)}\ ,$$

and it is strictly increasing (respectively, decreasing) on $[-1, b]$ (respectively, $[b, a]$). It follows that[8]

$$\mathfrak{N}^{\pm}_{s,\max/\min}(\lambda) \le \int_0^a f(x)\, dx + 2f(b) + a + 1\ .$$

But

$$f(b) = \left(\frac{\lambda - C_3 + C_1}{(1+u) C_2 C_0}\right)^{\frac{1}{\theta_0}} \left(\frac{u(\lambda - C_3 + C_1)}{(1+u) C_1}\right)^{\frac{u}{\theta_0}}\ ,$$

and

$$\begin{aligned}
\int_0^a f(x)\, dx &\le \left(\int_0^{\frac{\lambda - C_3}{C_1}} \left(\frac{\lambda - C_3 - C_1 x}{C_2 C_0}\right)^{\frac{2}{\theta_0}} dx\right)^{\frac{1}{2}} \left(\int_0^{\frac{\lambda - C_3}{C_1}} (x+1)^{\frac{2u}{\theta_0}}\, dx\right)^{\frac{1}{2}}\\
&\le \left(\frac{\theta_0 (\lambda - C_3)^{\frac{2}{\theta_0}+1}}{(2+\theta_0)(C_2 C_0)^{\frac{2}{\theta_0}} C_1}\right)^{\frac{1}{2}} \left(\frac{\theta_0 (\lambda - C_3 + C_1)^{\frac{2u}{\theta_0}+1}}{(2u+\theta_0) C_1^{\frac{2u}{\theta_0}+1}}\right)^{\frac{1}{2}}\\
&= \frac{\theta_0 (\lambda - C_3)^{\frac{1}{\theta_0}+\frac{1}{2}} (\lambda - C_3 + C_1)^{\frac{u}{\theta_0}+\frac{1}{2}}}{(2+\theta_0)^{\frac{1}{2}} (2u+\theta_0)^{\frac{1}{2}} (C_2 C_0)^{\frac{1}{\theta_0}} C_1^{1+\frac{u}{\theta_0}}}\ .
\end{aligned}$$

---

[8] A similar argument is made in the proof of [4, Corollary 12.13-(viii)]. In that case, the authors use a strictly decreasing function $f : (-\infty, a] \to [0,\infty)$. The resulting estimate should be

$$\mathfrak{N}^{\pm}_{s,\max/\min}(\lambda) \le \int_0^a f(x)\, dx + f(0) + a + 1\ ,$$

but the terms $f(0) + a + 1$ were missing in that publication. This correction does not affect the final estimate of $\mathfrak{N}^{\pm}_{s,\max/\min}(\lambda)$ obtained there.

So $\mathfrak{N}^{\pm}_{s,\max/\min}(\lambda) \le C\lambda^{\frac{1+u}{\theta_0}+1}$ for some $C > 0$ and all large enough λ, giving (v) with $\theta = \frac{1+u}{\theta_0} + 1$. □

Table 13 describes the above conditions on κ in terms of r.

$\kappa \ge \frac{1}{2}$	$r \le \frac{n-1}{2} - \frac{1}{2u}$
$\lvert\kappa\rvert < \frac{1}{2}$	$\lvert r - \frac{n-1}{2}\rvert < \frac{1}{2u}$
$\kappa \le -\frac{1}{2}$	$r \ge \frac{n-1}{2} + \frac{1}{2u}$

Table 13. Correspondence between conditions on κ and r

7. Relatively local model of the Witten's perturbation

Let $m \in \mathbb{N}$, and let $L_1, \dots, L_a$ be compact stratifications. For each $i = 1, \dots, a$, let N_i be a dense stratum of L_i, let $k_i = \dim N_i + 1$, and let $*_i$ and ρ_i be the vertex and radial function of $c(L_i)$. Then $M := \mathbb{R}^m \times \prod_{i=1}^a (N_i \times \mathbb{R}_+)$ is a dense stratum of $A := \mathbb{R}^m \times \prod_{i=1}^a c(L_i)$. For any relatively compact open neighborhood O of $x := (0, *_1, \dots, *_a)$, all general adapted metrics on M are quasi-isometric on $M \cap O$ to a metric of the form $g = g_0 + \sum_{i=1}^a \rho_i^{2u_i}\tilde g_i + (d\rho_i)^2$, where g_0 is the Euclidean metric on $\mathbb{R}^m$, every $\tilde g_i$ is a general adapted metric on N_i, and $u_i > 0$. Suppose that g is good; i.e., the metrics $\tilde g_i$ are good, and $u_i \le 1$. We can assume that every N_i is connected, which means that the fiber of $\lim : \widehat{M} \to \overline{M}$ over x consists of a unique point, which can be identified to x (see [4, Proof of Proposition 3.20]). According to Section 1.4, the rel-local model of a rel-Morse function around a rel-critical point is of the form $f = \frac{1}{2}(\rho_+^2 - \rho_-^2)$, where $\rho_\pm$ is the radial function of $\mathbb{R}^{m_\pm} \times \prod_{i\in I_\pm} c(L_i)$, for some decomposition $m = m_+ + m_-$ ($m_\pm \in \mathbb{N}$), and some partition of $\{1, \dots, a\}$ into sets $I_\pm$. The rel-critical set of f consists only of x. Let d_s, δ_s, D_s and Δ_s be the Witten's perturbations of d, δ, D and Δ on $\Omega(M)$ induced by f. Let $\mathcal{H}_{s,\max/\min} = \bigoplus_r \mathcal{H}^r_{s,\max/\min} = \ker \Delta_{s,\max/\min}$, with the induced grading. The following result is a direct consequence of Corollary 6.8 and [4, Example 9.1 and Lemma 5.1], taking also into account Table 13.

Corollary 7.1. *(i)* $\Delta_{s,\max/\min}$ *has a discrete spectrum.*
(ii) We have

$$\mathcal{H}^r_{s,\max/\min} \cong \bigoplus_{(r_1,\dots,r_a)} \bigotimes_{i=1}^a H^{r_i}_{\max/\min}(N_i) ,$$

where $(r_1, \dots, r_a)$ *runs in the subset of* $\mathbb{N}^a$ *defined by the conditions*

$$r = m_- + \sum_{i=1}^a r_i + |I_-| ,$$

$$\left.\begin{array}{ll} r_i < \frac{k_i-1}{2} + \frac{1}{2u_i} & \text{if } i \in I_+ \\ r_i \ge \frac{k_i-1}{2} + \frac{1}{2u_i} & \text{if } i \in I_- \end{array}\right\} \quad \text{for } \mathcal{H}^r_{s,\max} ,$$

$$\left.\begin{array}{ll} r_i \le \frac{k_i-1}{2} - \frac{1}{2u_i} & \text{if } i \in I_+ \\ r_i > \frac{k_i-1}{2} - \frac{1}{2u_i} & \text{if } i \in I_- \end{array}\right\} \quad \text{for } \mathcal{H}^r_{s,\min} .$$

(*iii*) *If* $e_s \in \mathcal{H}_{s,\max/\min}$ *with norm one for every* s*, and* h *is a bounded measurable function on* $\mathbb{R}_+$ *with* $h(\rho) \to 1$ *as* $\rho \to 0$*, then* $\langle he_s, e_s\rangle \to 1$ *as* $s \to \infty$*.*
(*iv*) *Let* $0 \leq \lambda_{s,\max/\min,0} \leq \lambda_{s,\max/\min,1} \leq \cdots$ *be the eigenvalues of* $\Delta_{s,\max/\min}$*, repeated according to their multiplicities. Given* $k \in \mathbb{N}$*, if* $\lambda_{s,\max/\min,k} > 0$ *for some* s*, then* $\lambda_{s,\max/\min,k} > 0$ *for all* s *and* $\lambda_{s,\max/\min,k} \in O(s)$ *as* $s \to \infty$*.*
(*v*) *There is some* $\theta > 0$ *such that* $\liminf_k \lambda_{s,\max/\min,k}\, k^{-\theta} > 0$*.*

For every $\rho > 0$, let B_ρ be the open ball of center 0 and radius ρ in $\mathbb{R}^m$, and let

$$U_{x,\rho} = B_\rho \times \prod_{i=1}^{a} (N_i \times (0,\rho)) \subset M\ .$$

Taking complex coefficients, by Propositions 6.2, 6.5 and 6.7, the following result clearly boils down to the case of Proposition 4.6.

Proposition 7.2. *For* $\alpha \in L^2\Omega(M)$*, let* $\alpha_t = \exp(itD_{s,\max/\min})\alpha$*. If* $\operatorname{supp}\alpha \subset \overline{U_{x,a}}$ *for some* $a > 0$*, then* $\operatorname{supp}\alpha_t \subset \overline{U_{x,a+|t|}}$ *for all* $t \in \mathbb{R}$*.*

8. Proof of Theorem 1.1

This theorem follows from Corollary 7.1 (i),(v) with the same arguments as [4, Theorem 1.1]. More precisely, [4, Propositions 14.2 and 14.3] are used to globalize the properties of the rel-local model, the min-max principle (see e.g. [35, Theorem XIII.1]) is used to show that the properties of the statement are invariant by taking Witten's perturbation defined by rel-admissible functions, and Remark 6 (iii),(iv) is used to produce rel-admissible cutoff functions and partitions of unity with bounded differential. These functions are needed for the Witten's perturbation and to apply [4, Propositions 14.2 and 14.3].

9. Functional calculus

Let M be a stratum of a compact stratification, equipped with a good general adapted metric g. Let f be any rel-admissible function on M, and let d_s, δ_s, D_s and Δ_s be the corresponding Witten's perturbations of d, δ, D and Δ. Since f is rel-admissible, for every s, $\Delta_s - \Delta$ is a homomorphism with uniformly bounded norm by (74). From (74) and the min-max principle (see e.g. [35, Theorem XIII.1]), it also follows that $\mathsf{D}(\Delta_{s,\max/\min}) = \mathsf{D}(\Delta_{\max/\min})$, $\mathsf{D}^\infty(\Delta_{s,\max/\min}) = \mathsf{D}^\infty(\Delta_{\max/\min})$, and that the properties stated in Theorem 1.1 can be extended to the perturbation $\Delta_{s,\max/\min}$.

For any rapidly decaying function ϕ on $\mathbb{R}$, $\phi(\Delta_{s,\max/\min})$ is a Hilbert-Schmidt operator on $L^2\Omega(M)$ by the version of Theorem 1.1 (ii) for $\Delta_{s,\max/\min}$. In fact, $\phi(\Delta_{s,\max/\min})$ is a trace class operator because ϕ can be given as the product of two rapidly decaying functions, $|\phi|^{1/2}$ and $\operatorname{sign}(\phi)\,|\phi|^{1/2}$, where $\operatorname{sign}(\phi)(x) = \operatorname{sign}(\phi(x)) \in \{\pm1\}$ if $\phi(x) \neq 0$.

Like in the case of closed manifolds (see e.g. [36, Chapters 5 and 8]), $\phi(\Delta_{s,\max/\min})$ is given by a Schwartz kernel K_s, and $\operatorname{Tr}\phi(\Delta_{s,\max/\min})$ equals the integral of the pointwise trace of K_s on the diagonal. But we do not know whether K_s is uniformly bounded because a "rel-Sobolev embedding theorem" is missing [4, Section 19]. Theorem 1.1 (ii) becomes important in our arguments to make up for this lack.

10. The wave operator

With the notation of Section 9, suppose that f is a rel-Morse function. Take a general chart $O \equiv O'$ around every $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$, like in Section 1.4. Let us add the subindex "x" to the notation of M', N_i, $m_\pm$ and $I_\pm$ in this case. Take a good adapted metric g'_x on M'_x of the form used in Section 7. Consider the Witten's perturbed operators $d'_{x,s}$, $\delta'_{x,s}$, $D'_{x,s}$ and $\Delta'_{x,s}$ on $\Omega(M'_x)$ defined by the function $f' := \frac{1}{2}(\rho_+^2 - \rho_-^2)$ (a prime and the subindex x is added to their notation). Add also a prime to the notation of the sets $U_{x,\rho}$ of Section 7, considered in M'_x. Let $\rho_0 > 0$ such that $\overline{U'_{x,\rho_0}} \subset O'$. Then, for $0 < \rho \leq \rho_0$, there is some open $U_{x,\rho} \subset M$ so that $U_{x,\rho} \equiv U'_{x,\rho}$. Moreover, according to Remark 3, we can assume $g|_{U_{x,\rho_0}} \equiv g'_x|_{U'_{x,\rho_0}}$.

Consider the wave equation

$$\frac{d\alpha_t}{dt} - iD_s\alpha_t = 0\,, \tag{100}$$

where $\alpha_t \in \Omega(M)$ depends smoothly on t. Given any $\alpha \in \mathsf{D}^\infty(\Delta_{s,\max/\min})$, its solution with the initial condition $\alpha_0 = \alpha$ is given by $\alpha_t = \exp(itD_{s,\max/\min})\alpha$. Moreover a usual energy estimate shows that such a solution is unique (see e.g. [36, Proposition 7.4]); in fact, given any $c > 0$, it is also unique for $|t| \leq c$.

Proposition 10.1. *Let $0 < a < b < \rho_0$ and $\alpha \in L^2\Omega(M)$. The following properties hold for $\alpha_t = \exp(itD_{s,\max/\min})\alpha$:*

(*i*) *If $\operatorname{supp}\alpha \subset M \smallsetminus U_{x,a}$, then $\operatorname{supp}\alpha_t \subset M \smallsetminus U_{x,a-|t|}$ for $0 < |t| \leq a$.*
(*ii*) *If $\operatorname{supp}\alpha \subset \overline{U_{x,a}}$, then $\operatorname{supp}\alpha_t \subset \overline{U_{x,a+|t|}}$ for $0 < |t| \leq b - a$.*

Proof. First, let us prove (ii). We can assume that $\alpha \in \mathsf{D}^\infty(\Delta_{s,\max/\min})$ because $\exp(itD_{s,\max/\min})$ is bounded. Since $\operatorname{supp}\alpha \subset \overline{U_{x,a}}$, we have $\alpha|_{U_{x,\rho_0}} \equiv \alpha'|_{U'_{x,\rho_0}}$ for a unique $\alpha' \in \Omega(M'_x)$ supported in $\overline{U'_{x,a}}$. We get $\alpha' \in \mathsf{D}^\infty(\Delta'_{x,s,\max/\min})$ because $\alpha \in \mathsf{D}^\infty(\Delta_{s,\max/\min})$. Let $\alpha'_t = \exp(itD'_{x,s,\max/\min})\alpha'$. By Proposition 7.2, we have $\operatorname{supp}\alpha'_t \subset \overline{U'_{x,a+|t|}}$ for $0 < |t| \leq b - a$. Then $\alpha'_t|_{U'_{x,\rho_0}} \equiv \beta_t|_{U_{x,\rho_0}}$ for a unique $\beta_t \in \Omega(M)$ supported in $\overline{U_{x,a+|t|}}$. Now, $\beta_t \in \mathsf{D}^\infty(\Delta_{s,\max/\min})$ because $\alpha'_t \in \mathsf{D}^\infty(\Delta'_{x,s,\max/\min})$. Moreover β_t satisfies (100) for $|t| \leq b - a$ with initial condition $\beta_0 = \alpha$. So $\beta_t = \alpha_t$ by the uniqueness of the solution of (100), obtaining $\operatorname{supp}\alpha_t \subset \overline{U_{x,a+|t|}}$.

Finally, (i) follows from (ii) in the following way. For any $\beta \in \Omega_0(M)$ with $\operatorname{supp}\beta \subset \overline{U_{x,a-|t|}}$, let $\beta_\tau = \exp(i\tau D_{s,\max/\min})\beta$ for $\tau \in \mathbb{R}$. By (ii), we get $\operatorname{supp}\beta_{-t} \subset \overline{U_{x,a}}$, and therefore $\langle\alpha_t, \beta\rangle = \langle\alpha, \beta_{-t}\rangle = 0$. This shows that $\operatorname{supp}\alpha_t \subset M \smallsetminus U_{x,a-|t|}$. □

Remark 12. The steps given to achieve Proposition 10.1 are simpler here than in [4]. In fact, it would be difficult to adapt the arguments of [4] since an expression of $\mathsf{D}^\infty(\Delta_{\max/\min})$ is missing in Section 4.2.2.

11. Proof of Theorem 1.2

This theorem now follows like [4, Theorem 1.2]. Thus the details are omitted.

Consider the notation of Section 10. By (71), the numbers $\beta^r_{\max/\min}$ are also given by the cohomology of $d_{s,\max/\min} = d_{\max/\min} + s\,df\wedge$ on $\mathsf{D}(d_{s,\max/\min}) = e^{-sf}\,\mathsf{D}(d_{\max/\min}) = \mathsf{D}(d_{\max/\min})$.

Let ϕ be a smooth rapidly decaying function on $\mathbb{R}$ with $\phi(0) = 1$. Then $\phi(\Delta_{s,\max/\min})$ is of trace class (Section 9), and let $\mu^r_{s,\max/\min} = \operatorname{Tr}(\phi(\Delta_{s,\max/\min,r}))$. Then the following result follows formally like [36, Proposition 14.3].

Proposition 11.1. *We have*

$$\sum_{r=0}^{k}(-1)^{k-r}\beta^r_{\max/\min} \le \sum_{r=0}^{k}(-1)^{k-r}\mu^r_{s,\max/\min} \quad (0 \le k < n)\ ,$$

$$\chi_{\max/\min} = \sum_{r=0}^{n}(-1)^r\,\mu^r_{s,\max/\min}\ .$$

For $\rho \le \rho_0$, let $U_\rho = \bigcup_x U_{x,\rho}$, with x running in $\operatorname{Crit}_{\mathrm{rel}}(f)$. Fix some $\rho_1 > 0$ such that $4\rho_1 < \rho_0$. Let $\mathfrak{G}$ and $\mathfrak{H}$ be the Hilbert subspaces of $L^2\Omega(M)$ consisting of forms essentially supported in $M \smallsetminus U_{\rho_1}$ and $M \smallsetminus U_{3\rho_1}$, respectively. Since

$$\Delta_{s,\max/\min} = \Delta_{\max/\min} + s\,\mathbf{Hess}f + s^2\,|df|^2$$

on $\mathsf{D}(\Delta_{s,\max/\min}) = \mathsf{D}(\Delta_{\max/\min})$ for all $s \ge 0$ by (74), it follows that there is some $C > 0$ so that, if s is large enough,

$$\Delta_{s,\max/\min} \ge \Delta_{\max/\min} + Cs^2 \quad \text{on} \quad \mathfrak{G} \cap \mathsf{D}(\Delta_{\max/\min})\ . \tag{101}$$

Let h be a rel-admissible function on M such that $h \ge 0$, $h \equiv 1$ on U_{ρ_1} and $h \equiv 0$ on $M \smallsetminus U_{2\rho_1}$ (Remark 6 (iii)). Then $T_{s,\max/\min} = \Delta_{s,\max/\min} + hCs^2$, with domain $\mathsf{D}(\Delta_{\max/\min})$, is self-adjoint in $L^2\Omega(M)$ with a discrete spectrum. Moreover

$$T_{s,\max/\min} \ge \Delta_{\max/\min} + Cs^2 \tag{102}$$

for s large enough by (101).

Take some $\phi \in \mathcal{S}_{\mathrm{ev}}$ such that $\phi \ge 0$, $\phi(0) = 1$, $\operatorname{supp}\hat\phi \subset [-\rho_1, \rho_1]$, and $\phi|_{[0,\infty)}$ is monotone [4, Section 18.2], where $\hat\phi$ denotes its Fourier transform. Write $\phi(x) = \psi(x^2)$ for some $\psi \in \mathcal{S}$. Using Proposition 10.1 (i), the argument of the first part of the proof of [36, Lemma 14.6] gives the following.

Lemma 11.2. $\psi(\Delta_{s,\max/\min}) = \psi(T_{s,\max/\min})$ *on* $\mathfrak{H}$*.*

Let $\Pi : L^2\Omega(M) \to \mathfrak{H}$ denote the orthogonal projection. According to Section 9, $\psi(\Delta_{s,\max/\min})$ is of trace class for all $s \ge 0$. Then the self-adjoint operator $\Pi\,\psi(\Delta_{s,\max/\min})\,\Pi$ is also of trace class (see e.g. [36, Proposition 8.8]).

Lemma 11.3. $\operatorname{Tr}(\Pi\,\psi(\Delta_{s,\max/\min})\,\Pi) \to 0$ *as* $s \to \infty$*.*

Proof. This follows like [4, Lemma 18.3], using (102), the min-max principle and Lemma 11.2, and expressing the trace as sum of eigenvalues. □

The following is a direct consequence of Corollary 7.1 (i)–(iv).

Corollary 11.4. *If* h *is a bounded measurable function on* $\mathbb{R}_+$ *such that* $h(\rho) \to 1$ *as* $\rho \to 0$*, then* $\operatorname{Tr}(h(\rho)\,\phi(\Delta'_{x,s,\max/\min,r})) \to \nu^r_{x,\max/\min}$ *as* $s \to \infty$*.*

For every $x \in \operatorname{Crit}_{\mathrm{rel}}(f)$, let $\widetilde{\mathfrak{H}}_x \subset L^2\Omega(M)$ and $\widetilde{\mathfrak{H}}'_x \subset L^2\Omega(M'_x)$ be the Hilbert subspaces of differential forms supported in $\overline{U_{x,3\rho_1}}$ and $\overline{U'_{x,3\rho_1}}$, respectively. We have $\widetilde{\mathfrak{H}}_x \equiv \widetilde{\mathfrak{H}}'_x$ because $g \equiv g'_x$ on $U_{x,\rho_0} \equiv U'_{x,\rho_0}$. Moreover $\Delta_s \equiv \Delta'_{x,s}$ on differential forms supported in $U_{x,\rho_0} \equiv U'_{x,\rho_0}$. By using Proposition 10.1 (ii), the argument of the first part of the proof of [36, Lemma 14.6] can be adapted to show the following.

Lemma 11.5. *$\phi(\Delta_{s,\max/\min}) \equiv \phi(\Delta'_{x,s,\max/\min})$ on $\widetilde{\mathfrak{H}}_x \equiv \widetilde{\mathfrak{H}}'_x$ for all $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$.*

For every $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$, let $\widetilde{\Pi}_x : L^2\Omega(M) \to \widetilde{\mathfrak{H}}_x$ and $\widetilde{\Pi}'_x : L^2\Omega(M'_x) \to \widetilde{\mathfrak{H}}'_x$ denote the orthogonal projections. Since the subspaces $\widetilde{\mathfrak{H}}_x$ are orthogonal to each other, $\widetilde{\Pi} := \sum_x \widetilde{\Pi}_x : L^2\Omega(M) \to \widetilde{\mathfrak{H}} := \sum_x \widetilde{\mathfrak{H}}_x$ is the orthogonal projection.

Lemma 11.6. *$\mathrm{Tr}(\widetilde{\Pi}\,\phi(\Delta_{s,\max/\min,r})\,\widetilde{\Pi}) \to \nu^r_{\max/\min}$ as $s \to \infty$.*

Proof. This follows like [4, Lemma 18.3], using Corollary 11.4 and Lemma 11.5, and, for all $x \in \mathrm{Crit}_{\mathrm{rel}}(f)$, considering $\widetilde{\Pi}'_x$ as the multiplication operator by the characteristic function of $U'_{x,3\rho_1}$. □

Since $\Pi + \widetilde{\Pi} = 1$, Theorem 1.2 follows from Proposition 11.1, and Lemmas 11.3 and 11.6.

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