Mathematical and Numerical Analysis of a Transient Magnetic Model with Voltage Drop Excitations

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Abstract

This paper deals with the mathematical and numerical analysis of a nonlinear 2D transient magnetic model when the source data are given in terms of the voltage drop excitations in conductors and the remanent magnetic flux for permanent magnets. The formulation consists of a distributed nonlinear magnetostatic model with time appearing as a parameter, and a circuit equation linking currents and voltage drops. This last equation is used to express the problem as an implicit ODE system whose operator involves the resolution of the distributed model. The model is spatially discretized using a finite element method and an implicit Euler scheme is employed for time discretization. We perform the mathematical analysis of the problem at both the continuous and discrete levels and obtain an error estimate that is illustrated with some numerical results.

Keywords: Transient magnetic, Nonlinear partial differential equation, Finite element approximation, Voltage drops

1. Introduction

The objective of this paper is the mathematical and numerical analysis of a nonlinear transient magnetic model defined in a two-dimensional domain, with sources given in terms of the potential drops in conductors and the remanent fluxes of permanent magnets. This model arises, for instance, in the simulation of electric machines and, in particular, of permanent magnet synchronous motors (PMSM). In this kind of devices, the magnetic core is usually laminated orthogonally to the direction of the currents traversing the coils. Moreover, eddy current losses are often neglected in permanent magnets, so that these regions are modelled as non-conducting; eventually, a posteriori formulas could be used to estimate such losses (see, for instance, [17, 21]).

Both of the above simplifications allow us to build a 2D transient magnetic model in a cross section of the device, the stator coils being the only conducting part. These coils are generally composed by stranded wires carrying a uniformly distributed current density. The mathematical model used to simulate these conductors strongly depends on the kind of the imposed source; see, for instance, [5]. Indeed, if the source data are given in terms of the current traversing the wires, the problem reduces to solving a nonlinear magnetostatic problem at each time step, and thus time appears as a parameter. However, in the case where the potential drops are given, the distributed magnetostatic model has to be coupled with a circuit equation linking currents and voltage drops. In this paper, we focus on this last case because the model offers challenges from a mathematical and numerical point of view, as detailed below.

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Here, we give a first step towards the analysis of the genuine physical problem, as we do not consider the motion of the machine, what would lead to a much more difficult problem; see, for instance, [8] for a case incorporating also induction effects. Our mathematical model will be obtained from the low-frequency approximation of Maxwell equations, without taking any eddy current effects into consideration. Therefore, we will deal with an integro-differential problem coupling an elliptic partial differential equation, written in terms of the magnetic vector potential, with the circuit equations relating currents and voltage drops in stranded conductors. The partial differential equations are nonlinear due to the presence of ferromagnetic materials in the cores which usually have a strongly nonlinear magnetic behavior.

In the literature, we can find several references dealing with the analysis of low-frequency electromagnetic models coupled with circuit equations. For example, in [15], the authors study the well-posedness of a three-dimensional field/circuit nonlinear problem in the presence of eddy currents and provide error estimates for time discretization. In [11], the authors deal with a 3D field/circuit linear model, focusing only on the continuous formulation. Alternatively, field/circuit models also fit in the framework of differential algebraic systems of equations (DAE), usually when using finite integration techniques for the spatial discretization; see, for instance, [4, 3]. Finally, we also highlight the results presented in [10], where we can find a study of some classes of differential algebraic systems of equations in an abstract framework, in particular covering the case of systems of DAE coupled with partial differential equations (PDE). However, we deal with a system of elliptic partial differential equations coupled with a vector ordinary differential equation in terms of time that is not covered by the previous results.

As discussed above, we focus on a model that does not consider eddy current effects. Firstly, we obtain an integro-differential problem arising from the coupling of the Maxwell system of equations with the circuit equations relating currents and voltage drops in stranded conductors. To perform its mathematical analysis, it is written as a nonlinear system of implicit ordinary differential equations in terms of the currents traversing the coils, which are functions of time. The operator defining this system expresses the so-called flux linkages per unit length in the coils in terms of the currents traversing them, via the resolution of some 2D magnetostatics problems. The properties of this operator are deduced directly from results already existing in the literature (specifically, those appearing in [14]). To perform the numerical approximation of the continuous problem we propose an Euler-implicit scheme for the ODE, combined with a finite element method for the approximation of the involved distributed operator. Some convergence results are obtained for this numerical scheme. However, for the numerical implementation, we use the alternative approach proposed in [5] which consists in eliminating the unknown currents from the system by means of the circuit equations. This idea is also exploited in the theoretical analysis of the eddy current model performed in [11]. As a consequence, we need to prove an equivalence result between the implemented scheme and the discrete problem theoretically analysed.

The paper is organized as follows. In Section 2 we present the 2D nonlinear transient magnetic model in a cross-section transversal to the device, written in terms of the magnetic vector potential; moreover, we express the problem as a system of implicit ODE, and perform its mathematical analysis in the continuous case. In Section 3 we introduce the finite element discretization of the magnetostatics problem involved in the definition of the ODE operator. In Section 4 we propose an implicit Euler scheme for the discretization of the system of ODE and prove an error estimate for its solution. In Section 5, we show some numerical results for a test with analytical solution to illustrate the obtained convergence results. Finally, in appendix Appendix A we prove the equivalence between the analysed problem and the implemented one; appendix Appendix B contains the analytical expression of the magnetic vector potential corresponding to the analytical test used for the numerical results.

2. Mathematical Analysis of the Continuous Problem

In this section we state a 2D transient magnetic problem that arises in the mathematical modeling of laminated magnetic media, with sources given in terms of the potential drops per unit length in conductors and the remanent fluxes in permanent magnets. A similar model, without permanent magnets, has been studied in [5] from a computational point of view. Here, we will also present and analyze the continuous formulation.

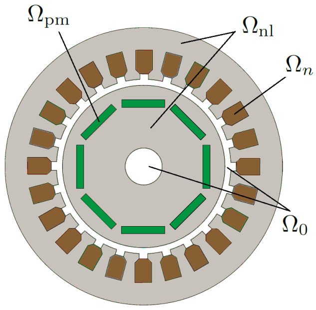

Figure 1: Illustration of the subdomains.

2.1. A two-dimensional transient magnetic model

Let us assume that the current density sources $\boldsymbol{J}$ have non-null component only in the z space direction and that this component does not depend on z, i.e., $\boldsymbol{J} = J_z \mathbf{e}_z$, with $J_z = J_z(x, y, t)$. We also assume that the geometry and the magnetic field $\boldsymbol{H}$ are invariant along the $z-$direction, and that all materials are magnetically isotropic. In this case, under an appropriate decay of fields at infinity (see [16]), the magnetic field $\boldsymbol{H}$, and then the magnetic induction $\boldsymbol{B}$, have only components on the $xy-$plane and both are independent of z.

Since we are interested in using a finite element method for the numerical solution, we will restrict ourselves to a bounded domain. Thus, let us consider a 2D convex bounded domain Ω, with Lipschitz continuous boundary, containing a cross-section transversal to the device. For a given current density $\boldsymbol{J} \in \mathrm{L}^2(\Omega)^3$, we seek $\boldsymbol{H} \in \mathrm{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{B} \in \mathrm{H}(\mathrm{div}, \Omega)$ such that:

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J} \quad \text{in } \Omega, \tag{2.1}$$

$$\mathrm{div}\, \boldsymbol{B} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{B} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where the boundary condition means there is no magnetic flux through the boundary. This model is completed below with the constitutive law relating the magnetic field to the flux density.

Let us assume that Ω is composed of the following open subsets (see fig. 1)

- a magnetically linear subdomain Ω_0,
- non-magnetic connected conductors Ω_n, $n = 1, \ldots, N_c$,
- a permanent magnet region Ω_{pm} and
- a nonlinear ferromagnetic core Ω_{nl}.

We further assume that the boundaries of the conductors, $\partial\Omega_n$, $n = 1, \ldots, N_c$, are mutually disjoint and do not touch the boundary of Ω, and also that the same is true for the boundaries of the connected components of the permanent magnet region. We notice that all parts of the domain are non-conducting except for the non-magnetic conductors, which support the current density $\boldsymbol{J}$. Moreover, we will use the notation $\Omega_{\mathrm{c}} := \cup_{n=1}^{N_c} \Omega_n$ and $\mathfrak{U} := \{\Omega_0, \Omega_{\mathrm{c}}, \Omega_{\mathrm{pm}}, \Omega_{\mathrm{nl}}\}$.

In this framework, vector fields $\boldsymbol{H}$ and $\boldsymbol{B}$ are linked by the constitutive relations:

$$\begin{aligned} \boldsymbol{H} &= \nu_0 \boldsymbol{B} && \text{in } \Omega_0 \cup \Omega_{\mathrm{c}}, \\ \boldsymbol{H} &= \nu_{\mathrm{pm}} \boldsymbol{B} - \nu_{\mathrm{pm}} \boldsymbol{B}^r && \text{in } \Omega_{\mathrm{pm}}, \\ \boldsymbol{H} &= \widetilde{\nu}(|\boldsymbol{B}|) \boldsymbol{B} && \text{in } \Omega_{\mathrm{nl}}, \end{aligned}$$

where ν_0 is the vacuum magnetic reluctivity, $\boldsymbol{B}^r$ is the remanent flux density in the permanent magnets and $\nu_{\rm pm} : \Omega_{\rm pm} \longrightarrow \mathbb{R}^+$ is the magnetic reluctivity in the permanent magnets. In principle, both the magnetically linear subdomain and the nonlinear core subdomain may have several parts with different magnetic reluctivities. However, for the sake of simplicity, we have assumed there is only one for each of these two subdomains and that the magnetic reluctivity of the linear one is that of the vacuum, ν_0. We assume that $\boldsymbol{B}^r$ has only components on the $xy-$plane and both are independent of the $z-$coordinate and that $\nu_{\rm pm} \in \mathrm{L}^\infty(\Omega_{\rm pm})$, also being uniformly bounded from below by a positive constant. Furthermore, we define the global magnetic reluctivity function $\nu : \Omega \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}^+$ as

$$\nu(\mathbf{x}; s) := \begin{cases} \nu_0 & \text{if } \mathbf{x} \in \Omega_0 \cup \Omega_{\rm c}, \\ \nu_{\rm pm}(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_{\rm pm}, \\ \widetilde{\nu}(s) & \text{if } \mathbf{x} \in \Omega_{\rm nl}. \end{cases}$$

Let us make the following assumptions on the nonlinear reluctivity $\widetilde{\nu} : \mathbb{R}_0^+ \longrightarrow \mathbb{R}^+$,

$$\exists\, \nu_1, \nu_2 > 0 \,:\, \nu_1 \le \widetilde{\nu}(s) \le \nu_2 \quad \text{a.e. in } \mathbb{R}_0^+, \tag{2.4}$$

$$\exists\, M_{\widetilde{\nu}} > 0 \,:\, |\widetilde{\nu}(p)p - \widetilde{\nu}(q)q| \le M_{\widetilde{\nu}}|p-q| \quad \forall\, p, q \in \mathbb{R}_0^+, \tag{2.5}$$

$$\exists\, \alpha_{\widetilde{\nu}} > 0 \,:\, (\widetilde{\nu}(p)p - \widetilde{\nu}(q)q)(p-q) \ge \alpha_{\widetilde{\nu}}|p-q|^2 \quad \forall\, p, q \in \mathbb{R}_0^+. \tag{2.6}$$

We notice that the above assumptions on the reluctivity can be derived from the natural properties of the physical BH-curves corresponding to nonlinear magnetic materials (see [12, Chapter 2]).

Finally, we will suppose that all conductors are stranded, which makes it possible to assume that the current density is uniform and given by

$$J_{z,n}(t) = \frac{i_n(t)}{\mathrm{meas}(\Omega_n)}, \quad n = 1, \ldots, N_c,$$

where $i_n(t)$ denotes the total current across Ω_n at time t. Actually, for each conductor, the source can be given in terms of either the current or the potential drop per unit length in the z-direction, and we will focus on the latter alternative.

In order to solve the described two-dimensional model, it is convenient to introduce a magnetic vector potential because it leads to solving a scalar problem instead of a vector one. Since $\boldsymbol{B}$ is divergence-free, there exists a so-called magnetic vector potential $\boldsymbol{A}$ such that $\boldsymbol{B} = \mathbf{curl}\,\boldsymbol{A}$. Under the assumptions above, we can choose a magnetic vector potential that does not depend on z and does not have either x or y components, i.e., $\boldsymbol{A} = A(x, y, t)\mathbf{e}_z$ (see, for instance, [7]).

Next, we will see how to include the potential drops per unit length as sources of our formulation. For the sake of simplicity, we will assume that the electric conductivity σ is constant for all conductors, but otherwise the development below can be applied with no significant change (see [5]). Let us denote by $\boldsymbol{E}$ the electric field. From Faraday's law in the conducting domain,

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } \Omega_{\rm c},$$

the invariance under translation in the z-direction hypothesis and the axial direction of the currents, we deduce that there exist N_c scalar potentials v_n, $n = 1, \ldots, N_c$, unique up to a constant, such that

$$\frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{E} = -\mathbf{grad}\, v_n \quad \text{in } \Omega_n \times \mathbb{R}, \quad n = 1, \ldots, N_c.$$

Taking into account the assumptions on $\boldsymbol{J}$ and Ohm's law, $\boldsymbol{J} = \sigma \boldsymbol{E}$, we deduce that $\boldsymbol{E}$ in conductors has non-null component only in the z space direction which is spatially constant in each Ω_n, $n = 1, \ldots, N_c$. Moreover, since $\boldsymbol{A} = A\mathbf{e}_z$, we have $-\mathbf{grad}\, v_n = -\dfrac{\partial v_n}{\partial z}\mathbf{e}_z$ in each Ω_n, $n = 1, \ldots, N_c$. As a consequence, the above equation reduces to

$$\frac{\partial A}{\partial t} + E_z = -c_n(t) \quad \text{in } \Omega_n, \quad n = 1, \dots, N_c, \tag{2.7}$$

where $c_n(t) := -\dfrac{\partial v_n}{\partial z}(t)$ is the potential drop per unit length in direction z in conductor Ω_n, $n = 1, \dots, N_c$. Multiplying eq. (2.7) by the electric conductivity, integrating on each Ω_n, $n = 1, \dots, N_c$, and taking Ohm's law into account we deduce

$$\frac{d}{dt}\int_{\Omega_n} \sigma A(x, y, t)\, dxdy + i_n(t) = -c_n(t)\sigma\, \mathrm{meas}(\Omega_n), \quad n = 1, \dots, N_c.$$

Thus, if voltage drops are given in conductors, then the problem to be solved is the following:

Problem 1. *Given $\mathbf{c}(t) \in \mathbb{R}^{N_c}$, $\boldsymbol{B}^r(x, y)$ and a vector of initial currents $\boldsymbol{i}_0 \in \mathbb{R}^{N_c}$, find $A(x, y; t)$ and $\boldsymbol{i}(t) \in \mathbb{R}^{N_c}$ for every $t \in [0, T]$ satisfying $\boldsymbol{i}(0) = \boldsymbol{i}_0$,*

$$-\operatorname{div}(\nu_0 \mathbf{grad}\, A) = 0 \quad \textit{in } \Omega_0, \tag{2.8}$$

$$-\operatorname{div}(\nu_0 \mathbf{grad}\, A) = \frac{i_n(t)}{\mathrm{meas}(\Omega_n)} \quad \textit{in } \Omega_n, n = 1, \dots, N_c, \tag{2.9}$$

$$-\operatorname{div}(\nu_{\mathrm{pm}} \mathbf{grad}\, A) = -\operatorname{div}\left(\nu_{\mathrm{pm}} \left(\boldsymbol{B}^r\right)^{\perp}\right) \quad \textit{in } \Omega_{\mathrm{pm}}, \tag{2.10}$$

$$-\operatorname{div}(\widetilde{\nu}(|\mathbf{grad}\, A|)\mathbf{grad}\, A) = 0 \quad \textit{in } \Omega_{\mathrm{nl}}, \tag{2.11}$$

$$[\nu(\cdot; |\mathbf{grad}\, A|)\mathbf{grad}\, A \cdot \mathbf{n}]_{\Gamma} = \begin{cases} \nu_{\mathrm{pm}} \left(\boldsymbol{B}^r\right)^{\perp} \cdot \mathbf{n}_{\mathrm{pm}}, & \textit{if } \Gamma \subset \partial\Omega_{\mathrm{pm}}, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.12}$$

$$A = 0 \quad \textit{on } \partial\Omega, \tag{2.13}$$

and, for every $t \in (0, T]$,

$$\frac{d}{dt}\int_{\Omega_n} \sigma A(x, y, t) dxdy + i_n(t) = -c_n(t)\sigma\, \mathrm{meas}(\Omega_n), \quad n = 1, \dots, N_c. \tag{2.14}$$

In the above equations, $[\,\cdot\,]_{\Gamma}$ denotes the jump across any interface Γ, $\left(\boldsymbol{B}^r\right)^{\perp} := -B^r_y \mathbf{e}_x + B^r_x \mathbf{e}_y$, $\mathbf{n}$ is a unit normal vector to interface Γ and $\mathbf{n}_{\mathrm{pm}}$ is a unit normal vector to $\partial\Omega_{\mathrm{pm}}$ pointing outside Ω_{pm}. We observe that eqs. (2.8) to (2.11) follow from eq. (2.1) and that boundary condition (2.13) implies eq. (2.3).

Remark 2. *Notice that the jump discontinuity in eq.* (2.12) *follows from the transmission condition $[\boldsymbol{H} \times \mathbf{n}]_{\Gamma} = \mathbf{0}$, which, at the same time, follows directly from the regularity of $\boldsymbol{H}$, as long as there are no surface currents on Γ.*

The variational formulation associated to eddy currents problem eqs. (2.8) to (2.13) can be obtained using classical techniques, resulting in

Problem 3. *Given $\mathbf{c}(t) \in \mathcal{C}([0, T])^{N_c}$, $\boldsymbol{i}_0 \in \mathbb{R}^{N_c}$ and $\boldsymbol{B}^r \in \mathrm{L}^2\left(\Omega_{\mathrm{pm}}\right)^3$, find $A(t) \in \mathrm{H}^1_0(\Omega)$ for every $t \in [0, T]$ and $\boldsymbol{i}(t) \in \mathcal{C}^{0,1}([0, T])^{N_c}$ satisfying $\boldsymbol{i}(0) = \boldsymbol{i}_0$,*

$$\int_{\Omega} \nu(\mathbf{x}; |\mathbf{grad}\, A(\mathbf{x}, t)|)\mathbf{grad}\, A(\mathbf{x}, t) \cdot \mathbf{grad}\, W(\mathbf{x})$$
$$= \sum_{n=1}^{N_c} \int_{\Omega_n} \frac{i_n(t)}{\mathrm{meas}(\Omega_n)} W(\mathbf{x}) + \int_{\Omega_{\mathrm{pm}}} \nu_{\mathrm{pm}}(\mathbf{x}) \left(\boldsymbol{B}^r\right)^{\perp}(\mathbf{x}) \cdot \mathbf{grad}\, W(\mathbf{x}),$$

for every $W \in \mathrm{H}_0^1(\Omega)$ *and* $t \in [0,T]$, *and*

$$\frac{d}{dt}\int_{\Omega_n} \sigma A(t) + i_n(t) = -c_n(t)\sigma\,\mathrm{meas}(\Omega_n), \quad n = 1,\dots,N_c \quad \textit{in } (0,T].$$

In the next section, we will write Problem 3 as a nonlinear implicit system of ordinary differential equations in order to prove that it is well-posed.

2.2. Transient magnetic problem as a system of ODE

Let $\boldsymbol{\mathcal{F}} : \mathbb{R}^{N_c} \longrightarrow \mathbb{R}^{N_c}$ be the nonlinear operator defined as

$$\boldsymbol{\mathcal{F}}(\boldsymbol{i}) := \left(\int_{\Omega_1} \sigma A, \dots, \int_{\Omega_{N_c}} \sigma A\right)^{\mathrm{T}} \in \mathbb{R}^{N_c},$$

with A the solution of the nonlinear magnetostatics problem:

Problem 4. *Given* $\boldsymbol{i} \in \mathbb{R}^{N_c}$ *and* $\boldsymbol{B}^r \in \mathrm{L}^2(\Omega_{\mathrm{pm}})^3$, *find* $A \in \mathrm{H}_0^1(\Omega)$ *such that*

$$\int_\Omega \nu(\mathbf{x}; |\mathbf{grad}\,A(\mathbf{x})|)\,\mathbf{grad}\,A(\mathbf{x}) \cdot \mathbf{grad}\,W(\mathbf{x})$$
$$= \sum_{n=1}^{N_c} \int_{\Omega_n} \frac{i_n}{\mathrm{meas}(\Omega_n)} W(\mathbf{x}) + \int_{\Omega_{\mathrm{pm}}} \nu_{\mathrm{pm}}(\mathbf{x})\,(\boldsymbol{B}^r)^\perp(\mathbf{x}) \cdot \mathbf{grad}\,W(\mathbf{x}),$$

for every $W \in \mathrm{H}_0^1(\Omega)$.

Let us notice that the integrals characterising the components of $\boldsymbol{\mathcal{F}}$ are related to the so-called *flux linkages* per unit length since the latter are defined, for conductor Ω_n, $n = 1,\dots,N_c$, as

$$\frac{1}{\sigma\,\mathrm{meas}(\Omega_n)}\int_{\Omega_n} \sigma A.$$

Moreover, we notice that we are using the following notation convention: we denote vector fields with uppercase bold letters, vectors in $\mathbb{R}^n$ with lowercase bold letters and vector operators with calligraphic bold letters.

Theorem 5. *Problem 4 has a unique solution.*

Proof. The proof of this theorem follows directly from the results presented in [14]. Indeed, let $\mathcal{B} : \mathrm{H}_0^1(\Omega) \longrightarrow \mathrm{H}^{-1}(\Omega)$ be the operator defined by

$$\langle \mathcal{B}(A), W\rangle = \int_\Omega \nu(\cdot; |\mathbf{grad}\,A|)\,\mathbf{grad}\,A \cdot \mathbf{grad}\,W$$

for every $W \in \mathrm{H}_0^1(\Omega)$. Under conditions (2.4) to (2.6), operator $\mathcal{B}$ is strongly monotone and Lipschitz continuous, with constants

$$\alpha = C_{PF}^{-2}\min\left\{\nu_0, \nu_{\mathrm{pm}}^1, \alpha_{\tilde{\nu}}\right\} \quad \text{and} \quad M = \max\left\{\nu_0, ||\nu_{\mathrm{pm}}||_{L^\infty(\Omega_{\mathrm{pm}})}, 3M_{\tilde{\nu}}\right\},$$

respectively, C_{PF} being the Poincaré-Friedrichs inequality constant (see [12]).
Concerning the right-hand side, since functions $\frac{i_n}{\mathrm{meas}(\Omega_n)}\chi_{\Omega_n}$ belong to $\mathrm{L}^2(\Omega)$ for $n = 1,\dots,N_c$, (χ_K being the indicator function of set K), and $\boldsymbol{B}^r \in \mathrm{L}^2(\Omega_{\mathrm{pm}})^3$, then the operator associated to the right-hand side is in $\mathrm{H}^{-1}(\Omega)$. □

From this theorem, we deduce that operator $\boldsymbol{\mathcal{F}}$ is well defined and therefore we can rewrite Problem 3 as

Problem 6. *Given* $\boldsymbol{c}(t) \in \mathcal{C}([0,T])^{N_c}$ *and* $\boldsymbol{i}_0 \in \mathbb{R}^{N_c}$*, find* $\boldsymbol{i}(t) \in \mathcal{C}^{0,1}([0,T])^{N_c}$ *such that* $\boldsymbol{i}(0) = \boldsymbol{i}_0$ *and*

$$\frac{d}{dt}\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}(t)\right) + \boldsymbol{i}(t) = -\left(c_1(t)\sigma\,\mathrm{meas}(\Omega_1), \ldots, c_{N_c}(t)\sigma\,\mathrm{meas}(\Omega_{N_c})\right)^{\mathrm{T}} \quad \text{in } (0,T].$$

Remark 7. *We notice that, due to the definition of operator* $\boldsymbol{\mathcal{F}}$*, it is obvious that Problems 3 and 6 are equivalent.*

Theorem 8. *Operator* $\boldsymbol{\mathcal{F}}$ *is strongly monotone and globally Lipschitz continuous in* $\mathbb{R}^{N_c}$ *with respective constants* C_{SM} *and* C_L *to be defined below.*

Proof. Let $\boldsymbol{i}^1, \boldsymbol{i}^2 \in \mathbb{R}^{N_c}$ be given and $A_1, A_2 \in \mathrm{H}^1_0(\Omega)$ be the associated solutions to Problem 4, respectively. Then, $\mathcal{F}_n\left(\boldsymbol{i}^j\right) = \int_{\Omega_n} \sigma A_j$ for $j = 1,2$ and $n = 1, \ldots, N_c$.
Let us consider the inner product in $\mathbb{R}^{N_c}$ defined as follows

$$\boldsymbol{k}^1 * \boldsymbol{k}^2 := \sum_{n=1}^{N_c} \frac{k_n^1 k_n^2}{\sigma\,\mathrm{meas}(\Omega_n)},$$

with $\|\cdot\|_*$ the associated norm.
First, we will prove that $\boldsymbol{\mathcal{F}}$ is strongly monotone:

$$\langle \mathcal{B}(A_1) - \mathcal{B}(A_2), A_1 - A_2\rangle = \sum_{n=1}^{N_c} \int_{\Omega_n} \frac{i_n^1 - i_n^2}{\mathrm{meas}(\Omega_n)}(A_1 - A_2) = \left(\boldsymbol{i}^1 - \boldsymbol{i}^2\right) * \left(\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^1\right) - \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^2\right)\right).$$

Since $\mathcal{B}$ is strongly monotone with constant α,

$$\alpha||A_1 - A_2||^2_{\mathrm{H}^1(\Omega)} \leq \langle \mathcal{B}(A_1) - \mathcal{B}(A_2), A_1 - A_2\rangle = \left(\boldsymbol{i}^1 - \boldsymbol{i}^2\right) * \left(\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^1\right) - \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^2\right)\right). \tag{2.15}$$

Now, taking $W \in \mathrm{H}^1_0(\Omega)$ such that

$$\int_{\Omega_n} \sigma W = i_n^1 - i_n^2,\ n = 1, \ldots, N_c, \qquad \text{and} \qquad ||W||_{\mathrm{H}^1(\Omega)} \leq C\left\|\boldsymbol{i}^1 - \boldsymbol{i}^2\right\|_* \tag{2.16}$$

for some $C > 0$ independent of $\boldsymbol{i}^1, \boldsymbol{i}^2$, we get

$$\langle \mathcal{B}(A_1) - \mathcal{B}(A_2), W\rangle = \sum_{n=1}^{N_c} \int_{\Omega_n} \frac{i_n^1 - i_n^2}{\mathrm{meas}(\Omega_n)} W = \left\|\boldsymbol{i}^1 - \boldsymbol{i}^2\right\|_*^2.$$

Thus, taking into account that $\mathcal{B}$ is Lipschitz continuous,

$$\left\|\boldsymbol{i}^1 - \boldsymbol{i}^2\right\|_*^2 \leq M||A_1 - A_2||_{\mathrm{H}^1(\Omega)}||W||_{\mathrm{H}^1(\Omega)} \leq MC||A_1 - A_2||_{\mathrm{H}^1(\Omega)}\left\|\boldsymbol{i}^1 - \boldsymbol{i}^2\right\|_*.$$

Replacing in eq. (2.15) we get

$$\frac{\alpha}{M^2C^2}\left\|\boldsymbol{i}^1 - \boldsymbol{i}^2\right\|_*^2 \leq \alpha||A_1 - A_2||^2_{\mathrm{H}^1(\Omega)} \leq \left(\boldsymbol{i}^1 - \boldsymbol{i}^2\right) * \left(\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^1\right) - \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^2\right)\right),$$

and then $\boldsymbol{\mathcal{F}}$ is a strongly monotone operator globally in $\mathbb{R}^{N_c}$ in the $||\cdot||_*$-norm with constant α/M^2C^2. We define C_{SM} the corresponding constant in the usual norm.
We notice that we can take $W \in \mathrm{H}^1_0(\Omega)$ verifying eq. (2.16). Indeed, let $\widetilde{\Omega} := \Omega \setminus \Omega_{\mathrm{c}}$, $C_n := \frac{i_n^1 - i_n^2}{\mathrm{meas}(\Omega_n)}$,

$n = 1, \dots, N_c$, and $g_n \in \mathrm{H}^{1/2}(\partial\Omega_n)$ with $g_n(\mathbf{x}) = C_n$ for every $\mathbf{x} \in \partial\Omega_n$, $n = 1, \dots, N_c$. Then,

$$\int_{\Omega_n} \sigma C_n = i_n^1 - i_n^2.$$

Moreover, let us consider the following Dirichlet problem:

$$\begin{cases} \text{Given } g_n \in \mathrm{H}^{1/2}(\partial\Omega_n),\ n = 1, \dots, N_c, \text{ find } \widetilde{W} \in \mathrm{H}_0^1(\widetilde{\Omega}) \text{ such that} \\ \qquad -\Delta \widetilde{W} = 0 \quad \text{in } \widetilde{\Omega}, \\ \qquad \widetilde{W} = g_n \quad \text{on } \partial\Omega_n,\ n = 1, \dots, N_c. \end{cases}$$

This problem is well-defined and

$$||\widetilde{W}||_{\mathrm{H}^1(\widetilde{\Omega})} \leq C \left\| \boldsymbol{i}^1 - \boldsymbol{i}^2 \right\|_*,$$

with C independent of $\boldsymbol{i}^1, \boldsymbol{i}^2$. We can define $W \in \mathrm{H}_0^1(\Omega)$ by

$$W := \begin{cases} \widetilde{W} & \text{in } \widetilde{\Omega}, \\ C_n & \text{in } \Omega_n, n = 1, \dots, N_c. \end{cases}$$

Now, we will show that $\boldsymbol{\mathcal{F}}$ is Lipschitz continuous. Indeed, since

$$\left(\int_{\Omega_n} \sigma(A_1 - A_2) \right)^2 \leq \sigma^2 ||A_1 - A_2||^2_{\mathrm{L}^1(\Omega_n)} \leq \sigma^2 \operatorname{meas}(\Omega_n) ||A_1 - A_2||^2_{\mathrm{L}^2(\Omega_n)^2} \\ \leq \sigma^2 \operatorname{meas}(\Omega_n) ||A_1 - A_2||^2_{\mathrm{H}^1(\Omega_n)},$$

then, for every $n = 1, \dots, N_c$,

$$\left\| \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^1\right) - \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^2\right) \right\|_* \leq \left(\sum_{n=1}^{N_c} \sigma ||A_1 - A_2||^2_{\mathrm{H}^1(\Omega_n)} \right)^{1/2} \leq C_1 ||A_1 - A_2||_{\mathrm{H}^1(\Omega)}.$$

Finally, since

$$||A_1 - A_2||^2_{\mathrm{H}^1(\Omega)} \leq \frac{1}{\alpha} \left(\boldsymbol{i}^1 - \boldsymbol{i}^2\right) * \left(\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^1\right) - \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^2\right)\right) \\ \leq \frac{1}{\alpha} \left\|\boldsymbol{i}^1 - \boldsymbol{i}^2\right\|_* \left\| \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^1\right) - \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^2\right) \right\|_* \leq \frac{C_1}{\alpha} \left\|\boldsymbol{i}^1 - \boldsymbol{i}^2\right\|_* ||A_1 - A_2||_{\mathrm{H}^1(\Omega)},$$

we conclude that

$$\left\| \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^1\right) - \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}^2\right) \right\|_* \leq \frac{C_1^2}{\alpha} ||\boldsymbol{i}^1 - \boldsymbol{i}^2||_*,$$

and therefore $\boldsymbol{\mathcal{F}}$ is Lipschitz continuous globally in $\mathbb{R}^{N_c}$ in the $||\cdot||_*$-norm with constant C_1^2/α. We define C_L the corresponding constant in the usual norm. □

From the last theorem, applying a result by E. H. Zarantonello (see [19], Theorem 25.B), we deduce that

Corollary 9. *$\boldsymbol{\mathcal{F}}$ is invertible and its inverse $\boldsymbol{\mathcal{F}}^{-1}$ is Lipschitz continuous with Lipschitz constant equal to $\frac{1}{C_{SM}}$.*

This result allows us to rewrite Problem 6 in the following way:

Problem 10. *Given* $\boldsymbol{c}(t) \in \mathcal{C}([0,T])^{N_c}$ *and* $\boldsymbol{i}_0 \in \mathbb{R}^{N_c}$*, find* $\boldsymbol{i}(t) \in \mathcal{C}^{0,1}([0,T])^{N_c}$ *such that* $\boldsymbol{i}(0) = \boldsymbol{i}_0$ *and*

$$\frac{d}{dt}\boldsymbol{\ell}(t) + \boldsymbol{\mathcal{F}}^{-1}\left(\boldsymbol{\ell}(t)\right) = -\left(c_1(t)\sigma\,\mathrm{meas}(\Omega_1), \dots, c_{N_c}(t)\sigma\,\mathrm{meas}(\Omega_{N_c})\right)^{\mathrm{T}} \quad \text{in } (0,T],$$

with $\boldsymbol{\ell}(t) = \boldsymbol{\mathcal{F}}\left(\boldsymbol{i}(t)\right)$ *for every* $t \in [0,T]$*.*

Remark 11. *Problems 6 and 10 can be defined with lower regularity assumptions on the source data* $\boldsymbol{c}(t)$*. For instance, if* $\boldsymbol{c}(t)$ *is Lebesgue-measurable in* $[0,T]$ *and* $|\boldsymbol{c}(t)|$ *is bounded by a Lebesgue integrable function, both problems have a unique absolutely continuous solution,* $\boldsymbol{i}(t) \in AC([0,T])$*, fulfilling the differential equation almost everywhere in* $[0,T]$*. Furthermore, most of the results presented in this paper can also be proved under these assumptions.*

Theorem 12. *Problem 6 has a unique solution* $\boldsymbol{i}(t) \in \mathcal{C}^{0,1}([0,T])^{N_c}$ *such that*

$$\|\boldsymbol{i}(t)\| \leq \|\boldsymbol{i}_0\| + \frac{T}{C_{SM}}\left(\|\boldsymbol{i}_0\| + \sigma \max_{n=1,\dots,N_c}\{\mathrm{meas}(\Omega_n)\}\,\|\boldsymbol{c}\|_{\mathrm{L}^2(0,T)}\right)e^{T/C_{SM}}$$

for every $t \in [0,T]$*.*

Proof. Since $\boldsymbol{\mathcal{F}}^{-1}$ is globally Lipschitz continuous in $\mathbb{R}^{N_c}$, from Theorem 2.15 in [2] we conclude that Problem 10 has a unique solution $\boldsymbol{i} = \boldsymbol{\mathcal{F}}^{-1}\left(\boldsymbol{\ell}\right)$, with $\boldsymbol{\ell} \in \mathcal{C}^1([0,T])^{N_c}$. Therefore, Problem 6 has a unique solution $\boldsymbol{i} \in \mathcal{C}^{0,1}([0,T])^{N_c}$.
Moreover, integrating the equation appearing in Problem 10 in $(0,t)$,

$$\boldsymbol{\ell}(t) - \boldsymbol{\ell}(0) = -\int_0^t \left(\boldsymbol{\mathcal{F}}^{-1}\left(\boldsymbol{\ell}(s)\right) + \left(c_1(s)\sigma\,\mathrm{meas}(\Omega_1), \dots, c_{N_c}(s)\sigma\,\mathrm{meas}(\Omega_{N_c})\right)^{\mathrm{T}}\right) ds$$

for every $t \in [0,T]$. Thus,

$$\begin{aligned}
\|\boldsymbol{\ell}(t) - \boldsymbol{\ell}(0)\| &\leq \int_0^t \left\|\boldsymbol{\mathcal{F}}^{-1}\left(\boldsymbol{\ell}(s)\right)\right\| ds + \int_0^t \left\|\left(c_1(s)\sigma\,\mathrm{meas}(\Omega_1), \dots, c_{N_c}(s)\sigma\,\mathrm{meas}(\Omega_{N_c})\right)^{\mathrm{T}}\right\| ds \\
&\leq \int_0^t \left\|\boldsymbol{\mathcal{F}}^{-1}\left(\boldsymbol{\ell}(s)\right) - \boldsymbol{\mathcal{F}}^{-1}\left(\boldsymbol{\ell}(0)\right)\right\| ds + T\left(\sigma \max_{n=1,\dots,N_c}\{\mathrm{meas}(\Omega_n)\}\,\|\boldsymbol{c}\|_{\mathrm{L}^2(0,T)} + \|\boldsymbol{i}_0\|\right) \\
&\leq \int_0^t \frac{1}{C_{SM}}\|\boldsymbol{\ell}(s) - \boldsymbol{\ell}(0)\|\,ds + T\left(\sigma \max_{n=1,\dots,N_c}\{\mathrm{meas}(\Omega_n)\}\,\|\boldsymbol{c}\|_{\mathrm{L}^2(0,T)} + \|\boldsymbol{i}_0\|\right).
\end{aligned}$$

Then, taking Gronwall's inequality into account (see, for instance, [13], Lemma 1.4.1),

$$\|\boldsymbol{\ell}(t) - \boldsymbol{\ell}(0)\| \leq T\left(\sigma \max_{n=1,\dots,N_c}\{\mathrm{meas}(\Omega_n)\}\,\|\boldsymbol{c}\|_{\mathrm{L}^2(0,T)} + \|\boldsymbol{i}_0\|\right)e^{T/C_{SM}}.$$

Now, since $\boldsymbol{i}(t) = \boldsymbol{\mathcal{F}}^{-1}\left(\boldsymbol{\ell}(t)\right)$,

$$\begin{aligned}
\|\boldsymbol{i}(t)\| - \|\boldsymbol{i}_0\| \leq \|\boldsymbol{i}(t) - \boldsymbol{i}_0\| = \left\|\boldsymbol{\mathcal{F}}^{-1}\left(\boldsymbol{\ell}(t)\right) - \boldsymbol{\mathcal{F}}^{-1}\left(\boldsymbol{\ell}(0)\right)\right\| &\leq \frac{1}{C_{SM}}\|\boldsymbol{\ell}(t) - \boldsymbol{\ell}(0)\| \\
&\leq \frac{T}{C_{SM}}\left(\sigma \max_{n=1,\dots,N_c}\{\mathrm{meas}(\Omega_n)\}\,\|\boldsymbol{c}\|_{\mathrm{L}^2(0,T)} + \|\boldsymbol{i}_0\|\right)e^{T/C_{SM}}.
\end{aligned}$$

□

3. Discretization of the ODE Operator

In this section we introduce an operator $\boldsymbol{\mathcal{F}}_h$ that will be constructed as an approximation of the ODE operator $\boldsymbol{\mathcal{F}}$. This new operator $\boldsymbol{\mathcal{F}}_h$ will be used in the next sections to introduce a numerical scheme.

To this end, in the sequel we will assume that Ω along with the connected components of its subdomains $\mathcal{U} \in \mathfrak{U}$ are Lipschitz polygons (we recall that $\mathfrak{U} = \{\Omega_0, \Omega_{\mathrm{c}}, \Omega_{\mathrm{pm}}, \Omega_{\mathrm{nl}}\}$). Moreover, we consider regular triangular meshes $\mathcal{T}_h$ of Ω such that each element $T \in \mathcal{T}_h$ is contained in the closure of one of its subdomains (h stands, as usual, for the corresponding mesh-size). Therefore, $\mathcal{T}_h(\mathcal{U}) := \left\{T \in \mathcal{T}_h : \ T \subset \overline{\mathcal{U}}\right\}$ are meshes of $\overline{\mathcal{U}}$, for any $\mathcal{U} \in \mathfrak{U}$.

Moreover, let $\mathcal{L}_h(\Omega)$ be the space of standard piecewise linear finite elements on $\mathcal{T}_h$:

$$\mathcal{L}_h(\Omega) := \left\{\psi_h \in \mathrm{H}^1(\Omega) \, : \, \psi_h|_T \in \mathbb{P}_1(T) \quad \forall T \in \mathcal{T}_h\right\},$$

and $\mathcal{L}_h^0(\Omega)$ the subespace

$$\mathcal{L}_h^0(\Omega) := \{\psi_h \in \mathcal{L}_h(\Omega) \, : \, \psi_h|_{\partial\Omega} = 0\}\,.$$

Let us define the nonlinear operator $\boldsymbol{\mathcal{F}}_h : \mathbb{R}^{N_c} \longrightarrow \mathbb{R}^{N_c}$ given by

$$\boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}\right) := \left(\int_{\Omega_1} \sigma A_h, \ldots, \int_{\Omega_{N_c}} \sigma A_h\right)^{\mathrm{T}} \in \mathbb{R}^{N_c},$$

with A_h being the solution of the discrete nonlinear magnetostatics problem:

Problem 13. *Given $\boldsymbol{i} \in \mathbb{R}^{N_c}$ and $\boldsymbol{B}^r \in \mathrm{L}^2\left(\Omega_{\mathrm{pm}}\right)^3$, find $A_h \in \mathcal{L}_h^0(\Omega)$ such that*

$$\int_\Omega \nu(\mathbf{x}; |\mathbf{grad}\, A_h|)\, \mathbf{grad}\, A_h \cdot \mathbf{grad}\, W_h = \sum_{n=1}^{N_c} \int_{\Omega_n} \frac{i_n}{\mathrm{meas}(\Omega_n)} W_h + \int_{\Omega_{\mathrm{pm}}} \nu_{\mathrm{pm}} \left(\boldsymbol{B}^r\right)^\perp \cdot \mathbf{grad}\, W_h,$$

for every $W_h \in \mathcal{L}_h^0(\Omega)$.

Remark 14. *Since $\mathcal{L}_h^0(\Omega) \subset \mathrm{H}_0^1(\Omega)$ for every $h > 0$, Problem 13 has a unique solution and then operator $\boldsymbol{\mathcal{F}}_h$ is well-defined in $\mathbb{R}^{N_c}$.*

Theorem 15. *Operator $\boldsymbol{\mathcal{F}}_h$ is strongly monotone and Lipschitz continuous globally in $\mathbb{R}^{N_c}$ and uniformly for $h > 0$. Then, $\boldsymbol{\mathcal{F}}_h$ is invertible and its inverse $\boldsymbol{\mathcal{F}}_h^{-1}$ is Lipschitz continuous globally in $\mathbb{R}^{N_c}$ for every $h > 0$, with Lipschitz constant independent of h.*

Proof. Let $\boldsymbol{i}^1, \boldsymbol{i}^2 \in \mathbb{R}^{N_c}$ be given and $A_h^1, A_h^2 \in \mathcal{L}_h^0(\Omega)$ be the associated solutions to Problem 13, respectively. Then, $\mathcal{F}_{h,n}\left(\boldsymbol{i}^j\right) = \int_{\Omega_n} \sigma A_h^j$ for $j = 1, 2$ and $n = 1, \ldots, N_c$. In order to prove the desired properties of $\boldsymbol{\mathcal{F}}_h$, the same steps as in Theorem 8 can be followed, replacing fields $A_1, A_2 \in \mathrm{H}_0^1(\Omega)$ with $A_h^1, A_h^2 \in \mathcal{L}_h^0(\Omega)$. Moreover, it can be shown that we can take $W_h \in \mathcal{L}_h^0(\Omega)$ such that

$$\int_{\Omega_n} \sigma W_h = I_n^1 - I_n^2,\, n = 1, \ldots, N_c, \qquad \text{and} \qquad ||W_h||_{\mathrm{H}^1(\Omega)} \leq \ C\left\|\boldsymbol{i}^1 - \boldsymbol{i}^2\right\|_*,$$

with $C > 0$ independent of h. For this purpose, we can use the well-posed weak problem:

$$\begin{cases} \text{Given } g_n \in \mathrm{H}^{1/2}(\partial\Omega_n),\ n = 1, \ldots, N_c, \text{ find } \widetilde{W}_h \in \mathcal{L}_h(\widetilde{\Omega}) \text{ such that } \widetilde{W}_h|_{\partial\Omega_n} = g_n,\ n = 1, \ldots, N_c, \\ \widetilde{W}_h|_{\partial\Omega} = 0 \text{ and} \\ \qquad \displaystyle\int_{\widetilde{\Omega}} \mathbf{grad}\, \widetilde{W}_h \cdot \mathbf{grad}\, V_h = 0 \\ \text{for every } V_h \in \mathcal{L}_h^0(\widetilde{\Omega}). \end{cases}$$

instead of the continuous one and define

$$W_h := \begin{cases} \widetilde{W}_h & \text{in } \widetilde{\Omega}, \\ C_n & \text{in } \Omega_n, n = 1, \dots, N_c. \end{cases}$$

By applying Zarantonello's theorem cited above, we deduce from last theorem that $\boldsymbol{\mathcal{F}}_h$ is invertible and that its inverse $\boldsymbol{\mathcal{F}}_h^{-1}$ is Lipschitz continuous with Lipschitz constant independent of $h > 0$. □

Then, we can define the semidiscrete versions of Problems 6 and 10 in the following way:

Problem 16. *Given* $\boldsymbol{c}(t) \in \mathcal{C}([0,T])^{N_c}$ *and* $\boldsymbol{i}_0 \in \mathbb{R}^{N_c}$*, find* $\boldsymbol{i}_h(t) \in \mathcal{C}^{0,1}([0,T])^{N_c}$ *such that* $\boldsymbol{i}_h(0) = \boldsymbol{i}_0$ *and*

$$\frac{d}{dt}\boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_h(t)) + \boldsymbol{i}_h(t) = -\left(c_1(t)\sigma \operatorname{meas}(\Omega_1), \dots, c_{N_c}(t)\sigma \operatorname{meas}(\Omega_{N_c})\right)^{\mathrm{T}} \quad \text{in } (0,T].$$

Problem 17. *Given* $\boldsymbol{c}(t) \in \mathcal{C}([0,T])^{N_c}$ *and* $\boldsymbol{i}_0 \in \mathbb{R}^{N_c}$*, find* $\boldsymbol{i}_h(t) \in \mathcal{C}^{0,1}([0,T])^{N_c}$ *such that* $\boldsymbol{i}_h(0) = \boldsymbol{i}_0$ *and*

$$\frac{d}{dt}\boldsymbol{\ell}_h(t) + \boldsymbol{\mathcal{F}}_h^{-1}(\boldsymbol{\ell}_h(t)) = -\left(c_1(t)\sigma \operatorname{meas}(\Omega_1), \dots, c_{N_c}(t)\sigma \operatorname{meas}(\Omega_{N_c})\right)^{\mathrm{T}} \quad \text{in } (0,T],$$

with $\boldsymbol{\ell}_h(t) = \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_h(t))$ *for every* $t \in [0,T]$.

Theorem 18. *Let* $A(t) \in \mathrm{H}_0^1(\Omega)$ *and* $A_h(t) \in \mathcal{L}_h^0(\Omega)$ *be the solutions to Problems 4 and 13, respectively, with data* $\boldsymbol{i}(t)$*. If* $\boldsymbol{i}$ *and* $\boldsymbol{i}_h$ *are the solutions to Problems 6 and 16, then,*

$$\|\boldsymbol{i} - \boldsymbol{i}_h\|_{\mathrm{L}^2(0,T)^{N_c}} \le C\left(||A - A_h||_{\mathrm{L}^2\left(0,T;\,\mathrm{L}^2\left(\cup_{n=1}^{N_c}\Omega_n\right)\right)} + T||A(0) - A_h(0)||_{\mathrm{L}^2\left(\cup_{n=1}^{N_c}\Omega_n\right)}\right). \tag{3.1}$$

Proof. Subtracting Problems 6 and 16 we obtain

$$\begin{cases} \dfrac{d}{dt}\left(\boldsymbol{\mathcal{F}}(\boldsymbol{i}) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_h)\right) + (\boldsymbol{i}(t) - \boldsymbol{i}_h(t)) = \boldsymbol{0}, \\ (\boldsymbol{i}(0) - \boldsymbol{i}_h(0)) = \boldsymbol{0}. \end{cases}$$

Now, integrating in time in $(0,t)$ for $t \in (0,T]$, and multiplying by $(\boldsymbol{i}(t) - \boldsymbol{i}_h(t))$, we deduce

$$\langle \boldsymbol{\mathcal{F}}(\boldsymbol{i}(t)) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_h(t)), \boldsymbol{i}(t) - \boldsymbol{i}_h(t)\rangle + \left\langle \int_0^t (\boldsymbol{i}(s) - \boldsymbol{i}_h(s))\, ds, \boldsymbol{i}(t) - \boldsymbol{i}_h(t)\right\rangle = \langle \boldsymbol{\mathcal{F}}(\boldsymbol{i}_0) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_0), \boldsymbol{i}(t) - \boldsymbol{i}_h(t)\rangle, \tag{3.2}$$

for every $t \in [0,T]$. We notice that the second term in the left-hand side of eq. (3.2) satisfies

$$\left\langle \int_0^t (\boldsymbol{i}(s) - \boldsymbol{i}_h(s))\, ds, \boldsymbol{i}(t) - \boldsymbol{i}_h(t)\right\rangle = \frac{1}{2}\frac{d}{dt}\left\| \int_0^t (\boldsymbol{i}(s) - \boldsymbol{i}_h(s))\, ds\right\|^2.$$

Hence, if we add and subtract the term $\boldsymbol{\mathcal{F}}_h(\boldsymbol{i}(t))$ in the first term of the left-hand side of eq. (3.2), we get

$$\frac{1}{2}\frac{d}{dt}\left\| \int_0^t (\boldsymbol{i}(s) - \boldsymbol{i}_h(s))\, ds\right\|^2 + \langle \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}(t)) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_h(t)), \boldsymbol{i}(t) - \boldsymbol{i}_h(t)\rangle = -\langle \boldsymbol{\mathcal{F}}(\boldsymbol{i}(t)) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}(t)), \boldsymbol{i}(t) - \boldsymbol{i}_h(t)\rangle + \langle \boldsymbol{\mathcal{F}}(\boldsymbol{i}_0) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_0), \boldsymbol{i}(t) - \boldsymbol{i}_h(t)\rangle, \tag{3.3}$$

for every $t \in [0,T]$. Integrating eq. (3.3) in $[0,T]$,

$$\frac{1}{2}\left\|\int_0^T (\boldsymbol{i}(t)-\boldsymbol{i}_h(t))\,dt\right\|^2 + \int_0^T \langle \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}(t)) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_h(t)), \boldsymbol{i}(t)-\boldsymbol{i}_h(t)\rangle\,dt$$
$$= -\int_0^T \langle \boldsymbol{\mathcal{F}}(\boldsymbol{i}(t)) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}(t)), \boldsymbol{i}(t)-\boldsymbol{i}_h(t)\rangle\,dt + \int_0^T \langle \boldsymbol{\mathcal{F}}(\boldsymbol{i}_0) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_0), \boldsymbol{i}(t)-\boldsymbol{i}_h(t)\rangle\,dt. \quad (3.4)$$

Moreover, since $\boldsymbol{\mathcal{F}}_h$ is a strongly monotone operator globally in $\mathbb{R}^{N_c}$ (and uniformly in $[0,T]$),

$$\|\boldsymbol{i}-\boldsymbol{i}_h\|^2_{\mathrm{L}^2(0,T)^{N_c}} \leq C\int_0^T \langle \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}(t)) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_h(t)), \boldsymbol{i}(t)-\boldsymbol{i}_h(t)\rangle\,dt.$$

Thus, from eq. (3.4) we get

$$\|\boldsymbol{i}-\boldsymbol{i}_h\|^2_{\mathrm{L}^2(0,T)^{N_c}} \leq C\|\boldsymbol{i}-\boldsymbol{i}_h\|_{\mathrm{L}^2(0,T)^{N_c}}\left(\|\boldsymbol{\mathcal{F}}(\boldsymbol{i}) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i})\|_{\mathrm{L}^2(0,T)} + T\|\boldsymbol{\mathcal{F}}(\boldsymbol{i}_0) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_0)\|\right). \quad (3.5)$$

Then, taking into account the definitions of $\boldsymbol{\mathcal{F}}$ and $\boldsymbol{\mathcal{F}}_h$, we have

$$\|\boldsymbol{\mathcal{F}}(\boldsymbol{i}(t)) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}(t))\| \leq C\|A(t)-A_h(t)\|_{\mathrm{L}^2\left(\cup_{n=1}^{N_c}\Omega_n\right)}, \quad (3.6)$$

with $C>0$ independent of $h>0$, and finally

$$\|\boldsymbol{i}-\boldsymbol{i}_h\|_{\mathrm{L}^2(0,T)^{N_c}} \leq C\left(\|A-A_h\|_{\mathrm{L}^2\left(0,T;\,\mathrm{L}^2\left(\cup_{n=1}^{N_c}\Omega_n\right)\right)} + T\|A(0)-A_h(0)\|_{\mathrm{L}^2\left(\cup_{n=1}^{N_c}\Omega_n\right)}\right).$$

□

Remark 19. *We notice that the error estimate obtained in Theorem 18 means that the convergence order of the solution to Problem 16 to the one of Problem 6 in* $\mathrm{L}^2(0,T)^{N_c}$ *is going to be determined by the spatial error made when approximating Problem 4 by Problem 13 in the* $\mathrm{L}^2\left(\cup_{n=1}^{N_c}\Omega_n\right)$*-norm. In Section 5, we will see that the numerical results seem to suggest that the optimal convergence order is* $O(h^2)$*. However, to the authors' knowledge, this can only be theoretically obtained under quite strong regularity assumptions (see [1, 18]). Therefore, we have chosen to work with a more reasonable hypothesis, leading to a suboptimal error estimate for* $\|\boldsymbol{i}-\boldsymbol{i}_h\|_{\mathrm{L}^2(0,T)^{N_c}}$*. In particular, we will bound the expressions in eq. (3.6) with the* $\mathrm{H}^1\left(\cup_{n=1}^{N_c}\Omega_n\right)$*-norm of the difference between the continuous and the discrete magnetic vector potentials.*

Consequently, we are going to give a sufficient condition that will allow us to express the error estimate in terms of the problem data.

Assumption 1. *Let* $A(t) \in \mathrm{H}^1_0(\Omega)$ *and* $A_h(t) \in \mathcal{L}^0_h(\Omega)$ *be the solutions to Problems 4 and 13, respectively, with data* $\boldsymbol{i}(t)$*. Let us assume there exists* $\varepsilon \in (0,1]$ *such that* $A(t)|_{\mathcal{U}} \in \mathrm{H}^{1+\varepsilon}(\mathcal{U})$ *and*

$$||A(t)||_{\mathrm{H}^{1+\varepsilon}(\mathcal{U})} \leq C\left(\|\boldsymbol{i}(t)\| + ||\boldsymbol{B}^r||_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3}\right)$$

for every $\mathcal{U} \in \mathfrak{U}$*, where* $C>0$ *depends only on* Ω*.*

Corollary 20. *Under Assumption 1, if* $\boldsymbol{i}$ *and* $\boldsymbol{i}_h$ *are the solutions to Problems 6 and 16, then,*

$$\|\boldsymbol{i}-\boldsymbol{i}_h\|_{\mathrm{L}^2(0,T)^{N_c}} \leq Ch^{\varepsilon}\left(||\boldsymbol{B}^r||_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3} + \|\boldsymbol{i}_0\| + \|\boldsymbol{c}\|_{\mathrm{L}^2(0,T)^{N_c}}\right), \quad (3.7)$$

with $C>0$ *independent of* h*.*

Proof. Notice that, under Assumption 1, following [20] we have

$$||A(t) - A_h(t)||_{\mathrm{H}^1(\Omega)} \leq Ch^{\varepsilon} \sum_{\mathcal{U}\in\mathfrak{U}} ||A(t)||_{\mathrm{H}^{1+\varepsilon}(\mathcal{U})},$$

with $A(t) \in \mathrm{H}_0^1(\Omega)$ and $A_h(t) \in \mathcal{L}_h^0(\Omega)$ being the solutions to Problems 4 and 13 with data $\boldsymbol{i}(t)$, respectively, and C a constant independent of h.
Therefore, taking into account Theorem 12, we have the following approximation result:

$$\begin{aligned} ||A(t) - A_h(t)||_{\mathrm{L}^2\left(\cup_{n=1}^{N_c}\Omega_n\right)} &\leq C||A(t) - A_h(t)||_{\mathrm{H}^1\left(\cup_{n=1}^{N_c}\Omega_n\right)} \\ &\leq Ch^{\varepsilon}\left(\|\boldsymbol{i}(t)\| + ||\boldsymbol{B}^r||_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3}\right) \leq Ch^{\varepsilon}\left(\|\boldsymbol{i}_0\| + ||\boldsymbol{B}^r||_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3} + \|\boldsymbol{c}\|_{\mathrm{L}^2(0,T)^{N_c}}\right), \end{aligned}$$

with $C > 0$ independent of $h > 0$. Thus, using eq. (3.1), we conclude

$$\|\boldsymbol{i} - \boldsymbol{i}_h\|_{\mathrm{L}^2(0,T)^{N_c}} \leq Ch^{\varepsilon}\left(||\boldsymbol{B}^r||_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3} + \|\boldsymbol{i}_0\| + \|\boldsymbol{c}\|_{\mathrm{L}^2(0,T)^{N_c}}\right).$$

□

Remark 21. *For the sake of simplicity, in order to avoid dealing with variational crimes, we have assumed the subdomains to be Lipschitz polygons. However, the theoretical results from [20] that we employ to prove the error estimate hold under more general assumptions (in particular, in the case of fig. 1).*

4. Numerical Analysis of a Fully Discrete Problem

In this section we propose a numerical scheme to approximate the solution to Problem 6. Let us consider a uniform partition $\{t_m := m\Delta t, m = 0, \dots, M\}$ of $[0, T]$ with step size $\Delta t := \frac{T}{M}$. Then, the fully-discrete version of Problem 6 reads as follows:

Problem 22. *Given $\boldsymbol{c}(t) \in \mathcal{C}([0,T])^{N_c}$ and $\boldsymbol{i}_0 \in \mathbb{R}^{N_c}$, find $\boldsymbol{i}_h^m \in \mathbb{R}^{N_c}$, $m = 0, \dots, M$, such that $\boldsymbol{i}_h^0 = \boldsymbol{i}_0$ and*

$$\boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}_h^m\right) + \Delta t\, \boldsymbol{i}_h^m = \boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}_h^{m-1}\right) - \Delta t\left(c_1(t_m)\sigma\operatorname{meas}(\Omega_1), \dots, c_{N_c}(t_m)\sigma\operatorname{meas}(\Omega_{N_c})\right)^{\mathrm{T}}, m = 1, \dots, M, \quad (4.1)$$

being $\boldsymbol{\mathcal{F}}_h$ the nonlinear operator defined in Section 3.

4.1. Well-Posedness of the Fully Discrete Problem

In order to prove the following theorem we will make use again of Zarantonello's theorem.

Theorem 23. *Problem 22 has a unique solution.*

Proof. Let us define the nonlinear operators $\boldsymbol{\mathcal{G}}_{h,\Delta t} : \mathbb{R}^{N_c} \longrightarrow \mathbb{R}^{N_c}$,

$$\boldsymbol{\mathcal{G}}_{h,\Delta t}\left(\boldsymbol{k}\right) := \boldsymbol{\mathcal{F}}_h\left(\boldsymbol{k}\right) + \Delta t\, \boldsymbol{k}, \quad h, \Delta t > 0.$$

Since $\boldsymbol{\mathcal{F}}_h$ is strongly monotone and Lipschitz continuous globally in $\mathbb{R}^{N_c}$ with constants C_{SM} and C_L, respectively, we deduce that $\boldsymbol{\mathcal{G}}_{h,\Delta t}$ are strongly monotone and Lipschitz continuous globally in $\mathbb{R}^{N_c}$ with constants $(C_{SM} + \Delta t)$ and $(C_L + \Delta t)$, respectively. Therefore, proceeding by induction over m, and using again the theorem by Zarantonello cited in Section 2, we conclude the proof. □

4.2. Error Estimate

For any function $\boldsymbol{f} \in \mathcal{C}([0,T])^{N_c}$, let us define $\boldsymbol{f}_{\Delta t}$ the piecewise constant approximation of $\boldsymbol{f}$

$$\boldsymbol{f}_{\Delta t}(t) := \begin{cases} \boldsymbol{f}(0) & \text{for } t = 0, \\ \boldsymbol{f}(t_m) & \text{for } t \in (t_{m-1}, t_m],\ m = 1, \dots, M. \end{cases}$$

Remark 24. *If $\boldsymbol{f} \in \mathcal{C}^{0,1}([0,T])$ with Lipschitz constant $L_{\boldsymbol{f}}$, then*

$$\|\boldsymbol{f} - \boldsymbol{f}_{\Delta t}\|^2_{\mathrm{L}^2(0,T)^{N_c}} = \sum_{m=1}^{M} \int_{t_{m-1}}^{t_m} \|\boldsymbol{f}(t) - \boldsymbol{f}(t_m)\|^2 \, dt \le \frac{L_{\boldsymbol{f}}^2 T}{3} \Delta t^2.$$

We have the following result:

Theorem 25. *Under Assumption 1, if $\boldsymbol{c}(t) \in \mathcal{C}^{0,1}([0,T])^{N_c}$ then the solutions to Problems 6 and 22, $\boldsymbol{i}(t)$ and $\{\boldsymbol{i}_h^m\}_{m=0}^M$, respectively, satisfy*

$$\left(\sum_{m=1}^{M} \Delta t \, \|\boldsymbol{i}(t_m) - \boldsymbol{i}_h^m\|^2 \right)^{1/2} \le C \left(\Delta t \, (L_{\boldsymbol{i}} + L_{\boldsymbol{c}}) + h^{\varepsilon} \left(\|\boldsymbol{i}_0\|^2 + ||\boldsymbol{B}^r||^2_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3} + \|\boldsymbol{c}\|^2_{\mathrm{L}^2(0,T)^{N_c}} \right) \right), \tag{4.2}$$

with $L_{\boldsymbol{i}}$ and $L_{\boldsymbol{c}}$ the Lipschitz constants of mappings $\boldsymbol{i}$ and $\boldsymbol{c}$, respectively, and $C > 0$ is independent of Δt.

Proof. Let us denote $\boldsymbol{g} : [0,T] \longrightarrow \mathbb{R}^{N_c}$, the mapping

$$\boldsymbol{g}(t) := -\left(c_1(t)\sigma \,\mathrm{meas}(\Omega_1), \dots, c_{N_c}(t)\sigma \,\mathrm{meas}(\Omega_{N_c})\right)^{\mathrm{T}}.$$

Firstly, integrating the equation appearing in Problem 6 between 0 and t_k, summing up equations eq. (4.1) for $m = 1, \dots, k$, and subtracting them we obtain

$$\boldsymbol{\mathcal{F}}(\boldsymbol{i}(t_k)) - \boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}_h^k\right) - \Delta t \sum_{m=1}^{k} \boldsymbol{i}_h^m = \boldsymbol{\mathcal{F}}(\boldsymbol{i}_0) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_0) - \int_0^{t_k} \boldsymbol{i}(t)\, dt + \int_0^{t_k} \boldsymbol{g}(t) - \boldsymbol{g}_{\Delta t}(t)\, dt.$$

Adding and subtracting the term $\int_0^{t_k} \boldsymbol{i}_{\Delta t}(t)\, dt$ in the above expression, multiplying it by $\left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k\right)$ and taking the strong monotonicity of $\boldsymbol{\mathcal{F}}_h$ into account, we have

$$\begin{aligned} C_{SM} \left\|\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k\right\|^2 &+ \Delta t \left\langle \sum_{m=1}^{k} \boldsymbol{i}(t_m) - \boldsymbol{i}_h^m, \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\rangle \\ &\le -\left\langle \boldsymbol{\mathcal{F}}(\boldsymbol{i}(t_k)) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}(t_k)), \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\rangle + \left\langle \boldsymbol{\mathcal{F}}(\boldsymbol{i}_0) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_0), \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\rangle \\ &\quad + \left\langle \int_0^{t_k} \boldsymbol{i}_{\Delta t}(t) - \boldsymbol{i}(t)\, dt, \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\rangle + \left\langle \int_0^{t_k} \boldsymbol{g}(t) - \boldsymbol{g}_{\Delta t}(t)\, dt, \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\rangle. \end{aligned}$$

Multiplying the above expression by Δt and summing up for $k=1,\ldots,\ell$, we get

$$C_{SM}\sum_{k=1}^{\ell}\Delta t\left\|\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\|^2+(\Delta t)^2\sum_{k=1}^{\ell}\left\langle\sum_{m=1}^{k}\left(\boldsymbol{i}(t_m)-\boldsymbol{i}_h^m\right),\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\rangle$$
$$\leq-\Delta t\sum_{k=1}^{\ell}\left\langle\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}(t_k)\right)-\boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}(t_k)\right),\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\rangle+\Delta t\sum_{k=1}^{\ell}\left\langle\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}_0\right)-\boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}_0\right),\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\rangle$$
$$+\Delta t\sum_{k=1}^{\ell}\left\langle\int_0^{t_k}\boldsymbol{i}_{\Delta t}(t)-\boldsymbol{i}(t)\,dt,\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\rangle+\Delta t\sum_{k=1}^{\ell}\left\langle\int_0^{t_k}\boldsymbol{g}(t)-\boldsymbol{g}_{\Delta t}(t)\,dt,\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\rangle \quad (4.3)$$

for every $\ell=1,\ldots,M$. Now, we are going to discuss every term in eq. (4.3) separately. Firstly, concerning the second term on the left-hand side, taking into account that $2\langle p,p-q\rangle=||p||^2+||p-q||^2-||q||^2$ and writing

$$\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k=\sum_{m=1}^{k}\left(\boldsymbol{i}(t_m)-\boldsymbol{i}_h^m\right)-\sum_{m=1}^{k-1}\left(\boldsymbol{i}(t_m)-\boldsymbol{i}_h^m\right), \quad (4.4)$$

we get

$$(\Delta t)^2\sum_{k=1}^{\ell}\left\langle\sum_{m=1}^{k}\left(\boldsymbol{i}(t_m)-\boldsymbol{i}_h^m\right),\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\rangle$$
$$=\frac{1}{2}\sum_{k=1}^{\ell}\left\{\left\|\Delta t\sum_{m=1}^{k}\left(\boldsymbol{i}(t_m)-\boldsymbol{i}_h^m\right)\right\|^2+\left\|\Delta t\left(\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right)\right\|^2-\left\|\Delta t\sum_{m=1}^{k-1}\left(\boldsymbol{i}(t_m)-\boldsymbol{i}_h^m\right)\right\|^2\right\}$$
$$=\frac{1}{2}\left\{\sum_{k=1}^{\ell}\left\|\Delta t\left(\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right)\right\|^2+\left\|\Delta t\sum_{k=1}^{\ell}\left(\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right)\right\|^2\right\}$$
$$\geq\frac{1}{2}\left\|\Delta t\sum_{k=1}^{\ell}\left(\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right)\right\|^2. \quad (4.5)$$

On the other hand, concerning the first term on the right-hand side of eq. (4.3) and using Young's inequality for each $k=1,\ldots,\ell$, we obtain

$$\Delta t\sum_{k=1}^{\ell}\left\langle\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}(t_k)\right)-\boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}(t_k)\right),\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\rangle$$
$$\leq\sum_{k=1}^{\ell}\Delta t\,\frac{\varepsilon_1}{2}\left\|\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}(t_k)\right)-\boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}(t_k)\right)\right\|^2+\sum_{k=1}^{\ell}\Delta t\,\frac{1}{2\varepsilon_1}\left\|\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\|^2,$$

for every $\varepsilon_1>0$. Following the same argument as in Section 3, from Assumption 1 we conclude that

$$\Delta t\sum_{k=1}^{\ell}\left\langle\boldsymbol{\mathcal{F}}\left(\boldsymbol{i}(t_k)\right)-\boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}(t_k)\right),\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\rangle$$
$$\leq\frac{1}{2\varepsilon_1}\sum_{k=1}^{\ell}\Delta t\left\|\boldsymbol{i}(t_k)-\boldsymbol{i}_h^k\right\|^2+Ch^{2\varepsilon}T\frac{\varepsilon_1}{2}\left(\|\boldsymbol{i}_0\|^2+||\boldsymbol{B}^r||^2_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3}+\|\boldsymbol{g}\|^2_{\mathrm{L}^2(0,T)^{N_c}}\right). \quad (4.6)$$

Similarly, for the second term on the right-hand side of eq. (4.3) we have

$$\begin{aligned}\Delta t \sum_{k=1}^{\ell} \left\langle \boldsymbol{\mathcal{F}}(\boldsymbol{i}_0) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_0), \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\rangle \\ &\le \frac{\varepsilon_2}{2} \|\boldsymbol{\mathcal{F}}(\boldsymbol{i}_0) - \boldsymbol{\mathcal{F}}_h(\boldsymbol{i}_0)\|^2 + \frac{1}{2\varepsilon_2} \left\| \Delta t \sum_{k=1}^{\ell} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\|^2 \\ &\le C h^{2\varepsilon} \frac{\varepsilon_2}{2} \left(\|\boldsymbol{i}_0\|^2 + \|\boldsymbol{B}^r\|^2_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3} \right) + \frac{1}{2\varepsilon_2} \left\| \Delta t \sum_{k=1}^{\ell} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\|^2 . \end{aligned} \quad (4.7)$$

Moreover, regarding the third term on the right-hand side of eq. (4.3), writing again eq. (4.4) and using summation by parts, we obtain

$$\begin{aligned}&\Delta t \sum_{k=1}^{\ell} \left\langle \int_0^{t_k} \boldsymbol{i}_{\Delta t}(t) - \boldsymbol{i}(t)\, dt, \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\rangle \\ &= \left\langle \int_0^{t_\ell} \boldsymbol{i}_{\Delta t}(t) - \boldsymbol{i}(t)\, dt, \Delta t \sum_{k=1}^{\ell} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\rangle - \sum_{k=1}^{\ell-1} \left\langle \int_{t_k}^{t_{k+1}} \boldsymbol{i}_{\Delta t}(t) - \boldsymbol{i}(t)\, dt, \Delta t \sum_{m=1}^{k} \left(\boldsymbol{i}(t_m) - \boldsymbol{i}_h^m \right) \right\rangle .\end{aligned} \quad (4.8)$$

Using Young's inequality in the first term of eq. (4.8), we deduce

$$\begin{aligned}\left\langle \int_0^{t_\ell} \boldsymbol{i}_{\Delta t}(t) - \boldsymbol{i}(t)\, dt, \Delta t \sum_{k=1}^{\ell} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\rangle \\ &\le \frac{\varepsilon_3}{2} \left\| \int_0^{t_\ell} \boldsymbol{i}_{\Delta t}(t) - \boldsymbol{i}(t)\, dt \right\|^2 + \frac{1}{2\varepsilon_3} \left\| \Delta t \sum_{k=1}^{\ell} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\|^2 \\ &\le \frac{\varepsilon_3 T}{2} \|\boldsymbol{i}_{\Delta t} - \boldsymbol{i}\|^2_{\mathrm{L}^2(0,T)^{N_c}} + \frac{1}{2\varepsilon_3} \left\| \Delta t \sum_{k=1}^{\ell} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\|^2 ,\end{aligned}$$

for every $\varepsilon_3 > 0$. Concerning the second term in eq. (4.8), we have

$$\begin{aligned}\sum_{k=1}^{\ell-1} \left\langle \int_{t_k}^{t_{k+1}} \boldsymbol{i}_{\Delta t}(t) - \boldsymbol{i}(t)\, dt, \Delta t \sum_{m=1}^{k} \left(\boldsymbol{i}(t_m) - \boldsymbol{i}_h^m \right) \right\rangle \\ &\le \sum_{k=1}^{\ell-1} \|\boldsymbol{i}_{\Delta t} - \boldsymbol{i}\|_{\mathrm{L}^2(t_k,t_{k+1})^{N_c}} \sqrt{\Delta t} \left\| \Delta t \sum_{m=1}^{k} \left(\boldsymbol{i}(t_m) - \boldsymbol{i}_h^m \right) \right\| \\ &\le \sum_{k=1}^{\ell-1} \frac{\beta_1}{2} \|\boldsymbol{i}_{\Delta t} - \boldsymbol{i}\|^2_{\mathrm{L}^2(t_k,t_{k+1})^{N_c}} + \Delta t \sum_{k=1}^{\ell-1} \frac{1}{2\beta_1} \left\| \Delta t \sum_{m=1}^{k} \left(\boldsymbol{i}(t_m) - \boldsymbol{i}_h^m \right) \right\|^2 \\ &\le \frac{\beta_1}{2} \|\boldsymbol{i}_{\Delta t} - \boldsymbol{i}\|^2_{\mathrm{L}^2(0,T)^{N_c}} + \Delta t \sum_{k=1}^{\ell-1} \frac{1}{2\beta_1} \left\| \Delta t \sum_{m=1}^{k} \left(\boldsymbol{i}(t_m) - \boldsymbol{i}_h^m \right) \right\|^2 ,\end{aligned} \quad (4.9)$$

for every $\beta_1 > 0$. Finally, we can bound analogously the fourth term on the right-hand side of eq. (4.3), obtaining

$$\Delta t \sum_{k=1}^{\ell} \left\langle \int_0^{t_k} \boldsymbol{g}(t) - \boldsymbol{g}_{\Delta t}(t)\, dt, \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\rangle \leq \left(\frac{\varepsilon_4 T + \beta_2}{2} \right) \|\boldsymbol{g} - \boldsymbol{g}_{\Delta t}\|^2_{\mathrm{L}^2(0,T)^{N_c}} + \frac{1}{2\varepsilon_4} \left\| \Delta t \sum_{k=1}^{\ell} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\|^2 + \Delta t \sum_{k=1}^{\ell-1} \frac{1}{2\beta_2} \left\| \Delta t \sum_{m=1}^{k} \left(\boldsymbol{i}(t_m) - \boldsymbol{i}_h^m \right) \right\|^2, \quad (4.10)$$

for every $\varepsilon_4, \beta_2 > 0$. Using eqs. (4.5) to (4.10) in eq. (4.3) and rearranging the terms appropriately, we conclude

$$\begin{aligned} &\left(C_{SM} - \frac{1}{2\varepsilon_1} \right) \sum_{k=1}^{\ell} \Delta t \left\| \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\|^2 + \left(\frac{1}{2} - \frac{1}{2\varepsilon_2} - \frac{1}{2\varepsilon_3} - \frac{1}{2\varepsilon_4} \right) \left\| \Delta t \sum_{k=1}^{\ell} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\|^2 \\ &\leq \frac{\varepsilon_3 T + \beta_1}{2} \|\boldsymbol{i}_{\Delta t} - \boldsymbol{i}\|^2_{\mathrm{L}^2(0,T)^{N_c}} + \frac{\varepsilon_4 T + \beta_2}{2} \|\boldsymbol{g} - \boldsymbol{g}_{\Delta t}\|^2_{\mathrm{L}^2(0,T)^{N_c}} \\ &\quad + h^{2\varepsilon} \left(\frac{C_1 \varepsilon_2 + C_2 T \varepsilon_1}{2} \right) \left(\|\boldsymbol{i}_0\|^2 + ||\boldsymbol{B}^r||^2_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3} \right) \\ &\quad + h^{2\varepsilon} \frac{C_2 \varepsilon_1 T}{2} \|\boldsymbol{g}\|^2_{\mathrm{L}^2(0,T)^{N_c}} + \Delta t \left(\frac{1}{2\beta_1} + \frac{1}{2\beta_2} \right) \sum_{k=1}^{\ell-1} \left\| \Delta t \sum_{m=1}^{k} \left(\boldsymbol{i}(t_m) - \boldsymbol{i}_h^m \right) \right\|^2, \end{aligned}$$

for every $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \beta_1, \beta_2 > 0$ and $\ell = 1, \ldots, M$. Thus, taking $\varepsilon_1 > \frac{1}{2C_{SM}} > 0$, $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \frac{3\varepsilon_1}{(1-C_{SM})\varepsilon_1 + 1} > 0$ and $\beta_1 = \beta_2 = \frac{2\varepsilon_1}{2C_{SM}\varepsilon_1 - 1} > 0$,

$$\begin{aligned} &\sum_{k=1}^{\ell} \Delta t \left\| \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\|^2 + \left\| \Delta t \sum_{k=1}^{\ell} \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\|^2 \\ &\leq C \left(\|\boldsymbol{i}_{\Delta t} - \boldsymbol{i}\|^2_{\mathrm{L}^2(0,T)^{N_c}} + \|\boldsymbol{g} - \boldsymbol{g}_{\Delta t}\|^2_{\mathrm{L}^2(0,T)^{N_c}} \right) + C h^{2\varepsilon} \left(\|\boldsymbol{i}_0\|^2 + ||\boldsymbol{B}^r||^2_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3} + \|\boldsymbol{g}\|^2_{\mathrm{L}^2(0,T)^{N_c}} \right) \\ &\quad + \Delta t \sum_{k=1}^{\ell-1} \left\| \Delta t \sum_{m=1}^{k} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\|^2. \end{aligned}$$

Now, using the discrete Gronwall inequality (see Lemma 1.4.2 in [13]), we conclude

$$\begin{aligned} \sum_{k=1}^{M} \Delta t \left\| \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\|^2 &\leq \sum_{k=1}^{M} \Delta t \left\| \boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right\|^2 + \left\| \Delta t \sum_{k=1}^{M} \left(\boldsymbol{i}(t_k) - \boldsymbol{i}_h^k \right) \right\|^2 \\ &\leq C \left(\|\boldsymbol{i}_{\Delta t} - \boldsymbol{i}\|^2_{\mathrm{L}^2(0,T)^{N_c}} + \|\boldsymbol{g} - \boldsymbol{g}_{\Delta t}\|^2_{\mathrm{L}^2(0,T)^{N_c}} + h^{2\varepsilon} \left(\|\boldsymbol{i}_0\|^2 + ||\boldsymbol{B}^r||^2_{\mathrm{L}^2(\Omega_{\mathrm{pm}})^3} + \|\boldsymbol{g}\|^2_{\mathrm{L}^2(0,T)^{N_c}} \right) \right). \end{aligned}$$

□

5. Numerical Results

In this section we report some numerical results obtained from a Fortran code that solves a problem equivalent to Problem 22, allowing us to confirm the convergence result stated in Theorem 25. This equivalence is proved in appendix Appendix A. At each time step, the nonlinearity is solved by means of the fixed-point algorithm proposed in [5]. To this end, we have solved an academic problem built from the analytical test presented in [6] for a linear case. In our setting, we consider sources given in terms of time-

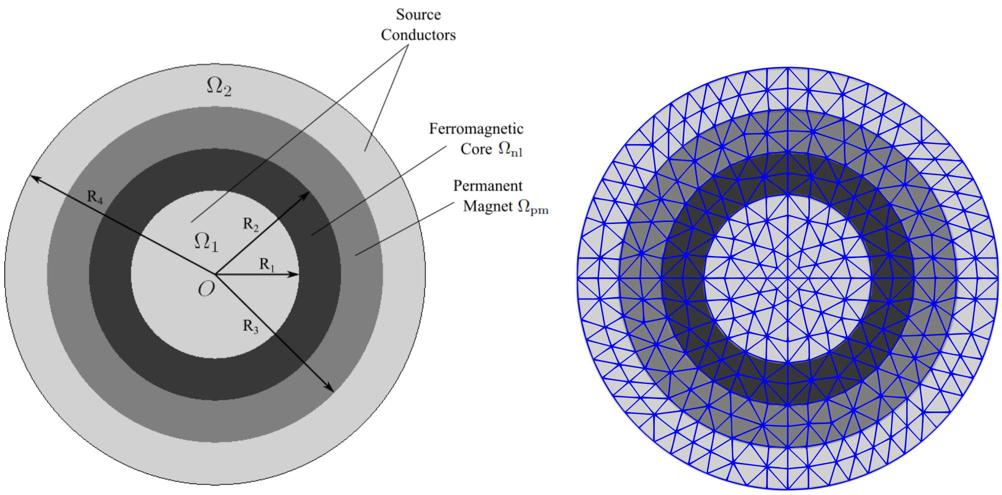

Figure 2: Sketch of the domain Ω (left). Coarsest mesh (right).

dependent voltage drops per unit length and replace the linear material with a permanent magnet and a nonlinear core.

In fig. 2-left we show the problem domain Ω that includes the cross sections of two coaxial copper wires, Ω_1 and Ω_2, separated by a permanent magnet, Ω_{pm}, and a ferromagnetic core, Ω_{nl}. We assume that the core and the magnet are non-conducting, and that the copper domains carry a uniformly distributed current density, i.e., they are stranded conductors.

Let us consider a cylindrical coordinate system (ρ, θ, z), with $\mathbf{e}_\rho$, $\mathbf{e}_\theta$ and $\mathbf{e}_z$ the corresponding local orthonormal basis. We assume that the z axis is orthogonal to the domain at point O. To apply the 2D transient magnetic model analysed in this paper, we suppose that the current density of the sources is uniform in each of them and orthogonal to the computational domain. More precisely,

$$\boldsymbol{J} = J_z(\rho, t)\mathbf{e}_z = \begin{cases} \dfrac{i(t)}{\pi R_1^2}\mathbf{e}_z & \text{in } (0, R_1) \times [0, T], \\ \mathbf{0} & \text{in } (R_1, R_3) \times [0, T], \\ -\dfrac{i(t)}{\pi(R_4^2 - R_3^2)}\mathbf{e}_z & \text{in } (R_3, R_4) \times [0, T]. \end{cases}$$

In this case, if the magnetic constitutive law in the permanent magnet is of the form $\boldsymbol{H} = \nu_{\text{pm}}\boldsymbol{B} - \nu_{\text{pm}}\boldsymbol{B}^r$ with a remanent flux $\boldsymbol{B}^r = B^r\mathbf{e}_\theta$, $B^r \in \mathbb{R}$, then all fields are independent of the azimuthal variable and the solution to the magnetostatics problem eqs. (2.1) to (2.3) is

$$\boldsymbol{H} = H_\theta(\rho, t)\mathbf{e}_\theta = \begin{cases} \dfrac{\rho\, i(t)}{2\pi R_1^2}\mathbf{e}_\theta & \text{in } (0, R_1) \times [0, T], \\ \dfrac{i(t)}{2\pi\rho}\mathbf{e}_\theta & \text{in } (R_1, R_3) \times [0, T], \\ i(t)\left(\dfrac{1}{2\pi\rho} + \dfrac{(R_3^2 - \rho^2)}{2\pi(R_4^2 - R_3^2)\rho}\right)\mathbf{e}_\theta & \text{in } (R_3, R_4) \times [0, T]. \end{cases}$$

In the ferromagnetic core, we will consider the nonlinear constitutive magnetic law given by

$$B_\theta = \mu_0 H_\theta + \frac{2J_s}{\pi}\mathrm{atan}\left(\frac{\pi(\mu_r-1)\mu_0 H_\theta}{2J_s}\right). \tag{5.1}$$

We notice that, from Corollary 2.2 in [12], it can be seen that the corresponding nonlinear reluctivity function satisfies eqs. (2.4) to (2.6).

Following the same arguments as in [5], and using the notation $\gamma := \frac{(\mu_r-1)\mu_0}{4J_s}$, the expression of the solution to the magnetostatics problem (2.8)–(2.13) can be obtained by integrating B_θ in space (see appendix Appendix B). In particular, it can be seen that the magnetic vector potential vanishes at R_4 for every $t \in [0,T]$. This property allows us to have a conductor, Ω_2, that touches the boundary of the whole domain. Indeed, if we had considered a domain Ω_0 representing the air surrounding the device, the solution A would be identically zero there.

Moreover, the expression of the potential drops per unit length in Ω_1 and Ω_2 can also be analytically computed using eq. (2.7), obtaining

$$c_1(t) = \frac{i'(t)}{8\pi\nu_0} - \frac{i'(t)}{2\pi\nu_0}\left\{\ln\left(\frac{R_2}{R_1}\right) + \frac{\nu_0}{\nu_{\mathrm{pm}}}\ln\left(\frac{R_3}{R_2}\right) + \frac{R_4^2}{R_4^2-R_3^2}\ln\left(\frac{R_4}{R_3}\right)\right\}$$
$$- \frac{J_s\gamma i'(t)}{\pi}\ln\left(\frac{\gamma^2 i(t)^2 + R_2^2}{\gamma^2 i(t)^2 + R_1^2}\right) - \frac{i(t)}{\sigma\pi R_1^2},$$

$$c_2(t) = \frac{i'(t)}{\pi(R_4^2-R_3^2)^2\nu_0}\left\{\frac{R_3^2R_4^2}{2}\ln\left(\frac{R_4}{R_3}\right) - \frac{R_4^4-R_3^4}{8}\right\} + \frac{i(t)}{\sigma\pi(R_4^2-R_3^2)}.$$

For the numerical computations, we have used the geometrical data $R_1 = 0.5$ m, $R_2 = 0.75$ m, $R_3 = 1$ m and $R_4 = 1.25$ m. Moreover, the copper coils electrical conductivity σ is equal to 5.7×10^7 (Ohm m)$^{-1}$ and the magnetic reluctivity of the vacuum $\nu_0 = \frac{1}{4\pi}\times10^7$ H^{-1} m; the material of the permanent magnet is characterised by $\nu_{\mathrm{pm}} = 0.95\nu_0$ and the remanent flux density $\boldsymbol{B}^r = B^r\mathbf{e}_\theta$ by $B^r = 1.3$ T. Moreover, we have considered $\mu_0 = \frac{1}{\nu_0}$, $\mu_r = 5000$ and $J_s = 1.75$ T in the nonlinear material law of the ferromagnetic core. The considered source in the coils is the potential drop per unit length obtained for a current $i(t) = 3000\cos(2\pi f t)$ A with a frequency $f = 50$ Hz. Finally, the initial currents in the conductors are $i_{0,1} = 3000$ A and $i_{0,2} = -3000$ A, respectively. These initial conditions allow us to obtain the steady state current $\boldsymbol{i}(t)$ from the beginning of the simulation.

We solve the problem in a source cycle (that is, in the time interval $[0,T] = [0, 0.02]$ seconds) with several successively refined meshes and time steps, starting from the mesh shown in fig. 2-right and a step size $\Delta t = \frac{T}{40}$. We have computed the errors by comparing the numerical solutions with the analytical one given by $\boldsymbol{i}(t) = \big(i(t), -i(t)\big)^{\mathrm{T}}$. Specifically, we have computed the relative error for currents $\{\boldsymbol{i}_h^m\}_{m=1}^M$ in the $\mathrm{L}^2(0,T)$-norm, that is,

$$\mathcal{E}_h^{\Delta t} := \frac{\left(\sum_{m=1}^M \Delta t\,|\boldsymbol{i}(t_m) - \boldsymbol{i}_h^m|^2\right)^{1/2}}{\left(\sum_{m=1}^M \Delta t\,|\boldsymbol{i}(t_m)|^2\right)^{1/2}}.$$

Table 1 shows these relative errors at different levels of discretization. We notice that, when we take a time step small enough, an $O(h^2)$ error decay can be observed (see last row in table 1). On the other hand, considering a mesh size small enough allows us to show the expected convergence order in time $O(\Delta t)$ (see single-framed values in the last column in table 1). We notice that the continuous solution is such that $A(t)|_{\Omega_n} \in \mathrm{H}^2(\Omega_n)$, $n = 1,2$, for every $t \in [0,T]$. Thus, the corresponding part of the convergence order proved in Theorem 25 is $O(h)$, which is less than the one numerically obtained. The improvement of this

	h	$\frac{h}{2}$	$\frac{h}{4}$	$\frac{h}{8}$	$\frac{h}{16}$
Δt	0.1138	0.0806	0.0774	0.0772	0.0771
$\frac{\Delta t}{2}$	0.0895	0.0438	0.0390	0.0388	0.0388
$\frac{\Delta t}{4}$	0.0831	0.0279	0.0199	0.0195	0.0195
$\frac{\Delta t}{8}$	0.0815	0.0222	0.0105	0.0098	0.0098
$\frac{\Delta t}{16}$	0.0812	0.0206	0.0062	0.0050	0.0049
$\frac{\Delta t}{32}$	0.0811	0.0201	0.0045	0.0027	0.0025
$\frac{\Delta t}{64}$	0.0811	0.0200	0.0040	0.0016	0.0012
$\frac{\Delta t}{128}$	0.0811	0.0200	0.0039	0.0012	0.0006
$\frac{\Delta t}{256}$	0.0811	0.0200	0.0038	0.0011	0.0004
$\frac{\Delta t}{512}$	0.0811	0.0200	0.0038	0.0011	0.0003

Table 1: Relative errors $\mathcal{E}_h^{\Delta t}$.

order is due to the fact that the norm used in eq. (3.6) is $||\cdot||_{\mathrm{H}^1(\cup_{n=1}^{N_c}\Omega_n)}$, while the $\mathrm{L}^2\left(\cup_{n=1}^{N_c}\Omega_n\right)$-norm could have been used. In the $\mathrm{L}^2\left(\cup_{n=1}^{N_c}\Omega_n\right)$-norm, the magnetostatics problem converges with order $O(h^2)$ for this particular example.

Once the convergence order is checked, we illustrate in one single figure the simultaneous dependence on h and Δt of the error for current $\boldsymbol{i}$ in the $\mathrm{L}^2(0,T)$-norm by choosing initial coarse values for both discretization step-sizes and, for each successively refined mesh, we take the value of Δt proportional to h^2 (see the double-framed values in table 1). fig. 3 shows a log-log plot of the corresponding relative errors $\mathcal{E}_h^{\Delta t}$ versus the number of degrees of freedom (d.o.f.). The slope of the curve shows again the convergence order $O(h^2+\Delta t)$.

Appendix A. An Equivalence Result between Two Fully Discrete Schemes

In Section 4, we analysed the numerical convergence for Problem 22. However, following [5], we have implemented a different numerical scheme for the same problem. In this section, we will prove the equivalence between the two discretizations.

Let us consider the following discrete problem, which is the one used for the implementation:

Problem 26. *Given* $\boldsymbol{c}(t)\in\mathcal{C}([0,T])^{N_c}$, $\boldsymbol{i}^0\in\mathbb{R}^{N_c}$ *and* $\boldsymbol{B}^r\in\mathrm{L}^2\left(\Omega_{\mathrm{pm}}\right)^3$, *find* $A_h^m\in\mathcal{L}_h^0(\Omega)$, $m=1,\ldots,M$,

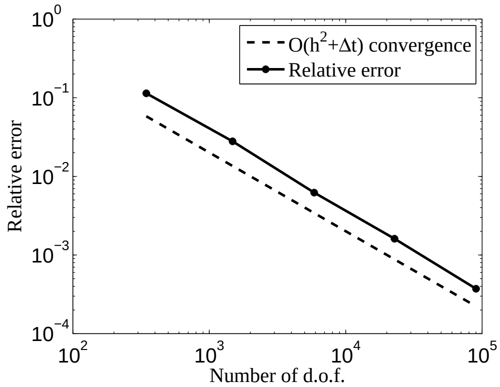

Figure 3: $\mathcal{E}_h^{\Delta t}$ versus d.o.f. (log-log scale); $\Delta t = Ch^2$.

such that

$$\int_\Omega \nu(\cdot;|\mathbf{grad}\, A_h^m|)\,\mathbf{grad}\, A_h^m \cdot \mathbf{grad}\, W_h + \frac{1}{\Delta t}\sum_{n=1}^{N_c}\int_{\Omega_n}\left(\int_{\Omega_n}\sigma A_h^m\right)\frac{1}{\mathrm{meas}(\Omega_n)}W_h$$
$$= \frac{1}{\Delta t}\sum_{n=1}^{N_c}\int_{\Omega_n}\left(\int_{\Omega_n}\sigma A_h^{m-1}\right)\frac{1}{\mathrm{meas}(\Omega_n)}W_h - \sum_{n=1}^{N_c}\int_{\Omega_n}\sigma c_n(t_m)W_h + \int_{\Omega_{\mathrm{pm}}}\nu_{\mathrm{pm}}\,(\boldsymbol{B}^r)^\perp\cdot\mathbf{grad}\, W_h,$$

for every $W_h \in \mathcal{L}_h^0(\Omega)$, with $A_h^0 \in \mathcal{L}_h^0(\Omega)$ the solution to the weak formulation

$$\int_\Omega \nu(\cdot;|\mathbf{grad}\, A_h^0|)\,\mathbf{grad}\, A_h^0\cdot\mathbf{grad}\, W_h = \sum_{n=1}^{N_c}\int_{\Omega_n}\frac{i_{0,n}}{\mathrm{meas}(\Omega_n)}W_h + \int_{\Omega_{\mathrm{pm}}}\nu_{\mathrm{pm}}\,(\boldsymbol{B}^r)^\perp\cdot\mathbf{grad}\, W_h,$$

for every $W_h \in \mathcal{L}_h^0(\Omega)$.

Theorem 27. *Let $\boldsymbol{i}_h^m \in \mathbb{R}^{N_c}$, $m = 0,\dots,M$, be the solution to Problem 22 and $A_h^m \in \mathcal{L}_h^0(\Omega)$, $m = 1,\dots,M$, defined as the solution to Problem 13. Then, $A_h^m \in \mathcal{L}_h^0(\Omega)$, $m = 1,\dots,M$, are a solution to Problem 26.*

Proof. Since Problem 13 has a unique solution, we deduce that

$$\mathcal{F}_{h,n}\left(\boldsymbol{i}_h^m\right) = \int_{\Omega_n}\sigma A_h^m, \quad n = 1,\dots,N_c,\ m = 1,\dots,M.$$

Furthermore, since $\boldsymbol{i}_h^m$, $m = 0,\dots,M$, is the solution to Problem 22,

$$i_{h,n}^m = -\frac{1}{\Delta t}\mathcal{F}_{h,n}\left(\boldsymbol{i}_h^m\right) + \frac{1}{\Delta t}\mathcal{F}_{h,n}\left(\boldsymbol{i}_h^{m-1}\right) - c_n(t_m)\sigma\,\mathrm{meas}(\Omega_n)$$
$$= -\frac{1}{\Delta t}\int_{\Omega_n}\sigma A_h^m + \frac{1}{\Delta t}\int_{\Omega_n}\sigma A_h^{m-1} - c_n(t_m)\sigma\,\mathrm{meas}(\Omega_n)$$

for $n = 1,\dots,N_c$, $m = 1,\dots,M$. Replacing these expressions in Problem 13 we conclude that $A_h^m \in \mathcal{L}_h^0(\Omega)$, $m = 1,\dots,M$, are a solution to Problem 26. □

Theorem 28. *Let $A_h^m \in \mathcal{L}_h^0(\Omega)$, $m = 1,\dots,M$, be a solution to Problem 26. Let us define $\boldsymbol{i}_h^0 := \boldsymbol{i}_0$ and*

$\boldsymbol{i}_h^m \in \mathbb{R}^{N_c}$, $m = 1, \ldots, M$, *such that*

$$i_{h,n}^m := -\frac{1}{\Delta t}\int_{\Omega_n} \sigma A_h^m + \frac{1}{\Delta t}\int_{\Omega_n} \sigma A_h^{m-1} - c_n(t_m)\sigma\,\mathrm{meas}(\Omega_n), \tag{A.1}$$

$n = 1, \ldots, N_c$. *Then,* $\boldsymbol{i}_h^m$, $m = 0, \ldots, M$, *are the solution to Problem 22.*

Proof. Let $\widetilde{A}_h^m \in \mathcal{L}_h^0(\Omega)$, $m = 0, \ldots, M$, be the solution to Problem 13 with the currents defined in eq. (A.1). In particular, $\widetilde{A}_h^0 = A_h^0$. Furthermore, taking the definitions of $\boldsymbol{i}_h^m$, $m = 0, \ldots, M$, into account, fields $\widetilde{A}_h^m \in \mathcal{L}_h^0(\Omega)$, $m = 1, \ldots, M$, are also the solutions to the following problems:

$$\int_\Omega \nu(\cdot; |\mathbf{grad}\,\widetilde{A}_h^m|)\,\mathbf{grad}\,\widetilde{A}_h^m \cdot \mathbf{grad}\,W_h + \frac{1}{\Delta t}\sum_{n=1}^{N_c}\int_{\Omega_n}\left(\int_{\Omega_n}\sigma A_h^m\right)\frac{1}{\mathrm{meas}(\Omega_n)}W_h$$
$$= \frac{1}{\Delta t}\sum_{n=1}^{N_c}\int_{\Omega_n}\left(\int_{\Omega_n}\sigma A_h^{m-1}\right)\frac{1}{\mathrm{meas}(\Omega_n)}W_h - \sum_{n=1}^{N_c}\int_{\Omega_n}\sigma c_n(t_m)W_h + \int_{\Omega_{\mathrm{pm}}}\nu_{\mathrm{pm}}\left(\boldsymbol{B}^r\right)^\perp \cdot \mathbf{grad}\,W_h,$$

for every $W_h \in \mathcal{L}_h^0(\Omega)$. By subtracting to the above equalities those in Problem 26, we deduce that

$$\int_\Omega \left(\nu(\cdot; |\mathbf{grad}\,\widetilde{A}_h^m|)\,\mathbf{grad}\,\widetilde{A}_h^m - \nu(\cdot; |\mathbf{grad}\,A_h^m|)\,\mathbf{grad}\,A_h^m\right)\cdot \mathbf{grad}\,W_h = 0$$

for every $W_h \in \mathcal{L}_h^0(\Omega)$, $m = 1, \ldots, M$. Since $\mathcal{L}_h^0(\Omega) \subset \mathrm{H}_0^1(\Omega)$, we can rewrite the last equality in the following way:

$$\left\langle \mathcal{B}\left(\widetilde{A}_h^m\right) - \mathcal{B}(A_h^m), W_h\right\rangle = 0$$

for every $W_h \in \mathcal{L}_h^0(\Omega)$, $m = 1, \ldots, M$. In particular, taking $W_h = \widetilde{A}_h^m - A_h^m$, and since operator $\mathcal{B}$ is strongly monotone, we obtain

$$0 = \left\langle \mathcal{B}\left(\widetilde{A}_h^m\right) - \mathcal{B}(A_h^m), \widetilde{A}_h^m - A_h^m\right\rangle \geq M\left\|\widetilde{A}_h^m - A_h^m\right\|_{\mathrm{H}^1(\Omega)}^2,$$

and therefore $\widetilde{A}_h^m = A_h^m$ for $m = 1, \ldots, M$.
Finally, taking the definition of $\boldsymbol{\mathcal{F}}_h$ into account, we have,

$$\mathcal{F}_{h,n}\left(\boldsymbol{i}_h^m\right) = \int_{\Omega_n}\sigma\widetilde{A}_h^m = \int_{\Omega_n}\sigma A_h^m, \quad n = 1, \ldots, N_c,\ m = 1, \ldots, M,$$

and then

$$\boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}_h^m\right) + \Delta t\,\boldsymbol{i}_h^m = \boldsymbol{\mathcal{F}}_h\left(\boldsymbol{i}_h^{m-1}\right) - \left(c_1(t_m)\sigma\,\mathrm{meas}(\Omega_1), \ldots, c_{N_c}(t_m)\sigma\,\mathrm{meas}(\Omega_{N_c})\right)^{\mathrm{T}},$$

$m = 1, \ldots, M$. Hence $\boldsymbol{i}_h^m$ are the solution to Problem 22. □

Remark 29. *In Theorems 27 and 28, we have seen that given a solution to Problem 26 we can compute the corresponding solution to Problem 22, and vice versa. Moreover, from the proof of the last theorem it can be deduced that Problem 26 has a unique solution. Indeed, given two solutions* $A_h^{m,1}, A_h^{m,2} \in \mathcal{L}_h^0(\Omega)$, $m = 1, \ldots, M$, *to Problem 26, let* $\boldsymbol{i}_h^{m,1}, \boldsymbol{i}_h^{m,2} \in \mathbb{R}^{N_c}$, $m = 0, \ldots, M$, *be the corresponding solutions to Problem 22, built as indicated in Theorem 28. Moreover, let* $\widetilde{A}_h^{m,1}, \widetilde{A}_h^{m,2} \in \mathcal{L}_h^0(\Omega)$, $m = 1, \ldots, M$, *be the solutions to Problem 13 corresponding to these currents. In the above proof, we have seen that* $\widetilde{A}_h^{m,1} = A_h^{m,1}$ *and* $\widetilde{A}_h^{m,2} = A_h^{m,2}$. *Therefore, since Problem 22 is well-posed,* $\boldsymbol{i}_h^{m,1} = \boldsymbol{i}_h^{m,2}$, $m = 1, \ldots, M$. *Consequently,* $\widetilde{A}_h^{m,1} = \widetilde{A}_h^{m,2}$, *and thus* $A_h^{m,1} = A_h^{m,2}$, $m = 1, \ldots, M$.

Appendix B. Analytical Expression of the Solution to the Numerical Example

In this appendix we state the analytical expression of the solution to the magnetostatics problem appearing in Section 5, obtained from the magnetic flux density given in eq. (5.1):

In $(0, R_1) \times [0, T]$,

$$A(\rho, t) = \frac{I(t)}{2\pi\nu_0} \left\{ -\frac{\rho^2}{2R_1^2} + \ln\left(\frac{R_2}{R_1}\right) + \frac{\nu_0}{\nu_{\text{pm}}} \ln\left(\frac{R_3}{R_2}\right) + \frac{R_4^2}{R_4^2 - R_3^2} \ln\left(\frac{R_4}{R_3}\right) \right\}$$

$$+ \frac{2J_s}{\pi} \left\{ R_2 \text{atan}\left(\frac{\gamma I(t)}{R_2}\right) - R_1 \text{atan}\left(\frac{\gamma I(t)}{R_1}\right) + \frac{\gamma I(t)}{2} \ln\left(\frac{\gamma^2 I(t)^2 + R_2^2}{\gamma^2 I(t)^2 + R_1^2}\right) \right\} + B^r (R_3 - R_2).$$

In $(R_1, R_2) \times [0, T]$,

$$A(\rho, t) = \frac{I(t)}{2\pi\nu_0} \left\{ -\frac{1}{2} + \ln\left(\frac{R_2}{\rho}\right) + \frac{\nu_0}{\nu_{\text{pm}}} \ln\left(\frac{R_3}{R_2}\right) + \frac{R_4^2}{R_4^2 - R_3^2} \ln\left(\frac{R_4}{R_3}\right) \right\}$$

$$+ \frac{2J_s}{\pi} \left\{ R_2 \text{atan}\left(\frac{\gamma I(t)}{R_2}\right) - \rho \, \text{atan}\left(\frac{\gamma I(t)}{\rho}\right) + \frac{\gamma I(t)}{2} \ln\left(\frac{\gamma^2 I(t)^2 + R_2^2}{\gamma^2 I(t)^2 + \rho^2}\right) \right\} + B^r (R_3 - R_2).$$

In $(R_2, R_3) \times [0, T]$,

$$A(\rho, t) = \frac{I(t)}{2\pi\nu_0} \left\{ -\frac{1}{2} + \frac{\nu_0}{\nu_{\text{pm}}} \ln\left(\frac{R_3}{\rho}\right) + \frac{R_4^2}{R_4^2 - R_3^2} \ln\left(\frac{R_4}{R_3}\right) \right\} + B^r (R_3 - \rho).$$

In $(R_3, R_4) \times [0, T]$,

$$A(\rho, t) = \frac{I(t)}{2\pi\nu_0} \left\{ \frac{\rho^2 - R_4^2}{2(R_4^2 - R_3^2)} + \frac{R_4^2}{R_4^2 - R_3^2} \ln\left(\frac{R_4}{\rho}\right) \right\}.$$

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References

References

[1] Abdulle, A., Huber, M. E., 2016. Error estimates for finite element approximations of nonlinear monotone elliptic problems with application to numerical homogenization. Numer. Methods Partial Differential Eq. 32 (3), 955–969.

[2] Bagagiolo, F., 2016. Ordinary differential equations. `www.science.unitn.it/~bagagiol/noteODE.pdf`.

[3] Baumanns, S., 2012. Coupled electromagnetic field/circuit simulation: Modeling and numerical analysis. Ph.D. thesis, Mathematisch-Naturwissenschaftlichen Fakultät.

[4] Benderskaya, G., 2007. Numerical methods for transient field-circuit coupled simulations based on the finite integration technique and a mixed circuit formulation. Ph.D. thesis, Fachbereich Elektrotechnik und Informationstechnik.

[5] Bermúdez, A., Óscar Domínguez, Gómez, D., Salgado, P., 2013. Finite element approximation of nonlinear transient magnetic problems involving periodic potential drop excitations. Comput. Math. Appl. 65, 1200–1219.

[6] Bermúdez, A., Rodríguez, R., Salgado, P., 2008. A finite element method for the magnetostatic problem in terms of scalar potentials. SIAM J Numer Anal 46 (3), 1338–1363.

[7] Bossavit, A., 1999. Eddy currents in dimension 2: voltage drops. In: Int. Symp. on Theoret. Electrical Engineering (Proc. ISTET'99, W. Mathis, T. Schindler, eds). University Otto-von-Guericke (Magdeburg, Germany), pp. 103–107.

[8] Buffa, A., Maday, Y., Rapetti, F., 2001. A sliding mesh-mortar method for a two dimensional eddy current model of electric engines. ESAIM: Math Model Num Anal 35 (2), 191–228.
[9] Heise, B., 1994. Analysis of a fully discrete finite element method for a nonlinear magnetic field problem. SIAM J Numer Anal 31 (3), 745–759.
[10] Matthes, M., 2012. Numerical analysis of nonlinear partial differential-algebraic equations: A coupled and an abstract systems approach. Ph.D. thesis, Mathematisch-Naturwissenschaftlichen Fakultät.
[11] Nicaise, S., Tröltzsch, F., 2013. A coupled maxwell integrodifferential model for magnetization processes. Math Nachr 287 (4), 432–452.
[12] Pechstein, C., 2004. Multigrid-newton-methods for nonlinear magnetostatic problems. Ph.D. thesis, Johannes Kepler University Linz, Austria.
[13] Quarteroni, A., Valli, A., 1994. Numerical Approximation of Partial Differential Equations. Springer Verlag.
[14] Roubícek, T., 2013. Nonlinear Partial Differential Equations with Applications. Springer Basel.
[15] Slodivka, M., Vrábel', V., 2017. Existence and uniqueness of a solution for a field/circuit coupled problem. ESAIM: Mathematical Modelling and Numerical Analysis 51 (3), 1045–1061.
[16] Touzani, R., Rappaz, J., 2013. Mathematical Models for Eddy Currents and Magnetostatics with Selected Applications. Springer.
[17] Ugalde, G., Almandoz, G., Poza, J., González, A., 2009. Computation of iron losses in permanent magnet machines by multi-domain simulations. In: 2009 13th European Conference on Power Electronics and Applications. IEEE, pp. 1–10.
[18] Xu, J., 1996. Two-grid discretization techniques for linear and nonlinear pdes. SIAM J Numer Anal 33 (5), 1759–1777.
[19] Zeidler, E., 1990. Nonlinear Functional Analysis and its Applications II/B. Nonlinear Monotone Operators. Springer-Verlag, New York.
[20] Zenísek, A., 1990. The finite element method for nonlinear elliptic equations with discontinuous coefficients. Numerische Mathematik 58 (1), 51–77.
[21] Zhu, Z., Ng, K., Schofield, N., Howe, D., 2004. Improved analytical modelling of rotor eddy current loss in brushless machines equipped with surface-mounted permanent magnets. IEE Proceedings - Electric Power Applications 151 (6), 641–650.