# A UNIVERSAL RIEMANNIAN FOLIATED SPACE

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Abstract. It is proved that the isometry classes of pointed connected complete Riemannian $n$-manifolds form a Polish space, $\mathcal{M}_*^\infty(n)$, with the topology described by the $C^\infty$ convergence of manifolds. This space has a canonical partition into sets defined by varying the distinguished point into each manifold. The locally non-periodic manifolds define an open dense subspace $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n) \subset \mathcal{M}_*^\infty(n)$, which becomes a $C^\infty$ foliated space with the restriction of the canonical partition. Its leaves without holonomy form the subspace $\mathcal{M}_{*,\mathrm{np}}^\infty(n) \subset \mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ defined by the non-periodic manifolds. Moreover the leaves have a natural Riemannian structure so that $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ becomes a Riemannian foliated space, which is universal among all sequential Riemannian foliated spaces satisfying certain property called covering-determination. $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ is used to characterize the realization of complete connected Riemannian manifolds as dense leaves of covering-determined compact sequential Riemannian foliated spaces.

Contents

1. Introduction 1
2. Preliminaries 3
2.1. Foliated spaces 3
2.2. Riemannian geometry 5
3. Quasi-isometries 6
4. Partial quasi-isometries 11
5. The $C^\infty$ topology on $\mathcal{M}_*(n)$ 12
6. Convergence in the $C^\infty$ topology 13
7. $\mathcal{M}_*^\infty(n)$ is Polish 15
8. Some basic properties of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ 17
9. Canonical bundles over $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ 20
10. Center of mass 24
11. Foliated structure of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ 25
12. Saturated subspaces of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ 29
Acknowledgements 30
References 30

## 1. Introduction

For any $n \in \mathbb{N}$ (we adopt the convention that $0 \in \mathbb{N}$), let $\mathcal{M}_*(n)$ denote the set of isometry classes, $[M, x]$, of pointed complete connected Riemannian $n$-manifolds, $(M, x)$. The cardinality of each complete connected Riemannian $n$-manifold is less than or equal to the cardinality of the continuum, and therefore it may be assumed that its underlying set is contained in $\mathbb{R}$. With this assumption, $\mathcal{M}_*(n)$ is a well defined set. This set is only interesting for $n \geq 2$ because $\mathcal{M}_*(0) = \{[\{0\}, 0]\}$ and $\mathcal{M}_*(1) = \{[\mathbb{R}, 0], [\mathbb{S}^1, 1]\}$. The set $\mathcal{M}_*(n)$ can be considered as a subset of the Gromov space $\mathcal{M}_*$ of isometry classes of pointed proper metric spaces [14], [15, Chapter 3]. However it is interesting to consider a finer topology on $\mathcal{M}_*(n)$, taking the differentiable structure into account. For that purpose, the following notion of $C^\infty$ convergence was defined on $\mathcal{M}_*(n)$.

*Key words and phrases.* $C^\infty$ convergence of Riemannian manifolds; locally non-periodic Riemannian manifolds; Riemannian foliated space.

**Definition 1.1** (See e.g. [33, Chapter 10, Section 3.2])**.** For each $m \in \mathbb{N}$, a sequence $[M_i, x_i] \in \mathcal{M}_*(n)$ is said to be $C^m$ *convergent* to $[M, x] \in \mathcal{M}_*(n)$ if, for each compact domain $\Omega \subset M$ containing $x$, there are pointed $C^{m+1}$ embeddings $\phi_i : (\Omega, x) \to (M_i, x_i)$ for large enough $i$ such that $\phi_i^* g_i \to g|_\Omega$ as $i \to \infty$ with respect to the $C^m$ topology [22, Chapter 2]. If $[M_i, x_i]$ is $C^m$ convergent to $[M, x]$ for all $m$, then it is said that $[M_i, x_i]$ is $C^\infty$ *convergent* to $[M, x]$.

Here, a *domain* in $M$ is a connected $C^\infty$ submanifold, possibly with boundary, of the same dimension as $M$.

It is admitted that $C^\infty$ convergence defines a topology on $\mathcal{M}_*(n)$ [32]. However we are not aware of any proof in the literature showing that it satisfies the conditions to describe a topology [28], [17] (see also [26] and [27] if $C^\infty$ convergence were defined with nets or filters). This is only proved on subspaces defined by manifolds of equi-bounded geometry, where the $C^\infty$ convergence coincides with convergence in $\mathcal{M}_*$ [29] (see also [33, Chapter 10]). The first main theorem of the paper is the following.

**Theorem 1.2.** *The $C^\infty$ convergence in $\mathcal{M}_*(n)$ describes a Polish topology.*

Recall that a space is called *Polish* if it is separable and completely metrizable.

The topology given by Theorem 1.2 will be called the $C^\infty$ *topology* on $\mathcal{M}_*(n)$, and the corresponding space is denoted by $\mathcal{M}_*^\infty(n)$.

For each complete connected Riemannian $n$-manifold $M$, there is a canonical continuous map $\iota : M \to \mathcal{M}_*^\infty(n)$ given by $\iota(x) = [M, x]$, which induces a continuous injective map $\bar\iota : \mathrm{Iso}(M)\backslash M \to \mathcal{M}_*^\infty(n)$, where $\mathrm{Iso}(M)$ denotes the isometry group of $M$. The more explicit notation $\iota_M$ and $\bar\iota_M$ may be also used. The images of the maps $\iota_M$ form a natural partition of $\mathcal{M}_*^\infty(n)$, denoted by $\mathcal{F}_*(n)$.

A Riemannian manifold, $M$, is said to be *non-periodic* if $\mathrm{Iso}(M) = \{\mathrm{id}_M\}$, and is said to be *locally non-periodic* if each point $x \in M$ has a neighborhood $U_x$ such that

$$\{\, h \in \mathrm{Iso}(M) \mid h(x) \in U_x \,\} = \{\mathrm{id}_M\}\ .$$

Let $\mathcal{M}_{*,\mathrm{np}}(n)$ and $\mathcal{M}_{*,\mathrm{lnp}}(n)$ be the $\mathcal{F}_*(n)$-saturated subsets of $\mathcal{M}_*(n)$ defined by non-periodic and locally non-periodic manifolds, respectively. The notation $\mathcal{M}_{*,\mathrm{np}}^\infty(n)$ and $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ is used when these sets are equipped with the restriction of the $C^\infty$ topology. The restrictions of $\mathcal{F}_*(n)$ to $\mathcal{M}_{*,\mathrm{np}}(n)$ and $\mathcal{M}_{*,\mathrm{lnp}}(n)$ are respectively denoted by $\mathcal{F}_{*,\mathrm{np}}(n)$ and $\mathcal{F}_{*,\mathrm{lnp}}(n)$. Note that $\mathcal{M}_{*,\mathrm{np}}(0) = \{[\{0\}, 0]\}$ and $\mathcal{M}_{*,\mathrm{lnp}}(1) = \emptyset$.

On the other hand, let $\mathcal{M}_{*,\mathrm{c}}^\infty(n)$ (respectively, $\widehat{\mathcal{M}}_{*,\mathrm{o}}^\infty(n)$) be the $\mathcal{F}_*(n)$-saturated subspace of $\widehat{\mathcal{M}}_*(n)$ consisting of classes $[M, x]$ such that $M$ is compact (respectively, open). Observe that, if $[N, y]$ is close enough to any $[M, x] \in \mathcal{M}_{*,\mathrm{c}}^\infty(n)$, then $N$ is diffeomorphic to $M$. Thus $\mathcal{M}_{*,\mathrm{c}}^\infty(n)$ is open in $\mathcal{M}_*(n)$, and therefore $\mathcal{M}_{*,\mathrm{o}}^\infty(n)$ is closed. Hence these are Polish subspaces of $\mathcal{M}_*(n)$, as well as their intersections with any Polish subspace. The intersection of $\mathcal{M}_{*,\mathrm{c/o}}^\infty(n)$ and $\mathcal{M}_{*,(\mathrm{l})\mathrm{np}}^\infty(n)$ is denoted by $\mathcal{M}_{*,(\mathrm{l})\mathrm{np},\mathrm{c/o}}^\infty(n)$. The restrictions of $\mathcal{F}_*(n)$ to $\mathcal{M}_{*,\mathrm{c/o}}(n)$ and $\mathcal{M}_{*,(\mathrm{l})\mathrm{np},\mathrm{c/o}}(n)$ are denoted by $\mathcal{F}_{*,\mathrm{c/o}}(n)$ and $\mathcal{F}_{*,(\mathrm{l})\mathrm{np},\mathrm{c/o}}(n)$, respectively. The second main theorem of the paper is the following.

**Theorem 1.3.** *The following properties hold for $n \ge 2$:*

- *(i)* $\mathcal{M}_{*,\mathrm{lnp}}(n)$ *is Polish and dense in* $\mathcal{M}_*^\infty(n)$.
- *(ii)* $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n) \equiv (\mathcal{M}_{*,\mathrm{lnp}}^\infty(n), \mathcal{F}_{*,\mathrm{lnp}}(n))$ *is a foliated space of dimension* $n$.
- *(iii)* $\mathcal{F}_{*,\mathrm{lnp},\mathrm{o}}(n)$ *is transitive.*
- *(iv) The foliated space* $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ *has canonical* $C^\infty$ *and Riemannian structures such that* $\bar\iota : \mathrm{Iso}(M)\backslash M \to \iota(M)$ *is an isometry for every locally non-periodic, complete, connected Riemannian manifold* $M$.
- *(v) For any locally non-periodic complete connected Riemannian manifold* $M$, *the quotient map* $M \to \mathrm{Iso}(M)\backslash M$ *corresponds to the holonomy covering of the leaf* $\iota(M)$ *by* $\bar\iota : \mathrm{Iso}(M)\backslash M \to \iota(M)$. *In particular, the set* $\mathcal{M}_{*,\mathrm{np}}(n)$ *is the union of leaves of* $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ *with trivial holonomy groups.*

The following result states a universal property of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$, which involves certain property called covering-determination (Definition 12.1).

**Theorem 1.4.** *Let $X$ be a sequential Riemannian foliated space of dimension $n \ge 2$ whose leaves are complete. Then $X$ is isometric to a saturated subspace of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ if and only if it is covering-determined.*

Recall that a space $X$ is called *sequential* if a subset $A \subset X$ is open whenever each convergent sequence $x_n \to x \in A$ in $X$ eventually belongs to $A$. For instance, first countable spaces are sequential. This condition could be removed by using convergence of nets or filters instead of sequences.

$\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ is used to prove the following result about realizations of Riemannian manifolds as leaves. It involves the obvious Riemannian versions of the conditions of being aperiodic or repetitive, which are standard for tilings or graphs (see e.g. [12, 16, 35]), and a weak version of aperiodicity (Definitions 12.4 and 12.6).

**Theorem 1.5.** *The following properties hold for a complete connected Riemannian manifold $M$ of bounded geometry and dimension $n \geq 2$:*

(*i*) *$M$ is non-periodic and has a* (*repetitive*) *weakly aperiodic connected covering if and only if it is isometric to a dense leaf of a* (*minimal*) *covering-determined compact sequential Riemannian foliated space.*

(*ii*) *If $M$ is aperiodic* (*and repetitive*), *then it is isometric to a dense leaf of a* (*minimal*) *covering-determined compact sequential Riemannian foliated space whose leaves have trivial holonomy groups.*

## 2. Preliminaries

2.1. **Foliated spaces.** Standard references for foliated spaces are [30], [5, Chapter 11], [6, Part 1] and [13].

Let $Z$ be a space and let $U$ be an open set in $\mathbb{R}^n \times Z$ ($n \in \mathbb{N}$), with coordinates $(x, z)$. For $m \in \mathbb{N}$, a map $f : U \to \mathbb{R}^p$ ($p \in \mathbb{N}$) is of *class $C^m$* if its partial derivatives up to order $m$ with respect to $x$ exist and are continuous on $U$. If $f$ is of class $C^m$ for all $m$, then it is called of *class $C^\infty$*. Let $Z'$ be another space, and let $h : U \to \mathbb{R}^p \times Z'$ ($p \in \mathbb{N}$) be a map of the form $h(x, z) = (h_1(x, z), h_2(z))$, for maps $h_1 : U \to \mathbb{R}^p$ and $h_2 : \mathrm{pr}_2(U) \to Z'$, where $\mathrm{pr}_2 : \mathbb{R}^n \times Z \to Z$ is the second factor projection. It will be said that $h$ is of *class $C^m$* if $h_1$ is of class $C^m$ and $h_2$ is continuous.

For $m \in \mathbb{N} \cup \{\infty\}$ and $n \in \mathbb{N}$, a *foliated structure* $\mathcal{F}$ of *class $C^m$* and *dimension* $\dim \mathcal{F} = n$ on a space $X$ is defined by a collection $\mathcal{U} = \{(U_i, \phi_i)\}$, where $\{U_i\}$ is an open covering of $X$, and each $\phi_i$ is a homeomorphism $U_i \to B_i \times Z_i$, for a locally compact Polish space $Z_i$ and an open ball $B_i$ in $\mathbb{R}^n$, such that the coordinate changes $\phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ are locally $C^m$ maps of the form

$$\phi_j \phi_i^{-1}(x, z) = (g_{ij}(x, z), h_{ij}(z)) .$$

These maps $h_{ij}$ will be called the *local transverse components* of the changes of coordinates. Each $(U_i, \phi_i)$ is called a *foliated chart*, the sets $\phi_i^{-1}(B_i \times \{z\})$ ($z \in Z_i$) are called *plaques*, and the collection $\mathcal{U}$ is called a *foliated atlas* of *class $C^m$*. Two $C^m$ foliated atlases on $X$ define the same $C^m$ foliated structure if their union is a $C^m$ foliated atlas. If we consider foliated atlases so that the sets $Z_i$ are open in some fixed space, then $\mathcal{F}$ can be also described as a maximal foliated atlas of class $C^m$. The term *foliated space* (of *class $C^m$*) is used for $X \equiv (X, \mathcal{F})$. If no reference to the class $C^m$ is indicated, then it is understood that $X$ is a $C^0$ (or *topological*) foliated space. The concept of $C^m$ foliated space can be extended to the case *with boundary* in the obvious way, and the boundary of a $C^m$ foliated space is a $C^m$ foliated space without boundary.

The foliated structure of a space $X$ induces a locally Euclidean topology on $X$, the basic open sets being the plaques of all foliated charts, which is finer than the original topology. The connected components of $X$ in this topology are called *leaves*. Each leaf is a connected $C^m$ $n$-manifold with the differential structure canonically induced by $\mathcal{F}$. The leaf that contains each point $x \in X$ is denoted by $L_x$. The leaves of $\mathcal{F}$ form a partition of $X$ that determines the topological foliated structure. The corresponding quotient space, called *leaf space*, is denoted by $X/\mathcal{F}$.

The restriction of $\mathcal{F}$ to some open subset $U \subset X$ is the foliated structure $\mathcal{F}|_U$ on $U$ defined by the charts of $\mathcal{F}$ whose domains are contained in $U$. More generally, a subspace $Y \subset X$ is a $C^m$ *foliated subspace* when it is a subspace with a $C^m$ foliated structure $\mathcal{G}$ so that, for each $y \in Y$, there is a foliated chart of $\mathcal{F}$ defined on a neighborhood $U$ of $y$ in $X$, whose restriction to $U \cap Y$ can be considered as a chart of $\mathcal{G}$ in the obvious way; in particular, $\dim \mathcal{G} \leq \dim \mathcal{F}$. For instance, any saturated subspace is a $C^m$ foliated subspace.

A map between foliated spaces is said to be a *foliated map* if it maps leaves to leaves. A foliated map between $C^m$ foliated spaces is said to be of *class $C^m$* if its local representations in terms of foliated charts are of class $C^m$. A $C^m$ *foliated diffeomorphism* between $C^m$ foliated spaces is a $C^m$ foliated homeomorphism between them whose inverse is also a $C^m$ foliated map.

Any topological space is a foliated space whose leaves are its points. On the other hand, any connected $C^m$ $n$-manifold $M$ is a $C^m$ foliated space of dimension $n$ with only one leaf. The $C^m$ foliated maps $M \to X$ can be considered as $C^m$ maps to the leaves of $X$, and may be also called $C^m$ *leafwise maps*. They form a set denoted by $C^m(M, \mathcal{F})$, which can be equipped with the obvious generalization of the (*weak*) $C^m$ *topology*. In particular, for $m = 0$, we get the subspace $C(M, \mathcal{F}) \subset C(M, X)$ with the compact-open topology. For instance, $C(I, \mathcal{F})$ ($I = [0, 1]$) is the space of leafwise paths in $X$.

Many concepts of manifold theory readily extend to foliated spaces. In particular, if $\mathcal{F}$ is of class $C^m$ with $m \geq 1$, there is a vector bundle $T\mathcal{F}$ over $X$ whose fiber at each point $x \in X$ is the tangent space $T_xL_x$. Observe that $T\mathcal{F}$ is a foliated space of class $C^{m-1}$ with leaves $TL$ for leaves $L$ of $X$. Then we can consider a $C^{m-1}$ Riemannian structure on $T\mathcal{F}$, which is called a (*leafwise*) *Riemannian metric* on $X$. This is a section of the associated bundle over $X$ of positive definite symmetric bilinear forms on the fibers of $T\mathcal{F}$, which is $C^{m-1}$ as foliated map. In this paper, a *Riemannian foliated space* is a $C^\infty$ foliated space equipped with a $C^\infty$ Riemannian metric, and an *isometry* between Riemannian foliated spaces is a $C^\infty$ diffeomorphism between them whose restrictions to the leaves are isometries; in this case, the Riemannian foliated spaces are called *isomertric*.

A foliated space has a "transverse dynamics," which can be described by using a pseudogroup (see [18–20]). A *pseudogroup* $\mathcal{H}$ on a space is a maximal collection of homeomorphisms between open subsets of $Z$ that contains $\mathrm{id}_Z$, and is closed by the operations of composition, inversion, restriction to open subsets of their domains, and combination. This is a generalization of a dynamical system, and all basic dynamical concepts can be directly generalized to pseudogroups. For instance, we can consider its *orbits*, and the corresponding orbit space is denoted by $Z/\mathcal{H}$. It is said that $\mathcal{H}$ is *generated* by a subset $E$ when all of its elements can be obtained from the elements of $E$ by using the pseudogroup operations. Certain *equivalence* relation between pseudogroups was introduced [18], [19], and equivalent pseudogroups should be considered to represent the same dynamics; in particular, they have homeomorphic orbit spaces.

The *germ groupoid* of $\mathcal{H}$ is the topological groupoid of germs of maps in $\mathcal{H}$ at all points of their domains, with the operation induced by the composite of partial maps and the étale topology. Its subspace of units can be canonically identified with $Z$. For each $x \in Z$, the group of elements of this groupoid whose source and range is $x$ is called the *germ group* of $\mathcal{H}$ at $x$. The germ groups at points in the same orbit are conjugated in the germ groupoid, and therefore the *germ group* of each orbit is defined up to isomorphisms. Under pseudogroup equivalences, corresponding orbits have isomorphic germ groups.

Let $\mathcal{U} = \{U_i, \phi_i\}$ be a foliated atlas of $\mathcal{F}$, with $\phi_i : U_i \to B_i \times Z_i$, and let $p_i = \mathrm{pr}_2 \phi_i : U_i \to Z_i$. The local transverse components of the corresponding changes of coordinates can be considered as homeomorphisms between open subsets of $Z = \bigsqcup_i Z_i$, which generate a pseudogroup $\mathcal{H}$. The equivalence class of $\mathcal{H}$ depends only on $\mathcal{F}$, and is called its *holonomy pseudogroup*. There is a canonical homeomorphism between the leaf space and the orbit space, $X/\mathcal{F} \to Z/\mathcal{H}$, given by $L \mapsto \mathcal{H}(p_i(x))$ if $x \in L \cap U_i$.

The *holonomy groups* of the leaves are the germ groups of the corresponding $\mathcal{H}$-orbits. The leaves with trivial holonomy groups are called *leaves without holonomy*. The union of leaves without holonomy is denoted by $X_0$. If $X$ is second countable, then $X_0$ is a dense $G_\delta$ saturated subset of $X$ [11, 21].

Given a loop $\alpha$ in a leaf $L$ with base point $x$, there is a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of $I$ and there are foliated charts $(U_{i_1}, \phi_{i_1}), \ldots, (U_{i_k}, \phi_{i_k})$ such that $\alpha([t_{l-1}, t_l]) \subset U_{i_l}$ for $l \in \{1, \ldots, k\}$. We can assume $(U_{i_k}, \phi_{i_k}) = (U_{i_1}, \phi_{i_1})$ because $\alpha$ is a loop. Let $h_{i_{l-1}, i_l}$ be the local transverse component of each change of coordinates $\phi_{i_l}\phi_{i_{l-1}}^{-1}$ defined around $p_{i_{l-1}}c(t_{l-1})$ and with $h_{i_{l-1}, i_l}p_{i_{l-1}}\alpha(t_{l-1}) = p_{i_l}\alpha(t_l)$. The germ the composition $h_{i_{k-1}, i_k} \cdots h_{i_1, i_0}$ at $p_{i_0}(x) = p_{i_k}(x)$ depends only on $\mathcal{F}$ and the class of $\alpha$ in $\pi_1(L, x)$, obtaining a surjective homomorphism of $\pi_1(L, x)$ to the holonomy group of $L$. This homomorphism defines a connected covering $\widetilde{L}^{\mathrm{hol}}$ of $L$, which is called its *holonomy covering*.

Now, let $R$ be an equivalence relation on a topological space $X$. A subset of $X$ is called ($R$-) *saturated* if it is a union of ($R$-) equivalence classes. The equivalence relation $R$ is said to be (*topologically*)*transitive* if there is an equivalence class that is dense in $X$. A subset $Y \subset X$ is called an ($R$-) *minimal set* if it is a minimal element of the family of nonempty saturated closed subsets of $X$ ordered by inclusion; this is equivalent to the condition that all equivalence classes in $Y$ are dense in $Y$. In particular, $X$ (or $R$) is called *minimal* when all equivalence classes are dense in $X$. These concepts apply to foliated spaces with the equivalence relation whose equivalence classes are the leaves.

2.2. **Riemannian geometry.** Let $M$ be a Riemannian manifold possibly with boundary or corners (in the sense of [7], [10]). Connectedness of Riemannian manifolds is not assumed in Sections 2.2, 3 and 10 because it is not relevant for the concepts of these sections, but this property is assumed in the rest of the paper: it is needed in Section 4, and it is implicit in Sections 5–9 and 11–12 because the manifolds are given by elements of $\mathcal{M}_*(n)$. The following standard notation will be used. The metric tensor is denoted by $g$, the distance function on each of the connected components of $M$ by $d$, the tangent bundle by $\pi : TM \to M$, the $\mathrm{GL}(n)$-principal bundle of tangent frames by $\pi : PM \to M$, the $\mathrm{O}(n)$-principal bundle of orthonormal tangent frames by $\pi : QM \to M$, the Levi-Civita connection by $\nabla$, the curvature by $\mathcal{R}$, the exponential map by $\exp : TM \to M$ (if $M$ is complete and $\partial M = \emptyset$), the open and closed balls of center $x \in M$ and radius $r > 0$ by $B(x, r)$ and $\overline{B}(x, r)$, respectively, and the injectivity radius by inj (if $\partial M = \emptyset$). The *penumbra* around a subset $S \subset M$ of radius $r > 0$ is the set $\mathrm{Pen}(S, r) = \{\, x \in M \mid d(x, S) < r \,\}$. If needed, "$M$" will be added to all of the above notation as a subindex or superindex. When a family of Riemannian manifolds $M_i$ is considered, we may add the subindex or superindex "$i$" instead of "$M_i$" to the above notation. A covering of $M$ is assumed to be equipped with the lift of $g$.

For $m \in \mathbb{Z}^+$, let $T^{(m)}M = T \cdots TM$ ($m$ times). We also set $T^{(0)}M = M$. If $l < m$, $T^{(l)}M$ is sometimes identified with a regular submanifold of $T^{(m)}M$ via zero sections, and therefore, for each $x \in M$, the notation $x$ may be also used for the zero elements of $T_xM$, $T_xTM$, etc. When the vector space structure of $T_xM$ is emphasized, its zero element is denoted by $0_x$, or simply by 0, and the image of the zero section of $\pi : TM \to M$ is denoted by $Z \subset TM$. Let $\pi \colon T^{(m)}M \to T^{(l)}M$ be the vector bundle projection given by composing the tangent bundle projections; in particular, we have $\pi : T^{(m)}M \to M$. Given any $C^m$ map between Riemannian manifolds, $\phi : M \to N$, the induced map $T^{(m)}M \to T^{(m)}N$ will be denoted by $\phi_*^{(m)}$ (or simply $\phi_*$ if $m = 1$, $\phi_{**}$ if $m = 2$, and so on).

Banach manifolds are also considered in some parts of the paper, using analogous notation.

The Levi-Civita connection determines a decomposition $T^{(2)}M = \mathcal{H} \oplus \mathcal{V}$, as direct sum of the horizontal and vertical subbundles. The *Sasaki metric* on $TM$ is the unique Riemannian metric $g^{(1)}$ so that $\mathcal{H} \perp \mathcal{V}$ and the canonical identities $\mathcal{H}_\xi \equiv T_\xi M \equiv \mathcal{V}_\xi$ are isometries for every $\xi \in TM$.

Continuing by induction, for $m \geq 2$, the *Sasaki metric* on $T^{(m)}M$ is defined by $g^{(m)} = (g^{(m-1)})^{(1)}$. The notation $d^{(m)}$ is used for the corresponding distance function on the connected components, and the corresponding open and closed balls of center $\xi \in T^{(m)}M$ and radius $r > 0$ are denoted by $B^{(m)}(\xi, r)$ and $\overline{B}^{(m)}(\xi, r)$, respectively. We may add the subindex "$M$" to this notation if necessary, or the subindex "$i$" instead of "$M_i$" when a family of Riemannian manifolds $M_i$ is considered. From now on, $T^{(m)}M$ is assumed to be equipped with $g^{(m)}$.

*Remark* 1. The following properties hold for $l < m$ and $\pi : T^{(m)}M \to T^{(l)}M$:

(i) $g^{(m)}|_{T^{(l)}M} = g^{(l)}$.
(ii) The submanifold $T^{(l)}M \subset T^{(m)}M$ is totally geodesic and orthogonal to the fibers of $\pi$. This follows easily by induction on $m$, where the case $m = 1$ is proved in [36, Corollary of Theorem 13].
(iii) The projection $\pi$ is a Riemannian submersion with totally geodesic fibers. Again, this follows by induction on $m$, and the case $m = 1$ is proved in [36, Theorems 14 and 18].
(iv) For every $\xi \in T^{(m)}M$, its projection $\pi(\xi)$ is the only point $\zeta \in T^{(l)}M$ that satisfies $d^{(m)}(\xi, \zeta) = d^{(m)}(\xi, T^{(l)}M)$. To see this, it is enough to prove that $\pi(\xi)$ is the only critical point of the distance function $d^{(m)}(\cdot, \xi)$ on $T^{(l)}M$. These critical points are just the points $\zeta \in T^{(l)}M$ where the shortest $g^{(m)}$-geodesics $\gamma$ from $\zeta$ to $\xi$ are orthogonal to $T^{(l)}M$ at $\zeta$. Hence $\gamma$ is a geodesic in $\pi^{-1}(\zeta)$ by (iii), obtaining $\zeta = \pi(\xi)$.
(v) For all $\zeta, \zeta' \in T^{(l)}M$, the point $\zeta'$ is the only $\xi \in \pi^{-1}(\zeta')$ satisfying $d^{(m)}(\xi, \zeta) = d^{(m)}(\xi, \pi^{-1}(\zeta))$. This follows like (iv), using (ii) instead of (iii).

Let $(U; x^1, \dots, x^n)$ be a chart of $M$. The corresponding metric coefficients are denoted by $g_{ij}$, and the Christoffel symbols of the first and second kind are denoted by $\Gamma_{ijk}$ and $\Gamma_{ij}^k$, respectively. Using the Einstein notation, recall that

$$\Gamma_{ij}^\alpha g_{\alpha k} = \Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \,. \tag{1}$$

Identify the functions $x^i$, $g_{ij}$, $\Gamma_{ijk}$ and $\Gamma_{ij}^k$ with their lifts to $TU$. We get a chart $(U^{(1)};x_{(1)}^1,\dots,x_{(1)}^{2n})$ of $TM$ with $U^{(1)}=TU$, $x_{(1)}^i=x^i$ and $x_{(1)}^{n+i}=v^i$ for $1\le i\le n$, where the functions $v^i$ give the coordinates of tangent vectors with respect to the local frame $(\partial_1,\dots,\partial_n)$ of $TU$ induced by $(U;x^1,\dots,x^n)$. The coefficients of the Sasaki metric $g^{(1)}$ with respect to $(TU;x_{(1)}^1,\dots,x_{(1)}^{2n})$ are [36, Eq. (3.5)]:

$$\left.\begin{aligned} g_{ij}^{(1)} &= g_{ij}-g_{\alpha\gamma}\Gamma_{\mu\beta}^{\alpha}\Gamma_{\alpha\nu}^{\beta}v^\mu v^\nu\\ g_{n+i\,j}^{(1)} &= \Gamma_{j\mu i}v^\mu\\ g_{n+i\,n+j}^{(1)} &= g_{ij}\end{aligned}\right\} \tag{2}$$

for $1\le i,j\le n$. Thus the metric coefficients $g_{\alpha\beta}^{(1)}$ are given by universal fractional expressions of the functions $g_{ij}$, $\partial_k g_{ij}$ and $v^i$ $(1\le i,j,k\le n)$.

Using induction again, for $m\ge 2$, let $(U^{(m)};x_{(m)}^1,\dots,x_{(m)}^{2^m n})$ be the chart of $T^{(m)}M$ induced by the chart $(U^{(m-1)};x_{(m-1)}^1,\dots,x_{(m-1)}^{2^{m-1}n})$ of $T^{(m-1)}M$, and let $g_{\alpha\beta}^{(m)}$ be the corresponding coefficients of $g^{(m)}$.

**Lemma 2.1.** (*i*) *The coefficients $g_{\alpha\beta}^{(m)}$ are given by universal fractional expressions of the coordinates $x_{(m)}^{n+1},\dots,x_{(m)}^{2^m n}$ and the partial derivatives up to order $m$ of the coefficients $g_{ij}$.*

(*ii*) *For each $\rho>0$, the partial derivatives up to order $m$ of the coefficients $g_{ij}$ are given by universal linear expressions of the functions $(\sigma_{\rho,\mu}^{(m)})^*g_{\alpha\beta}^{(m)}$ for $n+1\le\mu\le 2^m n$, where $\sigma_{\rho,\mu}^{(m)}:U\to U^{(m)}$ is the section of $\pi:U^{(m)}\to U$ determined by $(\sigma_{\rho,\mu}^{(m)})^*x_{(m)}^\nu=\rho\delta_{\mu\nu}$ for $n+1\le\nu\le 2^m n$, using Kronecker's delta.*

*Proof.* We proceed by induction on $m$. For $m=1$, (i) holds by (1) and (2), and (ii) holds by the second and third equalities of (2), since $\partial_i g_{jk}=\Gamma_{ijk}+\Gamma_{ikj}$ by (1). For arbitrary $m\ge 2$, assuming that (i) and (ii) hold for the case $m-1$, we get both properties for $m$ by applying the above case to $(g^{(m-1)})^{(1)}=g^{(m)}$. □

Let $\Omega\subset M$ be a compact domain and $m\in\mathbb{N}$. Fix a finite collection of charts of $M$ that covers $\Omega$, $\mathcal{U}=\{(U_a;x_a^1,\dots,x_a^n)\}$, and a family of compact subsets of $M$ with the same index set as $\mathcal{U}$, $\mathcal{K}=\{K_a\}$, such that $\Omega\subset\bigcup_a K_a$, and $K_a\subset U_a$ for all $a$. The corresponding $C^m$ norm of a $C^m$ tensor $T$ on $\Omega$ is defined by

$$\|T\|_{C^m,\Omega,\mathcal{U},\mathcal{K}}=\max_a\max_{x\in K_a\cap\Omega}\sum_{|I|\le m}\sum_{J,K}\left|\frac{\partial^{|I|}T_{a,J}^K}{\partial x_a^I}(x)\right| ,$$

using the standard multi-index notation, where $T_{a,J}^K$ are the coefficients of $T$ on $U_a\cap\Omega$ with respect to the frame induced by $(U_a;x_a^1,\dots,x_a^n)$. With this norm, the $C^m$ tensors on $\Omega$ of a fixed type form a Banach space. By taking the projective limit as $m\to\infty$, we get the Fréchet space of $C^\infty$ tensors of that type equipped with the $C^\infty$ *topology* (see e.g. [22]). Observe that $\mathcal{U}$ and $\mathcal{K}$ are also qualified to define the norm $\|\ \|_{C^m,\Omega',\mathcal{U},\mathcal{K}}$ for any compact subdomain $\Omega'\subset\Omega$. It is well known that $\|\ \|_{C^m,\Omega,\mathcal{U},\mathcal{K}}$ is equivalent to the norm $\|\ \|_{C^m,\Omega,g}$ defined by

$$\|T\|_{C^m,\Omega,g}=\max_{0\le l\le m}\max_{x\in\Omega}|\nabla^l T(x)| ;$$

i.e., there is some $C\ge 1$, depending only on $M$, $m$, $\Omega$, $\mathcal{U}$, $\mathcal{K}$ and $g$, such that

$$\frac{1}{C}\|\ \|_{C^m,\Omega,\mathcal{U},\mathcal{K}}\le\|\ \|_{C^m,\Omega,g}\le C\|\ \|_{C^m,\Omega,\mathcal{U},g} . \tag{3}$$

When $\partial M=\emptyset$, it is said that $M$ is of *bounded geometry* if $\operatorname{inj}_M>0$ and the function $|\nabla^m\mathcal{R}|$ is bounded for all $m\in\mathbb{N}$; in particular, $M$ is complete since $\operatorname{inj}_M>0$. More precisely, given $r>0$ and a sequence $C_m>0$, if $\operatorname{inj}_M\ge r$ and $|\nabla^m\mathcal{R}|\le C_m$ for all $m\in\mathbb{N}$, then $(r,C_m)$ is called a *geometric bound* of $M$. A family $\mathcal{C}$ of Riemannian manifolds without boundary is called of *equi-bounded geometry* if all of them are of bounded geometry with a common geometric bound; i.e., their disjoint union is of bounded geometry.

## 3. Quasi-isometries

Let $\phi\colon M\to N$ be a $C^1$ map between Riemannian manifolds. Recall that $\phi$ is called a ($\lambda$-) *quasi-isometry*, or ($\lambda$-) *quasi-isometric*, if there is some $\lambda\ge 1$ such that $\frac{1}{\lambda}|\xi|\le|\phi_*(\xi)|\le\lambda|\xi|$ for every $\xi\in TM$. This $\lambda$ is called a *dilation bound* of $\phi$. The second of the above inequalities, $|\phi_*(\xi)| \leq \lambda\,|\xi|$ for all $\xi \in TM$, means that $|\phi_*| \leq \lambda$; i.e., $|\phi_{*x}| \leq \lambda$ for all $x \in M$.

*Remark* 2. (i) Every quasi-isometry is an immersion.
(ii) If $|\phi_*| \leq \lambda$, then $\phi$ is $\lambda$-Lipschitz; i.e., $d_N(\phi(x),\phi(y)) \leq \lambda\, d_M(x,y)$ for all $x,y \in M$.
(iii) If $\phi : M \to N$ is a $\lambda$-quasi-isometry, then $\phi$ is $\lambda$-bi-Lipschitz; i.e., for all $x,y \in M$,

$$\frac{1}{\lambda}\, d_M(x,y) \leq d_N(\phi(x),\phi(y)) \leq \lambda\, d_M(x,y)\ .$$

(iv) Let $\psi : N \to L$ be another $C^1$ map between Riemannian manifolds. If $|\phi_*| \leq \lambda$ and $|\psi_*| \leq \mu$, then $|(\psi\phi)_*| \leq \lambda\mu$.
(v) The composition of a $\lambda$-quasi-isometry and a $\mu$-quasi-isometry is a $\lambda\mu$-quasi-isometry.
(vi) The inverse of a $\lambda$-quasi-isometric diffeomorphism is a $\lambda$-quasi-isometric diffeomorphism.

Consider the subbundle $T^{\leq r}M = \{\, \xi \in TM \mid |\xi| \leq r \,\} \subset TM$ for each $r > 0$. If $M$ has no boundary, then $T^{\leq r}M$ is a manifold with boundary, being $\partial T^{\leq r}M = T^rM := \{\, \xi \in TM \mid |\xi| = r \,\}$; otherwise, $T^{\leq r}M$ is a manifold with corners. Also, define $T^{(m),\leq r}M$ by induction on $m \in \mathbb{Z}^+$, setting $T^{(1),\leq r}M = T^{\leq r}M$ and $T^{(m),\leq r}M = T^{\leq r}T^{(m-1),\leq r}M$. Note that $T^{(m),\leq r}T^{(m'),\leq r}M = T^{(m+m'),\leq r}M$.

**Definition 3.1.** (i) It is said that $\phi : M \to N$ is a ($\lambda$-) *quasi-isometry of order* $m \in \mathbb{N}$, or a ($\lambda$-) *quasi-isometric map of order* $m$, if it is $C^{m+1}$ and $\phi_*^{(m)} : T^{(m),\leq 1}M \to T^{(m)}N$ is a ($\lambda$-) quasi-isometry. This $\lambda$ is called a *dilation bound of order* $m$ of $\phi$. The infimum of all dilations bounds of order $m$ is called the *dilation of order* $m$. If $\phi$ is a quasi-isometry of order $m$ for all $m \in \mathbb{N}$, then it is called a *quasi-isometry of order* $\infty$.
(ii) A collection $\Phi$ of maps between Riemannian manifolds is called a family of *equi-quasi-isometries of order* $m \in \mathbb{N}$ if it is a family of quasi-isometries of order $m$ with some common dilation bound of order $m$, which is called an *equi-dilation bound of order* $m$. If $\Phi$ is a collection of equi-quasi-isometries of order $m$ for all $m \in \mathbb{N}$, then it is called a family of *equi-quasi-isometries of order* $\infty$.
(iii) A Riemannian manifold $M$ is said to be *quasi-isometric with order* $m$ to another Riemannian manifold $N$ when there is a quasi-isometric diffeomorphism of order $m$, $M \to N$. With more generality, a collection $\{M_i\}$ of Riemannian manifolds is called *equi-quasi-isometric with order* $m$ to another collection $\{N_i\}$ of Riemannian manifolds, with the same index set, when there is a collection of equi-quasi-isometric diffeomorphisms of order $m$, $\{M_i \to N_i\}$.

*Remark* 3. (i) The $\lambda$-quasi-isometries of order 0 are the $\lambda$-quasi-isometries.
(ii) By Remark 1-(i), if $\phi$ is a $\lambda$-quasi-isometry of order $m \geq 1$, then it is a $\lambda$-quasi-isometry of order $m-1$.
(iii) For integers $0 \leq m' \leq m$, if $\phi$ is a $\lambda$-quasi-isometry of order $m$, then $\phi_*^{(m')}$ is a $\lambda$-quasi-isometry of order $m - m'$.

To begin with, let us clarify the concept of quasi-isometry of order 1. Consider the splittings $T^{(2)}M = \mathcal{H} \oplus \mathcal{V}$ and $T^{(2)}N = \mathcal{H}' \oplus \mathcal{V}'$, where $\mathcal{H}$ and $\mathcal{H}'$ are the horizontal subbundles, and $\mathcal{V}$ and $\mathcal{V}'$ are the vertical subbundles. Fix any $x \in M$ and $\xi \in T_xM$, and let $x' = \phi(x)$ and $\xi' = \phi_*(\xi)$. We have the canonical identities

$$T_\xi TM = \mathcal{H}_\xi \oplus \mathcal{V}_\xi \equiv T_xM \oplus T_xM\ , \quad T_{\xi'}TN = \mathcal{H}'_{\xi'} \oplus \mathcal{V}'_{\xi'} \equiv T_{x'}N \oplus T_{x'}N\ . \tag{4}$$

The pull-back Riemannian vector bundle $\phi^*TN$ is endowed with the pull-back $\nabla'$ of the Riemannian connection of $N$, and let $\phi_* : TM \to \phi^*TN$ also denote the homomorphism over $\mathrm{id}_M$ induced by $\phi$. Let $X$ be a $C^\infty$ tangent vector field on some neighborhood of $x$ in $M$ so that $X(x) = \xi$; thus $\phi_*X$ is a $C^1$ local section of $\phi^*TN$ around $x$ satisfying $(\phi_*X)(x) = \xi' \in (\phi^*TN)_x \equiv T_{\phi(x)}N$. Then, for any $\zeta \in T_xM$ and each $C^\infty$ function $f$ defined on some neighborhood of $x$, we have

$$\begin{aligned}\nabla'_\zeta(\phi_*(fX)) - \phi_*(\nabla_\zeta(fX)) &= f(x)\,\nabla'_\zeta(\phi_*X) + df(\zeta)\,\phi_*\xi - f(x)\,\phi_*(\nabla_\zeta X) - df(\zeta)\,\phi_*\xi \\ &= f(x)\,(\nabla'_\zeta(\phi_*X) - \phi_*(\nabla_\zeta X))\end{aligned}$$

in $(\phi^*TN)_x \equiv T_{x'}N$. Therefore $A_\phi(\zeta \otimes \xi) := \nabla'_\zeta(\phi_*X) - \phi_*(\nabla_\zeta X)$ depends only on $\zeta \otimes \xi$, and this expression defines a continuous section $A_\phi$ of $TM^* \otimes TM^* \otimes \phi^*TN$. Observe that $X$ can be chosen so that $\nabla_\zeta X = 0$, giving $A_\phi(\zeta\otimes\xi)=\nabla'_\zeta(\phi_*X)$ in this case. Then, from the definitions of tangent map and covariant derivative, it easily follows that, according to (4),

$$\phi_{**\xi}(\zeta_1,\zeta_2)\equiv(\phi_*(\zeta_1),\phi_*(\zeta_2)+A_\phi(\zeta_1\otimes\xi)) \tag{5}$$

for all $\zeta_1,\zeta_2\in T_xM$.

*Remark* 4. If $TM$ were used instead of $T^{\leq 1}M$ in the definition of quasi-isometries of order 1, we would get $A_\phi=0$, which is too restrictive. On the other hand, it would be weaker to use $T^1M$ instead of $T^{\leq 1}M$.

**Lemma 3.2.** *Suppose that $\phi:M\to N$ is $C^2$. Then the following properties hold for $r>0$ and $\mu,\nu,K\geq 0$:*

*(i) If $|\phi_{**\xi}|\leq\mu$ for all $\xi\in T^{\leq r}M$, then $|\phi_*|\leq\mu$ and $|A_\phi|\leq\mu/r$.*
*(ii) If $|\phi_*|\leq\nu$ and $|A_\phi|\leq K$, then $|\phi_{**\xi}|\leq\sqrt{2}(\nu+Kr)$ for all $\xi\in T^{\leq r}M$.*

*Proof.* Assume that $|\phi_{**\xi}|\leq\mu$ for all $\xi\in T^{\leq r}M$. We get $|\phi_*|\leq\mu$ by Remark 1-(i). Furthermore, for all $x\in M$ and $\xi,\zeta\in T_xM$ with $|\xi|=r$, according to (4) and (5),

$$|A_\phi(\zeta\otimes\xi)|\leq|(\phi_{*x}(\zeta),A_\phi(\zeta\otimes\xi))|=|\phi_{**\xi}(\zeta,0)|\leq\mu\,|(\zeta,0)|=\mu\,|\zeta|=\frac{\mu}{r}\,|\zeta|\,|\xi|\;.$$

Now, suppose that $|\phi_*|\leq\nu$ and $|A_\phi|\leq K$. Fix all $x\in M$ and $\xi,\zeta_1,\zeta_2\in T_xM$ with $|\xi|\leq r$, according to (4) and (5),

$$\begin{aligned}|\phi_{**\xi}(\zeta_1,\zeta_2)|&\leq|\phi_*(\zeta_1)|+|\phi_*(\zeta_2)+A_\phi(\zeta_1\otimes\xi)|\leq\nu\,|\zeta_1|+\nu\,|\zeta_2|+K\,|\zeta_1|\,|\xi|\\&\leq\nu\,|\zeta_1|+\nu\,|\zeta_2|+Kr\,|\zeta_1|\leq(\nu+Kr)\,(|\zeta_1|+|\zeta_2|)\leq\sqrt{2}(\nu+Kr)\,|(\zeta_1,\zeta_2)|\;.\end{aligned}$$ □

**Lemma 3.3.** *Suppose that $\phi:M\to N$ is $C^2$. Then the following conditions are equivalent for $r>0$:*

*(i) $\phi_*:T^{\leq r}M\to TN$ is a quasi-isometry.*
*(ii) $\phi$ is a quasi-isometry and $|A_\phi|$ is uniformly bounded.*

*In this case, the constants involved in the above properties are related in the following way:*

*(a) If $\mu$ is a dilation bound of $\phi_*:T^{\leq r}M\to TN$, then $\mu$ is a dilation bound of $\phi$ and $|A_\phi|\leq\mu/r$.*
*(b) If $\nu$ is a dilation bound of $\phi$, $|A_\phi|\leq K$, and $0<\kappa<1$ with $\nu K\kappa r<1$, then*

$$\mu=\max\left\{\sqrt{2}(\nu+Kr),\frac{\sqrt{2}\nu}{1-\nu K\kappa r},\frac{\sqrt{2}\nu}{\kappa}\right\}$$

*is a dilation bound of $\phi_*:T^{\leq r}M\to TN$.*

*Proof.* Assume that (i) holds, and let $\mu$ be a dilation bound of order 1 of $\phi$. Then $\phi$ is a $\mu$-quasi-isometry by Remark 1-(i). This shows (ii) and (a) by Lemma 3.2-(i).

Now, suppose that (ii) holds, and take $\nu$, $K$, $\kappa$ and $\mu$ like in (b). For all $x\in M$ and $\xi,\zeta_1,\zeta_2\in T_xM$ with $|\xi|\leq r$, according to (4) and (5),

$$\begin{aligned}|\phi_{**\xi}(\zeta_1,\zeta_2)|&\geq\frac{1}{\sqrt{2}}\left(|\phi_*(\zeta_1)|+|\phi_*(\zeta_2)+A_\phi(\zeta_1\otimes\xi)|\right)\geq\frac{1}{\sqrt{2}}\left(|\phi_*(\zeta_1)|+\kappa\,|\phi_*(\zeta_2)+A_\phi(\zeta_1\otimes\xi)|\right)\\&\geq\frac{1}{\sqrt{2}}\left(|\phi_*(\zeta_1)|+\kappa(|\phi_*(\zeta_2)|-|A_\phi(\zeta_1\otimes\xi)|)\right)\geq\frac{1}{\sqrt{2}}\left(\left(\frac{1}{\nu}-K\kappa\,|\xi|\right)|\zeta_1|+\frac{\kappa}{\nu}\,|\zeta_2|\right)\\&\geq\frac{1}{\sqrt{2}}\left(\left(\frac{1}{\nu}-K\kappa r\right)|\zeta_1|+\frac{\kappa}{\nu}\,|\zeta_2|\right)\geq\frac{1}{\mu}\,(|\zeta_1|+|\zeta_2|)\geq\frac{1}{\mu}\,|(\zeta_1,\zeta_2)|\;.\end{aligned}$$

This gives (i) and (b) by Lemma 3.2-(ii). □

For $c>0$, let $h_c:TM\to TM$ be the $C^\infty$ diffeomorphism defined by $h_c(\xi)=c\xi$. Observe that $h_c(T^{\leq 1}M)=T^{\leq c}M$, and the following diagram is commutative:

$$\begin{array}{ccc}TM & \xrightarrow{\phi_*} & TN\\ {\scriptstyle h_c}\big\downarrow & & \big\downarrow{\scriptstyle h_c}\\ TM & \xrightarrow{\phi_*} & TN\end{array}$$

For each $m \in \mathbb{Z}^+$, let $\mathcal{H}^{(m+1)}$ and $\mathcal{V}^{(m+1)}$ denote the horizontal and vertical vector subbundles of $T^{(m+1)}M$ over $T^{(m)}M$. Thus, for $\xi \in T^{(m-1)}M$ and $\zeta \in T_\xi T^{(m-1)}M$,

$$T_\zeta T^{(m)}M = \mathcal{H}^{(m+1)}_\zeta \oplus \mathcal{V}^{(m+1)}_\zeta \equiv T_\xi T^{(m-1)}M \oplus T_\xi T^{(m-1)}M \ . \tag{6}$$

**Lemma 3.4.** *For all $m \in \mathbb{Z}^+$, there is an orthogonal vector bundle decomposition, $T^{(m+1)}M = \mathcal{P}^{(m+1)} \oplus \mathcal{Q}^{(m+1)}$, preserved by $h^{(m)}_{c*}$, such that, for $\xi \in T^{(m-1)}M$, $\zeta \in T_\xi T^{(m-1)}M$ and $\zeta' = h^{(m)}_{c*}(\zeta)$, the canonical identity $T_\zeta T^{(m)}M \equiv T_{\zeta'}T^{(m)}M$ given by (6) induces identities, $\mathcal{P}^{(m+1)}_\zeta \equiv \mathcal{P}^{(m+1)}_{\zeta'}$ and $\mathcal{Q}^{(m+1)}_\zeta \equiv \mathcal{Q}^{(m+1)}_{\zeta'}$, so that $h^{(m)}_{c*} : \mathcal{P}^{(m+1)}_\zeta \to \mathcal{P}^{(m+1)}_{\zeta'} \equiv \mathcal{P}^{(m+1)}_\zeta$ is the identity, and $h^{(m)}_{c*} : \mathcal{Q}^{(m+1)}_\zeta \to \mathcal{Q}^{(m+1)}_{\zeta'} \equiv \mathcal{Q}^{(m+1)}_\zeta$ is multiplication by $c$.*

*Proof.* The proof is by induction on $m$. By the definition of connection, $h_{c*}$ preserves the orthogonal decomposition $T^{(2)}M = \mathcal{H} \oplus \mathcal{V}$. Moreover, for $\zeta \in TM$ and $\zeta' = c\zeta$, $h_{c*} : \mathcal{H}_\zeta \to \mathcal{H}_{\zeta'} \equiv \mathcal{H}_\zeta$ is the identity, and $h_{c*} : \mathcal{V}_\zeta \to \mathcal{V}_{\zeta'} \equiv \mathcal{V}_\zeta$ is multiplication by $c$. Thus the statement is true in this case with $\mathcal{P}^{(2)} = \mathcal{H}$ and $\mathcal{Q}^{(2)} = \mathcal{V}$.

Now, suppose that $m \geq 2$ and the result holds for $m-1$. For $\xi \in T^{(m-1)}M$ and $\zeta \in T_\xi T^{(m-1)}M$, we have canonical identities

$$\mathcal{H}^{(m+1)}_\zeta \equiv \mathcal{V}^{(m+1)}_\zeta \equiv T_\xi T^{(m-1)}M = \mathcal{P}^{(m)}_\xi \oplus \mathcal{Q}^{(m)}_\xi \ , \tag{7}$$

obtaining orthogonal decompositions, $\mathcal{H}^{(m+1)} = \mathcal{H}\mathcal{P}^{(m)} \oplus \mathcal{H}\mathcal{Q}^{(m)}$ and $\mathcal{V}^{(m+1)} = \mathcal{V}\mathcal{P}^{(m)} \oplus \mathcal{V}\mathcal{Q}^{(m)}$, where $(\mathcal{H}\mathcal{P}^{(m)})_\zeta \equiv \mathcal{P}^{(m)}_\xi \equiv (\mathcal{V}\mathcal{P}^{(m)})_\zeta$ and $(\mathcal{H}\mathcal{Q}^{(m)})_\zeta \equiv \mathcal{Q}^{(m)}_\xi \equiv (\mathcal{V}\mathcal{Q}^{(m)})_\zeta$ according to (7). Then the result follows with $\mathcal{P}^{(m+1)} = \mathcal{H}\mathcal{P}^{(m)} \oplus \mathcal{V}\mathcal{P}^{(m)}$ and $\mathcal{Q}^{(m+1)} = \mathcal{H}\mathcal{Q}^{(m)} \oplus \mathcal{V}\mathcal{Q}^{(m)}$. □

**Corollary 3.5.** *For all $m \in \mathbb{Z}^+$ and $c, r > 0$, we have $h^{(m)}_{c*}(T^{(m+1),\leq r}M) \subset T^{(m+1),\leq \bar{c}r}M$, where $\bar{c} = \max\{c, 1\}$, and $h^{(m)}_{c*} : T^{(m+1)}M \to T^{(m+1)}M$ is a $\hat{c}$-quasi-isometry, where $\hat{c} = \max\{c, 1/c\}$.*

**Lemma 3.6.** *For all $m \in \mathbb{Z}^+$, $r, s > 0$ and $\lambda \geq 0$, there is some $\mu \geq 0$ such that, for any $C^{m+1}$ map between Riemannian manifolds, $\phi : M \to N$, if $|(\phi^{(m)}_*)_{*\xi}| \leq \lambda$ for all $\xi \in T^{(m),\leq r}M$, then $|(\phi^{(m)}_*)_{*\xi}| \leq \mu$ for all $\xi \in T^{(m),\leq s}M$. Moreover $\mu$ can be chosen so that $\mu s \to 0$ as $s \to 0$ for fixed $m$, $r$ and $\lambda$.*

*Proof.* We proceed by induction on $m$.

For $m = 1$, we have $|\phi_{**\xi}| \leq \lambda$ for all $\xi \in T^{\leq r}M$. Then $|\phi_*| \leq \lambda$ and $|A_\phi| \leq \lambda/r$ by Lemma 3.2-(i). Using Lemma 3.2-(ii), it follows that $|\phi_{**\xi}| \leq \sqrt{2}\lambda(1 + s/r) =: \mu$ for all $\xi \in T^{\leq s}M$. Note that $\mu s \to 0$ as $s \to 0$ for fixed $r$ and $\lambda$ in this case.

Now, assume that $m \geq 2$ and the result holds for $m-1$. For $c = r/s$ and $t = \min\{cr, r\}$, the diagram

$$\begin{array}{ccc} T^{(m),\leq r}M & \xrightarrow{\phi^{(m)}_*} & T^{(m)}N \\ h^{(m-1)}_{1/c*}\Big\uparrow & & \Big\downarrow h^{(m-1)}_{c*} \\ T^{(m-1),\leq t}T^{\leq s}M & \xrightarrow{\phi^{(m)}_*} & T^{(m)}N \end{array} \tag{8}$$

is defined and commutative. By Corollary 3.5 and Remark 2-(iv), it follows that $|(\phi^{(m)}_*)_{*\xi}| \leq \hat{c}^2\lambda$ for all $\xi \in T^{(m-1),\leq t}T^{\leq s}M$, where $\hat{c} = \max\{c, 1/c\}$. Then, by the induction hypothesis applied to the map $\phi_* : T^{\leq s}M \to TN$, there is some $\mu \geq 0$, depending only on $m-1$, $t$, $s$ and $\hat{c}^2\lambda$, such that $|(\phi^{(m)}_*)_{*\xi}| \leq \mu$ for all $\xi \in T^{(m-1),\leq s}T^{\leq s}M = T^{(m),\leq s}M$, and so that $\mu s \to 0$ as $s \to 0$ for fixed $m$, $t$ and $\hat{c}^2\lambda$. □

**Corollary 3.7.** *For all $m \in \mathbb{Z}^+$, $r > 0$ and $\lambda \geq 0$, there is some $s > 0$ such that, for any $C^{m+1}$ map between Riemannian manifolds, $\phi : M \to N$, if $|(\phi^{(m)}_*)_{*\xi}| \leq \lambda$ for all $\xi \in T^{(m),\leq 1}M$, then $\phi^{(m+1)}_*(T^{(m+1),\leq s}M) \subset T^{(m+1),\leq r}N$.*

*Proof.* This is also proved by induction on $m$. The statement is true for $m = 0$ because, if $|\phi_*| \leq \lambda$, then $\phi_*(T^{\leq s}M) \subset T^{\leq \lambda s}N$ for all $s > 0$, and therefore it is enough to take $s = r/\lambda$ in this case.

Now, assume that $m \geq 1$ and the result is true for $m-1$. By Remark 1-(i), if $|(\phi^{(m)}_*)_{*\xi}| \leq \lambda$ for all $\xi \in T^{(m),\leq 1}M$, then $|(\phi^{(m-1)}_*)_{*\xi}| \leq \lambda$ for all $\xi \in T^{(m-1),\leq 1}M$. Hence, by the induction hypothesis, for all $r > 0$, there is some $s > 0$, as small as desired, such that $\phi^{(m)}_*(T^{(m),\leq s}M) \subset T^{(m),\leq r}N$. On the other hand, by Lemma 3.6, there is some $\mu > 0$, depending on $m$, $r$, $s$ and $\lambda$, such that $|(\phi_*^{(m)})_{*\xi}| \le \mu$ for all $\xi \in T^{(m),\le s}M$, and satisfying $\mu s \to 0$ as $s \to 0$ for fixed $m$, $r$ and $\lambda$. Thus we can choose $s$, and the corresponding $\mu$, so that $\mu s \le r$. Then

$$\phi_*^{(m+1)}(T^{(m+1),\le s}M) \subset T^{\le \mu s}T^{(m),\le r}N \subset T^{(m+1),\le r}N \ . \quad \square$$

**Lemma 3.8.** *For $m \in \mathbb{Z}^+$, $r, s > 0$ and $\lambda \ge 1$, there is some $\mu \ge 1$ such that, for any $C^{m+1}$ map between Riemannian manifolds, $\phi : M \to N$, if $\phi_*^{(m)} : T^{(m),\le r}M \to T^{(m)}N$ is a $\lambda$-quasi-isometry, then $\phi_*^{(m)} : T^{\le s}M \to T^{(m)}N$ is a $\mu$-quasi-isometry.*

*Proof.* Again, we use induction on $m$. The case $m = 1$ is a direct consequence of Lemma 3.3.

Now, assume that $m \ge 2$ and the result holds for $m-1$. Consider the notation of the proof of Lemma 3.6. From the commutativity of (8), and using Corollary 3.5 and Remark 2-(v), it follows that the lower horizontal arrow of (8) is a $\hat{c}^2\lambda$-quasi-isometry. Then, by the induction hypothesis applied to the map $\phi_* : T^{\le s}M \to TN$, there is some $\mu > 0$, depending only on $m-1$, $t$, $s$ and $\hat{c}^2\lambda$, such that $\phi_*^{(m)} : T^{(m),\le s}M \to T^{(m)}N$ is a $\mu$-quasi-isometry. $\square$

*Remark* 5. According to Lemma 3.8, we could use any $T^{(m),\le r}M$ instead of $T^{(m),\le 1}M$ to define quasi-isometries of order $m$, but the dilation bounds of order $m$ would be different.

**Proposition 3.9.** (*i*) *For all $m \in \mathbb{N}$ and $\lambda, \mu \ge 1$, there is some $\nu \ge 1$ such that, if $\phi : M \to N$ and $\psi : N \to L$ are quasi-isometries of order $m$, and $\lambda$ and $\mu$ are respective dilation bounds of order $m$, then $\psi\phi$ is a $\nu$-quasi-isometry of order $m$.*

(*ii*) *For all $m \in \mathbb{N}$ and $\lambda \ge 1$, there is some $\mu \ge 1$ such that, if $\phi : M \to N$ is a $\lambda$-quasi-isometric diffeomorphism of order $m$, then $\phi^{-1}$ is a $\mu$-quasi-isometry of order $m$.*

*Proof.* Let us prove (i). By Corollary 3.7, there is some $r > 0$, depending on $m$ and $\lambda$, such that

$$\phi_*^{(m+1)}(T^{(m+1),\le r}M) \subset T^{(m+1),\le 1}N \ ,$$

and therefore $\phi_*^{(m)}(T^{(m),\le r}M) \subset T^{(m),\le 1}N$. On the other hand, by Lemma 3.8, there is some $\lambda' \ge 1$, depending on $m$, $r$ and $\lambda$, such that $\phi_*^{(m)} : T^{(m),\le r}M \to T^{(m),\le 1}N$ is a $\lambda'$-quasi-isometry. So

$$(\psi\phi)_*^{(m)} = \psi_*^{(m)}\phi_*^{(m)} : T^{(m),\le r}M \to T^{(m)}L$$

is a $\lambda'\mu$-quasi-isometry by Remark 2-(v). Thus, by Lemma 3.8, there is some $\nu \ge 1$, depending on $m$, $r$ and $\lambda'\mu$, so that $(\psi\phi)_*^{(m)} : T^{(m),\le 1}M \to T^{(m)}L$ is a $\nu$-quasi-isometry; i.e., $\psi\phi$ is a $\nu$-quasi-isometry of order $m$.

Now, let us prove (ii). By Corollary 3.7, there is some $r > 0$, depending on $m$ and $\lambda$, such that

$$(\phi^{-1})_*^{(m+1)}(T^{(m+1),\le r}N) \subset T^{(m+1),\le 1}M \ ,$$

and therefore $(\phi^{-1})_*^{(m)}(T^{(m),\le r}N) \subset T^{(m),\le 1}M$. So

$$\phi_*^{(m)} : (\phi^{-1})_*^{(m)}(T^{(m),\le r}N) \to T^{(m),\le r}N$$

is a $\lambda$-quasi-isometric diffeomorphism, obtaining that

$$(\phi^{-1})_*^{(m)} = (\phi_*^{(m)})^{-1} : T^{(m),\le r}N \to (\phi^{-1})_*^{(m)}(T^{(m),\le r}N)$$

is a $\lambda$-quasi-isometry by Remark 2-(vi). Thus, by Lemma 3.8, there is some $\mu \ge 1$, depending on $m$, $r$ and $\lambda$, so that $(\phi^{-1})_*^{(m)} : T^{(m),\le 1}N \to T^{(m)}M$ is a $\mu$-quasi-isometry; i.e., $\phi^{-1}$ is a $\mu$-quasi-isometry of order $m$. $\square$

**Corollary 3.10.** *"Being quasi-isometric with order $m$" is an equivalence relation.*

Let $M$ and $N$ be connected Riemannian manifolds. For every $m \in \mathbb{N} \cup \{\infty\}$, consider the weak $C^m$ topology on $C^m(M,N)$ (see [22]). For $x \in M$ and $\Phi \subset C^m(M,N)$, let $\Phi(x) = \{\, \phi(x) \mid \phi \in \Phi \,\} \subset N$.

**Proposition 3.11.** *Assume that $N$ is complete. Let $x_0 \in M$, and let $\Phi \subset C^{m+1}(M,N)$ be a family of equi-quasi-isometries of order $m \in \mathbb{N} \cup \{\infty\}$. Then $\Phi$ is precompact in $C^m(M,N)$ if and only if $\Phi(x_0)$ is bounded in $N$.*

*Proof.* The "only if" part follows because the evaluation map $C^m(M,N) \to N$, $\phi \mapsto \phi(x_0)$, is continuous.

For $m \in \mathbb{N}$, the "if" part is proved by induction. For $m = 0$, the assumption that $\Phi \subset C^1(M,N)$ is a family of equi-quasi-isometries implies that $\Phi$ is equi-continuous by Remark 2-(iii). On the other hand, $\Phi(x) \subset \operatorname{Pen}_N(\Phi(x_0), \lambda\, d(x,x_0))$ for any $x \in M$ by Remark 2-(iii), where $\lambda \geq 1$ is an equi-dilation bound of $\Phi$. So $\Phi(x)$ is precompact in $N$ because $\Phi(x_0)$ is bounded and $N$ is complete. Therefore $\Phi$ is precompact in $C(M,N)$ by the Arzelà-Ascoli theorem.

Now, take an integer $m \geq 1$ and assume that the result holds for $m-1$. The map $C^m(M,N) \to C^{m-1}(T^{\leq 1}M, TN)$, $\phi \mapsto \phi_*|_{T^{\leq 1}M}$, is an embedding. So it is enough to prove that the image $\Phi_*$ of $\Phi$ by this map is precompact in $C^{m-1}(T^{\leq 1}M, TN)$. This holds by the induction hypothesis because $\Phi_* \subset C^m(T^{\leq 1}M, TN)$ is a family of equi-quasi-isometries of order $m-1$ by Remark 3-(iii).

The "if" part for $m = \infty$ can be proved as follows. In this case, we have proved that $\Phi$ is precompact in $C^l(M,N)$ for every $l \in \mathbb{N}$. By the continuity of the inclusion maps $C^{l+1}(M,N) \hookrightarrow C^l(M,N)$, it follows that $\Phi$ has the same closure $\overline{\Phi}$ in $C^l(M,N)$ and $C^{l+1}(M,N)$, and the weak $C^l$ and $C^{l+1}$ topologies coincide on $\overline{\Phi}$. Therefore $\overline{\Phi}$ is the closure of $\Phi$ in $C^\infty(M,N)$ too, and the weak $C^\infty$ and $C^l$ topologies coincide on $\overline{\Phi}$ for any $l \in \mathbb{N}$. Thus $\Phi$ is precompact in $C^\infty(M,N)$. □

## 4. Partial quasi-isometries

Let $M$ and $N$ be connected complete Riemannian manifolds without boundary.

**Definition 4.1.** For $m \in \mathbb{N}$, a partial map $f : M \rightarrowtail N$ is called a $C^m$ *local diffeomorphism* if $\operatorname{dom} f$ and $\operatorname{im} f$ are open in $M$ and $N$, respectively, and $f : \operatorname{dom} f \to \operatorname{im} f$ is a $C^m$ diffeomorphism. If moreover $f(x) = y$ for distinguished points, $x \in \operatorname{dom} f$ and $y \in \operatorname{im} f$, then $f$ is said to be *pointed*, and the notation $f : (M,x) \rightarrowtail (N,y)$ is used. The term *local homeomorphism* is used in the $C^0$ case.

The term "$C^m$ local diffeomorfism" ($m \geq 1$) may be also used in the standard sense, referring to any $C^m$ map $M \to N$ whose tangent map is an isomorphism at every point of $M$. The context will always clarify this ambiguity.

**Definition 4.2.** For $m \in \mathbb{N}$, $R > 0$ and $\lambda \geq 1$, a $C^{m+1}$ pointed local diffeomorphism $\phi\colon (M,x) \rightarrowtail (N,y)$ is called an $(m,R,\lambda)$*-pointed local quasi-isometry*, or a *local quasi-isometry* of *type* $(m,R,\lambda)$, if the restriction $\phi_*^{(m)} : \Omega^{(m)} \to T^{(m)}N$ is a $\lambda$-quasi-isometry for some compact domain $\Omega^{(m)} \subset \operatorname{dom} \phi_*^{(m)}$ with $B_M^{(m)}(x,R) \subset \Omega^{(m)}$.

*Remark* 6. (i) Any pointed local quasi-isometry $(M,x) \rightarrowtail (N,y)$ of type $(m,R,\lambda)$ is also of type $(m',R',\lambda')$ for $0 \leq m' \leq m$, $0 < R' < R$ and $\lambda' > \lambda$ (using Remark 1-(i)).

(ii) For integers $0 \leq m' \leq m$, any pointed $C^{m+1}$ local diffeomorphism $\phi : (M,x) \rightarrowtail (N,y)$ is a pointed local quasi-isometry of type $(m,R,\lambda)$ if and only if $\phi_*^{(m')} : (T^{(m')}M, x) \rightarrowtail (T^{(m')}N, y)$ is a pointed local quasi-isometry of type $(m-m', R, \lambda)$.

(iii) If there is an $(m,R,\lambda)$-pointed local quasi-isometry $(M,x) \rightarrowtail (N,y)$, then, for all $R' < R$ and $\lambda' > \lambda$, there is a $C^\infty$ $(m,R',\lambda')$-pointed local quasi-isometry $(M,x) \rightarrowtail (N,y)$ by [22, Theorem 2.7].

**Lemma 4.3.** *The following properties hold:*

(i) *If $\phi : (M,x) \rightarrowtail (N,y)$ and $\psi : (N,y) \rightarrowtail (L,z)$ are pointed local quasi-isometries of types $(m,R,\lambda)$ and $(m,\lambda R,\lambda')$, respectively, then $\psi \circ \phi : (M,x) \rightarrowtail (L,z)$ is an $(m,R,\lambda\lambda')$-pointed local quasi-isometry.*

(ii) *If $\phi : (M,x) \rightarrowtail (N,y)$ is an $(m,\lambda R,\lambda)$-pointed local quasi-isometry, then $\phi^{-1} : (N,y) \rightarrowtail (M,x)$ is an $(m,R,\lambda)$-pointed local quasi-isometry.*

*Proof.* To prove (i), it is enough to show that $\overline{B}_M^{(m)}(x,R) \subset \operatorname{dom}(\psi \circ \phi)_*^{(m)}$ by Remark 2-(v). For $\xi \in \overline{B}_M^{(m)}(x,R)$, we have $\xi \in \operatorname{dom} \phi$ and $d_N^{(m)}(y, \phi_*^{(m)}(\xi)) \leq \lambda\, d_M^{(m)}(x,\xi) \leq \lambda R$ by Remark 2-(iii), obtaining that $\xi \in \operatorname{dom}(\psi \circ \phi)_*^{(m)}$ since $(\psi \circ \phi)_*^{(m)} = \psi_*^{(m)} \circ \phi_*^{(m)}$.

To prove (ii), it is enough to show that $\overline{B}_N^{(m)}(y,R) \subset \phi_*^{(m)}(\overline{B}_M^{(m)}(x,\lambda R))$ by Remark 2-(vi). Let $A = \overline{B}_N^{(m)}(y,R) \cap \operatorname{im} \phi_*^{(m)}$, which is open in $\overline{B}_N^{(m)}(y,R)$ and contains $y$. For any $\zeta \in A$, there is some $\xi \in \operatorname{dom} \phi_*^{(m)}$ so that $\phi_*^{(m)}(\xi) = \zeta$. Then $d_M^{(m)}(x,\xi) \leq \lambda d_N(y,\zeta) \leq \lambda R$ by Remark 2-(iii), obtaining that $\xi \in \overline{B}_M^{(m)}(x,\lambda R)$.

Thus $A = \phi_*^{(m)}(\overline{B}_M^{(m)}(x,\lambda R)) \cap \overline{B}_N^{(m)}(y,R)$, which is closed in $\overline{B}_N^{(m)}(y,R)$. Therefore $\overline{B}_N^{(m)}(y,R) = A \subset \phi_*^{(m)}(\overline{B}_M^{(m)}(x,\lambda R))$ because $\overline{B}_N^{(m)}(y,R)$ is connected. □

## 5. The $C^\infty$ topology on $\mathcal{M}_*(n)$

**Definition 5.1.** For $m \in \mathbb{N}$ and $R, r > 0$, let $U^m_{R,r}$ be the set of pairs $([M,x],[N,y]) \in \mathcal{M}_*(n) \times \mathcal{M}_*(n)$ such that there is some $(m,R,\lambda)$-pointed local quasi-isometry $(M,x) \rightarrowtail (N,y)$ for some $\lambda \in [1,e^r)$.

The following standard notation is used for a set $X$ and relations $U, V \subset X \times X$:

$$\begin{aligned} U^{-1} &= \{\, (y,x) \in X \times X \mid (x,y) \in U \,\}\ , \\ V \circ U &= \{\, (x,z) \in X \times X \mid \exists y \in X \text{ so that } (x,y) \in U \text{ and } (y,z) \in V \,\}\ . \end{aligned}$$

Moreover the diagonal of $X \times X$ is denoted by $\Delta$.

**Proposition 5.2.** *The following properties hold for all $m, m' \in \mathbb{N}$ and $R, S, r, s > 0$:*

(i) $(U^m_{e^r R,r})^{-1} \subset U^m_{R,r}$.
(ii) $U^{m_0}_{R_0,r_0} \subset U^m_{R,r} \cap U^{m'}_{S,s}$, *where* $m_0 = \max\{m,m'\}$, $R_0 = \max\{R,S\}$ *and* $r_0 = \min\{r,s\}$.
(iii) $\Delta \subset U^m_{R,r}$.
(iv) $U^m_{e^s R,r} \circ U^m_{R,s} \subset U^m_{R,r+s}$.

*Proof.* Properties (ii) and (iii) are elementary, and (i) and (iv) are consequences of Lemma 4.3. □

**Proposition 5.3.** $\bigcap_{R,r>0} U^m_{R,r} = \Delta$ *for all* $m \in \mathbb{N}$.

*Proof.* We only prove "$\subset$" because "$\supset$" is obvious. For $([M,x],[N,y]) \in \bigcap_{R,r>0} U^m_{R,r}$, there is a sequence of pointed local quasi-isometries $\phi_i : (M,x) \rightarrowtail (N,y)$, with corresponding types $(m,R_i,\lambda_i)$, such that $R_i \uparrow \infty$ and $\lambda_i \downarrow 1$ as $i \to \infty$. Let us prove that $[M,x] = [N,y]$.

First, we inductively construct a pointed isometric immersion $\psi : (M,x) \to (N,y)$.

The restrictions $\phi_i : (B_M(x,R_1),x) \to (N,y)$ are pointed equi-quasi-isometries of order $m$ ($\lambda_1$ is an equi-dilation bound of order $m$). By Proposition 3.11, there is some subsequence $\phi_{k(1,l)}$ whose restriction to $B_M(x,R_1)$ converges to some pointed $C^m$ function $\psi_1 : (B_M(x,R_1),x) \to (N,y)$ in the weak $C^m$ topology. Since $\lambda_i \downarrow 1$, it follows that $\psi_1$ is an isometric immersion.

Now assume that, for some $i \geq 1$, there is some subsequence $\phi_{k(i,l)}$ whose restriction to $B_M(x,R_i)$ converges to some pointed isometric immersion $\psi_i : (B_M(x,R_i),x) \to (N,y)$. As before, by Proposition 3.11, the sequence $\phi_{k(i,l)}$ has some subsequence $\phi_{k(i+1,l)}$ whose restriction to $B_M(x,R_{i+1})$ converges to some pointed isometric immersion $\psi_{i+1} : (B_M(x,R_{i+1}),x) \to (N,y)$ in the weak $C^m$ topology. Moreover $\psi_{i+1}|_{B_M(x,R_i)} = \psi_i$. Thus the maps $\psi_i$ can be combined to define the desired pointed isometric immersion $\psi : (M,x) \to (N,y)$.

Now, let us show that $\psi$ is indeed a pointed isometry, and therefore $[M,x] = [N,y]$, as desired. By Lemma 4.3-(ii), each inverse $\phi_i^{-1} : (N,y) \rightarrowtail (M,x)$ is an $(m,R'_i,\lambda_i)$-pointed local quasi-isometry, where $R'_i = R_i/\lambda_i \uparrow \infty$. By using Proposition 3.11 as above, we get a subsequence $\phi^{-1}_{k'(i,l)}$ of each sequence $\phi^{-1}_{k(i,l)}$, whose restriction to $B_N(y,R'_i)$ converges to a pointed isometric immersion $\psi'_i : (B_N(y,R'_i),y) \to (M,x)$ in the weak $C^m$ topology, and such that $\phi^{-1}_{k'(i+1,l)}$ is also a subsequence of $\phi^{-1}_{k'(i,l)}$. So $\psi'_{i+1}|_{B_N(y,R'_i)} = \psi'_i$ for all $i$, obtaining that the maps $\psi'_i$ can be combined to define a pointed isometric immersion $\psi' : (N,y) \to (M,x)$. Since the operation of composition is continuous with respect to the weak $C^m$ topology [22, p. 64, Exercise 10], we get $\psi_i\psi'_i = \mathrm{id}_{B_N(y,R'_i)}$ for all $i$, giving $\psi\psi' = \mathrm{id}_N$. Therefore $\psi'$ is injective. Moreover $\psi'$ is also surjective because $M$ and $N$ are complete. Hence $\psi'$ is an isometry whose inverse is $\psi$. □

By Propositions 5.2 and 5.3, the sets $U^m_{R,r}$ form a base of entourages of a separating uniformity on $\mathcal{M}_*(n)$, which is called the $C^\infty$ *uniformity*. It will be proved that the induced topology satisfies the statement of Theorem 1.2; thus it is called the $C^\infty$ *topology*, and the corresponding space is denoted by $\mathcal{M}^\infty_*(n)$. The notation $\mathrm{Cl}_\infty$ and $\mathrm{Int}_\infty$ will be used for the closure and interior operators in $\mathcal{M}^\infty_*(n)$.

The following lemma will be used.

**Lemma 5.4.** *For any open $U \subset \mathcal{M}_*^\infty(n)$, the map $\mathbf{d}_U : \mathcal{M}_*^\infty(n) \to [0,\infty]$, defined by*

$$\mathbf{d}_U([M,x]) = \inf\{\, d_M(x,x') \mid x' \in M,\ [M,x'] \in U \,\}\ ,$$

*is upper semicontinuous.*

Here, recall that $\inf\emptyset = \infty$ in $\mathbb{R}$.

*Proof.* To prove that $\mathbf{d}_U$ is upper semicontinuous at some $[M,x] \in \mathcal{M}_*^\infty(n)$, we can assume that $D := \mathbf{d}_U([M,x]) < \infty$. Given any $\varepsilon > 0$, there is some $x' \in B_M(x, D+\varepsilon)$ such that $[M,x'] \in U$. Since $U$ is open, we have $U^m_{R,r}(M,x') \subset U$ for some $m \in \mathbb{N}$ and $R,r > 0$ with $R \geq D+\varepsilon$ and $e^r d_M(x,x') < D+\varepsilon$. Given any $[N,y] \in U^m_{2R,r}(M,x)$, there is some $(m,2R,\lambda)$-pointed local quasi-isometry $\phi : (M,x) \rightarrowtail (N,y)$ for some $\lambda \in [1,e^r)$. Take some $\delta > 0$ such that $\lambda(d_M(x,x')+\delta) < D+\varepsilon$, and let $\alpha$ be a smooth curve in $B_M(x,D+\varepsilon)$ of length $< d_M(x,x')+\delta$ from $x$ to $x'$. Hence $\phi\alpha$ is a well defined $C^{m+1}$ curve in $N$ from $y$ to $y' := \phi(x')$ of length $< \lambda(d_M(x,x')+\delta) < D+\varepsilon$, obtaining that $d_N(y,y') < D+\varepsilon$. On the other hand, $\phi$ is also an $(m,R,\lambda)$-pointed local quasi-isometry $(M,x') \rightarrowtail (N,y')$, showing that $[N,y'] \in U^m_{R,r}(M,x') \subset U$. So $\mathbf{d}_U([N,y]) < D+\varepsilon$. □

## 6. Convergence in the $C^\infty$ topology

**Lemma 6.1.** *Let $g$ and $g'$ be positive definite scalar products on a real vector space $V$, and let $|\ |$ and $|\ |'$ denote the respective induced norms on the vector space of tensors over $V$. The following properties hold:*

*(i) If $\lambda \geq 1$ satisfies $\frac{1}{\lambda}|v|' \leq |v| \leq \lambda|v|'$ for all $v \in V$, then $|g-g'| \leq \lambda^2 - \lambda^{-2}$.*
*(ii) If $|g-g'| \leq \varepsilon$ for some $\varepsilon \in [0,1)$, then $\sqrt{1-\varepsilon}\,|v| \leq |v|' \leq \sqrt{1+\varepsilon}\,|v|$ for all $v \in V$.*
*(iii) If $\lambda \geq 1$ satisfies $\frac{1}{\lambda}|v|' \leq |v| \leq \lambda|v|'$ for all $v \in V$, then $\frac{1}{\lambda^2}|\omega|' \leq |\omega| \leq \lambda^2|\omega|'$ for all $\omega \in V^* \otimes V^*$.*

*Proof.* To prove (i), take arbitrary vectors $v,w \in V$ with $|v| = |w| = 1$. By polarization,

$$\begin{aligned}(g-g')(v,w) &= \frac{1}{4}\left(|v+w|^2 - |v-w|^2 - |v+w|'^2 + |v-w|'^2\right)\\ &\leq \frac{1}{4}\left(\left(1-1/\lambda^2\right)|v+w|^2 + (\lambda^2-1)|v-w|^2\right) \leq 1 - \frac{1}{\lambda^2} + \lambda^2 - 1 = \lambda^2 - \frac{1}{\lambda^2}\ .\end{aligned}$$

Interchanging $g$ and $g'$ in these inequalities, it also follows that $|(g-g')(v,w)| \leq \lambda^2 - \lambda^{-2}$.

Property (ii) follows because, for any $v \in V$,

$$(1-\varepsilon)|v|^2 \leq |v|^2 - \big||v|^2 - |v|'^2\big| \leq |v|'^2 \leq |v|^2 + \big||v|^2 - |v|'^2\big| \leq (1+\varepsilon)|v|^2\ .$$

Let us prove (iii). For all $v,w \in V \smallsetminus \{0\}$,

$$\frac{|\omega(v,w)|}{|v|'\,|w|'} \leq \lambda^2\,\frac{|\omega(v,w)|}{|v|\,|w|} \leq \lambda^2\,|\omega|\ ,$$

obtaining $|\omega|' \leq \lambda^2|\omega|$. Interchanging the roles of $|\ |$ and $|\ |'$, we also get $|\omega| \leq \lambda^2|\omega|'$. □

The following coordinate free description of $C^m$ convergence is a direct consequence of (3).

**Lemma 6.2** (Lessa [29, Lemma 7.1])**.** *For $m \in \mathbb{N}$, a sequence $[M_i,x_i] \in \mathcal{M}_*(n)$ is $C^m$ convergent to $[M,x] \in \mathcal{M}_*(n)$ if and only if, for every compact domain $\Omega \subset M$ containing $x$, there are pointed $C^{m+1}$ embeddings $\phi_i\colon (\Omega,x) \to (M_i,x_i)$, for $i$ large enough, such that $\|g_M - \phi_i^* g_{M_i}\|_{C^m,\Omega,g_M} \to 0$ as $i \to \infty$.*

**Definition 6.3.** For $R,r > 0$ and $m \in \mathbb{N}$, let $D^m_{R,r}$ be the set of pairs $([M,x],[N,y]) \in \mathcal{M}_*(n) \times \mathcal{M}_*(n)$ such that there is some $C^{m+1}$ pointed local diffeomorphism $\phi\colon (M,x) \rightarrowtail (N,y)$ so that $\|g_M - \phi^* g_N\|_{C^m,\Omega,g_M} < r$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x,R) \subset \Omega$.

Given a set $X$, for $U \subset X \times X$ and $x \in X$, let $U(x) = \{\, y \in Y \mid (x,y) \in U \,\}$. In the case of $U \subset \mathcal{M}_*(n) \times \mathcal{M}_*(n)$ and $[M,x] \in \mathcal{M}_*(n)$, we simply write $U(M,x)$.

*Remark* 7. By Lemma 6.2, a sequence $[M_i,x_i] \in \mathcal{M}_*(n)$ is $C^\infty$ convergent to $[M,x] \in \mathcal{M}_*(n)$ if and only if it is eventually in $D^m_{R,r}(M,x)$ for arbitrary $m \in \mathbb{N}$ and $R,r > 0$.

**Proposition 6.4.** *(i) For all $R,r > 0$, if $0 < \varepsilon \leq \min\{1-e^{-2r}, e^{2r}-1\}$, then $D^0_{R,\varepsilon} \subset U^0_{R,r}$.*
*(ii) For all $m \in \mathbb{Z}^+$, $R,r > 0$ and $[M,x] \in \mathcal{M}_*(n)$, there is some $\varepsilon > 0$ such that $D^m_{R,\varepsilon}(M,x) \subset U^m_{R,r}(M,x)$.*

*Proof.* Let us show (i). If $([M,x],[N,y]) \in D^0_{R,\varepsilon}$, then there is a $C^1$ pointed local diffeomorphism $\phi : (M,x) \rightarrowtail (N,y)$ such that $\varepsilon_0 := \|g_M - \phi^* g_N\|_{C^0,\Omega,g_M} < \varepsilon$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x,R) \subset \Omega$. Choose some $\lambda \in [1, e^r)$ such that $\varepsilon_0 \le \min\{1-\lambda^{-2}, \lambda^2 - 1\}$. Set $g = g_M$ and $g' = \phi^* g_N$, and let $|\ |$ and $|\ |'$ denote the respective norms. For $\xi \in T\Omega$, we have

$$\frac{1}{\lambda}\,|\xi| \le \sqrt{1-\varepsilon_0}\,|\xi| \le |\xi|' \le \sqrt{1+\varepsilon_0}\,|\xi| \le \lambda\,|\xi|$$

by Lemma 6.1-(ii). Thus $\phi$ is a $(0,R,\lambda)$-pointed local quasi-isometry, obtaining that $([M,x],[N,y]) \in U^0_{R,r}$.

Let us prove (ii). Take $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M,x] \in \mathcal{M}_*(n)$. Let $\mathcal{U}$ be a finite collection of charts of $M$ with domains $U_a$, and let $\mathcal{K} = \{K_a\}$ be a family of compact subsets of $M$, with the same index set as $\mathcal{U}$, such that $K_a \subset U_a$ for all $a$, and $\overline{B}_M(x,R) \subset \operatorname{Int}(K)$ for $K = \bigcup_a K_a$. Let $\varepsilon > 0$, to be fixed later. For any $[N,y] \in D^m_{R,\varepsilon}(M,x)$, there is a $C^{m+1}$ pointed local diffeomorphism $\phi\colon (M,x) \rightarrowtail (N,y)$ so that $\|g_M - \phi^* g_N\|_{C^m,\Omega,g_M} < \varepsilon$ for some compact domain $\Omega \subset \operatorname{dom}\phi \cap \operatorname{Int}(K)$ with $B_M(x,R) \subset \Omega$. By continuity, there is another compact domain $\Omega' \subset \operatorname{dom}\phi \cap \operatorname{Int}(K)$ such that $\Omega \subset \operatorname{Int}(\Omega')$ and $\|g_M - \phi^* g_N\|_{C^m,\Omega',g_M} < \varepsilon$. As before, let $g = g_M$ and $g' = \phi^* g_N$.

With the notation of Section 2.2, let $\mathcal{U}^{(m)}$ be the family of induced charts of $T^{(m)}M$ with domains $U^{(m)}_a$, let $\mathcal{K}^{(m)}$ be the family of compact subsets

$$K^{(m)}_a = \{\, \xi \in T^{(m)}M \mid \pi(\xi) \in K_a,\ d^{(m)}_M(\xi, \pi(\xi)) \le R' \,\} \subset U^{(m)}_a\ ,$$

for some $R' > R$, where $\pi : T^{(m)}M \to M$, and let $K^{(m)} = \bigcup_a K^{(m)}_a$. Since $\overline{B}^{(m)}_M(x,R) \subset \operatorname{Int}(K^{(m)})$ and $\pi(\overline{B}^{(m)}_M(x,R)) = \overline{B}_M(x,R) \subset \Omega \subset \operatorname{Int}(\Omega')$ by Remark 1-(iv),(v), there is some compact domain $\Omega^{(m)} \subset T^{(m)}M$ such that $B^{(m)}_M(x,R) \subset \Omega^{(m)} \subset K^{(m)}$ and $\pi(\Omega^{(m)}) \subset \Omega'$.

Choose the following constants:

- some $C \ge 1$ satisfying (3) with $\mathcal{U}$, $\mathcal{K}$, $\Omega'$ and $g$;
- some $C^{(m)} \ge 1$ satisfying (3) with $\mathcal{U}^{(m)}$, $\mathcal{K}^{(m)}$, $\Omega^{(m)}$ and $g^{(m)}$;
- some $\delta \in (0, \min\{1-e^{-2r}, e^{2r}-1\}]$; and,
- by Lemma 2.1-(i), some $\varepsilon' > 0$ such that

$$\|g - g'\|_{C^m,\Omega',\mathcal{U},\mathcal{K}} < \varepsilon' \implies \|g^{(m)} - g'^{(m)}\|_{C^0,\Omega^{(m)},\mathcal{U}^{(m)},\mathcal{K}^{(m)}} < \delta / C^{(m)}\ .$$

Suppose that $\varepsilon \le \varepsilon'/C$. Then

$$\begin{aligned} \|g - g'\|_{C^m,\Omega',g} < \varepsilon &\implies \|g - g'\|_{C^m,\Omega',\mathcal{U},\mathcal{K}} < C\varepsilon \le \varepsilon' \\ &\implies \|g^{(m)} - g'^{(m)}\|_{C^0,\Omega^{(m)},\mathcal{U}^{(m)},\mathcal{K}^{(m)}} < \delta/C^{(m)} \implies \delta_0 := \|g^{(m)} - g'^{(m)}\|_{C^0,\Omega^{(m)},g^{(m)}} < \delta\ . \end{aligned}$$

For any $\lambda \in [1, e^r)$ such that $\delta_0 \le \min\{1-\lambda^{-2}, \lambda^2-1\}$, we have $\frac{1}{\lambda}\,|\xi|^{(m)} \le |\xi|'^{(m)} \le \lambda\,|\xi|^{(m)}$ for all $\xi \in T\Omega^{(m)}$ by Lemma 6.1-(ii), where $|\ |^{(m)}$ and $|\ |'^{(m)}$ denote the norms defined by $g^{(m)}$ and $g'^{(m)}$, respectively. So $\phi$ is an $(m,R,\lambda)$-pointed local quasi-isometry $(M,x) \rightarrowtail (N,y)$, and therefore $[N,y] \in U^{(m)}_{R,r}(M,x)$. □

**Proposition 6.5.** (*i*) *For all $R, r > 0$, if $e^{2\varepsilon} - e^{-2\varepsilon} \le r$, then $U^0_{R,\varepsilon} \subset D^0_{R,r}$.*

(*ii*) *For all $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M,x] \in \mathcal{M}_*(n)$, there is some $\varepsilon > 0$ such that $U^m_{R,\varepsilon}(M,x) \subset D^m_{R,r}(M,x)$.*

*Proof.* Let us show (i). If $([M,x],[N,y]) \in U^0_{R,\varepsilon}$, then there is a $(0,R,\lambda)$-pointed local quasi-isometry $\phi : (M,x) \rightarrowtail (N,y)$ for some $\lambda \in [1, e^\varepsilon)$. Set $g = g_M$ and $g' = \phi^* g_N$, and let $|\ |$ and $|\ |'$ denote the respective norms. Thus there is some compact domain $\Omega \subset \operatorname{dom}\phi$ such that $B_M(x,R) \subset \Omega$ and $\frac{1}{\lambda}\,|\xi| \le |\xi|' \le \lambda\,|\xi|$ for all $\xi \in T\Omega$. By Lemma 6.1-(i), it follows that

$$\|g - g'\|_{C^0,\Omega,g} \le \lambda^2 - \lambda^{-2} < e^{2\varepsilon} - e^{-2\varepsilon} \le r\ .$$

So $([M,x],[N,y]) \in D^0_{R,r}$.

Let us prove (ii). Let $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M,x] \in \mathcal{M}_*(n)$. Take $\mathcal{U}$, $\mathcal{K}$, $K$, $\mathcal{U}^{(m)}$, $\mathcal{K}^{(m)}$ and $K^{(m)}$ like in the proof of Proposition 6.4-(ii). Let $\varepsilon > 0$, to be fixed later. For any $[N,y] \in U^m_{R,\varepsilon}(M,x)$, there is an $(m,R,\lambda)$-pointed local quasi-isometry $\phi : (M,x) \rightarrowtail (N,y)$ for some $\lambda \in [1, e^\varepsilon)$. Again, let $g = g_M$ and $g' = \phi^* g_N$. Thus there is a compact domain $\Omega^{(m)} \subset \operatorname{dom}\phi^{(m)}_* \cap \operatorname{Int}(K^{(m)})$ so that $B^{(m)}_M(x,R) \subset \Omega^{(m)}$ and $\frac{1}{\lambda}\,|\xi|^{(m)} \le |\xi|'^{(m)} \le \lambda\,|\xi|^{(m)}$ for all $\xi \in T\Omega^{(m)}$, where $|\ |^{(m)}$ and $|\ |'^{(m)}$ denote the norms defined by $g^{(m)}$ and $g'^{(m)}$, respectively. By continuity, given any $\lambda' \in (\lambda, e^\varepsilon)$, there is some compact domain $\Omega'^{(m)} \subset \operatorname{dom}\phi_*^{(m)} \cap K^{(m)}$ such that $\Omega^{(m)} \subset \operatorname{Int}(\Omega'^{(m)})$ and $\frac{1}{\lambda'}|\xi|^{(m)} \leq |\xi|'^{(m)} \leq \lambda' |\xi|^{(m)}$ for all $\xi \in \Omega'^{(m)}$. By Lemma 6.1-(i), it follows that

$$\|g^{(m)} - g'^{(m)}\|_{C^0,\Omega'^{(m)},g^{(m)}} \leq \lambda'^2 - \lambda'^{-2} < e^{2\varepsilon} - e^{-2\varepsilon} .$$

There is some compact domain $\Omega \subset M$ such that $\Omega^{(m)} \cap M \subset \Omega \subset \operatorname{Int}(\Omega'^{(m)})$. Thus $\Omega \subset \Omega'^{(m)} \cap M \subset K^{(m)} \cap M = K$, and

$$B_M(x,R) = B_M^{(m)}(x,R) \cap M \subset \Omega^{(m)} \cap M \subset \Omega$$

by Remark 1-(ii). Take some $C \geq 1$ satisfying (3) with $\mathcal{U}$, $\mathcal{K}$, $\Omega$ and $g$, and some $C^{(m)} \geq 1$ satisfying (3) with $\mathcal{U}^{(m)}$, $\mathcal{K}^{(m)}$, $\Omega'^{(m)}$ and $g^{(m)}$. For $\rho > 0$ and $n+1 \leq \mu \leq 2^m n$, let $\sigma_{a,\rho,\mu}^{(m)} : U_a \to U_a^{(m)}$ be the section of each projection $\pi : U_a^{(m)} \to U_a$ of the type used in Lemma 2.1-(ii). Since $\Omega \subset \operatorname{Int}(\Omega'^{(m)})$, there is some $\rho > 0$ so that $\sigma_{\rho,\mu}^{(m)}(K_a \cap \Omega) \subset K_a^{(m)} \cap \Omega'^{(m)}$ for all $a$ and $\mu$. Thus, by Lemma 2.1-(ii), there is some $\varepsilon' > 0$, depending on $r$ and $\rho$, such that

$$\|g^{(m)} - g'^{(m)}\|_{C^0,\Omega'^{(m)},\mathcal{U}^{(m)},\mathcal{K}^{(m)}} < \varepsilon' \implies \|g - g'\|_{C^m,\Omega,\mathcal{U},\mathcal{K}} < r/C .$$

Suppose that $e^{2\varepsilon} - e^{-2\varepsilon} \leq \varepsilon'/C^{(m)}$. Then

$$\begin{aligned}\|g^{(m)} - g'^{(m)}\|_{C^0,\Omega'^{(m)},g^{(m)}} < e^{2\varepsilon} - e^{-2\varepsilon} &\implies \|g^{(m)} - g'^{(m)}\|_{C^0,\Omega'^{(m)},\mathcal{U}^{(m)},\mathcal{K}^{(m)}} < C^{(m)}(e^{2\varepsilon} - e^{-2\varepsilon}) \leq \varepsilon' \\ &\implies \|g - g'\|_{C^m,\Omega,\mathcal{U},\mathcal{K}} < r/C \implies \|g - g'\|_{C^m,\Omega,g} < r ,\end{aligned}$$

showing that $[N,y] \in D_{R,r}^{(m)}(M,x)$. □

**Corollary 6.6.** *The $C^\infty$ convergence in $\mathcal{M}_*(n)$ describes the $C^\infty$ topology.*

*Proof.* This is a direct consequence of Remark 7 and Propositions 6.4 and 6.5. □

## 7. $\mathcal{M}_*^\infty(n)$ is Polish

**Proposition 7.1.** *$\mathcal{M}_*^\infty(n)$ is separable.*

*Proof.* The isometry classes of pointed compact Riemannian manifolds form a subspace, $\mathcal{M}_{*,c}^\infty(n) \subset \mathcal{M}_*^\infty(n)$, which is dense because, for all $[M,x] \in \mathcal{M}_*^\infty(n)$ and $R > 0$, the ball $B_M(x,R)$ can be isometrically embedded in a compact Riemannian manifold.

As a consequence of the finiteness theorems of Cheeger on Riemannian manifolds [9], it follows that there are countably many diffeomorphism classes of compact $C^\infty$ manifolds (see [33, Corollary 37, p. 320] or [8, Theorem IX.8.1]). Thus there is a countable family $\mathcal{C}$ of $C^\infty$ compact manifolds containing exactly one representative of every diffeomorphism class.

For every $M \in \mathcal{C}$, the set of metrics on $M$, $\operatorname{Met}(M)$, is an open subspace of the space of smooth sections, $C^\infty(M; T^*M \odot T^*M)$, with the $C^\infty$ topology, where "$\odot$" denotes the symmetric product. Then, since $C^\infty(M; T^*M \odot T^*M)$ is separable, we can choose a countable dense subset $\mathcal{G}_M \subset \operatorname{Met}(M)$. Choose also a countable dense subset $\mathcal{D}_M \subset M$.

Clearly, the countable set

$$\{\, [(M,g),x] \mid M \in \mathcal{C},\ g \in \mathcal{G}_M,\ x \in \mathcal{D}_M \,\}$$

is dense in $\mathcal{M}_{*,c}^\infty(n)$, and therefore it is also dense in $\mathcal{M}_*^\infty(n)$. □

*Remark* 8. Observe that the proof of Proposition 7.1 shows that $\mathcal{M}_{*,c}^\infty(n)$ is dense in $\mathcal{M}_*^\infty(n)$.

**Proposition 7.2.** *$\mathcal{M}_*^\infty(n)$ is completely metrizable.*

*Proof.* The $C^\infty$ uniformity on $\mathcal{M}_*(n)$ is metrizable because it is separating and has a countable base of entourages [39, Corollary 38.4]. Thus it is enough to check that the $C^\infty$ uniformity on $\mathcal{M}_*(n)$ is complete.

Consider an arbitrary Cauchy sequence $[M_i,x_i]$ in $\mathcal{M}_*(n)$ with respect to the $C^\infty$ uniformity. We have to prove that $[M_i,x_i]$ is convergent in $\mathcal{M}_*^\infty(n)$. By taking a subsequence if necessary, we can suppose that $([M_i,x_i],[M_{i+1},x_{i+1}]) \in U_{R_i,r_i}^{m_i}$ for sequences, $m_i \uparrow \infty$ in $\mathbb{N}$, and $R_i \uparrow \infty$ and $r_i \downarrow 0$ in $\mathbb{R}^+$, such that $\sum_i r_i < \infty$, and $R_{i+1} \geq e^{r_i} R_i$ for all $i$. Let $\bar{r}_i = \sum_{j \geq i} r_j$. Consider other sequences $R'_i, R''_i \uparrow \infty$ in $\mathbb{R}^+$ such that $R'_i < R''_i \leq e^{-\bar{r}_i} R_i$ and $R'_{i+1} \geq e^{r_i} R''_i$.

For each $i$, there is some $\lambda_i \in (1, e^{r_i})$ and some $(m_i, R_i, \lambda_i)$-pointed local quasi-isometry $\phi_i\colon (M_i, x_i) \rightarrowtail (M_{i+1}, x_{i+1})$, which can be assumed to be $C^\infty$ by Remark 6-(iii). Then $\bar\lambda_i := \prod_{j\geq i}\lambda_j < e^{\bar r_i} < \infty$. For $i<j$, the pointed local quasi-isometry $\psi_{ij} = \phi_{j-1}\cdots\phi_i : (M_i, x_i) \rightarrowtail (M_j, x_j)$ is of type $(m_i, R_i/\bar\lambda_i, \bar\lambda_i)$ by Lemma 4.3-(i).

For $i, m \in \mathbb{N}$, let

$$\begin{gathered} B_i = B_i(x_i, R_i)\ , \qquad B_i' = B_i(x_i, R_i')\ , \qquad B_i'' = B_i(x_i, R_i'')\ , \\ B_i^{(m)} = B_i^{(m)}(x_i, R_i)\ , \quad B_i'^{(m)} = B_i^{(m)}(x_i, R_i')\ , \quad B_i''^{(m)} = B_i^{(m_i)}(x_i, R_i'')\ . \end{gathered}$$

A bar will be added to this notation when the corresponding closed balls are considered. We have $\phi_i(\overline{B}_i) \subset B_{i+1}$ because $R_{i+1} > \lambda_i R_i$, and $\phi_{i*}^{(m_i)}(\overline{B}_i''^{(m_i)}) \subset B_{i+1}'^{(m_i)} \subset B_{i+1}'^{(m_{i+1})}$ since $R_{i+1}' > \lambda_i R_i''$ and by Remark 1-(i). Furthermore $B_i'' \subset \operatorname{dom}\psi_{ij}$ and $B_i''^{(m_i)} \subset \operatorname{dom}\psi_{ij*}^{(m_i)}$ for $i<j$ because $R'' \leq R_i/\bar\lambda_i$. Therefore $\psi_{ij}(B_i) \subset B_j$ and $\psi_{ij*}^{(m_i)}(B_i''^{(m_i)}) \subset B_j'^{(m_j)}$.

The restrictions $\psi_{ij} : B_i \to B_j$ form a direct system of spaces, whose direct limit is denoted by $\widehat{M}$. Let $\psi_i : B_i \to \widehat{M}$ be the induced maps, whose images, $\widehat{B}_i := \psi_i(B_i)$, form an exhausting increasing sequence of subsets of $\widehat{M}$. All points $\psi_i(x_i)$ are equal in $\widehat{M}$, and will be denoted by $\hat x$. The space $\widehat{M}$ is connected because it is the union of the connected subspaces $\widehat{B}_i$ whose intersection contains $\hat x$. By the definition of the direct limit and since the maps $\psi_{ij}$ are open embeddings, it follows that all maps $\psi_i$ are open embeddings, and therefore $\widehat{M}$ is a Hausdorff $n$-manifold. Equip each $\widehat{B}_i$ with the $C^\infty$ structure that corresponds to the $C^\infty$ structure of $B_i$ by $\psi_i$. These $C^\infty$ structures are compatible one another because the open embeddings $\psi_{ij}$ are $C^\infty$, and therefore they define a $C^\infty$ structure on $\widehat{M}$. Moreover let $\hat g_i$ be the Riemannian metric on each $\widehat{B}_i$ that corresponds to $g_i|_{B_i}$ via $\psi_i$.

Take some compact domains, $\Omega_i$ in every $M_i$ and $\Omega_i^{(m_i)}$ in $T^{(m_i)}M_i$, such that $B_i' \subset \Omega_i \subset \operatorname{Int}(\Omega_i^{(m_i)})$ and $B_i'^{(m_i)} \subset \Omega_i^{(m_i)} \subset B_i''^{(m_i)}$; thus $\Omega_i \subset B_i''$ by Remark 1-(ii). Let $\widehat\Omega_i = \psi_i(\Omega_i)$.

*Claim* 1. $\widehat{M} = \bigcup_i \widehat\Omega_i$.

This equality holds because, for each $i$, there is some $j$ so that $R_j' > \bar\lambda_i R_i$, obtaining

$$\psi_{ij}(B_i) \subset B_j(x_j, \bar\lambda_i R_i) \subset B_j' \subset \Omega_j\ ,$$

and therefore $\widehat{B}_i = \psi_j\psi_{ij}(B_i) \subset \psi_j(\Omega_j) = \widehat\Omega_j$.

*Claim* 2. For all $i$, the restrictions $\hat g_j|_{\widehat\Omega_i}$, with $j \geq i$, form a convergent sequence in the space of $C^{m_i}$ sections, $C^{m_i}(\widehat\Omega_i; T\widehat\Omega_i^* \odot T\widehat\Omega_i^*)$, with the $C^{m_i}$ topology, and its limit, $\hat g_{i,\infty}$, is positive definite at every point.

Clearly, Claim 2 follows by showing that the restrictions of the metrics $g_{ij} := \psi_{ij}^* g_j$ to $\Omega_i$, for $j \geq i$, form a convergent sequence in $C^{m_i}(\Omega_i; T\Omega_i^* \odot T\Omega_i^*)$, and its limit, $g_{i,\infty}$, is positive definite at every point. To begin with, let us show that $g_{ij}|_{\Omega_i}$ is a Cauchy sequence with respect to $\|\ \|_{C^{m_i},\Omega_i,g_i}$.

We have

$$\frac{1}{\bar\lambda_i}\,|\xi|_i^{(m_i)} \leq |\xi|_{ij}^{(m_i)} \leq \bar\lambda_i\,|\xi|_i^{(m_i)} \tag{9}$$

for all $\xi \in T\Omega_i^{(m_i)}$, where $|\ |_i^{(m_i)}$ and $|\ |_{ij}^{(m_i)}$ are the norms defined by $g_i^{(m_i)}$ and $g_{ij}^{(m_i)}$, respectively. By Lemma 6.1-(i), it follows that

$$\|g_i^{(m_i)} - g_{ij}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},g_i^{(m_i)}} \leq \bar\lambda_i^2 - \bar\lambda_i^{-2}\ .$$

Then, for $k \geq j$,

$$\begin{aligned} \|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},g_{ij}^{(m_i)}} &= \|g_j^{(m_i)} - g_{jk}^{(m_i)}\|_{C^0,\psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)}),g_j^{(m_i)}} \\ &\leq \|g_j^{(m_j)} - g_{jk}^{(m_j)}\|_{C^0,\Omega_j^{(m_j)},g_j^{(m_j)}} \leq \bar\lambda_j^2 - \bar\lambda_j^{-2} \end{aligned} \tag{10}$$

because

$$\psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)}) \subset \psi_{ij*}^{(m_i)}(B_i''^{(m_i)}) \subset B_j'^{(m_j)} \subset \Omega_j^{(m_j)}$$

and $g_{jk}^{(m_j)} = g_{jk}^{(m_i)}$ on $\Omega_j^{(m_j)} \cap B_j^{(m_i)} \supset \psi_{ij*}^{(m_i)}(\Omega_i^{(m_i)})$ (Remark 1-(i)). We get

$$\|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},g_i^{(m_i)}} \le \bar\lambda_i^2(\bar\lambda_j^2 - \bar\lambda_j^{-2}) \tag{11}$$

by (9), (10) and Lemma 6.1-(iii).

Let $\mathcal{U}_i$ be a finite collection of charts of $M_i$ with domains $U_{i,a}$, and let $\mathcal{K}_i = \{K_{i,a}\}$ be a family of compact subsets of $M_i$, with the same index set as $\mathcal{U}_i$, such that $K_{i,a} \subset U_{i,a}$ for all $a$, and $\overline{B}_i'' \subset \bigcup_a K_{i,a} =: K_i$. Thus $\Omega_i \subset K_i$. With the notation of Section 2.2, let $\mathcal{U}_i^{(m_i)}$ be the family of induced charts of $T^{(m_i)}M_i$ with domains $U_{i,a}^{(m_i)}$. Like in the proof of Proposition 6.4-(ii), let $\mathcal{K}_i^{(m_i)}$ be the family of compact subsets

$$K_{i,a}^{(m_i)} = \{\, \xi \in B_i^{(m_i)} \mid \pi(\xi) \in K_{i,a},\ d_i^{(m_i)}(\xi, \pi_i(\xi)) \le R_i''' \,\} \subset U_{i,a}^{(m_i)} ,$$

for some $R_i''' > R_i''$, where $\pi : B_i^{(m_i)} \to B_i$. We have $B_i''^{(m_i)} \subset \bigcup_a K_{i,a}^{(m_i)} =: K_i^{(m_i)}$. Hence $\Omega_i^{(m_i)} \subset K_i^{(m_i)}$.

Choose some $C_i \ge 1$ satisfying (3) with $\mathcal{U}_i$, $\mathcal{K}_i$, $\Omega_i$ and $g_i$, and some $C_i^{(m_i)} \ge 1$ satisfying (3) with $\mathcal{U}_i^{(m_i)}$, $\mathcal{K}_i^{(m_i)}$, $\Omega_i^{(m_i)}$ and $g^{(m_i)}$. For any $\rho > 0$ and $n+1 \le \mu \le 2^{m_i}n$, let $\sigma_{i,a,\rho,\mu}^{(m_i)} : U_{i,a} \to U_{i,a}^{(m_i)}$ be the section of each projection $\pi : U_{i,a}^{(m_i)} \to U_{i,a}$ of the type used in Lemma 2.1-(ii). Since $\Omega_i \subset \operatorname{Int}(\Omega_i^{(m_i)})$, there is some $\rho > 0$ so that $\sigma_{i,a,\rho,\mu}^{(m_i)}(K_{i,a} \cap \Omega_i) \subset K_{i,a}^{(m_i)} \cap \Omega_i^{(m_i)}$ for all $a$ and $\mu$. Thus, by Lemma 2.1-(ii), given any $\varepsilon > 0$, there is some $\delta > 0$, depending on $\varepsilon$ and $\rho$, such that

$$\|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},\mathcal{U}_i^{(m_i)},\mathcal{K}_i^{(m_i)}} < \delta \Longrightarrow \|g_{ij} - g_{ik}\|_{C^{m_i},\Omega_i,\mathcal{U}_i,\mathcal{K}_i} < \varepsilon/C_i . \tag{12}$$

Since $\bar\lambda_j \downarrow 1$, we have $\bar\lambda_i^2(\bar\lambda_j^2 - \bar\lambda_j^{-2}) < \delta/C_i^{(m_i)}$ for $j$ large enough, giving

$$\begin{aligned}\|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},g_i^{(m_i)}} < \delta/C_i^{(m_i)} &\Longrightarrow \|g_{ij}^{(m_i)} - g_{ik}^{(m_i)}\|_{C^0,\Omega_i^{(m_i)},\mathcal{U}_i^{(m_i)},\mathcal{K}_i^{(m_i)}} < \delta \\ &\Longrightarrow \|g_{ij} - g_{ik}\|_{C^{m_i},\Omega_i,\mathcal{U}_i,\mathcal{K}_i} < \varepsilon/C_i \Longrightarrow \|g_{ij} - g_{ik}\|_{C^{m_i},\Omega_i,g_i} < \varepsilon\end{aligned}$$

by (11), (12) and (3). This shows that $g_{ij}|_{\Omega_i}$ is a Cauchy sequence in the Banach space $C^{m_i}(\Omega_i; T\Omega_i^* \odot T\Omega_i^*)$ with $\|\ \|_{C^{m_i},\Omega_i,g_i}$, and therefore it has a limit $g_{i,\infty}$. For all nonzero $\xi \in T\Omega_i$, we have

$$g_{i,\infty}(\xi,\xi) = \lim_j g_{ij}(\xi,\xi) \ge \frac{1}{\bar\lambda_i}\, g_i(\xi,\xi) > 0 ,$$

obtaining that $g_{i,\infty}$ is positive definite. This completes the proof of Claim 2.

According to Claim 2, each $\hat g_{i,\infty}$ is a $C^{m_i}$ Riemannian metric on $\widehat\Omega_i$, and, obviously, $\hat g_{j,\infty}|_{\widehat\Omega_i} = \hat g_{i,\infty}$ for $j > i$. Hence the metric tensors $\hat g_{i,\infty}$ can be combined to define a $C^\infty$ Riemannian metric $\hat g$ on $\widehat M$ by Claim 1.

Let $|\ |_{i,\infty}^{(m_i)}$ be the norm defined by $g_{i,\infty}^{(m_i)}$ on $T\Omega_i^{(m_i)}$. By (9) and because $|\ |_{i,\infty}^{(m_i)} = \lim_j |\ |_{ij}^{(m_i)}$ on $T\Omega_i^{(m_i)}$, we get $\frac{1}{\bar\lambda_i}\, |\xi|_i^{(m_i)} \le |\xi|_{i,\infty}^{(m_i)} \le \bar\lambda_i\, |\xi|_i^{(m_i)}$ for all $\xi \in T\Omega_i^{(m_i)}$. Thus, by Remark 2-(iii), $\Omega_i$ contains the $g_{i,\infty}$-ball of center $x_i$ and radius $R_i'/\bar\lambda_i$ because it contains $B_i'$; in particular, $\widehat M$ is complete because $R_i'/\bar\lambda_i \to \infty$ and every $\Omega_i$ is compact. Since $g_{i,\infty} = \psi_i^*\hat g$, it also follows that $\psi_{i*}^{(m_i)} : \Omega_i^{(m_i)} \to T^{(m_i)}\widehat M$ is a $\bar\lambda_i$-quasi-isometry. So $\psi_i : (M_i, x_i) \rightarrowtail (\widehat M, \hat x)$ is an $(m_i, R_i', \bar\lambda_i)$-pointed local quasi-isometry, obtaining that $([M_i, x_i], [\widehat M, \hat x]) \in U_{R_i',s_i}^{m_i}$ for any sequence $s_i \downarrow 0$ with $\bar\lambda_i < e^{s_i}$, and therefore $[M_i, x_i] \to [\widehat M, \hat x]$ as $i \to \infty$ in $\mathcal{M}_*^\infty(n)$. □

**Corollary 7.3.** *$\mathcal{M}_*^\infty(n)$ is Polish.*

*Proof.* This is the content of Propositions 7.1 and 7.2 together. □

Corollaries 6.6 and 7.3 give Theorem 1.2.

## 8. Some basic properties of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$

For each closed $C^\infty$ manifold $M$ of dimension $\ge 2$, the non-periodic metrics on $M$ form a residual subset of $\operatorname{Met}(M)$ with the $C^\infty$ topology [3, Corollary 3.5], [38, Proposition 1]. Then, since $\mathcal{M}_{*,\mathrm{c}}^\infty(n)$ is dense in $\mathcal{M}_*^\infty(n)$ (Remark 8), it follows that $\mathcal{M}_{*,\mathrm{np}}^\infty(n)$ is dense in $\mathcal{M}_*^\infty(n)$, and therefore $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ is dense in $\mathcal{M}_*^\infty(n)$ too. On the other hand, $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ is $G_\delta$ in $\mathcal{M}_*^\infty(n)$ by Lemmas 8.1 and 8.3 below, and therefore it is a Polish subspace [25, Theorem I.3.11]. This proves Theorem 1.3-(i).

**Lemma 8.1.** *For every $n \in \mathbb{Z}^+$ and $[M,x] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, there is some $r>0$ such that, if*

$$\{\, h \in \operatorname{Iso}(M) \mid h(x) \in \overline{B}(x,r) \,\} = \{\mathrm{id}_M\}\ ,$$

*then there is some neighborhood $\mathcal{L}$ of $[M,x]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ so that*

$$\{\, h \in \operatorname{Iso}(L) \mid h(y) \in \overline{B}(y,r) \,\} = \{\mathrm{id}_L\}$$

*for all $[L,y] \in \mathcal{L}$.*

*Proof.* Suppose that the statement is false. Then there is some convergent sequence, $[M_i,x_i] \to [M,x]$, in $\mathcal{M}^\infty_*(n)$ so that, for each $i$, some $h_i \in \operatorname{Iso}(M_i) \smallsetminus \{\mathrm{id}_{M_i}\}$ satisfies $h_i(x_i) \in \overline{B}_i(x_i,r)$. Choose any sequence of compact domains $\Omega_q$ of $M$ such that $\overline{B}(x,2r) \subset \operatorname{Int}(\Omega_q)$ and $d(x,\partial\Omega_q) \to \infty$ as $q \to \infty$. For each $q$ and $i$ large enough, there is some pointed smooth embedding $\phi_{q,i} : (\Omega_q,x) \to (M_i,x_i)$ so that $\phi_{q,i}^* g_i \to g|_{\Omega_q}$ as $i \to \infty$ with respect to the $C^\infty$ topology. Thus $\overline{B}_i(x_i,2r) \subset \phi_{q,i}(\operatorname{Int}(\Omega_q))$ for $i$ large enough.

*Claim* 3. If $r$ is small enough, we can assume that there is some $\delta > 0$ such that, for $i$ large enough, the maps $h_i$ can be chosen so that $d_i(z_i,h_i(z_i)) \ge \delta$ for some $z_i \in B_i(x_i,r)$.

Given any index $i$, suppose first that there is some $k \in \mathbb{Z} \smallsetminus \{0\}$ such that $h_i^k(x_i) \notin \overline{B}_i(x_i,r/2)$. Then there is some $k \in \mathbb{Z} \smallsetminus \{0\}$ such that $h_i^k(x_i) \notin \overline{B}_i(x_i,r/2)$ and $h_i^\ell(x_i) \in \overline{B}_i(x_i,r/2)$ if $|\ell| < |k|$. If $k=1$, then $d_i(x_i,h_i(x_i)) \ge r/2$. If $k=-1$, then

$$d_i(x_i,h_i(x_i)) = d_i\left(h_i^{-1}(x_i),x_i\right) \ge r/2$$

as well. If $|k| \ge 2$, then there is some $\ell \in \mathbb{Z}$ such that $|\ell|,|k-\ell| < |k|$. Hence

$$d_i\left(x_i,h_i^k(x_i)\right) \le d_i\left(x_i,h_i^\ell(x_i)\right) + d_i\left(h_i^\ell(x_i),h_i^k(x_i)\right) = d_i\left(x_i,h_i^\ell(x_i)\right) + d_i\left(x_i,h_i^{k-\ell}(x_i)\right) \le r\ .$$

Therefore, by using $h_i^k$ instead of $h_i$, we can assume that $d_i(x_i,h_i(x_i)) \ge r/2$ in this case.

Now, suppose that $h_i^k(x_i) \in \overline{B}_i(x_i,r/2)$ for all $k \in \mathbb{Z}$. Consider the non-trivial abelian subgroup $A_i = \overline{\{\, h_i^k \mid k \in \mathbb{Z} \,\}} \subset \operatorname{Iso}(M_i)$. Since $a(x_i) \in \overline{B}_i(x_i,r/2)$ for any $a \in A_i$, it follows that $A_i$ is compact in the $C^\infty$ topology by Proposition 3.11, and thus $A_i$ is a non-trivial compact abelian Lie subgroup of $\operatorname{Iso}(M_i)$. Let $\mu_i$ be a bi-invariant probability measure on $A_i$, and let $f_i : A_i \to M$ be the mass distribution defined by $f_i(a) = a(x_i)$. By the $C^\infty$ convergence $\phi_{q,i}^* g_i \to g|_{\Omega_q}$, we can suppose that $r$ is so small that the ball $B_i(x_i,2r/3)$ of $M_i$ satisfies the conditions of Proposition 10.2 for $i$ large enough. Then, since $f_i(A_i) \subset \overline{B}_i(x_i,r/2) \subset B_i(x_i,2r/3)$, the center of mass $y_i = \mathcal{C}_{f_i}$ is defined in $B_i(x_i,2r/3)$. Moreover $y_i$ is a fixed point of the canonical action of $A_i$ on $M$ [24, Section 2.1]. Since there is a neighborhood of the identity in the orthogonal group $\mathrm{O}(n)$ which contains no non-trivial subgroup (simply because $\mathrm{O}(n)$ is a Lie group), it follows that there is some $K>0$ such that, for any non-trivial subgroup $A \subset \mathrm{O}(n)$, there is some $a \in A$ and some $v \in \mathbb{R}^n$ such that $|v| = 1$ and $|a(v) - v| \ge K$. In our setting, the subgroup $\{\, a_{*y_i} \mid a \in A_i \,\}$ of the orthogonal group $\mathrm{O}(T_{y_i}M_i) \equiv \mathrm{O}(n)$ is non-trivial because $M_i$ is connected and $A_i$ is non-trivial. Hence there is some $a_i \in A_i$ and some $\xi_i \in T_{y_i}M_i$ such that $|\xi_i| = 1$ and $|a_{i*}(\xi_i) - \xi_i| \ge K$. By the $C^\infty$ convergence $\phi_{q,i}^* g_i \to g|_{\Omega_q}$, we can also assume that $r$ is so small that there exists some $C \ge 1$ such that $\exp_{y_i} : B(0_{y_i},r) \to B(y_i,r)$ is $C$-quasi-isometric for $i$ large enough. Then, for $z_i = \exp_{y_i}(\frac{r}{3}\,\xi_i) \in \overline{B}_i(y_i,r/3) \subset B_i(x_i,r)$, we get

$$d_i(z_i,a_i(z_i)) \ge \frac{r}{3C}\,|\xi_i - h'_{i*}(\xi_i)| \ge \frac{rK}{3C}\ .$$

Thus, by using $a_i$ instead of $h_i$, we can assume in this case that $d_i(z_i,h_i(z_i)) \ge rK/3C$. Therefore Claim 3 follows with $\delta = \min\{r/2, rK/3C\}$.

For each $q$, we can assume that

$$\overline{B}(x,\operatorname{diam}(\Omega_q)+r) \subset \operatorname{Int}(\Omega_{q+1})\ ,$$

obtaining

$$\overline{B}_i(x_i,\operatorname{diam}(\phi_{q,i}(\Omega_q))+r) \subset \operatorname{Int}(\phi_{q+1,i}(\Omega_{q+1}))$$

for all $i$ large enough by the $C^\infty$ convergence $\phi_{q,i}^* g_i \to g|_{\Omega_q}$. Then $h'_{q,i} := \phi_{q+1,i}^{-1}\, h_i\, \phi_{q,i} : \Omega_q \to M$ is well defined for each $q$ and all $i$ large enough because $x_i \in \phi_{q,i}(\Omega_q)$ and $h_i(x_i) \in \overline{B}_i(x_i,r)$. On the one hand, from the $C^\infty$ convergence $\phi_{q,i}^* g_i \to g|_{\Omega_q}$ and since $h_i(x_i) \in \overline{B}_i(x_i,r)$, we get the $C^\infty$ convergence $h'^*_{q,i} g \to g|_{\Omega_q}$ and $\limsup_i d(x,h'_{q,i}(x)) \le r$; in particular, for each $q$, the maps $h'_{q,i}$ are equi-quasi-isometries of order $\infty$.

Therefore, by Proposition 3.11, some subsequence of $h'_{q,i}$ is $C^\infty$ convergent to some $C^\infty$ map $h'_q : \Omega_q \to M$, which is an isometric embedding satisfying $h'_q(x) \in \overline{B}(x,r)$.

For all $p \geq q$, the restrictions $h'_p|_{\Omega_q}$ form a sequence of isometric embeddings satisfying $h'_p(x) \in \overline{B}(x,r)$. Then, by Proposition 3.11, there is some sequence of positive integers $p(q,k)$ for each $q$ so that the subsequence $h'_{p(q,k)}|_{\Omega_q}$ of $h'_p|_{\Omega_q}$ is $C^\infty$ convergent as $k \to \infty$ to an isometric embedding $h''_q : \Omega_q \to M$ satisfying $h''_q(x) \in \overline{B}(x,r)$. We can assume that $p(q+1,k)$ is a subsequence of $p(q,k)$ for each $q$, yielding $h''_{q+1}|_{\Omega_q} = h''_q$. So the maps $h''_q$ can be combined to define an isometry $h : M \to M$ satisfying $h(x) \in \overline{B}(x,r)$.

Now, fix any $q$ and let $z'_{p,i} = \phi_{p,i}^{-1}(z_i)$ for each $p \geq q$ and all $i$ large enough. From $z_i \in B_i(x_i,r)$ and the $C^\infty$ convergence $\phi_{p,i}^* g_i \to g|_{\Omega_p}$, it follows that $z'_{p,i}$ approaches the compact set $\overline{B}(x,r)$ as $i \to \infty$. Then, for each $p \geq q$, there is a sequence $z_{p,i}$ in $B(x,r)$ so that $d(z_{p,i}, z'_{p,i}) \to 0$. Hence, by the $C^\infty$ convergence $\phi_{p,i}^* g_i \to g|_{\Omega_p}$ and Claim 3, we get

$$\begin{aligned}
\sup\{\, d(z,h(z)) \mid z \in B(x,r) \,\} &= \sup\{\, d(z,h''_q(z)) \mid z \in B(x,r) \,\} \\
&\geq \sup\left\{ \liminf_p d(z,h'_p(z)) \mid z \in B(x,r) \right\} \geq \sup\left\{ \liminf_p \liminf_i d(z,h'_{p,i}(z)) \mid z \in B(x,r) \right\} \\
&\geq \liminf_p \liminf_i d(z_{p,i},h'_{p,i}(z_{p,i})) = \liminf_p \liminf_i d(z'_{p,i},h'_{p,i}(z'_{p,i})) \geq \liminf_i d_i(z_i,h_i(z_i)) \geq \delta \;.
\end{aligned}$$

So $h \neq \mathrm{id}_M$, which is a contradiction because $h(x) \in \overline{B}(x,r)$. □

**Lemma 8.2.** *For $n \geq 2$ and each point $[M,x] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, there is some $r > 0$ such that, for each $\varepsilon \in (0,r)$, there is some neighborhood $\mathcal{N}$ of $[M,x]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ so that, if an equivalence class $\iota(L)$ of $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ meets $\mathcal{N}$ at points $[L,y]$ and $[L,z]$, then either $d_L(y,z) < \varepsilon$ or $d_L(y,z) > r$.*

*Proof.* Since $M$ is locally non-periodic, there is some $r > 0$ such that

$$\{\, h \in \mathrm{Iso}(M) \mid d(x,h(x)) \leq r \,\} = \{\mathrm{id}_M\} \;. \tag{13}$$

Suppose that the statement is false for this $r$. Then, given any $\varepsilon \in (0,r)$, there are sequences $[L_i,y_i]$ and $[L_i,z_i]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ converging to $[M,x]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ such that $\varepsilon \leq d_i(y_i,z_i) \leq r$ for all $i$.

Take a sequence of compact domains $\Omega_q$ of $M$ such that $x \in \Omega_q$ and $d(x,\partial\Omega_q) \to \infty$ as $q \to \infty$. For each $q$, there are $C^\infty$ embeddings $\phi_{q,i} : \Omega_q \to M_i$ and $\psi_{q,i} : \Omega_q \to M_i$ for $i$ large enough so that $\phi_{q,i}(x) = y_i$, $\psi_{q,i}(x) = z_i$, and $\phi_{q,i}^* g_i, \psi_{q,i}^* g_i \to g|_{\Omega_q}$ as $i \to \infty$ with respect to the $C^\infty$ topology. We can also assume that, for each $q$,

$$\overline{B}(x, \mathrm{diam}(\Omega_q) + r) \subset \mathrm{Int}(\Omega_{q+1}) \;,$$

giving

$$\phi_{q,i}(\Omega_q) \subset \overline{B}_i(y_i, \mathrm{diam}(\phi_{q,i}(\Omega_q))) \subset \overline{B}_i(z_i, \mathrm{diam}(\phi_{q,i}(\Omega_q)) + r) \subset \mathrm{Int}(\psi_{q+1,i}(\Omega_{q+1}))$$

for $i$ large enough by the $C^\infty$ convergence $\phi_{q,i}^* g_i, \psi_{q,i}^* g_i \to g|_{\Omega_q}$ and since $d_i(y_i,z_i) \leq r$. So $h_{q,i} := \psi_{q+1,i}^{-1}\, \phi_q : \Omega_q \to M$ is well defined for each $q$ and all $i$ large enough. From the $C^\infty$ convergence $\phi_{q,i}^* g_i, \psi_{q,i}^* g_i \to g|_{\Omega_q}$, we also get the $C^\infty$ convergence $h_{q,i}^* g \to g|_{\Omega_q}$, and moreover

$$\liminf_i d(x,h_{q,i}(x)) \geq \varepsilon \;, \qquad \limsup_i d(x,h_{q,i}(x)) \leq r \;,$$

because $\phi_{q,i}(x) = y_i$, $\psi_{q,i}(x) = z_i$ and $\varepsilon \leq d_i(y_i,z_i) \leq r$. Then, like in the proof of Lemma 8.1, an isometry $h : M \to M$ can be constructed so that $\varepsilon \leq d(x,h(x)) \leq r$, which contradicts (13). □

**Lemma 8.3.** *Let $n \in \mathbb{N}$ and $r > 0$. For any convergent sequence $[M_i,x_i] \to [M,x]$ in $\mathcal{M}^\infty_*(n)$ and each $y \in B(x,r)$, there are points $y_i \in B_i(x_i,r)$ such that $[M_i,y_i] \to [M,y]$ in $\mathcal{M}^\infty_*(n)$.*

*Proof.* Take a sequence of compact domains $\Omega_q$ of $M$ such that $x,y \in \Omega_q$ and $d(x,\partial\Omega_q) \to \infty$ as $q \to \infty$. For each $q$, there is some index $i_q$ such that, for each $i \geq i_q$ there is a $C^\infty$ embedding $\phi_{q,i} : \Omega_q \to M_i$ satisfying $\phi_{q,i}(x) = x_i$ and $\phi_{q,i}^* g_i \to g|_{\Omega_q}$ as $i \to \infty$ with respect to the $C^\infty$ topology. Let $y_{q,i} = \phi_{q,i}(y)$ for all $i \geq i_q$. Then, for each $q$ and every $m \in \mathbb{Z}^+$, there is some index $i_{q,m} \geq i_q$ such that $d_i(x_i,y_{q,i}) < r$ and $\|\phi_{q,i}^* g_i - g\|_{C^m,\Omega_q,g} < 1/m$ for all $i \geq i_{q,m}$. Moreover we can assume that $i_{q,q} < i_{q+1,q+1}$ for all $q$. Now, let $y_i$ be any point of $B_i(x_i,r)$ for $i < i_{0,0}$, and let $y_i = y_{q,i}$ for $i_{q,q} \leq i < i_{q+1,q+1}$. Let us check that $[M_i,y_i] \to [M,y]$ in $\mathcal{M}^\infty_*(n)$. Fix any compact domain $\Omega$ of $M$ containing $y$, and let $m \in \mathbb{N}$. We have

$d(y,\partial\Omega_q) \to \infty$ as $q \to \infty$ because $d(x,\partial\Omega_q) \to \infty$ and $d(x,y) < r$. So there is some $q_0 \ge m$ such that $\Omega \subset \Omega_q$ for all $q \ge q_0$. For $i \ge i_{q_0,q_0}$, let $\phi_i = \phi_{q,i}|_\Omega$ if $i_{q,q} \le i < i_{q+1,q+1}$ with $q \ge q_0$. Then $\phi_i(y) = y_i$ and

$$\|\phi_i^* g_i - g\|_{C^m,\Omega_q,g} \le \|\phi_{q,i}^* g_i - g\|_{C^q,\Omega_q,g} < \frac{1}{q}$$

for $i_{q,q} \le i < i_{q+1,q+1}$, obtaining $\phi_i^* g_i \to g|_\Omega$ as $i \to \infty$. □

**Lemma 8.4.** *For $n \in \mathbb{N}$, let $[M,x] \in \mathcal{M}_*^\infty(n)$, and let $\mathcal{N}$ be a neighborhood of $[M,x]$ in $\mathcal{M}_*^\infty(n)$. Then there is some $\delta > 0$ and some neighborhood $\mathcal{L}$ of $[M,x]$ in $\mathcal{M}_*^\infty(n)$ such that $[L,z] \in \mathcal{N}$ for all $[L,y] \in \mathcal{L}$ and all $z \in B_L(y,\delta)$.*

*Proof.* There are some $m \in \mathbb{Z}^+$ and $\varepsilon > 0$, and a compact domain $\Omega$ of $M$ containing $x$ such that, for all $[L,z] \in \mathcal{M}_*^\infty(n)$, if there is some $C^\infty$ embedding $\phi : \Omega \to L$ so that $\phi(x) = z$ and $\|\phi^* g_L - g_M\|_{C^m,\Omega,g_M} < \varepsilon$, then $[L,z] \in \mathcal{N}$. Take any compact domain $\Omega'$ of $M$ whose interior contains $\Omega$. There is some $\varepsilon_0 > 0$ and some neighborhood $\mathcal{H}$ of $\mathrm{id}_M$ in the group of diffeomorphisms of $M$ with the weak $C^m$ topology such that, for all $h \in \mathcal{H}$ and any metric tensor $g'$ on $\Omega'$ satisfying $\|g' - g_M\|_{C^m,\Omega',g_M} < \varepsilon_0$, we have $h(\Omega) \subset \Omega'$ and $\|h^* g' - g_M\|_{C^m,\Omega,g_M} < \varepsilon$. Moreover there is some $\delta' > 0$ such that, for each $z' \in B_M(x,\delta')$, there is some $h \in \mathcal{H}$ so that $h(x) = z'$. Let $\mathcal{L}$ be the neighborhood of $[M,x]$ in $\mathcal{M}_*^\infty(n)$ that consists of the points $[L,y] \in \mathcal{M}_*^\infty(n)$ such that there is some $C^\infty$ embedding $\psi : \Omega' \to L$ so that $\psi(x) = y$ and $\|\psi^* g_L - g_M\|_{C^m,\Omega',g_M} < \varepsilon_0$. There is some $\delta > 0$ such that $B_L(y,\delta) \subset \psi(\Omega')$ and $\psi^{-1}(B_L(y,\delta)) \subset B_M(x,\delta')$ for all $[L,y] \in \mathcal{L}$ and $\psi : \Omega' \to L$ as above. Hence $z' = \psi^{-1}(z) \in B_M(x,\delta')$ for each $z \in B_L(y,\delta)$, and therefore there is some $h \in \mathcal{H}$ such that $h(x) = z'$. Then $\phi := \psi h$ is defined on $\Omega$ and satisfies $\phi(x) = \psi(z') = z$. Moreover

$$\|\phi^* g_L - g_M\|_{C^m,\Omega,g_M} = \|h^* \psi^* g_L - g_M\|_{C^m,\Omega,g_M} < \varepsilon$$

because $\|\psi^* g_L - g_M\|_{C^m,\Omega',g_M} < \varepsilon_0$ and $h \in \mathcal{H}$. □

## 9. Canonical bundles over $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$

For each $n \in \mathbb{N}$, consider the set of pairs $(M,\xi)$, where $M$ is a complete connected Riemannian manifold without boundary of dimension $n$, and $\xi \in TM$. Like in the case of $\mathcal{M}_*(n)$, we can assume that the underlying set of each complete connected Riemannian $n$-manifold is contained in $\mathbb{R}$, obtaining that these pairs $(M,\xi)$ form a well defined set. Define an equivalence relation on this set by declaring that $(M,\xi)$ is equivalent to $(N,\zeta)$ if there is an isometric diffeomorphism $\phi : M \to N$ such that $\phi_*(\xi) = \zeta$. The class of a pair $(M,\xi)$ will be denoted by $[M,\xi]$, and the corresponding set of equivalence classes will be denoted by $\mathcal{T}_*(n)$. If orthonormal tangent frames are used instead of tangent vectors in the above definition, we get a set denoted by $\mathcal{Q}_*(n)$. Let $\pi_{\mathcal{T}_*(n)} : \mathcal{T}_*(n) \to \mathcal{M}_*(n)$ and $\pi_{\mathcal{Q}_*(n)} : \mathcal{Q}_*(n) \to \mathcal{M}_*(n)$ be the maps defined by $\pi([M,\xi]) = [M,\pi_M(\xi)]$ and $\pi([M,f]) = [M,\pi_M(f)]$ for $[M,\xi] \in \mathcal{T}_*(n)$ and $[M,f] \in \mathcal{Q}_*(n)$; the simpler notation $\pi$ will be used for $\pi_{\mathcal{T}_*(n)}$ and $\pi_{\mathcal{Q}_*(n)}$ if there is no danger of misunderstanding. For each $[M,x] \in \mathcal{M}_*(n)$, there are canonical surjections $T_xM \to \pi_{\mathcal{T}_*(n)}^{-1}([M,x])$, $\xi \mapsto [M,\xi]$, and $Q_xM \to \pi_{\mathcal{Q}_*(n)}^{-1}([M,x])$, $f \mapsto [M,f]$. Via the canonical surjection $Q_xM \to \pi_{\mathcal{Q}_*(n)}^{-1}([M,x])$, the canonical right action of $\mathrm{O}(n)$ on $Q_xM$ induces a right action on $\pi_{\mathcal{Q}_*(n)}^{-1}([M,x])$; in this way, we get a canonical action of $\mathrm{O}(n)$ on $\mathcal{Q}_*(n)$ whose orbits are the fibers of $\pi_{\mathcal{Q}_*(n)}$. The operation of multiplication by scalars on $T_xM$ also induces an action of $\mathbb{R}$ on $\pi_{\mathcal{T}_*(n)}^{-1}([M,x])$. However the sum operation of $T_xM$ may not induce an operation on $\pi_{\mathcal{T}_*(n)}^{-1}([M,x])$. The following definition is analogous to Definition 1.1.

**Definition 9.1.** For each $m \in \mathbb{N}$, a sequence $[M_i,\xi_i] \in \mathcal{T}_*(n)$ (respectively, $[M_i,f_i] \in \mathcal{Q}_*(n)$) is said to be $C^m$ *convergent* to $[M,\xi] \in \mathcal{T}_*(n)$ (respectively, $[M,f] \in \mathcal{Q}_*(n)$) if, with the notation $x = \pi(\xi)$ and $x_i = \pi_i(x_i)$ (respectively, $x = \pi(f)$ and $x_i = \pi_i(f_i)$), for each compact domain $\Omega \subset M$ containing $x$, there are pointed $C^{m+1}$ embeddings $\phi_i : (\Omega,x) \to (M_i,x_i)$ for large enough $i$ such that $\phi_{i*}(\xi) = \xi_i$ (respectively, $\phi_{i*}(f) = f_i$), and $\phi_i^* g_i \to g|_\Omega$ as $i \to \infty$ with respect to the $C^m$ topology. If $[M_i,\xi_i]$ (respectively, $[M_i,f_i]$) is $C^m$ convergent to $[M,\xi]$ (respectively, $[M,f]$) for all $m$, then it is said that $[M_i,\xi_i]$ (respectively, $[M_i,f_i]$) is $C^\infty$ *convergent* to $[M,\xi]$ (respectively, $[M,f]$).

**Theorem 9.2.** *The $C^\infty$ convergence in $\mathcal{T}_*(n)$ and $\mathcal{Q}_*(n)$ describes a Polish topology.*

To prove Theorem 9.2, we follow the steps of Sections 5–7.

**Definition 9.3.** For $m \in \mathbb{N}$ and $R, r > 0$, let $V^m_{R,r}$ (respectively, $W^m_{R,r}$) be the set of pairs $([M,\xi],[N,\zeta]) \in \mathcal{T}_*(n) \times \mathcal{T}_*(n)$ (respectively, $([M,f],[N,h]) \in \mathcal{Q}_*(n) \times \mathcal{Q}_*(n)$) such that there is some $(m,R,\lambda)$-pointed local quasi-isometry $\phi : (M,x) \rightarrowtail (N,y)$ for some $\lambda \in [1,e^r)$ so that $\phi_*(\xi) = \zeta$ (respectively, $\phi_*(f) = h$).

The following proposition is proved like Proposition 5.2.

**Proposition 9.4.** *The following properties hold for all $m, m' \in \mathbb{N}$ and $R, S, r, s > 0$:*

*(i)* $(V^m_{e^r R,r})^{-1} \subset V^m_{R,r}$ *and* $(W^m_{e^r R,r})^{-1} \subset W^m_{R,r}$.

*(ii)* $V^{m_0}_{R_0,r_0} \subset V^m_{R,r} \cap V^{m'}_{S,s}$ *and* $W^{m_0}_{R_0,r_0} \subset W^m_{R,r} \cap W^{m'}_{S,s}$*, where* $m_0 = \max\{m,m'\}$*,* $R_0 = \max\{R,S\}$ *and* $r_0 = \min\{r,s\}$.

*(iii)* $\Delta \subset V^m_{R,r}$ *and* $\Delta \subset W^m_{R,r}$.

*(iv)* $V^m_{e^{r+s}R,r} \circ V^m_{e^{r+s}R,s} \subset V^m_{R,r+s}$ *and* $W^m_{e^{r+s}R,r} \circ W^m_{e^{r+s}R,s} \subset W^m_{R,r+s}$.

**Proposition 9.5.** $\bigcap_{R,r>0} V^m_{R,r} = \Delta$ *and* $\bigcap_{R,r>0} W^m_{R,r} = \Delta$ *for all* $m \in \mathbb{N}$.

*Proof.* We only prove the first equality because the proof of the second one is analogous. The inclusion "$\supset$" is obvious; thus let us prove "$\subset$". Let $([M,\xi],[N,\zeta]) \in \bigcap_{R,r>0} V^m_{R,r}$, and let $x = \pi_M(\xi)$ and $y = \pi_N(\zeta)$. Then there is a sequence of pointed local quasi-isometries $\phi_i : (M,x) \rightarrowtail (N,y)$, with corresponding types $(m,R_i,\lambda_i)$, such that $\phi_{i*}(\xi) = \zeta$, and $R_i \uparrow \infty$ and $\lambda_i \downarrow 1$ as $i \to \infty$. According to the proof of Proposition 5.3, there is a pointed isometric immersion $\psi : (M,x) \to (N,y)$ so that, for any $i$, the restriction $\psi : B_M(x,R_i) \to N$ is the limit of the restrictions of a subsequence $\phi_{k(i,l)}$ in the weak $C^m$ topology. Hence $\psi_*(\xi) = \lim_l \phi_{k(i,l)*}(\xi) = \zeta$, obtaining $[M,\xi] = [N,\zeta]$. □

By Propositions 9.4 and 9.5, the sets $V^m_{R,r}$ (respectively, $W^m_{R,r}$) form a base of entourages of a Hausdorff uniformity on $\mathcal{T}_*(n)$ (respectively, $\mathcal{Q}_*(n)$), which is also called the $C^\infty$ *uniformity*. The corresponding topology is also called the $C^\infty$ *topology*, and the corresponding space is denoted by $\mathcal{T}^\infty_*(n)$ (respectively, $\mathcal{Q}^\infty_*(n)$).

*Remark* 9. (i) The maps $\pi : \mathcal{T}^\infty_*(n) \to \mathcal{M}^\infty_*(n)$ and $\pi : \mathcal{Q}^\infty_*(n) \to \mathcal{M}^\infty_*(n)$ are uniformly continuous and open because $(\pi\times\pi)(V^m_{R,r}) = (\pi\times\pi)(W^m_{R,r}) = U^m_{R,r}$ for all $m \in \mathbb{N}$ and $R, r > 0$.

(ii) The canonical right $\mathrm{O}(n)$-action on $\mathcal{Q}^\infty_*(n)$ is continuous. This follows easily by using that the composite of maps is continuous in the weak $C^\infty$ topology [22, p. 64, Exercise 10], and the following property that can be easily verified: for each $[M,f] \in \mathcal{Q}^\infty_*(n)$ and any neighborhood $\mathcal{N}$ of $\mathrm{id}_M$ in the space of $C^\infty$ diffeomorphisms of $M$ with the weak $C^\infty$ topology, there is a neighborhood $O$ of the identity element $e$ in $\mathrm{O}(n)$ such that, for all $a \in O$, there is some $\phi \in \mathcal{N}$ so that $\phi(x) = x$ and $\phi_*(f) = h$.

**Definition 9.6.** For $R, r > 0$ and $m \in \mathbb{N}$, let $E^m_{R,r}$ (respectively, $F^m_{R,r}$) be the set of pairs $([M,\xi],[N,\zeta]) \in \mathcal{T}_*(n)\times\mathcal{T}_*(n)$ (respectively, $([M,f],[N,h]) \in \mathcal{Q}_*(n)\times\mathcal{Q}_*(n)$) such that, with the notation $x = \pi_M(\xi)$ and $y = \pi_N(\zeta)$, there is some $C^{m+1}$ pointed local diffeomorphism $\phi\colon (M,x) \rightarrowtail (N,y)$ so that $\phi_*(\xi) = \zeta$ (respectively, $\phi_*(f) = h$), and $\|g_M - \phi^* g_N\|_{C^m,\Omega,g_M} < r$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x,R) \subset \Omega$.

Like in the case of relations on $\mathcal{M}_*(n)$, for $V \subset \mathcal{T}_*(n)\times\mathcal{T}_*(n)$, $W \subset \mathcal{Q}_*(n)\times\mathcal{Q}_*(n)$, $[M,\xi] \in \mathcal{T}_*(n)$ and $[M,f] \in \mathcal{Q}_*(n)$, the simpler notation $V(M,\xi)$ and $W(M,f)$ is used instead of $V([M,\xi])$ and $W([M,f])$.

*Remark* 10. By (3), a sequence $[M_i,\xi_i] \in \mathcal{T}_*(n)$ (respectively, $[M_i,f_i] \in \mathcal{Q}_*(n)$) is $C^\infty$ convergent to $[M,\xi] \in \mathcal{T}_*(n)$ (respectively, $[M,f] \in \mathcal{Q}_*(n)$) if and only if it is eventually in $E^m_{R,r}(M,\xi)$ (respectively, $F^m_{R,r}(M,f)$) for arbitrary $m \in \mathbb{N}$ and $R, r > 0$.

**Proposition 9.7.** *(i) For $R, r > 0$, if $0 < \varepsilon \le \min\{1-e^{-2r}, e^{2r}-1\}$, then $E^0_{R,\varepsilon} \subset V^0_{R,r}$ and $F^0_{R,\varepsilon} \subset W^0_{R,r}$.*

*(ii) For all $m \in \mathbb{Z}^+$, $R, r > 0$ and $[M,\xi] \in \mathcal{T}_*(n)$ (respectively, $[M,f] \in \mathcal{P}_*(n)$), there is some $\varepsilon > 0$ such that $E^m_{R,\varepsilon}(M,\xi) \subset V^m_{R,r}(M,\xi)$ (respectively, $F^m_{R,\varepsilon}(M,\xi) \subset W^m_{R,r}(M,\xi)$).*

*Proof.* Let us show (i) for the case of $V^0_{R,r}$, the case of $W^0_{R,r}$ being analogous. Let $([M,\xi],[N,\zeta]) \in E^0_{R,\varepsilon}$, and let $x = \pi_M(\xi)$ and $y = \pi_N(\zeta)$. Then there is a $C^1$ pointed local diffeomorphism $\phi : (M,x) \rightarrowtail (N,y)$ such that $\phi_*(\xi) = \zeta$, and $\varepsilon_0 := \|g_M - \phi^* g_N\|_{C^0,\Omega,g_M} < \varepsilon$ for some compact domain $\Omega \subset \operatorname{dom}\phi$ with $B_M(x,R) \subset \Omega$. According to the proof of Proposition 6.4-(i), $\phi$ is a $(0,R,\lambda)$-pointed local quasi-isometry if $1 \le \lambda < e^r$ and $\varepsilon_0 \le \min\{1-\lambda^{-2}, \lambda^2 - 1\}$, obtaining that $([M,\xi],[N,\zeta]) \in V^0_{R,r}$.

As above, let us prove (ii) only for the case of $V^m_{R,r}(M,\xi)$. Take $m\in\mathbb{Z}^+$, $R,r>0$ and $[M,\xi],[N,\zeta]\in\mathcal{T}_*(n)$, and let $x=\pi_M(\xi)$ and $y=\pi_N(\zeta)$. According to the proof of Proposition 6.4-(ii), there is some $\varepsilon>0$ such that, for every $C^{m+1}$ pointed local diffeomorphism $\phi\colon(M,x)\rightarrowtail(N,y)$, if $\|g_M-\phi^*g_N\|_{C^m,\Omega,g_M}<\varepsilon$ for some compact domain $\Omega\subset\operatorname{dom}\phi\cap\operatorname{Int}(K)$ with $B_M(x,R)\subset\Omega$, then $\phi$ is an $(m,R,\lambda)$-pointed local quasi-isometry $(M,x)\rightarrowtail(N,y)$ for some $\lambda\in[1,e^r)$. Therefore $[N,\zeta]\in V^m_{R,r}(M,\xi)$ if $[N,\zeta]\in E^m_{R,\varepsilon}(M,\xi)$. □

**Proposition 9.8.** *(i) For all $R,r>0$, if $e^{2\varepsilon}-e^{-2\varepsilon}\le r$, then $V^0_{R,\varepsilon}\subset E^0_{R,r}$ and $W^0_{R,\varepsilon}\subset F^0_{R,r}$.*

*(ii) For all $m\in\mathbb{Z}^+$, $R,r>0$ and $[M,\xi]\in\mathcal{T}_*(n)$ (respectively, $[M,f]\in\mathcal{Q}_*(n)$), there is some $\varepsilon>0$ such that $V^m_{R,\varepsilon}(M,\xi)\subset E^m_{R,r}(M,\xi)$ (respectively, $W^m_{R,\varepsilon}(M,f)\subset F^m_{R,r}(Mf)$).*

*Proof.* This result follows from the proof of Proposition 6.5 in the same way as Proposition 9.7 follows from Proposition 6.4. □

As a direct consequence of Remark 10, and Propositions 9.7 and 9.8, we get that the $C^\infty$ convergence in $\mathcal{T}_*(n)$ and $\mathcal{Q}_*(n)$ describes the $C^\infty$ topology.

**Proposition 9.9.** *$\mathcal{T}^\infty_*(n)$ and $\mathcal{Q}^\infty_*(n)$ are separable*

*Proof.* With the notation of Proposition 7.1, for every $M\in\mathcal{C}$, let $\mathcal{D}'_M$ and $\mathcal{D}''_M$ be countable dense subsets of $TM$ and $QM$, respectively. Then the countable sets

$$\{\,[(M,g),\xi]\mid M\in\mathcal{C},\ g\in\mathcal{G}_M,\ \xi\in\mathcal{D}'_M\,\}\quad\text{and}\quad\{\,[(M,g),f]\mid M\in\mathcal{C},\ g\in\mathcal{G}_M,\ f\in\mathcal{D}''_M\,\}$$

are dense in $\mathcal{T}^\infty_*(n)$ and $\mathcal{Q}^\infty_*(n)$, respectively. □

**Proposition 9.10.** *$\mathcal{T}^\infty_*(n)$ and $\mathcal{Q}^\infty_*(n)$ are completely metrizable*

*Proof.* Only the case of $\mathcal{T}^\infty_*(n)$ is proved, the other case being similar. The $C^\infty$ uniformity on $\mathcal{T}^\infty_*(n)$ is metrizable because it has a countable base of entourages. Thus it is enough to check that this uniformity is complete.

Consider an arbitrary Cauchy sequence $[M_i,\xi_i]$ in $\mathcal{T}_*(n)$ with respect to the $C^\infty$ uniformity, and let $x_i=\pi_i(\xi_i)\in M_i$. We have to prove that $[M_i,\xi_i]$ is convergent in $\mathcal{T}^\infty_*(n)$. By taking a subsequence if necessary, we can suppose that $([M_i,\xi_i],[M_{i+1},\xi_{i+1}])\in V^{m_i}_{R_i,r_i}$ for sequences $m_i$, and $R_i$ and $r_i$ satisfying the conditions of the proof of Proposition 7.2. Thus, for each $i$, there is some $\lambda_i\in(1,e^{r_i})$ and some $(m_i,R_i,\lambda_i)$-pointed local quasi-isometry $\phi_i\colon(M_i,x_i)\rightarrowtail(M_{i+1},x_{i+1})$, which can be assumed to be $C^\infty$ (Remark 6-(iii)), such that $\phi_{i*}(\xi_i)=\xi_{i+1}$. Then, with the notation of the proof of Proposition 7.2, we have $\psi_{ij*}(\xi_i)=\xi_j$ for $i<j$. Therefore there is some $\hat\xi\in T_{\hat x}\widehat{M}$ so that $\psi_{i*}(\xi_i)=\hat\xi$ for all $i$, obtaining that $([M_i,\xi_i],[\widehat{M},\hat\xi])\in U^{m_i}_{R'_i/\bar\lambda_i,s_i}$ for all $i$ according to the proof of Proposition 7.2. Hence $[M_i,\xi_i]\to[\widehat{M},\hat\xi]$ as $i\to\infty$ in $\mathcal{T}^\infty_*(n)$. □

Propositions 9.9 and 9.10 together mean that $\mathcal{T}^\infty_*(n)$ and $\mathcal{Q}^\infty_*(n)$ are Polish, completing the proof of Theorem 9.2.

Let $\mathcal{T}^\infty_{*,\mathrm{lnp}}(n)\subset\mathcal{T}^\infty_*(n)$ and $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)\subset\mathcal{Q}^\infty_*(n)$ be the subspaces defined by locally non-periodic manifolds.

**Proposition 9.11.** *(i) The projection $\pi:\mathcal{T}^\infty_{*,\mathrm{lnp}}(n)\to\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ admits the structure of a Riemannian vector bundle of rank $n$ so that the canonical map $T_xM\to\pi^{-1}([M,x])$ is a orthogonal isomorphism for each $[M,x]\in\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$.*

*(ii) The projection $\pi:\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)\to\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ admits the structure of a $\mathrm{O}(n)$-principal bundle canonically isomorphic to the $\mathrm{O}(n)$-principal bundle of orthonormal references of $\mathcal{T}^\infty_{*,\mathrm{lnp}}(n)$.*

*Proof.* Obviously, the canonical $\mathrm{O}(n)$-action on $\mathcal{Q}^\infty_*(n)$ preserves $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$, and the $\mathrm{O}(n)$-orbits in $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ are the fibers of $\pi:\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)\to\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$.

*Claim* 4. For all $[M,x]\in\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, the canonical maps $T_xM\to\pi^{-1}_{\mathcal{T}_*(n)}([M,x])$ and $Q_xM\to\pi^{-1}_{\mathcal{Q}_*(n)}([M,x])$ are bijections.

Let us show the case of the first map in Claim 4, the case of the second one being similar. It was already pointed out that the canonical map $T_xM\to\pi^{-1}_{\mathcal{T}_*(n)}([M,x])$ is surjective, and let us to prove that it is also injective. If $[M,\xi]=[M,\zeta]$ for some $\xi,\zeta\in T_xM$, then $\phi_*(\xi)=\zeta$ for some $\phi\in\operatorname{Iso}(M)$ with $\phi(x)=x$. But $\phi=\mathrm{id}_M$ because $M$ is locally non-periodic, obtaining $\xi=\zeta$.

Let $X$ be a completely regular space with a right action of a Lie group $G$, and let $G_x\subset G$ denote the isotropy subgroup at some point $x\in X$. Recall that a *slice* at $x$ is a subspace $S\subset X$ containing $x$ such that $S\cdot G$ is open in $X$, and there is a $G$-equivariant continuous map $\kappa:S\cdot G\to G_x\backslash G$ with $\kappa^{-1}(G_x)=S$ [31, Definition 2.1.1]. Since $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ is completely regular and $\mathrm{O}(n)$ is compact, the $\mathrm{O}(n)$-action on $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ has a slice $\mathcal{S}$ at each point $[M,f]\in\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ [31, Theorem 2.3.3] (see also [23], [34, Theorems 5.1 and 5.2] and [5, Theorems 11.3.9 and 11.3.14]). Then $\Theta:=\pi(\mathcal{S})=\pi(\mathcal{S}\cdot\mathrm{O}(n))$ is open in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ by Remark 9-(i).

*Claim* 5. $\pi:\mathcal{S}\to\Theta$ is a homeomorphism.

This is the restriction of a continuous map (Remark 9-(i)), and therefore it is continuous. This map is also open because, for every open $W\subset\mathcal{S}$, the set $W\cdot\mathrm{O}(n)$ is open in $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ [31, Corollary of Proposition 2.1.2], and thus $\pi(W)=\pi(W\cdot\mathrm{O}(n))$ is open in $\mathcal{M}^\infty_*(n)$ (Remark 9-(i)). Obviously, $\pi:\mathcal{S}\to\Theta$ is surjective, and let us show that it is also injective. Take $[N,p],[L,q]\in\mathcal{S}$ such that $\pi([N,p])=\pi([N,q])=:x$. Thus there is some $a\in\mathrm{O}(n)$ so that $[L,q]=[N,p]\cdot a$. Since the isotropy group at $[M,f]$ is trivial by Claim 4, there is an $\mathrm{O}(n)$-equivariant continuous map $\kappa:\mathcal{S}\cdot\mathrm{O}(n)\to\mathrm{O}(n)$ so that $\kappa^{-1}(e)=\mathcal{S}$. It follows that $e=\kappa([L,q])=\kappa([N,p]\cdot a)=\kappa([N,p])\,a=a$, obtaining $[L,q]=[N,p]$, which completes the proof of Claim 5.

According to Claim 5, the inverse of $\pi:\mathcal{S}\to\Theta$ defines a continuous local section $\sigma:\Theta\to\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ of $\pi:\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)\to\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$. By the existence of continuous local sections, and since the $\mathrm{O}(n)$-action on $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ is continuous and free (Remark 9-(ii) and Claim 4), it easily follows that $\pi:\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)\to\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ admits the structure of an $\mathrm{O}(n)$-principal bundle.

By Claim 4, $\pi^{-1}_{\mathcal{T}_*(n)}([M,x])$ canonically becomes an orthogonal vector space for each $[M,x]\in\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, and we can canonically identify $\pi_{\mathcal{Q}^{-1}_*(n)}([M,x])$ to the set of linear isometries $\pi^{-1}_{\mathcal{T}_*(n)}([M,x])\to\mathbb{R}^n$. The continuity of the mapping $([M,f],[M,\xi])\mapsto[M,f]([M,\xi])$ is easy to check. By using this identity, we get a homeomorphism $\theta:\pi^{-1}_{\mathcal{T}_*(n)}(\Theta)\to\mathbb{R}^n\times\Theta$ defined by $\theta([M,\xi])=(\sigma([M,x])([M,\xi]),[M,x])$, where $\pi([M,\xi])=[M,x]$, whose inverse map is given by $\theta^{-1}(v,[M,x])=[M,\sigma([M,x])^{-1}(v)]$. If $\sigma':\Theta'\to\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ is another local section of $\pi:\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)\to\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ defining a map $\theta':\pi^{-1}(\Theta')\to\mathbb{R}^n\times\Theta'$ as above, and $[M,x]\in\Theta\cap\Theta'$, then the composite

$$\mathbb{R}^n\equiv\mathbb{R}^n\times\{[M,x]\}\xrightarrow{\theta^{-1}}\pi^{-1}_{\mathcal{T}_*(n)}([M,x])\xrightarrow{\theta'}\mathbb{R}^n\times\{[M,x]\}\equiv\mathbb{R}^n$$

is the orthogonal isomorphism $\sigma'([M,x])\circ\sigma([M,x])^{-1}$. It follows that $\pi:\mathcal{T}^\infty_{*,\mathrm{lnp}}(n)\to\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, with these local trivializations, becomes an orthogonal vector bundle of rank $n$ so that the canonical map $T_xM\to\pi^{-1}([M,x])$ is a orthogonal isomorphism for all $[M,x]\in\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$. Moreover, by Claim 4, there is a canonical isomorphism between $\mathcal{Q}^\infty_{*,\mathrm{lnp}}(n)$ and the $\mathrm{O}(n)$-principal bundle of orthonormal frames of $\mathcal{T}^\infty_{*,\mathrm{lnp}}(n)$. ☐

By the compatibility of exponential maps and isometries, a map $\exp:\mathcal{T}^\infty_*(n)\to\mathcal{M}^\infty_*(n)$ is well defined by setting $\exp([M,\xi])=[M,\exp_M(\xi)]$. For each $[M,x]\in\mathcal{M}^\infty_*(n)$, the restriction $\exp:\pi^{-1}([M,x])\to\mathcal{M}^\infty_*(n)$ may be denoted by $\exp_{[M,x]}$.

**Lemma 9.12.** *Consider convergent sequences $[M_i,f_i]\to[M,f]$ and $[M_i,f'_i]\to[M,f']$ in $\mathcal{Q}^\infty_*(n)$ for some $n\in\mathbb{Z}^+$. Let $x=\pi(f)$, $x'=\pi(f')$, $x_i=\pi_i(f_i)$ and $x'_i=\pi_i(f'_i)$. Suppose that there is some $r>0$ such that*

$$\left\{h\in\operatorname{Iso}(M)\mid h(x)\in\overline{B}(x,2r)\right\}=\{\mathrm{id}_M\}\;,\tag{14}$$

*and $d(x,x'),d_i(x_i,x'_i)\le r$ for all $i$. Then there is some compact domain $\Omega$ in $M$ whose interior contains $x$ and $x'$, and there are $C^\infty$ embeddings $\phi_i:\Omega\to M_i$ for $i$ large enough so that $\phi_{i*}(f)=f_i$ and $\lim_i\phi^{-1}_{i*}(f'_i)=f'$ in $PM$, and $\lim_i\phi_i^*g_i=g|_\Omega$ with respect to the $C^\infty$ topology.*

*Proof.* Let $\Omega_q$ be a sequence of compact domains in $M$ such that

$$\overline{B}(x,r)\subset\operatorname{Int}(\Omega_q)\;,\quad\operatorname{Pen}(\Omega_q,\operatorname{diam}(\Omega_q))\subset\operatorname{Int}(\Omega_{q+1})\;;$$

in particular, $x'\in\operatorname{Int}(\Omega_q)$. By the convergence $[M_i,f_i]\to[M,f]$ and $[M_i,f'_i]\to[M,f']$ in $\mathcal{Q}^\infty_*(n)$, for each $q$, there are $C^\infty$ embeddings $\phi_{q,i},\psi_{q,i}:\Omega_q\to M_i$ for $i$ large enough so that $\phi_{q,i*}(f)=f_i$, $\psi_{q,i*}(f')=f'_i$, and $\lim_i\phi^*_{q,i}g_i=g|_{\Omega_q}$ and $\lim_i\psi^*_{q,i}g_i=g|_{\Omega_q}$ with respect to the $C^\infty$ topology; in particular, $\phi_{q,i}(x)=x_i$

and $\psi_{q,i}(x') = x'_i$. We have $x'_i \in \overline{B}_i(x_i, r) \subset \operatorname{Int}(\phi_{q,i}(\Omega_q))$ for $i$ large enough, depending on $q$, and therefore $\phi_{q,i}(\Omega_q) \cap \psi_{q,i}(\Omega_q) \neq \emptyset$. Hence

$$\psi_{q,i}(\Omega_q) \subset \operatorname{Pen}_i(\phi_{q,i}(\Omega_q), \operatorname{diam}(\phi_{q,i}(\Omega_q))) \subset \operatorname{Int}(\phi_{q,i}(\Omega_{q+1}))$$

for $i$ large enough, depending on $q$. It follows that $h_{q,i} := \phi_{q+1,i}^{-1}\psi_{q,i}$ is a well defined $C^\infty$ embedding $\Omega_q \to M$. Observe that $\lim_i h_{q,i}^* g = g|_{\Omega_q}$ with respect to the $C^\infty$ topology. Moreover

$$\begin{aligned}\limsup_i d(x, h_{q,i}(x)) = \limsup_i d(x, \phi_{q+1,i}^{-1}\psi_{q,i}(x)) = \limsup_i d_i(x_i, \psi_{q,i}(x))\\ \le \limsup_i d_i(x_i, x'_i) + \limsup_i d_i(x'_i, \psi_{q,i}(x)) \le r + d(x', x) \le 2r\ .\end{aligned}$$

If the statement is not true, then some neighborhood $U$ of $f'$ in $PM$ contains no accumulation point of the sequence $\phi_{q+1,i*}^{-1}(f'_i) = \phi_{q+1,i*}^{-1}\psi_{q,i*}(f') = h_{q,i*}(f')$ for each $q$. With the arguments of the proof of Lemma 8.1, it follows that there is some $h \in \operatorname{Iso}(M)$ such that $d(x, h(x)) \le 2r$ and $h_*(f') \notin U$, which contradicts (14). ☐

## 10. Center of mass

The main tool used to prove Theorem 1.3-(ii)–(v) is the Riemannian center of mass of a mass distribution on a Riemannian manifold $M$ [24], [8, Section IX.7]; especially, we will use the continuous dependence of the center of mass on the mass distribution and the metric tensor.

Recall that a domain $\Omega \subset M$ is said to be *convex* when, for all $x, y \in \Omega$, there is a unique minimizing geodesic segment from $x$ to $y$ in $M$ that lies in $\Omega$ (see e.g. [8, Section IX.6]). For example, sufficiently small balls are convex. For a fixed convex compact domain $\Omega$ in $M$, let $\mathcal{C}(\Omega)$ be the set of functions $f \in C^2(\Omega)$ such that the gradient $\operatorname{grad} f$ is an outward pointing vector field on $\partial\Omega$ and $\operatorname{Hess} f$ is positive definite on the interior $\operatorname{Int}(\Omega)$ of $\Omega$. Notice that $\mathcal{C}(\Omega)$ is open in the Banach space $C^2(\Omega)$ with the norm $\|\ \|_{C^2,\Omega,g}$, and thus it is a $C^\infty$ Banach manifold. Moreover $\mathcal{C}(\Omega)$ is preserved by the operations of sum and product by positive numbers. Any $f \in \mathcal{C}(\Omega)$ attains its minimum value at a unique point $\mathbf{m}(f) \in \operatorname{Int}(\Omega)$, defining a function $\mathbf{m} : \mathcal{C}(\Omega) \to \operatorname{Int}(\Omega)$.

**Lemma 10.1.** $\mathbf{m}$ *is continuous.*

*Proof.* Consider the map $\mathbf{v} : \mathcal{C}(\Omega) \times \operatorname{Int}(\Omega) \to T\Omega$ defined by $\mathbf{v}(f, x) = \operatorname{grad} f(x)$, and let $Z \subset T\Omega$ denote the image of the zero section. Since the graph of $\mathbf{m}$ is equal to $\mathbf{v}^{-1}(Z)$, it is enough to prove the following.

*Claim* 6. $\mathbf{v}$ is $C^1$ and transverse to $Z$.

Here, smoothness and transversality refer to $\mathbf{v}$ considered as a map between $C^\infty$ Banach manifolds [1, p. 45].

Let $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{V}}$ denote the orthogonal projections of $T^{(2)}\Omega$ onto $\mathcal{H}$ and $\mathcal{V}$, respectively. Let $\mathfrak{X}^1(\Omega)$ denote the Banach space of $C^1$ vector fields over $\Omega$ with the norm $\|\ \|_{C^1,\Omega,g}$, which is equivalent to the norm $\|\ \|_1$ defined by

$$\|X\|_1 = \sup\{\, |X(x)| + |\nabla X(x)| \mid x \in \Omega \,\}\ .$$

The gradient map, $\operatorname{grad} : C^2(\Omega) \to \mathfrak{X}^1(\Omega)$, is a continuous linear map between Banach spaces, and therefore it is $C^\infty$. The evaluation map, $\operatorname{ev} : \mathfrak{X}^1(\Omega) \times \Omega \to T\Omega$, is $C^1$ because, if $X \in \mathfrak{X}^1(\Omega)$, $Y \in T_X\mathfrak{X}^1(\Omega) \equiv \mathfrak{X}^1(\Omega)$, $x \in \Omega$ and $\xi \in T_x\Omega$, then $\operatorname{ev}_*(Y, \xi) \in T_\xi T\Omega$ is easily seen to be determined by the conditions $\pi_{\mathcal{H}}(\operatorname{ev}_*(Y, \xi)) \equiv \xi$ in $\mathcal{H}_\xi \equiv T_x\Omega$ and $\pi_{\mathcal{V}}(\operatorname{ev}_*(Y, \xi)) \equiv Y(x) + \nabla_\xi X$ in $\mathcal{V}_\xi \equiv T_x\Omega$. Therefore $\mathbf{v}$ is $C^1$ because it is the restriction to $\mathcal{C}(\Omega) \times \operatorname{Int}(\Omega)$ of the composition

$$C^2(\Omega) \times \Omega \xrightarrow{\operatorname{grad} \times \operatorname{id}_\Omega} \mathfrak{X}^1(\Omega) \times \Omega \xrightarrow{\operatorname{ev}} T\Omega\ .$$

Fix any $f \in \mathcal{C}(\Omega)$ and $x \in \operatorname{Int}(\Omega)$ with $\mathbf{v}(f, x) \in Z$; thus $\operatorname{grad} f(x) = 0_x$.

*Claim* 7. $\pi_{\mathcal{V}} : \mathbf{v}_*(\{0_f\} \times T_x\Omega) \to \mathcal{V}_{0_x}$ is an isomorphism.

For any $\xi \in T_x\Omega$,

$$\pi_{\mathcal{V}}\, \mathbf{v}_*(0_f, \xi) = \pi_{\mathcal{V}}\, (\operatorname{grad} f)_*(\xi) \equiv \nabla_\xi \operatorname{grad} f$$

in $\mathcal{V}_{0_x} \equiv T_x\Omega$. Then Claim 7 follows because the mapping $\xi \mapsto \nabla_\xi \operatorname{grad} f$ is an automorphism of $T_x\Omega$ since $\operatorname{Hess} f$ is positive definite at $x$ and $\operatorname{Hess} f(\xi, \cdot) = g(\nabla_\xi \operatorname{grad} f, \cdot)$ on $T_xM$.

From Claim 7, it follows that $\mathbf{v}_*(\{0_f\} \times T_x\Omega)$ is a linear complement to $\mathcal{H}_{0_x} = T_{0_x}Z$ in $T_{0_x}T\Omega$; in particular, it is closed in $T_{0_x}T\Omega$ because $T_{0_x}T\Omega$ is Hausdorff of finite dimension.

Since $\mathbf{v}_* : T_f\mathcal{C}(\Omega)\times T_x\Omega \to T_{0_x}T\Omega$ is linear and continuous, and $T_{0_x}T\Omega$ is Hausdorff of finite dimension, we get that the space $\left(\mathbf{v}_{*(f,x)}\right)^{-1}(T_{0_x}Z)$ is closed and of finite codimension in the Banach space $T_f\mathcal{C}(\Omega)\times T_x\Omega$, and therefore it has a closed linear complement in $T_f\mathcal{C}(\Omega)\times T_x\Omega$ (see e.g. [37, p. 22]), which completes the proof of Claim 6. □

*Remark* 11. (i) In the last part of the above proof, the space $\left(\mathbf{v}_{*(f,x)}\right)^{-1}(T_{0_x}Z)$ can be described as follows. Since $h \mapsto \operatorname{grad} h(x)$ defines a continuous linear map $C^2(\Omega) \to T_x\Omega$, we have $\mathbf{v}_*(T_f\mathcal{C}(\Omega)\times\{0_x\}) \subset \mathcal{V}_{0_x}$ and $\mathbf{v}_*(h, 0_x) \equiv \operatorname{grad} h(x)$ in $\mathcal{V}_{0_x} \equiv T_x\Omega$ for any $h \in C^2(\Omega) \equiv T_f\mathcal{C}(\Omega)$, giving

$$\left(\mathbf{v}_{*(f,x)}\right)^{-1}(T_{0_x}Z) \equiv \{\,(h,\xi) \in C^2(\Omega)\times T_x\Omega \mid \operatorname{grad} h(x) + \nabla_\xi \operatorname{grad} f = 0\,\}\ ,$$

which is obviously closed and of finite codimension in $C^2(\Omega)\times T_x\Omega$.

(ii) In Lemma 10.1, the map $\mathbf{m}$ is $C^m$ if the Banach space $C^{m+2}(\Omega)$ is used instead of $C^2(\Omega)$.

Suppose that the Riemannian manifold $M$ is connected and complete. Let $(A,\mu)$ be a probability space, $B$ a convex open ball of radius $r > 0$ in $M$, and $f : A \to B$ a measurable map, which is called a *mass distribution* on $B$. Consider the $C^\infty$ function $P_f : B \to \mathbb{R}$ defined by

$$P_f(x) = \frac{1}{2}\int_A d(x, f(a))^2\,\mu(a)\ .$$

**Proposition 10.2** (H. Karcher [24, Theorem 1.2])**.** *With the above notation and conditions, the following properties hold:*

(*i*) $\operatorname{grad} P_f$ *is an outward pointing vector field on the boundary* $\partial\overline{B}$*.*

(*ii*) *If* $\delta > 0$ *is an upper bound for the sectional curvatures of* $M$ *in* $B$*, and* $2r < \pi/2\sqrt{\delta}$*, then* $\operatorname{Hess} P_f$ *is positive definite on* $B$*.*

If the hypotheses of Proposition 10.2 are satisfied, then $P_f \in \mathcal{C}(\overline{B})$, and therefore $P_f$ reaches its minimum on $\overline{B}$ at a unique point $\mathcal{C}_f \in B$, which is called the *center of mass* of $f$. It is known that $\mathcal{C}_f$ depends continuously on $f$ with respect to the supremum distance when $(A,\mu)$ is fixed [24, Corollary 1.6]; indeed, the following result follows directly from Lemma 10.1.

**Corollary 10.3.** (*i*) $\mathcal{C}_f$ *depends continuously on* $f$ *and the metric tensor of* $M$*.*

(*ii*) *If* $A$ *is the Borel* $\sigma$*-algebra of a metric space, then* $\mathcal{C}_f$ *depends continuously on* $\mu$ *in the weak-*$*$ *topology.*

## 11. Foliated structure of $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$

The goal of this section is to prove Theorem 1.3-(ii)–(v).

For any point $[M,x] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, choose some $r, \varepsilon > 0$ and some neighborhood $\mathcal{N}_0$ of $[M,x]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ satisfying the statement of Lemma 8.2 with $\varepsilon \le r/5$. Using [33, Chapter 6, Theorem 3.6], we can assume that $\varepsilon$ and $\mathcal{N}_0$ are so small that $B_L(y,\varepsilon)$ satisfies the conditions of Proposition 10.2 in $L$ for all $[L,y] \in \mathcal{N}_0$. Take any continuous function $\lambda : \mathcal{M}^\infty_*(n) \to [0,1]$ supported in $\mathcal{N}_0$ and with $\lambda([M,x]) = 1$, whose existence is a simple consequence of the metrizability of $\mathcal{M}^\infty_*(n)$ (Theorem 1.2). For $[L,y] \in \mathcal{N}_0$, let $\omega_L$ denote the Riemannian density of $L$, and let $\lambda_{L,y} : L \to [0,1]$ be the function defined by

$$\lambda_{L,y}(z) = \begin{cases} \lambda([L,z]) & \text{if } d_L(y,z) \le \varepsilon \\ 0 & \text{if } d_L(y,z) \ge \varepsilon\ , \end{cases}$$

which is well defined and continuous by Lemma 8.2. Take another neighborhood $\mathcal{N} \subset \mathcal{N}_0$ of $[M,x]$ where $\lambda > 0$. For $[L,y] \in \mathcal{N}$, we have $\int_L \lambda_{L,y}\,\omega_L > 0$, and set

$$\bar\lambda_{L,y} = \frac{\lambda_{L,y}}{\int_L \lambda_{L,y}\,\omega_L}\ .$$

Then $\mu_{L,y} = \bar\lambda_{L,y}\,\omega_L$ is a continuous density defining a probability measure on $L$, and the identity map $(L,\mu_{L,y}) \to L$ is a distribution of mass on $L$ satisfying the conditions of Proposition 10.2 with $B_L(y,\varepsilon)$. Thus its center of mass, $\mathcal{C}_{L,y}$, is defined in $B_L(y,\varepsilon)$. Let $\mathbf{c} : \mathcal{N} \to \mathcal{M}^\infty_*(n)$ be the map given by $\mathbf{c}([L,y]) = [L, \mathcal{C}_{L,y}]$.

**Lemma 11.1.** *If $[L,y],[L,y']\in\mathcal{N}$ and $d_L(y,y')\le\varepsilon$, then $\mathbf{c}([L,y])=\mathbf{c}([L,y'])$.*

*Proof.* Take any point $z\in L$. If $[L,z]\notin\mathcal{N}_0$ or $d_L(y,z),d_L(y',z)>\varepsilon$, then $\lambda_{L,y}(z)=\lambda_{L,y'}(z)=0$. If $[L,z]\in\mathcal{N}_0$ and $d_L(y,z)\le\varepsilon$, then $d_L(y',z)\le 2\varepsilon$, obtaining $d_L(y',z)\le\varepsilon$ by Lemma 8.2 since $5\varepsilon\le r$, and therefore $\lambda_{L,y}(z)=\lambda_{L,y'}(z)=\lambda([L,z])$. If $[L,z]\in\mathcal{N}_0$ and $d_L(y',z)\le\varepsilon$, we similarly get $\lambda_{L,y}(z)=\lambda_{L,y'}(z)$. Thus $\lambda_{L,y}=\lambda_{L,y'}$, obtaining $\mathcal{C}_{L,y}=\mathcal{C}_{L,y'}$, and therefore $\mathbf{c}([L,y])=\mathbf{c}([L,y'])$. □

**Lemma 11.2.** $\mathbf{c}$ *is continuous.*

*Proof.* Take any convergent sequence $[L_i,y_i]\to[L,y]$ in $\mathcal{N}$. Let $\Omega$ be a compact domain in $L$ whose interior contains $\overline{B}_L(y,\varepsilon)$. Then there is a $C^\infty$ embedding $\phi_i:\Omega\to L_i$ for each $i$ large enough so that $\lim_i\phi_i^*g_i=g|_\Omega$ with respect to the $C^\infty$ topology. It follows that $\lim_i\phi_i^*\mu_{L_i,y_i}=\mu_{L,y}|_\Omega$ with respect to the $C^0$ topology by the continuity of $\lambda$, and thus this convergence also holds in the space of probability measures on $\Omega$ with the weak-$*$ topology. Since $\phi_i^{-1}(\mathcal{C}_{L_i,y_i})$ is the center of mass of the mass distribution on $\Omega$ defined by the probability measure $\phi_i^*\mu_{L_i,y_i}$, it follows from Corollary 10.3 that $\lim_i\phi_i^{-1}(\mathcal{C}_{L_i,y_i})=\mathcal{C}_{L,y}$ in $L$. Therefore $\lim_i\mathbf{c}([L_i,y_i])=\mathbf{c}([L,y])$ in $\mathcal{M}_*^\infty(n)$ because $\Omega$ is arbitrary. □

Let $\mathcal{Z}=\mathbf{c}(\mathcal{N})$, and let $\mathcal{N}'=\bigcup_{[L,c]\in\mathcal{Z}}\iota_L(B_L(c,\varepsilon))$, which contains $\mathcal{N}$ because $d_M(y,\mathcal{C}_{L,y})<\varepsilon$ for all $[L,y]\in\mathcal{N}$. Also, let $\mathbf{c}':\mathcal{N}'\to\mathcal{Z}$ be defined by the condition $\mathbf{c}'([L,z])=[L,c]$ if $[L,c]\in\mathcal{Z}$ and $d_L(c,z)<\varepsilon$. To prove that $\mathbf{c}'$ is well defined, take another point $c'\in L$ satisfying $[L,c']\in\mathcal{Z}$ and $d_L(c',z)<\varepsilon$, and let us check that $[L,c]=[L,c']$. Choose points $y,y'\in L$ such that $[L,y],[L,y']\in\mathcal{N}$, $\mathbf{c}([L,y])=[L,c]$ and $\mathbf{c}([L,y'])=[L,c']$. Then

$$d_L(y,y')\le d_L(y,c)+d_L(c,z)+d_L(z,c')+d_L(c',y')<4\varepsilon\ ,$$

giving $d_L(y,y')\le\varepsilon$ by Lemma 8.2 since $5\varepsilon\le r$, which implies $[L,c]=[L,c']$ by Lemma 11.1. Furthermore $\mathbf{c}'$ is an extension of $\mathbf{c}$ because $d_L(y,\mathcal{C}_{L,y})<\varepsilon$ for all $[L,y]\in\mathcal{N}$. Note also that $\mathbf{c}'([L,c])=[L,c]$ for all $[L,c]\in\mathcal{Z}$.

**Lemma 11.3.** *If $[L,z],[L,z']\in\mathcal{N}'$ and $d_L(z,z')\le 2\varepsilon$, then $\mathbf{c}'([L,z])=\mathbf{c}'([L,z'])$.*

*Proof.* Let $\mathbf{c}'([L,z])=[L,c]$ and $\mathbf{c}'([L,z'])=[L,c']$. Choose points $[L,y],[L,y']\in\mathcal{N}$ with $\mathbf{c}([L,y])=[L,c]$ and $\mathbf{c}([L,y'])=[L,c']$. Then

$$d_L(y,y')\le d_L(y,c)+d_L(c,z)+d_L(z,z')+d_L(z',c')+d_L(c',y')<5\varepsilon\ .$$

From Lemma 8.2 and since $5\varepsilon\le r$, it follows that $[L,c]=[L,c']$. □

**Lemma 11.4.** $\mathbf{c}'$ *is continuous.*

*Proof.* Take any convergent sequence $[L_i,z_i]\to[L,z]$ in $\mathcal{N}'$. Let $\mathbf{c}'([L_i,z_i])=[L_i,c_i]$ and $\mathbf{c}'([L,z])=[L,c]$, and choose points $[L_i,y_i],[L,y]\in\mathcal{N}$ so that $\mathbf{c}([L_i,y_i])=[L_i,c_i]$ and $\mathbf{c}([L,y])=[L,c]$. We have

$$d_i(y_i,z_i)\le d_i(y_i,c_i)+d_i(c_i,z_i)<2\varepsilon\ ,\quad d_L(y,z)\le d_L(y,c)+d_L(c,z)<2\varepsilon\ .$$

Then, by Lemma 8.3, there are points $y_i'\in B_i(z_i,2\varepsilon)$ such that $\lim_i[L_i,y_i']=[L,y]$ in $\mathcal{M}_*^\infty(n)$ as $i\to\infty$. Thus $[L_i,y_i']\in\mathcal{N}$ for $i$ large enough, and moreover

$$d_i(y_i,y_i')\le d_i(y_i,z_i)+d_i(z_i,y_i')<4\varepsilon\ ,$$

obtaining $d_i(y_i,y_i')\le\varepsilon$ by Lemma 8.2 since $5\varepsilon\le r$. By Lemma 11.1, it follows that $\mathbf{c}([L_i,y_i'])=\mathbf{c}([L_i,y_i])=[L_i,c_i]$ for $i$ large enough, giving $\lim_i[L_i,c_i]=[L,c]$ in $\mathcal{M}_*^\infty(n)$ by Lemma 11.2. □

We can assume that $\varepsilon$ and $\mathcal{N}$ are so small that the following properties hold for all $[L,y]\in\mathcal{N}$ and $z\in B_L(y,\varepsilon)$:

(a) $\exp_L:B_{T_zL}(0_z,\varepsilon)\to B_L(z,\varepsilon)$ is a diffeomorphism; and

(b) $\{h\in\operatorname{Iso}(L)\mid h(z)\in\overline{B}(z,4\varepsilon)\}=\{\mathrm{id}_L\}$.

Observe that (b) can be assumed by Lemma 8.1. Notice also that (a) and (b) hold for all $[L,z]\in\mathcal{Z}$. Let

$$\widehat{\mathcal{N}}'=\{\,[L,\xi]\in\mathcal{T}_*^\infty(n)\mid\pi([L,\xi])\in\mathcal{Z},\ |\xi|<\varepsilon\,\}\ .$$

**Lemma 11.5.** $\exp:\widehat{\mathcal{N}}'\to\mathcal{N}'$ *is a homeomorphism.*

*Proof.* This map is obviously surjective; we will prove that it also injective. For $i \in \{1,2\}$, take points $[L_i,\xi_i] \in \widehat{\mathcal{N}}'$; thus $\xi_i \in T_{c_i}L_i$ for some points $[L_i,c_i] \in \mathcal{Z}$, and we have $\exp([L_i,\xi_i]) = [L_i,z_i]$ for $z_i = \exp_i(\xi_i)$. Suppose that $[L_1,z_1] = [L_2,z_2]$, which means that there is a pointed isometry $\phi : (L_1,z_1) \to (L_2,z_2)$. Then

$$\exp_2 \phi_*(\xi_1) = \phi \exp_1(\xi_1) = \phi(z_1) = z_2 = \exp_2(\xi_2) , \tag{15}$$

$$d_2(\phi(c_1),c_2) \le d_2(\phi(c_1),z_2) + d_2(z_2,c_2) = d_1(c_1,z_1) + d_2(z_2,c_2) < 2\varepsilon . \tag{16}$$

We get

$$[L_1,c_1] = \mathbf{c}'([L_1,c_1]) = \mathbf{c}'([L_2,\phi(c_1)]) = [L_2,c_2]$$

by Lemma 11.3 and (16). So there is an isometry $\psi : L_1 \to L_2$ such that $\psi(c_1) = c_2$. Then the isometry $h = \psi^{-1}\phi : L_1 \to L_1$ satisfies

$$d_1(c_1,h(c_1)) = d_2(c_2,\phi(c_1)) < 2\varepsilon$$

by (16), obtaining $h = \mathrm{id}_{L_1}$ by (a). Hence $\phi(c_1) = \psi(c_1) = c_2$, giving $\phi_*(\xi_1) = \xi_2$ by (15) and (a) since $\xi_i \in T_{c_i}L_i$. Therefore $\exp : \widehat{\mathcal{N}}' \to \mathcal{N}'$ is bijective.

The continuity of $\exp^{-1} : \mathcal{N}' \to \widehat{\mathcal{N}}'$ is a simple exercise using lemma 11.4. □

By Proposition 9.11-(i), there is some neighborhood $\Theta$ of $[M,x]$ in $\mathcal{M}^\infty_*(n)$ and some local trivialization $\theta : \pi^{-1}(\Theta) \to \mathbb{R}^n \times \Theta$ of the Riemannian vector bundle $\pi : \mathcal{T}^\infty_*(n) \to \mathcal{M}^\infty_*(n)$; in particular, $\theta : \pi^{-1}([L,y]) \to \mathbb{R}^n \times \{[L,y]\} \equiv \mathbb{R}^n$ is a linear isometry for all $[L,y] \in \Theta$. More precisely, according to the proof of Proposition 9.11, we can suppose that there is a local section $\sigma : \Theta \to \mathcal{Q}^\infty_*(n)$ of $\pi : \mathcal{Q}^\infty_*(n) \to \mathcal{M}^\infty_*(n)$ so that $\theta([L,\xi]) = (\sigma([L,y])([L,\xi]),[L,y])$ if $\pi_L(\xi) = y$. We can assume that $\mathcal{Z} \subset \Theta$ by Lemma 8.4. Hence, by Lemma 11.5, the composite

$$\mathcal{N}' \xrightarrow{\exp^{-1}} \widehat{\mathcal{N}}' \xrightarrow{\theta} B^n_\varepsilon \times \mathcal{Z}$$

is a homeomorphism $\Phi : \mathcal{N}' \to B^n_\varepsilon \times \mathcal{Z}$, where $B^n_\varepsilon$ denotes the open ball of radius $\varepsilon$ centered at the origin in $\mathbb{R}^n$. This shows that $\mathcal{F}_{*,\mathrm{lnp}}(n)$ is a foliated structure of dimension $n$ on $\mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, completing the proof of Theorem 1.3-(ii).

Recall that a Riemannian manifold $M$ (or its metric tensor) is called *nowhere locally homogenous* if there is no isometry between distinct open subsets of $M$. It is easy to see that the proof of [38, Proposition 1] can be adapted to the case of open manifolds, obtaining the following.

**Proposition 11.6.** *For any $C^\infty$ manifold $M$, the set of nowhere locally homogenous metrics on $M$ is residual in* $\mathrm{Met}(M)$ *with the weak and strong $C^\infty$ topologies.*

**Lemma 11.7.** *There is a nowhere locally homogenous complete Riemannian manifold $M$ such that $\iota(M)$ is dense in $\mathcal{M}^\infty_{*,\mathrm{o}}(n)$.*

*Proof.* According to the proof of Proposition 7.1, there is a countable dense set of points $[M_i,x_i]$ in $\mathcal{M}^\infty_{*,\mathrm{lnp,c}}(n)$ ($i \in \mathbb{N}$). For each $i$, take some $y_i \in M_i$ so that $d_i(x_i,y_i) = \max_{y\in M_i} d_i(x_i,y)$. For all $i \in \mathbb{N}$ and $j,k \in \mathbb{Z}^+$ with $1/j, 1/k < \operatorname{diam} M_i$, let $(M_{ijk},x_{ijk},y_{ijk})$ be a copy of $(M_i,x_i,y_i)$, let $g_{ijk}$ be the metric of $M_{ijk}$, and let $\Omega_{ijk}$ be a compact domain in $M_{ijk}$ containing $y_{ijk}$ and with diameter $< 1/j$. Observe that $\widehat{\Omega}_{ijk} := M_{ijk} \smallsetminus \mathrm{Int}(\Omega_{ijk})$ is also a compact domain. Take also corresponding mutually disjoint compact domains $\Omega'_{ijk}$ in $\mathbb{R}^n$ so that every bounded subset of $\mathbb{R}^n$ only meets a finite number of them. Let $M$ be the $C^\infty$ connected sum of $\mathbb{R}^n$ with all manifolds $M_{ijk}$ so that the connected sum with each $M_{ijk}$ only involves perturbations inside the interiors of $\Omega_{ijk}$ and $\Omega'_{ijk}$. Let $g$ be any Riemannian metric on $M$ whose restriction to each $\widehat{\Omega}_{ijk}$ equals $g_{ijk}$, and whose restriction to $\mathbb{R}^n \smallsetminus \bigcup_{ijk}\Omega'_{ijk}$ equals the Euclidean metric. Then $g$ is complete and $\iota(M,g)$ is dense in $\mathcal{M}^\infty_*(n)$. With the strong $C^\infty$ topology, $C^\infty(M;TM^* \odot TM^*)$ is a Baire space by [22, Theorem 4.4-(b)]. Since $\mathrm{Met}(M)$ is open in $C^\infty(M;TM^* \odot TM^*)$, and the complete metrics on $M$ form an open subspace $\mathrm{Met}_{\mathrm{com}}(M) \subset \mathrm{Met}(M)$, it follows that $\mathrm{Met}_{\mathrm{com}}(M)$ is a Baire space with the strong $C^\infty$ topology. Hence, by Proposition 11.6, there is a nowhere locally homogenous complete metric $g'$ on $M$ so that $\|g-g'\|_{C^k,\widehat{\Omega}_{ijk},g} < 1/k$ for all $i$, $j$ and $k$. Then $\iota(M,g')$ is also dense in $\mathcal{M}^\infty_{*,\mathrm{o}}(n)$. □

By Lemma 11.7, $\mathcal{F}_{*,\mathrm{lnp,o}}(n)$ is transitive, showing Theorem 1.3-(iii).

Now, for $k \in \{1,2\}$, let $\Phi_k : \mathcal{N}'_k \to B^n_{\varepsilon_k} \times \mathcal{Z}_k$ be two homeomorphisms constructed as above with maps $\mathbf{c}'_k : \mathcal{N}'_k \to \mathcal{Z}_k$, $\exp : \widehat{\mathcal{N}}'_k \to \mathcal{N}'_k$ and $\sigma_k : \Theta_k \to \mathcal{Q}^\infty_*(n)$.

**Lemma 11.8.** $\Phi_2\Phi_1^{-1} : \Phi_1(\mathcal{N}'_1 \cap \mathcal{N}'_2) \to \Phi_1(\mathcal{N}'_1 \cap \mathcal{N}'_2)$ *is* $C^\infty$ *(in the sense of Section 2.1).*

*Proof.* This map has the expression

$$\Phi_2\Phi_1^{-1}(v, [L, c]) = (\Psi(v, [L, c]), \Gamma([L, c])) ,$$

where $\Gamma : \mathbf{c}'_1(\mathcal{N}'_1 \cap \mathcal{N}'_2) \to \mathbf{c}'_2(\mathcal{N}'_1 \cap \mathcal{N}'_2)$ is the corresponding holonomy transformation, and $\Psi : \Phi_1(\mathcal{N}'_1 \cap \mathcal{N}'_2) \to \mathbb{R}^n$ is defined by

$$\Psi(v, [L, c]) = \sigma_2([L, c']) \exp^{-1}_{[L,c']} \exp_{[L,c]} \sigma_1([L, c])^{-1}(v) ,$$

where $[L, c'] = \Gamma([L, c])$. Let $[L, f] = \sigma_1([L, c])$ and $[L, f'] = \sigma_2([L, c'])$. We can take $c'$ so that $d(c, c') < \varepsilon_1 + \varepsilon_2$, and then

$$\Psi(v, [L, c]) = f' \exp^{-1}_{c'} \exp_c f^{-1}(v) .$$

To prove that $\Psi$ is $C^\infty$ in the sense of Section 2.1, fix any $(v, [L, c]) \in \Phi_1(\mathcal{N}'_1 \cap \mathcal{N}'_2)$, and take $c'$, $f$ and $f'$ as above. Let $V$ and $\mathcal{O}$ be open neighborhoods of $v$ and $[L, c]$ in $\mathbb{R}^n$ and $\mathcal{Z}_1$, respectively, such that $\overline{V} \times \mathcal{O} \subset \Phi_1(\mathcal{N}'_1 \cap \mathcal{N}'_2)$. Take any convergent sequence $[L_i, c_i] \to [L, c]$ in $\mathcal{O}$, and define $c'_i$, $f_i$ and $f'_i$ as before for each $i$. Notice that $\Psi(v, [L, c])$ and $\Psi(v, [L_i, c_i])$ are defined for all $v \in V$, and let $\psi, \psi_i : V \to \mathbb{R}^n$ be the $C^\infty$ maps given by $\psi(v) = \Psi(v, [L, c])$ and $\psi_i(v) = \Psi(v, [L_i, c_i])$. We have to prove that $\lim_i \psi_i = \psi$ with respect to the weak $C^\infty$ topology.

Let $\Omega$ be any compact domain in $L$ such that $\overline{B}_L(c, \varepsilon_1 + 2\varepsilon_2) \subset \operatorname{Int}(\Omega)$, and thus $\overline{B}_L(c', \varepsilon_2) \subset \operatorname{Int}(\Omega)$ too. Since the sections $\sigma_1$ and $\sigma_2$ are continuous, there are $C^\infty$ embeddings $\phi_i : \Omega \to L_i$ for $i$ large enough so that $\phi_{i*}(f) = f_i$ and $\lim_i \phi_i^* g_i = g|_\Omega$; in particular, $\phi_i(c) = c_i$. Hence $c'_i \in \phi_i(\operatorname{Int}(\Omega))$ for $i$ large enough, and moreover $\lim_i \phi_{i*}^{-1}(f'_i) = f'$ by (b) and Lemma 9.12. Observe that $\hat\psi := \exp^{-1}_{c'} \exp_c$ is defined on $W = f^{-1}(V) \subset B_{T_cL}(0_c, \varepsilon_1)$. It follows that $\hat\psi_i := \phi_{i*}^{-1} \exp^{-1}_{c'_i} \exp_{c_i} \phi_{i*}$ is also defined on $W$ for $i$ large enough, and moreover $\lim_i \hat\psi_i = \hat\psi$ in the space of $C^\infty$ maps $W \to T_{c'}L$ with the weak $C^\infty$ topology. So

$$\lim_i \phi_{i*}^{-1}(f'_i)\, \hat\psi_i f^{-1} = f' \hat\psi f^{-1} = \psi$$

in the space of $C^\infty$ maps $V \to \mathbb{R}^n$ with the weak $C^\infty$ topology. Then the result follows because

$$\phi_{i*}^{-1}(f'_i)\, \hat\psi_i f^{-1} = \phi_{i*}^{-1}(f'_i)\, \hat\psi_i\, (\phi_{i*}^{-1}(f_i))^{-1} = f'_i \phi_{i*} \hat\psi_i \phi_{i*}^{-1} f_i^{-1} = f'_i \exp^{-1}_{c'_i} \exp_{c_i} f_i^{-1} = \psi_i . \quad \square$$

According to Lemma 11.8, $\mathcal{F}_{*,\mathrm{lnp}}(n)$ becomes $C^\infty$ with the above kind of charts. Thus we can consider the tangent bundle $T\mathcal{F}_{*,\mathrm{lnp}}(n)$. For each leaf $\iota(M)$ of $\mathcal{F}_{*,\mathrm{lnp}}(n)$, the canonical homeomorphism $\bar\iota : \operatorname{Iso}(M)\backslash M \to \iota(M)$ is a $C^\infty$ diffeomorphism, and $\iota_{*x} : T_xM \to T_{[M,x]}\mathcal{F}_{*,\mathrm{lnp}}(n)$ is an isomorphism for each $x \in M$. According to Proposition 9.11, we get a canonical bijection $T\mathcal{F}_{*,\mathrm{lnp}}(n) \to \mathcal{T}^\infty_{*,\mathrm{lnp}}(n)$ defined by $\iota_{*x}(\xi) \mapsto [M, \xi]$ for $[M, \xi] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ and $\xi \in T_xM$. It is an easy exercise to prove that this bijection is an isomorphism of vector bundles. So the Riemannian structure on $\mathcal{T}^\infty_{*,\mathrm{lnp}}(n)$ defined in Proposition 9.11 corresponds to a Riemannian structure on $T\mathcal{F}_{*,\mathrm{lnp}}(n)$, which can be easily proved to be $C^\infty$ by using the above kind of flow boxes of $\mathcal{F}_{*,\mathrm{lnp}}(n)$. It is elementary that each isomorphism $\iota_{*x} : T_xM \to T_{[M,x]}\mathcal{F}_{*,\mathrm{lnp}}(n)$ is an isometry. This completes the proof of Theorem 1.3-(iv).

Theorem 1.3-(v) follows from the following.

**Lemma 11.9.** *The following properties hold for any point* $[M, x] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$*, any path* $\alpha : I := [0, 1] \to M$ *with* $\alpha(0) = x$*, and any neighborhood* $\mathcal{U}$ *of* $\iota\alpha$ *in* $C(I, \mathcal{F}_{*,\mathrm{lnp}}(n))$*:*

- *(i) If* $\alpha(1) = x$ *then, for each* $[N, y] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$ *close enough to* $[M, x]$*, there is a path* $\beta \in \mathcal{U}$ *with* $\beta(0) = \beta(1) = [N, y]$*.*
- *(ii) If* $\alpha(1) \neq x$ *then there is some path* $\beta \in \mathcal{U}$ *with* $\beta(0) \neq \beta(1)$*.*

*Proof.* Let $\Omega$ be a compact domain in $M$ whose interior contains $\alpha(I)$, let $[N, y] \in \mathcal{M}^\infty_{*,\mathrm{lnp}}(n)$, and let $\phi : (\Omega, x) \to (N, y)$ be a pointed $C^m$ embedding with $\|g_M - \phi^* g_N\|_{\Omega, C^m, g_M} < \varepsilon$ for some $m \in \mathbb{Z}^+$ and $\varepsilon > 0$. Let $\beta = \iota\phi\alpha \in C(I, \mathcal{F}_{*,\mathrm{lnp}}(n))$; that is, $\beta(t) = [N, \phi\alpha(t)]$ for each $t \in I$. Observe that $\beta \in \mathcal{U}$ if $m$ and $\Omega$ are large enough, and $\varepsilon$ is small enough (i.e., if $[N, y]$ is close enough to $[M, x]$). When $\alpha(1) = x$, we get

$$\beta(0) = [N, \phi(x)] = [N, y] = [N, \phi\alpha(1)] = \beta(1) .$$

Suppose now that $\alpha(1) \neq x$. Since $\mathcal{M}^{\infty}_{*,\mathrm{np}}(n)$ is dense in $\mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$, with the above notation, we can choose $[N,y] \in \mathcal{M}^{\infty}_{*,\mathrm{np}}(n)$ as close as desired to $[M,x]$. Hence $\iota : N \to \mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$ is injective, giving

$$\beta(0) = \iota\phi(x) \neq \iota\phi\alpha(1) = \beta(1) . \quad \square$$

## 12. Saturated subspaces of $\mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$

Let $X$ be a sequential Riemannian foliated space with complete leaves.

**Definition 12.1.** It is said that $X$ is *covering-determined* when there is a connected pointed covering $(\widetilde{L}_x, \tilde{x})$ of $(L_x, x)$ for all $x \in X$ such that $x_i \to x$ in $X$ if and only if $[\widetilde{L}_{x_i}, \tilde{x}_i]$ is $C^\infty$ convergent to $[\widetilde{L}_x, \tilde{x}]$. When this condition is satisfied with $\widetilde{L}_x = \widetilde{L}^{\mathrm{hol}}_x$ for all $x \in X$, it is said that $X$ is *holonomy-determined*.

**Example 12.2.** (i) The Reeb foliation on $S^3$ is not covering-determined with any Riemannian metric.
(ii) [29, Example 2.5] is covering-determined but not holonomy-determined.
(iii) $\mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$ is holonomy-determined.

*Remark* 12. (i) The condition of being covering-determined is hereditary by saturated subspaces.
(ii) The example $X$ of [29, Example 2.5] can be easily realized as a saturated subspace of a Riemannian foliated space $Y$ where the holonomy coverings of the leaves are isometric to $\mathbb{R}$. Multiplying the leaves by $S^1$, all holonomy covers of $Y \times S^1$ become isometric to $\mathbb{R} \times S^1$. The metric on $Y \times S^1$ can be modified so that no pair of these holonomy covers are isometric, obtaining a holonomy-determined foliated space, however $X \times S^1$ is not holonomy-determined with any metric. So holonomy-determination is not hereditary by saturated subspaces.
(iii) If $X$ satisfies the covering-determination with the pointed coverings $(\widetilde{L}_x, \tilde{x})$ of $(L_x, x)$ for $x \in X$, then $x = y$ in $X$ if and only if $[\widetilde{L}_x, \tilde{x}] = [\widetilde{L}_y, \tilde{y}]$; in particular, the leaves of $X$ are non-periodic.
(iv) If $X$ is compact and the mapping $x \mapsto [\widetilde{L}_x, \tilde{x}]$ is injective, then the "if" part of Definition 12.1 can be deleted.

*Proof of Theorem 1.4.* Any saturated subspace of $\mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$ is covering-determined by Example 12.2-(iii) and Remark 12-(i).

Suppose that $X$ satisfies the covering-determination with the pointed covers $(\widetilde{L}_x, \tilde{x})$ of $(L_x, x)$ for $x \in X$. Then the map $\iota : X \to \mathcal{M}^{\infty}_{*,\mathrm{lnp}}(n)$, defined by $\iota(x) = [\widetilde{L}_x, \tilde{x}]$, is a $C^\infty$ foliated embedding whose restrictions to the leaves are isometries. $\square$

*Remark* 13. Like in the above proof, a map $\iota^{\mathrm{hol}} : X \to \mathcal{M}^{\infty}_{*}(n)$ is defined by $\iota^{\mathrm{hol}}(x) = [\widetilde{L}^{\mathrm{hol}}_x, \tilde{x}]$, where $\tilde{x} \in \widetilde{L}^{\mathrm{hol}}_x$ is over $x$. This map may not be continuous [29, Example 2.5], but its restriction to $X_0$ is continuous by the local Reeb stability theorem, and therefore $\iota^{\mathrm{hol}}$ is Baire measurable if $X$ is second countable.

Any family $\mathcal{C}$ of complete connected Riemannian $n$-manifolds defines a closed $\mathcal{F}_*(n)$-saturated subspace $X := \mathrm{Cl}_\infty(\bigcup_{M \in \mathcal{C}} \iota(M)) \subset \mathcal{M}^{\infty}_{*}(n)$. The obvious $C^\infty$ version of arguments of [9] (see also [33, Chapter 10, Sections 3 and 4]) gives the following.

**Theorem 12.3.** *A family $\mathcal{C}$ of complete connected Riemannian $n$-manifolds is of equi-bounded geometry if and only if the closed subspace of $\mathcal{M}^{\infty}_{*}(n)$ defined by $\mathcal{C}$ is compact.*

*Remark* 14. A version of Theorem 12.3 using the Ricci curvature instead of $\mathcal{R}$ can be also proved with the arguments of [2].

For instance, let $\mathcal{M}_*(n, r, C_m) \subset \mathcal{M}_*(n)$ denote the subspace defined by the manifolds of bounded geometry with geometric bound $(r, C_m)$. Each $\mathcal{M}_*(n, r, C_m)$ is compact by Theorem 12.3, and the notion of $C^\infty$ convergence in $\mathcal{M}_*(n, r, C_m)$ is equivalent to the convergence in the topology of the Gromov space $\mathcal{M}_*$ [29], [33, Chapter 10]. Nonetheless, this is not the case on the whole of $\mathcal{M}_*(n)$ [4, Section 7.1.4].

Let us study the case of closed subspaces of $\mathcal{M}^{\infty}_{*}(n)$ defined by a single manifold.

**Definition 12.4.** A complete connected Riemannian manifold $M$ is called:

(i) *aperiodic* if, for all $m_i \uparrow \infty$ in $\mathbb{N}$, compact domains $\Omega_i' \subset \Omega_i \subset M$, points $x_i \in \Omega_i'$ and $y_i \in \Omega_i$, and $C^{m_i}$ pointed embeddings $\phi_{ij} : (\Omega_i, x_i) \to (\Omega_j, x_j)$ $(i \le j)$ and $\psi_i : (\Omega_i', x_i) \to (\Omega_i, y_i)$ such that

$$\lim_i d(x_i, \partial\Omega_i')) = \infty \ , \quad \lim_{i,j} \|g - \phi_{ij}^* g\|_{C^{m_i},\Omega_i,g} = \lim_i \|g - \psi_i^* g\|_{C^{m_i},\Omega_i',g} = 0 \ ,$$

we have

$$\lim_i \max\{\, d(x, \psi_i(x)) \mid x \in \Omega_i' \cap \overline{B}(x_i, r) \,\} = 0 \tag{17}$$

for some $r > 0$; and

(ii) *weakly aperiodic* if, to get (17), besides the conditions of (i), it is also required that there is some $s > 0$ and there are points $z_i \in \Omega_i'$ such that $\phi_{ij}(z_i) = z_j$ and $d(z_i, \psi_i(z_i)) < s$.

**Lemma 12.5.** *The following properties hold for any complete connected Riemannian n-manifold M:*

(*i*) *$M$ is aperiodic if and only if* $\mathrm{Cl}_\infty(\iota(M)) \subset \mathcal{M}_{*,\mathrm{np}}^\infty(n)$.

(*ii*) *$M$ is weakly aperiodic if and only if* $\mathrm{Cl}_\infty(\iota(M)) \subset \mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$.

*Proof.* This is a consequence of Propositions 5.2, 6.4 and 6.5, and using also arguments from the proof of Proposition 5.3 for the "if" parts. □

**Definition 12.6.** A complete connected Riemannian manifold $M$ is called *repetitive* if, for every compact domain $\Omega$ in $M$, and all $\varepsilon > 0$ and $m \in \mathbb{N}$, there is a family of $C^m$ embeddings $\phi_i : \Omega \to M$ such that $\bigcup_i \phi_i(\Omega)$ is a net in $M$ and $\|g - \phi_i^* g\|_{C^m,\Omega,g} < \varepsilon$ for all $i$.

Here, the term *net* in $M$ is used for a subset $A \subset M$ satisfying $\mathrm{Pen}(A, S) = M$ for some $S > 0$.

**Lemma 12.7.** *Let $M$ be a complete connected Riemannian n-manifold of bounded geometry. Then $M$ is repetitive if and only if* $\mathrm{Cl}_\infty(\iota(M))$ *is* $\mathcal{F}_*(n)$*-minimal.*

*Proof.* The "only if" part follows easily from Propositions 5.2, 6.4 and 6.5.

To prove the "if" part, assume that $\mathrm{Cl}_\infty(\iota(M))$ is $\mathcal{F}_*(n)$-minimal. Let $\Omega$ be a compact domain in $M$, and take some $m \in \mathbb{N}$ and $\varepsilon > 0$. Take some $x \in M$ and $R > 0$ such that $\Omega \subset B(x, R)$. Let $U = \mathrm{Int}_\infty(D_{R,\varepsilon}^m(M, x))$. Since $\mathrm{Cl}_\infty(\iota(M))$ is compact because $M$ is of bounded geometry (Theorem 12.3), there is some $S > 0$ such that $\mathbf{d}_U \le S$ on $\mathrm{Cl}_\infty(\iota(M))$ by Lemma 5.4. Hence $[M, x_i] \in U$ for a net of points $x_i$ in $M$. Thus there are $C^{m+1}$ pointed local diffeomorphisms $\phi_i \colon (M, x) \rightarrowtail (M, x_i)$ so that $\|g - \phi_i^* g\|_{C^m,\Omega_i,g} < \varepsilon$ for some compact domain $\Omega_i \subset \operatorname{dom}\phi_i$ with $B(x, R) \subset \Omega_i$; in particular, $\Omega \subset \operatorname{dom}\phi_i$ and $\|g - \phi_i^* g\|_{C^m,\Omega,g} < \varepsilon$ for all $i$, and $\bigcup_i \phi_i(\Omega)$ is a net in $M$, showing that $M$ is repetitive. □

*Proof of Theorem 1.5.* Suppose that $M$ is non-periodic and has a weakly aperiodic connected covering $\widetilde{M}$. Then $Y = \mathrm{Cl}_\infty(\iota(\widetilde{M}))$ is a compact saturated subspace of $\mathcal{M}_{*,\mathrm{lnp}}^\infty(n)$ by Theorem 12.3 and Lemma 12.5-(ii), and $M \equiv \mathrm{Iso}(\widetilde{M})\backslash\widetilde{M} \xrightarrow{\bar\iota} \iota(\widetilde{M})$ is an isometry. Moreover any sequential covering-determined transitive compact Riemannian foliated space can be obtained in this way by Theorem 1.4. If $\widetilde{M}$ is also repetitive, then $X$ is minimal by Lemma 12.7, completing the proof of (i).

Asume now that $M$ is aperiodic. Then $X = \mathrm{Cl}_\infty(\iota(M))$ is a compact $\mathcal{F}_{*,\mathrm{np}}(n)$-saturated subspace of $\mathcal{M}_{*,\mathrm{np}}^\infty(n)$ by Theorem 12.3 and Lemma 12.5-(i), and moreover $\iota : M \to \iota(M)$ is an isometry. Furthermore the leaves of $X$ have trivial holonomy groups by Theorem 1.3-(v). As before, $X$ is minimal if $M$ is also repetitive, showing (ii). □


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## References


1. R. Abraham and J. Robbin, *Transversal mappings and flows*, W. A. Benjamin, Inc., New York-Amsterdam, 1967. MR 0240836 (39 #2181)
2. M.T. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), no. 2, 429–445. MR 1074481 (92c:53024)
3. S. Bando and H. Urakawa, *Generic properties of the eigenvalue of the Laplacian for compact Riemannian manifolds*, Tôhoku Math. J. (2) **35** (1983), no. 2, 155–172. MR 699924 (84h:58146)

4. D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, American Mathematical Society, Providence, RI, 2001. MR 1835418 (2002e:53053)
5. A. Candel and L. Conlon, *Foliations. I*, Graduate Studies in Mathematics, vol. 23, American Mathematical Society, Providence, RI, 2000. MR 1732868 (2002f:57058)
6. ———, *Foliations. II*, Graduate Studies in Mathematics, vol. 60, American Mathematical Society, Providence, RI, 2003. MR 1994394 (2004e:57034)
7. J. Cerf, *Topologie de certains espaces de plongements*, Bull. Soc. Math. France **89** (1961), 227–380. MR 0140120 (25 #3543)
8. I. Chavel, *Riemannian geometry. A modern introduction*, second ed., Cambridge Studies in Advanced Mathematics, vol. 98, Cambridge University Press, Cambridge, 2006. MR 2229062 (2006m:53002)
9. J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math. **92** (1970), 61–74. MR 0263092 (41 #7697)
10. A. Douady, *Variétés à bord anguleux et voisinages tubulaires*, Séminaire Henri Cartan, 1961/62, Exp. 1, Secrétariat mathématique, Paris, 1961/1962, p. 11. MR 0160221 (28 #3435)
11. D.B.A. Epstein, K.C. Millett, and D. Tischler, *Leaves without holonomy*, J. London Math. Soc. (2) **16** (1977), no. 3, 548–552. MR 0464259 (57 #4193)
12. N.P. Frank, *A primer of substitution tilings of the Euclidean plane*, Expo. Math. **26** (2008), no. 4, 295–326. MR 2462439 (2010d:52047)
13. É. Ghys, *Laminations par surfaces de Riemann*, Panoramas & Synthèses **8** (2000), 49–95. MR 1760843 (2001g:37068)
14. M. Gromov, *Groups of polynomial growth and expanding maps. Appendix by Jacques Tits*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 53, 53–73. MR 623534 (83b:53041)
15. ———, *Metric structures for Riemannian and non-Riemannian spaces*, Progress in Mathematics, vol. 152, Birkhäuser Boston Inc., Boston, MA, 1999, Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. MR 1699320 (2000d:53065)
16. B. Grünbaum and G.C. Shephard, *Tilings and patterns*, W.H. Freeman and Company, New York, 1987. MR 857454 (88k:52018)
17. G. Gutierres and D. Hofmann, *Axioms for sequential convergence*, Appl. Categ. Structures **15** (2007), no. 5-6, 599–614. MR 2365760 (2009a:54021)
18. A. Haefliger, *Pseudogroups of local isometries*, Differential geometry (Santiago de Compostela, 1984), Res. Notes in Math., vol. 131, Pitman, Boston, MA, 1985, pp. 174–197. MR 864868 (88i:58174)
19. ———, *Leaf closures in Riemannian foliations*, A fête of topology, Academic Press, Boston, MA, 1988, pp. 3–32. MR 928394 (89c:57033)
20. ———, *Foliations and compactly generated pseudogroups*, Foliations: Geometry and Dynamics, Warsaw 2000 (River Edge, NJ), World Sci. Publi., 2002, pp. 275–296. MR 1882774 (2003g:58029)
21. G. Hector, *Feuilletages en cylindres*, Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Springer, Berlin, 1977, pp. 252–270. Lecture Notes in Math., Vol. 597. MR 0451260 (56 #9547)
22. M.W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, vol. 33, Springer-Verlag, New York-Heidelberg, 1976. MR 0448362 (56 #6669)
23. K.H. Hofmann and P.S. Mostert, *Elements of compact semigroups*, Charles E. Merrill Books, Inc., Columbus, 1966.
24. H. Karcher, *Riemannian center of mass and mollifier smoothing*, Comm. Pure Appl. Math. **30** (1977), no. 5, 509–541. MR 0442975 (56 #1350)
25. A.S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)
26. J.L. Kelley, *Convergence in topology*, Duke Math. J. **17** (1950), 277–283. MR 0036982 (12,194f)
27. ———, *General topology*, 2nd ed., Graduate Texts in Mathematics, no. 27, Springer-Verlag, New York-Berlin, 1975. MR 0370454 (51 #6681)
28. V. Koutník, *Closure and topological sequential convergence*, Convergence structures, Proc. Conf., Bechyně/Czech. 1984, Math. Res., vol. 24, Akademie-Verlag, Berlin, 1985, pp. 199–204. MR 835486 (87e:54005)
29. P. Lessa, *Reeb stability and the Gromov-Hausdorff limits of leaves in compact foliations*, Asian J. Math. 19 (2015), 433–464. MR 3361278
30. C.C. Moore and C. Schochet, *Global analysis on foliated spaces*, Mathematical Sciences Research Institute Publications, vol. 9, Springer-Verlag, New York, 1988, With appendices by S. Hurder, Moore, Schochet and Robert J. Zimmer. MR 0918974 (89h:58184)
31. R.S. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. of Math. (2) **73** (1961), 295–323. MR 0126506 (23 #A3802)
32. P. Petersen, *Convergence theorems in Riemannian geometry*, Comparison geometry, Cambridge University, Cambridge, 1997. MR 1452874 (98k:53049)
33. ———, *Riemannian geometry*, Graduate Texts in Mathematics, vol. 171, Springer-Verlag, New York, 1998. MR 1480173 (98m:53001)
34. A. Ramsay, *Local product structure for group actions*, Ergodic Theory Dynam. Systems **11** (1991), 209–217. MR 1101091 (92g:22020)
35. E.A. Robinson, Jr., *Symbolic dynamics and tilings of* $\mathbb{R}^d$, Symbolic Dynamics and its Applications, Proc. Sympos. Appl. Math., Amer. Math. Soc., Providence, RI, 2004, pp. 81–119. MR 2078847 (2005h:37036)

36. S. Sasaki, *On the differentiable geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J. (2) **10** (1958), no. 3, 211–368. MR 0112152 (22 #3007)
37. H.H. Schaefer, *Topological vector spaces*, Graduate Texts in Mathematics, vol. 3, Springer-Verlag, New York, Heidelberg, Berlin, 1971. MR 0342978 (49 #7722)
38. T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. (2) **121** (1985), no. 1, 169–186. MR 782558 (86h:58141)
39. S. Willard, *General topology*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1970. MR 0264581 (41 #9173)

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