New Stability Criterion for Positive Impulsive Fuzzy Systems by Applying Polynomial Impluse-Time-Dependent Method

Likui Wang, Bo Zheng, Hak-Keung Lam

***Abstract*—The problem of exponential stability and L_1-gain for continuous-time positive impulsive Takagi-Sugeno (T-S) fuzzy systems is further studied in this paper. Different from the Lyapunov function in the existing literatures, where the Lyapunov matrices are time-invariant or only linear dependent on the impulse interval, in this paper, a novel polynomial impulse-time-dependent (ITD) copositive Lyapunov function is constructed by using polynomial impulse time function. In addition, the binomial coefficients are applied to derive new finite linear programming conditions. Since more impulse intervals information are contained in the polynomial ITD copositive Lyapunov function, less conservative results are obtained. The final three examples demonstrate the influence of the polynomial degree on the results and the effectiveness of the developed new results.**

***Index Terms*—positive impulsive systems, polynomial impulse-time function, T-S fuzzy modeling, L_1-gain**

I. Introduction

IMPULSIVE systems are a subclass of hybrid systems commonly used to characterize instantaneous jumps at discrete moments in dynamics and have many applications in network control [1], sampled-data control [2], and biomedical field [3]. Recently, positive impulsive systems have attracted much attention due to the description of practical processes such as ecosystems [4] and traffic congestion [5]. Unlike general impulsive systems, positive impulsive systems have non-negative states and outputs when initial states and inputs are non-negative. The positivity has to be considered in the stability and stabilization analysis, and make the research more challenge than general systems [6]-[10]. Linear positive impulsive systems have many research results [11]-[16]. However, nonlinear positive impulse system research still has many problems. As we all know, the T-S fuzzy model can accurately approximate the nonlinear systems through local linearization, and is a powerful tool to deal with nonlinear systems [17]-[20]. By the T-S fuzzy model, the research results in linear impulsive systems can be widely applied to nonlinear impulsive systems. Therefore, the positive impulsive T-S fuzzy systems have received great attentions and obtained a great deal of results [21]-[24].

The stability and stabilization of impulsive systems is always a hot issue [25]-[29]. Many scholars are committed to reducing the conservatism of stability analysis, the essential point is how to design the Lyapunov function [17]-[24]. The quadratic Lyapunov function is an important tool for studying general systems and has been used in the stability of systems [25]. However, the general time-invariant quadratic Lyapunov function does not reflect the impulse time information of the impulsive systems, therefore, the result is conservative [21], [30]. In [20], an ITD quadratic Lyapunov function is designed to study stability for T-S fuzzy systems with delayed impulses. Whereafter, the ITD Lyapunov function design method was widely used in the research of various impulsive systems, such as stability analysis of nonlinear neutral state delay systems with impulses [22] and stability analysis of positive fuzzy impulsive systems [23]. Based on the results of [20], an ITD discretized quadratic Lyapunov function is proposed in [31] by partition on impulse intervals for the stability research of impulsive delay systems. By increasing the number of partitions, the conservatism of stability results can be further reduced.

As a special kind of positive systems, the positive impulsive T-S fuzzy systems can also be analyzed by applying ITD discretized quadratic Lyapunov function. However, for positive impulsive systems, the quadratic Lyapunov function does not reflect the positive characteristics. Sometimes the results are non-convex because of the positive conditions and need to be solved by complex iterative algorithms or other techniques [32]-[34]. The copositive Lyapunov function (CLF) proposed by considering the positive characteristics of system states provides a powerful tool for positive systems. The stability conditions by the CLF are expressed by linear programming, which can generally avoid non-convex conditions and be easily solved by using existing optimization algorithms [21],[24]. In order to analyze the stability of positive impulse fuzzy systems, an ITD discretized copositive Lyapunov function is proposed in [21]. Based on the discretized copositive Lyapunov function, less conservative conditions are obtained. Then in [24], the internal contact information of state variables in the impulse intervals is improved by introducing states $x(t)-x(t_k)$ and $x(t_k^-)$. Compared to [21], the method in [24] reduces the number of partitions, but both the ITD method and the ITD discretized method are only linear dependent on the impulse intervals. Could we use the impulse intervals to construct a polynomial ITD Lyapunov function to further reduce the conservatism? This is the first motivation of this paper.

On the other hand, for controller design of positive impul-

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sive fuzzy systems, the time-varying impulse intervals along with the positivity of the system states often lead to non-convex conditions, then how to use the polynomial ITD Lyapunov function to study the stabilization problem and get convex conditions is the second motivation of this paper.

Based on the above discussion, the main contributions of this paper are as follows: 1. Different from the existing method that dividing the impulse intervals into small subintervals, a novel polynomial ITD copositive Lyapunov function is designed for the exponential stability and L_1-gain of positive impulsive T-S fuzzy systems. Since more impulse intervals information are contained in polynomial ITD copositive Lyapunov function, less conservative stability conditions are obtained by setting appropriate degree of the polynomial. 2. In order to avoid non-convex conditions caused by uncertaint impulse interval, a controller is designed based on the polynomial ITD copositive Lyapunov function to stabilize the positive impulsive T-S fuzzy system and maintain positivity.

Notation: $\mathbb{N}$ denotes the set of positive integers. $X \succeq 0 (\preceq 0)$ means all entries of X are nonnegative (nonpositive). $D^+ f(t) = \lim_{\varepsilon \to 0+} \sup \frac{f(t+\varepsilon)-f(t)}{\varepsilon}$. A real materix X is called $Metzler$, if there exist a scalar $\alpha > 0$ such that $X + \alpha I \succeq 0$. 1-norm of a vector $x \in \mathbb{R}^n$ is $\|x\|_1 = \sum_{i=1}^n |x_i|$. $\lambda_{\min}(\nu_i)$ and $\lambda_{\max}(\nu_i)$ are, respectively, the minimal and maximal elements in all vector ν_i. $\mathcal{C}_n^m = \frac{n!}{m!(n-m)!}$, $m \leq n$. If $f \in L_1$ (resp.$f \in \ell_1$), then its norm is defined by $\|f\|_{L_1} = \int_0^\infty \|f(t)\|_1 \, \mathrm{d}t$ (resp. $\|f\|_{\ell_1} = \sum_{k=0}^\infty \|f(k)\|_1$) . $\overline{n_0, n_1} = \{n_0, n_0+1, \cdots, n_1\}$, with $n_0 \leq n_1 \in \mathbb{N}$.

II. Problem formulation

Consider the following impulsive T-S fuzzy mode:

Plant rule i: If $\varrho_1(t)$ is Φ_{i1}, and $\cdots$ and $\varrho_g(t)$ is Φ_{ig}, THEN

$$\begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) + E_i \omega(t), & t \neq t_k \\ x(t_k) = J_i x(t_k^-) + F_i \omega_d(t_k), & k \in \mathbb{N} \\ z(t) = C_i x(t) + D_i \omega(t) \end{cases} \tag{1}$$

where ϱ_j is known premise variable, Φ_{ij} is the fuzzy set, $i = \overline{1, r}, j \in \overline{1, g}$, r is the number of fuzzy rules. $x(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^{n_z}$, $u(t) \in \mathbb{R}^{n_u}$, $\omega(t) \in \mathbb{R}^{n_\omega}$, and $\omega_d(t_k) \in \mathbb{R}^{n_d}$ denote the system state, the output, the control input, the continuous-time disturbance input, and the discrete-time disturbance input. A_i, B_i, E_i, J_i, F_i, C_i, and D_i are constant matrices. $\{t_k\}$ denotes the impulsive instant, which strictly increases and satisfies $t_0 = 0$ and $\lim_{k \to \infty} t_k = +\infty$; $\mathcal{T}(\sigma_0, \sigma_1)$ denote the class of impulse interval $\sigma_0 \leq T_k = t_k - t_{k-1} \leq \sigma_1$ for all $k \in \mathbb{N}$.

By fuzzy inference methods, the impulsive T-S fuzzy system (1) can be derived as follows [21], [24]:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^r h_i(\varrho(t)) \left(A_i x(t) + B_i u(t) + E_i \omega(t) \right), t \neq t_k \\ x(t_k) = \sum_{i=1}^r h_i(\varrho(t_k)) \left(J_i x(t_k^-) + F_i \omega_d(t_k) \right), k \in \mathbb{N} \\ z(t) = \sum_{i=1}^r h_i(\varrho(t)) \left(C_i x(t) + D_i \omega(t) \right) \end{cases} \tag{2}$$

where $\varrho(t) = [\varrho_1(t), \cdots, \varrho_g(t)]^T$ and $0 \leq h_i(\varrho(t)) \leq 1$ is membership function.

Lemma 1 ([21]): System (2) is said to be positive, if for all $x(t_0) \succeq 0$, disturbance input $\omega(\cdot) \succeq 0$, and $\omega_d(\cdot) \succeq 0$, there exists a control input $u(t)$ such that the trajectory $x(t) \succeq 0$, $\forall t \geq t_0$.

Lemma 2 ([24]): System (2) with $u(\cdot) \equiv 0$ is positive if and only if matrix A_i are Metzler, $E_i \succeq 0$, $J_i \succeq 0$, $F_i \succeq 0$, $C_i \succeq 0$, $D_i \succeq 0$ for any $i \in \overline{1, r}$.

Definition 1: The system (2) is said to be globally exponentially stable (GES) over $\mathcal{T}(\sigma_0, \sigma_1)$ and have an L_1-gain bound γ, if system (2) is GES when $\omega(\cdot) \equiv 0, \omega_d(\cdot) \equiv 0$ and $\int_0^\infty \|z(s)\|_1 ds \leq \gamma \int_0^\infty \|\omega(s)\|_1 ds + \gamma \sum_{k \in \mathbb{N}} \|\omega_d(t_k)\|_1$ is satisfied with the zero-initial condition for any nonnegative $\omega(t) \in L_1, \omega_d(t_k) \in \ell_1$.

III. Main results

A. Relaxed stability conditions

In this section, a new stability criteria for the positive implsive T-S fuzzy system (2) is obtained by constructing a polynomial ITD copositive Lyapunov function. Similar to previous studies such as [21], we can construct the following auxiliary functions for impulse intervals:

$$\rho(t) = \frac{1}{t_k - t_{k-1}}, \rho_{10}(t) = \frac{t_k - t}{t_k - t_{k-1}}, t \in [t_{k-1}, t_k).$$

Note that $\rho_{10}(t_{k-1}) = 1$, $\rho_{10}(t_k^-) = 0$, and $\dot{\rho}_{10}(t) = -\rho(t)$. Let $\rho_{11}(t) = 1 - \rho_{10}(t)$, we have $\rho_{11}(t_{k-1}) = 0$, $\rho_{11}(t_k^-) = 1$, and $\dot{\rho_{11}}(t) = \rho(t)$. Since $\sigma_0 \leq T_k \leq \sigma_1$, we have

$$\rho(t) = \frac{\rho_{20}(t)}{\sigma_0} + \frac{\rho_{21}(t)}{\sigma_1} \tag{3}$$

where $\rho_{20} \in [0, 1]$, $\rho_{21}(t) = 1 - \rho_{20}(t)$.

Generally, linear equation $\rho_{10}(t) + \rho_{11}(t) = 1$ is used to construct ITD Lyapunov function. On the other hand, it is noted $(\rho_{10}(t) + \rho_{11}(t))^N = 1$ contains more impulse intervals information such as the cross term and can be used for the construction of polynomial ITD Lyapunov functions. Based on the above, we design the following polynomial ITD copositive Lyapunov function:

$$V(t) = x^{\mathrm{T}}(t) \sum_{s=0}^N \mathcal{C}_N^s \rho_{10}^{N-s}(t) \rho_{11}^s(t) \nu_s, t \in [t_{k-1}, t_k) \tag{4}$$

where N is a prescribed positive integer, $\nu_s \in \mathbb{R}_+^n$. Let $\mathbb{V}_N(t) = \sum_{s=0}^N \mathcal{C}_N^s \rho_{10}^{N-s}(t) \rho_{11}^s(t) \nu_s$.

Obviously, $V(t) > 0$ is continuous inside $t \in [t_{k-1}, t_k)$, and $V(t_{k-1}) = x^{\mathrm{T}}(t_{k-1}) \nu_0$, $V(t_k^-) = x^{\mathrm{T}}(t_k^-) \nu_N$. To lighten the notation, we will use h_i and ρ_{10} instead of $h_i(\varrho(t))$ and $\rho_{10}(t)$, $t \neq t_k$.

Remark 1: The polynomial ITD copositive Lyapunov function (4) contains more impulse intrevals information than ITD copositive Lyapunov function utilized in the past study such as [23]. When $N = 0$ or $N = 1$, the designed Lyapunov function (4) will reduces to $V(t) = x^{\mathrm{T}}(t) \nu_0$ [33] or $V(t) = x^{\mathrm{T}}(t)(\rho_{10} \nu_0 + \rho_{11} \nu_1)$ [23], which means that the classic Lyapunov matrices time-invariant CLF and the ITD copositive Lyapunov function are special cases of (4).

Next, we will discuss $\dot{\mathbb{V}}_N(t)$ when $t \in [t_{k-1}, t_k)$. Note

$$
\begin{aligned}
\dot{\mathbb{V}}_N(t)=&\sum_{s=0}^{N}\mathcal{C}_N^s\big((N-s)\rho_{10}^{N-s-1}\rho_{11}^{s}\dot{\rho}_{10}+s\rho_{10}^{N-s}\rho_{11}^{s-1}\dot{\rho}_{11}\big)\nu_s\\
=&(\rho_{10}+\rho_{11})\rho\sum_{s=0}^{N}\mathcal{C}_N^s\bigg(\Big(-(N-s)\rho_{10}^{N-s-1}\rho_{11}^{s}\\
&+s\rho_{10}^{N-s}\rho_{11}^{s-1}\Big)\nu_s\bigg)\\
=&\rho\sum_{s=0}^{N}\mathcal{C}_N^s\bigg(\Big(-(N-2s)\rho_{10}^{N-s}\rho_{11}^{s}\\
&-(N-s)\rho_{10}^{N-s-1}\rho_{11}^{s+1}+s\rho_{10}^{N-s+1}\rho_{11}^{s-1}\Big)\nu_s\bigg). \quad (5)
\end{aligned}
$$

Let $\nu_{-1}\equiv 0$, $\nu_{N+1}\equiv 0$, $\varphi = s+1$, and $\psi = s-1$, one can obtained that

$$
\begin{aligned}
&\sum_{s=0}^{N}\mathcal{C}_N^s\left(-(N-s)\rho^{N-s-1}\widetilde{\rho}^{s+1}+s\rho^{N-s+1}\widetilde{\rho}^{s-1}\right)\nu_s\\
=&\sum_{s=0}^{N-1}\mathcal{C}_N^s\left(-(N-s)\rho^{N-s-1}\widetilde{\rho}^{s+1}\right)\nu_s\\
&+\sum_{s=1}^{N}\mathcal{C}_N^s\left(s\rho^{N-s+1}\widetilde{\rho}^{s-1}\right)\nu_s\\
=&\sum_{\varphi=1}^{N}\frac{N!}{(\varphi-1)!(N-\varphi+1)!}\left(-(N-\varphi+1)\rho^{N-\varphi}\widetilde{\rho}^{\varphi}\right)\nu_{(\varphi-1)}\\
&+\sum_{\psi=0}^{N-1}\frac{N!}{(\psi+1)!(N-\psi-1)!}\left((\psi+1)\rho^{N-\psi}\widetilde{\rho}^{\psi}\right)\nu_{\psi+1}\\
=&\sum_{\varphi=1}^{N}\frac{N!}{(\varphi)!(N-\varphi)!}\left(-\varphi\rho^{N-\varphi}\widetilde{\rho}^{\varphi}\right)\nu_{\varphi-1}\\
&+\sum_{\psi=0}^{N-1}\frac{N!}{(\psi)!(N-\psi)!}\left((N-\psi)\rho^{N-\psi}\widetilde{\rho}^{\psi}\right)\nu_{\psi+1}\\
=&\sum_{s=0}^{N}\mathcal{C}_N^s\left(-s\rho^{N-s}\widetilde{\rho}^{s}\nu_{s-1}+(N-s)\rho^{N-s}\widetilde{\rho}^{s}\nu_{s+1}\right). \quad (6)
\end{aligned}
$$

Substitute (6) into (5), we have

$$
\begin{aligned}
\dot{\mathbb{V}}_N=&\rho\sum_{s=0}^{N}\mathcal{C}_N^s\rho_{10}^{N-s}\rho_{11}^{s}\Big(-(N-2s)\nu_s-s\nu_{s-1}\\
&+(N-s)\nu_{s+1}\Big). \quad (7)
\end{aligned}
$$

For simplicity

$$
\begin{aligned}
\mathit{\Lambda}^N=&\rho\sum_{s=0}^{N}\mathcal{C}_N^s\rho_{10}^{N-s}\rho_{11}^{s}\Big(-(N-2s)\nu_s-s\nu_{s-1}\\
&+(N-s)\nu_{s+1}\Big), \quad (8)\\
\mathit{\Lambda}_s^N=&-(N-2s)\nu_s-s\nu_{s-1}+(N-s)\nu_{s+1}. \quad (9)
\end{aligned}
$$

Remark 2: In [35], a ITD discrete Lyapunov function with polynomial time dependent ν_s is proposed. However, the stability condition need to limit the time t. Obviously, for $t\in[t_{k-1},t_k)$, $\mathbb{V}_N(t)$ is continuous, and $D^+\mathbb{V}_N(t)=\dot{\mathbb{V}}_N(t)$, which means that we will not increase any constraints when using the Lyapunov function (4) for stability analysis. Compared to general linear impulse-time dependent in [21], only ν_s and ν_{s-1} are related. IN Equation (7), ν_s, ν_{s-1} and ν_{s+1} are all related. This shows that polynomial ITD strengthen the internal connection of the Lyapunov function inside an impulse interval.

Theorem 1: Given a integer $N\ge 0$, a scalar $\mu>0$, the system (2) with $u(\cdot)\equiv 0$, $\omega(\cdot)\equiv 0$, $\omega_d(\cdot)\equiv 0$ is GES over $\mathcal{T}(\sigma_0,\sigma_1)$, if exist vectors $\nu_s\in\mathbb{R}_+^n$, $s\in\overline{0,N}$, $i\in\overline{1,r}$, $\nu_{-1}\equiv 0$, $\nu_{N+1}\equiv 0$, such that (9) and the following inequalities hold:

$$
\left(A_i^{\mathrm{T}}+\frac{\ln\mu}{\sigma(\mu)}I\right)\nu_s+\frac{\mathit{\Lambda}_s^N}{\sigma_0}\prec 0 \quad (10)
$$

$$
\left(A_i^{\mathrm{T}}+\frac{\ln\mu}{\sigma(\mu)}I\right)\nu_s+\frac{\mathit{\Lambda}_s^N}{\sigma_1}\prec 0 \quad (11)
$$

$$
J_i^{\mathrm{T}}\nu_0-\mu\nu_N\preceq 0 \quad (12)
$$

where

$$
\sigma(\mu)=\begin{cases}\sigma_1 & \mu\in(0,1]\\ \sigma_0 & \mu>1\end{cases}. \quad (13)
$$

Proof: Using (4) and (7), for $t\in[t_k,t_{k+1})$, $k\in\mathbb{N}$, we have

$$
D^+V(t)=x^{\mathrm{T}}(t)\sum_{s=0}^{N}\mathcal{C}_N^s\rho_{10}^{N-s}\rho_{11}^{s}\left(\sum_{i=1}^{r}h_iA_i^{\mathrm{T}}\nu_s+\rho\mathit{\Lambda}_s^N\right).
$$

From (3), (10), and (11), we have

$$
\begin{aligned}
\mathit{\Omega}_s=&\left(\sum_{i=1}^{r}h_iA_i^{\mathrm{T}}+\frac{\ln\mu}{\sigma(\mu)}I\right)\nu_s+\rho\mathit{\Lambda}_s^N\\
=&\left(\sum_{i=1}^{r}h_iA_i^{\mathrm{T}}+\frac{\ln\mu}{\sigma(\mu)}I\right)\nu_s+\left(\frac{\rho_{20}}{\sigma_0}+\frac{\rho_{21}}{\sigma_1}\right)\mathit{\Lambda}_s^N\\
=&\rho_{20}\left(\left(\sum_{i=1}^{r}h_iA_i^{\mathrm{T}}+\frac{\ln\mu}{\sigma(\mu)}I\right)\nu_s+\frac{\mathit{\Lambda}_s^N}{\sigma_0}\right)\\
&+\rho_{21}\left(\left(\sum_{i=1}^{r}h_iA_i^{\mathrm{T}}+\frac{\ln\mu}{\sigma(\mu)}I\right)\nu_s+\frac{\mathit{\Lambda}_s^N}{\sigma_1}\right)\\
\prec& 0.
\end{aligned}
$$

Then we can conclude that there exists a positive scalar α, such that

$$
\begin{aligned}
&D^+V(t)+\left(\frac{\ln\mu}{\sigma(\mu)}+\alpha\right)V(t)\\
&=x^{\mathrm{T}}(t)\sum_{s=0}^{N}\mathcal{C}_N^s\rho_{10}^{N-s}\rho_{11}^{s}(\mathit{\Omega}_s+\alpha I\nu_s)\\
&<0. \quad (14)
\end{aligned}
$$

For the impulse moment $t=t_k$, combining (2), (4), and (12), we have

$$
\begin{aligned}
V(t_k)=&\sum_{i=1}^{r}h_i(\varrho(t_k))x^{\mathrm{T}}(t_k^-)J_i^{\mathrm{T}}\nu_0\\
\le&\sum_{i=1}^{r}h_i(\varrho(t_k))x^{\mathrm{T}}(t_k^-)\mu\nu_N=\mu V(t_k^-). \quad (15)
\end{aligned}
$$

Next, similar to [21], we need to distinguish $\mu \in (0,1]$ and $\mu > 1$ to complete the proof.

When $\mu \in (0,1]$, we have $\sigma(\mu) = \sigma_1$, $0 \leq -\frac{\ln\mu}{\sigma_1}(t-t_k) \leq -\ln\mu$. Then, from (14), one gets

$$\begin{aligned} V(t) &< \mathrm{e}^{-\alpha(t-t_k)}\mathrm{e}^{-\frac{\ln\mu}{\sigma_1}(t-t_k)}V(t_k) \\ &\leq \frac{1}{\mu}\mathrm{e}^{-\alpha(t-t_k)}V(t_k). \end{aligned} \tag{16}$$

Working on (16) recursively with (15), we have

$$\begin{aligned} V(t) &< \frac{1}{\mu}\mathrm{e}^{-\alpha(t-t_k)}\mu V(t_k^-) < \frac{1}{\mu}\mathrm{e}^{-\alpha(t-t_k)}\mathrm{e}^{-\alpha(t_k^- - t_{k-1})}V(t_{k-1}) \\ &< \cdots < \frac{1}{\mu}\mathrm{e}^{-\alpha(t-t_0)}V(t_0). \end{aligned} \tag{17}$$

Since $\lambda_{\min}(\nu_s)\|x(t)\|_1 \leq V(t) \leq \lambda_{\max}(\nu_s)\|x(t)\|_1$, $s \in \overline{0,N}$, we obtain from (17) that

$$\|x(t)\|_1 < \frac{\lambda_{\max}(\nu_s)}{\mu\lambda_{\min}(\nu_s)}\mathrm{e}^{-\alpha(t-t_0)}\|x(t_0)\|_1. \tag{18}$$

When $\mu > 1$, $\sigma(\mu) = \sigma_0$. For $t \in [t_k, t_{k+1})$, applying the inequality of $-\frac{\ln\mu}{\sigma_0}(t-t_k) < 0$ and (14), the following inequality holds:

$$V(t) < \mathrm{e}^{-\alpha(t-t_k)}\mathrm{e}^{-\frac{\ln\mu}{\tau_0}(t-t_k)}V(t_k) \leq \mathrm{e}^{-\alpha(t-t_k)}V(t_k). \tag{19}$$

For $t \in [t_{k-1}, t_k)$, by the inequality $-\frac{\ln\mu}{\tau_0}\left(t_k^- - t_{k-1}\right) \leq -\frac{\ln\mu}{\tau_0}\tau_0 = -\ln\mu$, and (14), we have

$$\begin{aligned} V\left(t_k^-\right) &< \mathrm{e}^{-\alpha\left(t_k^- - t_{k-1}\right)}\mathrm{e}^{-\frac{\ln\mu}{\tau_0}\left(t_k^- - t_{k-1}\right)}V(t_{k-1}) \\ &< \frac{1}{\mu}\mathrm{e}^{-\alpha\left(t_k^- - t_{k-1}\right)}V(t_{k-1}). \end{aligned} \tag{20}$$

Then, using (15), (19) and (20), yields

$$\begin{aligned} V(t) &< \mathrm{e}^{-\alpha(t-t_k)}\mu V(t_k^-) \\ &< \mathrm{e}^{-\alpha(t-t_k)}\mu\frac{1}{\mu}\mathrm{e}^{-\alpha\left(t_k^- - t_{k-1}\right)}\cdots\mu\frac{1}{\mu}\mathrm{e}^{-\alpha\left(t_1^- - t_0\right)}V(t_0) \\ &= \mathrm{e}^{-\alpha(t-t_0)}V(t_0) \end{aligned} \tag{21}$$

which implies that

$$\|x(t)\|_1 < \frac{\lambda_{\max}(\nu_s)}{\lambda_{\min}(\nu_s)}\mathrm{e}^{-\alpha(t-t_0)}\|x(t_0)\|_1. \tag{22}$$

Therefore, the proof is completed.

Remark 3: Theorem 1 provides a new condition for the exponential stability of impulsive positive fuzzy systems based on the Lyapunov function (4). Although the Lyapunov function (4) contains non-convex polynomials, the resulting stability conditions are expressed as convex linear programming, and can be solved by existing optimization algorithms. Obviously, the stability analysis results in Theorem 1 can be combined with the discrete method in paper [21] to perform equidistant partition on the impulse interval. However, according to our simulation results, the stability analysis results of Theorem 1 are identical to the eigenvalue analysis results for linear systems. Further discretization is unnecessary. Details are shown in Example 1.

Theorem 2: Given scalars $\gamma > 0$, $\mu > 0$, and a integer $N \geq 0$, the system (2) with $u(\cdot) \equiv 0$ is GES and has L_1 -gain γ over $\mathcal{T}(\sigma_0, \sigma_1)$, if there exists vectors $\nu_s \in \mathbb{R}_+^n$, $\nu_{-1} \equiv 0$, $\nu_{N+1} \equiv 0$, $s \in \overline{0,N}$, $i \in \overline{1,r}$, such that (9), (12), (13), and the following inequalities hold:

$$\left(A_i^{\mathrm{T}} + \frac{\ln\mu}{\sigma(\mu)}I\right)\nu_s + \frac{\Lambda_s^N}{\sigma_0} + C_i^{\mathrm{T}}\mathbf{1} \prec 0 \tag{23}$$

$$\left(A_i^{\mathrm{T}} + \frac{\ln\mu}{\sigma(\mu)}I\right)\nu_s + \frac{\Lambda_s^N}{\sigma_1} + C_i^{\mathrm{T}}\mathbf{1} \prec 0 \tag{24}$$

$$E_i^{\mathrm{T}}\nu_s + D_i^{\mathrm{T}}\mathbf{1} - \gamma\phi(\mu)\mathbf{1} \prec 0 \tag{25}$$

$$F_i^{\mathrm{T}}\nu_0 - \gamma\phi(\mu)\mathbf{1} \preceq 0 \tag{26}$$

where

$$\phi(\mu) = \begin{cases} \mu & \mu \in (0,1] \\ \mathrm{e}^{\left(1-\frac{\sigma_1}{\sigma_0}\right)\ln\mu} & \mu > 1 \end{cases}. \tag{27}$$

Proof: Note that (23) and (24) implies (10) and (11). When $\omega(\cdot) \equiv 0$ and $\omega_d(\cdot) \equiv 0$, if (23), (24), and (12) hold, the positive impulsive T-S fuzzy system (2) is GES by Theorem 1. Next, the L_1-gain performance will be achieved.

By using the convex combination method, for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, we can compute from (2), (4), (23), (24), and (25) that

$$\begin{aligned} &D^+V(t) + \frac{\ln\mu}{\sigma(\mu)}V(t) + \|z(t)\|_1 - \gamma\phi(\mu)\|\omega(t)\|_1 \\ =&\sum_{s=0}^{N}\mathcal{C}_N^s\rho_{10}^{N-s}\rho_{11}^s\sum_{i=1}^{r}h_i\Bigg(\omega^{\mathrm{T}}\left(E_i^{\mathrm{T}}\nu_s + D_i^{\mathrm{T}}\mathbf{1} - \gamma\phi(\mu)\mathbf{1}\right) \\ &+ x^{\mathrm{T}}\left(\left(A_i + \frac{\ln\mu}{\sigma(\mu)}\right)\nu_s + \rho\Lambda_s^N + C_i^{\mathrm{T}}\mathbf{1}\right)\Bigg) \\ <&0. \end{aligned} \tag{28}$$

For the impulse moment $t = t_k$, since $\omega_d(t_k) \succeq 0$, we can obtain from (2), (4), (12), and (26) that

$$\begin{aligned} V(t_k) &= \sum_{i=1}^{r}h_i(\varrho(t_k))\left(x^{\mathrm{T}}\left(t_k^-\right)J_i^{\mathrm{T}} + \omega_d^{\mathrm{T}}(t_k)F_i^{\mathrm{T}}\right)\nu_0 \\ &\leq \mu V\left(t_k^-\right) + \gamma\phi(\mu)\|\omega_d(t_k)\|_1. \end{aligned} \tag{29}$$

Based on (28), (29), and the results in [21], under the zero initial condition, when $\mu \in (0,1]$, we have

$$\begin{aligned} &\int_0^\infty \|z(s)\|_1 ds - \gamma\int_0^\infty \|\omega(s)\|_1 ds - \gamma\sum_{k=1}^{\infty}\|\omega_d(t_k)\|_1 \\ &\leq -V(\infty) \leq 0. \end{aligned}$$

When $\mu > 1$, we have

$$\begin{aligned} &\int_0^\infty \|z(s)\|_1 ds - \gamma\int_0^\infty \|\omega(s)\|_1 ds - \gamma\sum_{k=1}^{\infty}\|\omega_d(t_k)\|_1 \\ &\leq -\mathrm{e}^{\frac{\tau_1\ln\mu}{\tau_0}}V(\infty) \leq 0. \end{aligned}$$

Therefore, we get the conclusion.

Remark 4: Note that the stability condition of Theorem 2 is expressed by convex linear programming, which shows the importance of binomial coefficients $\mathcal{C}_N^s$.

B. State-feedback stabilization

The parallel-distributed compensation scheme is adopted to design a controller for the positive impulsive fuzzy systems (2) as follows:

$$u(t)=\sum_{i=1}^{r} h_i K_i x(t), t \neq t_k \tag{30}$$

where $K_i \in \mathbb{R}^{n_u \times n}$ is control gain and h_i is the same as the ith rule membership function of the positive impulsive systems (2). Then, we consider the closed-loop impulsive T-S fuzzy system obtained based on (2) and (30), which is shown as follows:

$$\begin{cases} \dot{x}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r} h_i h_j\Big(\left(A_i+B_i K_j\right) x(t)+E_i \omega(t)\Big), t \neq t_k \\ x\left(t_k\right)=\sum_{i=1}^{r} h_i\left(\varrho\left(t_k\right)\right)\left(J_i x\left(t_k^{-}\right)+F_i \omega_d\left(t_k\right)\right), k \in \mathbb{N} \\ z(t)=\sum_{i=1}^{r} h_i\left(C_i x(t)+D_i \omega(t)\right) \end{cases} \tag{31}$$

Theorem 3: Given scalars $\gamma>0$, $\mu>0$, and a integer $N \geq 0$. the closed-loop system (2) is GES and has L_1-gain γ over $\mathcal{T}\left(\sigma_0, \sigma_1\right)$, if exist vectors $\nu_s \in \mathbb{R}_{+}^{n}$, $\nu_{-1} \equiv 0$, $\nu_{N+1} \equiv 0$, $\eta \in \mathbb{R}_{+}^{n_z}$, $\zeta_j \in \mathbb{R}^n \preceq 0$, $\xi_j \in \mathbb{R}^n \succeq 0$, scalars $0<p \leq 1$, $1 \leq q$, $\beta>0$, $s \in \overline{0, N}$, $i, j \in \overline{1, r}$, such that(9), (12), (13), (25), (26), (27), and the following inequalities hold:

$$p \nu_s \preceq \vartheta_0 \preceq \nu_s \tag{32}$$

$$\nu_s \preceq \vartheta_1 \preceq q \nu_s \tag{33}$$

$$\Omega_{ijs}+\Omega_{jis}+\beta I \succeq 0 \tag{34}$$

$$\left(A_i^{\mathrm{T}}+\frac{\ln \mu}{\sigma(\mu)} I\right) \nu_s+\zeta_i+\xi_i+\frac{\Lambda_s^N}{\sigma_0}+C_i \mathbf{1} \prec 0 \tag{35}$$

$$\left(A_i^{\mathrm{T}}+\frac{\ln \mu}{\sigma(\mu)} I\right) \nu_s+\zeta_i+\xi_i+\frac{\Lambda_s^N}{\sigma_1}+C_i \mathbf{1} \prec 0 \tag{36}$$

$$\tag{37}$$

where

$$\Omega_{ijs}=\eta^T B_j^T \nu_s A_i+B_i \eta\left(\frac{1}{q} \xi_j^{\mathrm{T}}+\frac{1}{p} \zeta_j^{\mathrm{T}}\right)$$

under the control law (30), and the controller gain

$$K_j=\frac{1}{\eta^{\mathrm{T}} B_h^{\mathrm{T}} \vartheta_0} \eta \zeta_j^{\mathrm{T}}+\frac{1}{\eta^{\mathrm{T}} B_h^{\mathrm{T}} \vartheta_1} \eta \xi_j^{\mathrm{T}} .$$

Proof: By Theorem 2, the stability conditions of system (2) with the controller (30) can be formulated as inequalities (12), (25), (26) and

$$\left(A_h^{\mathrm{T}}+K_h^{\mathrm{T}} B_h^{\mathrm{T}}+\frac{\ln \mu}{\sigma(\mu)}+\alpha I\right) \mathbb{V}_N+\rho \Lambda_s^N+C_h^{\mathrm{T}} \mathbf{1} \prec 0 \tag{38}$$

where $A_h=\sum_{i=1}^r h_i A_i$, and B_h, K_h have the same definition. By the definitions of K_j, it is obtained that

$$K_j^{\mathrm{T}} B_h^{\mathrm{T}} \mathbb{V}_N=\frac{\eta^{\mathrm{T}} B_h^{\mathrm{T}} \mathbb{V}_N}{\eta^{\mathrm{T}} B_h^{\mathrm{T}} \vartheta_0} \zeta_j+\frac{\eta^{\mathrm{T}} B_h^{\mathrm{T}} \mathbb{V}_N}{\eta^{\mathrm{T}} B_h^{\mathrm{T}} \vartheta_1} \xi_j$$

TABLE I

MAXIMUM UPPER BOUND OF T FOR SYSTEM (41) WITH $\mu=0.5$

N	1	3	5	7	10
[21]	1.69	1.809	1.861	1.886	1.906
Theorem 1	1.69	1.875	1.943	1.959	1.9608

TABLE II

MINIMUM LOWER BOUND OF T FOR SYSTEM (42) WITH $\mu=3$

N	1	3	5	10
[21]	0.316	0.295	0.291	0.287
Theorem 1	0.316	0.2831	0.2829	0.2829

which, together with (32), (33), and $\eta \succeq 0$, $B_h \succeq 0$, $\mathbb{V}_N \succ 0$, we have

$$\begin{aligned} & p \eta^{\mathrm{T}} B_h^{\mathrm{T}} \mathbb{V}_N \leq \eta^T B_h^T \vartheta_0 \leq \eta^{\mathrm{T}} B_h^{\mathrm{T}} \mathbb{V}_N, \\ & \eta^{\mathrm{T}} B_h^{\mathrm{T}} \mathbb{V}_N \leq \eta^T B_h^T \vartheta_1 \leq q \eta^{\mathrm{T}} B_h^{\mathrm{T}} \mathbb{V}_N, \end{aligned}$$

and

$$\frac{1}{p} \zeta_j+\frac{1}{q} \xi_j \preceq K_j^{\mathrm{T}} B_h^{\mathrm{T}} \mathbb{V}_N \preceq \zeta_j+\xi_j . \tag{39}$$

Then we can see (38) is ensured by (35) and (36) Next, we consider the positivity of the closed-loop impulsive T-S fuzzy system (31). From (39), we have

$$\begin{aligned} & \left(\eta^T B_h^T \mathbb{V}_N A_h+B_h \eta\left(\frac{1}{q} \xi_h^{\mathrm{T}}+\frac{1}{p} \zeta_h^{\mathrm{T}}\right)\right) \frac{1}{\eta^T B_h^T \mathbb{V}_N} \\ & =A_h+B_h\left(\frac{1}{p \eta^T B_h^T \mathbb{V}_N} \eta \zeta_h^{\mathrm{T}}+\frac{1}{q \eta^T B_h^T \mathbb{V}_N} \eta \xi_h^{\mathrm{T}}\right) \\ & \preceq A_h+B_h\left(\frac{1}{\eta^{\mathrm{T}} B_h^{\mathrm{T}} \vartheta_0} \eta \zeta_h^{\mathrm{T}}+\frac{1}{\eta^{\mathrm{T}} B_h^{\mathrm{T}} \vartheta_1} \eta \xi_h^{\mathrm{T}}\right) \\ & =A_h+B_h K_h \end{aligned}$$

Obviously, if (34) holds, $A_h+B_h K_h$ is Metzler. The proof is completed.

Remark 5: Using positive characteristics and linear scaling, an impulse time independent controller gain is designed in Theorem 3 for stabilization analysis. We divide the controller gain K_j into positive part and negative part by ξ_j and ζ_j, and their values are determined by search the parameters p and q. It should be pointed out that only when $\xi_j+\zeta_j \preceq 0$, the controller (30) can be used to stability analysis for system (31).

IV. Numerical Example

In this section, we use three numerical examples to show the validity of the research results. Example 1 shows the application in linear positive impulsive systems GES analysis. Example 2 shows the applications in positive impulsive T-S fuzzy systems GES analysis and L_2-gain. Example 3 uses a two linked tanks system to show the application of the research results in the real system. These routines are implemented in MATLAB using YALMIP with MOSEK.

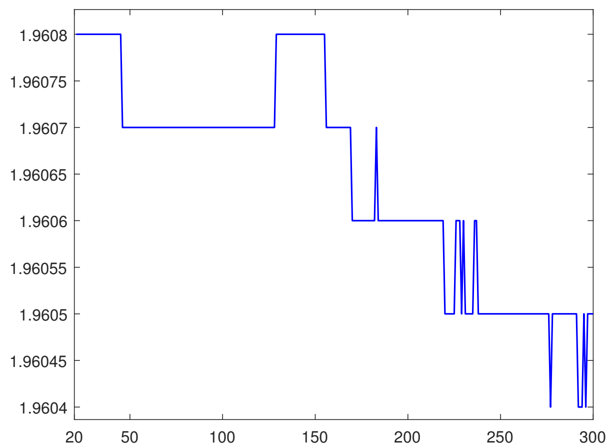

Fig. 1. $T_{\max}$ for system (41) with different values of N

Example 1: Consider the following linear impulsive model:

$$\begin{cases} \dot{x}(t) = Ax(t), & t \neq t_k \\ x(t_k) = Jx(t_k^-), & k \in \mathbb{N} \end{cases} \tag{40}$$

Case 1: $\sigma_0 = \sigma_1 = T$,

$$A = \begin{bmatrix} 0.1 & 1 \\ 1 & -1 \end{bmatrix}, J = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}. \tag{41}$$

Case 2: $\sigma_0 = \sigma_1 = T$,

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix}, J = \begin{bmatrix} 1.7 & 0.3 \\ 0.2 & 1.8 \end{bmatrix}. \tag{42}$$

Case 3: $\sigma_0 \leq T_k \leq \sigma_1$,

$$A = \begin{bmatrix} -1 & 0.1 \\ 0 & 1.2 \end{bmatrix}, J = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{43}$$

Case 1 and *Case 2*. Obviously, for the positive system (41) and (42), A is Metzler and $B \succeq 0$. By the eigenvalue analysis of $e^{AT}J$, for exponential stability, system (41) admits a maximum impulse interval $T \leq 1.9608$, and system (42) admits a minimum impulse interval $T \geq 0.2829$. Table I lists the maximum upper bound of T for system (41) and Table II lists the minimum upper bound of T for system (42) obtained by the method in [21] and Theorem 1 in this work. When N=1, the Lyapunov functions in this paper and in [21] both reduce to ITD copositive Lyapunov function, so the same results are obtained. Obviously, when $N = 10$, the same result as the eigenvalue analysis method can be obtained by using Theorem 1. Although the method in [21] can also reach 1.9602 for system (41) and 0.283 for system (42), the number of partitions needs to reach 500. In [21], the number of variables ν_s is N+1, and the number of linear programming inequalities is $(4N+1)r$. In Theorem 1, the number of variables ν_s is also $N+1$, but the number of linear programming inequalities is $(2N+1)r$. It can be seen that the method in this paper can use a lower number of variables to get better results than [21].

Fig. 1 shows the maximum upper bound of impulse interval $T_{\max}$ is calculated by different degree of polynomial $N \in [20, 300]$. This shows that, unlike in [21], the conservative degree of stability analysis results decreases gradually with the increase of the number of partitions, the conservative degree of stability analysis results can be effectively reduced by selecting the appropriate degree of polynomial in this paper. The influence of a high degree of polynomial on the results also appears in the paper [35], which we all consider a problem with the capability of the solver. However, since the Lyapunov function (4) is polynomial ITD and the stability condition is linear programming, it is difficult to give strict mathematical proof that the degree of conservatism will gradually decrease as N increases. Further explanation is provided in Example 2.

TABLE III
MAXIMUM UPPER BOUND ON σ_1 FOR SYSTEM (43) WITH $\mu > 1$

N	1	10	20	30
[21]	0.2712	0.4044	0.4239	0.4361
[24]	0.4857	0.5636	0.5701	0.5726
Theorem 1	0.2712	0.5774	0.5775	0.5776

Case 3. In this case, system (43) has time-varying impulse intervals, and the system stability range is $T_k \in [0.1824, 0.5776]$ according to the eigenvalue analysis. The GES results obtained by the methods in [21], [24] and this paper will vary depending on the parameter μ. In the previous *Case 1* and *Case 2*, the calculation is based on the fixed μ value. Let $\sigma_0 = 0.1824$, Table III lists the maximal upper bound of σ_1 obtained by [21], [24] and Theorem 1 in this paper when $\mu > 1$. It can be seen that our result is less conservative than in [21] and [24].

Example 2: Consider the following impulsive system in [21]:

$$\begin{cases} \dot{x}_1(t) = (-2 + 0.2\sin^2(x_1(t)))x_1(t) + 0.1x_2(t) + u_1(t) \\ \qquad + 0.5\omega(t) \\ \dot{x}_2(t) = (0.15 + 0.1\sin^2(x_1(t)))x_1(t) + u_2(t) \\ \qquad + (0.2 + 0.1\sin^2(x_1))x_2(t) \\ x_1(t_k) = (1.1 + 0.1\sin^2(x_1(t_k^-)))x_1(t_k^-) + 0.1x_2(t_k^-) \\ \qquad + 0.5\omega_d(t_k) \\ x_2(t_k) = 0.1x_1(t_k^-) + 0.8x_2(t_k^-) \\ z(t) = 0.2x_1(t) + 0.1x_2(t) + 0.1\omega(t) \end{cases}$$

By fuzzy inference methods, this nonlinear system can be represented by a two-rule impulsive T-S fuzzy model (2) with

$$A_1 = \begin{bmatrix} -2 & 0.1 \\ 0.15 & 0.2 \end{bmatrix}, A_2 = \begin{bmatrix} -1.8 & 0.1 \\ 0.25 & 0.3 \end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_1 = E_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} 1.1 & 0.1 \\ 0.1 & 0.8 \end{bmatrix}, J_2 = \begin{bmatrix} 1.2 & 0.1 \\ 0.1 & 0.8 \end{bmatrix},$$

$$F_1 = F_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}^{\mathrm{T}}, D_1 = D_2 = 0.1$$

and $h_1 = 1 - \sin^2(x_1(t))$, $h_2 = \sin^2(x_1(t))$.

Case (1). $\omega(\cdot) \equiv 0$, $\omega_d(\cdot) \equiv 0$ and $u(\cdot) \equiv 0$. For this example, the system is GES over the impulse interval $T_k \in [0.16, 0.58]$ in [21] and $T_k \in [0.1487, 0.5973]$ in [24].

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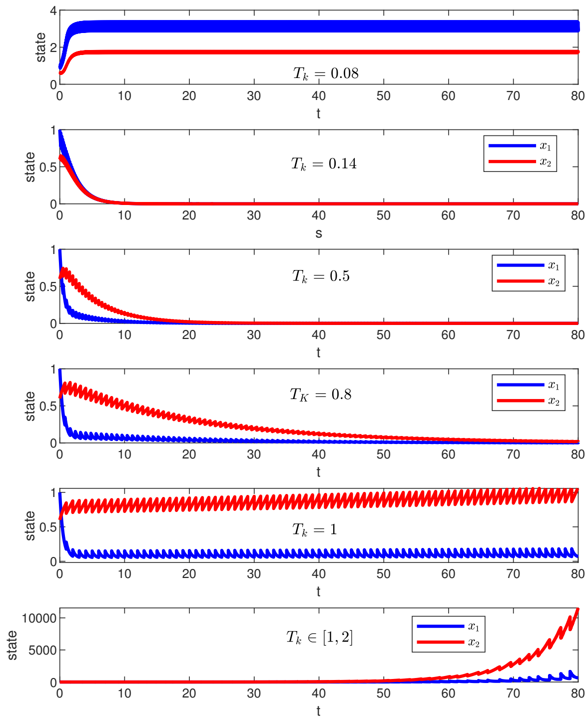

Fig. 2. State responses of the system with $u(\cdot) \equiv 0$ for Example 2 with different impulse intervals.

By Theorem 1 with $N = 10$, we have the system is GES over $T_k \in [0.1478, 0.604]$. Note that a linear impulsive system composed of A_2 and E_2 is GES over $T_k \in [0.1478, 0.604]$ by eigenvalue analysis. Fig. 2 shows the system state trajectories of this example under different impulse intervals. It can be seen that the impulsive system is still stable at $T = 0.14$ and $T = 0.8$, which proves the effectiveness of our results. Let $\sigma_0 = 0.2$, Table IV list the maximal upper bound of σ_1 obtained by [21], [24] and Theorem 1 in this paper. It is clear that our method can obtain better results with fewer parameters.

Case (2). $u(\cdot) \equiv 0$. Let $T_k = 0.21$, Table V lists the L_1-gain γ computed by [21], [24], and Theorem 2 in this paper. It can be seen that the L_1 performance γ changes with the polynomial degree N. It is worth noting that in general linear programming programs, there are two solvers: Feasp and Mincx. The Feasp can be used for Theorem 1, and the Mincx can be used for Theorem 2. For this example, when using the Mincx, $\min(\gamma) = 0.4837$ for all $N \geq 5$. This shows that in Example 1, the conservativeness of exponential stability analysis results slightly increases with the increase of N, which due to the problem of solver accuracy.

Case (3). As shown in Fig. 2, the system is unstable when impulse interval $T_k \in [1, 2]$ with $\omega(\cdot) \equiv 0$, $\omega_d(\cdot) \equiv 0$ and $u(\cdot) \equiv 0$. Assume that the continuous-time disturbance input $\omega(t) = 0.5\mathrm{e}^{-0.5t}$ and the discrete-time disturbance input $\omega_d(t_k) = \begin{cases} 2 * \sin(0.2t_k), t_k \in [0, 15] \\ 0, t_k > 8 \end{cases}$. We consider using the controller (30) such that the closed-loop impulsive fuzzy system is GES with L_1-gain and positive. Applying Theorem 3 with $N = 8$, $u = 1.2$, $p = 0.65$, $q = 1.69$, $\eta = [1, 8]^{\mathrm{T}}$, $\gamma = 0.26$, we have

$$K_1 = \begin{bmatrix} -0.0127 & -0.0772 \\ -0.1013 & -0.6178 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -0.0239 & -0.0817 \\ -0.1909 & -0.6537 \end{bmatrix}.$$

TABLE IV
MAXIMUM UPPER BOUND ON σ_1 OF EXAMPLE 2 WITH $\mu > 1$

N	1	3	5	7	10
[21]	0.2697	0.3369	0.3506	0.3564	0.3609
[24]	0.2841	0.3398	0.3522	0.3574	0.3616
Theorem 1	0.2697	0.3709	0.3713	0.3715	0.3715

TABLE V
VALUES OF L_1 OF EXAMPLE 2 WITH $\mu > 1$

N	1	2	5	7	10
[21]	Infeasible	0.88	0.63	0.58	0.54
[24]	0.83	/	/	0.55	0.54
Theorem 2	Infeasible	0.4860	0.4837	0.4837	0.4837

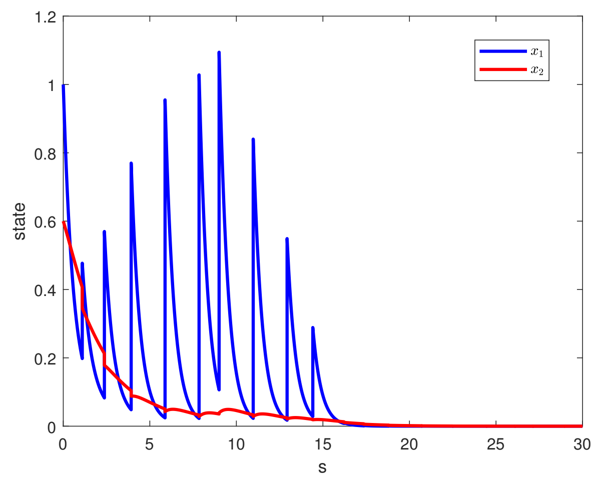

Fig. 3. State responses of the system for Example 2 with $T_k \in [1, 2]$

Let the initial states be $x(0) = [1, 0.6]$, The state trajectories of the system in this example with the controller (30) are shown in Fig. 3. Obviously, the controller (30) renders the impulse fuzzy system stable and maintain positivity.

Example 3: Consider the two linked tanks system in [24] and [31] described by the following T-S model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{4} h_i(A_i x(t) + B_i u(t)) \\ z(t) = \sum_{i=1}^{4} h_i C_i(x(t) - x_r) \end{cases} \tag{44}$$

where the membership functions $h_1 = f_{11}(t)f_{21}(t)$, $h_2 = f_{11}(t)f_{22}(t)$, $h_3 = f_{12}(t)f_{21}(t)$, $h_4 = f_{12}(t)f_{22}(t)$, with

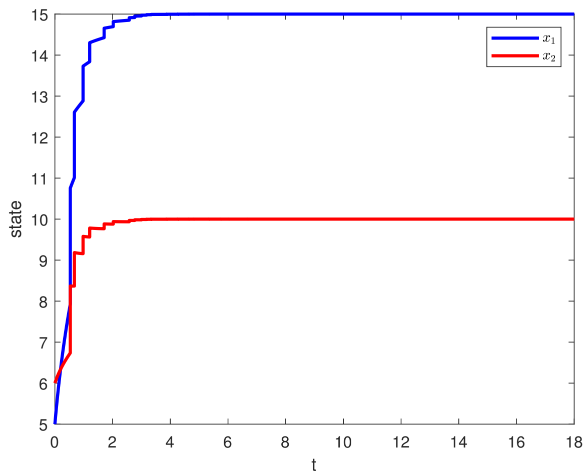

Fig. 4. State responses of the system without disturbance for Example 3

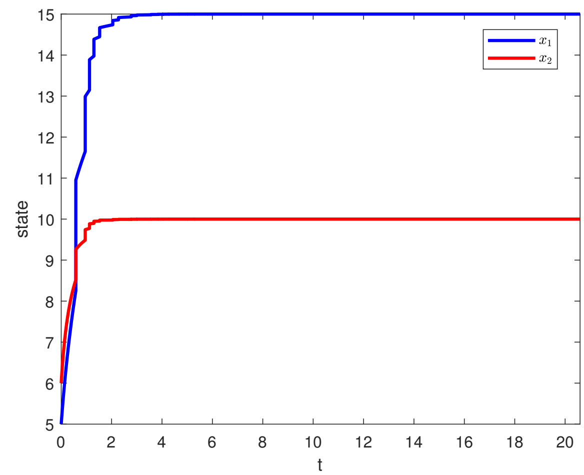

Fig. 5. State responses of the system with disturbance for Example 3

$f_{i1}(t)=\frac{z_i(t)-b_i}{a_i-b_i}$, $f_{i2}(t)=1-f_{i1}(t)$, $i=1,2$.

$$A_1=\begin{bmatrix} -R_1a_1-\frac{R_{12}a_1a_2}{\sqrt{|a_1^2-a_2^2|}} & \frac{R_{12}a_1a_2}{\sqrt{|a_1^2-a_2^2|}} \\ \frac{R_{12}a_1a_2}{\sqrt{|a_1^2-a_2^2|}} & -R_2a_2+\frac{R_{12}a_1a_2}{\sqrt{|a_1^2-a_2^2|}} \end{bmatrix},$$

$$A_2=\begin{bmatrix} -R_1a_1-\frac{R_{12}a_1b_2}{\sqrt{|a_1^2-b_2^2|}} & \frac{R_{12}a_1b_2}{\sqrt{|a_1^2-b_2^2|}} \\ \frac{R_{12}a_1b_2}{\sqrt{|a_1^2-b_2^2|}} & -R_2b_2+\frac{R_{12}a_1b_2}{\sqrt{|a_1^2-b_2^2|}} \end{bmatrix},$$

$$A_3=\begin{bmatrix} -R_1b_1-\frac{R_{12}b_1a_2}{\sqrt{|b_1^2-a_2^2|}} & \frac{R_{12}b_1a_2}{\sqrt{|b_1^2-a_2^2|}} \\ \frac{R_{12}b_1a_2}{\sqrt{|b_1^2-a_2^2|}} & -R_2a_2+\frac{R_{12}b_1a_2}{\sqrt{|b_1^2-a_2^2|}} \end{bmatrix},$$

$$A_4=\begin{bmatrix} -R_1b_1-\frac{R_{12}b_1b_2}{\sqrt{|b_1^2-b_2^2|}} & \frac{R_{12}b_1b_2}{\sqrt{|b_1^2-b_2^2|}} \\ \frac{R_{12}b_1b_2}{\sqrt{|b_1^2-b_2^2|}} & -R_2b_2+\frac{R_{12}b_1b_2}{\sqrt{|b_1^2-b_2^2|}} \end{bmatrix},$$

$$B_i=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C_i=\begin{bmatrix} 0.5 & 0.5 \end{bmatrix}.$$

In order to track the reference value x_r and render the closed-loop system positive, the following controller is designed:

$$\begin{aligned} u(t)=&\sum_{i=1}^{4}h_iK_i\left(x(t)-x_{\rm r}\right)-\sum_{i=1}^{4}h_iA_ix_{\rm r} \\ &+\sum_{k=1}^{\infty}\delta\left(t-t_k\right)K_d\left(x\left(t_k^-\right)-x_{\rm r}\right). \end{aligned} \tag{45}$$

where K_i and K_d are the controller gain. The closed-loop impulsive system can be represented as

$$\begin{cases} \dot{\Xi}(t)=\sum_{i=1}^{4}\sum_{j=1}^{4}h_ih_j(A_i+B_iK_j)\Xi(t), t\neq t_k \\ \Xi(t_k)=K_t\Xi(t_k^-) \\ z(t)=\sum_{i=1}^{4}h_i(\varrho(t))(C_i\Xi(t)). \end{cases} \tag{46}$$

where

$$\Xi(t)=\begin{cases} x(t)-x_r, x(t_0)\succeq x_r \\ x_r-x(t), x(t_0)\preceq x_r \end{cases}.$$

From Lemma 2, the system (46) is positive when $A_i+B_iK_j$ are Metzler, $K_t=I+BK_d\succeq 0$, $C_i\succeq 0$.

The parameters are given by [24]: $R_1=R_2=0.95$, $R_{12}=0.52$. $a_1=0.2236$, $b_1=0.4472$ ($x_1\in[5,20]$), $a_2=0.2582, b_2=0.4082$ ($x_2\in[6,15]$). Assume that $K_t=\begin{bmatrix} 0.6 & 0 \\ 0 & 0.5 \end{bmatrix}$, and the impulse interval $T_k\in[0.1,0.6]$.

First, we study the stability of closed-loop positive impulsive T-S fuzzy system (46). The solution is obtained by applying the Theorem 3 with $N=6$, $u=1.01$, $p=0.5$, $q=1.5$, $\eta=[1,5]^{\rm T}$:

$$K_1=\begin{bmatrix} 0.0109 & -0.0351 \\ 0.0544 & -0.1756 \end{bmatrix}, \tag{47}$$

$$K_2=\begin{bmatrix} 0.0216 & 0.0077 \\ 0.1079 & 0.0383 \end{bmatrix}, \tag{48}$$

$$K_3=\begin{bmatrix} 0.0267 & -0.0113 \\ 0.1335 & -0.0564 \end{bmatrix}, \tag{49}$$

$$K_4=\begin{bmatrix} -0.0147 & -0.1475 \\ -0.0736 & -0.7373 \end{bmatrix}. \tag{50}$$

Let $x(0)=[5,6]^{\rm T}$, $x_r=[15,10]$. The trajectories of state x_i for the closed-loop impulsive system (46) are shown in Fig. 4. which demonstrates that the controller (45) render the T-S positive impulsive fuzzy system (44) stable and track the reference value x_r.

Then, we consider the following system constructed by system (46) with disturbance:

$$\left\{\begin{aligned}\dot{\Xi}(t) =& \sum_{i=1}^{4}\sum_{j=1}^{4} h_i h_j (A_i + B_i K_j)\Xi(t) \\ &+ \sum_{i=1}^{r} h_i E_i \omega(t), t \neq t_k \\ \Xi\left(t_k\right) =& K_t \Xi(t_k^-) \\ z(t) =& \sum_{i=1}^{4} h_i (C_i \Xi(t) + D_i \omega(t)).\end{aligned}\right. \tag{51}$$

where $\omega(t) = \left(0.1\mathrm{e}^{-2t}, 0.1\mathrm{e}^{-2t}\right)^{\mathrm{T}}$,

$$E_i = \left[\begin{array}{cc} 0.1 & 0.2 \\ 0.2 & 0.1 \end{array}\right], D_i = \left[\begin{array}{cc} 0.5 & 0.5 \end{array}\right].$$

As $N = 6$, $u = 1.01$, $p = 0.6$, $q = 1.8$, $\eta = [1, 10]^{\mathrm{T}}$, applying Theorem 3, we have $\gamma = 0.689$

$$K_1 = \left[\begin{array}{cc} -0.0107 & -0.1255 \\ -0.1070 & -1.2552 \end{array}\right],$$
$$K_2 = \left[\begin{array}{cc} -0.0083 & -0.0834 \\ -0.0834 & -0.8339 \end{array}\right],$$
$$K_3 = \left[\begin{array}{cc} -0.0014 & -0.0987 \\ -0.0137 & -0.9867 \end{array}\right],$$
$$K_4 = \left[\begin{array}{cc} -0.0213 & -0.2475 \\ -0.2127 & -2.4750 \end{array}\right].$$

Figure 5 shows the trajectories of state x_i for the system (51). It can be seen that under the condition of disturbance $\omega(t)$, the controller render the system states positive and approach to the expected reference value x_r.

V. Conclusion

This paper provides a new sufficient conditions for exponential stability and L_1-gain of positive impulsive T-S fuzzy systems by designing the polynomial ITD Lyapunov function. In addition, based on parallel distributed compensation, an impulse-time independent controller is designed for stabilization. In the end, three numerical examples have been presented to validate the effectiveness of the obtained results. It is worth noting that the homogenous polynomially membership function dependent Lyapunov-Krasovskii functional in [17] and the polynomial ITD Lyapunov function in this paper have achieved low conservative results in the stability analysis. However, the dynamic mechanism needs to be clarified, which is our future research.

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