# Symmetries of Riemann Surfaces and Magnetic Monopoles

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# Declaration

I declare that this thesis was composed by myself, that the work contained therein is my own, except where explicitly stated otherwise in the text, and that the work has not been submitted for any other degree or professional qualification.

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# Abstract

This thesis studies, broadly, the role of symmetry in elucidating structure. In particular, I investigate the role that automorphisms of algebraic curves play in three specific contexts; determining the orbits of theta characteristics, influencing the geometry of the highly-symmetric Bring's curve, and in constructing magnetic monopole solutions. On theta characteristics, I show how to turn questions on the existence of invariant characteristics into questions of group cohomology, compute comprehensive tables of orbit decompositions for curves of genus 9 or less, and prove results on the existence of infinite families of curves with invariant characteristics. On Bring's curve, I identify key points with geometric significance on the curve, completely determine the structure of the quotients by subgroups of automorphisms, finding new elliptic curves in the process, and identify the unique invariant theta characteristic on the curve. With respect to monopoles, I elucidate the role that the Hitchin conditions play in determining monopole spectral curves, the relation between these conditions and the automorphism group of the curve, and I develop the theory of computing Nahm data of symmetric monopoles. As such I classify all 3-monopoles whose Nahm data may be solved for in terms of elliptic functions.

# Lay Summary

In this thesis I work with the concept of symmetry, a feature of our universe which humans learn to recognise at a very early age. Anyone who has doodled with pen and paper will know that amongst all the shapes that one can draw, those with symmetry are singled out, and the more symmetry we require the more restricted the drawing one can have. The same principle, that requiring symmetry enforces constraints and structure, applies to much of mathematics, and for mathematicians it is often extremely profitable to study configurations with lots of symmetry. The reason for this is twofold: (i) typically the more symmetry one requires, the easier it is to compute answers explicitly; (ii) it is often the most symmetric examples that are the most 'beautiful' in the abstract sense, with far reaching interconnections to other areas of mathematics. Motivated by these truths, in this thesis I investigate two realms where symmetry will be helpful on both frontiers.

The first realm is that of Riemann surfaces, objects which look like spheres or doughnuts with one or more holes. One can label points on these surfaces and ask how they are rearranged by the symmetries of the surface, for example as the Earth rotates about its axis points move along lines of constant latitude. As these points travel around they form orbits, and part of this thesis calculates these orbits both manually and numerically, identifying patterns in the numbers that emerge. I also consider one particular surface, called Bring's curve, special because of its manifold symmetries. Using a variety of perspectives I will give visualisations of this symmetry, studying its implications, including on the orbits previously mentioned.

The second realm is of magnetic monopoles, which can be thought of as magnetic analogues of electrons, but now with internal structure. Monopoles source a magnetic field, and they are distinguished from the bar magnets (or 'dipole' magnets) we see in everyday existence because the pole is isolated, not part of a North-South pair, hence the name monopole or one-pole. A remarkable development in the 1970s and 1980s showed that to the physical picture of a monopole one can associate the geometric data of a Riemann surface, and through this correspondence I study the ways of giving symmetry to the monopoles. I use computer calculations to simplify the process of translating between the two perspectives, and as such I am able to classify all the monopoles of a particular kind. In special cases the algebra simplifies significantly to the point where I am able to write down the answers explicitly, and in these situations I use computers to generate 3-dimensional pictures of the monopoles.

# Acknowledgements

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Amongst all those who have left footprints, a few deserve to be singled out. I want to thank Nils Bruin and Richard Houlston for their mentorship and guidance, not only with my research but with my career. Moreover, a special thanks must be extended to Andrew Beckett, Jacob Bradley, Bella Deutsch, and Josh Fogg. They have been not only my dear friends for four years now, but Josh has provided hours of kindness coupled to hours computing the circumference of an ellipse, Bella has been the keystone of party and planning in many an escapade, Andrew has been my greatest coconspirator in mathematics while my worst in board games, and Cob kept me sane as my longest and most liked flatmate.

Above all, I am most indebted to the support of my supervisor Harry Braden. He has been generous with his time well beyond the call of duty, pushed me to be the best mathematician I can be, advocated loudly for my work, been a cycling and climbing partner, and proven himself a fount of expertise on matters mathematical, grammatical, and geographical.

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# Chapter 1

# Introduction

> For the past several months, since the middle of April, he has dreamed many dreams about time. His dreams have taken hold of his research. His dreams have worn him out, exhausted him so that he sometimes cannot tell whether he is awake or asleep. But the dreaming is finished. Out of many possible natures of time, imagined in as many nights, one seems compelling. Not that the others are impossible. The others might exist in other worlds.
>
> – Alan Lightman
> Einstein's Dreams

This thesis, as the title may suggest, is structured around a simple concept: symmetry. In particular, I will demonstrate how imposing symmetry leads to geometric structure, and we will investigate the various forms in which this structure manifests itself.

First, in Chapter 2, we will look at Riemann surfaces. In the 18th and 19th century research into solutions of the quintic equation led to the parallel development of the 'theory of equations' and the 'theory of functions', a process intimately tied to the concept of symmetry through the work of Galois, and Riemann surfaces arose out of these fields jointly.

Combining both an algebraic and a geometric picture allowed for a dual perspective on the role the group of symmetries of the surface played, and we will explore these facets in a variety of ways. In §2.1 I will lay out much of the introductory material that will permeate the calculations herein. I will define Riemann surfaces and their related constructions, giving examples relevant to later sections, providing ample citations for the inquisitive reader. Most notably, I shall take a perspective especially tuned to pre-empt the rigours of later sections: I will spend time emphasising a triality of viewpoints on divisors, provide a strong foundation for visualising the geometry of $T\mathbb{P}^1$, and ensure that a variety of computational and theoretical tools available for scrutinising the automorphism group have been provided.

In §2.2 we will immediately put all of these techniques to good use. I will present different approaches to computing theta characteristics and their orbits under the action of the automorphism group, each method attuned to the varying levels of geometrical and computational data that may be available. I shall largely spurn the analytical aspects of theta functions, taking instead an algebraic path, but the theta aficionado may profit from retaining this point of view. Next, using the relation of theta characteristics to spin structures and thus to binary vectors, I will make two focused investigations: I will use group cohomology to rephrase the task of quantifying how many characteristics invariant under the automorphism group exist, and I will compute comprehensive tables of orbit decompositions for curves of genus 9 and less. While I will prove some important results, notably Proposition 2.2.26 and Theorem 2.2.35, I will leave many problems still left to be tackled in subsequent work, and to this end I finish the section with suggestions for directions of future study and conjectures to measure progress against.

In contrast to §2.2, in §2.3 I will take a much more narrow scope, choosing instead to study one particular exceptional Riemann surface: Bring's curve. In this section I will highlight the geometric aspects that make the curve worthy of such high esteem, including its large automorphism group, its intricate quotient structure, its geometric realisations, and the interplay between the three. By providing a complete realisation of the automorphism group of the Hulek-Craig model and the $\mathbb{P}^4$-model I will identify previously unknown quotients of the curve with interesting number-theoretic properties. This process will be aided by the unification of existing visualisations of the symmetric structure of Bring's curve. Moreover, I will prove analytical and numerical results about the theta characteristics on Bring's curve, determining their orbit decomposition. In particular, I will prove the existence of a unique invariant characteristic on the curve, write down the characteristic explicitly, and relate it to the Szegő kernel divisor. The work in this section has been submitted for publication and released as a preprint [BDH22]; my coauthor Harry Braden gave his consent for the work to be included in this thesis.

Secondly, in Chapter 3, we will look at magnetic monopoles. Monopoles have a rich history in mathematical physics, first seriously introduced by Dirac in 1931 as a possible explanation for the quantisation of electric charge, utilising the duality between electricity and magnetism in the source-free Maxwell's equations. The solutions Dirac found had singularities, and so were considered unphysical; their study remained somewhat of a novelty until the 1970s, when it was shown by 't Hooft and Polyakov that, when formulated as a symmetry breaking of a gauge theory with a higher gauge group, monopoles were inevitable in certain grand unified theories. As such their study exploded, peaking (by one metric) in 1984, including experiments attempting to observe monopoles such as the famous 'valentine's-day monopole'. This thesis will not focus on this experimental or theoretical aspect of monopoles, but instead on their mathematics.

In §3.1 I will introduce their formulation in terms of gauge theory, providing necessary background for later sections, including introducing the moduli space of monopoles, and writing down the hedgehog solution which will recur as an example throughout when describing the different guises of monopole data. I will also make passing comments on the choice of gauge group and the normalisation conventions on $\mathfrak{su}(2)$, the latter of which I will elaborate on in Appendix A.1.

In §3.2 I will introduce the link between Chapter 2 and Chapter 3, the monopole spectral curve, from two perspectives. From the view of Hitchin we will see the curve arise as a subset of Euclidean minitwistor space, and in fact as a Riemann surface. I will discuss in some detail the transcendental constraints imposed upon the curve, employing the machinery of divisors developed previously. In addition, we will see how the spectral curve arises from Nahm data, in particular as the characteristic polynomial of the associated Lax pair. Imposing the transcendental constraints on the spectral curve may be simpler from the Nahm viewpoint, and certainly the reconstruction of the monopole from this data is, and we will utilise these facts in later sections constructing explicit monopole spectral curves and visualising the corresponding field configurations.

In §3.3 I will consider the action of both extrinsic and intrinsic symmetries on the spectral curve. Extrinsically there is an action of the Euclidean group on minitwistor space, and I will write down the corresponding action on the coordinates of the spectral curve as well as on the associated Nahm data. By understanding the correspondence between the two, I will build upon work of Hitchin, Manton, and Murray showing how to construct Nahm matrices invariant under a rotational symmetry group, giving explicit examples. Intrinsically the full automorphism group of the Riemann surface acts on the spectral curve giving ramification data associated with the quotient, and I will prove how that ramification data constrains the dimension of the associated moduli space. Moreover, I will give conjectures of stricter bounds on this dimension related to the transcendental constraints on the spectral curve.

Finally, in §3.4 I will complete a partial classification of charge-3 monopoles based upon the symmetry conditions and constructions introduced in previous chapters. In particular, I will classify the spectral curves of charge-3 monopoles which quotient to an elliptic curve. Some of these curves are previously known, but we will find new curves and explicitly construct their Nahm data in two cases. Moreover, for a remaining spectral curve identified with an elliptic quotient, I will prove that one cannot solve for Nahm data in terms of elliptic functions generically. Part of the work in this section has been published as [BDH23], and my coauthor Harry Braden gave his consent for the work to be included in this thesis.

A key aspect of the work contributing to this thesis has been the use of computational tools. Throughout the course of this research I have used GAP, Macaulay2, Maple, Python, and Sage to compute examples, test conjectures, draw figures, and complete proofs. At the relevant sections I will provide references to code which one may use to reproduce the results and figures. Those relevant to §2.3 have previously been made available at `https://github.com/DisneyHogg/Brings_Curve`; new code is at `https://github.com/DisneyHogg/Riemann_Surfaces_and_Monopoles`.

As is natural, there are aspects of the story of symmetries, Riemann surfaces, magnetic monopoles, and my own work that have not been included. In addition to the two papers [BDH22, BDH23] previously mentioned in this introduction I have also released [DHBD22, BDHG22] as preprints during the course of my study. In [DHBD22], in collaboration with Andrew Beckett and Isabella Deutsch, I translated from German to English an important and oft-cited work of Wiman classifying genus-4 Riemann surfaces; in [BDHG22], in collaboration with Nils Bruin and Effie Gao, I developed a method for rigorously computing algebraic integrals, performing complexity analysis of this algorithm and implementing it in Sage. While both papers are related to the story of this thesis, indeed they will both be cited within, I have omitted them for the sake of brevity and focus.

Mathematically, there are two core perspectives I have neglected in this work, both related to the construction of monopoles. Though it is mentioned occasionally, I shall largely avoid working with the full ADHM construction of instantons using twistor theory, and so avoid working with solutions of the self-dual Yang-Mills equations, opting instead to focus on the minitwistor construction of monopoles. In addition, when constructing monopoles from the data of the spectral curve one can take the perspective of integrable systems, the inverse scattering method, and Baker-Akhiezer functions. As with the inclusion of my preprints, I have omitted the additional topics in the name of focus, but the loss is certainly sorely felt because of the beauty of the results to be found.

# Chapter 2

# Riemann Surfaces

"Should you just be an algebraist or a geometer?"
is like saying "Would you rather be deaf or blind?"

– Michael Atiyah
Mathematics in the 20th Century

The purpose of this chapter is to introduce Riemann surfaces, a beautiful area of algebraic geometry dating back to 1851, when Bernhard Riemann in his inaugural dissertation began to think of the theory of complex functions in terms of surfaces [Neu81]. Their study will not only be integral for our later discussion of minitwistor space and the spectral curve, but will also provide gems within their own right. There are many excellent sources to learn from with varying levels of sophistication, but [FK92, For91, GH78, Har77, Mir95, Vak10] have all been particularly influential during the course of this thesis.

In §2.1 I will lay down some of the notational and technical background required for this thesis. The style of presentation has been chosen to emphasise those results and concepts that I shall use in later sections. §2.2 will introduce theta characteristics from an algebraic perspective, develop the theory describing the orbit structure of theta characteristics under the automorphism group of a curve, and compile tables of orbits decompositions for many curves of genus 9 and less. Finally, in §2.3 I complete a comprehensive study of one particularly distinguished and highly symmetric curve of genus 4, Bring's curve. I will describe its quotient structure, unify visualisations of the automorphism group, and describe the orbit structure of the theta characteristics.

## 2.1 Background Material

> I care not to perform this part of my task methodically; but shall be content to produce the desired impression by separate citations of items, practically or reliably known to me as a whaleman; and from these citations, I take it - the conclusion aimed at will naturally follow of itself
>
> – Herman Melville
> Moby Dick

This section will be a review of many classical properties, able to be found in standard textbooks on Riemann surfaces. In any thesis one must expect and allow for a certain amount of prerequisite knowledge, and here I shall take that to be a familiarity with the concepts of manifolds, basic concepts in category theory, and most significantly a competence with sheaves and their cohomology. I have chosen to omit the details of these as they shall not be crucial to any arguments made later in the thesis, opting instead to make sure that the concepts and notation required from Riemann surfaces have been appropriately covered. Because the material in this chapter is the content of many excellent expositions, and is not the core theme of this thesis, I will omit proofs here opting instead to provide appropriate references.

### 2.1.1 The Category of Riemann Surfaces

#### Objects and Examples

Let us start with a simple definition of a Riemann surface in the language of manifolds.

**Definition 2.1.1** ([GH78], p. 15)**.** *A* ***Riemann surface*** *is a 1-dimensional complex manifold. Given a chart $\phi : U \to V$ on a Riemann surface $\mathcal{C}$ with $U \subset \mathcal{C}$ open, $V \subset \mathbb{C}$ open, and $P \in U$ such that $\phi(P) = 0$, for $Q \in U$ we call $z = \phi(Q)$ a* ***local coordinate*** *at $P$.*

Throughout, unless otherwise stated, I will assume that our Riemann surfaces are connected and compact. These assumptions will be used implicitly through much of the ensuing work, for example by forcing that the only global holomorphic functions on the Riemann surface are given by constant functions. Such compact Riemann surfaces are equivalently compact oriented real surfaces with a conformal structure [FK92, §IV], and so topologically a Riemann surface is classified by a single integer $g(\mathcal{C})$, the **genus**. Geometrically this can be thought of as the number of 'handles' of the surface. In situations where it is clear which Riemann surface is being referred to, I will drop the argument of $g$.

In general in algebraic geometry there are two concepts of genus, the **geometric genus** of a nonsingular[1] algebraic variety $p_g(\mathcal{C}) = \dim_{\mathbb{C}} H^0(\mathcal{C}, K_{\mathcal{C}})$ where

[1]At this first instance of a case of ambiguity in the 'correct' hyphenation (nonsingular or

$K_{\mathcal{C}}$ is the canonical sheaf (see Example 2.1.33) [Har77, p. 181], and the **arithmetic genus** of an $r$-dimensional projective variety $p_a(\mathcal{C}) = (-1)^r [P_{\mathcal{C}}(0) - 1]$ where $P_{\mathcal{C}}$ is the Hilbert polynomial of $\mathcal{C}$ [Har77, p. 54, Exercise I.7.2]. These two definitions agree for nonsingular projective varieties, and moreover agree with the topologically defined genus on a Riemann surface [Vak10, Exercise 21.7.I], so the difference shall not often be material in this thesis, hence I will not dwell on the details of these definitions. We shall however want to have a definition for compact connected projective curves when dealing with monopole spectral curves in §3.2, and there it will be sufficient to use that the arithmetic genus is in this situation given by $p_a(\mathcal{C}) = \dim_{\mathbb{C}} H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ [Har77, p. 230, Exercise III.5.3], where $\mathcal{O}_{\mathcal{C}}$ is the sheaf of holomorphic functions on $\mathcal{C}$ (see §2.1.2).

**Remark 2.1.2.** *In this thesis, I shall work with sheaf cohomology, but in all situations I will seek to enforce conditions whereby we have a* ***Leray cover*** *[For91, GH78], and so compute with Čech cohomology. For example, this is valid on any compact Riemann surface $\mathcal{C}$ computing cohomology of the sheaf $\mathcal{O}_{\mathcal{C}}(D)$ for some divisor $D$ when the cover is given by open sets isomorphic to a disk [For91, Exercise 16.3].*

**Example 2.1.3.** *The 1-dimensional complex projective space $\mathbb{P}^1$, otherwise known as the Riemann sphere, is a Riemann surface. As this space is topologically a sphere, $g = 0$. Taking a point $[\zeta_0 : \zeta_1] \in \mathbb{P}^1$, we will often use the affine coordinate $\zeta := \zeta_0/\zeta_1$ on the open patch $U_0 = \{\zeta_1 \neq 0\}$, thus denoting the point $[1:0]$ as $\infty$. On $U_1 = \{\zeta_0 \neq 0\}$ I will use the affine coordinate $\tilde{\zeta} = 1/\zeta$.*

**Example 2.1.4.** *Riemann surfaces of genus 1 are called* ***elliptic curves****, and are their own rich area of study. I will give a few of these details in §2.1.5, as they will be necessary for later results.*

**Definition 2.1.5** ([Mir95], p. 14)**.** *An* ***algebraic variety*** *is the vanishing locus in $\mathbb{P}^n$ of a set of homogeneous polynomials $\{F_i(X_0, \dots, X_n)\}$. We call a variety of dimension $r$ a* ***complete intersection*** *if the ideal $\langle F_i \rangle$ can be generated by $n-r$ elements. If there are $(n-1)$-many $F_i$ we call the corresponding algebraic variety an* ***algebraic curve*** *if it is a complete intersection, and if moreover the matrix of partial derivatives $\frac{\partial F_i}{\partial X_j}$ has full rank at every point contained in the vanishing locus we say it is* ***smooth****. Smooth algebraic curves are examples of Riemann surfaces. It is a consequence of Bezout's theorem [Har77, Theorem I.7.7] that each $F_i$ is necessarily irreducible for a smooth algebraic curve.*

*In the case of a singular (not smooth) algebraic curve coming from reducible $F_i$, we call an irreducible component* ***simple/multiple*** *if its multiplicity is one or greater than one respectively [Ful08, p. 53].*

*One important class of algebraic curves are* ***plane curves*** *given by the vanishing of a single homogeneous polynomial $F(X, Y, Z)$. For these, we often work*

non-singular) I shall provide two quotes which may be seen as guiding principles for usage throughout the rest of this thesis, "The hyphen is not an ornament; it should never be placed between two words that do not require uniting & can do their work equally well separate" [Fow26], and "If you take hyphens seriously, you will surely go mad" [Boo05].

*in affine coordinates $x = X/Z$, $y = Y/Z$, so we can consider the dehomogenised*[2] *polynomial $f(x, y) := F(x, y, 1) = 0$. A singular point of the curve (in the open set $Z \neq 0$ without loss of generality) is called a* ***node*** *if the Hessian of $f$ is nonsingular.*

*In addition, given a Riemann surface defined as a projective algebraic variety over $\mathbb{C}$, one may also consider the corresponding variety over different number fields. Such considerations are relevant for number theory and enumerative geometry, but aside from a few points I will almost always work over $\mathbb{C}$ in this thesis.*

**Example 2.1.6.** *Chow's theorem [Har77, Theorem B.2.2] states that any closed analytic subspace of complex projective space is algebraic, i.e. it is an algebraic subvariety of projective space.*

**Example 2.1.7.** *The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper-half plane $\mathcal{H}$ by Möbius transformation, and this extends to an action on $\mathcal{H}^* := \mathcal{H} \cup \mathbb{Q} \cup \{i\infty\}$ [Dol97, p. 101]. Given $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ a subgroup of finite index, $X(\Gamma) := \mathcal{H}^*/\Gamma$ is a compact Riemann surface called the* ***modular curve associated with*** *$\Gamma$. A common choice for $\Gamma$ is the group*[3]

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \equiv 0 \mod N \right\}$$

*for $N \in \mathbb{Z}_{>0}$, in which case one denotes $X(\Gamma) = X_0(N)$, the* ***modular curve of level*** *$N$ [Shi95]. Such Riemann surfaces will be seen in §2.3.*

Definition 2.1.5 is archetypal in a particular sense. It is shown in [GH78, p. 215] that, as a special case of the Kodaira embedding theorem, every compact Riemann surface can be thought as a smooth complex algebraic curve. Moreover, one can in fact show that $\mathbb{P}^3$ is always sufficient as an ambient space. In general, it is not possible to smoothly embed the curve in $\mathbb{P}^2$, but it can be done introducing node singularities. These can be removed by a process called **blowing up** the singularities to get a Riemann surface [Har77, p. 28] (also called the **normalisation** of the singular curve), the geometric genus of which is related to the arithmetic genus of the singular curve in a well defined way [Har77, Corollary V.3.7]. I will not give the details of this process here, but in practice this will mean we often think of Riemann surfaces as plane curves $F(X, Y, Z) = 0$.

### Morphisms

As complex manifolds, Riemann surfaces inherit the concept of a morphism between Riemann surfaces as a map that is holomorphic at the level of charts, and so isomorphisms as bijective holomorphic maps. As a smooth projective curve the compatible notion is that of (dominant) rational maps for morphisms, with

[2]On the use of '-ise' and '-ize' in verbs, Fowler writes on making '-ise' universal that "the sacrifice of significance to ease does not seem justified" [Fow26]. Nevertheless, in this thesis I shall make that sacrifice.

[3]Here, and throughout this thesis, I shall use $\equiv$ to denote equivalence under corresponding equivalence relation, which should be clear from context.

birational maps being isomorphisms. There is actually an equivalence of categories of compact Riemann surfaces, the category of compact complex projective curves, and the category of field extensions of $\mathbb{C}$ of transcendence degree 1 [Har77, Theorem I.6.12, Theorem B.2.2]. The field of transcendence degree 1 associated with $\mathcal{C}$ is the **field of (meromorphic) functions** $\mathrm{Hom}(\mathcal{C}, \mathbb{P}^1)$.

Taking the perspective of a manifold map for now, around a point $P \in \mathcal{C}$, taking local coordinates about $P$ and its image under $f : \mathcal{C} \to \mathcal{C}'$ we can write the map as as Taylor series, so its local behaviour is $z \mapsto z^m$ for some positive integer $m$. We call $m$ the **multiplicity** of $f$ at $P$, and denote it with $\mathrm{mult}_P(f)$. If $\mathrm{mult}_P(f) \geq 2$ we call $P$ a **ramification point**, in which case $f(P)$ is called a **branch point** [Mir95, Definition II.4.2, Definition II.4.5]. The set of all branch points is called the **branch locus**.

**Example 2.1.8** ([Mir95], Lemma II.4.6)**.** *Given a smooth projective plane curve $\mathcal{C}$ defined by $F(X, Y, Z) = 0$, the ramification points of the map $\mathcal{C} \to \mathbb{P}^1$ given by $[X : Y : Z] \mapsto [X : Z]$ are the points where $F = 0 = \partial F/\partial Y$.*

*As $F$ and $\partial F/\partial Y$ are both bivariate polynomials when written in affine coordinates, taking their resultant with respect to $y$ gives a single univariate polynomial in the coordinate of the $\mathbb{P}^1$ determining the branch locus. This is the computational approach to finding the branch locus taken in the Sage Riemann surfaces module [Sag21b, Sag21a].*

**Example 2.1.9** ([Mir95], Lemma II.4.7)**.** *The multiplicities corresponding to a meromorphic function on a Riemann surface are*

$$\mathrm{mult}_P(f) = \begin{cases} \mathrm{ord}_P(f), & P \textit{ a zero}, \\ -\mathrm{ord}_P(f), & P \textit{ a pole}, \\ \mathrm{ord}_P(f - f(P)), & \textit{otherwise}, \end{cases}$$

*where $\mathrm{ord}_P(f)$ is the order of a pole/zero as defined in complex analysis.*

It is clear in Examples 2.1.8 and 2.1.9 that the set of ramified points is discrete, hence finite in a compact Riemann surface, and this is in fact true for all morphisms of Riemann surfaces. As such we can define

$$d_Q(f) = \sum_{P \in f^{-1}(Q)} \mathrm{mult}_P(f),$$

and this function turns out to be constant in $Q \in \mathcal{C}'$ [Mir95, Proposition II.4.8], so associated with $f$ there is a positive integer $\deg f = d_Q(f)$ for any $Q$. This we call the **degree**.

**Example 2.1.10** ([Mir95], Proposition II.4.12)**.** *The constancy of the degree can be used to deduce that, for example, given a meromorphic function $f \neq 0$ on $\mathcal{C}$,*

$$\sum_{P \in \mathcal{C}} \mathrm{ord}_P(f) = 0.$$

**Example 2.1.11.** *A curve $\mathcal{C}$ is called **hyperelliptic** if $g(\mathcal{C}) \geq 2$ and there exists $f : \mathcal{C} \to \mathbb{P}^1$ a map such that $\deg(f) = 2$. Such curves can be written in affine* *coordinates as* $y^2 = p_{2g+2}(x)$ *for some degree-*$(2g+2)$ *polynomial. In fact, all genus-2 curves are hyperelliptic [Har77, Exercise IV.1.7].*

The degree of $f$ is related to the genera of $\mathcal{C}$, $\mathcal{C}'$ by the following powerful theorem.

**Theorem 2.1.12** (Riemann-Hurwitz Formula, [Mir95], Theorem II.4.16)**.** *Given* $f : \mathcal{C} \to \mathcal{C}'$ *a nonconstant morphism of Riemann surfaces, we have*

$$2\left[g(\mathcal{C}) - 1\right] = 2\deg(f)\left[g(\mathcal{C}') - 1\right] + \sum_{P \in \mathcal{C}} \left[\mathrm{mult}_P(f) - 1\right].$$

*This is called the* ***Riemann-Hurwitz (RH)*** *formula.*

**Example 2.1.13** (Degree-genus formula, also known as Plücker's formula, [Mir95], Proposition V.2.15)**.** *Given a smooth projective plane curve* $\mathcal{C}$ *defined by* $F(X, Y, Z) = 0$*, where* $F$ *is a degree-*$d$ *homogeneous polynomial, one can use the Riemann-Hurwitz formula to say* $g(\mathcal{C}) = \frac{1}{2}(d-1)(d-2)$*. As such, every smooth plane quartic has genus 3.*

### 2.1.2 Divisors, Line Bundles, and Riemann-Roch

#### Divisors

A fundamental tool in algebraic geometry and a concept that will be essential for work throughout this thesis is the concept of a divisor.

**Definition 2.1.14** ([Mir95], p, 129)**.** *A* ***(Weil) divisor*** *on* $\mathcal{C}$ *is a formal finite sum of points, i.e.* $D = \sum_i n_i P_i$ *for* $n_i \in \mathbb{Z}$, $P_i \in \mathcal{C}$*. The group of divisors under addition is denoted* $\mathrm{Div}(\mathcal{C})$*. The additive identity of this group, denoted by* $0$*, is the divisor where all* $n_i$ *are zero. The* ***degree*** *of* $D$ *is* $\deg D = \sum_i n_i$*. We say that* $D$ *is* ***effective*** *if all* $n_i$ *are nonnegative.*

**Remark 2.1.15.** *This is a special case of the generic definition of divisor for schemes, which are formal finite sums of codimension-1 subschemes, see [GH78, p. 130] and [Har77, p. 130].*

**Definition 2.1.16** ([Mir95], Definition V.1.16)**.** *Given* $f : \mathcal{C} \to \mathcal{C}'$*, we can define the pullback of divisors* $f^* : \mathrm{Div}(\mathcal{C}') \to \mathrm{Div}(\mathcal{C})$ *by*

$$f^*\left(\sum_i n_i Q_i\right) = \sum_i n_i \left(\sum_{P \in f^{-1}(Q_i)} \mathrm{mult}_P(f) P\right).$$

**Proposition 2.1.17** ([Mir95], Lemma V.1.17)**.** *The degree of the pullback of a divisor is given by* $\deg(f^* D) = \deg(f)\deg(D)$*.*

**Definition 2.1.18** ([Mir95], Definition V.1.3, Definition V.1.10)**.** *Given a meromorphic function* $f : \mathcal{C} \to \mathbb{P}^1$ *we define* $(f) \in \mathrm{Div}(\mathcal{C})$ *by*

$$(f) = \sum_{P \in \mathcal{C}} \mathrm{ord}_P(f) \cdot P.$$

*For $D \in \mathrm{Div}(\mathcal{C})$, if there exists $f$ such that $D = (f)$ we say $D$ is a* ***principal divisor***. *Likewise given a meromorphic differential $\omega$ define*

$$(\omega) = \sum_{P \in \mathcal{C}} \mathrm{ord}_P(\omega) \cdot P,$$

*where $\mathrm{ord}_P(\omega)$ is defined by writing $\omega$ in terms of a local coordinate $z$ at $P$ as $f(z)\mathrm{d}z$, and then $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f)$. For $D \in \mathrm{Div}(\mathcal{C})$, if there exists $\omega$ such that $D = (\omega)$ we say $D$ is a* ***canonical divisor***.

**Example 2.1.19.** *On $\mathbb{P}^1$, $(\zeta) = 0 - \infty$ and $(\mathrm{d}\zeta) = -2\infty$. We can then identify $(f) = f^*(0 - \infty)$.*

**Example 2.1.20.** *The* ***ramification divisor*** *of a map $f : \mathcal{C} \to \mathcal{C}'$ is $R_f := \sum_{P \in \mathcal{C}} [\mathrm{mult}_P(f) - 1]\, P$ [Mir95, Definition V.1.18]. Given $\omega$ a meromorphic differential on $\mathcal{C}'$, $(f^*\omega) = f^*(\omega) + R_f$ [Mir95, Lemma V.1.19].*

The map $f \mapsto (f)$ in fact gives a group homomorphism from the units in the field of meromorphic functions into $\mathrm{Div}(\mathcal{C})$, the image of which is the subgroup of principal divisors denoted by $\mathrm{PDiv}(\mathcal{C})$.

**Definition 2.1.21** ([Mir95], Definition V.2.1, Definition V.3.6)**.** *The* ***divisor class group*** *of $\mathcal{C}$ is the group quotient $\mathrm{Cl}(\mathcal{C}) = \mathrm{Div}(\mathcal{C}) / \mathrm{PDiv}(\mathcal{C})$. We say two divisors $D$, $D'$, which differ by a principal divisor are* ***linearly equivalent*** *and write $D \sim D'$. The restriction to effective divisors of the equivalence class of $D$ under the quotient is the* ***complete linear system*** *associated with $D$, denoted $|D|$.*

Any two nonzero meromorphic differentials are linearly equivalent, so we are able to talk about *the* canonical divisor, for which we use the notation $\mathcal{K}_{\mathcal{C}}$. Note linearly equivalent divisors have the same degree, so we can ask what the degree of the canonical divisor is.

**Lemma 2.1.22** ([Mir95], p. 132)**.** *The degree of the canonical divisor is $\deg(\mathcal{K}_{\mathcal{C}}) = 2g(\mathcal{C}) - 2$.*

**Proposition 2.1.23** ([Mir95], p. 147)**.** *On a compact Riemann surface, if $\deg(D) < 0$ then $|D| = \emptyset$.*

**Example 2.1.24.** *On $\mathbb{P}^1$, $|\mathrm{d}\zeta| = \emptyset$.*

Finally, one concept we will require later is a partial ordering on divisors.

**Definition 2.1.25** ([Mir95], p. 136)**.** *Given two divisors $D, D' \in \mathrm{Div}(\mathcal{C})$, we say $D \geq D'$ if $D - D'$ is effective.*

There is another formulation of divisors in terms of sheaf data. Denote by $\mathcal{O}_{\mathcal{C}}$ the sheaf of holomorphic function on $\mathcal{C}$, and by $\mathcal{O}_{\mathcal{C}}^{\times}$ those that are invertible under multiplication; likewise $\mathcal{M}_{\mathcal{C}}$, $\mathcal{M}_{\mathcal{C}}^{\times}$ for meromorphic functions.

**Definition 2.1.26** ([GH78], p. 131)**.** *A **(Cartier) divisor** is a global section of the quotient sheaf $\mathcal{M}_{\mathcal{C}}^{\times}/\mathcal{O}_{\mathcal{C}}^{\times}$, that is an open cover $\{U_\alpha\}$ of $\mathcal{C}$ with associated nonzero meromorphic functions $f_\alpha$ such that on $U_\alpha \cap U_\beta$, $f_\alpha = f_\beta$ up to a factor of a function in $\mathcal{O}_{\mathcal{C}}^{\times}(U_\alpha \cap U_\beta)$.*

**Remark 2.1.27.** *The quotient $\mathcal{M}_{\mathcal{C}}^{\times}/\mathcal{O}_{\mathcal{C}}^{\times}$ is not actually a sheaf but a presheaf; it can be made into a sheaf using the étale space [McM14, p. 78].*

We can get a Weil divisor from a Cartier divisor by the map

$$\{(U_\alpha, f_\alpha)\} \mapsto \sum_{P \in \mathcal{C}} \mathrm{ord}_P(f_\alpha) \cdot P, \quad \text{taking } \alpha \text{ so } P \in U_\alpha.$$

On a smooth projective variety this procedure gives all Weil divisors, and so the definitions are equivalent [Har77, Proposition II.6.11].

### The Picard Group and the Jacobian

In this thesis I will frequently work with holomorphic line bundles, which are rank-1 complex vector bundles over a complex manifold whose local trivialisations are holomorphic maps [GH78, p. 66-69].

**Definition 2.1.28** ([Har77], p. 143)**.** *The **Picard group** of a Riemann surface $\mathcal{C}$ is $\mathrm{Pic}(\mathcal{C})$, the group of isomorphism classes of holomorphic line bundles over $\mathcal{C}$, with the group operation being tensor product.*

Recall that line bundles are determined by the data of invertible transition functions subject to a cocycle condition, with the transition functions of a tensor product of line bundles being the product of the transition functions. A holomorphic line bundle thus has transition functions which are nonzero holomorphic functions, and as such we have the following result describing the Picard group.

**Proposition 2.1.29** ([Har77], Exercise III.4.5)**.** $\mathrm{Pic}(\mathcal{C}) \cong H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^{\times})$.

Divisors fit into a triality of perspectives, with the other two sides being line bundles and invertible sheaves, all united by the cohomology group $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^{\times})$. Miranda visualises the triality of perspectives on divisors/line bundles/invertible sheaves with a "commuting tetrahedron" [Mir95, p. 356].

To any Cartier divisor $D$ given by $\{(U_\alpha, f_\alpha)\}$, we can associate a holomorphic line bundle $[D]$ whose transition functions on $U_\alpha \cap U_\beta$ are $g_{\alpha\beta} = f_\alpha/f_\beta$. It can be shown that this gives a group isomorphism $\mathrm{Cl}(\mathcal{C}) \xrightarrow{\cong} \mathrm{Pic}(\mathcal{C})$ [Har77, Corollary II.6.16].

**Remark 2.1.30.** *The convention taken above for transition functions is such that $\{(U_\alpha, f_\alpha)\}$ defines a section of the line bundle as, on $U_\alpha \cap U_\beta$, $f_\alpha = g_{\alpha\beta} f_\beta$. I shall on occasion call $g_{\alpha\beta}$ the transition from $U_\beta$ to $U_\alpha$. This shall be the convention I will take throughout this thesis, and I shall reiterate this at various points.*

Letting $\mathcal{O}_{\mathcal{C}}([D])$ be the sheaf of holomorphic sections of $[D]$, we have that for all sections $s \in H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}([D]))$ that are not identically zero, $(s) \sim D$, and moreover there exists a meromorphic section $s_0$ such that $(s_0) = D$. The section $s_0$ is unique up to a constant scale factor, and so there is an isomorphism $|D| \cong \mathbb{P}H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}([D]))$. Moreover, up to tensoring with $s_0$ we can identify $\mathcal{O}_{\mathcal{C}}([D])$ with the sheaf $\mathcal{O}_{\mathcal{C}}(D)$ defined by [GH78, p. 136]

$$\mathcal{O}_{\mathcal{C}}(D)(U) = \{f \in \mathcal{M}_{\mathcal{C}}(U) \,|\, (f) + D \geq 0\}\,.$$

From now on in I shall be lax on the distinction between a line bundle $L \to \mathcal{C}$ and the sheaf of sections $\mathcal{O}_{\mathcal{C}}(L)$ when writing cohomology groups.

Note giving $|D|$ the structure of a projective space, we can now define a concept of subspace and dimension.

**Definition 2.1.31** ([Har77], p. 157-159)**.** *A **linear system** is a projective subspace of a complete linear system when identified with $\mathbb{P}H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}([D]))$, that is $\mathbb{P}V$ for some vector subspace $V \leq H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}([D]))$. We define the **dimension** of a linear system to be the (complex) dimension of the corresponding projective subspace, that is $\dim V - 1$, and the **degree** to be the degree of the corresponding divisor. The notation $g^r_d$ denotes any linear system of dimension $r$ and degree $d$.*

**Definition 2.1.32** ([BL04], p. 385)**.** *The **gonality** of an algebraic curve is the minimum $d$ such that the curve has a $g^1_d$.*

**Example 2.1.33.** *The line bundle $K_{\mathcal{C}} := [\mathcal{K}_{\mathcal{C}}]$ is called the **canonical bundle**, and it is the line bundle whose fibre at $P \in \mathcal{C}$ is spanned by $\mathrm{d}z$, where $z$ is a local coordinate at $P$ [GH78, p. 146]. The corresponding linear system (sometimes called the canonical linear system) has dimension $g(\mathcal{C}) - 1$, and degree $2g(\mathcal{C}) - 2$.*

*On a smooth curve the sheaf of sections of $K_{\mathcal{C}}$ is sometimes referred to as the **dualising sheaf** because of the isomorphism of **Serre duality** $H^1(\mathcal{C}, L) \cong H^0(\mathcal{C}, K_{\mathcal{C}} \otimes L^*)^*$ for any line bundle $L$ [Mir95, Theorem VI.3.3].*

Now from the Long Exact Sequence (LES) of cohomology associated with the exponential Short Exact Sequence (SES) of sheaves [GH78, p. 37]

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_{\mathcal{C}} \stackrel{\exp}{\to} \mathcal{O}^{\times}_{\mathcal{C}} \to 0,$$

one gets

$$0 \to {H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})}\big/{H^1(\mathcal{C}, \underline{\mathbb{Z}})} \to H^1(\mathcal{C}, \mathcal{O}^{\times}_{\mathcal{C}}) \stackrel{\delta}{\to} H^2(\mathcal{C}, \underline{\mathbb{Z}}) \to 0,$$

where $\delta$ is a connecting map. For a given $L \in \mathrm{Pic}(\mathcal{C})$, we call $\delta(L) := c_1(L)$ the **first Chern class** of $L$. Making the identification $H^2(\mathcal{C}, \underline{\mathbb{Z}}) \cong \mathbb{Z}$ using the fundamental class one finds that $c_1([D]) = \deg(D)$, and we use this to define the **degree** of a line bundle [GH78, p. 144]. With this concept we make two important definitions.

**Definition 2.1.34** ([Har77], p. 157)**.** *We call ${H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})}\big/{H^1(\mathcal{C}, \underline{\mathbb{Z}})} = \mathrm{Pic}^0(\mathcal{C})$ the **Picard variety** of $\mathcal{C}$. It is the group of isomorphism classes of degree-0 line bundles on $\mathcal{C}$.*

**Remark 2.1.35.** *For a line bundle $L \in \mathrm{Pic}(\mathcal{C})$ I will use the notation that for $n \in \mathbb{N}$,*

$$L^n := \underbrace{L \otimes \cdots \otimes L}_{\times n}.$$

*If $L$ is in the image of the exponential map, one can define $L^s$ for $s \in \mathbb{C}$ by multiplying the corresponding element of $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ by $s$.*

In general the set $\mathrm{Pic}^d(\mathcal{C})$ of isomorphism classes of degree-$d$ line bundles over $\mathcal{C}$ is a torsor[4] over $\mathrm{Pic}^0(\mathcal{C})$, and we immediately see that the tangent space to $\mathrm{Pic}(\mathcal{C})$ at any $L$ is $T_L \mathrm{Pic}(\mathcal{C}) \cong H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$.

**Definition 2.1.36** ([Mir95], Definition VIII.1.2)**.** *We define the* ***Jacobian variety*** *of $\mathcal{C}$ to be*

$$\mathrm{Jac}(\mathcal{C}) \cong {H^0(\mathcal{C}, K_{\mathcal{C}})^*}\big/{H_1(\mathcal{C}, \mathbb{Z})},$$

*where the embedding $H_1(\mathcal{C}, \mathbb{Z}) \hookrightarrow H^0(\mathcal{C}, K_{\mathcal{C}})^*$ is $[c] \mapsto \oint_c \omega$.*

Using Serre duality and Poincaré duality one gets the isomorphism $\mathrm{Pic}^0(\mathcal{C}) \cong \mathrm{Jac}(\mathcal{C})$ [Har77, p. 447]. It will be helpful for later to describe slightly more precisely the isomorphism $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \cong H^0(\mathcal{C}, K_{\mathcal{C}})^*$, which is induced by the pairing

$$\begin{aligned} H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) \times H^0(\mathcal{C}, K_{\mathcal{C}}) &\to \mathbb{C}, \\ (\{r_P\}, \omega) &\mapsto \sum_P \mathrm{Res}_P(r_P \omega), \end{aligned} \tag{2.1}$$

viewing an element of $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ as a collection of (equivalence classes of) Laurent tails, that is a Laurent tail divisor [Mir95, §VI.2-3]. Specifically here I am using the identification that for any effective divisor $D$ of sufficiently large degree there is an isomorphism between $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ and the cokernel of the map $H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(D)) \to H^0(\mathcal{C}, \mathcal{O}_D(D))$ coming from the SES [For91, §16.7]

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(D) \to \mathcal{O}_D(D) \to 0.$$

The connecting map is such that given Laurent tails $r_P$, $r_{P'}$ at $P \in U$, $P' \in U'$, with $U, U' \subset \mathcal{C}$ open sets, these determine meromorphic functions $f$, $f'$, on the respective open sets, and then on $U \cap U'$ the value of the corresponding element of $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ is $f - f'$. Taking a limit as the support of $D$ increases completes the picture relating Laurent tail divisors to elements of $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$. One helpful way to compute this residue we will see in §3.2.1 comes when there exists a differential of the second kind $\gamma_0$, that is a meromorphic differential with no poles of order exactly one, and higher order poles only at the $P$ in the support of the Laurent tails such that the function $g(P) := \int_{P_0}^P \gamma_0$ satisfies

$$\sum_P \mathrm{Res}_P(r_P \omega) = \sum_{P \in \mathcal{C}} \mathrm{Res}_P(g\omega).$$

[4]For any group $G$, a torsor over $G$ is a set $T$ with a group actions $a : G \times T \to T$ such that the map $G \times T \to T \times T$, $(g, t) \mapsto (a(g, t), t)$ is a bijection.

The latter sum can be calculated using the reciprocity law for differentials of the second kind [GH78, p. 241]

$$\sum_{P\in\mathcal{C}} \mathrm{Res}_P(g\omega) = \frac{1}{2\pi i}\sum_{j=1}^{g} \begin{vmatrix} \int_{a_j}\omega & \int_{a_j}\gamma_0 \\ \int_{b_j}\omega & \int_{b_j}\gamma_0 \end{vmatrix}.$$

**Remark 2.1.37** ([GH78], p. 42, [Mir95], p. 304)**.** *In making the identification* $\mathrm{Pic}^0(\mathcal{C}) \cong \mathrm{Jac}(\mathcal{C})$ *I have used that* $H_1(\mathcal{C},\mathbb{Z})$*, the singular homology group with values in* $\mathbb{Z}$*, and* $H^1(\mathcal{C},\underline{\mathbb{Z}})$*, the sheaf cohomology group valued in the constant sheaf* $\underline{\mathbb{Z}}$*, are equivalent. I shall herein be lax with notation and write* $H^1(\mathcal{C},\mathbb{Z})$ *for the sheaf cohomology group.*

**Example 2.1.38.** $\mathrm{Jac}(\mathbb{P}^1)$ *is trivial, and so line bundles over* $\mathbb{P}^1$ *are classified by their degree. We denote the line bundle of degree* $d$ *over* $\mathbb{P}^1$ *as* $\mathcal{O}(d)$*. We can identify* $K_{\mathbb{P}^1} \cong \mathcal{O}(-2)$*,* $T\mathbb{P}^1 \cong \mathcal{O}(2)$*.*

*The degree of the line bundle corresponds to the degree of the transition function, which will be a homogeneous polynomial. For example, on* $\mathbb{P}^1$ *take the affine coordinate* $\zeta = \zeta_0/\zeta_1$ *where* $\zeta_1 \neq 0$*, and* $\tilde{\zeta} = 1/\zeta$ *where* $\zeta_0 \neq 0$*. One can work out* $\frac{d}{d\tilde{\zeta}} = -\zeta^2\frac{d}{d\zeta}$*. This means that if we introduce the coordinate* $\eta$ *on the fibre of* $T\mathbb{P}^1$ *with* $\tilde{\eta}\frac{d}{d\tilde{\zeta}} = \eta\frac{d}{d\zeta}$*,* $\tilde{\eta} = -\eta/\zeta^2$*, and as such the transition function is up to a sign* $g_{01}(\zeta) = \zeta^2$*.*

*Sections of* $\mathcal{O}(d)$ *are given by homogeneous degree-d polynomials in* $\zeta_{0,1}$*, and so we immediately get* $\dim H^0(\mathbb{P}^1,\mathcal{O}(d)) = d+1$*.*

Picking a particular cohomology and homology basis $\{\omega_i\}$, $\{\gamma_j\}$ of $H^0(\mathcal{C},K_{\mathcal{C}})$, $H_1(\mathcal{C},\mathbb{Z})$ respectively, we see the Jacobian is a complex torus $\mathbb{C}^g/\Lambda$ where the lattice $\Lambda$ is generated by the vectors

$$\mathbf{\Omega}_j = \left(\int_{\gamma_j}\omega_1,\ldots,\int_{\gamma_j}\omega_g\right), \quad 1\leq j\leq 2g. \tag{2.2}$$

In fact, $\mathrm{Jac}(\mathcal{C})$ has a canonical principal polarisation coming from the intersection pairing of cycles $\circ$ on $\mathcal{C}$, corresponding to choosing a **canonical homology basis** $\{\gamma_j\} = \{a_j, b_j\}$ with $a_i \circ b_j = \delta_{ij}$, and so $\mathrm{Jac}(\mathcal{C})$ is a **Principally Polarised Abelian Variety (PPAV)** [BL04, p. 70, p. 317]. Torelli's theorem makes precise one way in which this definition is important.

**Theorem 2.1.39** (Torelli's Theorem, [BL04], Theorem 11.1.7, [GH78], p. 359)**.** $\mathrm{Jac}(\mathcal{C}) \cong \mathrm{Jac}(\mathcal{C}')$ *as principally polarised abelian varieties if and only if* $\mathcal{C} \cong \mathcal{C}'$ *(as algebraic curves).*

Being a PPAV gives us additional information about the Jacobian, for example we know that abelian subvarieties are in 1-1 correspondence with the idempotent elements of the $\mathbb{Q}$-endomorphism algebra, and moreover that up to isogeny the Jacobian decomposes into a product of irreducible abelian subvarieties [BL04, Theorem 5.3.2, Theorem 5.3.7]. Recall isogenies of abelian varieties are surjective homomorphisms with finite kernels. This lets one use algebra to constrain the structure of the Jacobian of the curve, as I will do explicitly in §3.4.3.

**Definition 2.1.40.** *The $g \times 2g$ matrix whose columns are the vectors $\mathbf{\Omega}_j$ of Equation 2.2 is called the* ***period matrix*** *and is denoted $\Omega \in M_{g,2g}(\mathbb{C})$. Choosing a canonical homology basis $\{\gamma_j\} = \{a_j, b_j\}$ such that $\Omega = (A, B)$ where $A_{ij} = \int_{a_j} \omega_i$ (and likewise for $B$) we define the* ***Riemann matrix*** *to be $\tau = A^{-1}B$.*

**Remark 2.1.41.** *Some authors would call the period matrix defined in Definition 2.1.40 the matrix of periods, reserving the name period matrix for what we call the Riemann matrix, see for example [BN10]. Moreover, sometimes the period matrix is defined to be a $2g \times g$ matrix instead. The convention taken in Definition 2.1.40 agrees with that used in the Riemann surfaces module of SageMath [Sag21b] where the period matrix and associated endomorphism ring can be computed numerically, as computations involving the code in this module have been invaluable during this thesis. In [BDHG22] Bruin, Gao, and I developed a method for computing the numerical integrals that occur when calculating the period matrix with rigorous error bounds.*

**Remark 2.1.42.** *The matrices $A, B$ used in the definition of the Riemann matrix are always invertible [Mir95, Lemma VIII.4.4], and so the Riemann matrix is well defined for a given choice of cohomology and canonical homology basis. A change of cohomology basis gives a $\mathrm{GL}_g(\mathbb{C})$ left action $T : \Omega \mapsto T\Omega$, $T : \tau \mapsto \tau$. A change of homology basis preserving the intersection pairing gives a $\mathrm{Sp}_{2g}(\mathbb{Z})$ right action $R : \Omega \mapsto \Omega R$, $\left(\begin{smallmatrix} \delta & \beta \\ \gamma & \alpha \end{smallmatrix}\right) : \tau \mapsto (\delta + \tau\gamma)^{-1}(\beta + \tau\alpha)$.*

The elements of the period matrix are typically transcendental numbers [BW08, Corollary 6.9], as also evidenced by the principle of Kontsevich and Zagier [KZ01]:

> "Whenever you meet a new number, and have decided (or convinced yourself) that it is transcendental, try to figure out whether it is a period".

This can be made very precise in the case of elliptic curves by the following result.

**Proposition 2.1.43** (Schneider-Lang Theorem, [BW08], p. 30)**.** *Letting $j$ be the elliptic $j$-invariant (see Equation 2.5), if $j(\tau)$ is rational then $\tau$ is either transcendental or an element of a quadratic imaginary field with class number 1.*

The $j$-invariants of elements of quadratic imaginary fields with class number 1 are known to be integers, and are enumerated in [Sil94, §A.3]. These results on transcendentality shall be relevant in §2.3 and §3.2 where they shall be used to make remarks about the transcendentality of specific curves or their periods.

The isomorphism between the Jacobian and $\mathrm{Pic}^0$ can be made explicit.

**Definition 2.1.44.** *The* ***Abel-Jacobi (AJ) map*** *based at $Q \in \mathcal{C}$ is*

$$\begin{aligned} \mathcal{A}_Q : \mathcal{C} &\to \mathrm{Jac}(\mathcal{C}), \\ P &\mapsto \left( \int_Q^P \omega_1, \dots, \int_Q^P \omega_g \right) \mod \Lambda. \end{aligned}$$

*The map $\mathcal{A}_Q$ is independent of the path of integration as we have quotiented by $\Lambda$. We can extend $\mathcal{A}_Q$ to a map of divisors by*

$$\begin{aligned}\mathcal{A}_Q : \mathrm{Div}(\mathcal{C}) &\to \mathrm{Jac}(\mathcal{C}),\\ \sum_i n_i P_i &\mapsto \sum_i n_i \mathcal{A}_Q(P_i).\end{aligned}$$

**Remark 2.1.45.** *As with the period matrix, computing the Abel-Jacobi map involves only computing integrals of algebraic integrands, and as such using methods of [BDHG22, Neu18] I implemented computation of the AJ map in Sage [Sag21b].*

There is no canonical choice of basepoint for the AJ map, and one can always calculate

$$\mathcal{A}_{Q'}(D) = \mathcal{A}_Q(D) - \deg(D)\cdot\mathcal{A}_Q(Q').$$

The properties of this map are described by the following theorem. Here we denote the image under the Abel-Jacobi map of the set of degree-$d$ effective divisors as $W_d$.

**Theorem 2.1.46** (Abel-Jacobi, [FK92], §III.6)**.** *Given effective divisors $D, D' \in \mathrm{Div}(\mathcal{C})$, $\mathcal{A}_*(D) = \mathcal{A}_*(D') \Leftrightarrow D \sim D'$. Moreover, $W_g = \mathrm{Jac}(\mathcal{C})$.*

**Corollary 2.1.47.** *For any basepoint on an elliptic curve $Q \in \mathcal{E}$, $\mathcal{A}_Q : \mathcal{E} \to \mathrm{Jac}(\mathcal{E})$ is an isomorphism with $\mathcal{A}_Q(Q) = 0$.*

This means that we have now understood the moduli space of degree-0 line bundles on $\mathcal{C}$ in terms of analytic divisor data.

**Remark 2.1.48.** *Abel (1802-1829) proved his theorem in his "Mémoire sur une propriété générale d'une classe très-étendue de fonctions transcendentes" (1826) in the language of complex analysis before the concept of Riemann surfaces or cohomology. It was "pronounced by Jacobi the greatest discovery of [the 19th] century on the integral calculus" [Caj94, p. 413].*

Given $f : \mathcal{C} \to \mathcal{C}'$, by pulling back line bundles and using Proposition 2.1.17 one gets a map $\mathrm{Pic}^0(\mathcal{C}') \to \mathrm{Pic}^0(\mathcal{C})$, and hence dually (in a way made precise in [BL04, §11.4]) $\mathrm{Jac}(\mathcal{C}) \to \mathrm{Jac}(\mathcal{C}')$. We can understand this map at the level of the complex torus as we have associated pullback and pushforward maps $f^* : H^0(\mathcal{C}', K_{\mathcal{C}'}) \to H^0(\mathcal{C}, K_{\mathcal{C}})$ and $f_* : H_1(\mathcal{C}, \mathbb{Z}) \to H_1(\mathcal{C}', \mathbb{Z})$. These are dual in the sense that for $\gamma \in H_1(\mathcal{C}, \mathbb{Z})$, $\omega' \in H^0(\mathcal{C}', K_{\mathcal{C}'})$,

$$\int_\gamma f^*\omega' = \int_{f_*\gamma} \omega'.$$

Picking bases gives matrices $T \in M_g(\mathbb{C})$, $R \in M_{2g'}(\mathbb{Z})$, for $f^*$, $f_*$ respectively, which we call the **analytic representation** $\rho_a(f)$ and the **rational representation** $\rho_r(f)$ (considered over the fields $\mathbb{C}$ and $\mathbb{Q}$ respectively, acting via left multiplication on vectors) [BL04, p. 10]. The duality condition thus says that $T\Omega = \Omega' R$, and moreover given either of $T$, $R$, one can recover the other by the following result.

**Proposition 2.1.49** ([BL04], Proposition 1.1.2). *$\begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} \in M_{2g}(\mathbb{C})$ is invertible, and hence $\rho_a \oplus \overline{\rho_a}$ is equivalent as a complex representation to $\rho_r$ with the conjugating matrix $\begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix}$.*

### Riemann-Roch and Weierstrass Points

We start by stating the Riemann-Roch theorem in a slightly unusual way.

**Theorem 2.1.50** (Riemann-Roch, [Har77], Theorem IV.1.3). *Given $L \to \mathcal{C}$ a holomorphic vector bundle,*

$$\dim H^0(\mathcal{C}, L) - \dim H^1(\mathcal{C}, L) = \deg(L) + (1-g).$$

*Equivalently, in terms of the associated divisors and letting $l(D) := \dim H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(D))$,*

$$l(D) - l(\mathcal{K}_{\mathcal{C}} - D) = \deg(D) + (1-g).$$

**Definition 2.1.51** ([Har77], Example IV.1.3.4). *The term $i(D) := l(\mathcal{K}_{\mathcal{C}} - D)$ in Riemann-Roch is often called the **index of speciality**. An effective divisor for which $i(D) > 0$ is called **special**.*

As a corollary of Riemann-Roch, one also gets the following helpful result.

**Proposition 2.1.52** (Clifford's Theorem, [Har77], Theorem IV.5.4). *Let $D$ be an effective special divisor, then $l(D) \leq \frac{1}{2}\deg(D) + 1$, with equality if and only if $D = 0$, $D = \mathcal{K}_{\mathcal{C}}$, or $D$ is a multiple of the $g^1_2$ when $\mathcal{C}$ is hyperelliptic.*

Using Riemann-Roch, we have that for $P \in \mathcal{C}$ the sequence $\{l(kP)\}_{k=0}^{\infty}$ behaves as

$$1, \underbrace{?, \dots, ?}_{1 \leq k \leq 2g-2}, g, g+1, \dots.$$

Moreover, the sequence can increase by a maximum of one from term to term, which leads to the following result.

**Theorem 2.1.53** (Weierstrass Gap Theorem, [FK92], §III.5.3). *Let $\mathcal{C}$ be a compact genus-$g$ Riemann surface, then $\forall P \in \mathcal{C}$, there exist unique integers $\{n_i\}_{i=1}^{g}$ such that*

$$1 = n_1 < n_2 < \dots < n_g < 2g,$$

*and there does not exist $f$ a global meromorphic function with $(f) = -n_i P$.*

**Definition 2.1.54.** *The **weight** of $P \in \mathcal{C}$ is*

$$w_P = \sum_{i=1}^{g} (n_i - i).$$

*A point $P$ is called a **Weierstrass point** if $w_P \neq 0$ (note that $w_P$ is always nonnegative, so at a Weierstrass point it is positive). We will sometimes denote the set of Weierstrass points as $W = W(\mathcal{C})$.*

**Remark 2.1.55.** *In §2.3.2, I will present two methods for calculating the Weierstrass points on Bring's curve. In general Weierstrass points can be computed algorithmically in Magma over exact fields[5] in any characteristic [Hes02a].*

We have results about the abundance of Weierstrass points.

**Proposition 2.1.56** ([FK92], §III.5)**.** *The set of Weierstrass points is discrete, with $\sum_{P\in\mathcal{C}} w_P = g^3 - g$. Moreover if $g \geq 2$ the number of Weierstrass points $|W|$ satisfies*

$$2g + 2 \leq |W| \leq g^3 - g,$$

*attaining the lower bound if and only if $\mathcal{C}$ is hyperelliptic, where the Weierstrass points are the branch points corresponding to the hyperelliptic involution. Generically a Riemann surface will have $|W| = g^3 - g$.*

### The Canonical Embedding

As part of §2.1.2, it was shown how for any divisor $D$ on $\mathcal{C}$ the complete linear system $|D|$ is parametrised by the projective space of sections $\mathbb{P}H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(D))$. In order to progress with this idea we make the following definition.

**Definition 2.1.57** ([McM14], p. 96)**.** *The **base locus** of a linear system $|D|$ on a curve $\mathcal{C}$ is the maximal divisor $B \geq 0$ such that $\forall E \in |D|$, $E \geq B$. A linear system is **basepoint-free** if $B = 0$.*

Parametrising[6] the linear system by sections, the base locus $B = \sum n_i P_i$ says that every section vanishes at $P_i$ with order at least $n_i$, and hence a linear system is basepoint-free if $\forall P \in \mathcal{C}$, $\exists s \in H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(D))$ such that $s(P) \neq 0$. Given such a basepoint-free linear system $|D|$, and picking a basis of sections $s_0, \ldots, s_N \in H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(D))$, one gets a well defined map [GH78, p. 176]

$$\begin{aligned} \iota_D : \mathcal{C} &\to \mathbb{P}^N, \\ P &\mapsto [s_0(P) : \cdots : s_N(P)]. \end{aligned}$$

Imposing conditions on $D$ one can ensure that this map is an embedding; such a divisor is called **very ample** [GH78, p. 180, p. 192]. The intersection of any hyperplane with the image in $\mathbb{P}^N$, counted with multiplicity, gives an effective divisor $H \sim D$, that is $H \in |D|$ [GH78, p. 176].

[5] A field is exact if computations in this field may be done exactly on a computer, e.g. $\mathbb{Q}$, finite fields, in contrast to fields where answers are only approximate on a computer, e.g. $\mathbb{R}$ where floating point numbers are used.

[6] In choosing the spelling "parametrising" over "parameterising" I am following the precedent of key textbooks [Har77, AH88, MS04].

**Remark 2.1.58.** *A corollary of the previous comment is that the degree of $\iota_D(\mathcal{C}) \subset \mathbb{P}^N$ as a projective curve is exactly equal to the degree of $D$, and that a linear system $g^1_d$ determines a degree-d morphism $\mathcal{C} \to \mathbb{P}^1$. The gonality of $\mathcal{C}$ can thus be reinterpreted as the smallest degree of a morphism $\mathcal{C} \to \mathbb{P}^1$*

One can prove that, for a compact Riemann surface $\mathcal{C}$ with $g \geq 2$, $\mathcal{K}_\mathcal{C}$ is very ample if and only if $\mathcal{C}$ is not hyperelliptic [Har77, Proposition IV.3.1]. This leads to the following natural definition.

**Definition 2.1.59** ([Har77], p. 341)**.** *Given a non-hyperelliptic curve $\mathcal{C}$ of genus $g \geq 3$, we call the embedding $\iota_{\mathcal{K}_\mathcal{C}} : \mathcal{C} \hookrightarrow \mathbb{P}^{g-1}$ given by the canonical divisor the **canonical embedding**, and denote its image $\mathcal{C}_{can}$. Given a singular plane curve we can define its canonical model to be the canonical embedding of its normalisation [KM09].*

**Example 2.1.60.** *The canonical embedding of a non-hyperelliptic genus-3 curve is a plane quartic. Moreover, all smooth plane quartics are genus-3 by Example 2.1.13 and non-hyperelliptic as they are complete intersections [Har77, Exercise IV.5.1].*

**Example 2.1.61** ([Har77], Example IV.5.2.2)**.** *The canonical embedding of a genus-4 curve is a degree-6 curve in $\mathbb{P}^3$. Every such curve can be written as the complete intersection of a unique irreducible quadric surface and a smooth irreducible cubic surface. Moreover, every such intersection is the canonical embedding of a genus-4 curve.*

The importance of the canonical embedding comes from the fact that the extrinsic geometry of the Riemann surface in the canonical embedding is reflected in the intrinsic structure of the Riemann surface itself. For example we get different ways to characterise Weierstrass points, namely by the following result.

**Proposition 2.1.62** ([McM14], Proposition 12.6, Theorem 12.7)**.** *Given a Weierstrass point $P \in \mathcal{C}$ with $g(\mathcal{C}) = g$, the following equivalent conditions are satisfied:*

1. *there exists a hyperplane $H \subset \mathbb{P}^{g-1}$ such that $H \cap \mathcal{C}_{can}$ has multiplicity at least $g$ at $P$,*
2. *there exists a holomorphic differential on $\mathcal{C}$ vanishing at $P$ with order at least $g$,*
3. *there exists a meromorphic function on $\mathcal{C}$ with poles just at $P$ of order at most $g$, and*
4. *the Wronskian determinant given by $\mathrm{Wr}(P) = \det\left(\frac{d^i\omega_j}{dz^i}\right)_{i,j=0,\dots,g-1}$, where $\{\omega_j\}$ is a basis of holomorphic differentials and $z$ is a local coordinate around $P$, vanishes.*

### 2.1.3 Surfaces in $\mathbb{P}^3$

As we will require some properties of quadric and cubic surfaces in $\mathbb{P}^3$ over the course of this thesis, thinking of Example 2.1.61 as the motivating example we shall now give a few properties.

**Quadrics and $T\mathbb{P}^1$**

One understands the possible quadric hypersurfaces of $\mathbb{P}^3$ very well. They are known to be either a double plane, the union of two planes, a singular cone given in some coordinates as $X_0X_1 - X_2^2 = 0$, or a smooth quadric given in some coordinates as $X_0X_3 - X_1X_2 = 0$ [Vak10, Exercise 19.8.B]. Only the latter two are irreducible. Example 2.1.61 has a partial generalisation by the following.

**Proposition 2.1.63** ([Har77], Exercise IV.5.1, Exercise V.2.9)**.** *The nonsingular complete intersection of a quadric cone and a degree-$k$ hypersurface in $\mathbb{P}^3$ is a non-hyperelliptic curve of genus $(k-1)^2$ and degree $2k$ for $k \geq 1$. Moreover, all such curves are given by such a complete intersection.*

The smooth quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, with the map $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$,

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^3, \\ ([\zeta_0 : \zeta_1], [\eta_0 : \eta_1]) &\mapsto [\zeta_0\eta_0 : \zeta_0\eta_1 : \zeta_1\eta_0 : \zeta_1\eta_1], \end{aligned}$$

a special case of the **Segre embedding**.

Over a field of characteristic not equal to 2, one can express the singular irreducible quadric as $Z_0^2 + Z_1^2 - X_2^2$ by letting $X_0 = Z_0 + iZ_1$, $X_1 = Z_0 - iZ_1$, showing clearly why it is called 'the' cone. The cone point $[0:0:0:1]$ is the unique singularity of the surface [Har77, Exercise I.5.2]. The cone is birational to the weighted projective space $\mathbb{P}(1:1:2)$ via a variation of the **Veronese embedding**,

$$\begin{aligned} \mathbb{P}(1:1:2) &\to \mathbb{P}^3, \\ [\zeta_0 : \zeta_1 : \eta] &\mapsto [\zeta_0^2 : \zeta_0\zeta_1 : \zeta_1^2 : \eta], \end{aligned}$$

where the preimage of the singular point is $[0:0:1]$ [Vak10, §8.2.11]. The automorphism group of this weighted projective space is given by the following result.

**Proposition 2.1.64** ([DI10], Proposition 7)**.** *The automorphism group of* $\mathbb{P}(1:1:2)$ *is* $\mathrm{Aut}(\mathbb{P}(1:1:2)) = \mathbb{C}^3 \rtimes [\mathrm{GL}_2(\mathbb{C})/\langle \pm \mathrm{Id}\rangle]$*. The* $\mathbb{C}^3$ *factor acts as translations* $[\zeta_0 : \zeta_1 : \eta] \mapsto [\zeta_0 : \zeta_1 : \eta + P(\zeta_0, \zeta_1)]$ *where $P$ is a homogeneous degree-2 polynomial, and the* $\mathrm{GL}_2(\mathbb{C})/\langle \pm \mathrm{Id}\rangle$ *factor acts via its linear representation on* $\zeta_{0,1}$*.*

An immediate consequence of Propositions 2.1.64 and 2.1.63 is that the automorphism group of any curve given by the intersection of the cone and a cubic is a subgroup of $\mathbb{C}^3 \rtimes [\mathrm{GL}_2(\mathbb{C})/\langle \pm \mathrm{Id}\rangle]$. The fact that the automorphism group of a genus $g \geq 2$ curve is finite (see Theorem 2.1.85) forces the group to actually be a finite subgroup of $\mathrm{GL}_2(\mathbb{C})/\langle \pm \mathrm{Id}\rangle$ (any nonzero translation would generate an infinite subgroup), which can equivalently be seen as fixing the embedding into $\mathbb{P}^3$ because the quadric on which the curve lies is unique.

Identifying $T\mathbb{P}^1$ with $\mathcal{O}(2)$ as in Example 2.1.38 gives the interpretation of the fibre coordinate as weight-2 with respect to the base $\mathbb{P}^1$ coordinate, and so the total space naturally embeds into $\mathbb{P}(1:1:2)$, missing the singular point.

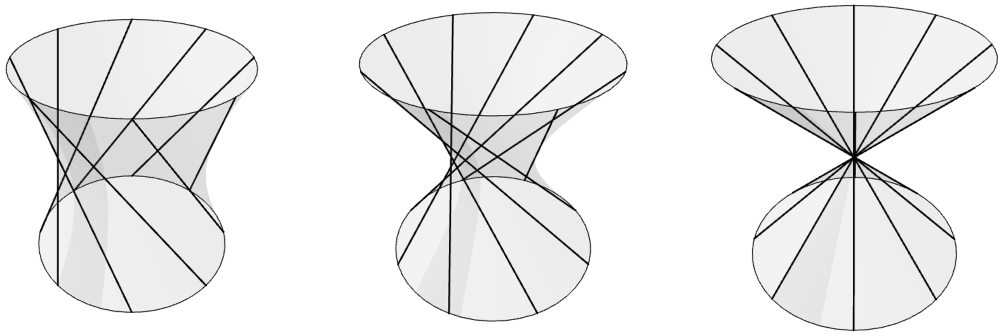


**Fig. 2.1** Limiting sequence of ruled hyperboloids

This gives a singular compactification of $T\mathbb{P}^1$; one can remove this singularity if required by blowing up the surface there, which yields the **Hirzebruch surface**[7] $\mathbb{F}_2 := \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(2))$ [Dol82, §1.2.3]. We can view this as the limit of the 'narrowing' process shown in Figure 2.1, where we demand continuity along the lines drawn.

In §3.2.1 we will require a robust knowledge of $T\mathbb{P}^1$ because of its interpretation as the **(Euclidean) minitwistor space** $\mathbb{MT}$, that is the space of oriented geodesics in Euclidean $\mathbb{R}^3$ with an appropriate complex structure [Hit82], and so I will take the opportunity to give a few more definitions here. I will fix the notation $\zeta, \eta$ for the base and fibre coordinate and $\pi$ for the projection $T\mathbb{P}^1 \to \mathbb{P}^1$ from here on in. Moreover, I will take $\tilde{U}_{0,1}$ to be the preimages under $\pi$ of the open sets $U_{0,1}$ covering $\mathbb{P}^1$ defined in Example 2.1.3. It is known that $\left\{\tilde{U}_{0,1}\right\}$ is a Leray cover of $T\mathbb{P}^1$ with respect to the sheaf $\mathcal{O}_{T\mathbb{P}^1}$, that is the Čech cohomology of the open cover $\left\{\tilde{U}_{0,1}\right\}$ is isomorphic to the sheaf cohomology of $T\mathbb{P}^1$ [AHH90].

**Definition 2.1.65.** *The* ***antiholomorphic involution*** *on* $T\mathbb{P}^1$ *is* $\tau(\zeta,\eta) = (-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)$.

It will be relevant for later work to note that this involution corresponds to the involution on $\mathbb{MT}$ coming from reversing the orientations of each geodesic. The involution $\tau$ induces an action on the space of holomorphic vector bundles over $T\mathbb{P}^1$ by $\sigma : E \to \tau^*\overline{E}$, where $\overline{E}$ is the conjugate bundle, that is the bundle whose transition functions are the complex conjugate of the transition functions of $E$. As this action will be recur frequently in §3.2 I shall introduce a simplified notation for it now.

**Definition 2.1.66.** *I will adopt the notation that for any complex function* $f = f(\zeta,\eta)$, $f^\tau := \overline{\tau^* f} = \overline{f\circ\tau}$.

A bundle is called real/quaternionic if the isomorphism $\sigma^2 : E \to \tau^*\overline{\tau^*\overline{E}} \cong E$ acts fibrewise as 1 or $-1$ respectively [AW77, Har78, BH11]. Note that, given bundles $E_i$, $i = 1,2$, with lifts $\sigma_i$ of $\tau$ such that $\sigma_i^2 = (-1)^{n_i}$, $n_i \in \mathbb{Z}$, $E_1 \otimes E_2$ has the lift of $\tau$ given by $\sigma_1 \otimes \sigma_2$ satisfying $(\sigma_1 \otimes \sigma_2)^2 = (-1)^{n_1+n_2}$.

[7]Not to be confused with the field of order 2.

**Example 2.1.67.** *The projection map $p : \mathbb{C}^2 \setminus 0 \to \mathbb{P}^1$ given by $p : (\zeta_0, \zeta_1) \mapsto [\zeta_0 : \zeta_1]$ determines the tautological line bundle $\mathcal{O}(-1)$. One can see this by trivialising the line bundle as*

$$\begin{aligned} \rho_0 = (p, \psi_0) : p^{-1}(U_0) &\to U_0 \times \mathbb{C}, \\ (\zeta_0, \zeta_1) &\to ([\zeta : 1], \zeta_1), \\ \rho_1 = (p, \psi_1) : p^{-1}(U_1) &\to U_1 \times \mathbb{C}, \\ (\zeta_0, \zeta_1) &\to ([1 : \zeta^{-1}], \zeta_0), \end{aligned}$$

*whereby $\rho_0 \circ \rho_1^{-1}\big|_{(U_0 \cap U_1)\times\mathbb{C}} : ([1 : \zeta^{-1}], \zeta_0) \to ([\zeta : 1], \zeta_1)$, so the associated transition function is $g_{01} = \zeta_1/\zeta_0 = \zeta^{-1}$ [For91, §29].*

*Now $\mathbb{P}^1$ has the antiholomorphic involution $[\zeta_0 : \zeta_1] \mapsto [-\bar{\zeta}_1 : \bar{\zeta}_0]$, or equivalently $\zeta \mapsto -1/\bar{\zeta}$. This naturally lifts to an antiholomorphic involution of the total space of the bundle given by $\sigma : (\zeta_0, \zeta_1) \mapsto (-\bar{\zeta}_1, \bar{\zeta}_0)$ such that $\sigma^2 : (\zeta_0, \zeta_1) \to (-\zeta_0, -\zeta_1)$. As such, $\mathcal{O}(-1)$ has a quaternionic structure, and one can determine whether $\mathcal{O}(k)$ has a real or quaternionic structure by the parity of $k$. Pulling back by $\pi : T\mathbb{P}^1 \to \mathbb{P}^1$ the same argument shows $\pi^*\mathcal{O}(k)$ has a real/quaternionic structure relative to the antiholomorphic involution $\tau$ depending on the parity of $k$.*

Using the known Leray cover we can determine the line bundles on $T\mathbb{P}^1$.

**Proposition 2.1.68** ([AHH90], Proposition 2.2)**.** *The cohomology group $H^1(T\mathbb{P}^1, \mathcal{O}_{T\mathbb{P}^1})$ is generated by monomials $\eta^i\zeta^j$ where $i > 0$ and $-2i < j < 0$.*

**Definition 2.1.69.** *We define a line bundle $L^s \to T\mathbb{P}^1$ for $s \in \mathbb{R}$ with transition function $g_{10} = e^{s\eta/\zeta}$. Denote $L := L^1$ and $L^s(k) := L^s \otimes \pi^*\mathcal{O}(k)$.*

**Remark 2.1.70.** *In Definition 2.1.69 I am using the convention that, when specifying transition functions, one has sections $f_i$ defined on an open cover $\{U_i\}$ related by $f_i = g_{ij} f_j$. With this convention, the bundle $\mathcal{O}(k)$ has transition function $g_{01} = \zeta^k$. This agrees with the conventions when they were set out in §2.1.2.*

**Remark 2.1.71.** *The line bundle $L$ has a more intrinsic definition, given by Hitchin in [Hit82, §5]. Namely, on any compact curve there is the natural function $1 \in H^0(\mathcal{C}, \mathcal{O}_\mathcal{C})$ hence by Serre duality a natural class $\omega \in H^1(\mathcal{C}, K_\mathcal{C})$, and given a vector bundle $E \xrightarrow{\pi} \mathcal{C}$ there is the tautological section of the pullback by $\pi$ of the total space $s_0 \in H^0(E, \pi^*E)$. Taking $E = K_\mathcal{C}^*$ these together define $s_0\pi^*\omega \in H^1(K_\mathcal{C}^*, \mathcal{O}_{K_\mathcal{C}^*})$ which by the exponential sequence will define a line bundle over $E$ (see §2.1.2). Taking $\mathcal{C} = \mathbb{P}^1$ gives $E = T\mathbb{P}^1$, $\omega$ is a differential-valued Laurent tail divisor with representative given solely by $d\zeta/\zeta$ at $\zeta = 0$, and the tautological section takes fibre value $\eta\frac{d}{d\zeta}$, so the class in $H^1(T\mathbb{P}^1, \mathcal{O}_{T\mathbb{P}^1})$ is a Laurent tail divisor with representative given solely by $\eta/\zeta$ at $\zeta = 0$.*

Realising the embedding of $T\mathbb{P}^1$ in the cone in $\mathbb{P}^3$ gives the following corollary of Proposition 2.1.63.

**Proposition 2.1.72.** *Let $\mathcal{C} \subset T\mathbb{P}^1$ be a curve in the linear system $|\pi^*\mathcal{O}(2k)|$ for $k \geq 1$. Then $g(\mathcal{C}) = (k-1)^2$, and $\mathcal{C}$ is not hyperelliptic for $k \geq 2$.*

### Cubics

Moreover, much is known for cubic surfaces $F \subset \mathbb{P}^3$, for example the famous result on 27 lines [Sal82, p. 500]. A classical result from [SR49, p. 122-124] gives that a generic cubic surface can be written as the vanishing of a determinant

$$\det \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} = 0,$$

where $u_1, \dots, w_3$ are linear homogeneous functions of the $\mathbb{P}^3$ coordinates, that is to say $P = [L_1 : L_2 : L_3 : L_4] \in F \subset \mathbb{P}^3$ if and only if there exists $P' = [X : Y : Z] \in \mathbb{P}^2$ such that

$$X u_i(P) + Y v_i(P) + Z w_i(P) = 0.$$

Thinking of $P'$ as a point in a plane $\Pi$ we get a birational transformation $\Pi \leftrightarrow F$, $P' \leftrightarrow P$. The map $\Psi : \Pi \to F$ will have the $L_a$ as homogeneous cubics in $X, Y, Z$. To see this rewrite the determinant equation as (for $i = 1, 2, 3$)

$$\sum_{a=1}^{4} a_{ia}(P') L_a = 0 \tag{2.3}$$

for some $a_{ia}$ linear homogeneous in the $X, Y, Z$. On each affine patch $L_a \neq 0$ solving Equation 2.3 involves inverting a $3 \times 3$ matrix whose entries are linear homogeneous polynomials in $X, Y, Z$. Likewise, given the $L_a$, we have a 3-parameter family of cubics given by

$$a L_1 + b L_2 + c L_3 + d L_4 = 0 \quad \text{for} \quad [a : b : c : d] \in \mathbb{P}^3.$$

A cubic in $\mathbb{P}^2$ has 10 projective coefficients, and so a 3-parameter family is defined by six constraints. Generically we can take those constraints to come in the form of intersection with six generic points $O_i \in \Pi$.

Finally, I shall set some definitions from classical geometry that will be used later in §2.3.

**Definition 2.1.73** ([Hir86], p. 192)**.** *An **Eckardt point** of a surface is a point where three lines contained within the surface intersect.*

**Definition 2.1.74** ([Hir86], p. 182, [Sal82], p. 500)**.** *A **double-six** is a collection of twelve lines $a_1, \dots, a_6, b_1, \dots, b_6$ in $\mathbb{P}^3$, arranged as*

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix}$$

*so that each line is disjoint from those in the same row and column, but intersects the other five lines.*

**Definition 2.1.75** ([Sal82], p. 104)**.** *The **osculating plane** $\Pi$ at a point $P$ on a curve $\mathcal{C}$ is the limiting plane through $P$, $P'$, $P''$ as $P', P'' \to P$ on the curve. Equivalently this is a plane such that $\Pi \cap \mathcal{C} \geq 3P$.*

### 2.1.4 The Automorphism Group

Given a compact Riemann surface $\mathcal{C}$, we have the associated group $\mathrm{Aut}(\mathcal{C})$ of automorphisms (i.e. self-isomorphisms) acting holomorphically on $\mathcal{C}$. The automorphism group of a curve is a powerful tool for studying the corresponding curves geometry, and will be fundamental for the rest of the thesis.

**Example 2.1.76** ([FK92], p. 277)**.** *The automorphism group of the projective line is* $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PSL}_2(\mathbb{C})$*, acting by Möbius transformations.*

**Example 2.1.77.** *An elliptic curve* $\mathcal{E}$ *has an automorphism group as an object in the category of projective varieties, and also an automorphism group as an object in the category of groups. To consider the group structure on an elliptic curve* $\mathcal{E}$ *we need to also give a base point* $O \in \mathcal{E}$ *which acts as the additive identity, which corresponds to picking a base point of the Abel-Jacobi map* $\mathcal{E} \to \mathrm{Jac}(\mathcal{E})$*. A morphisms of pairs* $(\mathcal{E}, O) \to (\mathcal{E}', O')$ *must map* $O \to O'$*. If we denote the group of automorphisms of* $\mathcal{E}$ *as a projective variety as* $\mathrm{Aut}(\mathcal{E})$*, and the group of* $O$*-fixing automorphisms as* $\mathrm{Aut}_O(\mathcal{E})$*, then we have the short exact sequence*

$$0 \to T_{\mathcal{E}} \to \mathrm{Aut}(\mathcal{E}) \to \mathrm{Aut}_O(\mathcal{E}) \to 0,$$

*where* $T_{\mathcal{E}}$ *is the group of translations of* $\mathcal{E}$*. It is a classical theorem that* $\mathrm{Aut}_O(\mathcal{E}) \in \{C_2, C_4, C_6\}$*, where* $C_n$ *is the cyclic group of order* $n$ *[Sil09, Theorem III.10.1]. Note that, because any elliptic curve has the hyperelliptic involution,* $\mathrm{Aut}_O(\mathcal{E})$ *is never the trivial group.*

**Definition 2.1.78.** *An automorphism of a projective plane curve is called a* ***collineation*** *if it acts on the projective coordinates via the natural action of* $\mathrm{PGL}_3(\mathbb{C})$.

**Example 2.1.79.** *The action of an automorphism on the canonical embedding of the curve corresponds to the analytic representation, that is the automorphism group is represented as a subgroup of* $\mathrm{PGL}_g(\mathbb{C})$*. The automorphisms of a smooth plane quartic are therefore always collineations.*

Given a finite group $G$ acting holomorphically and effectively (that is such that no nonidentity element acts trivially) on a Riemann surface $\mathcal{C}$ we can construct the **quotient Riemann surface** $\mathcal{C}/G$, whose points are orbits of the $G$ action[8] [Mir95, Theorem III.3.4]. This quotient comes with a canonical morphism $\pi : \mathcal{C} \to \mathcal{C}/G$ of degree $|G|$ ramified at the $P$ where the corresponding isotropy subgroup $G_P := \{g \in G \,|\, g \cdot P = P\}$ is nontrivial.

**Lemma 2.1.80** ([FK92], §III.7.7)**.** *The isotropy group* $G_P$ *is cyclic.*

As points in the same $G$ orbit of $\mathcal{C}$ have conjugate isotropy subgroups, we can label the finitely many nontrivial isotropy subgroups of the ramification points with conjugacy classes $C_i$ of subgroups of $G$, the elements of which must have constant order $c_i$ as the subgroups are cyclic.

[8] I am following the precedent of [Mir95, §III.3] in choosing the convention "$G$ action" over the more common choice "$G$-action". As such I shall also use "$G$ orbit" over "$G$-orbit".

**Definition 2.1.81** ([MSSV02], §2)**.** *We call the* ***ramification type*** *of the $G$ action on $\mathcal{C}$ the data of the tuple $(g(\mathcal{C}), G, (C_1, \dots, C_r))$. The vector $\boldsymbol{c} = (c_1, \dots, c_r)$ (or sometimes $(g_0; c_1, \dots, c_r)$ where $g_0 = g(\mathcal{C}/G)$) is called the* ***signature****. I will use exponents to indicate how many times a value of $c_i$ is repeated as in [Bro91]. A* ***generating vector*** *for the action is $\{\alpha_1, \beta_1, \dots, \alpha_{g_0}, \beta_{g_0}, \gamma_1, \dots, \gamma_r\} \subset G$ such that*

$$\gamma_1^{c_1} = \cdots = \gamma_r^{c_r} = \prod_{i=1}^{g_0}(\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}) \prod_{j=1}^{r} \gamma_j = 1.$$

**Example 2.1.82.** *The genus-2 hyperelliptic curve given by $y^2 - (x^5 - 1) = 0$ has a natural action of $C_5$. The two points at $x = 0$ and the one point at $x = \infty$ are fixed by this action, so the signature of the action is $(0; 5, 5, 5)$, which will also be denoted as $(0; 5^3)$, or just $(5^3)$ if omitting the quotient genus.*

The ramification type determines $g(\mathcal{C}/G)$ by Riemann-Hurwitz,

$$2\left[g(\mathcal{C}) - 1\right] = |G| \left\{ 2\left[g(\mathcal{C}/G) - 1\right] + \sum_{i=1}^{r} \left(1 - \frac{1}{c_i}\right) \right\}. \tag{2.4}$$

Considerations of the possible ramification type lead to results bounding the size of the automorphism group when $g(\mathcal{C}) \geq 2$.

**Definition 2.1.83** ([MSSV02], §1.3)**.** *If $G \leq \mathrm{Aut}(\mathcal{C})$ has order $|G| > 4\left[g(\mathcal{C}) - 1\right]$, the automorphism group $G$ is said to be* ***large****. If $G$ is large, $g(\mathcal{C}/G) = 0$.*

**Remark 2.1.84.** *Note that the property of having a large automorphism group is different to the concept of having* ***many*** *automorphisms, as defined in [Rau70]. Moreover, be aware that some authors use large to mean the condition $g(\mathcal{C}/G) = 0$, for example in [BRR13].*

One also gets results about the maximum order of a single automorphism on a curve, for example the maximal order of a single automorphism is $2(2g + 1)$ (the **Wiman bound**) [Wim95a, MSSV02], and one can get further stricter results considering odd automorphism groups [Wea03] or prime-order actions [RCR22].

**Theorem 2.1.85** (Hurwitz's Theorem, [Hur92])**.** *Given $\mathcal{C}$ with $g(\mathcal{C}) \geq 2$, $|\mathrm{Aut}(\mathcal{C})| \leq 84(g - 1)$. Curves that achieve this bound are called* ***Hurwitz curves****.*

**Example 2.1.86.** *Klein's curve has genus 3 and automorphism group $\mathrm{PSL}_2(\mathbb{F}_7)$ of order 168 [Kle79]. There are infinitely many curves achieving the Hurwitz bound [Mac61].*

Hurwitz's theorem is in fact a special case of a more general result.

**Theorem 2.1.87** (de Franchis' theorem, [Mar83])**.** *Given Riemann surfaces $\mathcal{C}, \mathcal{C}'$ with $g := g(\mathcal{C}) \geq g(\mathcal{C}') \geq 2$, the number of holomorphic maps $\mathcal{C} \to \mathcal{C}'$ is finite.*

The next result gives a link between Weierstrass points and symmetry.

**Theorem 2.1.88** ([FK92], p. 242)**.** *Automorphisms of compact Riemann surfaces permute Weierstrass points, that is we get a group homomorphism* $\mathrm{Aut}(\mathcal{C}) \to S_{W(\mathcal{C})}$*, and moreover the homomorphism is injective unless* $\mathcal{C}$ *is hyperelliptic in which case the kernel is generated by the hyperelliptic involution.*

Theorem 2.1.88 is trivial from the perspective of Weierstrass points as stalls[9] of the canonical embedding, showing the utility in the mantra that the intrinsic geometry of the curve is reflected in the extrinsic geometry of its canonical embedding. A useful result for showing a point is a Weierstrass point is the following.

**Proposition 2.1.89** ([Lew63, MV06, LS12])**.** *If an automorphism of a compact Riemann surface, genus* $g \geq 2$*, fixes more than 4 points then these fixed points are Weierstrass points.*

Automorphisms of $\mathcal{C}$ induce symplectic automorphisms of $\mathrm{Jac}(\mathcal{C})$, that is an automorphism of the complex torus that fixes the principal polarisation, which at the level of the rational representation requires that the matrix $R$ is symplectic. In the event that $\mathcal{C}$ is hyperelliptic the map gives an isomorphism $\mathrm{Aut}(\mathcal{C}) \cong \mathrm{Aut}(\mathrm{Jac}(\mathcal{C}))$; when $\mathcal{C}$ is non-hyperelliptic the isomorphism is $\mathrm{Aut}(\mathcal{C}) \cong \mathrm{Aut}(\mathrm{Jac}(\mathcal{C}))/\left\langle -1 \right\rangle$ [BSZ19, Theorem 4.11].

The classification of Riemann surfaces by their automorphism group is a deep field with a long history. The classification for genera 0 and 1 curves was done during the advent of Riemann surface theory, the case of genus-2 curves was completed by Bolza in [Bol87], the case of genera 3 and 4 by Wiman in [Wim95b] (see also [Bar12]). Separately, algorithms for computing all possible ramification types were created by Breuer in [Bre00], and developed by Paulhus for the LMFDB [LMF23].

The code of [BSZ19], implemented in Sage, gives a way to construct the rational representation of the automorphism group given a plane model of the curve. The code of [BRR13] can compute this from the data of ramification type and a generating vector provided the quotient genus is 0.

### 2.1.5 Elliptic Curves and Functions

Elliptic curves are Riemann surfaces of genus 1. Over $\mathbb{C}$, every elliptic curve can be written in terms of affine coordinates $x, y$ in **Weierstrass form**

$$\mathcal{E} : y^2 = 4x^3 - g_2 x - g_3$$

for some $g_2, g_3 \in \mathbb{C}$. In these coordinates, the differential $\frac{dx}{y}$ is globally holomorphic. In this brief section I will survey some of the basic results required in this thesis, namely in §2.3 and §3.4; more can be found in [DLMF, MM97, Sil09].

[9]A **stall** of a plane curve is a point where the osculating (hyper)plane has a 4-point intersection [Edg71]. According to Harris "This is a completely archaic, nineteenth-century word, and I suggest you forget it immediately."

### The Weierstrass $\wp$ Function and Lattice Invariants

As seen in Corollary 2.1.47, with a choice of basepoint the Abel-Jacobi map provides an isomorphism $\mathcal{E} \cong \mathrm{Jac}(\mathcal{E})$.

**Remark 2.1.90.** *As elliptic curves with a choice of identity are abelian varieties we can ask about isogenies and the equivalence classes under isogeny. It shall be helpful now to note a point that will be important later when discussing Bring's curve, namely that by a famous theorem of Shafarevich over any number field $\mathbb{L}$ the $\mathbb{L}$-isogeny class of any elliptic curve is finite [Sil94, Corollary IX.6.2].*

Written in Weierstrass form, it is natural to take the point at $x = \infty, y = +\infty$ as the basepoint, and so the AJ map is

$$(x(P), y(P)) \mapsto \int_\infty^P \frac{dx}{y}.$$

We can define the inverse function $\wp(z)$ by

$$z = \int_\infty^{\wp(z)} \frac{dx}{\sqrt{4x^3 - g_2 x - g_3}},$$

so equivalently $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ with $\wp(0) = \infty$. This is the **Weierstrass $\wp$ function**. It is a doubly-periodic meromorphic function, more properties about which can be read in [AS72, §18].

As $g = 1$, the period lattice for the elliptic curve is given by $\Lambda := 2\omega\mathbb{Z} + 2\omega'\mathbb{Z}$, with $\tau = \omega'/\omega$. The functions $g_2, g_3$ turn out to depend on $\omega, \omega'$ alone, satisfying the following:

(i) $g_k(\lambda\omega, \lambda\omega') = \lambda^{-2k} g_k(\omega, \omega')$,

(ii) given $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$, $g_k(1, (a\tau + b)/(c\tau + d)) = (c\tau + d)^{2k} g_k(1, \tau)$,

(iii) $\lim_{\mathrm{Im}\,\tau\to\infty} g_2(1, \tau) = \frac{4\pi^4}{3}$, $\lim_{\mathrm{Im}\,\tau\to\infty} g_3(1, \tau) = \frac{8\pi^6}{27}$,

(iv) when $\tau = i$, $g_2(1, \tau) = \frac{\Gamma(1/4)^8}{256\pi^2}$, $g_3(1, \tau) = 0$,

(v) when $\tau = e^{2\pi i/3}$, $g_2(1, \tau) = 0$, $g_3(1, \tau) = \frac{\Gamma(1/3)^{18}}{(2\pi)^6}$.

We shall call the $g_k$ the **lattice invariants**. We have [JS87, Theorem 3.16.2]

$$g_2, g_3 \in \mathbb{R} \Leftrightarrow \forall z \in \mathbb{C},\ \wp(\bar{z}; g_2, g_3) = \overline{\wp(z; g_2, g_3)} \Leftrightarrow \Lambda = \overline{\Lambda}.$$

Lattices for which $\Lambda = \overline{\Lambda}$ are called real lattices and they fall into two classes: rectangular lattices ($\omega \in \mathbb{R}, \omega' \in i\mathbb{R}$), and rhombic lattices ($\overline{\omega} = \omega'$). The rhombic lattices correspond to $\tau$ being on the boundary of the fundamental domain of the $\mathrm{SL}_2(\mathbb{Z})$ action on the upper half plane while the rectangular lattices correspond to $\tau$ on the imaginary axis with $\Im(\tau) \geq 1$. When restricted to rectangular or rhombic lattices we can say more about the values of $g_2(1, \tau)$ and $g_3(1, \tau)$. This is done by relating the $g_k$ to the roots $e_i$ of the corresponding cubic equation by

$$g_2(1, \tau) = 2(e_1^2 + e_2^2 + e_3^2), \quad g_3(1, \tau) = -4e_1e_2e_3.$$

(i) On a rectangular lattice have $e_i \in \mathbb{R}$ so $g_2 > 0$; further, $g_3 > 0$ if $|\tau| > 1$, $g_3 < 0$ if $|\tau| < 1$.

(ii) On a rhombic lattice, $e_1 \in \mathbb{R}$, $e_2 = \bar{e}_3$, and $\operatorname{sgn}(e_1) = \operatorname{sgn}(g_3)$.

A related concept that will occur multiple times in §3.2 and §3.4 is the **complete elliptic integral of the first kind** defined by

$$K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad k \in (0,1).$$

This is very often denoted as $K(m)$ where $m = k^2$, and the notation $k'$ is used for $\sqrt{1-k^2}$; in this thesis I shall use both. The complete elliptic integral can be used to express the periods of an elliptic curve when the corresponding modulus $k$ is defined in terms of the $e_i$; vast numbers of relations of this kind are given in [AS72, §17, §18].

### The $j$-Invariant

From the $g_k$ one can construct the **modular $j$-invariant**

$$j(\tau) := 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}. \tag{2.5}$$

This is invariant under modular transformations of $\tau$ as the name suggests. The $j$-invariant classifies elliptic curves over $\mathbb{C}$ up to isomorphism [Har77, Theorem 4.1]. Given a specific value of $j$, there exist methods to invert $j$ to find the corresponding value of $\tau$ [BBG95]. In particular, one that shall be used later in constructing monopole solutions is the following: solve the quadratic equation $4\alpha(1-\alpha) = 1728/j$ for $\alpha \in \mathbb{C}$, and then $\tau$ is given by

$$\tau = \tau(\alpha) = i\frac{{}_2F_1(1/6, 5/6, 1; 1-\alpha)}{{}_2F_1(1/6, 5/6, 1; \alpha)}. \tag{2.6}$$

This is multi-valued when $\alpha < 0$ [DLMF, 15.2.3], with a principal branch $\tau_p$ and second branch $\tau_p + 1$, but for our purposes this difference will not be important. Picking the other root of the quadratic gives $-1/\tau$, which is clearly just as valid a period. I will require some properties of $\tau(\alpha)$ when computing monopole spectral curves in §3.4, and so I shall briefly describe them now. The specific properties we require are that

(i) $\forall \alpha \in (0,1)$, $\tau(\alpha) \in i\mathbb{R}_{>0}$,

(ii) $\tau(0^+) = +i\infty$, $\tau(1/2) = i$, $\tau(1^-) = 0$,

(iii) $\forall \alpha < 0$, $\mathrm{Re}(\tau(\alpha)) \equiv 1/2 \bmod 1$,

(iv) $\tau(-\infty) = e^{2\pi i/3}$, $\tau(0^-) = \frac{1}{2} + i\infty$.

Evaluated at the specific $\tau(\alpha)$ in Equation 2.6 we find that

$$\operatorname{sgn}(g_3(1,\tau(\alpha))) = \begin{cases} 1, & \alpha < 1/2, \\ -1, & \alpha \in (1/2, 1). \end{cases}$$

Here we provide the necessary definitions and proofs. We can understand the behaviour of $\tau$ using known results about hypergeometric functions (see for example [AS72, §15]). First, in the region $\alpha \in (0,1)$, we may use the series expression for ${}_2F_1(a,b,c;z)$ when $|z| < 1$:

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(a)_n$ is the rising Pochhammer symbol

$$(a)_n = \begin{cases} 1, & n = 0, \\ a(a+1)\dots(a+n-1), & n \geq 1. \end{cases}$$

This means we have (i) that for all $\alpha \in (0,1)$, $\tau(\alpha) \in i\mathbb{R}_{>0}$. This is important as it makes the lattice rectangular, which forces the Weierstrass $\wp$ function to be real on the real axis [DLMF, §23.5]. Moreover, as ${}_2F_1(a,b,c;z)$ is increasing in $z \in (0,1)$, $\operatorname{Im}\tau(\alpha)$ is strictly decreasing in $\alpha$. We can calculate the limits to be

$$\tau(0^+) = +i\infty, \quad \tau(1^-) = 0,$$

so giving (ii). We may use [AS72, 15.3.10] which says that when $|1-z| < 1$, $|\arg(1-z)| < \pi$,

$${}_2F_1(a,b,a+b;z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \left[2\psi(n+1) - \psi(a+n) \right. \\ \left. -\psi(n+b) - \log(1-z)\right] (1-z)^n,$$

where $\psi$ is the digamma function [AS72, 6.3.1], to understand exactly this limiting behaviour, namely that the divergence is logarithmic.[10] We can also highlight a special value in this region, namely $\tau(1/2) = i$. For $\alpha \notin [0,1]$ we no longer have that $\tau$ lies on the imaginary axis, and we would thus need to get a rhombic lattice (that is $\operatorname{Re}\tau = 1/2$) for the reality of $\wp$. Numerical tests suggest that while this happens for $\alpha < 0$; for $\alpha > 1$ we instead get $\operatorname{Re}(-1/\tau) = 1/2$. Indeed we may

[10]In this thesis I will use log to denote the natural logarithm.

use [DLMF, 15.10.29] to say[11]

$$\begin{aligned}
{}_2F_1(1/6,5/6,1;1-\alpha) &= e^{5\pi i/6}\frac{\Gamma(1)\Gamma(1/6)}{\Gamma(1)\Gamma(1/6)}{}_2F_1(1/6,5/6,1;\alpha) \\
&\quad + e^{-\pi i/6}\frac{\Gamma(1)\Gamma(1/6)}{\Gamma(5/6)\Gamma(1/3)}\alpha^{-1/6}{}_2F_1(1/6,1/6,1/3;1/\alpha), \\
&= e^{5\pi i/6}{}_2F_1(1/6,5/6,1;\alpha) \\
&\quad + (-\alpha)^{-1/6}\frac{\Gamma(1/6)}{\Gamma(5/6)\Gamma(1/3)}{}_2F_1(1/6,1/6,1/3;1/\alpha),
\end{aligned}$$

and hence when $\alpha < 0$ (and taking the principal branch of the hypergeometric function) we get

$$\tau(\alpha) = i\left[e^{5\pi i/6} + T(\alpha)\right]$$

with

$$T(\alpha) = (-\alpha)^{-1/6}\frac{\Gamma(1/6)}{\Gamma(5/6)\Gamma(1/3)}\frac{{}_2F_1(1/6,1/6,1/3;1/\alpha)}{{}_2F_1(1/6,5/6,1;\alpha)} \in \mathbb{R}.$$

This means $\mathrm{Re}(\tau(\alpha)) \equiv 1/2 \mod 1$, which yields (iii). To get the asymptotics as $\alpha \to -\infty$, we use [AS72, 15.3.7]

$$\begin{aligned}
{}_2F_1(a,b,c;z) &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,a+1-c,a+b-1;z^{-1}) \\
&\quad + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,b+1-c,b+a-1;z^{-1}).
\end{aligned}$$

Taking $\alpha = -\epsilon^{-1}$, this gives that as $\epsilon \to 0^+$,

$${}_2F_1(a,b,c;-\epsilon^{-1}) \sim \frac{\Gamma(2/3)}{\Gamma(5/6)^2}\epsilon^{1/6}, \quad {}_2F_1(a,b,c;1+\epsilon^{-1}) \sim \frac{\Gamma(2/3)}{\Gamma(5/6)^2}(-\epsilon)^{1/6},$$

and so $\tau(-\infty) = e^{2\pi i/3} = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$. To get the remaining asymptotics of (iv), as $\alpha \to 0^-$ we write $\alpha = -\epsilon$. Then

$${}_2F_1(a,b,c;-\epsilon) \sim 1, \quad {}_2F_1(a,b,c;1+\epsilon) \sim \frac{-\Gamma(1)}{\Gamma(1/6)\Gamma(5/6)}\log(-\epsilon) = \frac{-1}{2\pi}(i\pi + \log\epsilon),$$

and so $\tau(0^-) = \frac{1}{2} + i\infty$. To get the asymptotics as $\alpha \to 1^+$ we recognise that $\tau(1-\alpha) = -1/\tau(\alpha)$ and so $-1/\tau(1^+) = \frac{1}{2} + i\infty$. Finally to get the asymptotics as $\alpha \to \infty$ we do the same, so $-1/\tau(\infty) = \frac{1}{2} + \frac{i\sqrt{3}}{2}$.

[11] I am very grateful to Adri Olde Daalhuis for this argument.

## 2.2 Theta Characteristics

> Now the various species of whales need some sort of popular comprehensive classification, if only an easy outline one for the present, hereafter to be filled in all its departments by subsequent laborers. As no man better advances to take this matter in hand, I hereupon offer my own poor endeavors. I promise nothing complete; because any human thing supposed to be complete, must for that very reason infallibly be faulty. I shall not pretend to a minute anatomical description of the various species, or - in this place at least - to much of any description. My object here is simply to project the draught of a systemization of cetology. I am the architect, not the builder.
>
> – Herman Melville
> Moby Dick

In this section I will devote some time to studying theta characteristics, and specifically their orbit structure. In [Ati71] theta characteristics were shown to be equivalent to spin structures on compact Riemann surfaces (and more generally on compact complex manifolds), and as such I will use the terms interchangeably. The term theta characteristic itself comes historically from the relations between theta characteristics and the transcendental theta function on the curve; in this thesis I will take a more modern algebraic treatment, which has the benefit of a slightly cleaner approach, though I will touch upon the relation to theta functions in §2.2.1. For a brief, but more complete, history of different approaches to theta characteristics see [Far12].

**Definition 2.2.1.** *A **square root** of $L \in \mathrm{Pic}(\mathcal{C})$ is $\tilde{L} \in \mathrm{Pic}(\mathcal{C})$ such that $\tilde{L}^2 = L$. Equivalently by the equivalence of §2.1.2 this is a solution in the divisor class group to $2D_{\tilde{L}} = D_L$.*

The question of how many, if any, square roots of a line bundle exist is neatly governed by the following result.

**Proposition 2.2.2.** *Given $L \in \mathrm{Pic}(\mathcal{C})$,*

- *$L$ has a square root if and only if $\deg L$ is even, and*
- *if $L$ has a square root it has exactly $2^{2g(\mathcal{C})}$ square roots.*

*Proof.* Use the snake lemma [Wei95, Lemma 1.3.2] applied to the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}^0(\mathcal{C}) \arrow[r] \arrow[d, "\otimes n"] & \mathrm{Pic}(\mathcal{C}) \arrow[r] \arrow[d, "\otimes n"] & \mathbb{Z} \arrow[r] \arrow[d, "\times n"] & 0 \\ 0 \arrow[r] & \mathrm{Pic}^0(\mathcal{C}) \arrow[r] & \mathrm{Pic}(\mathcal{C}) \arrow[r] & \mathbb{Z} \arrow[r] & 0, \end{tikzcd}$$

taking $n = 2$. For the second point, note that if we had two square roots $\tilde{L}_1, \tilde{L}_2$, then $M = \tilde{L}_1 \otimes \tilde{L}_2^{-1}$ satisfies $M^2 = \mathcal{O}_{\mathcal{C}}$. Hence $M$ is a 2-torsion element of the Picard group, and moreover $\deg M = 0$ so $M \in \mathrm{Pic}^0(\mathcal{C}) \cong \mathrm{Jac}(\mathcal{C})$. Square roots are thus in 1-1 correspondence to the half-period elements of the Jacobian lattice, of which there are $2^{2g}$. □

**Definition 2.2.3.** *A* ***theta characteristic*** *on $\mathcal{C}$ is a square root of $K_{\mathcal{C}}$. Denote the set of all theta characteristics as $S(\mathcal{C})$.*

It is a consequence of the proof of Proposition 2.2.2 that $S(\mathcal{C})$ is an affine space[12] over $\mathbb{Z}_2$ modelled on $H^1(\mathcal{C}, \mathbb{Z}_2)$.

**Definition 2.2.4** ([Dol12], p. 195, p. 210)**.** *A theta characteristic is called* ***odd/even*** *based on the parity of its index of speciality. An even theta characteristic $D$ is called* ***vanishing*** *if $i(D) > 0$ and* ***nonvanishing*** *otherwise.*

**Lemma 2.2.5** ([Fay73], p. 11, [Ati71], Theorem 3)**.** *On a Riemann surface of genus $g$ there are $2^{g-1}(2^g - 1)$ odd theta characteristics and $2^{g-1}(2^g + 1)$ even theta characteristics.*

There are many equivalent definitions of parity for a theta characteristic, and while Definition 2.2.4 is not computationally the easiest, it is one of the easiest to state. [Far12] provides a nice overview of different definitions and their connections. Note that for a theta characteristic $\dim H^0(\mathcal{C}, L) = \dim H^1(\mathcal{C}, L)$ by Riemann-Roch.

**Example 2.2.6.** *On $\mathbb{P}^1$, there is only one square root of $K_{\mathbb{P}^1} \cong \mathcal{O}(-2)$, given by $\mathcal{O}(-1)$. In this instance $\dim H^0(\mathbb{P}^1, \mathcal{O}(-1)) = 0$ and hence the characteristic is even.*

**Example 2.2.7** ([Mum71, KS10])**.** *Suppose we have a hyperelliptic curve $\mathcal{C}$ given by $y^2 = \prod_{i=1}^{2g+2}(x - e_i)$. We have a natural degree-2 map $\pi : \mathcal{C} \to \mathbb{P}^1$ given by $(x, y) \mapsto x$ and we can label with $p_i$ the points $(e_i, 0)$ such that $\pi(p_i) = e_i$. The canonical divisor class on $\mathbb{P}^1$ is represented by $\mathcal{K}_{\mathbb{P}^1} = -2q$ for any $q \in \mathbb{P}^1$, as can be seen as it is the divisor of $d\phi$ where $\phi(x) = \frac{x-q}{x}$ is a Möbius transformation of $\mathbb{P}^1$, and pulling back by $\pi$ we get*

$$\pi^*(-2q) = \begin{cases} -2(a+b), & \{a, b\} = \pi^{-1}(q),\ q \notin \{e_i\}, \\ -4p_i, & q = e_i. \end{cases}$$

*Moreover, the ramification divisor of the cover is clearly given by $R_\pi = \sum_i p_i$, and hence a classical description of the canonical divisor on $\mathcal{C}$ is given by*

$$\mathcal{K}_{\mathcal{C}} = -2D + \sum_i p_i$$

*where*

$$D = \begin{cases} a+b, & a, b \notin \{p_i\}, \\ 2p_i, & \textit{otherwise}. \end{cases}$$

[12] Equivalently $S(\mathcal{C})$ is a torsor for the $\mathbb{Z}_2$ vector space $H^1(\mathcal{C}, \mathbb{Z})$. This means that, given a choice of any theta characteristic to act as the 'origin', the set of theta characteristics is isomorphic to $H^1(\mathcal{C}, \mathbb{Z}_2)$.

*Note $D$ corresponds to the divisor of $\pi^*\mathcal{O}(1)$ (this is perhaps clearer by thinking of the divisor of $\mathcal{O}(1)$ being the intersection of a hyperplane with $\mathbb{P}^1$). Moreover, one can calculate that*

$$(y) = -(g+1)(\infty_+ + \infty_-) + \sum_i p_i,$$

*where $\infty_\pm$ are the two preimages of $x = \infty$ under $\pi$. We can then write $\infty_+ + \infty_- \sim D$, and as such one gets the additional divisor relation $(g+1)D \sim \sum_i p_i$. This lets us write $\mathcal{K}_\mathcal{C} \sim (g-1)D$, and so we can construct a class of theta characteristics given by*

$$\Delta = \sum_{j=1}^{g-1} p_{i_j},$$

*as then $2\Delta = \sum_j 2p_{i_j} \sim \sum_j D = (g-1)D \sim \mathcal{K}_\mathcal{C}$. If we have some of the $i_j$ equal then we can replace these terms with a copy of $D$, so we equivalently get a class of characteristics*

$$\Delta_l = lD + \sum_{j=1}^{g-1-2l} p_{i_j}$$

*where now all the $i_j$ are distinct and $0 \le l \le \left\lfloor \frac{g-1}{2} \right\rfloor$. Considering the function on $\mathcal{C}$ given by $\phi \circ \pi$ where $\phi(x) = \frac{x-e_i}{x-e_j}$ we have $(\phi\circ\pi) = 2(p_i - p_j)$ (when $i \neq j$) so $p_i - p_j$ is not a principal divisor, and in general*

$$\sum_{k=1}^{r} (p_{i_k} - p_{j_k}) \sim 0 \Leftrightarrow r = g+1$$

*(the equivalence when $r = g+1$ coming from the divisor of $y$, and again assuming all the indices are unique). Hence our presentation of the effective characteristics is unique. Moreover, we can extend to allow $l = -1$, and the condition with $r = g+1$ halves the number of characteristics this allows. Characteristics with $l < -1$ are equivalent to those with larger $l$ as we can tell by counting how many characteristics we have already found. The parity of these characteristics is given by the parity of*

$$\dim H^0(\mathcal{C}, \mathcal{O}_\mathcal{C}(\Delta_l)) = l+1.$$

### 2.2.1 Methods of Computation

We have seen in Examples 2.2.6 and 2.2.7 how to compute the theta characteristics on simple Riemann surfaces. I will now provide some details on four other methods used to compute theta characteristics.

#### The Riemann Constant Vector

**Definition 2.2.8** ([FK92], p. 290)**.** *Letting $\tau$ be the Riemann matrix of the genus-$g$ curve $\mathcal{C}$ and $\{a_j, b_j\}$ a choice of canonical homology basis, the **Riemann Con-***

***stant Vector (RCV)*** *based at $Q \in \mathcal{C}$ is given by*

$$(K_Q)_j = -\frac{1+\tau_{jj}}{2} + \sum_{k\neq j}^{g} \oint_{a_k} \omega_k(P)(\mathcal{A}_Q(P))_j, \quad j = 1, \dots, g.$$

The RCV is related to the canonical divisor by $2K_Q \equiv \mathcal{A}_Q(\mathcal{K}_\mathcal{C})$. We then have the following theorem about the **Szegő kernel divisor** $\Delta_\mathcal{C}$, defined in [Fay73, p. 7].

**Theorem 2.2.9** ([Fay73], p. 7-8)**.** *$K_Q = \mathcal{A}_Q(\Delta_\mathcal{C})$, and as such $\Delta_\mathcal{C}$ is a theta characteristic.*

**Remark 2.2.10.** *The choice of sign of $K_Q$ varies widely between authors, and hence one should be alert. The convention taken in Definition 2.2.8 is chosen to ensure that $K_Q$ is the image of a theta characteristic.*

The definition of the RCV also requires that the differentials are normalised with respect to the homology basis taken $\{a_j, b_j\}$ such that $\oint_{a_j} \omega_i = \delta_{ij}$. As such, hidden in Theorem 2.2.9 is the fact that the Szegő kernel divisor is homology dependent, because the RCV is. Because $S(\mathcal{C})$ is affine, modelled on $H^1(\mathcal{C}, \mathbb{Z}_2)$, we know that the image under the Abel-Jacobi map based at $Q \in \mathcal{C}$ of $S(\mathcal{C})$ is now the set $\{K_Q + h \mid h \in \mathrm{Jac}(\mathcal{C}), 2h \equiv 0\}$. Numerical computational methods exist to calculate the RCV and the period matrix, for example [DPS15] and [BSZ19] respectively, and thus this gives a way to numerically calculate the image in the Jacobian of all the theta characteristics.

With the RCV one can define a $(g-1)$-dimensional subvariety of the Jacobian $W_{g-1} - K_Q := \Theta$ called the **theta divisor**. This definition is independent of the basepoint chosen for the Abel-Jacobi map. The Riemann Vanishing Theorem states that $\Theta$ is exactly the subvariety of the Jacobian on which the **theta function**

$$\theta(z;\tau) = \sum_{n\in\mathbb{Z}^g} \exp\left[2\pi i \left(\frac{1}{2} n^T \tau n + n^T z\right)\right] \tag{2.7}$$

vanishes, and moreover if $D$ is a degree-$(g-1)$ effective divisor then the multiplicity of the zero of $\theta$ at $\mathcal{A}_Q(D) - K_Q$ is equal to $i(D)$ [FK92, p. 298]. As such, if $\Delta_\mathcal{C}$ were effective, $\theta(0;\tau) = 0$, which is not true for a generic Riemann surface [Fay73, p. 7] hence $\Delta_\mathcal{C}$ is generically an even characteristic.

### Odd Characteristics and Tangent Hyperplanes

Recall that when we consider the canonical embedding of a curve of genus $g \geq 3$ in $\mathbb{P}^{g-1}$, the intersection with a hyperplane gives an effective element of the canonical divisor class of the curve, as it corresponds to the zero-locus of a holomorphic differential on the curve. Clearly then, given a hyperplane $H$ which is tangent to the canonical embedding at $g-1$ points, each intersection has multiplicity two. Writing

$$H \cap \mathcal{C}_{can} = 2(P_1 + \dots + P_{g-1}) \sim \mathcal{K}_\mathcal{C},$$

we see that $\Delta := P_1 + \dots + P_{g-1}$ is an effective theta characteristic. Note effectiveness is equivalent to $l(\Delta) \geq 1$.

In the case of small genera, even more can be said. By Clifford's theorem (Proposition 2.1.52) we know $l(\Delta) < (g+1)/2$, and so when $g = 3$ we must have $l(\Delta) = 1$ hence the theta characteristic is odd. In fact this remains true for curves of genus 4 when the quadric the canonical embedding lies in (recall Example 2.1.61 and §2.1.3) is nonsingular [HL17, Theorem 2.2]. Conversely, when the quadric is singular the curve has a $g^1_3$ corresponding to the tangent plane to the cone through the cone point [Har77, Example IV.5.5.2], and this yields a vanishing even theta characteristic [Dol12, p. 210].

Moreover, from the discussion in §2.1.2, we know that for any $\Delta$ an odd theta characteristic $l(\Delta) > 0$ and hence there exists an effective divisor $E$ linearly equivalent to $\Delta$. As such, each odd theta characteristic determines a tangent hyperplane.

These two discussions taken together give us the following results characterising odd theta characteristics.

**Proposition 2.2.11** ([Dol12], p. 251, [HL17], Theorem 2.2)**.** *For any smooth plane quartic, the 28 bitangents are in 1-1 correspondence with the 28 odd theta characteristics.*

*Moreover, for any sextic that is the intersection of a smooth quadric and smooth cubic in $\mathbb{P}^3$, the 120 tritangent planes are in 1-1 correspondence with the 120 odd theta characteristics.*

These tangent hyperplanes are important to the curve, as they have been shown to determine the curve itself [CS03b, CS03a], and algorithms to do this reconstruction have been given for certain genera [CKRSN19, Leh22]. [Guà02] gives equations for the tangent hyperplanes in terms of the period matrix and the theta function on the curve when $l(\Delta) = 1$.

### Even Characteristics and Scorza Theory

Even theta characteristics are harder to study than odd characteristics, as the line bundle corresponding to a generic even theta characteristic has no sections, which is an obstacle to their study [TZ11]. In the case of genus 3, one particular technique exists using **Scorza theory**. To introduce this, we first make some definitions.

**Definition 2.2.12** ([Dol12], p. 5)**.** *Given $F = F(X_0, \dots, X_r)$ a homogeneous polynomial and $a = (a_0, \dots, a_r) \in \mathbb{C}^{r+1}$, the* ***polarisation*** *of $F$ at $a$ is $P_a(F) := \sum_i a_i \frac{\partial}{\partial X_i} F$.*

**Definition 2.2.13.** *The* ***rank*** *of a quadric $F(X_0, \dots, X_r)$ is the rank of the Hessian matrix $H$ given by $H_{ij} = \frac{\partial^2 F}{\partial X_i \partial X_j}$.*

**Definition 2.2.14** ([Stu08], p. 161, [Dol12], p. 155)**.** *The quartic and sextic invariants $I_4, I_6$ generating the ring of invariants of ternary cubics (that is homogeneous degree-3 polynomials $F(X_0, X_1, X_2)$) under the natural action of* $\mathrm{GL}_3(\mathbb{C})$ *are called the* ***Aronhold invariants***

**Definition 2.2.15** ([Dol12], p. 279)**.** *The* ***Clebsch covariant quartic*** *of a ternary quartic $F$ is the plane quartic curve*

$$\mathfrak{C}(F) := \left\{a \in \mathbb{P}^2 \,|\, I_4(P_a F) = 0\right\}.$$

We can now define two correspondences (in the sense of [Dol12, §5.5.1]) which we will soon equate, namely to a plane quartic curve given by $F = 0$ for which $\mathfrak{C}(F)$ is nonsingular assign the symmetric $(3,3)$-correspondence without united points [DK93, Proposition 6.8.1]

$$T_F = \{(a,b) \in \mathfrak{C}(F) \times \mathfrak{C}(F) \,|\, \mathrm{rank}(P_a P_b F) = 1\},$$

and to a nonvanishing theta characteristic $\Delta$ on a genus-$g$ curve $\mathcal{C}$ the symmetric $(g,g)$-correspondence without united points [DK93, §7.1]

$$T_\Delta = \{(P,Q) \in \mathcal{C} \times \mathcal{C} \,|\, l(\Delta + P - Q) > 0\}.$$

The latter is called the **Scorza correspondence**. These are related by the following key theorem.

**Theorem 2.2.16** ([DK93], Lemma 7.7.1, Theorem 7.8)**.** *Given $F$ defining a plane quartic such that $\mathfrak{C}(F)$ is nonsingular, there exists a unique nonvanishing theta characteristic $\Delta$ on $\mathfrak{C}(F)$ such that $T_F = T_\Delta$. This defines the* ***Scorza map***

$$\mathrm{Sc} : F \mapsto (\mathfrak{C}(F), \Delta),$$

*which is an injective birational isomorphism from the space of such $F$ to the space of nonsingular quartics with an even theta characteristic. Projecting to $\mathfrak{C}(F)$, the map is an unramified* $36:1$ *cover.*

Clearly $\mathfrak{C}(F)$ can be computed efficiently from knowledge of $F$, and moreover given explicit knowledge of the correspondence $T_F$, $\Delta = b_1 + b_2 + b_3 - a$ where $\{b_1, b_2, b_3\} \subset \mathfrak{C}(F)$ is the image of $a \in \mathfrak{C}(F)$ under the correspondence [DK93, Theorem 7.6]. This can be done explicitly assuming that at $a \in \mathfrak{C}(F)$ we can write $P_a F = \sum_{i=1}^{3} l_i^3$ for linear forms $l_i$, namely $\Delta = a_{12} + a_{13} + a_{23} - a$ where $a_{ij} = \{l_i = 0 = l_j\}$. The Scorza map is in fact $\mathrm{PSL}_3(\mathbb{C})$-equivariant, and this shall be relevant when considering orbits of even characteristics on genus-3 curves.

[DK93] also gives an implicit prescription for assigning a plane quartic curve, the **Scorza quartic**, to any pair $(\mathcal{C}_{can}, \Delta)$ satisfying some conditions where $\mathcal{C}_{can}$ is the canonical embedding of a genus-$g$ curve and $\Delta$ is a nonvanishing theta characteristic on it. It was shown there that when $\mathcal{C}_{can}$ is genus 3 any such pair satisfies the required conditions and this process is the inverse of the Scorza map. More recently it has been shown that a generic pair of any genus satisfies the required conditions, and the Scorza quartic has been found explicitly for some trigonal curves coming from Hilbert schemes [TZ11].

### Spin Structures

On a Riemann surface $\mathcal{C}$ denote with $U\mathcal{C}$ the unit tangent bundle with the SES

$$0 \to S^1 \xrightarrow{i} U\mathcal{C} \xrightarrow{\pi} \mathcal{C} \to 0.$$

This gives two corresponding SESs of singular (co)homology

$$\begin{aligned} 0 \to H_1(S^1, \mathbb{Z}_2) &\xrightarrow{i_*} H_1(U\mathcal{C}, \mathbb{Z}_2) \xrightarrow{\pi_*} H_1(\mathcal{C}, \mathbb{Z}_2) \to 0, \\ 0 \leftarrow H^1(S^1, \mathbb{Z}_2) &\xleftarrow{i^*} H^1(U\mathcal{C}, \mathbb{Z}_2) \xleftarrow{\pi^*} H^1(\mathcal{C}, \mathbb{Z}_2) \leftarrow 0, \end{aligned}$$

related to each other by Poincaré duality. Within this context, a spin structure is a cohomology class in $H^1(U\mathcal{C}, \mathbb{Z}_2)$ which restricts under $i^*$ to the generator 1 of $H^1(S^1, \mathbb{Z}_2) \cong \mathbb{Z}_2 = \{0, 1\}$. Given a basis $\{\gamma_1, \dots, \gamma_{2g}\}$ of $H_1(\mathcal{C}, \mathbb{Z})$ which I will always take to be in canonical form $\{a_i, b_i\}$, Kallel and Sjerve [KS10] construct a basis $\{\eta, \zeta_1, \dots, \zeta_{2g}\}$ of $H^1(U\mathcal{C}, \mathbb{Z}_2)$ such that the $\zeta_i$ are dual to the $\gamma_i$ and the set of spin structures is given by

$$\mathrm{Spin}(\mathcal{C}) = \left\{ s(\boldsymbol{x}) = \eta + \sum_{i=1}^{2g} x_i \zeta_i \,|\, \boldsymbol{x} \in \mathbb{Z}_2^{2g} \right\}. \tag{2.8}$$

As it will be relevant for proving a subsequent lemma, I shall give the details of how this basis is constructed. We shall think of the space $H^1(U\mathcal{C}, \mathbb{Z})$ as the space of framed curves, that is a smooth curve in $\mathcal{C}$ with a smooth vector field along the curve. There is a distinguished element $z = i_*(1) \in H_1(U\mathcal{C}, \mathbb{Z})$ corresponding to a small tangentially framed curve. Moreover, to a cycle $a \in H_1(\mathcal{C}, \mathbb{Z})$, [Joh80] defined $\tilde{a} \in H_1(U\mathcal{C}, \mathbb{Z}_2)$ such that for any two cycles $a, b \in H_1(\mathcal{C}, \mathbb{Z})$, we have that $\widetilde{a + b} = \tilde{a} + \tilde{b} + (a \circ b)z$. Note this is well defined with coefficients in $\mathbb{Z}_2$. Together $\left\{ z, \tilde{\zeta}_1, \dots, \tilde{\zeta}_{2g} \right\}$ give a basis of $H_1(U\mathcal{C}, \mathbb{Z}_2)$, and then $\{\eta, \zeta_1, \dots, \zeta_{2g}\}$ is the Poincaré dual basis of $H^1(U\mathcal{C}, \mathbb{Z}_2)$.

In [Ati71, Proposition 3.2], Atiyah proved the equivalence between theta characteristics and spin structures on a Riemann surface such that the corresponding actions of $\mathrm{Aut}(\mathcal{C})$ are equivalent. Moreover, we get the following result about the parity.

**Lemma 2.2.17.** *Writing $\boldsymbol{x} = (\boldsymbol{u}, \boldsymbol{v})$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}_2^g$, the parity of $s(\boldsymbol{x})$ is $q(\boldsymbol{x}) := \boldsymbol{u} \cdot \boldsymbol{v}$.*

*Proof.* This is proven in [Joh80, §5], but not in the notation of Equation 2.8, so I shall briefly be specific here. Namely Johnson shows that the parity of $\xi \in \mathrm{Spin}(\mathcal{C})$, which he calls the Atiyah invariant, is given by $\sum_{i=1}^{g} \langle \xi, \tilde{a}_i \rangle \langle \xi, \tilde{b}_i \rangle$. We are then done using the fact that

$$\langle s(\boldsymbol{x}), \tilde{a}_i \rangle = \langle \eta, \tilde{a}_i \rangle + \sum_i x_i \langle \zeta_i, \tilde{a}_i \rangle = u_i$$

and likewise for $\tilde{b}_i$ and $v_i$. $\square$

As a result of Lemma 2.2.17 we see the parity of a spin structure is given by a quadratic form on $H_1(\mathcal{C}, \mathbb{Z}_2)$ such that the associated bilinear form $H(\boldsymbol{x}, \boldsymbol{y}) := q(\boldsymbol{x}+\boldsymbol{y}) - q(\boldsymbol{x}) - q(\boldsymbol{y})$ is the reduction mod 2 of the intersection pairing on $H_1(\mathcal{C}, \mathbb{Z})$.

### 2.2.2 Orbits of Theta Characteristics

Having defined theta characteristics and seen a few ways of computing them, we are able to move on to discuss their orbits under the automorphism group of the corresponding curve. By considering the ramification divisor associated with $f \in \mathrm{Aut}(\mathcal{C})$, one can see that the canonical divisor $\mathcal{K}_{\mathcal{C}}$ pulls back to itself under $f$ and hence the pullback of a theta characteristic by an automorphism is again a theta characteristic. What we shall want to know is the orbit structure of the characteristics under $\mathrm{Aut}(\mathcal{C})$, and especially the existence of characteristics invariant under the whole group.

**Example 2.2.18.** *By our explicit expressions for the theta characteristics on a hyperelliptic curve in Example 2.2.7, we can see that the hyperelliptic involution fixes every theta characteristic. In fact the hyperelliptic involution is the unique automorphism to fix every theta characteristic [BGS07, KS10], as we will see later.*

#### Computing Orbits

Given $f \in \mathrm{Aut}(\mathcal{C})$, recall the notation $T = \rho_a(f)$ and $R = \rho_r(f)$ for the image under the analytic and rational representations respectively. It is worth noting that these can be computed in Sage using the method of [BSZ19].

If one is content with inexact numerical computations of the orbit structure, one may use the definition of the image of theta characteristics in the Jacobian using the RCV to calculate the orbits, namely given $x \in \mathbb{C}^g/\Lambda$ such that $2x \equiv \mathcal{A}_P(\mathcal{K}_{\mathcal{C}})$, this transforms as

$$x \mapsto Tx + (g-1)\mathcal{A}_P(f(P)).$$

This does allow one to identify the orbit of a given characteristic explicitly, but overall the method is slow and its correctness relies on the correctness of numerical computations. For this reason I shall not discuss it further, but an implementation may be seen in the code available at `https://github.com/DisneyHogg/Brings_Curve` which corresponds to §2.3.4.

When one is considering effective theta characteristics in terms of tangent hyperplanes the orbits can be computed in terms of $T$ again recalling Example 2.1.79. While the equations of the contact points may be computed numerically, in some cases this may be done exactly giving exact orbits, and an example of this may be seen for the odd characteristics on Bring's curve in Corollary 2.3.29.

In the special case of a non-hyperelliptic genus-3 curve one can compute the orbits of the even characteristics analytically using Scorza theory. Because of the equivariance of the Scorza map, it is known $\mathrm{Aut}(F) \leq \mathrm{Aut}(\mathfrak{C}(F))$. Hence on

$\mathcal{C}$ with theta characteristic $\Delta$, letting $F$ be such that $\mathrm{Sc}(F) = (\mathcal{C}, \Delta)$, we have $|O_\Delta| = \frac{|\mathrm{Aut}(\mathcal{C})|}{|\mathrm{Aut}(F)|}$ determining the size of the corresponding orbit [Ott16].

**Example 2.2.19.** *In [DK93, §8] the orbit decomposition of the even characteristics on Klein's curve is shown to be* $36 = 1 + 7 + 7 + 21$*.*

*Moreover, it is given in [DK93, §8.1] that, letting*

$$F_a := X^4 + Y^4 + Z^4 + 6a(X^2Y^2 + Y^2Z^2 + Z^2X^2)$$

*for* $a \in \mathbb{C}$*,* $\mathfrak{C}(F_a) \propto F_b$ *where* $b = \frac{1-2a+a^2}{6a^2}$*. Each* $F_b$ *has automorphism group* $S_4$ *(generically), and so the curve given by* $F_b = 0$ *has exactly two invariant even theta characteristics corresponding to the two roots of* $(1-6b)a^2 - 2a + 1 = 0$*.*

Because of the current limitations in extending Scorza theory to higher genus curves, this method is in practice not helpful for calculating most orbits decompositions.

The final approach to calculate the orbit decomposition I shall mention, which will be used to construct tables of orbit decompositions in §2.2.2 uses the connection to spin structures. These transform as $s(\boldsymbol{x}) \mapsto s(\tilde{\boldsymbol{x}})$ where

$$\tilde{\boldsymbol{x}} = R^T\boldsymbol{x} + \boldsymbol{v} \mod 2. \tag{2.9}$$

Here the vector $\boldsymbol{v}$ is computed as $v_i = \sum_{j<j'} R_{ji}R_{j'i}J_{jj'}$, $J = \begin{pmatrix} 0 & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0 \end{pmatrix}$.

**Remark 2.2.20.** *Note Igusa [Igu72, §V.1] derives this behaviour independently from the perspective of theta functions, taking characteristics to be vectors* $\boldsymbol{x}$ *in* $\mathbb{R}^{2g}$ *such that* $2\boldsymbol{x} \equiv 0 \mod 1$*.*

One can think of this action on $\boldsymbol{x}$ as matrix multiplication

$$\begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} R^T & \boldsymbol{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}.$$

We note that multiplying two of these matrices together we get

$$(R^T, \boldsymbol{v}) * ((R')^T, \boldsymbol{v}') = (R^T(R')^T, R^T\boldsymbol{v}' + \boldsymbol{v}).$$

The fact that this is an affine representation of the automorphism group reduces down to the types of calculations done in [Igu72, II.§5]. This method is fast and exact to implement as it only uses binary computations, and the computation of parity of a given characteristic is simple. Again an implementation of this method may be seen in the code available at `https://github.com/DisneyHogg/Brings_Curve` which corresponds to §2.3.4.

**Remark 2.2.21.** *The computation of orbit decompositions using Equation 2.9 was vital for the later tables in §2.2.2, and so I shall make a few remarks about how it works here. The computation of the rational representation of the automorphism group following [BSZ19] takes two stages; compute a basis of the endomorphism ring of the Jacobian, and find the symplectic elements of this ring.*

*Though the authors do not complete a complexity analysis, in practice I find that the first step is the slowest part of the algorithm in general. The computation of the endomorphism basis uses the LLL algorithm [LLL82], so one option to speed up this calculation is to use a parallelisation of LLL. Alternatively, as all the computation of the orbit decompositions requires is the rational representation mod 2, one might wonder whether calculating over $\mathbb{Z}_2$ the computation can be sped up; this may be an interesting direction for future research. The second part of the computation of the orbit decomposition which takes a significant time is the partition of the characteristics into orbits using the group action calculated via the rational representation. The method I have implemented naïvely loops over characteristics computing their orbits to partition the set of characteristics (sometime called a union-find approach), but recent work has described exponentially faster approaches in the case of linear actions on binary vectors as considered here [AKPW23]. Any further computation at higher genera will want to implement both of these approaches to immediately improve the performance of the computation.*

**Example 2.2.22.** *Suppose $R = I$ is the identity matrix, then*

$$v_i = \sum_{j<j'} \delta_{ji}\delta_{j'i}J_{jj'} = 0.$$

Suppose instead we had parametrised our characteristics as the set

$$S(\mathcal{C}) = \left\{ \eta_y + \sum_i x'_i \zeta_i \right\}$$

where $\eta_y = \eta + \sum_i y_i \zeta_i$ for some $\boldsymbol{y} \in \mathbb{Z}_2^{2g}$. One can derive the corresponding action on $\boldsymbol{x}'$ by noting

$$\begin{aligned} \boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{y} &\mapsto R^T(\boldsymbol{x}' + \boldsymbol{y}) + \boldsymbol{v}, \\ &= R^T\boldsymbol{x}' + \left[\boldsymbol{v} + (R^T - I)\boldsymbol{y}\right] + \boldsymbol{y}, \end{aligned}$$

and so

$$\boldsymbol{x}' \mapsto R^T\boldsymbol{x}' + \boldsymbol{v}_y,$$

where $\boldsymbol{v}_y = \boldsymbol{v} + (R^T - I)\boldsymbol{y}$. If $\boldsymbol{y}$ were fixed by the action of $R$, that is $\boldsymbol{y} = \tilde{\boldsymbol{y}}$, then $\boldsymbol{v}_y = 0$ and moreover the converse is true. As such, we have seen the following proposition.

**Proposition 2.2.23.** *There is an invariant characteristic on a curve $\mathcal{C}$ if and only if the corresponding affine representation on $\mathbb{Z}_2^{2g}$*

$$\begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} R^T & \boldsymbol{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}$$

*is equivalent by a translation $\boldsymbol{x} \to \boldsymbol{x}' := \boldsymbol{x} - \boldsymbol{y}$ to the linear action*

$$\begin{pmatrix} \boldsymbol{x}' \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} R^T & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}' \\ 1 \end{pmatrix}$$

*for some* $\boldsymbol{y} \in \mathbb{Z}_2^{2g}$.

### Invariant Characteristics

To make progress from Proposition 2.2.23, we now use that the question of whether an affine representation can be reduced to a linear one can be presented as a cohomology problem[13]. In particular, given a group $G$ and a (left) $G$-representation $\rho : G \to \mathrm{GL}(V)$ we have the following results.

**Proposition 2.2.24.** *An affine representation of $G$ on $V$ which acts multiplicatively via $\rho$ determines a 1-cocycle in the group cohomology[14] $H^1_{\mathrm{Grp}}(G, V)$ (making $V$ into a $G$-module in the natural way with $\rho$) with the standard linear representation $G \times V \to V$, $(g, x) = \rho(g)x$, corresponding to the zero 1-cocycle.*

*Proof.* An affine representation acting multiplicatively by $\rho$ is defined by

$$\begin{aligned} &G \times V \to V, \\ &(g, x) \mapsto g \cdot x := \rho(g)x + v(g). \end{aligned}$$

for some set map $v : G \to V$. By definition the set map $v$ is exactly a 1-cochain in group cohomology, with the linear representation giving the zero 1-cochain. Moreover, to truly get an action we require $\forall\, g, h \in G$, $x \in V$, $g \cdot (h \cdot x) = (gh) \cdot x$. We can compute from the definition

$$\begin{aligned} g \cdot (h \cdot x) &= \rho(g)(h \cdot x) + v(g), \\ &= \rho(g)[\rho(h)x + v(h)] + v(g), \\ &= \rho(gh)x + \rho(g)v(h) + v(g), \\ &= (gh) \cdot x + \rho(g)v(h) - v(gh) + v(g), \end{aligned}$$

and so we must have

$$\forall\, g, h \in G,\ 0 = \rho(g)v(h) - v(gh) + v(g). \tag{2.10}$$

In particular, setting $g = e$ in Equation 2.10 shows $v(e) = 0$. Equation 2.10 is exactly the condition that the 1-cochain $v$ is in fact a 1-cocycle [Wei95, Example 6.5.6]. □

**Proposition 2.2.25.** *Two affine representations as defined in Proposition 2.2.24 are equivalent under a translation of $V$ if and only if the associated 1-cocycle is a 1-coboundary.*

*Proof.* Fixing $y \in V$ and $v : G \to V$ defining an affine representation we have that

$$\begin{aligned} g \cdot (x + y) &= \rho(g)(x + y) + v(g), \\ &= \{\rho(g)x + [v(g) + (\rho(g) - I)y]\} + y. \end{aligned}$$

[13] I am grateful to Andrew Beckett for highlighting this to me.

[14] For an introduction to group cohomology with necessary definitions see [Wei95, §6]. I shall from here on in drop the Grp subscript as through context it shall not cause confusion with any other cohomologies in this thesis.

This defines a different affine action on $V$ given by 1-cocycle $v_y(g) := v(g) + (\rho(g) - I)y$. This new affine action is actually linear if and only if

$$\forall g \in G,\ (\rho(g) - I)y + v(g) = 0 \Leftrightarrow \forall g \in G,\ g \cdot y = y \Leftrightarrow y \in V^G, \tag{2.11}$$

where I have used $V^G$ to denote the subset of $V$ invariant under $G$. The condition that $v(g) = \rho(g)y - y$ for some $y \in V$ is exactly the condition that $v$ is a 1-coboundary [Wei95, Example 6.5.6]. □

To fix notation, I will follow Weibel to use $Z^1$ to denote the 1-cocycles (also called the derivations or **crossed homomorphisms**), and $B^1$ to denote the 1-coboundaries (also called the principal derivations).

In the case at hand of considering the group action of the automorphism group on theta characteristics the representation will be the reduction mod 2 (with the mod 2 reduction of $R$ denoted by $\overline{R}$) of the transpose of the rational representation $\rho = \overline{\rho}_r^T$ acting on $V = H_1(\mathcal{C}, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$. Moreover, we can count the number of invariant characteristics as the size of $H^0(G, V)$, as $H^0(G, V) = V^G$ is exactly the submodule of invariants. This fact gives us an immediate refinement of [KS10, Corollary 1.3].

**Proposition 2.2.26.** *The number of characteristics invariant under the action of the whole group is either 0 or $2^k$, where $k = \dim H^0(G, V)$ is the dimension of the subspace of invariants.*

*Proof.* This is immediate from the fact $H^0(G, V)$ is a vector space over $\mathbb{Z}_2$. □

**Example 2.2.27.** *I shall do a simple example, namely suppose we just have $G = C_2 = \{\pm 1\}$, where the generator of the $C_2$ is the hyperelliptic involution for which the rational representation is given by $R = -I$. Indeed the hyperelliptic involution is the only nonidentity automorphism $\tau$ for which $\rho_r(\tau) = I \mod 2$ [KS10]. An element of $Z^1$ defined by $v : G \to V$ must satisfy $v(1) = 0 \in V$, and then $v(-1) := x$ is arbitrary. An element of $B^1$ has the additional condition that $\exists y \in V$, $v(-1) = y - y = 0$, and so with the hyperelliptic representation the cohomology group is given by $H^1(C_2, V) = V$. Note this is what we would expect, as if the representation is trivial, then $\forall \boldsymbol{y}, \boldsymbol{v}_y = \boldsymbol{v}$, meaning that the action is always given by the specific shift $\boldsymbol{v}$.*

*At this point we recall a lemma of Atiyah whose proof will be useful.*

**Lemma 2.2.28** ([Ati71], Lemma 5.1)**.** *Let $V$ be a finite dimension $\mathbb{Z}_2$ vector space and $q : V \to \mathbb{Z}_2$ a quadratic function fixed under an affine transformation $x \mapsto Ax + b$ whose associated bilinear $H$ defined by*

$$H(x, y) = q(x + y) - q(x) - q(y)$$

*is non-degenerate. Then the affine transformation has a fixed point.*

*Proof.* As the transform preserves $q$ we get

$$q(x) = q(Ax + b) = q(Ax) + q(b) + H(Ax, b).$$

Setting $x = 0$ gives $q(b) = 0$ and hence $q(x) = q(Ax) + H(Ax, b)$. We can thus say

$$\begin{aligned} H(x,y) &= q(x+y) - q(x) - q(y), \\ &= [q(A(x+y)) + H(A(x+y),b)] - [q(Ax) + H(Ax,b)] \\ &\quad - [q(Ay) + H(Ay,b)], \\ &= [q(Ax+Ay) - q(Ax) - q(Ay)], \\ &= H(Ax,Ay), \end{aligned}$$

and so defining $A^*$ to be the dual of $A$ with respect to the inner product $H$ we have $A^*A = I$. Suppose we have $x \in \operatorname{Ker}(A-I)^*$, then

$$A^*x = x \Rightarrow Ax = x \Rightarrow H(x,b) = 0$$

and hence we know $b \perp \operatorname{Ker}(A-I)^*$. It is an exercise in undergraduate linear algebra that $(\operatorname{Ker}(A-I)^*)^\perp = \operatorname{Im}(A-I)$, so

$$b \in \operatorname{Im}(A-I) \Rightarrow \exists y \in V,\ b = (A-I)y \Rightarrow \exists y \in V,\ Ay + b = y.$$

□

*This shows us how to restrict the ongoing hyperelliptic example further. The quadratic function on $H_1(\mathcal{C}, \mathbb{Z}_2)$ given by the parity is preserved under the affine action of $\operatorname{Aut}(\mathcal{C})$, so the affine transformation given by the hyperelliptic involution has a fixed point. The vector $b$ in the proof of Lemma 2.2.28 is the value $v(-1)$, and so we must have $v(-1) \in \operatorname{Im} 0 = 0$. Hence we have an invariant spin structure, and moreover we have $2^{2g} = |V|$ of them.*

**Example 2.2.29** (Cyclic groups)**.** *Let us now try and understand [KS10, Theorem 1.1] in this language of group cohomology. Suppose we take a cyclic automorphism group $\langle a \rangle = G$, and let $n$ be the order of $a$. A cocycle $v \in Z^1$ must have $v(1) = 0$ as before, and then it is specified by $v(a)$, which is subject only to the condition that*

$$(I + A + \cdots + A^{n-1})v(a) = v(a^n) = v(1) = 0, \tag{2.12}$$

*where $A = \rho(a)$ (already reduced mod 2). Provided $A \neq I$, given that we have $(A-I)(\sum_{k=0}^{n-1} A^k) = A^n - I = 0$, the sum $\sum_{k=0}^{n-1} A^k$ has nontrivial kernel in which $v(a)$ must lie. A coboundary is given by $v(a) = (A-I)y$ for some $y \in V$, and hence*

$$H^1(G,V) \cong \operatorname{Ker}(I + \cdots + A^{n-1}) / \operatorname{Im}(A-I).$$

*Note this result is contained in [Wei95, Theorem 6.2.2].*

*Certainly if $A - I$ is invertible, then there is necessarily an invariant spin structure as $H^1 = 0$, and it is unique as then $H^0(G,V) = \operatorname{Ker}(A-I) = 0$. Moreover the converse is true that if there is a unique invariant characteristic then $A - I$ is invertible, as $\operatorname{Ker}(A-I) = 0$ is exactly the condition for invertibility. By the argument required in [BN12, p. 17], we can see that for $A - I$ to be invertible, it is necessary that $\sum_{k=0}^{n-1} A^k = 0$, and in fact this is sufficient when $n$ is odd as we* *can simply write down the inverse which is* $n^{-1}\sum_{k=1}^{n} kA^{k-1}$*. Note also certainly if* $n = 2$ *then* $A - I$ *cannot be invertible as*

$$(A - I)^2 = A^2 - 2A + I = 0.$$

*This in fact generalises; if* $n = 2^k$ *then* $A - I$ *cannot be invertible as*

$$(A - I)^n = \sum_{i=0}^{n} \binom{n}{i} A^i = A^n - I = 0$$

*using Lucas' theorem for the binomial coefficient of a prime power mod that prime (see for example [Fin47, Theorem 3], or alternatively prove this with induction).*

*Now, as in Example 2.2.27, the quadratic function given by the parity is preserved under the affine transformation generating the group action on* $H_1(\mathcal{C}, \mathbb{Z}_2)$*, so the transformation has a fixed point. As such, the remaining question is just how many invariant characteristics there are. This is given by* $\dim_{\mathbb{Z}_2} \ker(A - I)$ *and Proposition 2.2.26, and [KS10] shows how to compute whether this is zero using cyclotomic polynomials. Importantly they show that a necessary and sufficient condition when* $n$ *is odd is that the genus of the quotient* $g = g(\mathcal{C}/\langle a\rangle)$*, related to* $a$ *by* $\dim_{\mathbb{Z}} \ker(\rho_r(a) - I) = 2g$*, is 0.*

**Example 2.2.30** (Direct product of cyclic groups)**.** *Suppose next we consider the action of a group*

$$\langle a, b, | ab = ba, a^n = 1 = b^m\rangle = G.$$

*For* $v \in Z^1$*,* $v(a), v(b)$ *are subject to the same condition of Equation 2.12, and then we also have*

$$v(b^l) + b^l v(a^k) = v(a^k b^l) = v(a^k) + a^k v(b^l)$$

*and so we get the additional condition that*

$$(A - I)v(b) = (B - I)v(a),$$

*taking* $A = \rho(a)$*,* $B = \rho(b)$*. The conditions on* $v \in B^1$ *for* $v(a), v(b)$ *are again the same as for a single cyclic group, but now note the vector* $y$ *must be the same for both, that is* $\exists y \in Y$ *such that* $v(a) = (A - I)y$ *and* $v(b) = (B - I)y$*. We can then write out the cohomology in the somewhat laborious way*

$$H^1(G, V) = \frac{\left\{(x, x') \in \mathrm{Ker}\left(\sum_k A^k\right) \times \mathrm{Ker}\left(\sum_k B^k\right) \mid (A - I)x' = (B - I)x\right\}}{\left\{(x, x') \in \mathrm{Im}(A - I) \times \mathrm{Im}(B - I) \mid \text{“}(A - I)^{-1}x = (B - I)^{-1}x'\text{”}\right\}}$$

*where I have used quotes to indicate an approximate condition as* $(A - I)$ *and* $(B - I)$ *do not necessarily have inverses. Indeed if we have either of* $A - I$ *or* $B - I$ *invertible then these spaces would be the same and we would have* $H^1(G, V) = 0$*, and so a spin structure invariant under the whole group would exist. Assume without loss of generality that* $A - I$ *is invertible, then the number of invariant spin structures is* $1$*, as* $V^G \subset V^{\langle a\rangle} = 0$*.*

*By Lemma 2.2.28, we know that the class corresponding to the action on characteristics lies in the subgroup*

$$\frac{\{(x,x')\in \mathrm{Im}(A-I)\times\mathrm{Im}(B-I)\,|\,(A-I)x'=(B-I)x\}}{\{(x,x')\in \mathrm{Im}(A-I)\times\mathrm{Im}(B-I)\,|\,\text{“}(A-I)^{-1}x=(B-I)^{-1}x'\text{”}\}}.$$

*This we can then reduce to write as (breaking the symmetry between a and b)*

$$\mathrm{Im}(A-I)\cap\mathrm{Ker}(B-I)\cong H^1(G,V)\cong\mathrm{Ker}(A-I)\cap\mathrm{Im}(B-I).$$

*which is to say*

$$H^1(G,V)\cong\mathrm{Ker}(A-I)(B-I)/\,\mathrm{Ker}(B-I).$$

*Suppose we had for our closed cycle $v(a)=(A-I)y_1$, $v(b)=(B-I)y_2$ for $y_1\neq y_2$. Then we know $y_1-y_2\in\mathrm{Ker}((A-I)(B-I))$.*

*At this stage we have said all we can without using extra information of the exact representation $\rho$. A glance ahead at Tables 2.1 and 2.2 shows that the number of invariants under a $V_4=C_2\times C_2$ action depends on the signature of the action.*

A key question I shall want to address using group cohomology is when the action of $G$ has a **Unique Invariant Characteristic (UIC)**. To make more progress, we use the inflation-restriction exact sequence. That is for subgroup $N\triangleleft G$ and abelian group $V$ with $G$ action the SES

$$0\to H^1(G/N,V^N)\to H^1(G,V)\to H^1(N,V)^{G/N}\to H^2(G/N,V^N)\to H^2(G,V),$$

so named because the 3 inner maps are **inflation**, **restriction** (or **coinflation**), and **transgression** [Wei95, 6.8.3]. Suppose we know $H^1(N,V)=0$ and $H^0(N,V)=V^N=0$ (as we have when $N$ is an odd-order cyclic group), then denoting $K=G/N$ we have

$$0\to H^1(K,0)\to H^1(G,V)\to 0\to H^2(K,0)\to H^2(G,V).$$

This trivially gives $H^1(G,V)=0$, and moreover because we have $(V^N)^{(G/N)}\cong V^G$, we get $H^0(G,V)=0$.

In fact we do not need the restrictive condition that $H^1(N,V)=0$ in the situation we care about, as we want the case where there is a unique characteristic invariant under $N$, which is the case when $H^0(N,V)=V^N=0$ but also when the specific affine representation of $G$ as an element in $H^1(G,V)$ is the zero class when restricted to the action as an element in $H^1(N,V)$. This means that the class in $H^1(G,V)$ is in the kernel of the restriction map, and as we have said that the image of the inflation map is trivial, this means the class in $H^1(G,V)$ must be the zero class.

Read together these tell us that if we have a normal subgroup given with a UIC, then the action of the whole group has a UIC. Moreover, this extends to if we have a subgroup which is subnormal in the original group (that is $H\leq G$ such that $\exists H_i$ with $H\triangleleft H_1\triangleleft\cdots\triangleleft H_k\triangleleft G$). In summation this gives the following proposition.

**Proposition 2.2.31.** *If there exists $f \in G \leq \mathrm{Aut}(\mathcal{C})$ such that $f$ has odd order, $\langle f \rangle$ is subnormal in $G$, and $g(\mathcal{C}/\langle f \rangle) = 0$, then $\mathcal{C}$ has a unique theta characteristic invariant under the action of $G$. We will call this property of a curve SubNormal Odd cyclic Gonality (SNOG).*

### Tables of Orbits

I shall now give table of orbit decompositions for many curves of different genera, giving those for all possible curves of genus 2, 3, and 4. This I shall do computationally, using the representation in terms of binary vectors via spin structures as in [KS10]· In doing these computations, we are aided by the fact that we need only choose one representative curve from each class because the decomposition is uniquely determined by the rational representation and two curves in the interior of an equisymmetric family (in the sense of [Bro90]) have equivalent rational representations [RCR22, p. 896]. One can understand this intuitively; if the coefficients of the curve are varied only slightly (and generically) then as the rational representation is given in terms of integer valued matrices these would not be expected to change.

From Examples 2.2.18 and 2.2.27 we know that the only nonidentity automorphism of a curve that fixes every theta characteristic on a curve is the hyperelliptic involution $\iota$ (if it exists), hence we know that the **reduced automorphism group** [Rau70, Pop72]

$$\overline{\mathrm{Aut}}(\mathcal{C}) := \begin{cases} \mathrm{Aut}(\mathcal{C})/\langle \iota \rangle\,, & \mathcal{C} \text{ is hyperelliptic,} \\ \mathrm{Aut}(\mathcal{C}), & \text{otherwise,} \end{cases}$$

acts faithfully on the theta characteristics. There is no a priori reason to expect $\overline{\mathrm{Aut}}(\mathcal{C})$ to have an action on $\mathcal{C}$ when $\mathcal{C}$ is hyperelliptic [Rau70].

**Example 2.2.32.** *For Burnside's curve given by $y^2 = x(x^4-1)$, the full automorphism group is $\mathrm{GL}_2(3)$ generated by the automorphisms*

$$f(x,y) = \left(\frac{x+1}{x-1}, \frac{2\sqrt{2}\,y}{(x-1)^3}\right), \quad g(x,y) = (ix, \exp(i\pi/4)\,y)\,, \quad h(x,y) = \left(\frac{-1}{x}, \frac{-y}{x^3}\right).$$

*These satisfy $f^2 = 1$ and $g^4 = \iota = h^2$, whence the reduced automorphism group is $S_4$. By [Bre00, Table 9], $S_4$ has no action on a genus-2 curve.*

I shall subsequently give tables of curves of a given genus as a plane curve $f(x,y) = 0$, their reduced automorphism group (with the GAP group ID if the presentation given of the group is not specific [GAP22]), the quotient genus $g_0$ and signature $\boldsymbol{c}$ of the $\overline{\mathrm{Aut}}$ action on the curve (recall Definition 2.1.81) when it exists and is known, the $\overline{\mathrm{Aut}}$ orbits of the odd and even characteristics respectively presented as a list of values $a_b$ indicating $b$ orbits of size $a$, and the total number of characteristics invariant under the group action. When giving $f$ I shall leave in free parameters where possible, not specifying values that must be avoided.

Moreover, I shall use the notation of [Bar12, BB16] that $L_d(x, y, \dots)$ is a generic homogeneous degree-$d$ polynomial in the arguments. The code to recreate this data (except for the signature) is given in the Sage notebooks `list_of_plane_curves.ipynb` and `theta_characteristic_orbit.ipynb`.

**Remark 2.2.33.** *Throughout signatures are computed with the help of the LMFDB [LMF23], and in cases of ambiguity were verified by comparing the character of the rational representation found using this character and the Eichler trace formula [Bre00, p. 41] to that found by computing directly with Sage.*

**Remark 2.2.34.** *As part of the computation we employ (a wrapper of) GAP's* `StructureDescription` *method. Note GAP warns about this method because it is not an isomorphism invariant, and moreover it does not specify the map giving a semidirect product. Semidirect products will always be given in the form $N : H = N \rtimes H$. Moreover, throughout I will use the convention that the dihedral group $D_n$ be of size $2n$.*

Table 2.1: Orbit decomposition, all genus-2 curves

| $f$ | $\overline{\text{Aut}}$, $\boldsymbol{c}$ | Odd | Even | I |
|---|---|---|---|---|
| $y^2-(x^2-1)(x^2-a)(x^2-b)$ | $C_2$, $(1;2^2)$ | $2_3$ | $1_4, 2_3$ | 4 |
| $y^2-(x^2-1)(x^2-a)(x^2-a^{-1})$ | $V_4$, $(0;2^5)$ | $2_1, 4_1$ | $1_2, 2_2, 4_1$ | 2 |
| $y^2-(x^5-1)$ | $C_5$, $(0;5^3)$ | $1_1, 5_1$ | $5_2$ | 1 |
| $y^2-(x^6-ax^3+1)$ | $S_3$, $(0;2^2,3^2)$ | $6_1$ | $1_1, 3_3$ | 1 |
| $y^2-(x^6-1)$ | $D_6$, $(0;2^3,3)$ | $6_1$ | $1_1, 3_1, 6_1$ | 1 |
| $y^2-x(x^4-1)$ | $S_4$ | $6_1$ | $4_1, 6_1$ | 0 |

There are a few relevant comments to make about Table 2.1, which provides the orbit decomposition for genus-2 curves.

- The complete list of genus-2 curves with nontrivial reduced automorphism group comes from [Bol87].
- Every genus-2 curve with a unique invariant characteristic satisfied the conditions of Proposition 2.2.31, that is it is SNOG.

Table 2.2: Orbit decomposition, all non-hyperelliptic and hyperelliptic genus-3 curves

| $f$ | $\overline{\text{Aut}}$, $\boldsymbol{c}$ | Odd | Even | I |
|---|---|---|---|---|
| $1+L_2(x,y)+L_4(x,y)$ | $C_2$, $(1;2^4)$ | $1_4$, $2_{12}$ | $1_{12}$, $2_{12}$ | 16 |
| $L_1(x,y)+L_3(x,y)$ | $C_3$, $(0;3^5)$ | $1_1$, $3_9$ | $3_{12}$ | 1 |
| $1+y^4+x^4+(ay^2+bx^2)+cx^2y^2$ | $V_4$, $(0;2^6)$ | $2_6, 4_4$ | $1_8, 2_6, 4_4$ | 8 |
| $bx^2y^2+x^3+y^3+axy+1$ | $S_3$, $(0;2^4,3)$ | $1_1, 3_3, 6_3$ | $1_3, 3_9, 6_1$ | 4 |
| $y^4+x^3+ay^2+1$ | $C_6$, $(0;2,3^2,6)$ | $1_1, 3_1, 6_4$ | $3_4, 6_4$ | 1 |
| $1+y^4+x^4+a(x^2+y^2)+bx^2y^2$ | $D_4$, $(0;2^5)$ | $4_5, 8_1$ | $1_4, 2_4, 4_4, 8_1$ | 4 |

| | | | | |
|---|---|---|---|---|
| $xy^3+x^3+1$ | $C_9$, $(0;3,9^2)$ | $1_1, 9_3$ | $9_4$ | 1 |
| $y^4+ay^2+x^4+1$ | $(C_4 \times C_2) \rtimes C_2 \cong (16,13)$, $(0;2^3,4)$ | $4_1, 8_3$ | $2_6, 8_3$ | 0 |
| $1+y^4+x^4+a(y^2+x^2+y^2x^2)$ | $S_4$, $(0;2^3,3)$ | $4_1, 12_2$ | $1_2$, $3_2$, $4_1$, $6_2$, $12_1$ | 2 |
| $x^4+y^4+x$ | $((C_4 \times C_2) \rtimes C_2) \rtimes C_3$ $\cong (48,33)$, $(0;2,3,12)$ | $4_1, 24_1$ | $6_2, 24_1$ | 0 |
| $y^4+x^4+1$ | $(C_4^2 \rtimes C_3) \rtimes C_2 \cong (96,64)$, $(0;2,3,8)$ | $12_1, 16_1$ | $4_2, 12_1, 16_1$ | 0 |
| $xy^3+x^3+y$ | $\mathrm{PSL}_3(\mathbb{F}_2)$, $(0;2,3,7)$ | $28_1$ | $1_1, 7_2, 21_1$ | 1 |
| $y^2-(x^8+ax^6+bx^4+cx^2+1)$ | $C_2$, $(1;2^4)$ | $1_4$, $2_{12}$ | $1_{12}$, $2_{12}$ | 16 |
| $y^2-x(x^2-1)(x^4+ax^2+b)$ | $C_2$ | $1_4$, $2_{12}$ | $1_4$, $2_{16}$ | 8 |
| $y^2-(x^4+ax^2+1)(x^4+bx^2+1)$ | $V_4$, $(0;2^6)$ | $2_6$, $4_4$ | $1_8$, $2_6$, $4_4$ | 8 |
| $y^2-(x^4-1)(x^4+ax^2+1)$ | $V_4$ | $1_2$, $2_3$, $4_5$ | $1_2$, $2_7$, $4_5$ | 4 |
| $y^2-x(x^6+ax^3+1)$ | $S_3$, $(0;2^4,3)$ | $1_1$, $3_3$, $6_3$ | $1_3$, $3_9$, $6_1$ | 4 |
| $y^2-(x^8+ax^4+1)$ | $D_4$, $(0;2^2,4^2)$ | $4_5$, $8_1$ | $1_4$, $2_4$, $4_4$, $8_1$ | 4 |
| $y^2-(x^7-1)$ | $C_7$, $(0;7^3)$ | $7_4$ | $1_1$, $7_5$ | 1 |
| $y^2-x(x^6-1)$ | $D_6$ | $1_1$, $3_1$, $6_2$, $12_1$ | $1_1$, $2_1$, $3_1$, $6_5$ | 2 |
| $y^2-(x^8-1)$ | $D_8$ | $4_1$, $8_3$ | $1_2$, $2_1$, $4_2$, $8_3$ | 2 |
| $y^2-(x^8+14x^4+1)$ | $S_4$, $(0;3,4^2)$ | $4_1, 12_2$ | $1_2$, $3_2$, $4_1$, $6_2$, $12_1$ | 2 |

Again, I shall make remarks about Table 2.2.

- The complete list of non-hyperelliptic genus-3 curves with nontrivial (reduced) automorphism group comes from [Bar12]. Bars attributes the first work completing this to Henn in 1976, but Wiman appears to have completed the calculation earlier in [Wim95a]. Bars and Dolgachev disagree on the automorphism group of the curve given by $f = 1+y^4+x^4+a(x^2+y^2)+bx^2y^2$; using Sage I find agreement with Bars.

- The list of hyperelliptic curves with many automorphisms comes from [Sha07, MP21]. The latter reference will also be used for higher genera.

- [Sha07, Table 4] was used to verify the signatures of the non-hyperelliptic actions.

- At genus 3, not every curve with a unique invariant characteristic is SNOG. There is a single exception, Klein's curve, whose automorphism group $\mathrm{PSL}_3(\mathbb{F}_2)$ is simple and so cannot have a nontrivial subnormal cyclic group. There is a $C_7$ subgroup of the automorphism group quotienting to $\mathbb{P}^1$ (as clearly seen by writing Klein's curve in Lefschetz form [Lef21a, Lef21b, Zom10]), but it is not subnormal.

- The non-hyperelliptic curve with $\overline{\mathrm{Aut}} = S_4$ is the first example of a curve with large automorphism group without $I \leq 1$.

Table 2.3: Orbit decomposition, all non-hyperelliptic genus-4 curves stratified by whether the corresponding quadric is singular, and separately some hyperelliptic genus-4 curve with many automorphisms

| $f$ | $\overline{\mathrm{Aut}}$ | Odd | Even | I |
|---|---|---|---|---|
| $y^3+y(ax^4+bx^2+c)+(dx^6+ex^4+fx^2+g)$ | $C_2$, $(1;2^6)$ | $1_{24}$, $2_{48}$ | $1_{40}$, $2_{48}$ | 64 |
| $y^3+y(ax^4+bx^2+c)+dx(x^4+ex^2+f)$ | $C_2$, $(2;2^2)$ | $2_{60}$ | $1_{16}$, $2_{60}$ | 16 |
| $y^3+y[a(x^4+1)+bx^2]+x[c(x^4+1)+dx^2]$ | $V_4$, $(1;2^3)$ | $2_{24}$, $4_{18}$ | $1_{16}$, $2_{24}$, $4_{18}$ | 16 |
| $y^3+y[a(x^4+1)+bx^2]+x(x^4-1)$ | $V_4$, $(1;2^3)$ | $4_{30}$ | $1_4$, $2_{18}$, $4_{24}$ | 4 |
| $y^3+ayx^2+x(x^4+1)$ | $D_4$, $(0;2^4,4)$ | $4_{12}$, $8_9$ | $1_4$, $2_6$, $4_{18}$, $8_6$ | 4 |
| $y^3+y(x^4+a)+(bx^4+c)$ | $C_4$, $(0;2,4^4)$ | $1_4$, $2_{10}$, $4_{24}$ | $1_4$, $2_{18}$, $4_{24}$ | 8 |
| $y^3+ayx^2+(x^6+bx^3+1)$ | $S_3$, $(0;2^6)$ | $1_6$, $3_{18}$, $6_{10}$ | $1_{10}$, $3_{30}$, $6_6$ | 16 |
| $y^3+ayx^2+(x^6+1)$ | $D_6$, $(0;2^5)$ | $2_3$, $6_{13}$, $12_3$ | $1_4$, $2_3$, $3_{12}$, $6_9$, $12_3$ | 4 |
| $y^3+y(ax^3+b)+(x^6+cx^3+d)$ | $C_3$, $(1;3^3)$ | $1_3$, $3_{39}$ | $1_1$, $3_{45}$ | 4 |
| $y^3+ay(x^3+1)+(x^6+20x^3-8)$ | $A_4$, $(0;2,3^3)$ | $4_3$, $12_9$ | $1_1$, $3_1$, $6_6$, $12_8$ | 1 |
| $y^3+ay+(x^6+b)$ | $C_6$, $(0;2,6^3)$ | $1_3$, $3_7$, $6_{16}$ | $1_1$, $3_{13}$, $6_{16}$ | 4 |
| $y^3+y+x^6$ | $C_{12}$, $(0;4,6,12)$ | $1_1$, $2_1$, $3_1$, $6_3$, $12_8$ | $1_1$, $3_1$, $6_6$, $12_8$ | 2 |
| $y^3+ay+(x^5+b)$ | $C_5$, $(0;5^4)$ | $5_{24}$ | $1_1$, $5_{27}$ | 1 |
| $y^3+y+x^5$ | $C_{10}$, $(0;5,10^2)$ | $10_{12}$ | $1_1$, $5_3$, $10_{12}$ | 1 |
| $y^3-(x^6+ax^5+bx^4+cx^3+dx^2+ex+f)$ | $C_3$, $(0;3^6)$ | $3_{40}$ | $1_1$, $3_{45}$ | 1 |
| $y^3-(x^6+ax^4+bx^2+1)$ | $C_6$, $(0;2^2,3^3)$ | $3_8$, $6_{16}$ | $1_3$, $3_{13}$, $6_{16}$ | 1 |
| $y^3-x(x^4+ax^2+1)$ | $C_6\times C_2$, $(0;2^2,3,6)$ | $6_8$, $12_6$ | $1_1$, $3_5$, $6_8$, $12_6$ | 1 |
| $y^3-(x^6+ax^3+1$ | $C_3\times S_3$, $(0;2^2,3^2)$ | $3_2$, $6_1$, $9_6$, $18_3$ | $1_1$, $3_3$, $9_{10}$, $18_2$ | 1 |
| $y^3-(x^5+1)$ | $C_{15}$, $(0;3,5,15)$ | $15_8$ | $1_1$, $15_9$ | 1 |
| $y^3-(x^6+1)$ | $C_6\times S_3$, $(0;2,6^2)$ | $6_2$, $18_4$, $36_1$ | $1_1$, $3_1$, $6_1$, $9_4$, $18_3$, $36_1$ | 1 |
| $y^3-x(x^4+1)$ | $C_3\times S_4$, $(0;2,3,12)$ | $12_2$, $24_1$, $36_2$ | $1_1$, $3_1$, $12_2$, $18_2$, $36_2$ | 1 |

| | | | | |
|---|---|---|---|---|
| $y^4(x+1)+y^3(x^2+ax+1)+y^2[b(x^3+1)+cx(x+1)]+y[dx(x^2+1)+ex^2]+fx^2(x+1)$ | $C_2$, $(1;2^6)$ | $1_{24}$, $2_{48}$ | $1_{40}$, $2_{48}$ | 64 |
| $y^6+y^4(x^2+ax+1)+y^2x(dx^2+bx+e)+cx^3$ | $C_2$, $(2;2^2)$ | $2_{60}$ | $1_{16}$, $2_{60}$ | 16 |
| $y^6+y^4(x^2+ax+1)+y^2x[d(x^2+1)+bx]+cx^3$ | $V_4$, $(1;2^3)$ | $2_{24}$, $4_{18}$ | $1_{16}$, $2_{24}$, $4_{18}$ | 16 |
| $y^6+y^2[cx(xy^2+1)+b(x^3+y^2)+ax(y^2+x)]+x^3$ | $V_4$, $(1;2^3)$ | $4_{30}$ | $1_4$, $2_{18}$, $4_{24}$ | 4 |
| $by^2(y^2-x)(x^2-1)-y^6-axy^2(y^2+x^2)-x^3$ | $D_4$, $(0;2^4,4)$ | $4_{12}$, $8_9$ | $1_4$, $2_6$, $4_{18}$, $8_6$ | 4 |
| $y^6+ay^3(x^3+1)+bxy^4+cx^2y^2+x^3$ | $S_3$, $(0;2^6)$ | $1_6$, $3_{18}$, $6_{10}$ | $1_{10}$, $3_{30}$, $6_6$ | 16 |
| $y^6+ay^3(x^3+1)+bxy^4+bx^2y^2+x^3$ | $D_6$, $(0;2^5)$ | $2_3$, $6_{13}$, $12_3$ | $1_4$, $2_3$, $3_{12}$, $6_9$, $12_3$ | 4 |
| $by^2(y^2-x)(x^2-1)-(y^2+x)^3$ | $S_4$, $(0;2^3,4)$ | $4_6$, $12_6$, $24_1$ | $1_4$, $4_6$, $6_6$, $12_6$ | 4 |
| $x^3y^3+y^6+(a+b+1)y^2(x^3-y^3)+(ab+a+b)y(x^3-y^3)+ab(x^3-y^3)$ | $S_3$, $(0;2^2,3^3)$ | $6_{20}$ | $1_1$, $3_{15}$, $6_{15}$ | 1 |
| $y^4(a+y^2)+x^3(1+ay^2)$ | $D_6$, $(0;2^2,3,6)$ | $12_{10}$ | $1_1$, $3_3$, $6_{13}$, $12_4$ | 1 |
| $x^3y^3+y^6+ax^3+y^3$ | $S_3\times S_3$, $(0;2^3,3)$ | $6_2$, $18_6$ | $1_1$, $3_6$, $9_9$, $36_1$ | 1 |
| $x^3y^3+y^6-x^3+y^3$ | $(S_3\times S_3)\rtimes C_2\cong(72,40)$, $(0;2,4,6)$ | $12_1$, $36_3$ | $1_1$, $6_3$, $9_3$, $18_3$, $36_1$ | 1 |
| $x^2y^3+y^4+a^5x^3+xy$ | $D_5$, $(0;2^2,5^5)$ | $10_{12}$ | $1_1$, $5_{15}$, $10_6$ | 1 |
| $xy-x^3+y^4+x^2y^3$ | $S_5$, $(0;2,4,5)$ | $20_3$, $60_1$ | $1_1$, $5_3$, $10_3$, $30_3$ | 1 |
| $y^2-(x^9-1)$ | $C_9$, $(0;9^3)$ | $3_1$, $9_{13}$ | $1_1$, $9_{15}$ | 1 |
| $y^2-x(x^4-1)(x^4+2i\sqrt{3}x^2+1)$ | $A_4$ | $4_3$, $6_4$, $12_7$ | $4_1$, $6_4$, $12_9$ | 0 |
| $y^2-x(x^8-1)$ | $D_8$ | $8_5$, $16_5$ | $2_2$, $4_1$, $8_{10}$, $16_3$ | 0 |
| $y^2-(x^{10}-1)$ | $D_{10}$ | $10_4$, $20_4$ | $1_1$, $5_3$, $10_8$, $20_2$ | 1 |

Again, I shall make remarks about Table 2.3.

- The complete list of non-hyperelliptic genus-4 curves with nontrivial (reduced) automorphism group comes from [Wim95b]. Wiman distinguishes his curves by whether they lie on the nonsingular quadric or the cone, and I have followed this putting those that lie on the cone in the first portion of the table. For the curves which lie on the nonsingular quadric Wiman described the curve by providing the quadric and cubic, hence one must use the resultant to get a single plane equation, and this may require a projective linear transformation to find a nondegenerate coordinate system. We find a typo in Wiman's curve (8) with octahedral symmetry.

- At genus 4 one sees the first example of curves of the same genus with isomorphic reduced automorphism groups but distinct orbit decompositions. It remains an open question as to whether the pair $(\overline{\mathrm{Aut}}, \boldsymbol{c})$ completely determines the orbit decomposition. This is not immediately obvious, as the signature does not fully the determine the rational representation, one must also pick a generating vector [Bro91, Definition 2.2].

- At this genus not all curves with a unique invariant characteristic are SNOG. The exceptions are

  1. the curve with $(\overline{\mathrm{Aut}}, \boldsymbol{c}) = (A_4, (0; 2, 3^3))$,
  2. the curve with $(\overline{\mathrm{Aut}}, \boldsymbol{c}) = (S_5, (0; 2, 4, 5))$, Bring's curve.

  More will be said on Bring's curve in 2.3.4, here I only note that similarly to Klein's curve there is an odd order cyclic group quotienting to $\mathbb{P}^1$ (here a $C_5$) that is not subnormal. On the $A_4$ curve, we note that the quotient of the curve by the $C_3$ action has genus 1, and hence presents the first case where the existence of a UIC is not clearly governed by an odd cyclic quotient to $\mathbb{P}^1$.

At this point, above genus 4 I am unaware of any complete lists classifying curves by their automorphism groups and giving plane models of the curves. Curves of genus $(d-1)(d-2)/2$ for $d \geq 3$ stand out because of the Plücker formula (Example 2.1.13). In order to implement the numerical method for computing the rational representation it is also necessary to have a model of the curve with coefficients in $\mathbb{Q}$, and so this further limits the possible curves we may investigate. I am unaware of any plane models of non-hyperelliptic curves of genus 8 and so examples in this genus are sadly missing; one can in principle get such models from the methods of [Shi95], using Sage's modular symbol functionality, but in practice the process of going from a canonical embedding to a plane model becomes infeasible. Likewise, one could use the methods of [Swi16] to get the canonical embedding, but the problem of finding a plane form from this remains.

I shall subsequently give some particular curves of interest; as the LMFDB does not contain the data of signatures with quotient genus $> 0$ at these genera I shall sometimes omit the signature of the action where it is unknown.

Table 2.4: Orbit decomposition, two non-hyperelliptic genus-5 curves, and separately all hyperelliptic genus-5 curves with many automorphisms

| $f$ | $\overline{\mathrm{Aut}}$, $\boldsymbol{c}$ | Odd | Even | I |
|---|---|---|---|---|
| $y^4-4(x^4-ax^2+1)y^2+b^2x^4$ | $C_2^4$ | $4_{40}$, $8_{30}$, $16_6$ | $1_{32}$, $4_{40}$, $8_{30}$, $16_6$ | 32 |
| $4x^8+36x^4y^4+81y^8+8x^6+30x^2y^4+5x^4+14y^4+2x^2+1$ | $(((C_4\times C_2)\rtimes C_4)\rtimes C_3)\rtimes C_2 \cong (192,181)$, $(0;2,3,8)$ | $16_1$, $24_2$, $48_1$, $96_4$ | $1_1$, $3_1$, $4_1$, $6_2$, $12_1$, $16_1$, $24_4$, $48_4$, $96_2$ | 1 |
| $y^2-(x^{11}-1)$ | $C_{11}$, $(0;11^3)$ | $1_1$, $11_{45}$ | $11_{48}$ | 1 |
| $y^2-x(x^{10}-1)$ | $D_{10}$ | $1_1$, $5_3$, $10_{12}$, $20_{18}$ | $1_1$, $2_1$, $5_3$, $10_{27}$, $20_{12}$ | 2 |
| $y^2-(x^{12}-1)$ | $D_{12}$ | $1_1$, $3_1$, $6_2$, $12_{12}$, $24_{14}$ | $1_1$, $2_1$, $3_1$, $6_5$, $12_{25}$, $24_8$ | 2 |
| $y^2-(x^{12}-33x^8-33x^4+1)$ | $S_4$ | $1_1$, $3_1$, $6_2$, $12_8$, $24_{16}$ | $3_2$, $4_1$, $6_3$, $8_1$, $12_{13}$, $24_{14}$ | 1 |
| $y^2-x(x^{10}+11x^5-1)$ | $A_5$, $(0;3^2,5)$ | $1_1$, $15_1$, $30_6$, $60_5$ | $6_3$, $10_3$, $15_4$, $30_{12}$, $60_1$ | 1 |

The first non-hyperelliptic curve of genus 5 used in Table 2.4 is the family of Humbert curves given in [KR89, (5.9)], the second comes from taking resultants of the polynomials provided in [Wim95b], hence one may choose to call it the Wiman octic for want of a better name. The Wiman octic is not SNOG; there is a unique characteristic invariant under the $((C_4\times C_2)\rtimes C_4)\rtimes C_3$ normal subgroup, but not under any subnormal cyclic group. Moreover, again we find that the quotient by $C_3$ has genus 1, and so the Wiman octic presents the second case where the existence of a UIC is not clearly governed by an odd cyclic quotient to $\mathbb{P}^1$.

Table 2.5: Orbit decomposition, some non-hyperelliptic genus-6 curves, and separately all hyperelliptic genus-6 curves with many automorphisms

| $f$ | $\overline{\mathrm{Aut}}$, $\boldsymbol{c}$ | Odd | Even | I |
|---|---|---|---|---|
| $L_5(x,y)+L_3(x,y)+L_1(x,y)$ | $C_2$ | $1_{96}$, $2_{960}$ | $1_{160}$, $2_{960}$ | 256 |
| $x^5+ax^2y^3+bx^3y+y^4+cxy^2+dx^2+ey$ | $C_3$ | $1_6$, $3_{670}$ | $1_{10}$, $3_{690}$ | 16 |
| $L_5(x,y)+L_1(x,y)$ | $C_4$ | $1_{16}$, $2_{40}$, $4_{480}$ | $2_{80}$, $4_{480}$ | 16 |
| $x^5+axy^4+bx^2y^2+cx^3+dy^2+ex$ | $C_4$ | $1_8$, $2_{44}$, $4_{480}$ | $1_8$, $2_{76}$, $4_{480}$ | 16 |
| $1+L_5(x,y)$ | $C_5$ | $1_1$, $5_{403}$ | $5_{416}$ | 1 |
| $x^5+ax^2y^3+bx^3y+y^4+cxy^2+dx^2+y$ | $S_3$ | $1_6$, $3_{90}$, $6_{290}$ | $1_{10}$, $3_{150}$, $6_{270}$ | 16 |
| $x^5+y^4+ax^3+x$ | $C_8$ | $1_4$, $2_6$, $4_{20}$, $8_{240}$ | $4_{40}$, $8_{240}$ | 4 |

| | | | | |
|---|---|---|---|---|
| $x^5+y^5+axy^3+bx^2y+1$ | $D_5$ | $1_6$, $5_{90}$, $10_{156}$ | $1_{10}$, $5_{150}$, $10_{132}$ | 16 |
| $x^5+y^5+ax^3+x$ | $C_{10}$ | $1_1$, $5_{19}$, $10_{192}$ | $5_{32}$, $10_{192}$ | 1 |
| $x^5+y^4+x$ | $C_{16}$ | $1_2$, $2_1$, $4_3$, $8_{10}$, $16_{120}$ | $8_{20}$, $16_{120}$ | 2 |
| $x^5+y^5+x$ | $C_{20}$ | $1_1$, $5_3$, $10_8$, $20_{96}$ | $10_{16}$, $20_{96}$ | 1 |
| $x^5+y^4+y$ | $C_5\times S_3$, $(0;2,10,15)$ | $1_1$, $5_1$, $15_{18}$, $30_{58}$ | $5_2$, $15_{30}$, $30_{54}$ | 1 |
| $x^4y+y^4+x$ | $C_{13}\rtimes C_3$ $\cong(39,1)$, $(0;3^2,13)$ | $1_1$, $13_5$, $39_{50}$ | $13_{10}$, $39_{50}$ | 1 |
| $y^3-(x^4-2i\sqrt{3}x^2+1)(x^4+2i\sqrt{3}x^2+1)^2$ | $(V_4\rtimes C_9)\rtimes C_2\cong(72,15)$, $(0;2,4,9)$ | $18_4$, $36_6$, $72_{24}$ | $1_1$, $9_7$, $18_4$, $36_{30}$, $72_{12}$ | 1 |
| $(x^6+y^6+1)+(x^2+y^2+1)(x^4+y^4+1)-12x^2y^2$ | $S_5$, $(0;2,4,6)$ | $6_2$, $12_2$, $20_3$, $30_2$, $60_{17}$, $120_7$ | $1_2$, $2_1$, $10_2$, $12_3$, $15_6$, $20_2$, $30_9$, $60_{21}$, $120_3$ | 2 |
| $x^5+y^5+1$ | $C_5^2\rtimes S_3$ $\cong(150,5)$, $(0;2,3,10)$ | $1_1$, $15_1$, $25_5$, $75_{13}$, $150_6$ | $15_2$, $25_{10}$, $75_{20}$, $150_2$ | 1 |
| $y^2-(x^{13}-1)$ | $C_{13}$, $(0;13^3)$ | $1_1$, $13_{155}$ | $13_{160}$ | 1 |
| $y^2-x(x^{12}-1)$ | $D_{12}$ | $2_1$, $4_1$, $6_1$, $12_{17}$, $24_{75}$ | $2_1$, $4_2$, $6_3$, $12_{43}$, $24_{64}$ | 0 |
| $y^2-x(x^4-1)(x^8+14x^4+1)$ | $S_4$ | $6_4$, $8_3$, $12_6$, $24_{79}$ | $4_4$, $6_4$, $8_3$, $12_{30}$, $24_{69}$ | 0 |
| $y^2-(x^{14}-1)$ | $D_{14}$ | $14_{16}$, $28_{64}$ | $1_1$, $7_7$, $14_{41}$, $28_{52}$ | 1 |

Some remarks on Table 2.5 are appropriate.

- The non-hyperelliptic genus-6 curves come from a list of all nonsingular plane quintics in [BB16]. The curve with automorphism group of order 150 is the Fermat quintic curve which has maximal automorphism group for a genus-6 curve [MSSV02], the curve with automorphism group $S_5$ is the Wiman sextic [Wim95b, Edg81a], and the curve with automorphism group of order 72 is given in the LMFDB with label 6.72-15.0.2-4-9.1. The curve with automorphism group of order 39 is attributed to Snyder in [Lef21b, p. 464], where it is constructed in a manner similar to Klein's curve.

- At this genus every curve written down with a UIC is SNOG.

Table 2.6: Orbit decomposition, some non-hyperelliptic genus-7 curves, and separately some hyperelliptic genus-7 curves including all with many automorphisms

| $f$ | $\overline{\mathrm{Aut}}$, $\boldsymbol{c}$ | Odd | Even | I |
|---|---|---|---|---|
| $(x^3+y^3)^2-x^2y^2-1$ | $D_6$ | $1_4,\ 2_{12},\ 3_{12},\ 6_{232},\ 12_{556}$ | $1_{12},\ 2_{12},\ 3_{36},\ 6_{336},\ 12_{508}$ | 16 |
| $x^6+y^6-x^3-y^3$ | $C_3\times S_3$ | $1_1,\ 3_1,\ 6_7,\ 9_{20},\ 18_{439}$ | $3_4,\ 6_6,\ 9_{60},\ 18_{426}$ | 1 |
| $x^6+y^4-1$ | $C_{12}\times C_2$, $(0;4,6,12)$ | $1_1,\ 3_1,\ 6_6,\ 12_{42},\ 24_{316}$ | $1_1,\ 2_1,\ 3_1,\ 6_{11},\ 12_{66},\ 24_{308}$ | 2 |
| $(x^4+y^4)^2-x^3y^3-x^2y^2$ | $C_8\rtimes V_4$ $\cong(32,43)$, $(0;2^3,8)$ | $8_{16},\ 16_{84},\ 32_{208}$ | $2_4,\ 4_{22},\ 8_{42},\ 16_{129},\ 32_{180}$ | 0 |
| $x^7+y^7-x^2y^2$ | $C_3\times D_7$ | $1_1,\ 21_{21},\ 42_{183}$ | $3_1,\ 21_{63},\ 42_{165}$ | 1 |
| $y^{21}-x(x+1)^{13}(x-1)^7$ | $C_3\times D_7$, $(0;2,6,21)$ | $1_1,\ 21_{21},\ 42_{183}$ | $3_1,\ 21_{63},\ 42_{165}$ | 1 |
| $x^9+y^9-x^6$ | $C_3\times D_9$, $(0;2,6,9)$ | $1_1,\ 18_3,\ 27_{21},\ 54_{139}$ | $3_1,\ 18_4,\ 27_{63},\ 54_{120}$ | 1 |
| $x^9+y^9-x^3y^3$ | $C_3\times D_9$, $(0;2,6,9)$ | $1_1,\ 18_3,\ 27_{21},\ 54_{139}$ | $3_1,\ 18_4,\ 27_{63},\ 54_{120}$ | 1 |
| $y^8-(x^2-1)(x^2+1)^3$ | $(C_{16}\rtimes C_2)\rtimes C_2\cong(64,41)$, $(0;2,4,16)$ | $16_8,\ 32_{42},\ 64_{104}$ | $4_2,\ 8_9,\ 16_{15},\ 32_{64},\ 64_{92}$ | 0 |
| $y^{16}-x(x-1)^9(x+1)^6$ | $(C_8\rtimes C_4)\rtimes C_2\cong(64,41)$, $(0;2,4,16)$ | $16_8,\ 32_{42},\ 64_{104}$ | $4_2,\ 8_9,\ 16_{15},\ 32_{64},\ 64_{92}$ | 0 |
| $y^9-6y^6+3(9x^4-5)y^3-8$ | $((C_4\times S_3)\rtimes C_2)\rtimes C_3$ $\cong(144,127)$, $(0;2,3,12)$ | $4_1,\ 12_1,\ 24_2,\ 36_2,\ 72_{41},\ 144_{35}$ | $6_2,\ 12_2,\ 18_6,\ 24_2,\ 36_{28},\ 72_{28},\ 144_{35}$ | 0 |
| $28x^4y^4+2x^7+2y^7+35x^3y^3+21x^2y^2+7xy+1$ | $\mathrm{PSL}_2(\mathbb{F}_8)$, $(0;2,3,7)$ | $28_1,\ 36_1,\ 252_{14},\ 504_9$ | $28_3,\ 36_3,\ 126_{16},\ 252_{18},\ 504_3$ | 0 |
| $y^2-x(x^6-1)(x^8-2)$ | $C_2$ | $1_{64},\ 2_{4032}$ | $1_{64},\ 2_{4096}$ | 128 |
| $y^2-x(x^7-1)(x^7-2)$ | $D_7$ | $1_1,\ 7_{63},\ 14_{549}$ | $1_3,\ 7_{189},\ 14_{495}$ | 4 |
| $y^2-(x^{15}-1)$ | $C_{15}$, $(0;15^3)$ | $3_1,\ 5_2,\ 15_{541}$ | $1_1,\ 5_1,\ 15_{550}$ | 1 |
| $y^2-(x^8-1)(x^8-2)$ | $D_8$ | $8_{72},\ 16_{472}$ | $1_4,\ 2_4,\ 4_{25},\ 8_{170},\ 16_{424}$ | 4 |
| $y^2-x(x^{14}-1)$ | $D_{14}$ | $1_1,\ 7_7,\ 14_{57},\ 28_{260}$ | $1_1,\ 2_1,\ 7_7,\ 14_{120},\ 28_{233}$ | 2 |
| $y^2-(x^{16}-1)$ | $D_{16}$ | $8_4,\ 16_{62},\ 32_{222}$ | $1_2,\ 2_1,\ 4_3,\ 8_{14},\ 16_{112},\ 32_{198}$ | 2 |

| | | | | |
|---|---|---|---|---|
| $y^2-(x^{16}+1)$ | $D_{16}$ | $8_4$, $16_{62}$, $32_{222}$ | $1_2$, $2_1$, $4_3$, $8_{14}$, $16_{112}$, $32_{198}$ | 2 |

I shall make some comments on Table 2.6.

- The curve with automorphism group $\mathrm{PSL}_2(\mathbb{F}_8)$ is the Fricke-Macbeath curve [Fri99, Mac65], the unique Hurwitz curve of genus 7, the rational plane model of which is a attributed to Brock in [Hid17]. The remaining curves come from [Zom10, Table 2, Table 5]. Table 5 in Zomorrodian gives all possible automorphism groups of non-hyperelliptic genus-7 curves where $|\mathrm{Aut}| < 65$, and a plane curve form for each; this list contains some errors, for example a typo in curve 4 and the fact that curve 8 is hyperelliptic (as checked with Maple [Map22]).
- Edge describes the orbits of some of the odd characteristics of the Fricke-Macbeath curve in terms of tangent hyperplanes [Edg67].
- All the curves written with a unique invariant characteristic are SNOG

Table 2.7: Orbit decomposition, all hyperelliptic curves of genus 8 with many automorphisms

| $f$ | $\overline{\mathrm{Aut}}$, $\boldsymbol{c}$ | Odd | Even | I |
|---|---|---|---|---|
| $y^2-(x^{17}-1)$ | $C_{17}$, $(0;17^3)$ | $17_{1920}$ | $1_1$, $17_{1935}$ | 1 |
| $y^2-x(x^4-1)(x^{12}-33x^8-33x^4+1)$ | $S_4$ | $4_2$, $8_2$, $12_{94}$, $24_{1312}$ | $4_2$, $6_8$, $8_4$, $12_{90}$, $24_{1322}$ | 0 |
| $y^2-x(x^{16}-1)$ | $D_{16}$ | $16_{72}$, $32_{984}$ | $2_2$, $4_1$, $8_7$, $16_{188}$, $32_{932}$ | 0 |
| $y^2-(x^{18}-1)$ | $D_{18}$ | $6_1$, $18_{63}$, $36_{875}$ | $1_1$, $3_1$, $6_1$, $9_{14}$, $18_{182}$, $36_{819}$ | 1 |

There is little to say about Table 2.7, except for that the two curves which have a UIC are SNOG.

Table 2.8: Orbit decomposition, two non-hyperelliptic curves of genus 9, and all hyperelliptic curves of genus 9 with many automorphisms

| $f$ | $\overline{\mathrm{Aut}}$, $\boldsymbol{c}$ | Odd | Even | I |
|---|---|---|---|---|
| $x^5y^2+y^5+x^2$ | $C_{19}\rtimes C_3 \cong (57,1)$, $(0;3^2,19)$ | $1_1$, $19_{27}$, $57_{2286}$ | $19_{36}$, $57_{2292}$ | 1 |
| $y-x^3-x^4y^3+xy^4+3x^2y^2$ | $S_5$, $(0;2,5,6)$ | $6_1$, $10_4$, $20_{12}$, $30_{35}$, $60_{300}$, $120_{929}$ | $1_2$, $5_6$, $6_1$, $10_{10}$, $15_{24}$, $20_6$, $30_{85}$, $60_{402}$, $120_{867}$ | 2 |

| | | | | |
|---|---|---|---|---|
| $y^2-(x^{19}-1)$ | $C_{19}$, $(0;19^3)$ | $1_1$, $19_{6885}$ | $19_{6912}$ | 1 |
| $y^2-(x^{12}-33x^8-33x^4+1)(x^8+14x^4+1)$ | $S_4$ | $1_2$, $2_1$, $3_2$, $4_2$, $6_{13}$, $8_{11}$, $12_{172}$, $24_{5357}$ | $3_4$, $4_4$, $6_{18}$, $8_{16}$, $12_{290}$, $24_{5316}$ | 2 |
| $y^2-x(x^{18}-1)$ | $D_{18}$ | $1_1$, $3_1$, $6_2$, $9_{14}$, $12_1$, $18_{245}$, $36_{3507}$ | $1_1$, $2_1$, $3_1$, $6_5$, $9_{14}$, $18_{497}$, $36_{3395}$ | 2 |
| $y^2-(x^{20}-1)$ | $D_{20}$ | $1_1$, $5_3$, $10_{12}$, $20_{246}$, $40_{3144}$ | $1_1$, $2_1$, $5_3$, $10_{27}$, $20_{504}$, $40_{3024}$ | 2 |
| $y^2-(x^{20}-228x^{15}+494x^{10}+228x^5+1)$ | $A_5$, $(0;3,5^2)$ | $1_1$, $5_3$, $10_3$, $15_4$, $20_9$, $30_{117}$, $60_{2117}$ | $6_3$, $10_{12}$, $15_{16}$, $20_{12}$, $30_{345}$, $60_{2006}$ | 1 |

Finally some comments on Table 2.8.

- The curve with $\overline{\mathrm{Aut}} = S_5$ is the Fricke octavic curve, defined in [Edg84], constructed similarly to Bring's curve in $\mathbb{P}^3$ and so a plane form of the curve is found using resultants and a judicious choice of projective transformation to find a well conditioned coordinate system.
- The curve with automorphism group of order 57 is a generalisation of Klein's curve and Snyder's curve [Lef21b, p. 464].
- Not all the curves written here with a UIC are SNOG, the hyperelliptic curve with $\overline{\mathrm{Aut}} \cong A_5$ cannot have a subnormal cyclic group.

At this genus the computations were becoming prohibitively slow, with the calculation of the symplectic automorphism group of the Fricke octavic curve in Sage taking just under three hours on a laptop.

Leaving behind the criteria of requiring a plane model of the curve, one can compute additional examples of theta characteristic decompositions using the code from [BRR13], available at `https://sites.google.com/a/u.uchile.cl/mat-ciencias-prof-anita-rojas/home/proyectos`,[15] from the data of a group, signature, and choice of generating vector provided the quotient genus is 0. The Sage notebook `genus_order_invariants_data.ipynb` shows how this can be done.

Having computed now many examples in low genera, we can pick out some families of curves with unique invariant characteristics, giving us the following theorem.

**Theorem 2.2.35.** *There are infinitely many curves, both hyperelliptic and non-hyperelliptic, with a unique invariant characteristic.*

*Proof.* It is sufficient to consider only curves of Lefschetz type, in particular the two families we shall consider are one of the Wiman hyperelliptic curves

[15] I am grateful to Anita Rojas for her correspondence on the workings of this code.

$y^2 = x^{2g+1} - 1$ and the Lefschetz curves of the form $x^m y^n + y^m + x^n = 0$ for coprime $m, n$ where $p := m^2 - mn + n^2 > 7$ is a prime congruent to 1 mod 3.

The former has automorphism group $C_{2g+1} \times C_2$, which contains the normal subgroup $C_{2g+1}$ of odd order which quotients to $\mathbb{P}^1$.

The latter has automorphism group of the form $C_p \rtimes C_3$, which contains the normal subgroup $C_p$ of odd order which quotients to $\mathbb{P}^1$, which is easiest seen by writing the curve in the from $\tilde{y}^p + \tilde{x}^a(1 + \tilde{x}) = 0$ as can always be done for some $a$ [Lef21b, p. 464]. □

For the hyperelliptic family in the proof of Theorem 2.2.35 we can say more in the case that $2g + 1$ is a prime $p$, as all characteristics that are not invariant are in orbits of order $p$. In this we must have that

$$2^{p-1} = 2^{2g} \equiv 1 \mod p,$$

a fact that is an immediate consequence of Fermat's little theorem for odd primes. As $p$ is an odd prime, this means $2^g \equiv \pm 1 \mod p$ ($p | a^2 - 1 \Rightarrow p | a - 1$ or $p | a + 1$). Moreover, we know that orbits decompose into those of even and odd characteristics, hence we must have one of $2^{g-1}(2^g \pm 1)$ congruent to 0 mod $p$ and the other congruent to 1. We can identify the two cases as follows:

1. if $2^g \equiv 1$, $2^{g-1}(2^g - 1) \equiv 0$, and $2^{g-1}(2^g + 1) \equiv 2^g \equiv 1$,
2. if $2^g \equiv -1$, $2^{g-1}(2^g - 1) \equiv -2^g \equiv 1$, and $2^{g-1}(2^g + 1) \equiv 0$,

Hence the question reduces down to what value $2^g = 2^{\frac{p-1}{2}}$ is mod $p$. Suppose first we have that 2 is a square mod $p$, that is $\exists a$ such that $a^2 \equiv 2 \mod p$. Then we immediately get $2^g \equiv a^{p-1} \equiv 1$ by another application of the little theorem. The converse, that if 2 is not a square then $2^g \equiv -1$, was proven in [Isr19]. One can look at the OEIS to get a characterisation of the values of $g$ which give these cases, by considering what value $g$ is mod 4. We have

$$\begin{aligned} g \equiv 3, 0 \mod 4 &\Rightarrow p \equiv \pm 1 \mod 8 \Rightarrow 2 \text{ is a square}, \\ g \equiv 1, 2 \mod 4 &\Rightarrow p \equiv \pm 3 \mod 8 \Rightarrow 2 \text{ not a square}. \end{aligned}$$

which correspond to the cases where the invariant characteristic is even/odd respectively.

### 2.2.3 Future Directions

In this section I have begun an investigation of the orbits of theta characteristics, providing a group cohomology framework with which to frame the question, and ample data to begin investigating conjectures. Nevertheless, the structure of a theorem determining when a group action on a curve will have a unique invariant characteristic remains unclear. To this end I shall speculate on possible directions of study that may be fruitful in at least describing the correct results to prove.

- The algorithm ran to get the orbit decomposition under the reduced automorphism group can be trivially extended to give the orbit decomposition under any subgroup of this. It is possible that this richer dataset will provide more insight, and so the construction of an effective way to store and present this data would be a stepping stone towards understanding the full theory.

- [MSSV02] defines **normal homocyclic covers of** $\mathbb{P}^1$ to be those curves $\mathcal{C}$ with a $G \triangleleft \mathrm{Aut}(\mathcal{C})$ isomorphic to a product of cyclic groups such that $\mathcal{C}/G \cong \mathbb{P}^1$. For a subset of these, where $G$ is cyclic, we have already classified when there is a unique invariant characteristic, and so perhaps this subset of curves presents a more simple classification task.

- We saw that there was only one exception to the rule that every curve with a unique invariant characteristics that was not SNOG still had an odd-order cyclic subgroup quotienting to $\mathbb{P}^1$; this came from a curve in genus 4 with $A_4$ automorphism group. This is perhaps therefore a sensible condition to investigate.

- Machine classification (namely a pipeline using a Standard Scaler followed by a Random Forest classifier[16]) using features built from the group data and the signature alone achieved an accuracy of approximately 93% in cross-validation when predicting if the corresponding group action gave a unique invariant characteristic when trained on data of over 1000 group actions, a far higher accuracy than the approximately 54% one would expect if choosing randomly with prior knowledge of the frequency of actions with a UIC in the dataset. Specifically, I provided the features of genus, group order, whether the group action was large, the maximum power of 2 dividing the group order, the number of involutions in the group, the number of involutions up to conjugacy, the number of entries in the signature, the number of even entries in the signature, the maximum entry of the signature, and the dimension of the corresponding family for 1326 group actions in genus 12 or less. This suggests that from very basic heuristics along one should be able to get strong results understanding the behaviour of UICs. Moreover, estimating feature importance using these methods showed that whether or not a group action was large for a given genus was unimportant in determining whether a given action led to a UIC. I expect that such computational methods could lead to better understanding of which features to include when formulating theorems. To lay a benchmark for future work I provide in Table 2.9 the prediction of the classifier given the corresponding data for all the simple Hurwitz group with order $< 10^6$, provided in [Con87] (the $J_i$ are the first two Janko groups). These results are correct for the two Hurwitz curves known, though removing their data from the training set makes the classifier less accurate. Running a similar pipeline which instead predicts whether a group action has 0, 1, or many invariants characteristics confirms the predictions of Table 2.9, predicting those without a unique invariant characteristic have zero.

[16] I am very grateful to Jacob Bradley for showing me how to do this.

- Every finite group has a composition series, which provides one way of finding subnormal groups. The smallest nontrivial subnormal group in the composition series is a simple group, and so if $G$ is a finite group of odd order then the smallest nontrivial entry in the composition series is $H \cong C_p$ for $p$ some odd prime, and each composition factor is likewise isomorphic to $C_{p_i}$ for some odd primes $p_i$. The computed data shows that for any curve with $\mathrm{Aut}(\mathcal{C})$ odd (necessarily non-hyperelliptic) with quotient genus $g_0 = 0$, there is a unique invariant characteristic. One might suspect that this can be proven with the machinery of the inflation-restriction exact sequence. This would generalise the infinite family of non-hyperelliptic curves with a UIC given in Theorem 2.2.35 by results of [Wea03].

Table 2.9: Machine prediction of whether $I = 1$, all simple Hurwitz groups of order $< 10^6$

| $G$ | $g$ | $I = 1$ |
|---|---|---|
| $\mathrm{PSL}(2,7)$ | 3 | True |
| $\mathrm{PSL}(2,8)$ | 7 | False |
| $\mathrm{PSL}(2,13)$ | 14 | True |
| $\mathrm{PSL}(2,27)$ | 118 | True |
| $\mathrm{PSL}(2,29)$ | 146 | True |
| $\mathrm{PSL}(2,41)$ | 411 | False |
| $\mathrm{PSL}(2,43)$ | 474 | True |
| $J_1$ | 2091 | False |
| $\mathrm{PSL}(2,71)$ | 2131 | False |
| $\mathrm{PSL}(2,83)$ | 3404 | True |
| $\mathrm{PSL}(2,97)$ | 5433 | False |
| $J_2$ | 7201 | False |
| $\mathrm{PSL}(2,113)$ | 8589 | False |
| $\mathrm{PSL}(2,125)$ | 11626 | True |

## 2.3 Bring's Curve

> However, no method belonging to the study of Mathematics should have been left behind, untried and extended, from this lovable science by its untiring lovers and worshippers
>
> – Erland Samuel Bring
> Transformation of Algebraic Equations

This section shows how the theory laid out in §2.1 and §2.2 may be used to gain insight into specific algebraic curves, and in the process discover beautiful geometry. In particular, in this section I will use the geometry of cubic surfaces to identify two models of Bring's curve, explicitly identify the automorphism group of Bring's curve to complete the picture of its quotient structure, determine the orbits of its theta characteristics using Weierstrass points, and connect its unique invariant theta characteristic with the Riemann constant vector. This work, completed in collaboration with Harry Braden, has been submitted for publication and is available as a preprint [BDH22]. Jupyter notebooks containing code used for calculations are available at `https://github.com/DisneyHogg/Brings_Curve`.

Let us start by defining Bring's curve as a projective curve.

**Definition 2.3.1.** ***Bring's Curve*** $\mathcal{B}$ *is defined in* $\mathbb{P}^4$ *by the (homogeneous) equations*[17]

$$H_k := \sum_{i=1}^{5} x_i^k = 0, \quad k = 1, 2, 3, \tag{2.13}$$

*where we have taken the coordinates* $[x_1 : x_2 : x_3 : x_4 : x_5] \in \mathbb{P}^4$.

One can easily check that this is indeed a smooth projective curve. Bring's curve was introduced by Erland Bring in 1786 [Kle88, p. 157] in relation to solutions of the quintic equation, namely given a quintic polynomial $\prod_{i=1}^{5}(x-x_i)$, the equations of Bring's curve determine when the quintic is in reduced Bring-Jerrard form.[18]

As said in §2.1.1, one knows Bring's curve must have a (possibly singular) plane curve model, which we give now.

**Definition 2.3.2.** *The* ***Hulek-Craig (HC) model*** *of Bring's curve is the (singular) plane curve in* $\mathbb{P}^2$ *given by*

$$F(X, Y, Z) := X(Y^5 + Z^5) + (XYZ)^2 - X^4YZ - 2(YZ)^3 = 0, \tag{2.14}$$

*taking homogeneous coordinates* $[X : Y : Z] \in \mathbb{P}^2$. *We denote its normalisation by* $\overline{\mathcal{B}}$.

This model was used in [Hul87, Cra02] where they studied the curves modular properties, but I shall not discuss these aspects in this thesis. Another plane model of Bring's curve giving in [Web05, Proposition 3.1] is given by $(x-1)y^5 - (x+1)x^2 = 0$, found by considering the curve as a cyclic cover of $\mathbb{P}^1$. I will call Equation 2.13 the $\mathbb{P}^4$-model of Bring's curve, Equation 2.14 the HC model of Bring's curve; in Proposition 2.3.7 I will prove their equivalence.

### 2.3.1 Basic Properties

In order to study Bring's curve we will first need some introductory properties. Taking affine coordinates $(x, y) = (X/Z, Y/Z)$ such that the HC model is[19] $0 = f(x, y) = F(x, y, 1)$, $\mathrm{Res}_y(f(x, y), \partial_y f(x, y)) = x^4(x^5-1)^2(256x^{10} - 837x^5 + 3456)$

[17] Note here we use the notation $H_k$ of [Bur83] for the symmetric sum, whereas [Edg78] uses $S_k$.

[18] Bring, unlike Jerrard, never made the mistake of suggesting that the reduction to Bring-Jerrard form gives a general method for solving quintics [Caj94, p. 349].

[19] Note the coordinates $x, y$ will also interchangeably be used as coordinates on the normalisation away from the preimages of singular points.

determines the branch locus of the curve (recall Example 2.1.8). Within this, one can determine the singular points.

**Lemma 2.3.3** ([BN12]). *The only singular points in the HC model of the curve are* $V_k = [\zeta^k : \zeta^{2k} : 1]$ *for* $k = 0, \dots, 4$, *where* $\zeta = \exp(2\pi i/5)$, *and* $V_5 = [1 : 0 : 0]$.

We shall return to these points in due course; each of these singular points desingularises to two points. We also have nonsingular points $a = [0 : 0 : 1]$, $b = [0 : 1 : 0]$, about which we can take a local coordinate $t$ such that nearby points are to leading order $[2t^3 : t : 1]$ and $[2t^3 : 1 : t]$ respectively. The point $V_5$ desingularises to two points on $\overline{\mathcal{B}}$, which we denote

$$c = [1 : 0 : 0]_2, \quad d = [1 : 0 : 0]_1,$$

which in the vicinity of these points have local behaviour $[1 : t : t^4]$ and $[1 : t^4 : t]$ respectively.

**Corollary 2.3.4.** *On* $\overline{\mathcal{B}}$ *we have the divisors*

$$(x) = 3a + 2b - 4c - d, \quad (y) = a - b - 3c + 3d, \quad R_x = 2a + b + 3c + \sum_i r_i,$$

*where* $R_x$ *is the ramification divisor corresponding to the map* $x : \overline{\mathcal{B}} \to \mathbb{P}^1$ *and the* $r_i$ *are the roots of the polynomial* $256x^{10} - 837x^5 + 3456$ *appearing in the resultant.*

**Lemma 2.3.5** ([BN12], p. 18). *The genus of Bring's curve is* $g = 4$ *and we have the ordered basis of (unnormalised) differentials on* $\overline{\mathcal{B}}$

$$v_1 = \frac{(y^3 - x)\mathrm{d}x}{\partial_y f(x, y)}, \quad v_2 = \frac{(y^2 x - 1)\mathrm{d}x}{\partial_y f(x, y)}, \quad v_3 = \frac{(y - x^2)\mathrm{d}x}{\partial_y f(x, y)}, \quad v_4 = \frac{y(x^2 - y)\mathrm{d}x}{\partial_y f(x, y)}.$$

One can compute such differentials algorithmically from the HC model (as Braden and Northover do), and thus find the genus to be four this way, but another route which will be conceptually helpful will be to note that in the $\mathbb{P}^4$-model, using $H_1$ to eliminate one of the coordinates, Bring's curve is written as the intersection of a quadric and cubic in $\mathbb{P}^3$. Such a complete intersection is known, as mentioned in Example 2.1.61, to be the canonical embedding of a genus-4 curve. Moreover, because the quadric surface containing $\mathcal{B}$ is unique, it must be the case that the differential $v_i$ satisfy a quadratic relation and that the corresponding quadric is nonsingular hence isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ by §2.1.3. Specifically, here one finds the quadric is given by

$$\mathcal{Q} : v_1 v_2 + v_3 v_4 = 0, \tag{2.15}$$

with the map $\mathcal{Q} \to \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$\begin{aligned} \varphi : \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathcal{Q} \subset \mathbb{P}^3, \\ ([u : v], [z : w]) &\mapsto [uz : vw : vz : -uw] := [v_1 : v_2 : v_3 : v_4]. \end{aligned} \tag{2.16}$$

The inverse map $\varphi^{-1} : \mathcal{Q} \to \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$\varphi^{-1}([v_1 : v_2 : v_3 : v_4]) = \begin{cases} ([v_1 : v_3], [v_3 : v_2]), & \{v_1, v_3\} \neq \{0\} \neq \{v_2, v_3\}, \\ ([-v_4 : v_2], [v_3 : v_2]), & \{v_1, v_3\} = \{0\} \neq \{v_2, v_3\}, \\ ([v_1 : v_3], [v_1 : -v_4]), & \{v_1, v_3\} \neq \{0\} = \{v_2, v_3\}, \\ ([-v_4 : v_2], [v_1 : -v_4]), & \{v_1, v_3\} = \{0\} = \{v_2, v_3\}. \end{cases}$$

**Proposition 2.3.6** ([BN12])**.** *The canonical divisor class on Bring's curve $\overline{\mathcal{B}}$ is $[\mathcal{K}_{\overline{\mathcal{B}}}] = [a + 2b + 3c]$.*

*Proof.* This may be shown analytically or (as given in the accompanying notebook) entirely using computer algebra. To show this analytically we start by considering the numerator of $v_3$. We can calculate

$$\mathrm{Res}_y(f(x, y), y - x^2) = -x(x^5 - 1)^2.$$

Thus to determine the divisor we need to consider the finite points $x = 0$ (i.e. $a$, $b$), $x = \zeta^k$ and the infinite points $c$, $d$. We have a simple zero at $a$, a simple pole at $b$, a pole of order 8 at $c$ and a pole of order 2 at $d$, which can be worked out by using the local expansions defining $a, b, c, d$, e.g at $a$ we have

$$y - x^2 \sim t - (2t^3)^2,$$

so our simple zero. As noted previously the singular points $V_k$ desingularise to two points $P_k$, $P'_k$; at these points the function $y - x^2$ has a simple zero. Thus we obtain

$$\begin{aligned} (y - x^2) &= a - b - 8c - 2d + \sum_{k=0}^{4} [P_k + P'_k], \\ (\partial_y f) &= 2a - 2b - 16c - 4d + \sum_k [P_k + P'_k] + \sum_i r_i, \end{aligned}$$

both of which can be checked to be degree 0. Finally, use the ramification divisor $R_x = 2a + b + 3c + \sum_i r_i$ to get

$$(\mathrm{d}x) = -2(4c + d) + R_x \Rightarrow (v_3) = (y - x^2) + (\mathrm{d}x) - (\partial_y f) = a + 2b + 3c.$$

□

We now establish the main result of this subsection.

**Proposition 2.3.7.** *The HC model (Equation 2.14) is a model of Bring's curve (Equation 2.13).*

*Proof.* We will prove this by explicitly constructing the birational transform. To do so we make use of the proof of [Dye95, Theorem 3], where the author considers a particular Clebsch hexagon[20], constructs a pencil of plane sextics from this, and

[20] For the purposes of this definition a hexagon is a set of 6 points in $\mathbb{P}^2$, no three of which are collinear, called the vertices. A **Brianchon point** of a hexagon is a vertex point through

finds Bring's curve as the canonical model of a distinguished point in this pencil. By assuming that the HC model is already the distinguished curve in a pencil, we can construct the birational map. Note this is a fundamentally different approach to the that originally taken with the HC model, which was derived from considerations of the modular theory of the curve.

Explicitly, Dye introduces $j$ as a solution to $j^2 - j - 1 = 0$ and then defines the pencil of curves[21] $S_\lambda = S + \lambda \left|\mathcal{C}\right|^3$ where

$$\begin{aligned}
&S(x,y,z) = (x+jy)^6 + (x-jy)^6 + (y+jz)^6 + (y-jz)^6 + (z+jx)^6 + (z-jx)^6, \\
&\mathcal{C}(x,y,z) = x^2 + y^2 + z^2.
\end{aligned} \tag{2.17}$$

Next Dye considers the Clebsch hexagon $H$ with vertices

$$(1, \pm j, 0), \quad (0, 1, \pm j), \quad (\pm j, 0, 1),$$

for which the corresponding 10 Brianchon points are at

$$(\pm j^2, 1, 0), \quad (0, \pm j^2, 1), \quad (1, 0, \pm j^2), \quad (1, \pm 1, \pm 1).$$

Dye shows that there is a unique member of the pencil, which he calls $\Gamma$, that contains the vertices. Moreover, $\Gamma$ has the vertices and only the vertices as double points.

To get a canonical model for $\Gamma$, which Dye shows has genus 4, we now need that a generic cubic surface in $\mathbb{P}^3$ is birational to the vanishing condition for a system of plane cubics through 6 base points in generic positions, as seen in §2.1.3. We apply this taking these six points to be the vertices of the hexagon $H$. We shall see a posteriori that these are indeed in general position.

We shall now proceed as follows.

1. We construct the cubic which corresponds to the vanishing condition for the cubics through the 6 points $V_k$ on the HC model.

2. We verify that the constructed map is birational from $\mathbb{P}^2$ to the cubic.

3. Motivated by the geometry of Bring's curve, we find a collineation which maps the found cubic to the standard Clebsch surface (defined below) in $\mathbb{P}^4$.

4. We verify that restricting to the HC model in $\mathbb{P}^2$ corresponds to restricting to Bring's curve in the Clebsch surface.

5. We give examples of this new birational map from the (normalisation) of the HC model to Bring's curve on some particular points.

We do this now.

which 3 edges (the lines joining two distinct vertices) pass that is not a vertex. A **Clebsch hexagon** is a hexagon with 10 Brianchon points [Dye91].

[21] We are using the coordinates $[x:y:z]$ here, distinct from $[X:Y:Z]$, to highlight that these are not those of the HC model.

1. For our purposes, the six points we want to intersect with are the vertices of the Clebsch hexagon, which are the double points of the exceptional curve $\Gamma$. Assuming the HC model gives such an exceptional curve, we take the points $V_k$ identified in Lemma 2.3.3. Hence, if we write a generic cubic in $X, Y, Z$ as

   $$a_0X^3+a_1X^2Y+a_2X^2Z+a_3XY^2+a_4XYZ+a_5XZ^2+a_6Y^3+a_7Y^2Z+a_8YZ^2+a_9Z^3$$

   the equations on the coefficients we get (coming from intersecting with $V_k$ $(k = 0, \dots, 4)$ and $V_5$ respectively) are

   $$a_0\zeta^{3k} + a_1\zeta^{4k} + a_2\zeta^{2k} + a_3 + a_4\zeta^{3k} + a_5\zeta^k + a_6\zeta^k + a_7\zeta^{4k} + a_8\zeta^{2k} + a_9 = 0,$$
   $$a_0 = 0.$$

   Setting the coefficients of $\zeta^{nk}$ to be zero gives us the 3-parameter family of cubics

   $$\begin{aligned} 0 &= aX^2Y + bX^2Z + cXY^2 + dXZ^2 - dY^3 - aY^2Z - bYZ^2 - cZ^3, \\ &:= aL_1 + bL_2 + cL_3 + dL_4. \end{aligned}$$

   Precisely because the resulting family of cubics is 3-parameter, we know that the six points from $H$ must have been sufficiently general.

   Comparing the coefficients we see that our map into $\mathbb{P}^3$ is (essentially) the canonical embedding. as

   $$[v_1 : v_2 : v_3 : v_4] = [-L_4 : L_3 : -L_2 : L_1].$$

   One can check, using for example Gröbner bases, that the $L_a$ satisfy the equation
   $$L_2L_4^2 - L_1^2L_4 - L_1L_3^2 + L_2^2L_3 = 0.$$
   This is the cubic we call $F$.

2. One can use the package `Cremona` in Macaulay2 [Sta18, GS] to check that the map $\Psi$ is birational. Note one needs to make sure that the range is chosen such that the map is explicitly surjective, not just use the implicit knowledge that the map is surjective onto its image. Doing so and asking for the inverse map gives

   $$[X : Y : Z] = [L_2^2 - L_1L_3 : L_1L_4 : L_2L_4].$$

   For example, we can see

   $$\begin{aligned} \frac{L_2^2 - L_1L_3}{L_1L_4} &= \frac{(X^2Z - YZ^2)^2 - (X^2Y - Y^2Z)(XY^2 - Z^3)}{(X^2Y - Y^2Z)(XZ^2 - Y^3)}, \\ &= \frac{X^4Z^2 - X^2YZ^3 - X^3Y^3 + XY^4Z}{X^3YZ^2 - XY^2Z^3 - X^2Y^4 + Y^5Z} = \frac{X}{Y}, \end{aligned}$$

and
$$\frac{L_1L_4}{L_2L_4} = \frac{L_1}{L_2} = \frac{X^2Y - Y^2Z}{X^2Z - YZ^2} = \frac{(X^2Z - YZ^2)(Y/Z)}{X^2Z - YZ^2} = \frac{Y}{Z}.$$

3. We now have a cubic surface in $\mathbb{P}^3$ corresponding to the system of curves intersecting the $V_i$. From [Hir86, p. 198-201] we know there exists a coordinate system in which this cubic can be written as the subset of $\mathbb{P}^4$

$$H_3 = 0 = H_1$$

(this is called 'the' Clebsch surface). To look for such a coordinate system, we write this surface in $\mathbb{P}^3$ as

$$\begin{aligned} 0 = {} & x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + x_1^2x_4 + x_2^2x_4 + x_3^2x_4 + x_1x_4^2 \\ & + x_2x_4^2 + x_3x_4^2 + 2x_1x_2x_3 + 2x_1x_2x_4 + 2x_1x_3x_4 + 2x_2x_3x_4. \end{aligned}$$

From here, one can use the fact that the 10 Eckardt points of the cubic in $L_a$ form the 10 vertices of a pentahedron [Hir86, p. 199], and that in the coordinate system of the Clebsch surface in $\mathbb{P}^4$ these lie at the permutations of $[1 : -1 : 0 : 0 : 0]$ [Edg78]. This gives us a possible way of spotting the transform if we can calculate the Eckardt points of $F$. To do this, we use the identification from [Dye95], that the 3 lines in $F$ intersecting to give an Eckardt point come from the 3 edges of $H$ intersecting at a Brianchon point. To this end we find the images of the $V_iV_j$ for which we give a generating set of the ideal corresponding to the line, for example

$$\begin{aligned} \Psi(V_0V_5) : & \left\langle L_4 - L_3, L_2 - L_1 \right\rangle, \\ \Psi(V_1V_2) : & \big\langle L_4 + (\zeta^2 + \zeta + 1)L_2 + (\zeta^4 - \zeta^2 - \zeta)L_1, \\ & \quad L_3 + (\zeta^3 + 2\zeta^2 + \zeta)L_2 + (\zeta^3 + \zeta^2 + \zeta)L_1 \big\rangle, \\ \Psi(V_3V_4) : & \big\langle L_4 + (-\zeta^2 - \zeta)L_2 + (\zeta^3 + 2\zeta^2 + \zeta)L_1, \\ & \quad L_3 + (\zeta^3 - \zeta - 1)L_2 + (\zeta^2 + \zeta + 1)L_1 \big\rangle. \end{aligned}$$

Then

$$\begin{aligned} \Psi(V_0V_5) \cap \Psi(V_1V_2) \cap \Psi(V_3V_4) &= [-\zeta^3 - \zeta^2 - 1 : -\zeta^3 - \zeta^2 - 1 : 1 : 1], \\ &= \Psi(V_0V_5 \cap V_1V_2 \cap V_3V_4). \end{aligned}$$

One can do likewise to find the other Eckardt points.

Armed now with the knowledge of the Eckardt points, we can find appropriate projective transforms $A$ that biject the sets of Eckardt points by acting $(x_1, x_2, x_3, x_4)^T = A(L_1, L_2, L_3, L_4)^T$, for example

$$A = \begin{pmatrix} \zeta^3 & -1 & -\zeta^2 & \zeta \\ 1 & -\zeta^3 & -\zeta & \zeta^2 \\ \zeta^2 & -\zeta & -1 & \zeta^3 \\ \zeta & -\zeta^2 & -\zeta^3 & 1 \end{pmatrix}.$$

We can then use again Macaulay2 to check that this transform then gives us the correct cubic in $\mathbb{P}^3$.

4. Moreover, if we now restrict to the HC model in the system of curves that intersect the $V_i$, that is we impose the condition that $F(X, Y, Z) = 0$, we get the final degree-2 polynomial on the $x_i$ ($H_2 = 0$). This quadric is in fact the Schur quadric corresponding to a distinguished double-six of lines that necessarily exist on a cubic surface constructed as above [Dye95]. An equivalent rational map is given in [Dol12, p. 557], but the inverse is not provided.

5. To see this working, let us consider $c$ and $d$, and verify that these are desingularised on the smooth canonical embedding. Taking $[X : Y : Z] = [1 : t : t^4]$ we get

$$[L_1 : L_2 : L_3 : L_4] = [1 : t^3 : t + t^6 : -t^2],$$

and so taking the limit we get $[1 : 0 : 0]_2 \mapsto [1 : 0 : 0 : 0]$ in $L$ coordinates. Acting with $A$, we get the point in $\mathbb{P}^4$ given by

$$[x_1 : x_2 : x_3 : x_4 : x_5] = [\zeta^3 : 1 : \zeta^2 : \zeta : \zeta^4].$$

Repeating the process taking $[X : Y : Z] = [1 : t^4 : t]$ gives $[1 : 0 : 0]_1 \mapsto [0 : 1 : 0 : 0]$ in $L$ coordinates, which is equivalently

$$[x_1 : x_2 : x_3 : x_4 : x_5] = [1 : \zeta^3 : \zeta : \zeta^2 : \zeta^4].$$

This makes sense, as the change $Y \leftrightarrow Z$ corresponds to $L_1 \leftrightarrow L_2$, $L_3 \leftrightarrow L_4$.

Moreover, to clarify our last point on the imposition of $H_2 = 0$, we consider the point $[X : Y : Z] = [0 : 1 : 1]$ which does not lie on the HC model of the curve. Under our birational map this corresponds to the point

$$\begin{aligned} &[L_1 : L_2 : L_3 : L_4] = [1 : 1 : 1 : 1], \\ \Rightarrow &[x_1 : x_2 : x_3 : x_4 : x_5] = [2 - \sqrt{5} : -2 + \sqrt{5} : -1 : 1 : 0]. \end{aligned}$$

This point does lie on the Clebsch diagonal surface given by $H_1 = 0 = H_3$, but does not satisfy $H_2 = 0$.

□

Note that in the above proof of the equivalence of the Riemann surfaces, the birational map we constructed was defined over $\mathbb{Q}[\zeta]$. As such, to equate the two algebraic curves we need to be working over a field containing $\mathbb{Q}[\zeta]$. We will later see when looking at quotients of Bring's curve that it is insufficient to work over $\mathbb{Q}$. Indeed, note that over $\mathbb{Q}$ there are no solutions to the equations defining Bring's curve in the $\mathbb{P}^4$-model because the quadric term forces each $x_i$ to be 0.

### The Period Matrix

In this short subsection we introduce the notation of the Riemann matrix of Bring's curve, indicating how it has been calculated multiple ways in the past.

**Theorem 2.3.8** ([RR92, GAR00, BDH22]). *Define the matrices $M$, $M_S$ by*

$$M = \begin{pmatrix} 4 & 1 & -1 & 1 \\ 1 & 4 & 1 & -1 \\ -1 & 1 & 4 & 1 \\ 1 & -1 & 1 & 4 \end{pmatrix}, \quad M_S = \begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}.$$

*A Riemann matrix for Bring's curve is given by $\tau_{\mathcal{B}} = \tau_0 M$, where the complex number $\tau_0$ is given by the conditions*

$$j(\tau_0) = -\frac{29^3 \times 5}{2^5}, \quad j(5\tau_0) = -\frac{25}{2}, \tag{2.18}$$

*with $j$ the elliptic-$j$ function on the upper half plane (Equation 2.5). Then $\tau_0 = -0.5 + 0.186676i\,(6.d.p)$. Further, there exists a symplectic transformation such that $\tau_{\mathcal{B}} = \tau_0 M_S$.*

*Proof.* This is known, and has been proven in multiple different ways. We summarise these and highlight a numerical approach.

1. This was first shown in [RR92], but the equations for $j(\tau_0)$, $j(5\tau_0)$ were incorrectly swapped as first noted[22] in [BN12].

2. It was calculated in [BN12] going via the HC model.

3. In [Web05], by viewing the curve as a cyclic cover of $\mathbb{P}^1$, a period matrix is constructed with respect to a homology basis with intersection matrix

$$I_W = \begin{pmatrix} 0 & 1 & 1 & -1 & -1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \end{pmatrix},$$

that is given as $\Omega = (A, B)$ where the columns[23] $A_k, B_k$ are

$$A_k = \begin{pmatrix} \zeta^k(1-\zeta^2) \\ \zeta^{2k+3/2}(1-\zeta^4)(l\Phi - 1) \\ \zeta^{4k+3}(1-\zeta^3)\Phi(1-l) \\ \zeta^{3k+2}(1-\zeta)l \end{pmatrix}, \quad B_k = \begin{pmatrix} \zeta^{2k+3/2}(1-\zeta^4)(l\Phi - 1) \\ \zeta^{4k+3}(1-\zeta^3)\Phi(1-l) \\ \zeta^{3k+2}(1-\zeta)l \\ \zeta^k(1-\zeta^2) \end{pmatrix},$$

[22] Note the approximation of $\tau_0$ in [BN12] contains a typographic error.

[23] Note [Web05, Lemma 5.1] uses $\zeta = \exp(2\pi i/10)$, whereas we take $\zeta = \exp(2\pi i/5)$. As such, the columns of $\Omega$ look different to those of Weber in terms of the exponent of $\zeta$, but they do indeed agree.

for $k = 0, \ldots, 3$, $\Phi = \frac{1}{2}(1+\sqrt{5})$, and $l = \left|\frac{I(-1,0)}{I(-\infty,-1)}\right| \approx 0.848641$ where

$$I(a,b) = \int_a^b (t-1)^{-1/5} t^{-3/5} (t+1)^{-4/5} dt.$$

One can find (using Sage) the matrix

$$C = \begin{pmatrix} 0 & 1 & -2 & 1 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & -2 & 2 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

such that $C^T I_W C = J_g = \begin{pmatrix} 0 & \mathrm{Id}_g \\ -\mathrm{Id}_g & 0 \end{pmatrix}$, i.e. $C$ transforms the homology basis to one which is canonical. This means we get a Riemann matrix $\tau_W = (AC)^{-1}(BC) = C^{-1}A^{-1}BC$, and one can numerically find that the matrix

$$R = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix} = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 0 & -1 & -2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 & -1 & -1 & 1 & 2 \\ 1 & 0 & 1 & -1 & 1 & 2 & 1 & -1 \end{array}\right)$$

relates $\tau_W$ to $\tau_{\mathcal{B}}$.

4. For a numerical approach, we may consider the Riemann matrix calculated by SageMath, and find a transform to $\tau_{\mathcal{B}}$, as is done in the corresponding notebook.

□

Riera and Rodriguez (hereafter abbreviated to R&R) constructed the constraints on $\tau_0$ via $j$-invariants by considering the quotients by group actions of the curve to elliptic curves [RR92]. They show that these constraints give a unique value of $\tau_0$ modulo $\Gamma_0(5)$, or equivalently in the language of [GAR00] that $\tau_0$ gives a distinguished point in the modular curve $X_0(5)$. As we will see later in §2.3.3, there are additional quotients to elliptic curves not considered by R&R, which have $j$-invariants

$$j(15\tau_0) = -\frac{5^2 \times 241^3}{2^3}, \quad j(3\tau_0) = \frac{5 \times 211^3}{2^{15}}. \tag{2.19}$$

The latter is identified in relation to Bring's curve in [Ser08, Exercise 8.3.2c]. Serre says that this curve (50H) and 50E (using the naming convention of [BK75, Table 1]) with $j(5\tau_0)$ are 15-isogenous over $\mathbb{Q}$. This isogeny is not too mysterious when we think on the level of the corresponding elliptic curves over $\mathbb{C}$ as $\mathbb{C}/\langle 1, 3\tau_0\rangle$ and $\mathbb{C}/\langle 1, 5\tau_0\rangle$, wherein the isogeny $\mathbb{C}/\langle 1, 3\tau_0\rangle \to \mathbb{C}/\langle 1, 5\tau_0\rangle$ is the composition of the quotients by the maps $z \mapsto z + \tau_0$ and $z \mapsto z + 1/5$ respectively. There is a complete $\mathbb{Q}$-isogeny class of elliptic curves of order four with periods $\tau_0, 3\tau_0, 5\tau_0$ and $15\tau_0$ [BK75, Table 1].

It follows from Equation 2.18 and Proposition 2.1.43 that $\tau_0$ is transcendental. In Weber's form of the period matrix, the transcendentality comes about because of the constant $l$, which is a ratio of Schwarz-Christoffel integrals which arise from the map of a Euclidean quadrilateral to a hyperbolic quadrilateral.

### The Automorphism Group

This subsection is devoted to the following result.

**Proposition 2.3.9** ([Wim95b]). $\mathrm{Aut}(\mathcal{B}) = S_5$. *This is the maximal possible automorphism group for a genus 4 surface, and Bring's curve is the only curve to achieve it.*

Wiman's proof was enumerative and produced equations for the curves with a given automorphism group, as we have already seen in §2.2.2. One can certainly see from the form of the equations determining the $\mathbb{P}^4$-model of the curve that $S_5 \leq \mathrm{Aut}(\mathcal{B})$, acting as permutations of the $x_i$ coordinates fixing the subspace $\sum_i x_i = 0$. Moreover, [Hir86, p. 201] gives that $\mathrm{Aut}(\mathcal{B}) \leq S_5$ from the perspective of the Clebsch surface, thus fixing the automorphism group of Bring's curve. Other different proofs of Wiman's result may be found in [KK90, MSSV02].

The uniqueness of the quadric $\mathcal{Q}$ in which Bring's curve lays means that automorphisms of the curve become automorphisms of $\mathcal{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$, hence we know $\mathrm{Aut}(\mathcal{B})$ is isomorphic to a finite subgroup of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) = C_2 \ltimes (\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}))$. Moreover, by Example 2.1.79 we know $\mathrm{Aut}(\mathcal{B})$ is isomorphic to a finite subgroup of $\mathrm{PGL}_4(\mathbb{C})$. What is nontrivial is the following fact.

**Theorem 2.3.10** ([Dye95]). *The $A_5$ subgroup of $\mathrm{Aut}(\mathcal{B})$ can be realised as a group of collineations in the HC model, that is, can be realised as a subgroup of $\mathrm{PGL}_3(\mathbb{C})$.*

*Proof.* We will be explicit about the construction here as this will be profitable later but the result is classical; [BN12] explains how this representation follows from [Dye95], and more details may be found in [CS16]. The group $A_5$ has two inequivalent irreducible 3-dimensional representations, one of which is given by $\langle R, S\rangle$ where

$$R := \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 & 2 \\ 1 & \zeta^2+\zeta^{-2} & \zeta+\zeta^{-1} \\ 1 & \zeta+\zeta^{-1} & \zeta^2+\zeta^{-2} \end{pmatrix},\ R^2 = I,\ S := \begin{pmatrix} 1 & & \\ & \zeta & \\ & & \zeta^{-1} \end{pmatrix},\ S^5 = I,\ (RS)^3 = I. \tag{2.20}$$

Note the other inequivalent irreducible 3-dimensional representation comes from replacing $\zeta$ in Equation 2.20 with $\zeta^2$ or $\zeta^3$. The invariants of the representation $\langle R, S\rangle$ (when acting on $(X,Y,Z)^T$ via left multiplication) are

$$\begin{aligned}
i_2 &= \left(\frac{X}{2}\right)^2 + \sum_{k=0}^{4}\left(\frac{\frac{X}{2} + Y\zeta^k + Z\zeta^{-k}}{\sqrt{5}}\right)^2 = \frac{1}{2}X^2 + 2YZ,\\
i_6 &= \left(\frac{X}{2}\right)^6 + \sum_{k=0}^{4}\left(\frac{\frac{X}{2} + Y\zeta^k + Z\zeta^{-k}}{\sqrt{5}}\right)^6,\\
i_{10} &= \left(\frac{X}{2}\right)^{10} + \sum_{k=0}^{4}\left(\frac{\frac{X}{2} + Y\zeta^k + Z\zeta^{-k}}{\sqrt{5}}\right)^{10},\\
i_{15} &= \left|\frac{\partial\{i_2, i_6, i_{10}\}}{\partial\{X,Y,Z\}}\right|.
\end{aligned}$$

There is a polynomial relation between $i_{15}^2$ and $i_2$, $i_6$, $i_{10}$. In particular the vanishing of $i_6 - \lambda i_2^3$ gives us Dye's 1-parameter family of $A_5$-invariant sextics in $\mathbb{P}^2$; this pencil[24] appears to have first been studied by Winger [Win25]. This pencil yields curves of genus 10 for generic $\lambda$ [Dye95]. The special value[25] of $\lambda = 13/100$ yields a genus-4 curve, namely Bring's curve, as we have that

$$\frac{1}{12}\left(100 i_6 - 13 i_2^3\right) = X(Y^5 + Z^5) + (XYZ)^2 - X^4YZ - 2(YZ)^3 = F(X,Y,Z).$$

□

To complete our picture, we use the following result.

**Proposition 2.3.11.** *The map*

$$U : (x,y) \mapsto \left(-\frac{y^5 + x^3y - 3xy^2 + 1}{(y - x^2)(y^3 - x)}, -\frac{y^2x - 1}{y^3 - x}\right)$$

*is an automorphism of the HC model, and which together with $R$ and $S$ generates the entire automorphism group $S_5$.*

*Proof.* The proof that $U$ is an automorphism is simple algebraic verification. In order to find this map, we adapted the methods of [BSZ19]. To see that $\langle R, S, U\rangle \cong S_5$, note that $U$ is of order 4, for example by checking that it has the orbit $a \mapsto c \mapsto b \mapsto d \mapsto a$. As such, $U$ corresponds to an odd permutation under the isomorphism $\mathrm{Aut}(\mathcal{B}) \cong S_5$, and a single odd permutation and all of $A_5$ together generate $S_5$. □

By fixing a map from (the normalisation of) the HC model to the $\mathbb{P}^4$-model as we did in the proof of Proposition 2.3.7, we have fixed an isomorphism from

[24] A variation on this pencil has been used to explain why Bring's curve is uniquely defined as an $A_5$-invariant curve of genus 4, but there is a 1-parameter family of 4-dimensional $A_5$-invariant principally polarised abelian varieties, deforming $\mathrm{Jac}(\mathcal{B})$ [Zi21, LZ23]. The paper [Mel03] gives further interesting representation theoretic perspectives on Bring's curve.

[25] For $\lambda = 17/180$ the curve is of genus 0; for $\lambda = 1/10$ the curve is reducible.

the automorphism group of the curve (in the HC model) to $S_5$, which we shall denote $\psi : \langle R, S, U\rangle \to S_5$. For example, it is simple to verify that we have $U^2([X:Y:Z]) = [X:Z:Y]$. We see $U^2([L_1:L_2:L_3:L_4]) = [L_2:L_1:L_4:L_3]$, and so $U^2([x_1:x_2:x_3:x_4:x_5]) = [x_2:x_1:x_4:x_3:x_5]$, that is $(12)(34)$. Through similar calculation we can find

$$\psi(R) = (13)(24), \quad \psi(S) = (13425), \quad \psi(U) = (1324).$$

There are myriad choices that can be made in constructing the birational transformation (such as the labelling of the coordinates in $\mathbb{P}^3$ and the ordering of the rows of $A$, or indeed composing with any automorphism of the $\mathbb{P}^4$-model), and changing these would give different isomorphisms to $S_5$.

As we have previously described in this section, the uniqueness of the quadric $\mathcal{Q}$ whose intersection with a cubic yields the canonical model of the curve leads to an isomorphism of the automorphism group of the curve and a subgroup of $\mathrm{Aut}(\mathcal{Q}) = C_2 \ltimes (\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C}))$ which we now write down. Let $([u:v],[z:w])$ be the coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, then using the birational map constructed in Proposition 2.3.7 one can conjugate the standard irreducible 4-dimensional representation of $S_5$ on $[x_1:x_2:x_3:x_4]$ to an action on $[v_1:v_2:v_3:v_4]$. The resulting action of $(12)$ is (projectively)

$$\begin{pmatrix} v \\ u \end{pmatrix} \mapsto \begin{pmatrix} j & -1 \\ -1 & -j \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} := A \begin{pmatrix} w \\ z \end{pmatrix}, \quad \begin{pmatrix} w \\ z \end{pmatrix} \mapsto \begin{pmatrix} -1 & j-1 \\ j-1 & 1 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} := B \begin{pmatrix} v \\ u \end{pmatrix}, \tag{2.21}$$

where $j = -\zeta^3 - \zeta^2$ satisfies $j^2 - j - 1 = 0$; that is, it is the $j$ defined by Dye. One can show that $(34)$ has the same action, where the other root $j' = \zeta^3 + \zeta^2 + 1$ is taken. One can check that $AB = 1 \in \mathrm{PGL}_2(\mathbb{C})$, consistent with the fact that $(12)^2 = 1 \in S_5$. Note the transposition interchanges the two copies of $\mathbb{P}^1$, which is the action of the semi-direct product with $C_2$. Combining the two transforms one gets that $(12)(34)$ acts as

$$[u:v] \mapsto [-v:u], \quad [z:w] \mapsto [-w:z].$$

This fixes each copy of $\mathbb{P}^1$ and acts via a rotation on each. Moreover, we can calculate the action of $(145)$ on the copies to be

$$\begin{pmatrix} v \\ u \end{pmatrix} \mapsto \begin{pmatrix} \zeta & -\zeta^3 - \zeta \\ -\zeta^2 - 1 & -\zeta^2 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}, \quad \begin{pmatrix} w \\ z \end{pmatrix} \mapsto \begin{pmatrix} \zeta^3 + 1 & \zeta^2 \\ \zeta^4 & -\zeta^3 - \zeta \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix}. \tag{2.22}$$

As $\langle (12)(34), (145)\rangle \cong A_5$, we discover that the action of $A_5$ does not interchange the two copies of $\mathbb{P}^1$, but odd-parity elements in $S_5$ do. Here $A_5$ is given by the diagonal embedding in $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$.

### 2.3.2 Geometric Points

We now have a good understanding of how the automorphism group acts on the curve, and so before looking at quotient Riemann surfaces in §2.3.3 we want to first consider orbits of points that have geometric significance on the curve. These points will have important connections to the function theory of the curve; they are also related to physical aspects of Euclidean realisations (i.e. immersions in Euclidean 3-space) of the curve. Such orbits are characterised by the following result from Wiman.

**Proposition 2.3.12** ([Wim95b])**.** *There are only 3 orbits of points of size less than* 120 *on* $\mathcal{B}$ *and these have sizes 24, 30, and 60 respectively.*

*Proof.* By Lemma 2.1.80 we know the stabiliser of a point must be a cyclic group. The cyclic subgroups of $S_5$ are $C_2$, $C_3$, $C_4$, $C_5$, $C_6$; the corresponding orbits would thus be of (respective) sizes 60, 40, 30, 24 or 20. To obtain Wiman's result we must show that $C_3$ does not fix a point on Bring's curve. Using Riemann-Hurwitz we have

$$3 = 120(g-1) + 30a_{60} + 40a_{40} + 45a_{30} + 48a_{24} + 50a_{20},$$

where $a_k \geq 0$ are the number of $S_5$ orbits of size $k$. There are no solutions to this for $g \geq 1$. For $g = 0$ we have the unique solution $1 = a_{60} = a_{30} = a_{24}$. This shows there can be no points with $C_3$ stabiliser. □

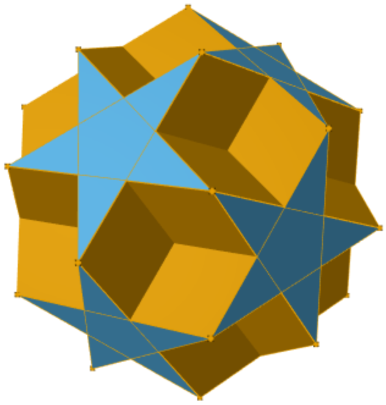
**Remark 2.3.13.** *What we have shown in Proposition 2.3.12 is that the signature of the action of $S_5$ on Bring's curve is* $(0; 2, 4, 5)$*, a fact we have seen previously in Table 2.3.*

These points and corresponding geometric structures are important when relating Clebsch's diagonal surface to Hilbert modular surfaces [Hir76, Bel04]. We identify these orbits as the geometric points[26] on the curve as defined in [SW97]. Explicitly they are the vertices, face-centres, and edge-centres of the universal map $\{5,4\}_6$ - the Petrie polygon (as defined in [CM80, §8.6]) of degree 6 coming from the tiling of the hyperbolic disk by pentagons, where four pentagons meet at a vertex [Sin88]. It is noted in [Web05] that this tessellation has a Euclidean realisation as a dodecadodecahedron (Figure 2.2a). This has 30 vertices, 60 edges, and 24 faces, giving genus

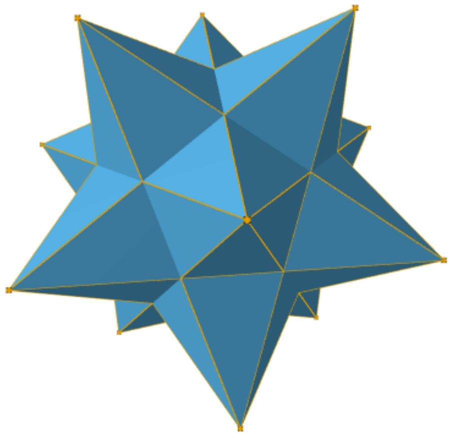
$$g = 1 - \frac{V - E + F}{2} = 4,$$

as we expect. In a recent paper, this connection to the dodecadodecahedron was used to identify Bring's curve as the moduli space of equilateral plane pentagons up to the action of the conformal group [Ram22]. The (small) stellated dodecahedron (Figure 2.2b) also has genus 4 (having $V = 12 = F$, $E = 30$), coming from the tessellation $\{5/2, 5\} \cong \{5, 5|3\}$, which can be interpreted as adding

[26]Being a geometric point on a curve is a priori not an interesting statement unless we know the corresponding map is regular, as we have in this case.



(a) Dodecadodecahedron



(b) Small Stellated Dodecahedron

**Fig. 2.2** Geometric realisations

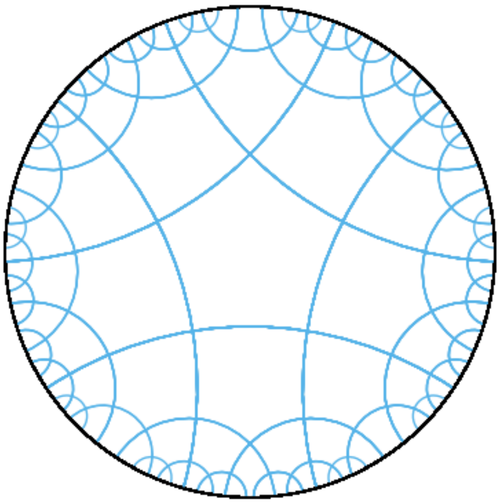
three 'holes' to the $\{5,5\}$ tessellation [CM80, §8.5]. This $\{5,5|3\}$ tessellation has automorphism group $C_2 \times A_5$, which is an index-2 subgroup of $C_2 \times S_5$, the automorphism group of $\{5,4\}_6$. This is due to the map $D_1$ defined in [Hen13, §3.1], which maps the dodecadodecahedron to the small stellated dodecahedron. We include both these tessellations in Figure 2.3 below. Klein connected the small stellated dodecahedron to Bring's curve through a degree-3 covering $\mathcal{B} \to \mathbb{P}^1$ constructed from the hyperbolic triangles giving the tessellation [Web05], and we will see this map later in §2.3.4 in a different context.

#### Weierstrass Points

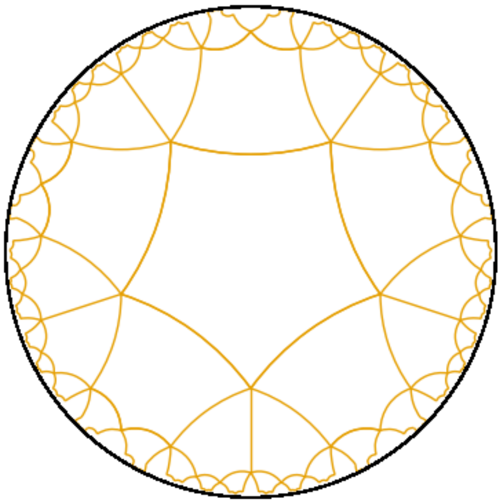
Recall we defined Weierstrass points in §2.1.2. In [SW97] the Weierstrass points are shown (implicitly) to correspond to the edge-centres of the universal map (defined within). This shows that the order-2 rotation that permutes the vertices and face-centres adjacent to an edge-centre will preserve the corresponding Weierstrass point. They can also be interpreted geometrically as pairwise-symmetric distributed along the edges of the small stellated dodecahedron. As Weber identifies the order-3 symmetry as the rotation about the axis through opposite vertices of the *unstellated* dodecahedron, one might wonder from this picture whether Weierstrass points are fixed points of the order-3 permutations in the group. This turns out not be the case and a counting argument helps elucidate: the small stellated dodecahedron has 12 faces, which in turn means we want to have $\frac{60}{12} = 5$ Weierstrass points per face. Hence where three faces overlap there must be three Weierstrass points 'stacked' there, which are invisibly permuted by the action.

Having now identified the Weierstrass points as some of the geometric points, we give a concrete result about what the Weierstrass points are.

**Proposition 2.3.14** ([Edg78]). *Bring's curve has 60 Weierstrass points, on*



(a) $\{5,4\}$ tessellation



(b) $\{5,5\}$ tessellation

**Fig. 2.3** Hyperbolic tilings

*which* Aut($\mathcal{B}$) *acts transitively. Letting* $\{\alpha, \beta, \gamma\}$ *be the roots of the cubic* $x^3 + 2x^2 + 3x + 4$*, these are given in the* $\mathbb{P}^4$*-model by* $W_{ijk}$ *where, for example,*

$$W_{345} = [1 : 1 : \alpha : \beta : \gamma].$$

*Proof.* Edge, working in the $\mathbb{P}^4$-model, identifies the Weierstrass points with the 60 intersections of the curve with the 10 planes $\Pi_{ij} = \{x_i = x_j\}$. To do this, Edge quotes [Wim95b] to show that these intersection points are stalls, i.e. inflection points of certain linear series, and for the canonical embedding these are exactly the Weierstrass points by Proposition 2.1.62. Simple algebra then gives the exact expression we write down. This viewpoint makes it clear that the Weierstrass points at the intersection with $\Pi_{ij}$ are preserved by the transposition[27] $(ij)$ only, so have orbits of size $\frac{120}{2} = 60$, and as the automorphism restricts to a permutation of the Weierstrass points, the action is then transitive. □

With our naming convention, note $W_{ijk}$ is defined by $x_i = \alpha$, $x_j = \beta$, $x_k = \gamma$. If we choose a different labelling of the roots of the cubic, this would give a different labelling of the Weierstrass points. The Weierstrass points split as $60 = 6 \times 10$, with 6 Weierstrass points being fixed by each of the 10 involutions in $S_5$.

The property that the automorphism group acts transitively on the Weierstrass points is very rare, as characterised by the following result.

**Theorem 2.3.15** ([LS12], Theorem 15)**.** *If* $X$ *is a Riemann surface of genus* $g > 2$ *with* $g^3 - g$ *Weierstrass points on which* Aut $X$ *acts transitively then either*

- $g = 4$ *and* $X$ *is Bring's curve,*

[27] In [Dye95, Theorem 4], these transpositions are associated with the Brianchon points of the Clebsch hexagon $H$ from the proof of Proposition 2.3.7.

- *$g = 3$ and $X$ is Klein's curve, or*
- *$g = 3$ and $\operatorname{Aut} X \cong S_4$.*

**Remark 2.3.16.** *As we have an explicit birational map from our plane model to our canonical embedding of the curve, we can get the explicit forms of the Weierstrass points in the HC model using Edge's identification of the Weierstrass points in the $\mathbb{P}^4$-model. If one does not have this information, it is still possible to calculate the Weierstrass points using Sage and some educated guesswork. Using computer algebra, and the characterisation of Weierstrass points as zeros of the Wronskian determinant, one can check that in the HC model the Weierstrass points have base coordinates at the 60 roots of the polynomial equations*

$$\begin{aligned}
0 &= x^{12} - 32x^{11} - 114x^{10} - 200x^9 + 100x^8 + 48x^7 - 936x^6 + 1728x^5 - 2000x^4 \\
&\quad + 3200x^3 - 2624x^2 + 768x - 64, \\
0 &= x^{24} - 24x^{23} + 1306x^{22} - 2864x^{21} + 10096x^{20} - 32704x^{19} - 5704x^{18} - 41824x^{17} \\
&\quad + 43056x^{16} + 831616x^{15} + 837856x^{14} + 992256x^{13} + 2603136x^{12} + 1238016x^{11} \\
&\quad + 1560576x^{10} + 5584896x^9 + 3357696x^8 + 3838976x^7 + 5856256x^6 + 2543616x^5 \\
&\quad + 2200576x^4 + 1355776x^3 + 454656x^2 + 65536x + 4096, \\
0 &= x^{24} + 56x^{23} + 1176x^{22} - 1784x^{21} - 3904x^{20} + 36096x^{19} + 12776x^{18} - 211904x^{17} \\
&\quad + 304736x^{16} + 431616x^{15} + 339456x^{14} - 1985664x^{13} - 625344x^{12} + 1034496x^{11} \\
&\quad + 3512576x^{10} - 584704x^9 - 3572224x^8 - 2018304x^7 + 3303936x^6 + 3055616x^5 \\
&\quad + 1099776x^4 + 45056x^3 + 229376x^2 - 16384x + 4096.
\end{aligned}$$

*One can check that the Galois group of each polynomial is $C_4 \times S_3$. We have seen that in the HC model the automorphisms have coefficients in $\mathbb{Z}[\zeta]$, and so we know the splitting field must be an extension of $\mathbb{Q}[\zeta]$, which accounts for the $C_4$ factor in the Galois group. $S_3$ has a subgroup of order 2, corresponding to an extension of degree 2, and a brute force calculation shows that we also wish to adjoin $i\sqrt{2}$. This has already nearly reduced the problem, and then one needs a small moment of inspiration to find the last thing to adjoin. Looking at [RR92] then may direct one to adjoin the real root of the polynomial $x^3 + 7x^2 + 8x + 4$, say $\xi$, and this gives the full splitting field. We can solve the cubic explicitly using Cardano's formula to find*

$$\xi = -\frac{1}{3}\left\{7 + \left[\sqrt[3]{145 + 30\sqrt{6}} + \sqrt[3]{145 - 30\sqrt{6}}\right]\right\},$$

*and as such we could also take our splitting field to be*

$$\mathbb{Q}\left[\zeta, i\sqrt{2}, \sqrt[3]{145 + 30\sqrt{6}}\right].$$

*We observe by Cardano's formula that $\alpha, \beta, \gamma \in \mathbb{Q}[i\sqrt{2}, \sqrt[3]{30 + 15\sqrt{6}}]$, and this latter field is isomorphic to $\mathbb{Q}[i\sqrt{2}, \sqrt[3]{145 + 30\sqrt{6}}]$. With these expression for the Weierstrass points, and the explicit knowledge of the automorphism group as an* *action on affine coordinates, we can find explicitly the transposition that preserves a given Weierstrass point.*

We now have seen how our Weierstrass points satisfy characterisations 1 and 4 of Proposition 2.1.62, and we complete the picture with the following result which we believe new.

**Proposition 2.3.17.** *Define the points $P_{345}$, $P'_{345}$ corresponding to the Weierstrass point $W_{345}$ (and similarly for the other Weierstrass points) by*

$$\begin{aligned}P_{345} = [\delta' : \delta : (-43\alpha^2 - 113\alpha - 92)\beta/112 \\ - (13\alpha^2 - 27\alpha - 20)/28 : (43\alpha^2 + 25\alpha - 4)\beta/112 + (13\alpha^2 + 3\alpha - 24)/28 : 1],\\ P'_{345} = [\delta : \delta' : (-43\alpha^2 - 113\alpha - 92)\beta/112 \\ - (13\alpha^2 - 27\alpha - 20)/28 : (43\alpha^2 + 25\alpha - 4)\beta/112 + (13\alpha^2 + 3\alpha - 24)/28 : 1].\end{aligned}$$

*Here $\alpha$, $\beta$ were defined in Proposition 2.3.14 and $\delta$, $\delta'$ are the roots of*

$$x^2 - \left[\frac{(11\alpha + 12)\beta + 4(3\alpha + 2)}{14}\right] x + \left[\frac{23(155\alpha + 388)\beta + 92(97\alpha + 172)}{6272}\right].$$

*Then there is a holomorphic differential $\nu_{345}$ on Bring's curve with divisor*

$$4W_{345} + P_{345} + P'_{345},$$

*and a meromorphic function on Bring's curve with divisor*

$$P_{145} + P'_{145} + P_{245} + P'_{245} - 4W_{345}.$$

*Proof.* Note that algorithmically it is possible to construct these through the work in [Hes02b], the tools for which are partially implemented in Sage but not with enough generality for us to use out-the-box. As such we need a different approach.

We know that for any Weierstrass point $W$ there is a hyperplane $H \subset \mathbb{P}^3$ intersecting the canonical embedding with multiplicity $g = 4$ at $W$ (that is $H \cap \mathcal{B} = 4W + P + P'$ for some $P, P' \in \mathcal{B}$). Such a hyperplane gives a holomorphic differential $\nu_{ijk}$ with $(\nu_{ijk}) = 4W_{ijk} + P_{ijk} + P'_{ijk}$. From [Edg78] we know that for Bring's curve the osculating plane at $W$ intersects four times, and so this is the plane we are looking for. As Edge gives a formula for the osculating plane (attributed to Hesse), we can explicitly calculate the remaining two intersections with the curve in terms of a polynomial roots. This gives us the first result for the divisor of the meromorphic differential.

Furthermore, from [Edg81b] we have a tritangent plane which has intersection with Bring's curve

$$2(W_{145} + W_{245} + W_{345}).$$

We will discuss this plane (and others like it) more in §2.3.4, but for now all we need is that this means there is a holomorphic differential[28] $\omega^{(1)}_{45\alpha}$ on $\mathcal{B}$ with

[28] Here I am using notation to pre-empt Proposition 2.3.28.

$(\omega^{(1)}_{45\alpha}) = 2(W_{145} + W_{245} + W_{345})$. As such we get the divisor of the meromorphic function

$$\left(\frac{\nu_{145}\nu_{245}}{\left(\omega^{(1)}_{45\alpha}\right)^2}\right) = P_{145} + P'_{145} + P_{245} + P'_{245} - 4W_{345}.$$

As we can calculate the formula for all these planes explicitly if we wish, we could (in principle) construct the corresponding function and differential. □

**Remark 2.3.18.** *We are able to verify the results in Proposition 2.3.17 using the Abel-Jacobi map implemented in SageMath [DH21]; see the Bring's curve notebooks.*

### Vertices and Face-Centres

Using Figure 2 of [BN12] we are able to link the Petrie polygon to the R&R model of the curve, and this gives us a concrete expression for the remaining geometric points.

**Proposition 2.3.19.** *The face-centres are exactly the points fixed by an order-5 automorphism of the curve. They are given in the $\mathbb{P}^4$-model by the permutations of $[1 : \zeta : \zeta^2 : \zeta^3 : \zeta^4]$. The corresponding points in the normalisation $\overline{\mathcal{B}}$ of the HC model are the desingularisations of $V_k$, together with $a$, $b$, $c$, $d$, and*

1. $[-2\zeta^3 - 2\zeta^2 : \zeta^3 + \zeta^2 - 1 : 1] = [\sqrt{5} + 1 : -\sqrt{5}/2 - 3/2 : 1] = [2j : j' - 2 : 1]$,
2. $[-2\zeta^3 - 2 : -2\zeta^3 - \zeta - 1 : 1]$,
3. $[-2\zeta^2 - 2\zeta : -\zeta^3 + \zeta + 1 : 1]$,
4. $[2\zeta^3 + 2\zeta^2 + 2\zeta : \zeta^3 + 2\zeta^2 + 2\zeta + 1 : 1]$,
5. $[-2\zeta^2 - 2 : \zeta^3 - \zeta^2 + \zeta : 1]$,
6. $[2\zeta^3 + 2\zeta + 2 : \zeta^2 - \zeta + 1 : 1]$,
7. $[-2\zeta^3 - 2\zeta : 2\zeta^3 + \zeta^2 + \zeta + 2 : 1]$
8. $[2\zeta^3 + 2\zeta^2 + 2 : -\zeta^3 - \zeta^2 - 2 : 1] = [-\sqrt{5} + 1 : \sqrt{5}/2 - 3/2 : 1] = [2j' : j - 2 : 1]$,
9. $[2\zeta^2 + 2\zeta + 2 : -\zeta^3 - 2\zeta^2 - \zeta : 1]$,
10. $[-2\zeta - 2 : -\zeta^2 - 2\zeta - 1 : 1]$,

*where $j, j'$ are the roots identified by Dye mentioned in §2.3.1.*

*Proof.* Certainly the order-5 rotation about a face-centre fixes that centre, and so a Riemann-Hurwitz counting argument gives us that the 24 face-centres are the fixed points of order-5 automorphisms. It is a simple matter of computation to verify the given expressions are fixed; this may be done in Sage. □

The Galois group $\mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q}) \cong C_4$ acts element-wise on the face-centres and the orbits partition the set of face-centres into six sets of four. Each set of four face-centres form the vertices of a quadrilateral whose edges lie in $\{H_2 = 0\}$ [Edg78]. Moreover, the faces of the dodecadodecahedron corresponding to the face-centres in each quadrilateral are parallel. We have a similar result for the vertices.

**Proposition 2.3.20.** *The vertices are exactly the points fixed by an order-4 automorphism of the curve. They are given in the $\mathbb{P}^4$-model by the permutations of $[1 : i : -1 : -i : 0]$.*

We could use our birational map as above to give the vertices in HC coordinates, but these are not very illuminating. For example, the vertex $[1 : i : -1 : -i : 0]$ maps to

$$[X : Y : Z] = [(3\zeta^3 - \zeta^2 + 2\zeta + 1)i + 3\zeta^3 + 3\zeta^2 + 5 : (2\zeta^3 + 2\zeta + 1)i + 2\zeta^3 + 2\zeta^2 + 2 : 1]$$

though $[1 : -1 : i : -i : 0]$ maps to $[1 : (-1 + i)/2 : (-1 + i)/2]$.

### 2.3.3 Quotients by Subgroups

Both [RR92] and [Web05] consider the quotient of $\mathcal{B}$ by the action of subgroups of $S_5$. In this section we shall study the various quotients of Bring's curve of nonzero genus and the relations between them, both clarifying and extending previous work. In §2.3.3 we will use the Riemann-Hurwitz theorem to describe possible quotients. In §2.3.3 we shall note the various relationships we expect between the quotients just on group theoretic grounds, while in §2.3.3-§2.3.3 we turn to their explicit construction. In so doing we discover a number of curious isomorphisms beyond those expected. In §2.3.3 we summarise our calculations and relate them to known isogeny results. Throughout we will use the following: for any subgroup $H$ of the automorphisms $\mathrm{Aut}(\mathcal{C})$ of a curve $\mathcal{C}$, $H \leq \mathrm{Aut}(\mathcal{C})$, then the normaliser $N_{\mathrm{Aut}(\mathcal{C})}(H)$ acts on the $H$ orbits and $N_{\mathrm{Aut}(\mathcal{C})}(H)/H \leq \mathrm{Aut}(\mathcal{C}/H)$; if $g \in N_{\mathrm{Aut}(\mathcal{C})}(H)$ we will denoted by $\overline{g}$ the $H$-coset of $N_{\mathrm{Aut}(\mathcal{C})}(H)$ containing this. The quotient curves of this section are summarised in Figures 2.4 and 2.6.

#### Genera of Quotients

Our first step is to know the topology of the quotients we are going to find. To this end, we give the following result.

**Proposition 2.3.21.** *The data of the quotients of $\mathcal{B}$ by subgroups $\langle\sigma\rangle \leq S_5$ is summarised by the following table:*

| $\sigma$ | *Example* | $\lvert cl(\sigma)\rvert$ | $\lvert\mathrm{Fix}(\sigma)\rvert$ | *Fixed Points* | $N_{S_5}(\langle\sigma\rangle)/\langle\sigma\rangle$ | $g(\mathcal{B}/\langle\sigma\rangle)$ |
|---|---|---|---|---|---|---|
| (12) | - | *10* | *6* | *Edges* | $S_3$ | *1* |
| (123) | *RS* | *20* | *0* | - | $V_4$ | *2* |
| (12)(34) | *R* | *15* | *2* | *Vertices* | $V_4$ | *2* |
| (1234) | *U* | *30* | *2* | *Vertices* | $C_2$ | *1* |
| (123)(45) | - | *20* | *0* | - | $C_2$ | *1* |
| (12345) | *S* | *24* | *4* | *Face-centres* | $C_4$ | *0* |

*Here $cl(\sigma)$ is the conjugacy class of $\sigma$,* $\mathrm{Fix}(\sigma) = \{P \in \mathcal{B} \,|\, \sigma(P) = P\}$*.*

*Proof.* As conjugate elements yield isomorphic quotients we need only to give one $\sigma$ per conjugacy class. The first three columns follow from the group theory we have previously shown in §2.3.1, the fourth and fifth follow from §2.3.2 and the sixth is elementary group theory. The final column remaining is then a Riemann-Hurwitz argument for genus, which will we demonstrate for the quotient by (2345) as in [RR92].

For general $\sigma$, denoting by $\pi$ the projection $\mathcal{B} \to \mathcal{B}/\langle\sigma\rangle := \mathcal{C}$, Riemann-Hurwitz says

$$g_{\mathcal{B}} - 1 = (\deg \pi)(g_{\mathcal{C}} - 1) + \frac{1}{2}B,$$

where $B$ is the degree of ramification of $\pi$, which in the case of a quotient by a group action corresponds to the fixed point structure of $\sigma$.

Consider (24)(35). A fixed point of $\mathcal{B}$ under this, given by projective coordinates $x_i, i = 1, \ldots, 5$, must have

$$(x_1, x_4, x_5, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5).$$

From this we can see $\lambda = \pm 1$. Taking $\lambda = 1$ gives no solutions, but taking $\lambda = -1$ one finds the equations

$$x_1 = 0, \quad x_1^2 + 2x_2^2 + 2x_3^2 = 0, \quad x_1^3 = 0,$$

which gives the 2 fixed points $[0 : 1 : \pm i : -1 : \mp i]$. Hence we have

$$4 - 1 = 2(g_{\mathcal{C}} - 1) + \frac{1}{2}(1 + 1) \Rightarrow g_{\mathcal{C}} = 2.$$

Moving now to (2345), thinking about the possible branching structure we get from Riemann-Hurwitz

$$3 = 4(g_{\mathcal{C}} - 1) + \frac{1}{2}\left[\sum_{P \in \mathrm{Fix}(\sigma)} 3 + \sum_{P \in \mathrm{Fix}(\sigma^2) \setminus \mathrm{Fix}(\sigma)} 1\right].$$

Fixed points of (2345) will correspond to points such that

$$(x_1, x_3, x_4, x_5, x_2) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5),$$

and we get the constraint $\lambda^4 = 1$. The equations of the curve become

$$\begin{aligned} x_1 + x_2(1 + \lambda + \lambda^2 + \lambda^3) &= 0, \\ x_1^2 + x_2^2(1 + \lambda^2 + 1 + \lambda^2) &= 0, \\ x_1^3 + x_2^3(1 + \lambda^3 + \lambda^2 + \lambda) &= 0, \end{aligned}$$

and checking the possible cases one finds that the only fixed points of (2345) are $[0 : 1 : \pm i : -1 : \mp i]$. These are also the only fixed points of $(24)(35) = (2345)^2$ and so the second sum vanishes, hence we find $g_{\mathcal{C}} = 1$. $\square$

R&R [RR92] provides a nice visual interpretation of the quotient by a 4-cycle, namely think of a sphere with four handles attached around an equator, then (2345) is the cycle rotating these handles to each other by a quarter turn about the axis through the centre of the equator. The fixed points are then where this axis intersects the sphere. Edge [Edg78], citing Wiman, says that Bring's curve "is, in ten different ways, in $(2,1)$ correspondence with a plane curve of genus 1". The ten $(2,1)$ correspondences noted by Edge are exactly the ten quotients by a transposition giving a $2:1$ map $\mathcal{B} \to \mathcal{E}$, where $\mathcal{E}$ is an elliptic curve. Such a map is called a bielliptic structure, and Bring's curve is the unique genus-4 curve to have 10 such structures [CDC05]. This table also lets us reconstruct the results about the gonality of Bring's curve from [GWW10].

**Relations between Quotients**

We now consider the various relationships we might expect between the quotient curves of Bring's curve. Recall Definition 2.1.83 that if $\mathcal{C}$ is a curve of genus $g \geq 2$ and $H \leq \mathrm{Aut}(\mathcal{C})$ is such that $|H| > 4(g-1)$, a so called 'large automorphism group', then the genus of the quotient curve $g_{\mathcal{C}/H} = 0$. For Bring's curve we are therefore interested in subgroups of $S_5$ of order less than or equal to 12, and as we have seen the quotient by a 5-cycle leads to quotient genus 0 we may exclude subgroups containing such. The relevant conjugacy classes of subgroups are then[29]

(a) $\langle(12)(34)\rangle \cong C_2$, $\langle(12),(34)\rangle \cong V_4$, $\langle(1324)\rangle \cong C_4$, $\langle(1324),(12)\rangle \cong D_4$. Each of these groups $H$ have the same normaliser: $N_{S_5}(H) = \langle(1324),(12)\rangle \cong D_4$.

(b) $\langle(12)\rangle \cong C_2$, $\langle(345)\rangle \cong C_3$, $\langle(345),(12)\rangle \cong C_6$, $\langle(345),(34)\rangle \cong S_3$, $\langle(345),(12)(34)\rangle \cong S_3$ (we shall call this subgroup $S_3'$), $\langle(345),(12),(34)\rangle \cong D_6 \cong S_3 \times C_2$. Each of these groups $H$ have the same normaliser: $N_{S_5}(H) = \langle(345),(12),(34)\rangle \cong D_6$.

(c) $\langle(12)(34),(13)(24)\rangle \cong V_4$, $\langle(234),(12)(34)\rangle \cong A_4$. Each of these groups $H$ have the same normaliser: $N_{S_5}(\langle(234),(12)(34)\rangle) = \langle(234),(12),(34)\rangle \cong S_4$.

[29] Note that here we are using the Sage convention for the dihedral group that $|D_n| = 2n$, the same convention as used in §2.2.2.

The groupings here are such that if $H$, $H'$ are from the same item then they share the same normaliser $N = N_{S_5}(H) = N_{S_5}(H')$. In particular each of $H$, $H'$ and $HH' = H'H$ are normal in $N$ and we have commutativity of the following diagram of quotients

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "H"] \arrow[d, "H'"'] \arrow[dr, "HH'"] & \mathcal{B}/H \arrow[d, "HH'/H"] \\
\mathcal{B}/H' \arrow[r, "HH'/H'"'] & \mathcal{B}/HH'
\end{tikzcd}$$

where the arrows are labelled by the symmetry being quotiented; by an isomorphism theorem we have of course that $HH'/H \cong H'/(H \cap H')$ and $HH'/H' \cong H/(H \cap H')$. If, say $H \leq H'$, then $HH' = H'$ and one side of this diagram collapses.

Now a Riemann-Hurwitz calculation shows that the each of the quotients by $\langle (12), (34) \rangle$, $\langle (1324), (12) \rangle$, $\langle (345), (34) \rangle$ and $\langle (345), (12), (34) \rangle$ is of genus 0 and so not being considered. Thus from the preceding discussion we have the following relations amongst quotients for the subgroups of (a), (b) and (c):

$$\begin{tikzcd}
\mathcal{B} \arrow[d, "(12)(34)"'] \arrow[dr, "(1324)"] & \\
\mathcal{C} \arrow[r] & \mathcal{E}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \mathcal{B} \arrow[r, "(12)"] \arrow[dl, "S_3'"'] \arrow[d, "(345)"'] \arrow[dr, "C_6"] & \mathcal{E}' \arrow[d, "(345)"] \\
\mathcal{E}'' & \mathcal{C}' \arrow[l, "(12)(34)"] \arrow[r, "(12)"'] & \mathcal{E}'''
\end{tikzcd}$$

$(a)$ $(b)$

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dl, "(12)(34)"'] \arrow[d, "V_4"'] \arrow[dr, "A_4"] & \\
\mathcal{C} \arrow[r] & \mathcal{E}^{(iv)} \arrow[r, "(234)"'] & \mathcal{E}^{(v)}
\end{tikzcd}$$

$(c)$

Here $\mathcal{C}, \mathcal{C}'$ are genus-2 curves and $\mathcal{E}, \ldots, \mathcal{E}^{(v)}$ elliptic curves that we cannot yet specify purely on group theoretic grounds. We turn now to their specification and indeed we shall find some interesting identities between them, which will then be summarised in §2.3.3.

### Quotients by a 4- and 2,2-cycle

Armed with the knowledge of the genus of the quotients we expect we shall now write them explicitly. We will begin with the 2,2-cycle corresponding to $U^2$ with the 4-cycle $U$ arising in the discussion. Recall that we have

$$U^2 : [X, Y, Z] \to [X, Z, Y], \quad \psi(U^2) = (12)(34).$$

The normaliser of $\langle(12)(34)\rangle$ in $S_5$ is $N_{S_5}(\langle\langle(12)(34)\rangle\rangle) = \langle(12),(1324)\rangle \cong D_4$. As we remarked earlier, $N_{S_5}(\langle\langle(12)(34)\rangle\rangle)/\langle(12)(34)\rangle \cong V_4$ are symmetries which remain when we go to the quotient $\mathcal{B}/\langle U^2\rangle$ and we expect the quotient genus-2 curve to have (at least) this $\overline{V_4}$ symmetry group. One of these involutions, which we will later see[30] to be $\overline{(12)} = \overline{(34)}$, is the hyperelliptic involution of the quotient curve and so further quotienting by this symmetry gives $\mathbb{P}^1$. Quotienting by the other two involutions will yield elliptic curves, and we will complete this construction now, both from the perspective of the HC model, and the $\mathbb{P}^4$-model.

Starting in the HC model, in order to get the first quotient $\mathcal{B}/\langle U^2\rangle$ we express our curve in terms of the invariants of $U^2$: $X$, $T := Y+Z$, and $V := YZ$. Then

$$\begin{aligned} 0 &= X(Y^5+Z^5) + (XYZ)^2 - X^4YZ - 2(YZ)^3, \\ &= X(T^5 - 5T^3V + 5TV^2) + X^2V^2 - X^4V - 2V^3. \end{aligned}$$

In $\mathbb{P}^{1,1,2}$ this is our genus 4 curve. Setting[31] $T = 1$ and viewing

$$0 = X(1 - 5V + 5V^2) + X^2V^2 - X^4V - 2V^3$$

as the affine part of a projective curve we have (after a not-very-illuminating transformation, for which Maple was used, see the corresponding notebooks) the hyperelliptic curve

$$\mathcal{C}_1 : \; B^2 = A^6 + 4A^5 + 10A^3 + 4A + 1. \tag{2.23}$$

This is the genus-2 curve of [RR92] with automorphism group $V_4$. Calculating the Igusa-Clebsch invariants (which give the $\overline{\mathbb{Q}}$-isomorphism class of a curve) and searching the LMFDB [LMF23] shows that this is a model for the modular curve $X_0(50)$, which we verify by directly calculating a model for $X_0(50)$ over $\mathbb{Q}$ using [Shi95]. The substitutions $A = (2+A')/(2-A')$, $B = 4B'/(2-A')^3$ make the $V_4$ symmetry clearer, giving

$$(B')^2 + \left[(A')^6 - 5(A')^4 - 40(A')^2 - 80\right] = 0,$$

where we have the hyperelliptic involution $B' \to -B'$ and the map $A' \to -A'$ generating $V_4$. We shall now be explicit about how these work.

As previously mentioned, quotienting $\mathcal{C}_1$ by either of the two non-hyperelliptic involutions yields an elliptic curve, with each involution having two fixed points from Riemann-Hurwitz. Quotienting by $(B', A') \to (B', -A')$ by introducing $A'' = -(A')^2$ yields

$$\mathcal{E}_1 : (B')^2 = (A'')^3 - 5(A'')^2 - 40(A'') - 80 \;\text{ with } j\text{-invariant } j_{\mathcal{E}_1} = -\frac{5 \times 29^3}{2^5} := j(\tau_0). \tag{2.24}$$

The fixed points are $(A', B') = (0, \pm\sqrt{-80})$, corresponding to the two points

[30] The $\langle(12)(34)\rangle$ cosets of the normaliser are $\{e,(12)(34)\}$, $\{(12),(34)\}$, $\{(13)(24),(14)(23))\}$, and $\{(1324),(1423)\}$.

[31] Equally one may set $X = 1$ but we note that setting $V = 1$ yields a genus-4 curve as here $V$ is not of weight 1.

$(X, V) = ([1 \pm \sqrt{5}]/2, -1 \pm \sqrt{5}/2)$, which are the images of the four vertices $[1 : \pm i : \mp i : -1 : 0]$ and $[1 : \pm i : -1 \mp i : 0]$ respectively, depending on sign. We recognise $\mathcal{E}_1$ to be the elliptic curve $E_1$ in [RR92].

We may also quotient $\mathcal{C}_1$ by $(B', A') \to (-B', -A')$. To do this write $\mathcal{C}_1$ as

$$(B')^2C^4 + \left[(A')^6 - 5(A')^4C^2 - 40(A')^2C^4 - 80C^6\right] = 0,$$

from where we see the same automorphism acts as $C \to -C$. Quotienting by this action by introducing $C' = -C^2$ yields

$$(B')^2(C')^2 = (A')^6 - 5(A')^4(C') - 40(A')^2(C')^2 - 80(C')^3.$$

Setting $A' = 1$ and taking $B'' = B'C'$ gives the standard elliptic form

$$\mathcal{E}_2 : (B'')^2 = 1 - 5(C') - 40(C')^2 - 80(C')^3, \quad j_{\mathcal{E}_2} = -\frac{25}{2} = j(5\tau_0). \tag{2.25}$$

The fixed point of this involution is $[B' : A' : C] = [1 : 0 : 0]$; this is a singular point where the desingularisation corresponds to the two points at infinity, or correspondingly $(X, V) = (\mp i/2 - 1/2, 1/4)$, which are the images of the two vertices $[1 : -1 : \pm i : \mp i : 0]$. We recognise[32] $\mathcal{E}_2$ to be $E_2$ in [RR92].

Next let us quotient the $\mathbb{P}^4$-model directly by the action of $(12)(34)$ and compare with these quotients just obtained from the HC model. To this end we introduce semi-invariants of the action of $(1324) = \psi(U)$, defined by

$$(s_1, s_2, s_3, s_4)^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{pmatrix} (x_1, x_2, x_3, x_4)^T.$$

These are constructed such that $U \cdot s_j = i^{j-1}s_j$, and so we have[33] $U^2$-invariants $[s : t : u : v] := [s_1 : s_2s_4 : s_3 : s_4^2] \in \mathbb{P}(1 : 2 : 1 : 2)$. In terms of these invariants we have

$$\begin{aligned} H_2 &= \frac{1}{4}\left(5s^2 + u^2 + 2t\right), \\ H_3 &= \frac{3}{16}\left(-5s^3 + su^2 + \frac{t^2u}{v} + 2st + uv\right). \end{aligned}$$

Eliminating $t$ from these equations, and setting $s = 1$, we can use Maple to get the genus-2 curve in Weierstrass form

$$\mathcal{C}_1 : y^2 = 100 - 25x^2 - 10x^4 - x^6, \tag{2.26}$$

where $x = u$ and $y = -10 + 2uv$. The roots of the sextic here are a Möbius

[32] There is a typo in the constant term of R&R's $E_2$ if it is to have the stated $j$-invariant.

[33] Note one could have instead taken $v = s_2^2$ for the last invariant, but the choice $v = s_4^2$ happens to be better around the vertices that are the fixed points of the 4-cycle. Moreover, recognise that the invariants taken are defined over the field extension $\mathbb{Q}[i]$ and not $\mathbb{Q}$.

transform (namely $x \mapsto \sqrt{20}/x$) of that in the curve $\mathcal{C}_1$ previously given in Equation 2.23, so these two curves are isomorphic over $\mathbb{Q}[\sqrt{5}]$. Note the reason we see $\mathbb{Q}[\sqrt{5}]$ here is that it is the degree-2 subfield of $\mathbb{Q}[\zeta]$, the field required to have equivalence of the Hulek-Craig and $\mathbb{P}^4$-models of Bring's curve. We now aim to identify the $V_4$ of this curve described earlier with the quotient of the normaliser $\langle (12), (1324) \rangle / \langle (12)(34) \rangle$. One can check that $(12) : [s : t : u : v] \to [s : t : u : t^2/v]$. This fixes $x$, and so must be the automorphism $y \to -y$; that is $\overline{(12)} = \overline{(34)}$ is the hyperelliptic involution of $\mathcal{C}_1$. Indeed one can check

$$\begin{aligned} y = -10 + 2uv \mapsto -10 + 2\frac{t^2 u}{v}&, \\ &= -10 - 2\left(-5 + u^2 + 2t + uv\right), \\ &= -10 - 2\left(-10 + uv + (5 + u^2 + 2t)\right), \\ &= 10 - 2uv = -y. \end{aligned}$$

Likewise, $(1324) : [s : t : u : v] \to [s : t : -u : -v]$, and so leads to the automorphism $(x, y) \to (-x, y)$. Now if $\mathsf{x} = x^2$, the elliptic curve $y^2 = 100 - 25\mathsf{x} - 10\mathsf{x}^2 - \mathsf{x}^3$ has $j$-invariant $-25/2$ and so we may identify $\mathcal{E}_2 \cong \mathcal{B}/\langle (12)(34), (1324) \rangle = \mathcal{B}/\langle (1324) \rangle$. Similarly $(13)(24) : [s : t : u : v] \to [s : t : -u : -t^2/v]$ leads to $(x, y) \to (-x, -y)$ and we may identify $\mathcal{E}_1 \cong \mathcal{B}/\langle (12)(34), (13)(24) \rangle$. Note when we compare with [RR92], we differ from R&R in the identification of which quotient is being taken and their ascribing of $\tau_0$ and $5\tau_0$. Our results agree with those of [Web05], wherein the author describes an order-4 rotation (which he calls $\phi$, but we shall call $\tilde{U}$), and calculates the quotient by its action $T = \mathcal{B}/\langle \tilde{U} \rangle$ to be

$$T: \; y^2 = 4x^3 - 75x - 1475, \quad j_T = -\frac{25}{2} = j(5\tau_0).$$

Note that our strategy of semi-invariants can be used directly to calculate the quotient by the 4-cycle (1324), something we were unable to do in the HC model because of the nonlinearity of the automorphism $U$. To do so introduce new variables invariant under (1324) (and so necessarily under (12)(34)) given by $u' = uv$, $v' = v^2$. These let us rewrite the defining equations of Bring's curve as

$$\begin{aligned} H_2 &= \frac{1}{4}\left(5s^2 + \frac{(u')^2}{v'} + 2t\right), \\ H_3 &= \frac{3}{16}\left(-5s^3 + \frac{s(u')^2}{v'} + \frac{t^2 u'}{v'} + 2st + u'\right), \end{aligned}$$

and we can apply Maple to find a Weierstrass form of the resulting elliptic curve. This is exactly the process of quotienting by $(x, y) \mapsto (-x, -y)$ as above, but in a different language.

Furthermore, from our previous investigation using group theory, we know that the quotient $\mathcal{B}/V_4$, where this $V_4$ is the one containing only 2,2-cycles, can further be quotiented by (234) to give $\mathcal{B}/A_4$. Rather than attempt to construct this quotient using the invariants previously calculated on the 2,2-quotient, we step back and recall that the invariant ring $\mathbb{Q}[x_1, \ldots, x_n]^{A_n}$ is generated by the symmetric polynomials $s_k = \sum_{i=1}^{n} x_i^k$ for $k = 1, \ldots, n$ and the Vandermonde polynomial $V = \prod_{1\leq i<j\leq n}(x_j - x_i)$. As generators of the invariant algebra we know that there must be a relation between $V^2$ and the other generators as $V^2$ is an $S_n$-invariant, and the $S_n$-invariant algebra is generated by the $s_k$. Taking $n = 4$ in the case of Bring's curve, the relations imposed from the curve are $s_2 + s_1^2 = 0 = s_3 - s_1^3$, and this gives for the additional relation,

$$4s_4^3 - \frac{373}{16}s_1^4 s_4^2 + \frac{431}{8}s_1^8 s_4 - \frac{701}{16}s_1^{12} + V^2 = 0. \tag{2.27}$$

Setting $s_1 = 1$ (as we may do at all points on the quotient except those points coming from the vertices on Bring's curve) we see this is clearly an elliptic curve $\mathcal{E}_3$ with hyperelliptic involution $V \to -V$, and for which we can calculate[34] the $j$-invariant to be $\frac{211^3\times 5}{2^{15}} = j(3\tau_0)$. Further quotienting by the hyperelliptic involution then corresponds to the quotient $\mathcal{B}/S_4$, which is $\mathbb{P}^1$ as expected, being the quotient by a large automorphism group.

### Quotients by a 3-cycle

We may utilise the same methods illustrated above for the 2,2-cycles to calculate the quotient of Bring's curve by a 3-cycle. We will work with the 3-cycle (345) for purely aesthetic reasons. The normaliser of $\langle(345)\rangle$ in $S_5$ is $N_{S_5}(\langle 345\rangle)) = \langle(12),(34),(345)\rangle \cong D_6$ and we expect the quotient genus-2 curve $\mathcal{B}/\langle(345)\rangle$ to have $D_6/C_3 \cong V_4$ symmetry group. Again one of these involutions, which we will later see to be[35] $\overline{(34)}$, is the hyperelliptic involution on the curve and so further quotienting by this symmetry gives $\mathbb{P}^1$. Quotienting by the other two involutions again yields elliptic curves we shall now describe. As the quotients of both the HC model and $\mathbb{P}^4$-model via semi-invariants proceed analogously we shall present here only the $\mathbb{P}^4$-model calculations.

Letting $\rho$ be a primitive cube-root, we take semi-invariants of the action of (345)

$$(s_1, s_2, s_3, s_4)^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & \rho & \rho^2 \\ 0 & 1 & \rho^2 & \rho \end{pmatrix} (x_2, x_3, x_4, x_5)^T.$$

We correspondingly take invariants[36] $[s : t : u : v] = [s_1 : s_2 : s_3 s_4 : s_3^3] \in \mathbb{P}(1 : 1 : 2 : 3)$. In terms of these variables we have

$$H_2 = \frac{2}{3}\left(3s^2 + 3st + 2t^2 + u\right),$$
$$H_3 = \frac{1}{9}\left(-27s^2 t - 27st^2 - 8t^3 + v + 6tu + \frac{u^3}{v}\right).$$

Eliminating $u$ from these and setting $s = 1$, we can use Maple to get the genus-2

[34] This is the curve 15-isogenous to $\mathcal{E}_2$ noted by Serre, see §2.3.1.

[35] The cosets of $\langle(345)\rangle$ in the normaliser are now $\{e,(345),(354)\}$, $\{(12),(12)(345),(12)(354)\}$, $\{(34),(45),(35))\}$, and $\{(12)(34),(12)(45),(12)(35))\}$.

[36] Note in this case that the invariant $v$ is not defined over $\mathbb{Q}$, but over the cyclotomic extension $\mathbb{Q}[\rho]$.

curve in Weierstrass form

$$\mathcal{C}_2 : y^2 = 108\left(4 + 12x + 95x^2 + 170x^3 + 155x^4 + 72x^5 + 16x^6\right) \tag{2.28}$$

where $x = t$ and $y = -90t - 90t^2 - 40t^3 + 4v$. Examining the roots of the sextic confirms that $\mathcal{C}_2$ is a genuinely distinct genus-2 curve from $\mathcal{C}_1$. Indeed, this curve does not currently exist[37] in the LMFDB, but one can check that over $\mathbb{Q}[\sqrt{5}]$ it is isomorphic to the curve 2500.a.400000.1 given by $y^2 = -7x^6 - 8x^5 + 10x^3 - 8x - 7$. Moreover, introducing $x', y'$ by $x = -2/(1+x')$, $y = y'/(1+x')^3$, yields

$$(y')^2 = 432\left[(x')^6 + 80(x')^4 + 125(x')^2 + 50\right].$$

This makes the $V_4$ symmetry evident, being generated by $x' \to -x'$ and $y' \to -y'$. The elliptic curve obtained from quotienting by the action $x' \to -x'$ is

$$\mathcal{E}_4 : (y')^2 = 432\left[(x'')^3 + 80(x'')^2 + 125x'' + 50\right], \quad j_{\mathcal{E}_4} = -\frac{5^2 \times 241^3}{2^3} = j(15\tau_0). \tag{2.29}$$

To our knowledge this elliptic curve has not been previously noted in discussions of Bring's curve. The elliptic curve obtained from quotienting by the action $(x', y') \to (-x', -y')$ is the previously seen

$$\mathcal{E}_1 : (y'')^2 = 432\left[1 + 80z' + 125(z')^2 + 50(z')^3\right], \quad j_{\mathcal{E}_1} = -\frac{5 \times 29^3}{2^5} = j(\tau_0).$$

We now wish to identify the quotient $\langle (12), (34), (345)\rangle / \langle (345)\rangle \cong V_4$ with the $V_4$ just described. It requires a little effort to see $(12) : [s : t : u : v] \to [-s - t : t : u : v]$, which corresponds to $x' \to -x'$, and that $(34) : [s : t : u : v] \to [s : t : u : u^3/v]$, which fixes $x'$ and so must be the map $y' \to -y'$. This means that the remaining involution $(x', y') \to (-x', -y')$ comes from the group element $(12)(34)$. As such we identify $\mathcal{E}_4 \cong \mathcal{B}/\langle (12), (345)\rangle$, and we have a curious isomorphism, namely

$$\mathcal{B}/\langle (12)(34), (345)\rangle \cong \mathcal{B}/\langle (12)(34), (13)(24)\rangle \cong \mathcal{E}_1 \text{ corresponding to } \begin{tikzcd} \mathcal{B} \arrow[r, "(345)"] \arrow[d, "(12)(34)"] & \mathcal{C}_2 \arrow[d, "\overline{(12)(34)}"] \\ \mathcal{C}_1 \arrow[r, "(13)(24)"'] & \mathcal{E}_1 \end{tikzcd}.$$

As before we can also consider distinguished points on the quotients coming from fixed points. Whereas $(345)$ has no fixed points when acting on $\mathcal{B}$ each involution on $\mathcal{C}_2$ that gives a quotient to an elliptic curve has two fixed points. The fixed points of $\overline{(12)}$ are the orbits under $(345)$ of the six fixed points of $(12)$ that are Weierstrass points. The fixed points of $\overline{(12)(34)}$ are the orbits under $(345)$ of the $3 \times 2$ fixed points of $(12)(34)$, $(12)(45)$, and $(12)(35)$ that are vertices.

[37]This is to be expected, as it does not satisfy the conditions to be included in curves investigated in [BSS+16].

**Quotients by a Transposition**

The strategy of quotienting from the $\mathbb{P}^4$-model using semi-invariants from the previous sections can also be used to calculate the quotient of Bring's curve by a transposition. We take semi-invariants

$$(s_1, s_2, s_3, s_4)^T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} (x_1, x_2, x_3, x_4)^T,$$

and in terms of these variables the defining equations of Bring's curve become

$$\begin{aligned} H_2 &= \frac{1}{2}(s_1^2 + s_2^2) + s_3^2 + s_4^2 + (s_1 + s_3 + s_4)^2, \\ H_3 &= \frac{1}{4}(s_1^3 + 3s_1 s_2^2) + s_3^3 + s_4^3 - (s_1 + s_3 + s_4)^3. \end{aligned}$$

Taking invariants $[s : t : u : v] = [s_1 : s_2^2 : s_3 : s_4]$ and eliminating $t$ yields the elliptic curve

$$\mathcal{E}_2 : \; 4s^3 + 8s^2u + 7su^2 + u^3 + sv^2 - uv^2 = 0, \quad j_{\mathcal{E}_2} = -\frac{25}{2} = j(5\tau_0). \tag{2.30}$$

We observe that (also curiously) the quotient of $\mathcal{B}$ by a transposition is isomorphic to the quotient by a 4-cycle. It is not clear that there is any a priori reason for this to be the case. We are however able to relate the elliptic curves $\mathcal{E}_2$ and $\mathcal{E}_4$ as follows.

The normaliser[38] of $\langle(12)\rangle$ in $S_5$ is $N_{S_5}(\langle(12)\rangle) = \langle(12), (34), (345)\rangle \cong D_6$ where now $N_{S_5}(\langle(12)\rangle)/\langle(12)\rangle \cong S_3$. We may immediately identify the action of $\overline{(34)}$ as the hyperelliptic involution on $\mathcal{E}_2$ as it acts as $v \to -v$, but we also retain another group action, that of $\overline{(345)}$. To understand this action recall that there are six fixed points of the action of $(12)$ on $\mathcal{B}$, which are all Weierstrass points. This gives six more distinguished points on the curve $\mathcal{E}_2$ in addition to the two images of the vertices we have seen previously. The six Weierstrass points fixed by $(12)$ break into two orbits of three under $(345)$ which we denote by $\{H_i\}$, $\{H_i'\}$ $(i = 1, 2, 3)$. Just as $(345)$ gives Bring's curve as an unramified cover of the genus-2 curve $\mathcal{C}_2$, quotienting each by a transposition has $\overline{(345)}$ yielding an unramified automorphism of the quotient curves. Hence there exists a quotient from $\mathcal{E}_2 = \mathcal{B}/\langle(12)\rangle$ to another elliptic curve $\mathcal{E}_4 = \mathcal{B}/\langle(12), (345)\rangle$ such that the following diagram commutes,

$$\begin{tikzcd} \mathcal{B} \arrow[r, "(345)"] \arrow[d, "(12)"] & \mathcal{C}_2 \arrow[d, "\overline{(12)}"] \\ \mathcal{E}_2 \arrow[r, "\overline{(345)}"'] & \mathcal{E}_4 \end{tikzcd}$$

[38] The cosets of $\langle(12)\rangle$ in the normaliser are $\{e, (12)\}$, $\{(345), (12)(345)\}$, $\{(354), (12)(354)\}$, $\{(34), (12)(34)\}$, $\{(35), (12)(35)\}$, and $\{(45), (12)(45)\}$.

To view this action on the elliptic curve $\mathcal{E}_2$, we calculate that $(345) : [s : t : u : v] \to [s : t : -s - u/2 - v/2 : s + 3u/2 - v/2]$. In the $s = 1$ affine chart $4 + 8u + 7u^2 + u^3 + (1-u)v^2 = 0$ we can take as a cohomology basis

$$\eta := \frac{du}{2v(1-u)},$$

and it is a simple algebraic calculation to see that this differential is invariant under the action of $\overline{(345)}$. If $\{\overline{H}_i\}$, $\{\overline{H'}_i\}$ are the images of the Weierstrass points on $\mathcal{E}_2$ such that $(345)(H_i) = H_{i+1}$, $\overline{(345)}(\overline{H}_i) = \overline{H}_{i+1}$ and so forth, the invariance of $\eta$ tells us that

$$\int_{\overline{H}_1}^{\overline{H}_2} \eta = \int_{\overline{H}_2}^{\overline{H}_3} \eta = \dots, \quad \int_{\overline{H'}_1}^{\overline{H'}_2} \eta = \int_{\overline{H'}_2}^{\overline{H'}_3} \eta = \dots, \quad \int_{\overline{H}_1}^{\overline{H'}_1} \eta = \int_{\overline{H}_2}^{\overline{H'}_2} \eta = \dots$$

This means that if $\Lambda = \langle 1, 5\tau_0 \rangle$ is the period lattice of the elliptic curve $\mathcal{E}_2$ then $\int_{\overline{H}_i}^{\overline{H}_{i+1}} \eta$ and $\int_{\overline{H'}_i}^{\overline{H'}_{i+1}} \eta$ are the same fixed element of $\Lambda/3$. By choosing an appropriate basis we may take $\overline{(345)} : z \to z + 1/3$ for $z \in \mathbb{C}/\langle 1, 5\tau_0 \rangle$. Then the quotient map will be a 3:1 isogeny of elliptic curves and we have the period of $\mathcal{E}_4$ to be $15\tau_0$ and $j_{\mathcal{E}_4} = j(15\tau_0)$. Under this isogeny the six images on $\mathcal{E}_2$ of the Weierstrass points $[1 : 1 : \alpha : \beta : \gamma]$ are mapped to two on $\mathcal{E}_4$. Indeed, one may construct the isogeny explicitly using the Weierstrass $\wp$ function and the Abel-Jacobi map via the diagram

$$\begin{tikzcd}
\mathcal{E}_2 \arrow[r] \arrow[d, "P \mapsto \int_\infty^P \eta"'] & \mathcal{E}_4 \\
J(\mathcal{E}_2) \arrow[r, "z \mapsto 3z"'] & J(\mathcal{E}_4) \arrow[u, "z \mapsto (\wp(z), \wp'(z))"']
\end{tikzcd}$$

where $J(\mathcal{E})$ represents the Jacobian of the elliptic curve viewed as $\mathbb{C}/\langle 1, \tau \rangle$.

**Remark 2.3.22.** *This calculation may be verified numerically, as shown in the Bring's curve notebooks.*

**Remark 2.3.23.** *Using the language of Example 2.1.77,* $\overline{(345)}$ *is a translation. To see this take variables $x, y$ defined by*

$$u = \frac{-70s^3 + 6sx}{2(25s^2 + 3x)}, \quad v = \frac{-3y}{25s^2 + 3x},$$

*which in the affine patch where $s = 1$ give $\mathcal{E}_2$ as the curve $x^3 - 25/3x + 2950/27 + y^2 = 0$. In these coordinates* $(345)$ *acts as*

$$x \mapsto \frac{5(275 - 3x + 15y)}{3(65 + 15x - 3y)}, \quad y \mapsto \frac{20(55 - 15x - 3y)}{(65 + 15x - 3y)}.$$

*Indeed for the projective coordinates $[X : Y : Z]$ of our curve $X^3 - 25/3XZ^2 +$*

$$\begin{tikzcd}
 & & \mathcal{B} \arrow[dl] \arrow[d] \arrow[dr] & & & \\
 & \mathcal{E}_{5\tau_0} \arrow[dl] \arrow[dr] \arrow[drrrr, leftrightsquigarrow] & \mathcal{C}_2 \arrow[l, rightsquigarrow] \arrow[dll] \arrow[dl] \arrow[dr] & \mathcal{C}_1 \arrow[dl] \arrow[dr] \arrow[drr] & & \\
\mathcal{E}_{15\tau_0} \arrow[dr] & \mathbb{P}^1 \arrow[d] & \mathbb{P}^1 \arrow[d] & \mathcal{E}_{\tau_0} \arrow[dll] \arrow[r, leftrightsquigarrow] & \mathcal{E}_{\tau_0} \arrow[dl] & \mathcal{E}_{5\tau_0} \arrow[dlll] \\
 & \mathbb{P}^1 & \mathbb{P}^1 & \mathcal{E}_{3\tau_0} \arrow[dl] & & \\
 & & \mathbb{P}^1 & & &
\end{tikzcd}$$

**Fig. 2.4** Quotient structure of Bring's curve

$2950/27Z^3 + Y^2Z = 0$ *we have under* $\overline{(345)}$ *that*

$$\begin{aligned}
[X : Y : Z] &\mapsto [5(275Z/3 - X + 5Y) : 20(55Z - 15X - 3Y) : 65Z + 15X - 3Y], \\
&\mapsto [275Z/3 - X - 5Y : -220Z + 60X - 12Y : 13Z + 3X + (3Y)/5], \\
&\mapsto [X : Y : Z].
\end{aligned}$$

*When working with a Weierstrass model of an elliptic curve it is standard to take the distinguished basepoint to be* $\infty = [0 : 1 : 0]$ *and we see in terms of* $(x, y)$ *that* $\infty \mapsto (-25/3, 20) \mapsto (-25/3, -20) \mapsto \infty$.

#### Summarising

We can collect the information of the quotients we have seen into Figure 2.4. Solid arrows represent a covering map coming from a quotient, whereas 'squiggly' arrows indicate isomorphisms that we were unable to explain by group theory alone.

Figure 2.5 shows the corresponding groups which are quotiented by, where an arrow now indicates that a source group is normal in the target, whereas a dashed line just indicates that a group is a subgroup. The label used corresponds to the list in §2.3.3, except where that label is ambiguous and the exact group must be specified.

Finally, we recall from a Riemann-Hurwitz argument that each elliptic curves that arises as a quotient from a genus-2 curve with $V_4$ symmetry has two marked points which are the branch points of the covering map, equivalently the images of the fixed points of the involution being quotiented by. We have shown that the preimages of these on Bring's curve are geometric points. In Figure 2.6 we

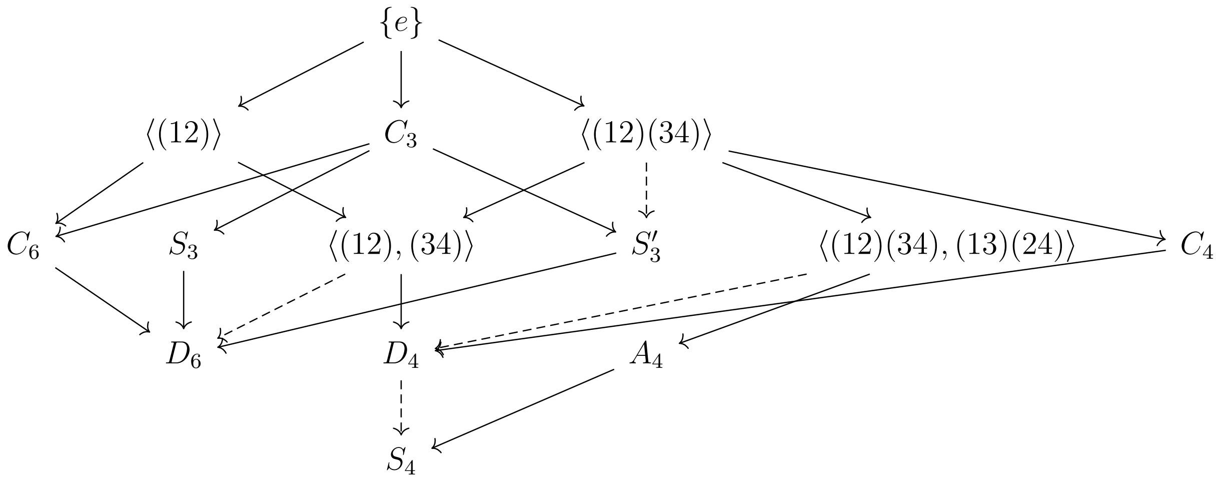



**Fig. 2.5** Subgroup structure of $S_5$ corresponding to the quotients of Bring's curve

illustrate the quotient structure highlighting these points. (We do not decorate the curve $\mathcal{B}/A_4 \cong \mathcal{E}_{3\tau_0}$, but include it so as to show every quotient with genus greater than 0.) Note the nodes of Figure 2.6 do not correspond directly to the placements of those in Figures 2.4 and 2.5.

With the information of the quotients, we can now discuss the isogeny class and isomorphism class of the Jacobian of Bring's curve with the following results.

**Proposition 2.3.24** ([RR92], §4, [Web05], Corollary 5.5)**.** *The $\mathbb{C}$-isogeny class of the Jacobian of Bring's curve is $\mathcal{E}_3^4$.*

*Proof.* As we will later want to strengthen this result, we will use methods which relate subvarieties of the Jacobian to idempotents, and thus to subgroups of the automorphism group. Namely we will use [LR04, §4.2], which gives the isogeny decomposition in terms of Prym varieties of the Jacobian of a curve with an $A_5$ action. Note, because of comments made before in §2.3.1, Bring's curve is the unique genus-4 curve for which we could apply this argument. The result follows as, using the notation of Lange, $X_{A_4} = \mathcal{E}_3$, $X_{D_5} = X_{Z_5} = Y = \mathbb{P}^1$.

This proof also follows from [LR04, Proposition 5.1], which uses only the action of $S_4$ on the curve. Moreover, one could use [KR89, Theorem C] taking the subgroups to be $\langle(12)\rangle, \langle(34)\rangle$, $C_4$ and $A_4$. Alternatively, using the isomorphism $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$ and [KR89, Example 2] one can find $\operatorname{Jac}\mathcal{B} \sim \operatorname{Jac}\mathcal{C}_1 \times \operatorname{Jac}\mathcal{C}_2$ and proceed from there. These proof strategies all follow the same approach of looking for idempotents.

One may also obtain the result following the same method as R&R. We use that fact that isogenies act on the period matrix by right multiplication by matrices $R \in M_4(\mathbb{Z})$, and so we have the required isogeny by taking $\lambda = 1$ in the identity

$$\left(\lambda^{-1}\,\mathrm{Id}\right)\begin{pmatrix}1 & \tau M\end{pmatrix}\begin{pmatrix}\lambda\,\mathrm{Id} & 0\\ 0 & M'\end{pmatrix} = \begin{pmatrix}\mathrm{Id} & \dfrac{5}{\lambda}\tau\,\mathrm{Id}\end{pmatrix},$$

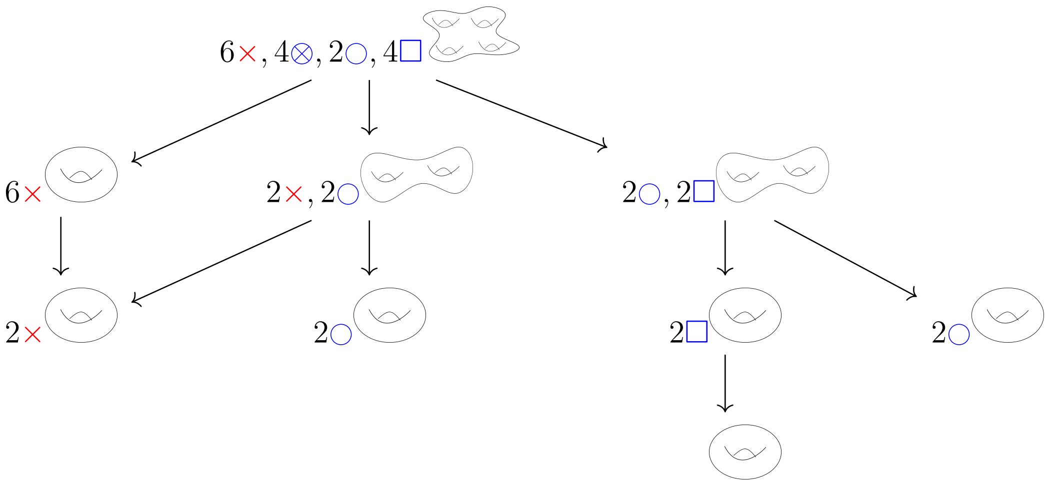



**Fig. 2.6** Quotient structure including marked points $\times$ (Weierstrass points $[1 : 1 : \alpha : \beta : \gamma]$ and permutations), $\otimes$ (vertices $[1 : -1 : 0 : \pm i : \mp i]$ and $[1 : -1 : \pm i : 0 : \mp i]$), $\bigcirc$ (vertices $[1 : -1 : \pm i : \mp i : 0]$), and $\square$ (vertices $[1 : \pm i : \mp i : -1 : 0]$ and $[1 : \pm i : -1 : \mp : 0]$)

where

$$M' := 5M^{-1} = \begin{pmatrix} 2 & -1 & 1 & -1 \\ -1 & 2 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ -1 & 1 & -1 & 2 \end{pmatrix}.$$

$\square$

Note that we are able to construct the quotient $\mathcal{B}/A_4$ directly from the $\mathbb{P}^4$-model using transformations with coefficients in $\mathbb{Q}$, and the resulting curve is defined over $\mathbb{Q}$, so by [KR89, Remark 6] Proposition 2.3.24 can be strengthened to a statement about the $\mathbb{Q}$-isogeny class of the Jacobian of Bring's curve. Calculating using Sage we find that the $\mathbb{Q}$-isogeny class of $\mathcal{B}/A_4$ is 50a using the Cremona labels for elliptic curve $\mathbb{Q}$-isogeny classes. Hence the following result holds.

**Proposition 2.3.25** ([Ser08], Exercise 8.3.2(b))**.** *The $\mathbb{Q}$-isogeny class of the Jacobian of the $\mathbb{P}^4$-model of Bring's curve is* $(50a)^4$.

Note in Proposition 2.3.25 we had to be careful to specify the Jacobian of the $\mathbb{P}^4$-model of Bring's curve, as we have seen that HC model is not birationally equivalent over a field that does not contain $\mathbb{Q}[\zeta]$. The $\mathbb{Q}$-isogeny class of the elliptic curve $\mathcal{B}/\langle(12)(34),(13)(24)\rangle$ calculated via the HC model is 50b. Similarly, the $\mathbb{Q}$-isogeny class of the two elliptic curves covered by $\mathcal{C}_2$ is 450b, and the computation of the quotient required the coefficient field to be $\mathbb{Q}[\rho]$. In order to not have a contradiction with Proposition 2.3.25, we must have that the $\mathbb{Q}$-isogeny classes 50b and 50a merge over $\mathbb{Q}[\sqrt{5}]$, and that the isogeny classes 450b and 50a merge over $\mathbb{Q}[\rho]$, which is indeed the case.[39]

[39]The isogeny classes 450b and 50b merge over $\mathbb{Q}[\sqrt{-15}]$.

One could use computational tools such as those in [BSS+16, Lom18][40] to numerically find the $\mathbb{Q}$-isogeny class of the Jacobian of the HC model of Bring's curve, as we did in the notebooks. Such computational results using idempotents can be helpful for developing our understanding, for example one can use computer algebra to search for relations between the characters $\mathrm{Ind}_H^{S_5}(1_H)$, that is the characters of $S_5$ induced from the trivial representation of $H \leq S_5$. Doing so gives relations between subvarieties of the Jacobian of Bring's curve following [KR89, Theorem 3], for example

$$\begin{gathered} J_{\langle(12)\rangle} \times J_{\langle(12)(34),(13)(24)\rangle} \sim J_{\langle(12)(34)\rangle} \times J_{\langle(12),(34)\rangle}, \\ J_{\langle(12)(34),(13)(24)\rangle} \times J_{S_3} \sim J_{\langle(12),(34)\rangle} \times J_{S_3'}, \end{gathered} \tag{2.31}$$

where I have used the shorthand notation $J_H := \mathrm{Jac}(\mathcal{B}/H)$. Using Riemann-Hurwitz arguments these would let us say that

$$\mathcal{B}/\langle(12)\rangle \sim \mathcal{B}/C_4 \quad \text{and} \quad \mathcal{B}/\langle(12)(34),(13)(24)\rangle \sim \mathcal{B}/S_3'$$

without having to do any calculation, entirely from group theory. The reason why these isogenies are actually isomorphisms is not clear.

Proposition 2.3.24 can also be strengthened in a different direction.

**Proposition 2.3.26** ([GAR00], Theorem 4.1)**.** *The Jacobian of Bring's curve is isomorphic as a complex torus to $\mathcal{E}_2^3 \times \mathcal{E}_1$.*

*Proof.* The proof in [GAR00] is very general, considering Jacobians whose period matrix is invariant under $S_n$ for any $n$. The isomorphism from the period matrix given in [RR92] is

$$C\begin{pmatrix}1 & \tau M\end{pmatrix}\begin{pmatrix}D & 0 \\ 0 & E\end{pmatrix} = \big(1, \tau\,\mathrm{diag}(5,5,5,1)\big)\,,$$

where

$$C = \begin{pmatrix} -1 & 1 & -1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}, E = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -2 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

□

Note it would not be possible that the Jacobian is isomorphic to the product of the elliptic curves as a principally polarised abelian variety, as it is well known that the Jacobian of any smooth compact Riemann surface with canonical principal polarisation is irreducible [GAR00]. In particular this means the matrix $\begin{pmatrix}D & 0 \\ 0 & E\end{pmatrix}$ of the proposition is not symplectic.

[40] To make the reconstruction process of Lombardo more accessible, we recreated in Sage some of the functions implemented by Lombardo in Magma.

### 2.3.4 Theta Characteristics

We now investigate the theta characteristics of Bring's curve, of which we have already seen some details in Table 2.3. Among other results we shall identify the unique invariant spin structure of Bring's curve. Note that because Bring's curve has genus 4, there are 120 odd and 136 even characteristics on the curve.

**Example 2.3.27.** *It is a computational exercise using the methods of Proposition 2.3.6 to verify the divisors of the differentials* $(v_2) = 3b + c + 2d$ *and* $(v_4) = 2a + b + 3d$*, hence* $\mathcal{K}_{\mathcal{B}} \sim \left(\frac{-xv_3v_4}{v_2}\right) = 2(3a + b - c)$*. As such* $\Delta = 3a + b - c$ *is a theta characteristic on Bring's curve. A simple calculation in Sage shows that it is even.*

#### Tritangent Planes and Odd Characteristics

Recalling Proposition 2.2.11, we know that on Bring's curve the tritangent planes are in 1-1 correspondence with the odd theta characteristics, and as such to understand the orbit structure of the odd characteristics we need only understand the tritangent planes, about which we have the following result.

**Proposition 2.3.28** ([Edg81b])**.** *The 120 tritangent planes on Bring's curve split into two classes, 60 in each class:*

1. *those where all three contact points are Weierstrass points, and*
2. *those where only one contact points is a Weierstrass point.*

*Planes in the first and second class respectively have equations (recalling notation from §2.3.2)*

$$\Pi^{(1)}_{\alpha jk} := \left\{ \frac{x_j}{\beta} - \frac{x_k}{\gamma} = 0 \right\},$$
$$\Pi^{(2)}_{ijk} := \left\{ (\alpha - 1)(\alpha + 4)x_i + (\beta - 1)(\beta + 4)x_j + (\gamma - 1)(\gamma + 4)x_k = 0 \right\},$$

*where* $[x_i] \in \mathbb{P}^4$ *and* $i, j, k$ *distinct.*

To clarify the notation we have used for the planes, recall that the position of the indices on Weierstrass points $W_{ijk}$ indicates which root was equal to $x_i$. The same principle holds for $\Pi^{(1)}_{\alpha jk}, \Pi^{(1)}_{i\beta k}$, and $\Pi^{(1)}_{ij\gamma}$.

**Corollary 2.3.29.** *The orbit decomposition of odd theta characteristics on Bring's curve is*

$$120 = 20 + 20 + 20 + 60. \tag{2.32}$$

*Proof.* The characteristics coming from the tritangent planes are

$$T^{(1)}_{\alpha jk} = \sum_{\substack{i=1 \\ i \neq j,k}}^{5} W_{ijk},$$
$$T^{(2)}_{ijk} = W_{ijk} + O^{+}_{ijk} + O^{-}_{ijk}.$$

Here we define the points $O^{\pm}_{ijk}$ by, for example,

$$O^{\pm}_{345} = \left[ \frac{1 \pm i\sqrt{15}}{2} : \frac{1 \mp i\sqrt{15}}{2} : \alpha^2 + \alpha + 1, \beta^2 + \beta + 1, \gamma^2 + \gamma + 1 \right].$$

A simple orbit-stabiliser argument then gives the orbit decomposition as, for example, $T^{(1)}_{\alpha 45}$ is stabilised by the symmetric group $S_{\{1,2,3\}}$ and $T^{(2)}_{345}$ is stabilised by $S_{\{1,2\}}$. □

**Even Characteristics**

In Corollary 2.3.29 we were able to fully characterise the odd theta characteristics on Bring's curve without too much difficulty through the use of existing work giving the stalls of the canonical embedding. The story for even characteristics is different because Scorza theory cannot be used here. In [Bur83], motivated by its realisation via an elliptic modular surface, Burns identified a theta characteristic on Bring's curve invariant under the $A_5$ subgroup of the automorphism group, though he described this characteristic only in terms of two line bundles on the curve, not directly in terms of points on the curve. We shall now fully classify the orbits of the even characteristics and give an explicit description of Burns' divisor in the process.

We have two distinct methods of probing the orbit decomposition of the theta characteristics on Bring's curve as described in §2.2.2.

1. Use the method of [KS10], wherein theta characteristics are identified with vectors in $\mathbb{Z}_2^{2g}$, and the action of automorphisms given by the homology representation of the automorphism group of the curve as found using the methods of [BSZ19].

2. Identify theta characteristic with the $2^{2g}$ translates by half-lattice vectors of a half-canonical vector in the Jacobian of the curve, done using the implementation of the Abel-Jacobi map I developed [DH21], and the action of automorphisms given by the analytic representation of the automorphism group of the curve as found using the methods of [BSZ19].

Both methods not only verify Corollary 2.3.29 but also gives the following result.

**Theorem 2.3.30.** *The orbit decomposition of even theta characteristics on Bring's curve is*

$$136 = 1 + 5 + 5 + 5 + 10 + 10 + 10 + 30 + 30 + 30. \tag{2.33}$$

**Corollary 2.3.31.** *Bring's curve has a unique theta characteristic invariant under the action of the automorphism group, which is also the theta characteristic invariant under the $A_5$ subgroup found in [Bur83].*

**Remark 2.3.32.** *Note in both the orbit decomposition of the odd characteristics and the even characteristics there is an as of yet unexplained 'threeness' whereby if an orbit of a certain size occurs, it occurs either exactly once or exactly thrice.*

The existence of the unique invariant theta characteristic was known in [BN12], but it had not been identified. This we rectify with the following result.

**Theorem 2.3.33.** *The theta characteristic $\Delta$ (defined in Example 2.3.27) is the unique invariant theta characteristic on Bring's curve.*

*Proof.* We first consider the action of $S$ on $a$, $b$, $c$, $d$. We have

$$\begin{aligned}
a &:= [0:0:1] \simeq [2t^3 : t : 1] \xrightarrow{S} [2t^3 : \zeta t : \zeta^{-1}] = [2\zeta t^3 : \zeta^2 t : 1] = [2(\zeta^2 t)^3 : \zeta^2 t : 1] \\
&\qquad\qquad\quad = [2\epsilon^3 : \epsilon : 1], \\
b &:= [0:1:0] \simeq [2t^2 : 1/t : 1] \xrightarrow{S} [2t^2 : \zeta/t : \zeta^{-1}] = [2\zeta t^2 : \zeta^2/t : 1] = [2(t/\zeta^2)^2 : \zeta^2/t : 1] \\
c &:= [1:0:0]_2 \simeq [1 : t : t^4] \xrightarrow{S} [1 : \zeta t : \zeta^{-1} t^4] = [1 : \zeta t : (\zeta t)^4], \\
d &:= [1:0:0]_1 \simeq [1 : t^4 : t] \xrightarrow{S} [1 : \zeta t^4 : \zeta^{-1} t] = [1 : (\zeta^{-1} t)^4 : \zeta^{-1} t].
\end{aligned}$$

Thus $a$, $b$, $c$ and $d$ are invariant under the symmetry $S$ and consequently $\Delta = 3a + b - c$ is also invariant.

Similarly, as mentioned in Proposition 2.3.11, the action of $U$ on $a, b, c, d$ can be calculated as

$$a \mapsto c \mapsto b \mapsto d \mapsto a,$$

and so

$$\Delta = 3a + b - c \mapsto 3c + d - b = 3a + b - c - (3a + 2b - 4c - d) = \Delta - (x) \sim \Delta.$$

We have thus shown that $\Delta$ is invariant under $\langle S, U\rangle$. To complete the proof that $\Delta$ is the invariant theta characteristic, one could attempt to show that $\Delta$ is invariant under the action of $R$ by direct computation, but this proves to be difficult. It is instead better to check in Sage that the unique spin structure invariant under the whole automorphism group is actually also the unique spin structure invariant under the subgroup generated by $S$ and $U$. In fact, by [KS10, Theorem 1.2] and our work in §2.3.3, we know there is a unique theta characteristic invariant[41] under $\langle S\rangle$, and this completes the proof. □

This proof strategy[42] is similar to the identification of the unique invariant theta characteristic on Klein's curve in [KS10], proven to exist in [Bur83].

We now want to make the connection to [Bur83]. Recall Equations 2.15 and 2.16. This gives us two degree-3 maps $f_i : \mathcal{B} \to \mathbb{P}^1$, namely $f_i = \pi_i \circ \varphi^{-1}|_{\mathcal{B}}$, $i = 1, 2$, where $\varphi$ was the isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathcal{Q}$, and $\pi_i$ the projection to the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$. These are the two $g^1_3$ on any non-hyperelliptic genus-4 curve whose associated quadric is nonsingular [Har77, Example IV.5.5.2]. What are the corresponding divisors? Working in the $L_a$ coordinates, we can use Sage

[41] Under the action of $S$, the orbit decompositions are $120 = 24 \times 5$ and $136 = 1 + 27 \times 5$.

[42] The final part of the proof in that paper, showing invariance under the order-2 generator of $\mathrm{PSL}_2(\mathbb{F}_7)$, is in our opinion incomplete; Kallel (private correspondence) believes our reasoning correct. The theorem remains correct nevertheless, as one can check using their methodology that invariance under the order-7 generator is enough to specify the unique invariant spin structure, as the decompositions are $28 = 4 \times 7$ and $36 = 1 + 5 \times 7$. It is a curious coincidence that in both cases it was a generator of order 2 for which the direct calculation was difficult.

to find that $f_1^{-1}([1:0]) = 2[0:0:0:1] + [1:0:0:0] = 2b + c := L'$, while $f_2^{-1}([1:0]) = 2[0:1:0:0] + [0:0:0:1] = 2d + b := L$.

**Proposition 2.3.34.** *The divisors $L$, $L'$ satisfy the properties described in [Bur83], namely*

$$\Delta \sim 3(L' - L) + L, \quad \mathcal{K}_{\mathcal{B}} \sim L + L', \quad 0 \sim 5(L' - L).$$

*Proof.* This is straightforward verification from the definitions. □

Note we can connect this back to the degree-3 map given by Klein. We know from [Web05] that we expect this map to be branched at face-centres of the $\{5, 5|3\}$ tessellation, and these come from face-centres of the $\{5, 4\}_6$ tessellation, and indeed recall we saw that $a, b, c, d$ were face-centres.

An additional characterisation of the invariant theta characteristic is given by the following result.

**Proposition 2.3.35.** *In the homology basis of [RR92], the RCV satisfies*

$$K_a = \frac{1}{10}(3, 2, -2, -3) + \Im(\tau_0)(1, -2, -2, 1)i = \mathcal{A}_a(\Delta).$$

*As such, in the R&R homology basis, the unique invariant theta characteristic is the divisor (class) of the Szegő kernel, i.e. $\Delta = \Delta_{\mathcal{B}}$.*

*Proof.* The first equality is shown analytically in [BN12]. The second is shown numerically in the corresponding notebooks, using the Abel-Jacobi map developed by Disney-Hogg [DH21]. To verify the RCV we followed the procedure laid out in §2.2.1, implementing the methodology of [DPS15] using the theta function in Sage developed by Bruin and Ganjian [Bru21]. While these calculations are numerical in nature, the calculations can be done with arbitrary binary precision. We were satisfied by calculating with 400 binary digits of precision, giving an absolute error of less than $10^{-118}$ for the first equality, and of less than $10^{-23}$ for the second equality. □

# Chapter 3

# Magnetic Monopoles

> Particularly I am glad to see the *Monopoleon* again in Cambridge. This title shall indicate that I have a friendlier view to his theory of 'monopoles' than earlier: There is some mathematical beauty in this theory
>
> Wolfgang Pauli
> Letter to Niels Bohr, March 5th, 1949

I will now move onto a different topic, and one motivated by fundamental physics; that of magnetic monopoles. In the 20th Century and earlier magnetic fields had been discovered, and it was observed that sources of the magnetic field only came in pairs, in contrast to the behaviour of the electric field which is sourced by electrons. The first serious suggestion of the possibility of sources of the magnetic field, now called magnetic monopoles, was in a 1931 paper by Dirac in which he showed that, in the presence of a monopole, electric charge had to be quantised. For a more complete review of the literature up until 1990 including historical sources, see [GT90], and for specific sources on experimental evidence for monopoles see [Mil06]. One particular reference deserving of being singled out is [Cab82], describing how a "single candidate event, consistent with one Dirac unit of magnetic charge" was observed on the 14th of February, 1982, thus earning the moniker the 'valentines-day monopole'. Such an event has never been replicated.

This chapter shall not be on the physical aspects of monopoles, but instead their modern treatment in terms of gauge theory, and how to construct solutions using methods from algebraic geometry. §3.1 will describe the gauge-theoretical formulation of monopoles, laying down a concrete definition of what indeed a monopole is, and briefly describing some aspects of the moduli space of all monopole solutions. I will not provide any background or preliminaries on gauge theory as they shall largely be immaterial, but for a general introduction see [FO06]. §3.2 will introduce the breakthrough tool from algebraic geometry used to study monopoles: the spectral curve. I will discuss two approaches to constructing this data, how it may be used to reconstruct the monopole gauge fields themselves, and in particular I shall give new interpretations and visualisations of existing constraints which are required of monopole spectral curves. In §3.3 I will discuss how symmetries act on the many presentations of monopole data and describe how this can be used to construct monopole data invariant under certain symmetry groups, computations for which code is provided (see `nahm_data.py`). Finally in §3.4 I will complete a partial classification of charge-3 monopoles by identifying a subset of all possible monopole spectral curves amenable to solutions in terms of elliptic functions.

In the remainder of this section I will lay some ground work for the coming study of monopoles by highlighting some basic properties of the Lie group SU(2) that we shall require throughout.

### Properties of SU(2)

SU(2) is the group of all complex $2\times 2$ unitary matrices with determinant 1, for which we have the useful parametrisation

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} p & q \\ -\bar{q} & \bar{p} \end{pmatrix} \mid p, q \in \mathbb{C}, |p|^2 + |q|^2 = 1 \right\}.$$

Given the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which satisfy[1] $\sigma_j \sigma_k = \delta_{jk} + i\epsilon_{jkl}\sigma_l$, the matrices $T_j = -\frac{i}{2}\sigma_j$ form a basis of the associated Lie algebra $\mathfrak{su}(2)$. Moreover, taking the inner product on $\mathfrak{su}(2)$ to be $\langle X, Y\rangle = -2\,\mathrm{Tr}(XY) = 2\,\mathrm{Tr}(X^\dagger Y)$, this basis is orthonormal.

**Remark 3.0.1.** *The choice of normalisation of the inner product is arbitrary, but later conventions must be chosen to be compatible with this (for example the asymptotic behaviour of a monopoles Higgs field, see Remark 3.1.6). Effort will be made to highlight these choices when they occur.*

It is a classical result that the irreducible representations of SU(2) are classified by their dimension [Hal15, §4.2], that is every $r$-dimensional irreducible representation is equivalent to $\mathbb{S}^{r-1}$, the action of SU(2) on degree-$(r-1)$ homogeneous bivariate polynomials, which is equivalently the action on sections $H^0(\mathbb{P}^1, \mathcal{O}(r-1))$. Because of this, I will use $\zeta_{0,1}$ to denote the two variables. Moreover, the irreducible representations of SO(3) are exactly the pushforwards under the projection $\pi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ of the even degree spaces, $\pi_* \mathbb{S}^{2n}$ [Hal15, §C.1]. The following theorem describes the decomposition of a tensor product of these representations.

**Proposition 3.0.2** ([Hal15], Theorem C.1)**.**

$$\mathbb{S}^r \otimes \mathbb{S}^s = \bigoplus_{i=0}^{\min(r,s)} \mathbb{S}^{r+s-2i}.$$

[1] Here I am using summation notation to implicitly sum over the repeated index $l$. At various points in this thesis I shall use summation notation, and it should be clear from context when this occurs.

**Example 3.0.3.** *The natural representation of* $\mathrm{SO}(3)$ *acting on* $\mathbb{R}^3$ *via matrix multiplication is equivalent to* $\mathbb{S}^2$*.*

We may take the derivative of these Lie group representations to get representations of the Lie algebras $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. I will fix notation by letting $\mathfrak{su}(2) = \langle X, Y, H \rangle$ subject to the commutation relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

**Lemma 3.0.4** ([Hal15], §4.2)**.** *The derivative Lie algebra representation*[2] $\Pi : \mathfrak{su}(2) \to \mathfrak{gl}(\mathbb{S}^{2r})$ *is given by*

$$\Pi(X) = \zeta_1 \frac{\partial}{\partial \zeta_0}, \quad \Pi(Y) = \zeta_0 \frac{\partial}{\partial \zeta_1}, \quad \Pi(H) = -\zeta_0 \frac{\partial}{\partial \zeta_0} + \zeta_1 \frac{\partial}{\partial \zeta_1}.$$

Recall now a standard definition.

**Definition 3.0.5.** *A* ***highest-weight vector*** $v$ *of weight* $2r$ *in an* $\mathfrak{su}(2)$ *representation space is defined by the conditions*

$$X \cdot v = 0, \quad H \cdot v = 2rv.$$

*The vector space then spanned by* $\left\{ (Y \cdot)^k v \right\}$ *is a* ***highest-weight subspace*** *of dimension* $2r + 1$*.*

**Example 3.0.6.** *The highest-weight vector of weight* $2r$ *in* $\mathbb{S}^{2r}$ *is* $\zeta_1^{2r}$*, and so any* $P \in \mathbb{S}^{2r}$ *can be written as*

$$P(\zeta_0, \zeta_1) = \tilde{P}(\zeta_0 \partial_{\zeta_1}) \zeta_1^{2r}$$

*for some univariate polynomial* $\tilde{P}$*. Moreover, the highest-weight vectors* $v_i \in \mathbb{S}^2 \otimes \mathbb{S}^{2(r+i)}$ *of weight* $2r$ *can be given, e.g.* $v_{-1} = \zeta_1^2 \otimes \zeta_1^{2r-2}$*.*

**Remark 3.0.7.** *Note the symmetry between* $\zeta_0$ *and* $\zeta_1$ *has been broken in the definition of the derivative representation. We will see the impact of this later in* §*3.3.2*.

## 3.1 Monopoles in Gauge Theory

> F. Klein wrote that "... a physicist, for his problems, can extract from these theories only very little, and an engineer nothing." The development of the sciences in the following years decisively disproved this remark
>
> – Vladimir Arnold
> Mathematical Methods of Classical Mechanics

We will now want to write down explicitly the equations we will consider when

[2] I will not decorate the representation $\Pi$ with $r$ to indicate which image space is being used, as this should be clear from the context.

defining monopoles. Historically it was the physicists 't Hooft and Polyakov who simultaneously and independently ('t Hooft submitted his paper just 35 days before Polyakov) showed that monopoles could exist in gauge theories whose gauge group was compact and contained U(1) as a subgroup [tH74, Pol74], though Polyakov did not use this language.

### 3.1.1 The Yang-Mills-Higgs Equations

Given some (simple, compact, [JT80, §IV.18]) gauge group $G$, let $P = G\times\mathbb{R}^{d+1}$ be the trivial principal $G$-bundle over Minkowski space (taking the mostly-positive metric), with associated adjoint bundle $\mathrm{ad}(P) = \mathfrak{g}\times\mathbb{R}^{d+1}$. I will throughout this thesis use the notation of $x$ for the coordinates on $\mathbb{R}^{d+1}$ taking in index notation $x_\mu$ for the spacetime coordinates, with $x_0$ the time coordinate and the $x_i$ the space coordinates.

$P$ is the only possible $G$-bundle over $\mathbb{R}^{d+1}$ because the base is contractible. The bundle $\mathrm{ad}(P)$ is flat and so we may view a connection $A \in \Gamma(T^*\mathbb{R}^{d+1}\otimes\mathrm{ad}(P))$ and adjoint scalar field (also called the **Higgs field**) $\phi \in \Gamma(\mathrm{ad}(P))$ as a $\mathfrak{g}$-valued one-form and scalar respectively. Letting $\langle\cdot,\cdot\rangle$ be an ad-invariant inner product on $\mathfrak{g}$, and extending it to an inner product on $\mathfrak{g}$-values forms using the induced inner product on $T^*\mathbb{R}^{d+1}$, we can make the following definition.

**Definition 3.1.1** ([JT80], I.1.9a)**.** *The* ***Yang-Mills-Higgs (YMH) action*** *is*

$$S_{YMH}[A,\phi] = \int_{\mathbb{R}^{d+1}} \left[-\left|F\right|^2 - \left|D\phi\right|^2 - V(\phi)\right] \mathrm{d}^{d+1}x, \tag{3.1}$$

*where $F = dA + A\wedge A$ and $D = d + A$ are the curvature and covariant derivative associated with $A$, and $V(\phi) = \lambda\left(1-|\phi|^2\right)^2$ is the* ***$\phi^4$-potential****. By our choice of signs, we want $\lambda \geq 0$. If we remove the terms containing $\phi$, this is just the pure Yang-Mills action.*

Equation 3.1 is, up to conventions on factors which will not be important, the action taken by 't Hooft in [tH74]. There it was argued that in order to get magnetic monopoles one would want $G$ to be nonabelian, compactly covering U(1), and so the simplest choice is to take $G = \mathrm{SU}(2)$ which I will take herein. The case $G = \mathrm{U}(1)$ is of particular importance historically, as it is related to the Dirac monopole [Dir31, MS04].

Through the standard principle of least action one can get equations associated with the action relevant to mathematical physics.

**Proposition 3.1.2** ([JT80], I.1.17, I.2.2)**.** *The variational equations corresponding to $S_{YMH}$ in Minkowski $\mathbb{R}^{d+1}$ are the* ***Yang-Mills-Higgs equations***

$$\begin{aligned} DF &= 0 \quad \textit{(Bianchi)},\\ (-1)^{d+1}\star D\star F &= -\left[\phi, D\phi\right],\\ \star D\star D\phi &= -\frac{1}{2\left|\phi\right|}V'(\phi)\phi, \end{aligned} \tag{3.2}$$

*where $\star$ is the Hodge star operator on the corresponding Minkowski space.*

We will want to look for static solutions, i.e. time-independent solutions with $A_0 = 0$, of Equations 3.2. For such solutions the action $S_{YMH}$ is infinite, but for a constant-time hyperplane we may define the energy to be the integral of the Lagrangian density over this hypersurface and search only for finite-energy static solutions. This is equivalent to asking for finite-action solutions to the Yang-Mills-Higgs equations on Euclidean $\mathbb{R}^d$, called **solitons** [JT80, §I.2].

**Remark 3.1.3.** *One can ask for finite-action solutions to the Yang-Mills-Higgs equations on manifolds different from $\mathbb{R}^d$. While I will not go into this subject in any details, at various points I will mention one such case of particular interest to current research, namely hyperbolic 3-space $H^3$.*

The existence of such finite-action solutions is not guaranteed a priori, and indeed they do not for $d > 4$; $d = 4$ solutions are called **instantons** and are gauge equivalent to pure Yang-Mills solitons; $d = 3$ solutions are called **monopoles**; and $d = 2$ solutions are **vortices** [JT80, p. 10, Corollary II.2.3]. In this thesis I will only want to consider monopoles, and so now restrict to considering $d = 3$.

It is clearly a necessity that tending towards the sphere at infinity $|D\phi| \to 0$, and for $\lambda \neq 0$, $|\phi| \to 1$. This process is called **symmetry breaking** as at infinity the $\mathrm{SU}(2)$ gauge symmetry is broken and only a $\mathrm{U}(1)$ symmetry remains. Symmetry breaking is relevant in phenomenology [Hig64], but we will interpret it as giving a topological charge to the soliton.

**Remark 3.1.4.** *The symmetry breaking works as the bundle $P$ of which $F$ is the curvature decomposes into a direct sum of eigenbundles of $\phi$ on the sphere at infinity, breaking the gauge group from $\mathrm{SU}(2)$ to $S(\mathrm{U}(1) \times \mathrm{U}(1)) \cong \mathrm{U}(1)$. Starting with the more generic $\mathrm{SU}(n)$ gauge group there are different possibilities for the symmetry breaking depending on the eigenvalues of $\phi$; **maximal** symmetry breaking is when all the eigenvalues are distinct and the gauge group breaks to $\mathrm{U}(1)^{n-1}$, **minimal** symmetry breaking is when all but one of the eigenvalues are the same and the gauge group breaks to $\mathrm{U}(n-1)$ [MS04, BN22].*

### BPS Limit

The variational equations we have found so far are second order, but we want to apply the classic strategy when working with topological solitons: write the energy functional of a static configuration as the integral of a square term plus a topological term, and then we locally must have a minimising solution by setting the squared term to 0. This will be possible if we set $\lambda = 0$ but retain that $|\phi| = 1$ at infinity. We are more specific about the conditions we want.

**Definition 3.1.5** ([Hit82], §6, [Hit83], §1)**.** *We define the **monopole boundary conditions** to be that as $r := |x| \to \infty$,*

1. $|\phi| = 1 - \frac{k}{r} + \mathcal{O}(r^{-2})$,
2. $\frac{\partial|\phi|}{\partial\Omega} = \mathcal{O}(r^{-2})$,
3. $|D\phi| = \mathcal{O}(r^{-2})$.

**Remark 3.1.6.** *This may seem to contradictory to [AH88, §2], where the first condition is $|\phi| \sim 1 - \frac{k}{2r}$. Note however that there the convention used is that the norm on $\mathfrak{su}(2)$ is $|X|^2 = -\frac{1}{2}\operatorname{Tr}(X^2)$. We will see a concrete example of this in Example 3.1.17. The dependence of asymptotics on the choice of normalisation will ultimately come down to the fact that the asymptotics of $\phi$ are governed by a topological condition, and so any change in the definition of the norm comes with a corresponding change of the apparent $k$. Indeed, [Hit83] cites work of Taubes that shows that all the monopole boundary conditions are automatic for a solution of $F = \star D\phi$ with $|\phi| \to 1$. The conventions used by other authors are collected in §A.1.*

**Remark 3.1.7.** *Recall that the angular derivative is defined as*

$$\frac{\partial |\phi|}{\partial \Omega} = \sqrt{\left(\frac{\partial |\phi|}{\partial \theta}\right)^2 + \sin^2\theta \left(\frac{\partial |\phi|}{\partial \varphi}\right)^2}.$$

*The second monopole boundary condition evaluated at large distances means $k$ is a constant.*

With $\lambda = 0$ we can rewrite the energy functional of static configurations at any time using the following lemma

**Lemma 3.1.8** ([Bog76, Ati87])**.** *Taking $\star$ to be the Hodge star operator on Euclidean $\mathbb{R}^3$ and $S^2_R$ a sphere of radius $R$ centred at the origin,*

$$E = \int_{\mathbb{R}^3} \left(|F|^2 + |D\phi|^2\right) d^3x = \int_{\mathbb{R}^3} |F \mp \star D\phi|^2 \, d^3x \mp 2 \lim_{R\to\infty} \int_{S^2_R} \langle \phi \wedge (d\phi \wedge d\phi)\rangle \, .$$

This boundary term turns out to be a topological contribution[3] taking a value in $8\pi\mathbb{Z}$ which can be interpreted in two ways, either as the degree of $\phi$ as a continuous map of 2-spheres $S^2_\infty \to \{X \in \mathfrak{su}(2) \mid |X| = 1\}$, or as the Chern class of the associated eigenvalue-1 complex eigenbundle $L \to S^2_\infty$. This first interpretation can also be viewed from the perspective of symmetry breaking as a map into the coset space $\mathrm{SU}(2)/\mathrm{U}(1)$ [MS04, §8.13]. For any given value of the topological term, the energy functional is made stationary (actually locally minimal) when the square term is zero. This gives us the following definition.

**Definition 3.1.9** ([Bog76, PS75])**.** *We define the* ***Bogomol'nyi-Prasad-Sommerfield (BPS) equation(s)*** *to be*

$$F = \pm \star D\phi.$$

*The sign is chosen to saturate the positive bound $E \geq \pm 8\pi k$ for some $k \in \mathbb{Z}$. We will want to have $k > 0$ and so take the equation $F = \star D\phi$ (this sign is also motivated by a reduction of an anti-self-dual field of higher dimension, see §3.1.1).*

[3] By a topological contribution, I mean a term which depends only on the fields up to a homotopy.

**Remark 3.1.10.** *Historically, the name BPS comes about because of the two separate contributions cited, with Bogomol'nyi providing the 'trick' for rewriting the energy functional in Lemma 3.1.8, and Prasad-Sommerfield considering the limiting case where the potential is zero, i.e. $\lambda = 0$. The limit $\lambda \to 0$ was first called the Prasad-Sommerfield limit in [Man77].*

**Lemma 3.1.11** ([JT80], p. 5)**.** *Given one solution $(A(\boldsymbol{x}), \phi(\boldsymbol{x}))$ to the BPS equation, there is a corresponding 1-parameter family of solutions for $\mu \in \mathbb{R}^\times$, $(A_\mu(\boldsymbol{x}), \phi_\mu(\boldsymbol{x})) = (\mu A(\mu\boldsymbol{x}), \mu\phi(\mu\boldsymbol{x}))$.*

As a result of how we derived the BPS equation, it is clear any solution of the BPS equation solves the YMH equations with $\lambda = 0$.

**Definition 3.1.12.** *An* SU(2) ***Euclidean BPS monopole*** *is a smooth solution to the BPS equation with gauge group* SU(2) *on $\mathbb{R}^3$ satisfying the monopole boundary conditions. The integer $k$ determining the energy to be $E = 8\pi k$ is called the* ***charge*** *of the solution.*

**Remark 3.1.13.** *Schematically, for a solution of the BPS equation where $B = \star F \sim D\phi$ determines the magnetic field, taking the first monopole boundary condition we expect $D\phi \sim \frac{k}{r^2}$, which looks like the electric field of a classical electron. This gives a partial interpretation to the understanding of a solution to the BPS as a magnetic analogue of the electron.*

**Proposition 3.1.14** ([War81c], p. 317-318, [Hit83], p. 155)**.** *Given a BPS monopole, $c_1(L) = k = -\deg\phi$ where $k$ is the integer in the asymptotic expansion of $|\phi|$ and $L \to S^2_\infty$ is the $(1)$-eigenbundle of $\phi$ over the sphere at infinity.*

**Remark 3.1.15.** *The proof of this fact uses* ***Ward's formula*** *that $\nabla^2 |\phi|^2 = 2|D\phi|^2$, which is related to $k$ by the fact that the energy of a BPS solution is $E = \int_{\mathbb{R}^3} 2|D\phi|^2 d^3x$. As such we can think of $\nabla^2 |\phi|^2$ as the energy density $\mathcal{E}(\boldsymbol{x})$ normalised such that $\int \mathcal{E} d^3x = 8\pi k$.*

**Remark 3.1.16.** *The connection between the degree of a map $\mathbb{P}^1 \to \mathbb{P}^1$ and the Chern class of a line bundle $L \to \mathbb{P}^1$ is exactly that seen in §2.1.2.*

**Example 3.1.17** (Prasad-Sommerfield solution [PS75])**.** *We can make an ansatz of spherical symmetry for a static monopole in $\mathbb{R}^3$ to assume our solution has the form*

$$\phi = ih(r)\frac{x^j\sigma_j}{r} = \frac{-2h(r)x^jT_j}{r} = \frac{H(r)x^iT_i}{r^2},$$
$$A_j = -\frac{i}{2}\left[1 - k(r)\right]\frac{\epsilon_{jk}{}^{l}x^k\sigma_l}{r^2} = \left[1 - k(r)\right]\frac{\epsilon_{jkl}x^kT^l}{r^2},$$

*that is, $H = -2rh$. The functions $h, k$ are those used in [MS04, (8.79)], $k$ is not to be confused with the charge. It is a computational task to determine that the BPS equation reduces to the ODEs*

$$rH' - H = k^2 - 1, \qquad rk' = kH,$$

*and these give the solutions*

$$H(r) = 1 - r\coth r, \qquad\qquad k(r) = \frac{r}{\sinh(r)}.$$

*Note we can verify here that* $\phi_\infty = -\hat{\boldsymbol{x}} \cdot \boldsymbol{T} \Rightarrow \deg\phi = -1$ *as the corresponding map of spheres is the antipodal map. We can verify Proposition 3.1.14 here, as we know from our degree calculation that* $k = 1$*, and*

$$\begin{aligned} |\phi|^2 &= \frac{H^2}{r^4} x^i x^j \langle T_i, T_j \rangle = \frac{H^2}{r^2}, \\ \Rightarrow |\phi| &= 1 - \frac{1}{r} + \mathcal{O}(r^{-2}). \end{aligned}$$

*This spherically symmetric solution with* $\phi$ *pointing outwards gets the name the* ***hedgehog solution*** *from Polyakov, who wrote down the equations (in second order form) in [Pol74], or the name* ***Prasad-Sommerfield solution****, from the first authors to solve the equations exactly [PS75]. Somewhat amusingly they describe the discovery of such a solution via "shimmying" trial solutions involving hyperbolic trigonometric functions.*

*If we substitute back in* $H = -2rh$ *into the ODEs for the BPS solution we get*

$$-2r(h + rh') + 2rh = k^2 - 1, \qquad\qquad rk' = -2rkh$$

*or equivalently*

$$-2r^2 h' = k^2 - 1, \qquad\qquad k' = -2kh$$

*as they are written in [MS04, (8.87), (8.88)]. Consider now the 1-parameter family of functions*

$$h_\mu(r) = \frac{\mu}{2}\left[\coth(\mu r) - \frac{1}{\mu r}\right], \qquad\qquad k_\mu(r) = \frac{\mu r}{\sinh(\mu r)},$$

*which one can verify also gives a solution to the BPS equation if we define analogously* $\phi_\mu = i h_\mu(r) \frac{x^j \sigma_j}{r}$*, which is nothing but the scaling of Lemma 3.1.11. Now*

$$\mathrm{Tr}(\phi_\mu^2) = -2h_\mu(r)^2 \sim_{r\to\infty} -\frac{\mu^2}{2}\left[1 - \frac{2}{\mu r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right].$$

*Hence if we had instead defined the norm on our Lie algebra as* $|X|^2 = -\alpha\,\mathrm{Tr}(X^2)$*, to have* $\phi_\mu \to 1$ *we need* $\mu = \sqrt{2/\alpha}$*. As the first order to the correction of* $|\phi_\mu|^2$ *is* $\frac{2}{\mu r}$*, if we always want the Prasad-Sommerfield solution to be a monopole then we know that our asymptotic conditions in Definition 3.1.5 must be defined relative to our choice of norm on* $\mathfrak{su}(2)$*, as discussed in Remark 3.1.6.*

**Remark 3.1.18.** *Note the hedgehog has a singular zero of* $\phi$ *at* $r = 0$*, and no poles. In general the charge of the monopole will correspond to the number of zeros of* $\phi$ *counted with multiplicity as we have already seen through Proposition 3.1.14 [AFG75]; we need to count with multiplicity, as the Higgs field can* *have an 'anti-zero' [Sut96a]. As mentioned in §3.1.1, the existence of a charge-$k$ monopole for any $k$ is not immediately clear from the variational approach, and indeed because of the repulsive magnetic Coulomb force an attractive force is required. It was argued heuristically by Manton [Man77] (using the asymptotic forces between monopoles with the Higgs field providing the attractive force) and proven by Taubes [Tau81, JT80] that there exists a charge-$k$ monopole which can be written as $k$ well-separated charge-1 monopoles, with the positions of the monopoles corresponding to the zeros of $\phi$ [War81a]. For gauge groups different from* SU(2) *this identification between charge and zeros of $\phi$ need not occur [War81b].*

#### Self-Dual Reduction

We mentioned briefly in §3.1.1 how solutions to the YMH equations on Euclidean $\mathbb{R}^3$ come from static solutions to the YMH equations on Minkowski $\mathbb{R}^4$. It is worthwhile to also note that one can get solutions to the BPS equation from time-independent pure Yang-Mills solutions on Euclidean $\mathbb{R}^4$, which give instantons. In particular, given that the connection ${}^4A = {}^4A_\mu dx^\mu$ is time-independent, we can write it as ${}^4A = \phi dx^0 + {}^3A_i dx^i$ where $\phi, {}^3A_i$ are functions of $x_i$ only[4], and then one has the following result.

**Proposition 3.1.19** ([Man77], attributed to J. M. Cervero)**.** *The electromagnetic tensor ${}^4F$ is (anti-)self-dual if and only if $({}^3F, \phi)$ satisfy the BPS equation, that is*

$$\star_4 {}^4F = \pm {}^4F \Leftrightarrow {}^3F \pm \star_3 D\phi = 0.$$

*Here I have made explicit which dimension to consider for the Hodge star.*

Proposition 3.1.19 is unsurprising when viewed through the lens that both the self-duality equation and the BPS equation can be formulated in terms of a quaternionic moment map equation [Hit87] (see Proposition 3.1.23) though this development followed later. For a thorough review of solutions to the Yang-Mills equations including the relation to monopoles see [Act79].

**Remark 3.1.20.** *Here we have written $\mathbb{R}^4 \cong \mathbb{R} \times \mathbb{R}^3$, and then asked for $\mathbb{R}$-invariance giving a Euclidean monopole on $\mathbb{R}^3$. One also has that $\mathbb{R}^4 \setminus \mathbb{R}^2$ is conformally equivalent to $S^1 \times H^3$ ($H^3$ being hyperbolic 3-space) and so asking for $S^1$-invariance gives hyperbolic monopoles on $H^3$ [Ati87, §5]. Note we need to remove the $\mathbb{R}^2$ that is fixed by an $S^1$ action on $\mathbb{R}^4$.*

### 3.1.2 The Moduli Space

We have now described the solutions we want to investigate, and a starting point would be to try and understand the moduli space of such solutions. Having introduced the concept of the moduli space of monopoles in this section we will

[4] Here I have used ${}^4A$ and ${}^3A$ to distinguish between the connection on $\mathbb{R}^4$ independent of $x^0$ and the corresponding induced connection on $\mathbb{R}^3$.

see later in §3.3.1 how to bound the dimension of certain submanifolds of the moduli space corresponding to monopoles with symmetries.

**Remark 3.1.21.** *On a historical note, it was Riemann who introduced the concept of moduli in his 1857 paper [Ji15].*

The specific moduli space one initially wants to consider is $N_k$, the space of monopoles gauge fields up to gauge equivalence. Using an index theorem calculation applied to a Dirac operator constructed from the monopole data Weinberg [Wei79] showed that $\dim_{\mathbb{R}} N_k = 4k - 1$, with the interpretation being that a charge-$k$ monopole comes with four parameters corresponding to the position and phase of each of the $k$ 1-monopole constituents and that an overall phase can be factored out by gauge transform. As the monopole fields are naturally acted on by the Euclidean group $\mathrm{E}(3)$, $N_k$ inherits an action. One typically enlarges $N_k$ by a $\mathrm{U}(1)$ phase to $M_k$, the moduli space of charge-$k$ framed monopoles, by restricting possible gauge transforms to those which tend to the identity along a fixed direction [AH88, HMM95]. As such $\dim_{\mathbb{R}} M_k = 4k$, and moreover it now has an action of $\mathrm{U}(1)$ in addition to the inherited action of $\mathrm{E}(3)$. Associated with $M_k$ is the submanifold of (strongly-) centred charge-$k$ monopoles $M_k^0 \subset M_k$ with an action of the orthogonal group $\mathrm{O}(3)$ which parametrises monopoles up to gauge transform with fixed centre, or equivalently framed monopoles with fixed centre and phase [AH88, MS04]. $M_k^0$ is a totally geodesic manifold of real dimension $4(k-1)$.

**Example 3.1.22.** *We saw in Example 3.1.17 that there was a unique spherically-symmetric 1-monopole parametrised by its position. We will see in §3.2.2 using the spectral curve and Nahm matrices that these are in fact the only 1-monopoles, so* $M_1^0 = *$ *the one-point space and* $M_1 = \mathbb{R}^3 \times S^1$.

### Hyperkähler Metric

We have seen now that 4 divides the dimension of $M_k$ and $M_k^0$, which is a necessary condition for them to be hyperkähler manifolds as the tangent space becomes a quaternionic vector space. In fact $M_k$ is hyperkähler, as made clear by the following sequence of results.

**Proposition 3.1.23** ([Hit87], p. 23)**.** *The BPS equation* $F = \star D\phi$ *is equivalent to the vanishing of a hyperkähler moment map on the space of* $L^2$ *fields*[5] $(A, \phi)$ *with the quaternionic group action being the* $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ *gauge transformations which preserve normalisability.*

The gauge transformations which preserve normalisability with respect to the $L^2$ norm are exactly those which we restricted to in defining $M_k$ as $S^1$-bundle over $N_k$ [AH88, Chapter 3], yielding the following result.

**Corollary 3.1.24** ([Hit87], p. 76)**.** *The monopole moduli space* $M_k$ *is a hyperkähler quotient, with metric inherited from the* $L^2$ *metric.*

[5]By "$L^2$ fields" I am referring to fields on $\mathbb{R}^3$ whose $L^2$ norm $\|f\| = \int_{\mathbb{R}^3} |f|^2 \, \mathrm{d}^3x$ is finite.

Monopole moduli spaces thus provide important examples of hyperkähler manifolds with an SO(3)-invariant metric.

**Example 3.1.25.** *The hyperkähler metric on $M_1 \cong \mathbb{R}^3 \times S^1$ is the flat metric [AH88]. $M_2^0$ is the simplest nontrivial example of the monopole moduli space, and the metric was calculated explicitly in [AH85, AH88] in terms of elliptic functions. As such, $M_2^0$ is often called the* ***Atiyah-Hitchin manifold****.*

**Remark 3.1.26.** *$M_2^0$ as a hyperkähler manifold has been discovered in other areas of mathematical physics, for example as the Coulomb branch of the vacuum moduli space of pure* SU(2) *(2+1)-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory [SW96, CH97, HW97].*

In this thesis I shall not go into any detail on the hyperkähler geometry of the moduli space, but it remains an active area of research because of attempts to understand the geometry of the moduli space of hyperbolic monopoles. For these monopoles the $L^2$ metric diverges, and so various attempts have been made to understand what the most natural structure and metric are [BA90, Nas07, Hit08, BCS15, Sut22a, Sut22b, FR23].

### Rational Maps

One way to understand the monopole moduli space was given in terms of rational maps.

**Theorem 3.1.27** ([Don84, Hur85b])**.** *The framed monopole moduli space $M_k$ and the moduli space $R_k$ of degree-$k$ rational maps $S : \mathbb{P}^1 \to \mathbb{P}^1$ such that $S(\infty) = 0$ are diffeomorphic.*

**Remark 3.1.28.** *The condition in Theorem 3.1.27 that $S(\infty) = 0$ is present to fix an orientation of the target $\mathbb{P}^1$ relative to the source $\mathbb{P}^1$. Such rational maps are sometimes called* ***based****, for example in [HS96a]. Note that without this condition the space of degree-$k$ rational maps has real dimension $2 \times 2 \times (k+1) - 2 = 4k+2$ (there are $k + 1$ complex coefficients in the numerator and denominator of a degree-$k$ rational map, and we remove and overall scale), whereas $\dim_{\mathbb{R}} R_k = 2(k + 1) + 2k - 2 = 4k$ as desired.*

The submanifold of $R_k$ corresponding to $M_k^0$ can also be identified, namely given a rational map given by $p(\zeta)/q(\zeta)$ and labelling the roots of $q(\zeta)$ as $\beta_i$, $i = 1, \dots, k$, one gets that the framed monopole corresponding to the rational map is centred if and only if [Hur83, AH88, HMM95]

$$\sum_i \beta_i = 0, \qquad \left| \prod_i p(\beta_i) \right| = 1.$$

Note computing these quantities does not require explicitly finding the roots of $q$; the sum $\sum_i \beta_i$ is proportional to the coefficient of $\zeta^{k-1}$ in $q$, and the product $\prod_i p(\beta_i)$ is proportional to the resultant of $p$ and $q$. To actually get $M_k^0$ we need to fix a phase of the framed monopole so we are considering strongly-centred framed monopoles not just centred framed monopoles, which we can do without loss of generality by imposing the condition $\prod_i p(\beta_i) = 1$. Note the condition that $\prod_i p(\beta_i) = 1$ is invariant under $p \mapsto e^{2\pi i/k}p$, and $M_k^0$ is actually the quotient of $\widetilde{M_k^0} := \{\sum_i \beta_i = 0, \prod_i p(\beta_i) = 1\}$ by this $C_k$ action. As such $\widetilde{M_k^0}$ is a $k$-fold cover of $M_k^0$, and it turns out to be the universal cover [AH88, p. 20].

**Example 3.1.29.** *The rational maps corresponding to $M_1$ are given by $S(\zeta) = \frac{a}{\zeta - b}$ for $a, b \in \mathbb{C}$. The submanifold corresponding to $M_1^0$ is given just by $S(\zeta) = \frac{1}{\zeta}$ [AH88, p. 18].*

**Example 3.1.30.** *The rational maps corresponding to monopoles in $M_2^0$ are given by*

$$S(\zeta) = \frac{a_0 + a_1\zeta}{\zeta^2 + b_0}$$

*where $a_0, a_1, b_0 \in \mathbb{C}$ satisfy $a_0^2 + a_1^2 b_0 = 1$ [AH88, p. 20].*

Theorem 3.1.27 was initially conjectured based upon work of Atiyah [Ati87, Ati84] finding rational maps associated with hyperbolic monopoles. Donaldson proved the theorem from the perspective of Nahm data (§3.2.2), and Hurtubise showed the connection to spectral curves and the scattering picture of Hitchin (§3.2.1). The assignment of a rational map to a given monopole is such that $R_k$ inherits a natural group action, though the choice of map breaks some of the symmetry of the monopole moduli space by choosing a distinguished direction [AH88, HMM95].

Jarvis [Jar98] also gave a construction of a rational map associated with a monopole in a way that naturally generalised in the case of non-maximal symmetry breaking and arbitrary gauge group. Jarvis' construction has the added benefit that rather than fixing a direction, it requires just a fixing of origin, which makes the associated rational maps much more useful when studying symmetries of monopoles which fix an origin. The construction associates to each $k$-monopole an equivalence class of degree-$k$ rational maps, where two rational maps are equivalent if they are related by an SU(2) Möbius transformation [IS99].

The rational map approach has a number of uses for understanding monopoles. Numerically, given a rational map one can construct the Higgs field, allowing for the plotting of approximate energy density isosurfaces [IS99]. This process still involves solving a PDE, so lacks the computational gains that come from plotting energy density isosurfaces via Nahm data (see §3.4.2). Moreover, as described in the appendix of [HMS98], understanding the rational maps symmetric under finite groups $G \leq \mathrm{SU}(2)$ can give existence results for $G$-symmetric monopoles (see Propositions 3.3.6 and 3.3.8).

### Geodesic Approximation

The monopole solutions we consider are static, but one can ask about time-dependent YMH solutions which are 'close' to monopole solutions at any point in time, interpreted as a scattering process. It was suggested by Manton in [Man82] and proven rigorously in [Stu94] that such dynamical processes are well-approximated in the low-energy limit by geodesics in the moduli space $M_k$. As such, if one can find a 4-dimensional geodesic submanifold of the moduli space this generically corresponds to the SO(3) orbits of a geodesic in $M_k$, and so one can obtain a scattering of monopoles. It was in this way that scattering corresponding to $M_1$ and $M_2^0$ was first understood [AH85].

A long standing question is whether the geodesic motion in $M_2^0$ is integrable. The geodesic does have three known conserved quantities, total angular momentum, a generalised momentum, and energy. Moreover, the asymptotic region of the moduli space limits to the asymptotic region of Taub-NUT space, gaining an extra SO(2) symmetry and so an additional conserved quantity corresponding to relative electric charge, and as such the corresponding geodesic motion is Liouville integrable [Sch91]. As this quantity is not conserved away from the limiting region of the moduli space, the geodesic motion in the full moduli space $M_2^0$ is conjectured to be nonintegrable, a notion supported by numerical investigations involving Poincaré recurrence plots [TR88, TR89]. At present attempts to prove the nonintegrability of the geodesic motion using differential Galois theory have been unsuccessful [MPV23].

## 3.2 Monopole Spectral Curves

> Says Plowden, the whale so caught belongs to the King and Queen, "because of its superior excellence." And by the soundest commentators this has ever been held a cogent argument in such matters.
>
> – Herman Melville
> Moby Dick

Now, over half way through this thesis, I shall introduce the rallying banner uniting the study of Riemann surface and magnetic monopoles, namely the spectral curve. An object of "superior excellence", it is a key tool of study which allows us to use the machinery of algebraic geometry to understand monopoles, and of itself the spectral curve provides gems in the realms of pure mathematics. We will approach the spectral curve from two equivalent perspectives, Hitchin's scattering approach, and through Nahm's equations.

### 3.2.1 Hitchin's Scattering Approach

In [Hit82], Hitchin will define the spectral curve by considering a scattering picture which I will briefly cover now. This will involve the construction of a vector bundle over minitwistor space, and in fact can be viewed as $\mathbb{R}$-invariant instanton bundles over twistor space, as monopoles are $\mathbb{R}$-invariant instantons [CG81, WW91]. The distinction between the instanton bundles and the monopole bundles will be that for the latter there is a canonically defined algebraic curve, and this will be the spectral curve we seek. Note the scattering discussed here is of particles scattering in a monopole background, distinct from the picture of monopole scattering discussed in §3.1.2.

### Monopole Bundles

Let $\mathbb{MT}$ be the Euclidean minitwistor space introduced in §2.1.3, and fix some gauge and Higgs field $(A, \phi)$ associated with the trivial SU(2) bundle $P$. Moreover let $\tilde{E} \cong \mathbb{R}^3 \times \mathbb{C}^2$ be the rank-2 complex vector bundle associated with $P$ by the standard[6] matrix representation of SU(2). Over an oriented line $l \in \mathbb{MT}$ with direction $\boldsymbol{u}$ one can define a 2-dimensional complex vector space

$$E_l = \left\{ s \in \Gamma(l, \tilde{E}) \,|\, (D_{\boldsymbol{u}} - i\phi)s = 0 \right\},$$

where $\Gamma(l, \tilde{E})$ is the space of (smooth) sections of $\tilde{E}$ over the line $l$, and $D_{\boldsymbol{u}}$ is the covariant derivative in the direction of $l$. This gives a corresponding complex vector bundle $E \to \mathbb{MT}$ called the **monopole bundle**. $\mathbb{MT}$ has an involution $l \to -l$ coming from reversing the orientation of a line, and this lifts to a map $E_l \to E_{-l}$.

**Remark 3.2.1.** *In [Hit82, Hit83] Hitchin uses the notation $E, \tilde{E}$ the other way around.*

**Remark 3.2.2.** *The operator $(D_{\boldsymbol{u}} - i\phi)$ is the Dirac operator considered in [Wei79] when Weinberg gave the dimension of the moduli space of monopoles.*

Now we have seen in §2.1.3 that $\mathbb{MT}$ can be given the structure of a complex manifold, whereby it is isomorphic to $T\mathbb{P}^1$, and the orientation-reversing involution on $\mathbb{MT}$ corresponds to an antiholomorphic involution $\tau$ of $T\mathbb{P}^1$ (see Definition 2.1.65). Moreover, when $(A, \phi)$ corresponds to a BPS monopole, it is a consequence of the hyperkähler moment map picture discussed in §3.1.2 that the operator defining $E$ is in fact holomorphic [Hit87, §II.2], and so we get the following theorem.

**Theorem 3.2.3** ([Hit82], Theorem 4.2)**.** *If $A, \phi$, satisfy the* SU(2) *BPS equation then $E \to T\mathbb{P}^1$ is a holomorphic vector bundle such that*

- *$E$ is trivial along the image of a real section of $T\mathbb{P}^1 \to \mathbb{P}^1$, that is the image of a section invariant under $\tau$,*
- *$E$ has a symplectic structure, that is a symplectic form $\omega_z$ on each fibre $E_z$ varying holomorphically with $z$, and*
- *$E$ has a quaternionic structure (in the sense of §2.1.3), that is an antilinear map $\sigma : E_z \to E_{\tau z}$ such that $\sigma^2 = -1$ varying antiholomorphically in $z$.*

*Moreover, every such $E$ gives a solution to the BPS equations.*

**Remark 3.2.4.** *Hitchin [Hit82, p. 587] remarks that nothing about the proof of Theorem 3.2.3 is specific to* SU(2)*, and so can be rephrased for any real form of a complex Lie group.*

[6] Calling this representation the 'standard' representation is using terminology from [Hal15, §4.2].

**Remark 3.2.5.** *Hitchin [Hit82] remarks that the approach of using $T\mathbb{P}^1$ to study objects in $\mathbb{R}^3$ was first done by Weierstrass in his study of minimal surfaces. Small takes this further in [Sma02] to consider the minimal surfaces determined by monopoles.*

### The Spectral Curve

To include all the information for a monopole solution we also need to consider boundary conditions. The following result shows that this is not immediate.

**Lemma 3.2.6** ([Hit87], p. 37)**.** *Not every line admits nontrivial $L^2$ solutions to $(D_{\boldsymbol{u}} - i\phi)s = 0$.*

This leads us to make the following definition.

**Definition 3.2.7** ([Hit87], p. 37)**.** *A **spectral line** is $l \in \mathbb{MT}$ in the direction $\boldsymbol{u}$ along which*

$$(D_{\boldsymbol{u}} - i\phi)s = 0$$

*admits a nontrivial $L^2$ solution. The **spectral curve** $\mathcal{C} \subset T\mathbb{P}^1$ is the collection of spectral lines after identifying $\mathbb{MT}$ and $T\mathbb{P}^1$.*

At this point the name "spectral curve" is unjustified, as we only know that that the spectral curve is some subset of $T\mathbb{P}^1$. In fact, it is a consequence of the proof of Lemma 3.2.6 that the spectral curve lies in some compact subset of $T\mathbb{P}^1$, and so provided it is closed it is also compact [Hit82, p. 595]. To get a geometric description of the curve, we first note that along any given line parametrised by $t$, using the asymptotic form of $A, \phi$ in the 't Hooft gauge[7] [tH74]

$$\phi \sim \left(1 - \frac{k}{2r}\right)\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \mathcal{O}(r^{-2}),$$

there is a unique (up to scale) solution $s_\pm$ that is normalisable with respect to the $L^2$ norm as $t \to \pm\infty$. As such we define the holomorphic line bundles $L^\pm \subset E$ corresponding to such solutions. In terms of these we can write the spectral curve as $\mathcal{C} = \{z \in T\mathbb{P}^1 \mid L^+_z = L^-_z\}$. Recalling Definition 2.1.69 the key result is then the following.

**Proposition 3.2.8** ([Hit82], Theorem 6.3)**.** *$L^+ \cong L(-k)$, and $E$ can be given as an extension of line bundles*

$$0 \to L^+ \to E \to (L^+)^* \to 0,$$

*equivalently*

$$0 \to L(-k) \to E \to L^*(k) \to 0.$$

The proof proceeds by showing that $L^+$ is a holomorphic subbundle of $E$, whereby one knows there is an SES $0 \to L^+ \to E \to L' \to 0$ for some line bundle $L'$. The symplectic form on $E$ ensures that $L' = (L^+)^*$, and then Hitchin goes

[7] This is called the 'Abelian gauge' in [AFG75].

on to identify $L^+ \cong L(-k)$. In this context one should think of $L$ as being the monopole bundle corresponding to the trivial solution of the U(1) BPS equation. Clearly this process could be carried out using $L^-$ instead so one also has that $E$ can be written as an extension $0 \to L^- \to E \to (L^-)^* \to 0$.

Moreover, as $\tau$ corresponds to reversing the orientation of geodesics in minitwistor space, it must be the case that $\tau^*\overline{L^+} = L^-$. This allows us to identify $L^-$ from the identification $L^+ \cong L(-k)$, recalling that $g_{01} = e^{-\eta/\zeta}\zeta^{-k}$ is the transition function for $L(-k)$, hence $\tau^*\overline{g_{01}} = e^{-\eta/\zeta}(-\zeta)^k$ is the transition function for $\tau^*\overline{L(-k)}$, and it must be the transition function from $\tilde{U}_0$ to $\tilde{U}_1$. Up to a factor of $(-1)^k$, the term $e^{-\eta/\zeta}(-\zeta)^k$ is equal to the transition function from $\tilde{U}_0$ to $\tilde{U}_1$ of $L^*(-k)$, so $L^- \cong L^*(-k)$.

As such, restricting to $\mathcal{C}$ where $L^+ = L^-$ one has $L \cong L^*$ and so the line bundle $L^2 \to \mathcal{C}$ is trivial. Hence, temporarily just assuming that $\mathcal{C}$ is a smooth genus-$g$ curve, $L \in \mathrm{Pic}^0(\mathcal{C})$ and the corresponding vector in the Jacobian is a 2-torsion point, which imposes $g$ constraints on the period matrix of $\mathcal{C}$. Moreover, because $\tau$ maps $L \to L^* \cong L^{-1}$, $L$ is an imaginary point in $\mathrm{Jac}(\mathcal{C})$ under the complex structure induced by $\tau$. As such, the $g$ constraints imposed on $\mathcal{C}$ are *real* constraints.

The $k$ occurring in Proposition 3.2.8 is the charge of the corresponding monopole, and the construction of $E$ as an extension is the equivalent of the $\mathcal{A}_k$ ansatz for the construction of charge-$k$ instantons [AW77]. It is a result of homological algebra that extensions of holomorphic vector bundles over a complex manifold $X$ of the form

$$0 \to L_1 \to E \to L_2 \to 0$$

are classified by sheaf cohomology $H^1(X, L_1 \otimes L_2^{-1})$. To see this we require two facts:

1. given $R$-modules $A$, $B$, equivalence classes of extensions of $A$ by $B$ are in 1-1 correspondence with $\mathrm{Ext}^1_R(A, B)$ [Wei95, Theorem 3.4.3],

2. fixing a sheaf of $R$-modules $F$ on a ringed space $(X, R)$, $H^1(X, F) = \mathrm{Ext}^1_R(R, F)$, which follows as the sheaf cohomology and Ext functors are the right-derived functors of the same global section functor $\Gamma(X, \cdot) = \mathrm{Hom}_R(R, \cdot)$.

The result then follows by thinking of the line bundles as sheaves of $\mathcal{O}_X$-modules and tensoring the SES to get

$$0 \to L_1 \otimes L_2^{-1} \to E \otimes L_2^{-1} \to \mathcal{O}_X \to 0.$$

This can also be interpreted as saying that the transition function of $E$ is upper triangular, with the diagonal entries being the transition functions of $L_{1,2}$, and the off-diagonal entry giving maps in $\mathrm{Hom}(L_2, L_1)$ (see Theorem 3.2.33). As such, the monopole bundle is classified by a cohomology class $\Gamma \in H^1(T\mathbb{P}^1, L^2(-2k))$.

We can now express the spectral curve as an algebraic curve concretely.

**Theorem 3.2.9** ([Hit82], Proposition 7.3)**.** *The spectral curve is represented in*

*$\tilde{U}_0$ by the polynomial*

$$P(\zeta,\eta) = \eta^k + a_1(\zeta)\eta^{k-1} + \cdots + a_k(\zeta) = 0, \tag{3.3}$$

*where $a_r$ is a polynomial of degree $2r$, i.e. $a_r \in H^0(\mathbb{P}^1, \mathcal{O}(2r))$.*

*Proof.* Recall that we have a symplectic form $\omega$ on $E$. As we want to find the $z \in T\mathbb{P}^1$ such that $L_z^+ = L_z^-$, this is equivalent to finding the $z$ such that $\omega(L_z^+, L_z^-) = 0$. The concept of projection in $E$ can be defined using $\omega$, and so the spectral curve is now given by the $z \in T\mathbb{P}^1$ such that the projection map $P_z : L_z^- \to (L_z^+)^*$ is the zero map (viewed as a projection of subspaces of $E_z$). We can view $P_z \in (L_z^- \otimes L_z^+)^*$, so we are getting the zeros of the section

$$P \in H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(2k)),$$

recalling $\pi$ is the projection $T\mathbb{P}^1 \to \mathbb{P}^1$. As $\tilde{U}_{0,1}$ give a Leray cover of $T\mathbb{P}^1$ with respect to the sheaf $\pi^*\mathcal{O}(2k)$, such a section is given by $(f_0, f_1)$ with the $f_i$ holomorphic on $\tilde{U}_i$ satisfying

$$f_0(\zeta,\eta) = \zeta^{2k} f_1(1/\zeta, \eta/\zeta^2),$$

on $\tilde{U}_0 \cap \tilde{U}_1$. For the $f_i$ to be holomorphic we must have that the Laurent series for $f_0$ contains no $\zeta^{-1}$ terms, and that the series for $f_1$ likewise contains no $\zeta$ terms. This limits the range of their expansions so we must have

$$f_0(\zeta,\eta) = \sum_{i=0}^{k} a_i(\zeta)\eta^{k-i},$$

and because of the $\zeta^{2k}$ term we must have that $a_i$ has degree at most $2i$. Moreover, in order to get compactness we know that as $\eta \to \infty$ we cannot have any solutions, and so as $a_0$ is a constant it must be the case $a_0 \neq 0$. Hence we rescale to get

$$f_0(\zeta,\eta) = \eta^k + a_1(\zeta)\eta^{k-1} + \cdots + a_k(\zeta).$$

I will often abuse notation by writing $f_0 = P$ and calling this the section. □

**Remark 3.2.10.** *By thinking of the spectral curve as the zero locus of a section of the line bundle $\pi^*\mathcal{O}(2k)$, we can equivalently think of it as a (Weil) divisor in the linear system $|\pi^*\mathcal{O}(2k)|$ recalling the line bundle/divisor correspondence from §2.1.2. As such, by Proposition 2.1.72 we know $g(\mathcal{C}) = (k-1)^2$. This result only applies when the curve is nonsingular; one must use [Hit83, Proposition 3.1] and the arithmetic genus when the curves are reducible.*

*This approach to getting the genus is succinct, but it presents two issues:*

1. *it does not give an intuition as to what the differentials on the curve are, and*

2. *it is rather nonelementary.*

*Tackling the second point first using the machinery of Riemann-Hurwitz, suppose we wrote*

$$\eta^k + \sum_{i=1}^{k} a_i(\zeta)\eta^{k-i} = \prod_{j=1}^{k} \left[\eta - \eta_j(\zeta)\right],$$

*whereby $a_i$ is the elementary symmetric polynomial of degree $i$ in the $\eta_j$. A generic $\zeta$ has $k$ preimages, and the branch points are the $\zeta_*$ where some $\eta_j$ coincide. Generically we may ensure these branch points are not at $\zeta = \infty$. Around such a $\zeta_*$, the part of the curve contributing to multiplicity will look like $\eta^2 - p(\zeta - \zeta_*) = 0$ where $p$ is a quartic polynomial, so one gets a contribution of 4 to the ramification index. Hence*

$$g(\mathcal{C}) = 1 - k + \frac{1}{2} \cdot 4 \cdot \binom{k}{2} = (k-1)^2.$$

*Tackling now the first point, one can check that the differentials*

$$\omega_{i,j} := \frac{\zeta^i \eta^j \mathrm{d}\zeta}{\partial_\eta P}, \quad i, j \geq 0,$$

*are holomorphic if $i + 2j \leq 2k - 4$, which gives the correct number of differentials. I will denote the basis $\{\omega_{i,j}\}$ as $\left\{\Omega^{(l)} \,|\, 1 \leq l \leq g\right\}$, ordering lexicographically in $j, i$ [HMR00, (23)], that is*

$$\Omega^{(1)} = \frac{\eta^{k-2}\mathrm{d}\zeta}{\partial_\eta P}, \quad \Omega^{(2)} = \frac{\eta^{k-3}\mathrm{d}\zeta}{\partial_\eta P}, \quad \Omega^{(3)} = \frac{\eta^{k-3}\zeta\mathrm{d}\zeta}{\partial_\eta P}, \quad \ldots, \quad \Omega^{(g)} = \frac{\zeta^{2k-4}\mathrm{d}\zeta}{\partial_\eta P}. \tag{3.4}$$

The action of $\tau$ preserves $\mathcal{C}$, that is the variety corresponding to $P(\zeta, \eta)$ is preserved, as is clear by considering it as the set of spectral lines. This means $P$ must transform in a predictable way under $\tau$, and this is made precise by the following lemma (recalling the notation of Definition 2.1.66).

**Lemma 3.2.11.** $(-1/\zeta^2)^k P = P^\tau$.

*Proof.* Note first that, as we know $\mathcal{C}$ is preserved by the pullback of $\tau$, it must be the case that $\overline{P \circ \tau} = fP$ for some function $f$. Recall that when we write $P \in H^0(T\mathbb{P}^1, \pi^*\mathcal{O}(2k))$, we are using $\eta$ to denote the tautological section $\eta \frac{d}{d\zeta}$, which pulls back under $\tau$ to $-\overline{\eta}\frac{d}{d\overline{\zeta}}$. Hence, as we fixed that $P$ was monic of degree-$k$ in $\eta$, we find $\overline{\tau^* P} = (-1/\zeta^2)^k P$. □

**Corollary 3.2.12.** *If $(\zeta, \eta_j(\zeta))$ is a root of $P$, $(-1/\overline{\zeta}, -\overline{\eta_j(\zeta)}/\overline{\zeta}^2)$ is a root of $\overline{P}$. More specifically, if $\zeta$ is a branch point, so is $-1/\overline{\zeta}$, and the monodromy*[8] *at each has the same cycle type.*

We can check that this condition implies $a_i(\zeta) = (-1)^i \zeta^{2i} \overline{a_i(-1/\overline{\zeta})}$, and so each $a_i$ can be described by $2i + 1$ real parameters as [Bra11, p. 306]

$$a_i(\zeta) = \chi_i \prod_{l=1}^{i} \left(\frac{\overline{\alpha_{i,l}}}{\alpha_{i,l}}\right)^{\frac{1}{2}} (\zeta - \alpha_{i,l}) \left(\zeta + \frac{1}{\overline{\alpha_{i,l}}}\right), \quad \chi_i \in \mathbb{R}, \quad \alpha_{i,l} \in \mathbb{C}. \tag{3.5}$$

[8] I did not cover monodromy in §2.1, but for an introduction to the fundamental concepts see [Mir95, §III.4].

This gives a total of $\sum_{i=1}^{k} 2i+1 = k^2+2k$ real parameters.

**Remark 3.2.13.** *The data $(\mathcal{C}, \tau)$ of a Riemann surface and an antiholomorphic involution on the surface defines a* ***real Riemann surface****, or equivalently a* ***Klein surface*** *[Sch16].*

We have thus seen that the monopole determines the spectral curve, and in fact the converse is true.

**Theorem 3.2.14** ([Hit82], Theorem 7.6)**.** *The cohomology class corresponding to $E$ is $\Gamma = \delta(\rho)$ for $\rho \in H^0(\mathcal{C}, L^2)$ a trivialisation and*

$$\delta : H^0(\mathcal{C}, L^2) \to H^1(T\mathbb{P}^1, L^2(-2k))$$

*the connecting map of a long exact sequence of cohomology. In particular, $\mathcal{C}$ determines $E$.*

The particular short exact sequence of sheaves of sections inducing the long exact sequence is

$$0 \to \mathcal{O}_{T\mathbb{P}^1}(L^2(-2k)) \stackrel{\otimes P}{\to} \mathcal{O}_{T\mathbb{P}^1}(L^2) \to \mathcal{O}_{\mathcal{C}}(L^2).$$

Coupled to Theorem 3.2.3, this says that given the spectral curve of a monopole, one can in principle determine the original monopole.

**Remark 3.2.15.** *Historically, it was noticed that imposing $g = (k-1)^2$ real constraints on the $k^2+2k$ parameters leaves $4k-1$ degrees of freedom, which was the maximal number of degrees of freedom found of a solution 'constructed' using twistor methods and the $\mathcal{A}_k$ ansatz in [CG81], which suggested that the degrees of freedom of the spectral curve could fully encode the degrees of freedom in the monopole solution.*

**Example 3.2.16** ([Hit82], Example 7.7)**.** *The most generic form of a curve given by Equation 3.5 when $k=1$ is*

$$\eta + \left[(ix_1 - x_2) - 2x_3\zeta + (ix_1 + x_2)\zeta^2\right] = 0,$$

*where $(x_1, x_2, x_3) := \boldsymbol{x} \in \mathbb{R}^3$. This has three real parameters, so one may expect this to be a monopole spectral curve, and indeed we will see in Example 3.2.51 that this is the spectral curve for a hedgehog centred at $\boldsymbol{x}$.*

### The Hitchin Conditions and the Ercolani-Sinha Constraints

Having seen some properties of the spectral curve, we make the following definition.

**Definition 3.2.17** ([Hit83], p. 146)**.** *The* ***Hitchin conditions*** *on a compact algebraic curve $\mathcal{C} \subset T\mathbb{P}^1$, $\mathcal{C} \in |\pi^*\mathcal{O}(2k)|$, are*

1. *$\mathcal{C}$ has no multiple components,*

2. *$\mathcal{C}$ is real with respect to the real structure $\tau$,*

3. *$L^2 \to \mathcal{C}$ is trivial and $L(k-1) \to \mathcal{C}$ is real (in the sense of §2.1.3), and*

4. *$\forall s \in (0,2)$, $H^0(\mathcal{C}, L^s(k-2)) = 0$.*

These conditions are necessary and sufficient conditions on a curve for it to be a monopole spectral curve [Hit83]; as we will see later in Theorem 3.2.47 this fits into a triality with monopole gauge data and Nahm data. Hitchin interprets the fourth condition as "equivalent to the non-singularity of the monopole". Hurtubise remarks that the condition that $L(k-1) \to \mathcal{C}$ is real can be rephrased as saying that the natural pairing on sections of $L^2$ given by $\langle s, s' \rangle = s^\tau \otimes s'$, coming from the fact $\overline{\tau^* L^2} = L^{-2}$, is $(-1)^{k-1}$-definite [Hur83].

**Example 3.2.18.** *The final condition on $H^0(\mathcal{C}, L^s(k-2))$ was not initially identified in [Hit82], Hitchin's first paper defining the monopole spectral curve. There the spectral curves corresponding to the axially-symmetric monopoles found in [War81c, Pra81, PR81, FHP81] were written down as*

$$P(\zeta, \eta) = \begin{cases} \prod_{i=1}^{l} \left[\eta^2 + (k_0/2 + k_i)^2 \pi^2 \zeta^2\right], & k = 2l, \\ \eta \prod_{i=1}^{l} \left[\eta^2 + k_i^2 \pi^2 \zeta^2\right], & k = 2l+1, \end{cases}$$

*for some $k_i \in \mathbb{Z}$. This was fixed in [Hit83, Theorem 8.2], where it was shown that the condition on $H^0$ forces $k_0 = 1$, $k_i = i$.*

**Remark 3.2.19.** *Using the adjunction formula in the form of [Har77, Proposition II.8.20], or equivalently by viewing the spectral curve as the nonsingular complete intersection of a quadric and cubic in $\mathbb{P}^3$ (as $T\mathbb{P}^1$ is the smooth part of a singular quadric) and using [Har77, Exercise II.8.4], one finds that the canonical line bundle on the curve is $K_\mathcal{C} = \pi|_\mathcal{C}^* \mathcal{O}(2k-4)$ [Bie07].*

*Therefore, imposing the Hitchin conditions one finds that $L(k-2)$ is a non-vanishing even theta characteristic. Moreover, from [Hit83, Proposition 4.5] we know*

$$\dim^0(\mathcal{C}, \pi^*\mathcal{O}(k-2)) = \sum_{i=0}^{\lfloor k/2-1 \rfloor} (k-2i-1) = \begin{cases} j^2, & k = 2j, \\ j(j+1), & k = 2j+1. \end{cases}$$

*This lets us calculate the parity of the characteristic $\pi^*\mathcal{O}(k-2)$; for example when $k$ is odd $\pi^*\mathcal{O}(k-2)$ is a vanishing even characteristic. Indeed, when $k=3$ it is the unique such characteristic.*

By Proposition 2.1.17 we know $\deg L^s(k-2) = k(k-2) = g(\mathcal{C}) - 1$, so we get the picture that $L^s(k-2)$ is a straight line curve in $W_{g-1}$ of period 2 in the direction $[\eta/\zeta] \in T_{L^s(k-2)} \operatorname{Pic}^{g-1}(\mathcal{C}) \cong H^1(\mathcal{C}, \mathcal{O}_\mathcal{C})$, not intersecting the theta divisor for any $s \in (0,2)$. Changing the basepoint of the AJ map which maps the degree-$(g-1)$ divisor into $W_{g-1} \subset \operatorname{Jac}(\mathcal{C})$ just corresponds to a translation, so this statement is basepoint independent. The condition of not intersecting the theta divisor is hard to impose explicitly on a curve, though as described in

Example 3.2.18 it was done for axially symmetric monopoles, and so now we shall discuss different ways of understanding the Hitchin conditions.

One approach is to identify $[\eta/\zeta] \in H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ as a vector in $\mathrm{Jac}(\mathcal{C})$, or choosing a basis as $\boldsymbol{U} \in \mathbb{C}^g$. Recalling the notation from Remark 3.2.10 we call $(\zeta, \eta_j(\zeta))$ the preimages of $\zeta$ in $\mathcal{C}$ and let $0_j = (0, \eta_j(0))$, $\infty_j$ the image of $0_j$ under $\tau$. By Remark 2.1.71 $[\eta/\zeta]$ corresponds to the set of Laurent tails $\left\{r_{0_j}\right\}$, $r_{0_j} = \frac{\eta_j(0)}{\zeta}$, and under Serre duality this gives the linear map on holomorphic differentials

$$\omega \mapsto \sum_j \mathrm{Res}_{0_j}\left(\frac{\eta}{\zeta}\omega\right). \tag{3.6}$$

**Remark 3.2.20.** *Labelling $0_j = (0, \eta_j(0))$ gives an explicit ordering to the sheets above $\zeta = 0$. We can then order the sheets above a general $\zeta$ by analytically continuing along a path from 0 to $\zeta$. In general, because of monodromy of the curve $\mathcal{C}$, this ordering will not be path independent and so we should be aware that any time we use $\eta_j(\zeta)$ in an expression, either the expression should be invariant under permutations of the index $j$ (as with the Equation 3.6) or be defined on the curve with branch cuts.*

To get the coordinates of a vector in the Jacobian viewed at $\mathbb{C}^g/\Lambda$, we first fix a canonical homology basis $\{a_j, b_j\}$ and basis of $a$-canonically normalised differentials[9] $\{\nu_j\}$ (i.e. $\int_{a_k} \nu_j = \delta_{jk}$, $\int_{b_k} \nu_j = \tau_{jk}$, where $\tau$ is the Riemann matrix). The $j$th entry of the vector in $\mathbb{C}^g$ corresponding to a linear map $f \in H^0(\mathcal{C}, K_{\mathcal{C}})^*$ is now $f(\nu_j)$. Hence the $j$th entry of the vector in $\mathbb{C}^g$ corresponding to $[\eta/\zeta]$ is

$$U_j = \sum_l \mathrm{Res}_{0_l}\left(\frac{\eta}{\zeta}\nu_j\right).$$

If one can introduce a differential of the second kind $\gamma_0$ such that $\gamma_0 \sim d(\eta/\zeta)$ around $0_j$ and $\gamma_0$ is holomorphic everywhere else we may use the reciprocity law for differentials of §2.1.2 to get that

$$U_j = \frac{1}{2\pi i}\sum_l \begin{vmatrix} \oint_{a_l}\nu_j & \oint_{a_l}\gamma_0 \\ \oint_{b_l}\nu_j & \oint_{b_l}\gamma_0 \end{vmatrix} = \left[\frac{1}{2\pi i}\oint_{b_j}\gamma_0\right] - \tau_{jl}\left[\frac{1}{2\pi i}\oint_{a_l}\gamma_0\right].$$

Such a differential exists, and moreover if we have two such differentials $\gamma_0, \gamma_0'$ then their difference is a everywhere holomorphic differential, so fixing the $g$ values $\oint_{a_k}\gamma_0$ uniquely defines $\gamma_0$ [GH78, p. 244]. I shall set $\oint_{a_k}\gamma_0 = 0$ following standard precedent.

**Definition 3.2.21.** *Defining $\gamma_0$, we call the vector $\boldsymbol{U} \in \mathbb{C}^g$ defined by*

$$U_j = \frac{1}{2\pi i}\oint_{b_j}\gamma_0$$

*the **winding vector**.*

[9] With regards to monopoles I shall use $\tau$ for both the antiholomorphic involution and the Riemann matrix. Context shall hopefully be sufficient to distinguish each.

**Remark 3.2.22.** *There is nothing distinguished about the choice of $\zeta = 0$ for the location of the residues here. As $\eta/\zeta$ is a global meromorphic function, the corresponding Laurent tail divisor gives the zero cohomology class in $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$, and adding this to $[\eta/\zeta]$ would cancel the residues at $\zeta = 0$, shifting them to be taken at $\zeta = \infty$. One could analogously define a differential $\gamma_\infty \sim d(-\eta/\zeta)$ here, which would be related to $\gamma_0$ by $\gamma_\infty = d(-\eta/\zeta) + \gamma_0$ [ES89, BE10b].*

**Proposition 3.2.23.** $\tau : \boldsymbol{U} \mapsto -\boldsymbol{U}$

*Proof.* Applying $\tau$ to the set of Laurent tails we get $\{r_{\infty_j}\}$ where $r_{\infty_j} = -\overline{\eta_j(0)}\zeta = \lim_{P\to\infty_j}\left(\frac{\eta}{\zeta}\right)(P)$ [ES89, p. 395]. As such by our definition of $\gamma_\infty$ we would have

$$\tau : U_j \mapsto \frac{-1}{2\pi i}\oint_{b_j}\gamma_\infty = \frac{1}{2\pi i}\oint_{b_j}\left[-\gamma_0 + d\left(\frac{\eta}{\zeta}\right)\right] = -U_j.$$

□

The triviality of $L^2 \to \mathcal{C}$ can now be understood in terms of the winding vector.

**Proposition 3.2.24** ([HMR00], §3, [BE10b], Lemma 2.1)**.** *Let $\mathcal{C} \subset T\mathbb{P}^1$ be a nonsingular curve in $|\pi^*\mathcal{O}(2k)|$, $\Omega^{(l)}$ be the ordered holomorphic differentials as per Equation 3.4, and $\Omega = (A, B)$ the period matrix with respect to cycles $a_i, b_i$. The triviality of $L^s|_{\mathcal{C}}$ for $s \in \mathbb{R}$ is equivalent to the existence of a solution for integers $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^g$ to the equation*

$$\sum_{j=1}^{g} A_{lj}n_j + B_{lj}m_j = -s\,\delta_{1l}, \tag{3.7}$$

*equivalently the existence of a homology cycle $\mathfrak{es}$ satisfying*

$$\oint_{\mathfrak{es}}\Omega^{(l)} = -s\,\delta_{1l}. \tag{3.8}$$

*The $\boldsymbol{n}, \boldsymbol{m}$ found are such that $\mathfrak{es} = \sum_i a_i n_i + b_i m_i$ and the winding vector corresponding to $L$ is given by*

$$\boldsymbol{U} = \frac{1}{s}\left[\boldsymbol{n} + \tau\boldsymbol{m}\right],$$

*where $\tau = A^{-1}B$ is the Riemann matrix.*

*Proof.* It will be informative to write out the proof of [HMR00]. $L^s$ is trivial if we have nowhere-zero holomorphic functions $\beta_i$ on $\tilde{U}_i \cap \mathcal{C}$ respectively such that $\beta_0 = e^{-s\eta/\zeta}\beta_1$ on the intersection $\tilde{U}_0 \cap \tilde{U}_1 \cap \mathcal{C}$. As $\zeta \to 0$ on the $j$th sheet we have

$$d\left(\frac{\eta}{\zeta}\right) \sim \left[-\frac{\eta_j(0)}{\zeta^2} + \mathcal{O}(1)\right]d\zeta \Rightarrow d\log\beta_1 \sim \left[-\frac{s\eta_j(0)}{\zeta^2} + \mathcal{O}(1)\right]d\zeta$$

as we cannot have an essential singularity in $\beta_0$. If we take $\omega$ to be a holomorphic

1-form on $\mathcal{C}$ then

$$\omega = \frac{\left[\sum_{i=0}^{k-2} \alpha_i(\zeta)\eta^{k-2-i}\right] d\zeta}{\partial_\eta P} \tag{3.9}$$

for some polynomials $\alpha_i$ of degree $2i$. Introducing the notation $g_j$ such that $\omega|_{0_j} = g_j \, d\zeta|_{\zeta=0}$, then thinking of $d\log\beta_1$ as a differential of the second kind the reciprocity law says

$$\frac{1}{2\pi i}\sum_{i=1}^{g}\begin{bmatrix} \oint_{a_i}\omega & \oint_{a_i} d\log\beta_1 \\ \oint_{b_i}\omega & \oint_{b_i} d\log\beta_1 \end{bmatrix} = -s\sum_{j=1}^{k}\eta_j(0)g_j.$$

Now for any cycle $c \in H_1(\mathcal{C}, \mathbb{Z})$ have that $\oint_c d\log\beta_i \in 2\pi i\mathbb{Z}$, and so we may introduce the notation

$$m_j = -\frac{1}{2\pi i}\oint_{a_j} d\log\beta_1, \quad n_j = \frac{1}{2\pi i}\oint_{b_j} d\log\beta_1.$$

Note the integrals for $\beta_0$ will be exactly the same as $d\log\beta_0 = -s\, d\left(\frac{\eta}{\zeta}\right) + d\log\beta_1$, and $d\left(\frac{\eta}{\zeta}\right)$ will have zero periods as it is an exact differential. We can thus define the cycle $\mathfrak{es} = \sum_j n_j a_j + m_j b_j$ such that

$$\oint_{\mathfrak{es}} \omega = -s\sum_{i=1}^{k}\eta_i(0)g_i.$$

We can now calculate

$$(\partial_\eta P)(\zeta,\eta) = \sum_{i=1}^{k}\prod_{j\neq i}^{k}[\eta - \eta_j(\zeta)] \Rightarrow (\partial_\eta P)(\zeta, \eta_i(\zeta)) = \prod_{j\neq i}^{k}[\eta_i(\zeta) - \eta_j(\zeta)],$$

and hence

$$\begin{aligned} g_i &= \frac{\sum_{j=0}^{k-2}\alpha_j(0)\eta_i^{k-2-j}(0)}{\prod_{j\neq i}^{k}[\eta_i(0)-\eta_j(0)]}, \\ \Rightarrow \sum_{i=1}^{k}\eta_i(0)g_i &= \sum_{i=1}^{k}\frac{\sum_{j=0}^{k-2}\alpha_j(0)\eta_i^{k-1-j}(0)}{\prod_{j\neq i}^{k}[\eta_i(0)-\eta_j(0)]}, \\ &= \sum_{j=0}^{k-2}\alpha_j(0)\left[\sum_{i=1}^{k}\frac{\eta_i^{k-1-j}(0)}{\prod_{l\neq i}^{k}[\eta_i(0)-\eta_l(0)]}\right] = \alpha_0, \end{aligned}$$

using the Vandermonde identity, or equivalently Lagrange Interpolation (LI), and that $\alpha_0$ is a constant. The result in terms of $A, B$ then follows from expanding out the integral.

In order to get the expression for $\boldsymbol{U}$, note that $d\log\beta_1$ is a differential of the second kind satisfying $\frac{1}{s}d\log\beta_1 \sim d(\eta/\zeta)$ around $\zeta = 0$, but $\oint_{a_j}\frac{1}{s}d\log\beta_1 \neq 0$. As such we can get the differential of the second kind $\gamma_0$ as $\gamma_0 = \frac{1}{s}\left[d\log\beta_1 + 2\pi i \sum_j m_j\nu_j\right]$. $\square$

**Definition 3.2.25.** *We call the requirement of a solution to Equation 3.7 (equivalently 3.8) in the case $s=2$ the **Ercolani-Sinha (ES) constraints**, and we call $\mathfrak{es}$ the **Ercolani-Sinha cycle**.*

**Proposition 3.2.26.** *If the ES cycle $\mathfrak{es}$ exists it is unique.*

*Proof.* Suppose we have two cycles $\mathfrak{es}, \mathfrak{es}'$ both satisfying the ES constraints, then for any holomorphic differential $\omega$ we would have $\int_{\mathfrak{es}-\mathfrak{es}'}\omega = 0$. As such, by the nonsingularity of the period matrix this implies $\mathfrak{es} = \mathfrak{es}'$. $\square$

**Remark 3.2.27.** *We would expect a solution to the ES constraints to exist only if the curve had sufficient symmetry for the components of the period matrix to be related to each other such that radicals and transcendentals cancel out of the resulting equations. In fact, the satisfying of the ES constraints is enough to argue that a monopole spectral curve must be transcendental, i.e. its coefficients cannot be contained in $\bar{\mathbb{Q}}$ [Bra21].*

**Remark 3.2.28.** *Note the statement of Proposition 3.2.24 is written to emphasise the fact that we required none of the Hitchin conditions.*

Suppose now a curve $\mathcal{C}$ satisfies the ES constraints, and moreover is real with respect to the antiholomorphic involution $\tau$ on $T\mathbb{P}^1$. Here $\tau$ restricts to a antiholomorphic involution of $\mathcal{C}$, so induces a linear map $\tau_* : H_1(\mathcal{C},\mathbb{Z}) \to H_1(\mathcal{C},\mathbb{Z})$ satisfying $\tau_*^2 = \mathrm{Id}$ and

$$\forall a, b, \in H_1(\mathcal{C},\mathbb{Z}), \quad a \circ b = -(\tau_* a)\circ(\tau_* b),$$

that is $\tau$ is orientation-reversing. We can describe how the ES cycle behaves under $\tau$.

**Proposition 3.2.29** ([HMR00], §3)**.** $\tau_*\mathfrak{es} = -\mathfrak{es}$.

*Proof.* Houghton et al. prove this using the action of $\tau$ on the differentials. It is also an immediate consequence of Proposition 3.2.23 and Proposition 3.2.24 for us. I shall also give an additional proof from a different perspective in Remark 3.2.34. $\square$

**Remark 3.2.30.** *One may verify this in Sage as done for the tetrahedral 3-monopole [HMM95] in `Ercolani-Sinha_vector_tetrahedral_3-monopole.ipynb`. The action of $\tau_*$ is found computationally using [KK14, Proposition 3.1].*

**Remark 3.2.31.** *One can ask about the action of $\mathrm{Aut}(\mathcal{C})$ on $\mathfrak{es}$, or equivalently the action on $\boldsymbol{U}$. It was shown in [HMR00] that $\mathfrak{es}$ is invariant under the $A_4$ tetrahedral subgroup of the $S_4 \times C_3$ full automorphism group of the tetrahedral 3-monopole spectral curve, and more generally it was shown in [HMR00, Bra11] that $\mathfrak{es}$ is invariant under the action of any rotation (see §3.3.1). Braden proved this algebraically, whereas Houghton et al. used a continuity argument.*

*Computations in the Sage notebook* ***Ercolani-Sinha_ vector_ tetrahedral_ 3-monopole. ipynb*** *show that, on the tetrahedral 3-monopole, the ES cycle is invariant under the $S_4$ subgroup of the automorphism group, but not under the action of the $C_3$.*

*This discussion shall be important in §3.4 as we can hope to simplify the Hitchin conditions when we have a quotient map $\pi : \mathcal{C} \to \mathcal{C}/G := \mathcal{C}'$. Such a map gives an isogeny decomposition of the Jacobian*

$$\mathrm{Jac}(\mathcal{C}) \sim \mathrm{Jac}(\mathcal{C}') \times \mathrm{Prym}(\mathcal{C} \to \mathcal{C}'),$$

*where* Prym *is the* ***Prym variety****, the orthogonal complement to* $\mathrm{Jac}(\mathcal{C}')$ *with respect to the polarisation on* $\mathrm{Jac}(\mathcal{C})$ *[Rie83, RCR19]. In the case where $\mathfrak{es}$ is invariant under the group action, it pushes down to a cycle on the quotient curve, and so the corresponding vector $\boldsymbol{U}$ is a pullback from the quotient, hence lies in the* $\mathrm{Jac}(\mathcal{C}')$ *factor. We can then hope to impose the ES condition on this subvariety.*

Finally, supposing on $\mathcal{C}$ one also imposes the nonsingularity condition of the Hitchin constraints, one gets a further condition on the ES cycle.

**Proposition 3.2.32** ([HMR00], §3)**.** *Let $k > 1$, then the Hitchin condition that $\forall s \in (0, 2)$, $H^0(\mathcal{C}, L^s(k-2)) = 0$ implies $\mathfrak{es}$ is primitive, i.e. $\mathfrak{es}$ is not an integer multiple of another cycle.*

The proof of this result uses two steps,

1. $\forall s \in (0, 2)$, $H^0(\mathcal{C}, L^s(k-2)) = 0 \Rightarrow \forall s \in (0, 2)$, $H^0(\mathcal{C}, L^s) = 0$,
2. $\forall s \in (0, 2)$, $H^0(\mathcal{C}, L^s) = 0 \Leftrightarrow \mathfrak{es}$ is primitive.

As such, the proposition does not have an immediate converse, though in the case of some curves the existence of a primitive ES cycle is in fact sufficient to impose the fourth Hitchin condition. Denoting with $M_{k,NS}$ the moduli space of nonsingular $k$-monopole spectral curves satisfying the Hitchin constraints, and $M_{k,ES}$ the moduli space of nonsingular $\tau$-invariant curves in $|\pi^*\mathcal{O}(2k)|$ satisfying the ES constraints with $\mathfrak{es}$ primitive, we then have $M_{k,NS} \subseteq M_{k,ES}$. Clearly in the case $k = 2$ they are equal, and moreover in [BDE11, p. 646] the authors will argue using the known dimension of the moduli space that in their case of certain $C_3$-invariant 3-monopoles, imposing the ES constraints with a primitive cycle is itself sufficient. [HMR00, p. 242] and [BE06, p. 53] give examples of curves which satisfy the ES constraints with a primitive cycle, but not the full Hitchin condition on $H^0(\mathcal{C}, L^s(k-2))$, and hence show that the inclusion can be proper. One can ask about the dimension of $M_{k,ES}$; to do so it is helpful to take the adapted cohomology basis of Vinnikov [Vin93] defined such that the differentials satisfy $\overline{\tau^*\nu_l} = \nu_l$. This basis can be related to that of Equation 3.4, whereby one finds $\nu_1 = i\Omega^{(1)}$, and hence the ES constraints become

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{pmatrix} = \boldsymbol{v} := \begin{pmatrix} -2i \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.10}$$

In [KK14] the authors take a homology bases adapted to the cohomology basis of Vinnikov and show $\overline{A} = A$, $\overline{B} = -B + AH$, where $H$ is some integer matrix fixed by the topological type of the action of $\tau$, and hence in our case fixed simply by the genus $g$. Taking the sum and difference of Equation 3.10 and its complex conjugate respectively, we get the following real equations

$$\begin{aligned} A(2\boldsymbol{n} + H\boldsymbol{m}) &= 0, \\ (2B - AH)\boldsymbol{m} &= 2\boldsymbol{v}. \end{aligned} \tag{3.11}$$

The first of Equations 3.11 imposes no constraints on the period matrix and is simply a discrete condition on $\boldsymbol{n}, \boldsymbol{m}$ corresponding to $\left(\begin{smallmatrix} 1 & H \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{smallmatrix}\right) = -\left(\begin{smallmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{smallmatrix}\right)$ i.e. $\tau_* \mathfrak{es} = -\mathfrak{es}$, but the second equation imposes $g = (k-1)^2$ real constraints on $A, B$. Given that the real dimension of the space of nonsingular curves in $|\pi^* \mathcal{O}(2k)|$ is $k^2 + 2k$, we find $\dim_{\mathbb{R}} M_{k,ES} = k(k+2) - (k-1)^2 = 4k - 1 = \dim_{\mathbb{R}} N_k = \dim_{\mathbb{R}} M_{k,NS}$, so $M_{k,NS}$ is a codimension-0 subset of $M_{k,ES}$.

An alternative way of interpreting the triviality of $L^2$ comes from the work of Corrigan & Goddard, who work directly with the cohomology class $\Gamma \in H^1(T\mathbb{P}^1, L^2(-2k))$, interpreting it as a component of the transition function of the monopole bundle. In [CG81] they use the twistor perspective to write

$$\Gamma = \left[ \frac{e^{\gamma + \chi} + (-1)^k e^{-\gamma - \chi^\tau}}{\zeta^{-k} P} \right], \tag{3.12}$$

where $\gamma = \eta/\zeta$, $\chi(\zeta, \eta) = \sum_{i=0}^{k-1} \eta^i \chi_i(\zeta)$ for $\chi_i(\zeta)$ holomorphic on $\tilde{U}_0$ (in the sense that is has a Laurent series in $\zeta$), and $P = P(\zeta, \eta)$ a polynomial which turns out to be that which defines $\mathcal{C}$. With this data they define

$$\Theta := 2\gamma + \chi + \chi^\tau. \tag{3.13}$$

This object $\Theta$ shall be key in the ensuing discussion of the work of Corrigan & Goddard.

**Theorem 3.2.33.** *The form of Equation 3.12 can be deduced assuming one has a monopole spectral curve and unpacking the connecting map of Theorem 3.2.14.*

*Proof.* As per Remark 2.1.2 I shall compute using Čech cohomology with the cochain differential $\check{\delta} : \check{C}^k \to \check{C}^{k+1}$, using that $\tilde{U}_{0,1}$ is a Leray cover of $T\mathbb{P}^1$ with respect to the sheaves $L^2$, $L^2(-2k)$.

It will be helpful to identify exactly what component of the transition function is being spoken about. Given a short exact sequence $0 \to L(-k) \xrightarrow{\alpha} E \xrightarrow{\beta} L^*(k) \to 0$ we know there is a choice of vector subspace $V_z \leq E_z$ varying holomorphically in $z \in T\mathbb{P}^1$ such that $E_z = \alpha(L(-k)_z) \oplus V_z$ and $\beta(V_z) = L^*(k)_z$. Writing $g_{01}$ for the transition function of $L(-k)$, $F_{01}$ for the transition function of $E$, and taking sections $f_{0,1} \in \Gamma(\tilde{U}_{0,1}, E)$ respectively we know

$$\beta(F_{01} f_1) = \beta(f_0) = g_{01}^{-1} \beta(f_1) \Rightarrow \begin{pmatrix} 0 & 1 \end{pmatrix} F_{01} = g_{01}^{-1} \begin{pmatrix} 0 & 1 \end{pmatrix} \Rightarrow F_{01} = \begin{pmatrix} * & * \\ 0 & g_{01}^{-1} \end{pmatrix}.$$

Moreover, if we instead denote with $f_{0,1}$ sections of $L(-k)$ we have $\alpha(g_{01} f_1) =$

$\alpha(f_0) = F_{01}\alpha(f_1)$ and hence

$$\begin{pmatrix}1\\0\end{pmatrix} g_{01} = F_{01}\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow F_{01} = \begin{pmatrix} g_{01} & * \\ 0 & g_{01}^{-1}\end{pmatrix} \Rightarrow F_{10} = \begin{pmatrix} g_{01}^{-1} & -* \\ 0 & g_{01}\end{pmatrix}.$$

Recall for $L(-k)$ we have $g_{01} = e^{-\gamma}\zeta^{-k}$. It is the $*$ term that is the component of the transition function Corrigan & Goddard write down. The cohomology class $\Gamma \in H^1(T\mathbb{P}^1, L^2(-2k))$ determining $E$ was really classifying the bundle $E\otimes L(-k)$ as an extension $0 \to L^2(-2k) \to E\otimes L(-k) \to \mathcal{O}_{T\mathbb{P}^1} \to 0$. As such, $E\otimes L(-k)$ has a transition function

$$F'_{01} = \begin{pmatrix} g_{01}^2 & *' \\ 0 & 1\end{pmatrix},$$

and so $*' = g_{01}* = e^{-\gamma}\zeta^{-k}*$.

An alternative way to connect the cohomology class in $H^1(T\mathbb{P}^1, L^2(-2k))$ to the component of the transition function is pick a basis $\left\{v\otimes (d\zeta)^{k/2}\right\}$ of (the fibres of) $L(-k)\cong L^+$ and the dual basis $\left\{v^*\otimes\left(\frac{d}{d\zeta}\right)^{k/2}\right\}$ of $L^*(k)$. Here $v$ represents a choice of basis trivialising $L$ over $\tilde{U}_0$. Then the component of the transition function given by an element $fl \in \check{C}^1(T\mathbb{P}^1, \mathrm{Hom}(L^*(k), L(-k)))$, where $l$ is the linear map specified by $v^*\otimes\left(\frac{d}{d\zeta}\right)^{k/2} \to v\otimes(d\zeta)^{k/2}$, is simply $f$. To go between $fl \in \check{C}^1(T\mathbb{P}^1, \mathrm{Hom}(L^*(k), L(-k)))$ and $f'l' \in \check{C}^1(T\mathbb{P}^1, \mathrm{Hom}(\mathcal{O}, L^2(-2k)))$, where $l'$ is the linear map determined by $1 \to v\otimes v\otimes (d\zeta)^k$, we must tensor with the identity map $v\otimes(d\zeta)^{k/2} \to v\otimes(d\zeta)^{k/2}$ in $\check{C}^1(T\mathbb{P}^1, \mathrm{Hom}(L(-k), L(-k)))$. On the overlap this identity map is given by the function $e^{-\gamma}\zeta^{-k}$, the transition function of $L(-k)$, because the basis vectors are different on the two open sets; as such we have $f\zeta^k e^\gamma = f'$.

Write now the SES of sheaves

$$0 \to \mathcal{O}_{T\mathbb{P}^1}(L^2(-2k)) \overset{\otimes P}{\to} \mathcal{O}_{T\mathbb{P}^1}(L^2) \overset{i_*}{\to} \mathcal{O}_{\mathcal{C}}(L^2) \to 0,$$

where $i_*$ is the direct image functor of the inclusion map $i : \mathcal{C} \hookrightarrow T\mathbb{P}^1$. This induces the LES of cohomology

$$\cdots \to H^0(T\mathbb{P}^1, L^2) \to H^0(\mathcal{C}, L^2) \overset{\delta}{\to} H^1(T\mathbb{P}^1, L^2(-2k)) \to H^1(T\mathbb{P}^1, L^2) \to \dots,$$

and we are told by Hitchin that $\delta(\rho) = \Gamma$ is the class defining the monopole bundle for any $\rho \in H^0(\mathcal{C}, L^2)$. Abusing notation, take a class $[\rho] \in H^0(\mathcal{C}, L^2)$ for $\rho \in \check{C}^0(\mathcal{C}, L^2)$, and by exactness we have $\rho = i_*\tilde{\rho}$ for some $\tilde{\rho} \in \check{C}^0(T\mathbb{P}^1, L^2)$. Then we have, unpacking the proof of the Snake lemma,

$$\underbrace{0 = \check{\delta}(\rho)}_{\text{as } H^0\ \check{\delta}\text{-closed}} = \check{\delta}(i_*\tilde{\rho}) = i_*\check{\delta}(\tilde{\rho}) \Rightarrow \check{\delta}(\tilde{\rho}) \in \ker(i_*)\cap\check{C}^1(T\mathbb{P}^1, L^2) = \mathrm{Im}(\otimes P)\cap\check{C}^1(T\mathbb{P}^1, L^2).$$

Hence we know there exists an element of $H^1(T\mathbb{P}^1, L^2(-2k))$ which, when tensored with $P$, gives $[\check{\delta}(\tilde{\rho})]$. The connecting differential $\delta$ is then defined to give this element, i.e. $\delta([\rho]) = [\check{\delta}(\tilde{\rho})\otimes P^{-1}]$.

As $\tilde{U}_{0,1}$ is a Leray cover of $T\mathbb{P}^1$ an element $\tilde{\rho} \in \check{C}^0(T\mathbb{P}^1, L^2)$ is represented by a pair $(\beta_0, \beta_1)$ where $\beta_i$ is holomorphic on $\tilde{U}_i$. The Čech differential of this element is $\check{\delta}((\beta_0, \beta_1))_{10} = \beta_0 - e^{-2\gamma}\beta_1 \in \check{C}^1(T\mathbb{P}^1, L^2)$ (implicitly writing the function restricted to $\tilde{U}_0 \cap \tilde{U}_1$ in $\tilde{U}_0$ coordinates) by definition [GH78, §0.3], hence we get that the transition function for $E$ is

$$F_{10} = \begin{pmatrix} \zeta^k e^{\gamma} & \Gamma \\ 0 & \zeta^{-k}e^{-\gamma} \end{pmatrix}, \quad \Gamma = -\zeta^k e^{\gamma} \frac{\check{\delta}((\beta_0, \beta_1))_{01}}{P} = \left[ \frac{e^{\gamma}\beta_0 - e^{-\gamma}\beta_1}{\zeta^{-k}P} \right]. \tag{3.14}$$

The condition that there exists $[\rho] \in H^0(\mathcal{C}, L^2)$ such that $\rho = i_*\tilde{\rho}$ means $\beta_1 = e^{2\gamma}\beta_0$ restricted to $\tilde{U}_0 \cap \tilde{U}_1 \cap \mathcal{C}$, and as such we know the numerator vanishes whenever the denominator vanishes.

As $(\beta_0|_{\mathcal{C}}, \beta_1|_{\mathcal{C}})$ determines a nonzero global holomorphic section, i.e. an element of $H^0(\mathcal{C}, L^2)$, we know that if $D$ is the divisor of the section $l(D) > 0$ and so $D$ is linearly equivalent to an effective divisor. Moreover, because $L^2$ is trivial on $\mathcal{C}$ we in fact know $l(D) = 1$ and so $D$ is effective. As $\deg(D) = 0$, it follows that $D = 0$, and the $\beta_i$ must be nonzero everywhere on $\mathcal{C}$. Write $\beta_i|_{\mathcal{C}} = e^{\delta_i}$ for some (possibly multivalued) function $\delta_i$ which must be holomorphic on $\mathcal{C} \cap \tilde{U}_i$. Note $\delta_i$ plays the role of $\log \beta_i$ in the proof of Proposition 3.2.24. From the SES of sheaves [Hit82]

$$0 \to \mathcal{I}_{\mathcal{C}/T\mathbb{P}^1} \otimes \mathcal{O}_{T\mathbb{P}^1}(L^2) \to \mathcal{O}_{T\mathbb{P}^1}(L^2) \to i_*\mathcal{O}_{\mathcal{C}}(L^2) \to 0,$$

where $\mathcal{I}_{\mathcal{C}/T\mathbb{P}^1}$ is defined to be the kernel of the map $\mathcal{O}_{T\mathbb{P}^1} \to i_*\mathcal{O}_{\mathcal{C}}$, we get

$$i_*\mathcal{O}_{\mathcal{C}}(L^2) \cong \mathcal{O}_{T\mathbb{P}^1}(L^2) \Big/ \left[\mathcal{I}_{\mathcal{C}/T\mathbb{P}^1} \otimes \mathcal{O}_{T\mathbb{P}^1}(L^2)\right]. \tag{3.15}$$

The ideal sheaf $\mathcal{I}_{\mathcal{C}/T\mathbb{P}^1}$ is generated by $P$ and so Equation 3.15 says we can extend from $\tilde{U}_i \cap \mathcal{C}$ to all of $\tilde{U}_i$ to write

$$\beta_i = e^{\chi_i} + P\beta_{i,\text{coset}},$$

with $\beta_{i,\text{coset}}, \chi_i$ holomorphic on $\tilde{U}_i$ and $\chi_i|_{\mathcal{C}} = \delta_i$. We fix the ambiguity in the $\chi_i$ extending $\delta_i$ by asking that it contains no terms of degree $k$ or higher in $\eta$, i.e. we take the residue mod $P$, as $P$ is of degree $k$. Hence

$$\Gamma = \left[ \frac{e^{\chi_0+\gamma} - e^{\chi_1-\gamma}}{\zeta^{-k}P} \right].$$

The $\beta_{i,\text{coset}}$ term has disappeared as it corresponds to a contribution in the image of $\check{C}^0(T\mathbb{P}^1, L^2(-2k)) \to \check{C}^1(T\mathbb{P}^1, L^2(-2k))$ and so cohomologically it is zero.

We finish by proving a reality condition, namely that $e^{\chi_1} = (-1)^{k-1}e^{-\chi_0^{\tau}}$. To do so recall that we also had the extension $0 \to L^*(-k) \to E \to L(k) \to 0$ defining $E$. The transition function for $E$ coming from the trivialisation $\rho$ is then

$$F_{10} = \begin{pmatrix} \zeta^k e^{-\gamma} & \Gamma' \\ 0 & \zeta^{-k}e^{\gamma} \end{pmatrix}, \quad \Gamma' = \left[ \frac{e^{-\chi_0-\gamma} - e^{-\chi_1+\gamma}}{\zeta^{-k}P} \right]. \tag{3.16}$$

This comes from the fact that the element $(\beta_0', \beta_1') \in \check{C}^0(T\mathbb{P}^1, L^{-2})$ which restricts to $\rho$ must be $(\beta_0^{-1}, \beta_1^{-1})$, which exists as $\beta_i$ is nowhere-zero on $\tilde{U}_i$.

We also get the cohomology class in $H^1(T\mathbb{P}^1, L^{-2}(-2k))$ defining $E$ by applying $\sigma$, the lift of $\tau$ to the bundle, to $\Gamma$ as noted in [Hit82, §8]. Applying $\sigma$ to the (10) class $\check{\delta}((\beta_0, \beta_1))_{10} P^{-1} \in H^1(T\mathbb{P}^1, L^{-2}(-2k))$ we get the (01) class $\check{\delta}((\beta_1^\tau, \beta_0^\tau))_{01} (P^\tau)^{-1}$ as $(\beta_1^\tau, \beta_0^\tau) \in \check{C}^0(T\mathbb{P}^1, L^{-2})$ is the class restricting to $\rho$. Now recalling from §3.2.1 the (01) transition function of $\tau^* \overline{L(-k)}$ is $g_{10}^\tau = (-\zeta)^{-k} e^\gamma$, $\sigma$ maps the SES defining $E$ as

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, bend right, "\sigma"'] & L(-k) \arrow[r] & E \arrow[r] & (L(-k))^* \arrow[r] & 0, \\
0 \arrow[r] & (\mathbf{-1})^k L^*(-k) \arrow[r] & \tau^* \overline{E} \arrow[r] & (\mathbf{-1})^k (L^*(-k))^* \arrow[r] & 0,
\end{tikzcd}$$

where $\mathbf{-1}$ represents the line bundle on $T\mathbb{P}^1$ with transition function $-1$. This gives transition function for $\tau^* \overline{E} \cong E$

$$F_{01} = \begin{pmatrix} (-\zeta)^{-k} e^\gamma & \sigma\Gamma \\ 0 & (-\zeta)^k e^{-\gamma} \end{pmatrix}, \quad \sigma\Gamma = (-\zeta)^k e^{-\gamma} \frac{\check{\delta}((\beta_1^\tau, \beta_0^\tau))_{01}}{(-1)^k P} = \left[ \frac{e^{\chi_1^\tau - \gamma} - e^{\chi_0^\tau + \gamma}}{\zeta^{-k} P} \right].$$

Multiplying by $(-1)^k$ we find $F_{01} = \begin{pmatrix} \zeta^{-k} e^\gamma & (-1)^k \sigma\Gamma \\ 0 & \zeta^k e^{-\gamma} \end{pmatrix}$ and so comparing to Equation 3.16 we get $\Gamma' = (-1)^{k-1} \sigma\Gamma$. As such we get the equations

$$e^{-\chi_1^\tau} = (-1)^{k-1} e^{\chi_0}, \quad e^{-\chi_0^\tau} = (-1)^{k-1} e^{\chi_1},$$

so we can write, taking $\chi = \chi_0$,

$$\Gamma = \left[ \frac{e^{\chi + \gamma} + (-1)^k e^{-\chi^\tau - \gamma}}{\zeta^{-k} P} \right].$$

Note the condition that $e^{\chi_0} = (-1)^{k-1} e^{-\chi_1^\tau}$ is exactly the requirement that the pairing of sections on $L^2$, $\langle s, s' \rangle = s^\tau \otimes s'$, is $(-1)^{k-1}$-definite as described by Hurtubise [Hur83]. □

**Remark 3.2.34.** *We can now look at the proof of Proposition 3.2.24 again as saying that*

$$m_j = \frac{-1}{2\pi i} \int_{a_j} d\chi = \frac{1}{2\pi i} \int_{a_j} d\chi^\tau, \quad n_j = \frac{1}{2\pi i} \int_{b_j} d\chi = \frac{-1}{2\pi i} \int_{b_j} d\chi^\tau,$$

*provided $d \log(-1)^{k-1}$ has zero periods (otherwise the periods of $d\chi^\tau$ are augmented by subtracting the periods of $d\log(-1)^{k-1}$). Using that for any $c \in H_1(\mathcal{C}, \mathbb{Z})$, $\int_c d\chi^\tau = \int_c \overline{d(\tau^* \chi)} = \overline{\int_{\tau_* c} d\chi}$ we can again derive in a new way the fact $\tau_* \mathfrak{es} = -\mathfrak{es}$. In particular taking the Kalla and Klein canonical homology basis so $\tau_* b_j = -b_j + H_{jl} a_l$ we find*

$$n_j = \frac{-1}{2\pi i} \int_{b_j} d\chi^\tau = \overline{\frac{1}{2\pi i} \int_{-b_j + H_{jl} a_l} d\chi} = -n_j - H_{jl} m_l \Rightarrow 2\boldsymbol{n} + H\boldsymbol{m} = 0.$$

*An alternative perspective is to view the periods of $d\chi$, $d\chi^\tau$ as the summands of automorphy of the corresponding multivalued functions [For91, Theorem 10.13]. The triviality of $L^2$ forces these two multivalued functions to have summands of the opposite sign, but the summands are related by the action of $\tau$, and this constrains the possible values of the summands.*

**Remark 3.2.35.** *Note how $\Gamma$ in Equation 3.12 differs from the component of the transition function of [War81a, p. 568, Case (b)]*

$$\Gamma_{Ward} = \frac{e^f + (-1)^k e^{-f}}{\psi},$$

*where $f = f^\tau$, $\psi = \psi^\tau$. Ward shows such a transition function satisfies the required reality condition as*

$$\begin{pmatrix} \zeta^k e^f & \Gamma_{Ward} \\ 0 & \zeta^{-k} e^{-f} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & \zeta^k \psi \end{pmatrix} = \begin{pmatrix} \Gamma_{Ward} & (-\zeta)^k e^{-f} \\ \zeta^{-k} e^{-f} & \psi e^{-f} \end{pmatrix}.$$

*From [CG81, (3.5)], supposing we can write $\gamma = \mu - \nu$ as a splitting over $\tilde{U}_{0,1}$, we can decompose the transition function*

$$\begin{pmatrix} \zeta^k e^\gamma & \Gamma \\ 0 & \zeta^{-k} e^{-\gamma} \end{pmatrix} = \begin{pmatrix} e^{-\nu} & 0 \\ 0 & e^\nu \end{pmatrix} \begin{pmatrix} \zeta^k & \rho \\ 0 & \zeta^{-k} \end{pmatrix} \begin{pmatrix} e^\mu & 0 \\ 0 & e^{-\mu} \end{pmatrix}$$

*where $\rho = e^{\mu+\nu}\Gamma$. As such combining two transformations of this type we can say that*

$$\begin{pmatrix} \zeta^k e^\gamma & \Gamma \\ 0 & \zeta^{-k} e^{-\gamma} \end{pmatrix} \sim \begin{pmatrix} \zeta^k e^{\Theta/2} & \Gamma_\Theta \\ 0 & \zeta^{-k} e^{-\Theta/2}, \end{pmatrix}$$

*where $\Gamma_\Theta = \frac{e^{\Theta/2} + (-1)^k e^{-\Theta/2}}{\zeta^{-k} P}$ is of the form given by Ward with $\psi = \zeta^{-k} P$; $\sim$ indicates gauge equivalence, and this equivalence ends up not relying on the ability to split $\gamma$. This computation is also given in different notation in [Cor82].*

On the data of $\Theta$ satisfying $\Theta^\tau = \Theta$ Corrigan & Goddard will impose two conditions necessary for the nonsingularity of the monopole.

**Definition 3.2.36.** *The* ***Corrigan-Goddard (CG) constraints*** *on $\Theta$ are*

1. *that $e^{\Theta|_{\mathcal{C}}} + (-1)^k = 0$, i.e. $\Theta|_{\mathcal{C}} \in \pi i \left[(k-1) + 2\mathbb{Z}\right]$, such that the numerator vanishes wherever the denominator vanishes and moreover that,*

2. *defining the functions $\Theta_r(\zeta)$ by $\Theta(\zeta, \eta) = 2\pi i \sum_{r=0}^{k-1} \Theta_r(\zeta) \left(\frac{\eta}{\zeta}\right)^r$,*

$$\begin{aligned} \forall\, 2 \le r \le k-1,\, |s| < r, \qquad \oint_{|\zeta|=1} \Theta_r(\zeta) \zeta^s \frac{d\zeta}{\zeta} &= 0, \\ \oint_{|\zeta|=1} \Theta_1(\zeta) \frac{d\zeta}{\zeta} &= 2. \end{aligned} \tag{3.17}$$

Equation 3.17 imposes $\sum_{r=1}^{k-1} 2r - 1 = (k-1)^2$ constraints, which one can check are actually real constraints, and moreover they (generically) are independent

[ORS82]. We recognise this as imposing $g(\mathcal{C})$ constraints, and so we might expect that there is a relation to the $g$ ES constraints.

A more cohomology-minded approach is to think of the function $\Theta$ as a representative of a class $[\Theta] \in H^1(T\mathbb{P}^1, \mathcal{O}_{T\mathbb{P}^1})$, real as $\Theta^\tau = \Theta$. The first CG condition on $\Theta$ simply says the restriction $[\Theta|_{\mathcal{C}}] \in H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ lies in the subset $H^1(\mathcal{C}, \mathbb{Z})$. Now [Hit83, Proposition 3.1] gives that any class in $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ can be written uniquely in the form $\sum_{r=1}^{k-1} \eta^r \pi^* c_r(\zeta)$ where $c_r \in H^1(\mathbb{P}^1, \mathcal{O}(-2r))$. In this representation for $[\Theta|_{\mathcal{C}}]$ we can equate $c_r(\zeta) = 2\pi i \zeta^{-r}\Theta_r(\zeta)$. Furthermore, Serre duality on $\mathbb{P}^1$ comes from the existence of the nondegenerate pairing on $H^0(\mathbb{P}^1, \mathcal{O}(n-2)) \times H^1(\mathbb{P}^1, \mathcal{O}(-n))$ [Vak10, Example 18.5.4]

$$\langle f, c \rangle = \mathrm{Res}_{\zeta=0}(fc) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \zeta f(\zeta) c(\zeta) \frac{d\zeta}{\zeta}.$$

As such we can rephrase the second set of CG conditions as

$$\left\langle \zeta^{s+r-1}, 2\pi i \zeta^{-r} \Theta_r(\zeta) \right\rangle = \begin{cases} 0, & r > 1, |s| < r, \\ 2, & r = 1, s = 0. \end{cases}$$

Because of the nondegeneracy of the pairing, and the fact that $\{\zeta^{s+r-1} \mid |s| < r\}$ is a basis of $H^0(\mathbb{P}^1, \mathcal{O}(2r-2))$, these conditions say that $[c_r] = [0] \in H^1(\mathbb{P}^1, \mathcal{O}(-2r))$ for $r > 1$ and $[c_1] = [2/\zeta]$. Combined this means $[\Theta] = [2\eta/\zeta] \in H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$, and as such the CG conditions are exactly equivalent to the triviality of $L^2 \to \mathcal{C}$.

**Remark 3.2.37.** *Clearly $[(-1)^{k-1} e^{\Theta|_{\mathcal{C}}}] = [e^{2\gamma}] \in H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^\times)$, so denote with $[\Theta_{(\boldsymbol{u},\boldsymbol{v})}] \in H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ the classes in the preimage under the exponential map such that the vector in $\mathbb{C}^g$ associated with $[\Theta_{(\boldsymbol{u},\boldsymbol{v})}]$ is $2\boldsymbol{U} + \boldsymbol{u} + \tau\boldsymbol{v}$, $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{Z}^g$. Now for any logarithm of $(-1)^{k-1}$ in $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ there must be a corresponding $\boldsymbol{u}, \boldsymbol{v}$ such that $[\Theta_{(\boldsymbol{u},\boldsymbol{v})}] = [\log(-1)^{k-1} + \Theta|_{\mathcal{C}}]$. Furthermore, by the argument laid out interpreting Equation 3.17 cohomologically, if the CG conditions hold then $[\Theta|_{\mathcal{C}}] = [2\gamma]$, and as such it must be the case that the vector corresponding to $[\log(-1)^{k-1}]$ is exactly the corresponding $\boldsymbol{u} + \tau\boldsymbol{v}$. As such that summands of automorphy of $d\chi^\tau$ are augmented as described in Remark 3.2.34, and hence $\Theta$ is a multivalued function. It is worth remarking we always knew the vector associated with $[\log(-1)^{k-1}]$ was some lattice vector as the exponential $[(-1)^{k-1}] \in H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^\times)$ has zero Chern class.*

This final point, that the CG conditions are equivalent to the triviality of $L^2 \to \mathcal{C}$ and so to the ES constraints, appears in contradiction to the work of [HMR00] where they prove that the CG constraints imply the ES constraints but the converse only holds true when the ES cycle $\mathfrak{es}$ can be written as the sum of cycles on $\mathcal{C}$ lifting the equator $|\zeta| = 1$. Such a condition does not hold for the tetrahedral 3-monopole spectral curve. To prove this the authors, following [CG81], write $\Theta$ using Lagrange interpolation as

$$\Theta(\zeta, \eta) = \pi i \sum_{r=1}^{k} \left[ \nu_r \prod_{s \neq r} \frac{\eta - \eta_s(\zeta)}{\eta_r(\zeta) - \eta_s(\zeta)} \right]$$

where $\eta_r(\zeta)$ is a labelling of the values of $\eta$ at $\zeta \in \tilde{U}_0 \cap \tilde{U}_1 \cap \mathcal{C}$ and the $\nu_r$ are determined by $\Theta(\zeta, \eta_r(\zeta)) = \pi i \nu_r$. The first CG condition on $\Theta$ forces that the $\nu_r$ are integers, odd when $k$ is even and even when $k$ is odd; the reality condition then ensures that if $(-1/\bar{\zeta}, -\bar{\eta}_r(\zeta)/\bar{\zeta}^2) = (-1/\bar{\zeta}, \eta_s(-1/\bar{\zeta}))$ then $\nu_r = -\nu_s$. In order to be able to consistently make such a labelling of the fibre values one requires that the $\eta_r$ are distinct on $\tilde{U}_0 \cap \tilde{U}_1 \cap \mathcal{C}$ and moreover that the monodromy of the overlap is trivial. The former requirement can be ensured by rotating the monopole such that are no branch points on the equator $|\zeta| = 1$ and shrinking the overlap to be a small annulus containing the equator. It is not clear that the latter condition above on monodromy can always be imposed, but it holds generically for all known spectral curves checked, namely the 2-monopole, the tetrahedral 3-monopole, and the $D_6$ and $C_4$ 3-monopoles of §3.4.1 and §3.4.2 (see `Ercolani-Sinha_vector_2-monopole.ipynb` and `Ercolani-Sinha_vector_tetrahedral_3-monopole.ipynb` for examples of how to compute this in Sage). In fact, random sampling suggests that the monodromy of the equator is trivial for any irreducible[10] $\tau$-invariant curve in the linear system $|\pi^*\mathcal{O}(2k)|$.

Having written $\Theta$ using Lagrange interpolation, Houghton et al. then deduce that if the CG conditions hold then the ES cycle is[11] $\mathfrak{es} = \frac{1}{2}\sum_i \nu_i E_i$, where $E_i$ is the lift of the equator in the annulus to the sheet $\eta = \eta_i(\zeta)$.

**Example 3.2.38.** *On the 2-monopole spectral curve given by $\eta^2 + a_2(\zeta) = 0$ label the roots as $\eta_i = (-1)^i\sqrt{-a_2(\zeta)}$, picking some branch of the square root. One then finds $\nu_1 = -\nu_2 := \nu$ and so $\Theta = \frac{\pi i \nu \eta}{\sqrt{-a_2(\zeta)}}$ [War81a]. The simplest choice $\nu = 1$ suffices to get a solution to Equations 3.17. To see this, take the form of the 2-monopole spectral curve given in [FHP83] where $a_2(\zeta) = A(\zeta^4+1) + B\zeta^2$, $A, B \in \mathbb{R}$. Picking then the square root such that $\sqrt{-a_2(\zeta)} \in i\mathbb{R}_{>0}$ when $|\zeta| = 1$,*

$$\begin{aligned}
\int_{|\zeta|=1} \Theta_1(\zeta)\frac{d\zeta}{\zeta} &= \int_0^{2\pi} \frac{\nu}{2i}\frac{id\theta}{\sqrt{A(e^{2i\theta}+e^{-2i\theta})+B}}, \\
&= \frac{\nu}{2\sqrt{B}}\int_0^{2\pi}\frac{d\theta}{\sqrt{1+(2A/B)\cos(2\theta)}}, \\
&= \frac{\nu}{2\sqrt{B}\sqrt{1+\beta}}\int_0^{2\pi}\frac{d\theta}{\sqrt{1-\frac{2\beta}{1+\beta}\sin^2(\theta)}} \quad \textit{letting} \quad \beta = \frac{2A}{B}, \\
&= \frac{2\nu}{\sqrt{B}\sqrt{1+\beta}}K\left(\frac{2\beta}{1+\beta}\right),
\end{aligned}$$

*where $K = K(m)$ is the complete elliptic integral of the first kind. Hence for $\nu = 1$ with appropriately chosen $\beta$, one can find $A, B$ such that the CG conditions are satisfied. It is worth remarking here that the two lifted equators give cycles $E_i \in H_1(\mathcal{C}, \mathbb{Z})$ satisfying $E_1 + E_2 = 0$, and the ES cycle for this monopole is*

[10] The reducible curve given by $P(\zeta,\eta) = (\eta^2 + 2i\eta\zeta - \zeta^2 + \zeta/4)(\eta^2 - 2i\eta\zeta - \zeta^2 - \zeta^3/4)$, constructed by Derek Harland, shows this cannot be true for all $\tau$-invariant curves.

[11] [HMR00] in fact incorrectly deduces that $\sum_i \nu_i E_i$ is the ES cycle, but this factor of 2 error comes from taking $\gamma = \frac{\eta}{2\zeta}$.

*$\mathfrak{es} = \pm E_1 = \pm\frac{1}{2}(E_1 - E_2)$ (with the sign depending on the branch of the square root taken), as can be seen in the corresponding Sage notebook* **`Ercolani-Sinha_vector_2-monopole.ipynb`** *or deduced from [FHP83, HMR00]. The easiest way to get this is to observe*

$$\int_{|\zeta|=1} \Theta_1(\zeta)\frac{d\zeta}{\zeta} = \int_{|\zeta|=1} \frac{\nu}{2}\frac{\zeta}{(-1)^{i-1}\eta_i}\frac{d\zeta}{\zeta} = \int_{\frac{\nu}{2}(E_1-E_2)} \frac{d\zeta}{2\eta}.$$

*Moreover, [FHP83] will calculate here*

$$\chi(\zeta,\gamma) = \frac{2\gamma}{\tilde{\beta}\sqrt{-A}}\left[K(\tilde{\beta}^{-4}) - \Pi(\zeta^2\tilde{\beta}^{-2}/\delta)\right],$$

*where $\tilde{\beta} = \sqrt{-\beta^{-1} + \sqrt{\beta^{-2} - 1}}$, $\sin\delta = \tilde{\beta}^{-2}$, and $\Pi$ is the complete elliptic integral of the third kind [AS72, §17.7]. The multivaluedness comes from the elliptic integral; an easier way to see this is to use [Hur83] which gives that $d\chi \sim \wp \Rightarrow \chi \sim \zeta$ the Weierstrass $\zeta$ function [AS72, §18] which has simple summands of automorphy.*

**Example 3.2.39.** *On the 3-monopole spectral curve given by $\eta^3 + \alpha_2\eta\zeta^2 + \beta(\zeta^6 - 1)$ label the roots at $\zeta = 1$ by $\eta_1 = 0$, $\eta_i = (-1)^i\sqrt{-\alpha_2}$, $i = 2, 3$. One finds $\nu_1 = 0$, $\nu_2 = -\nu_3 = \nu$. [OR82] shows that for suitable parameter choices one can get a solution to the CG conditions (see also [Sop82]). Again it is worth remarking here that the equators lift to cycles satisfying $E_1 + E_2 + E_3 = 0$, and the ES cycle is $\mathfrak{es} = \pm(E_1 - E_2)$ (with the sign depending on the branch of the square root taken).*

**Example 3.2.40.** *In contrast to Examples 3.2.38 and 3.2.39, consider the tetrahedral 3-monopole given by $\eta^3 + \chi(\zeta^6 + 5\sqrt{2}\zeta^3 - 1) = 0$ where $\chi^{1/3} = -\frac{1}{6}\frac{\Gamma(1/6)\Gamma(1/3)}{2^{1/6}\pi^{1/2}}$ [HMR00, BE10a].[12] Define $\eta_f(\zeta)$ on an annulus around the equator by $\eta_f(1) = 5\sqrt{2}\,|\chi|^{1/3}$ and then analytic continuation, and so the values $\eta_i(\zeta) = \omega^i\eta_f(\zeta)$ where $\omega$ is a primitive 3rd root of unity. These $\eta_i(\zeta)$ are the roots of the polynomial defining the spectral curve at any given $\zeta$ in the annulus. By acting on the $\eta_i$ with $\tau$ one finds for the $\nu_i$ that $\nu_3 = 0$, $\nu_1 = -\nu_2 := \nu$, and so expanding our the Lagrange interpolation formula*

$$\Theta = \frac{2\pi i\nu}{3}(\omega - \omega^2)y(y-1), \quad y = \eta/\eta_f.$$

*As such we get*

$$\int_{|\zeta|=1} \Theta_2(\zeta)\frac{d\zeta}{\zeta} = \frac{-\nu}{\sqrt{3}\,|\chi|^{1/3}}\int_0^{2\pi}\left[5\sqrt{2} + 2i\sin(3\theta)\right]^{-2/3} d\theta,$$

*the latter integral of which is strictly positive as one can check numerically. To impose the CG conditions one would need $\int_{|\zeta|=1}\Theta_2(\zeta)\frac{d\zeta}{\zeta} = 0 \Rightarrow \nu = 0$, but then*

[12] [HMR00] determined the value of $\chi$ numerically, and the value was later proven correct analytically in [BE10a].

$\Theta = 0$ *identically and one cannot solve* $\int_{|\zeta|=1} \Theta_1(\zeta)\frac{d\zeta}{\zeta} = 2$. *The CG conditions can therefore not be consistently solved on the tetrahedral 3-monopole curve.*

Example 3.2.40 leads to an apparent contradiction, with the ES constraints being imposable on the tetrahedral 3-monopole, but not the CG constraints; indeed in [HMR00] it is noted that $\mathfrak{es}$ is not a sum of equators for the tetrahedral 3-monopole. The resolution to this seeming contradiction is a subtle point, that the Lagrange interpolation formula is not the most general form of a function associated with a class in $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$ whose exponential takes value $(-1)^{k-1}$ around the equator. The easiest way to see this is with a counting argument: the Lagrange interpolation formula gives $k$ integer parameters with which to describe an element of $H^1(\mathcal{C}, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, which is clearly insufficient to parametrise all elements unless $k = 2(k-1)^2 \Rightarrow k = 2$ (and in fact even then it is insufficient, as there is a linear relation between the equators in $H_1(\mathcal{C}, \mathbb{Z})$). What terms are missing then?

To elucidate this, let us unpack the isomorphism between the homology group $H_1(\mathcal{C}, \mathbb{Z})$ and the (Čech) cohomology group $H^1(\mathcal{C}, \mathbb{Z})$. Taking a simplicial complex associated with $\mathcal{C}$, to each vertex (i.e. a 0-simplex) $v$ in the complex assign an open set $\mathrm{St}(v)$ called the **star** of $v$ given by the union of all simplices[13] containing $v$, and take an open covering of $\mathcal{C}$ by taking the star of all vertices[14], that is $\{U_v := \mathrm{St}(v)\}$. Two vertices $v, v'$ have $\mathrm{St}(v) \cap \mathrm{St}(v') \neq \emptyset$ if and only if there is a 1-simplex containing them, thus the module of Čech 1-cochains $\prod_{v,v'} \mathbb{Z}(U_v \cap U_{v'})$ obtains a $\mathbb{Z}$ factor exactly where there is a $\mathbb{Z}$ factor coming from the 1-simplex $\{v, v'\}$ in the module of simplicial 1-chains. Writing $\Theta$ using LI as done specifies the values of an element of $H^1(\mathcal{C}, \mathbb{Z})$ only at the overlaps $U_v \cap U_{v'}$ where $\{v, v'\}$ is a portion of a lift of the equator, assigning them the value $\pi i \nu_j$ when they form the cycle $E_j$. This implicitly then assigns the value 0 to any other cycle, and as such it follows directly that assuming the LI form of $\Theta$ gives $\mathfrak{es} = \sum_i \frac{1}{2}\nu_i E_i$. To give a completely general form of $\Theta$ would be to give LI formula also at the other overlaps, taking into account the necessary relations between these in the cohomology group, and then there would be a single expression in $\zeta, \eta$ which is equivalent as a cohomology class in $H^1(T\mathbb{P}^1, \mathcal{O}_{T\mathbb{P}^1})$. It is worth pointing out just how easy it is to overlook this point: $\tilde{U}_{0,1} \cap \mathcal{C}$ is a Leray cover for the sheaf $\mathcal{O}_{\mathcal{C}}$ but not the constant sheaf $\underline{\mathbb{Z}}$, and this is why the more refined open cover is required when specifying an element of $H^1(\mathcal{C}, \mathbb{Z})$.

In order to be able to best utilise the perspective of the CG constraints in future research on monopoles, it would be helpful to be able to classify the monopole spectral curves for which the ES cycle is a sum of equators, and moreover in the cases where it is not provide an explicit representative of the function $\Theta$. This may be especially relevant when starting to understand the equivalent of the ES constraints for hyperbolic monopoles [NR07]. In better understanding the connection, it may helpful to use the perspective of the generalised Legendre transform [Hou00, Bie09].

A final way of interpreting the condition on the triviality of $L^2$ is given in [Nas07]. I shall not go into this approach in detail, but briefly it works as follows:

[13] I am following [GH78] in taking the plural of simplex to be simplices and not simplexes.

[14] Note this open cover is such that the nerve of the cover is the original simplicial complex.

for a generic curve $\mathcal{C}$ in the linear system $|\pi^*\mathcal{O}(2k)|$, fixing a trivialisation of $\rho \in H^0(\mathcal{C}, L^2)$ by normalising it uniquely up to a U(1) factor, Nash considers the deformation space of $\hat{\mathcal{C}} := \rho(\mathcal{C}) \subset L^2$. As is standard the deformation space is identified with the space of global sections of the normal bundle, so deformations of $\hat{\mathcal{C}}$ are governed by $H^0(\hat{\mathcal{C}}, \hat{N})$, where $\hat{N}$ is the normal bundle to $\hat{\mathcal{C}}$ in $L^2$; likewise the deformations of $\mathcal{C}$ are governed by $H^0(\mathcal{C}, N)$ where $N$ is the normal bundle of $\mathcal{C}$ in $T\mathbb{P}^1$. Using $\rho$ to view $\hat{N}$ as a vector bundle over $\mathcal{C}$ and restricting $L^2$, Nash shows that there is a SES of bundles

$$0 \to L^2 \to \hat{N} \to N \to 0,$$

as so using the main vanishing theorem [Nas07, Theorem 2.8, Corollary 2.9] and that $N \cong \pi|_{\mathcal{C}}^* \mathcal{O}(2k)$ this gives the portion of the LES

$$0 \to H^0(\mathcal{C}, L^2) \to H^0(\mathcal{C}, \hat{N}) \to H^0(\mathcal{C}, \pi|_{\mathcal{C}}^* \mathcal{O}(2k)) \to H^1(\mathcal{C}, L^2) \to 0.$$

Identifying $H^0(\mathcal{C}, \hat{N})$ with $H^0(\hat{\mathcal{C}}, \hat{N})$ we see that requiring the triviality of $L^2$ restricts the deformations in $|\pi^*\mathcal{O}(2k)|$ that are valid, precisely by $g$ degrees of freedom.

### 3.2.2 Nahm Data

Hitchin's scattering approach to the spectral curve makes clear how the curve determines the original monopole at the level of cohomology, but it is not convenient for the reconstruction of the gauge fields from this data. An alternative approach bridging this difficulty is given by Nahm in [Nah83], described in [Hit83] by Hitchin as "a bold adaptation of the ADHM construction of instantons". I shall not give details of the ADHM construction [AHDM78] here, but as a brief overview it involves the construction of a linear operator between quaternionic vector spaces. Using the identification of monopoles as certain time-invariant instantons, Nahm was able to give the data of a monopole in terms of a triple of matrices.

**Definition 3.2.41.** *The data* $\{T_i(s) \mid T_i \in M_k(\mathbb{C}_\infty),\ s \in [0, 2],\ i = 1, 2, 3\}$ *is called* ***Nahm data*** *if*

1. *the* $T_i$ *satisfy* ***Nahm's equations****,*

$$\frac{dT_i}{ds} := T_i' = \frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} \left[T_j, T_k\right],$$

2. *the* $T_i(s)$ *are regular for* $s \in (0, 2)$*, and moreover they have simple poles at* $s = 0, 2$ *with residues that form the irreducible* $k$*-dimensional representation of* $\mathfrak{su}(2)$*, and*

3. $T_i(s) = -T_i^\dagger(s)$*,* $T_i(s) = T_i^T(2 - s)$*.*

*The* $T_i$ *are called* ***Nahm matrices****.*

**Remark 3.2.42.** *One often sees the parameter $s$ having range $[-1,1]$ instead, and this is equally valid, so I will be lax on which I choose in a given scenario. It will turn out when we connect Nahm data to a solution of the BPS equation, defining the Nahm data on $[-a/2, a/2]$ is required for the Higgs field $\phi = -(i/2)\boldsymbol{\phi}\cdot\boldsymbol{\sigma} = \boldsymbol{\phi}\cdot\boldsymbol{T}$ to have $\sqrt{-2\operatorname{Tr}(\phi^2)} = |\phi| \to a$ [CG84, §4].*

The data of the triple of matrices $T_i(s)$ at any fixed $s$ can be thought of as an element $\mathcal{T} \in \mathbb{R}^3 \otimes \mathfrak{u}(k)$. We also could have here included another matrix $T_4 = -T_4^\dagger$ by modifying Nahm's equations to

$$T_i' = [T_4, T_i] + \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j, T_k],$$

but this can always be gauged away by the transform

$$\begin{aligned} T_i &\mapsto UT_iU^{-1}, \\ T_4 &\mapsto UT_4U^{-1} - \frac{dU}{ds}U^{-1}, \end{aligned}$$

for $U : (0,2) \to \mathrm{U}(k)$ satisfying $U(2-s) = \left(U^T(s)\right)^{-1}$ [Don84, §1]. Restricting to $T_4 = 0$ one still retains the gauge freedom to transform $T_j \mapsto UT_jU^{-1}$ for fixed $U \in \mathrm{U}(k)$, which I shall denote as $\mathcal{T} \mapsto {}_U\mathcal{T}$.

**Remark 3.2.43.** *In analogy to Proposition 3.1.19, the equations for a self-dual connection independent of $x_i$, $i = 1,2,3$ are equivalent to Nahm's equations. This was observed in [CG84], where it was given as an example of a "reciprocity" between the self-duality equations in $d$ dimensions and $4-d$ dimensions (with the ADHM equations being the example for $d=0$). Even more generically, given a subgroup $\Lambda \le \mathbb{R}^4$, $\Lambda$-independent instantons on $\mathbb{R}^4$ correspond to $\Lambda^*$-invariant instantons on $(\mathbb{R}^4)^*$ under a generalised interpretation of the Nahm transform [Jar04].*

**Remark 3.2.44.** *When constructing Nahm data, rather than strictly impose the condition $T_i(s) = T_i(2-s)^T$ themselves, [HMM95, HS96d, HS96c] appeal to the argument of [Hit83, p. 181] which says (equivalently) that a basis always exists for which $T_i(s) = T_i(2-s)^T$ provided on the corresponding spectral curve $\mathcal{C}$ (to be defined subsequently) we have that $L^2 \to \mathcal{C}$ is trivial. This is equivalent to the existence of the Ercolani-Sinha vector, and so they claim that this can be done.*

*The correct interpretation of this statement comes from thinking about $\mathrm{SU}(2)$ monopoles within the wider context of $G$-monopoles where $G$ is a classical group [HM89]. There the transpose condition (up to conjugacy) is only required for SO- or Sp-monopoles. The fact that this condition is then present for Nahm data that arises for a $\mathrm{SU}(2)$ monopole is down to the isomorphism $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$. In [Hit83] Hitchin uses the transpose condition to show that it makes the operator required for the ADHMN construction*

$$\Delta = \left(\mathrm{Id}_k \otimes \sum_\mu x_\mu \sigma_\mu\right) + \left(\mathrm{Id}_{2k}\frac{d}{ds} + \sum_\mu T_\mu \otimes \sigma_\mu\right)$$

*quaternionic-linear, but reading [AHDM78] shows that in the case of* $G = \mathrm{SU}(2)$ *the operator need only be complex linear, not quaternionic linear. [CFTG78] describes this difference explicitly.*

**Example 3.2.45.** *In the case* $k = 1$*, Nahm data is*[15] *given by* $T_j(s) = ix_j$ *for* $(x_1, x_2, x_3) \in \mathbb{R}^3$*.*

**Example 3.2.46.** *In the case* $k = 2$*, taking the ansatz that* $T_j(s) = \frac{1}{2i} f_j(s)\sigma_j$ *(no sum), Nahm's equations give*

$$\frac{1}{2i} f_j' \sigma_j = \frac{1}{2} \sum_{k,l} \epsilon_{jkl} \left[ \frac{1}{2i} f_k \sigma_k, \frac{1}{2i} f_l \sigma_l \right] = \frac{-i}{4} \sum_{k,l,m} \epsilon_{jkl} f_k f_l \epsilon_{klm} \sigma_m,$$

*which become the Euler top equations* $f_1' = f_2 f_3$ *(and cycles). These may be solved with the correct boundary conditions to get Nahm data, and in fact they give the most general Nahm data in charge 2 [BE21, §3.6].*

The circle of ideas connecting monopoles, spectral curves, and Nahm data, was proven by Hitchin in the form of the following key theorem.

**Theorem 3.2.47** ([Hit83])**.** *The following data are equivalent:*

- *gauge fields* $(A, \phi)$ *satisfying the BPS equation (Definition 3.1.12) subject to the monopole boundary conditions (Definition 3.1.5) up to gauge equivalence,*
- *a spectral curve* $\mathcal{C}$ *satisfying the Hitchin conditions (Definition 3.2.17), and*
- *Nahm data (Definition 3.2.41) up to gauge equivalence.*

It is a consequence of the proof of Theorem 3.2.47 that the $T_i(s)$ gain the interpretation as endomorphisms of the vector space $H^0(\mathcal{C}, L^s(k-1))$, whose dimension is $k$ for $s \in (0, 2)$, but whose dimension jumps when $L^s$ is trivial, i.e. $s = 0, 2$, unless $k = 1$. This is the reason for the distinguished behaviour of the matrices at the endpoint. Moreover, this identifies the two instances of the parameter $s$, giving the interpretation that Nahm's equations linearise on $\mathrm{Jac}(\mathcal{C})$ (i.e. the solution to Nahm's equation corresponds to a straight line in the Jacobian). This we shall discuss slightly more in the next section.

**Remark 3.2.48.** *Keeping* $T_4$*, one can see that Nahm's equations come from a quaternionic moment map equation [Hit87], essentially as an avatar of the reciprocity observed in [CG84]. With this realisation the connection to the BPS equation for monopoles is unsurprising but not complete, as we need to deal with the boundary conditions. Moreover, it means there is a construction of the moduli space of solutions to Nahm's equations with the correct boundary behaviour up to gauge transform as a hyperkähler quotient, and so the moduli space has a hyperkähler metric [HKLR87]. The transform between Nahm data and monopole gauge fields is an analogue of a Fourier transform (namely the Fourier-Mukai transform), and so it was conjectured in [AH88], proven in [Nak93], that this transform is a hyperkähler isometry. This metric on the space of Nahm data can be computed explicitly, for example as done in [BS97].*

[15] On the use of data in the singular, I direct the reader to [Joh22].

### The Integrability of Nahm's Equations

In the second half of the 20th century Peter Lax gave a principle for associating commuting linear operators, a **Lax pair**, to a nonlinear evolution equation such that the eigenvalues of the linear operators were conserved quantities of the original system i.e. the operators are isospectral [Lax68]. Krichever subsequently showed how solutions to the nonlinear system could be constructed in terms of function theory on a Riemann surface given by the eigenvalues [Kri76, Kri77]. While I will not develop this perspective in this thesis, the interested reader can see [FHP83, BBT03, BE18, BE21]. The following proposition will give such a Lax formulation of Nahm's equation, and thus make the link to the spectral curve.

**Proposition 3.2.49** ([Hit83], Proposition 4.16)**.** *Nahm's equations are equivalent to the Lax equation*

$$\left[\frac{d}{ds} + M, L\right] = 0$$

*where the Lax matrices with spectral parameter $\zeta$ are given by*

$$\begin{aligned} L(\zeta) &= (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2, \\ M(\zeta) &= -iT_3 + (T_1 - iT_2)\zeta. \end{aligned} \tag{3.18}$$

**Remark 3.2.50.** *Here I am overloading the notation $L$, now meaning both the line bundle $L \to T\mathbb{P}^1$ and the Lax matrix built out of Nahm matrices, under the assumption that both are sufficiently distinct such that one can work out the correct meaning from context.*

As a result of the isospectrality of the matrix $L$, the characteristic polynomial defined by[16] $\det[\eta \operatorname{Id}_k + L(\zeta)] = 0$ defines a algebraic curve independent of $s$. A consequence of the proof of Theorem 3.2.47 is that interpreting $\zeta, \eta$ as coordinates on $T\mathbb{P}^1$, the characteristic polynomial is exactly the corresponding spectral curve $\mathcal{C}$, and so we shall consider them the same from now on. This now gives another interpretation for the fact that Nahm's equations linearise on the Jacobian of $\mathcal{C}$, which is a generic occurrence for Lax integrable systems satisfying a certain cohomological condition, described in [Gri85, Corollary 7.8].

**Example 3.2.51.** *Using the $1 \times 1$ Nahm data from Example 3.2.45, we see that the spectral curve for a 1-monopole is*

$$\eta + \left[(ix_1 - x_2) + 2x_3\zeta + (ix_1 + x_2)\zeta^2\right] = 0.$$

*This recreates the result of Example 3.2.16.*

**Example 3.2.52.** *Knowing now that the spectral curve associated with the Nahm Lax pair is the same as the spectral curve of the monopole, we can simply calculate*

[16] The unusual choice of sign in the characteristic polynomial here ensures that the corresponding curve in $T\mathbb{P}^1$ is exactly the monopoles spectral curve [Hit83, Proposition 4.16].

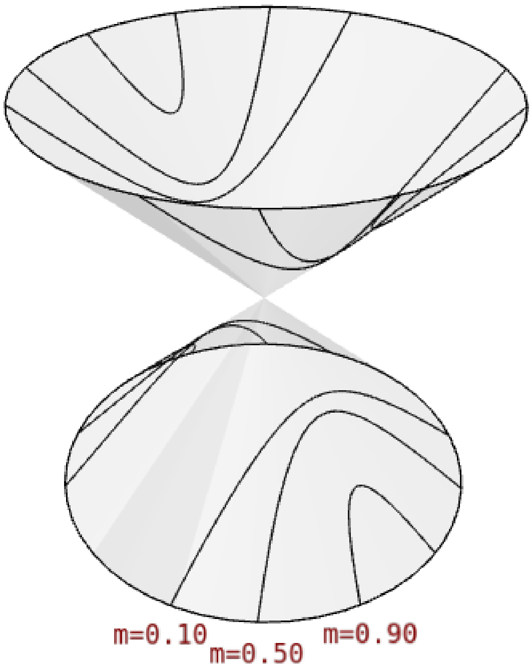



**Fig. 3.1** Charge-2 monopole spectral curve for $m = 0.1, 0.5$, and $0.9$, plotted on the cone

*the spectral curve for all 2-monopoles using the Nahm solution of Example 3.2.46 to get [ES89, BE21]*

$$\mathcal{C}: \quad \eta^2 + \frac{K^2}{4}\left[(\zeta^4+1) + 2(k^2 - k'^2)\zeta^2\right] = 0.$$

*Here $k, k', K(k)$ are the standard concepts of elliptic functions seen in §2.1.5. Note that $\mathrm{Aut}(\mathcal{C}) = C_2$ (ignoring translations, see Example 2.1.77), and so every 2-monopole spectral curve has nontrivial automorphism group.*

*I will take this opportunity to provide a visualisation of these curves I have not previously encountered, namely by plotting them on the cone described by the embedding of $T\mathbb{P}^1$ in a cone in $\mathbb{P}^3$. Recalling the transform of §2.1.3 and writing the resulting cone as $x^2 + y^2 - z^2 = 0$, the monopole spectral curves are given by the intersection of the cone and*

$$1 + \frac{K(m)^2}{2}\left[(x^2 - y^2) + (2m-1)z^2\right] = 0,$$

*where $m = k^2$. I have plotted this for three values of $m$ in Figure 3.1. The associated Sage notebook for producing this figure is* `plotting_curve_on_cone.ipynb`.

As a further consequence of the Lax formalism of Nahm's equations, one can use integrable systems techniques to construct Nahm data from the function theory on $\mathcal{C}$. I shall not discuss this in any detail in this thesis, for more see [ES89, BE10b].

### Monopoles from Nahm Data

In this section we will state how to go from Nahm data to a solution of the BPS equation.

**Theorem 3.2.53** ([Nah83], [CG84], §4, [BE18], Theorem 2.1)**.** *Given $k \times k$ Nahm data on the interval $[-1,1]$, a nonsingular, charge-$k$ solution to the BPS equation is given by*

$$\begin{aligned}(\phi(\boldsymbol{x}))_{ab} &= i\int_{-1}^{1} s\boldsymbol{v}_a^\dagger(s,\boldsymbol{x})\boldsymbol{v}_b(s,\boldsymbol{x})\,ds,\\ (A_i(\boldsymbol{x}))_{ab} &= \int_{-1}^{1} \boldsymbol{v}_a^\dagger(s,\boldsymbol{x})\partial_i\boldsymbol{v}_b(s,\boldsymbol{x})\,ds,\end{aligned} \tag{3.19}$$

*for $\boldsymbol{x} \in \mathbb{R}^3$, where the two vectors $\boldsymbol{v}_{0,1}(s,\boldsymbol{x}) \in \mathbb{C}^{2k}$ form an orthonormal basis of normalisable solutions to the Weyl equation*

$$\Delta^\dagger \boldsymbol{v} := \left[i\frac{d}{ds} - \sum_{j=1}^{3}\sigma_j \otimes (T_j + ix_j\,\mathrm{Id}_k)\right]\boldsymbol{v} = 0$$

*on the interval $[-1,1]$ with respect to the inner product*

$$\langle \boldsymbol{v},\boldsymbol{w}\rangle = \int_{-1}^{1}\boldsymbol{v}^\dagger(s,\boldsymbol{x})\boldsymbol{w}(s,\boldsymbol{x})\,ds.$$

*Moreover, all solutions can be obtained as such.*

**Remark 3.2.54.** *Implicit in Equation 3.19 is that, if $n = \dim\ker\Delta^\dagger$, then the matrices for $\phi, A_i$ constructed by the integrals are $n \times n$. As such to get $\mathrm{SU}(2)$ gauge fields it is necessary that $\dim\ker\Delta^\dagger = 2$. This is ensured by fixing the boundary conditions; different boundary conditions give rise to different numbers of normalisable solutions and hence correspond to monopoles with different gauge groups and amounts of symmetry breaking [HM89, Dan92].*

The fact that the nonsingularity of the monopole is immediate from the existence of Nahm data is an important appeal of this method, as is the fact that the construction of the monopole from the gauge fields from the Nahm data just involves solving a matrix ODE and computing integrals. For almost all known Nahm data this process is too hard to do analytically, but it may be done numerically [HS96d], and this is how I will plot energy density isosurfaces of monopole solutions in Figure 3.4. If one calculated with a gauge-equivalent triple of Nahm matrices ${}_U\mathcal{T}$, one can check that the corresponding normalisable solutions are given by $\boldsymbol{v}_U = \left(\begin{smallmatrix} U & 0 \\ 0 & U\end{smallmatrix}\right)\boldsymbol{v}$, which does not change $\phi$ or $A$.

**Remark 3.2.55.** *An inverse transform may be used to show that all monopoles have corresponding Nahm data. One can also do so by completing the circle of* *idea mentioned at the beginning of this part. See [CG84, §1] for a more detailed history.*

**Example 3.2.56** (The Hedgehog Solution, [MS04], §8.6)**.** *As with all the avatars with which one can interpret monopoles, I will demonstrate how the Prasad-Sommerfield monopole calculated in Example 3.1.17 can be recovered from Nahm data. The case of the hedgehog is the only known example where the full gauge field data can be recovered analytically in this way.*

*As seen in Example 3.2.45, when $k = 1$ the Nahm data is given by $T_j(s) = ic_j$ for $c_j \in \mathbb{R}$. For this example the Weyl equation is*

$$i \left[ \frac{d}{ds} - (\boldsymbol{x} + \boldsymbol{c}) \cdot \boldsymbol{\sigma} \right] \boldsymbol{v} = 0,$$

*which has solutions*

$$\boldsymbol{v}_a(s, \boldsymbol{x}) = e^{s[(\boldsymbol{x}+\boldsymbol{c})\cdot\boldsymbol{\sigma}]} \boldsymbol{v}_a(0) = [\cosh(\rho z) + \widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \sinh(\rho z)] \, \boldsymbol{v}_a(0), \tag{3.20}$$

*where we have set*

$$\widehat{\boldsymbol{n}} = \frac{\boldsymbol{x} + \boldsymbol{c}}{|\boldsymbol{x} + \boldsymbol{c}|}, \;\; \rho = |\boldsymbol{x} + \boldsymbol{c}|, \tag{3.21}$$

*and made use of the fact $(i\widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma})^2 = -1 \Rightarrow e^{ia\widehat{\boldsymbol{n}}\cdot\boldsymbol{\sigma}} = \cos(a) + i\widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \sin(a)$. As such*

$$\boldsymbol{v}_a^\dagger(s) \boldsymbol{v}_b(s) = \boldsymbol{v}_a^\dagger(0) \left[\cosh(2\rho s) + \widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \sinh(2\rho s)\right] \boldsymbol{v}_b(0)$$

*and for orthonormality we require*

$$\delta_{ab} = \int_{-1}^{1} \boldsymbol{v}_a^\dagger(s) \boldsymbol{v}_b(s) \, ds = \frac{\sinh(2\rho)}{\rho} \boldsymbol{v}_a^\dagger(0) \boldsymbol{v}_b(0) \tag{3.22}$$

*and so we may take*

$$\boldsymbol{v}_a(0) = \sqrt{\frac{\rho}{\sinh(2\rho)}} \boldsymbol{e}_a \Rightarrow \boldsymbol{v}_a(s) = \sqrt{\frac{\rho}{\sinh(2\rho)}} \left[\cosh(\rho s) + \widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma} \sinh(\rho s)\right] \boldsymbol{e}_a$$

*where*

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \;\; \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

*Thus the Higgs field is then given by*

$$\begin{aligned} \phi_{ab}(\boldsymbol{x}) &= i \frac{\rho}{\sinh(2\rho)} \int_{-1}^{1} s \left[\delta_{ab} \cosh(2\rho s) + (\widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma})_{ab} \sinh(2\rho s)\right] \, ds, \\ &= i \frac{1}{2\sinh(2\rho)} \, (\widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma})_{ab} \left[2\cosh(2\rho) - \frac{1}{\rho} \sinh(2\rho)\right], \\ &= i \left[\coth(2\rho) - \frac{1}{2\rho}\right] (\widehat{\boldsymbol{n}} \cdot \boldsymbol{\sigma})_{ab} \, . \end{aligned} \tag{3.23}$$

*This solution has the right asymptotic behaviour taking $|\phi|^2 = -\frac{1}{2} \operatorname{Tr} \phi^2$. One can also recover $A_i$ via this method, and thus check this is the solution of Example*

*3.1.17 with $\mu = 2$, translated such that the origin is at $-\boldsymbol{c}$. Hence in this case the centre of the monopole, which for the hedgehog is the point where $\phi$ vanishes, is given by $(i\,\mathrm{Tr}(T_j)) \in \mathbb{R}^3$.*

## 3.3 Symmetries of Monopoles

> This method does not help in finding general solutions, only finding symmetric solutions. Still, knowing only a few symmetric solutions is better than knowing nothing at all.
>
> – Sidney Coleman
> Aspects of Symmetry

The purpose of this section will be to introduce the necessary machinery for constructing Nahm data with certain rotational symmetries, a technique that will be vital in §3.4 facilitating the construction of new charge-3 spectral curves.

### 3.3.1 Group Actions on Monopole Data

#### Translations and Rotations

The 3-dimensional Euclidean group $\mathrm{E}(3) = \mathbb{R}^3 \rtimes \mathrm{O}(3)$ has an induced left action on monopole gauge fields from the corresponding left action on $\mathbb{R}^3$. It will be helpful in constructing monopole solutions to understand how this group action translates to an action on the corresponding Nahm data and spectral curve.

I will start with Nahm data, where Theorem 3.2.53 tells us how the $T_j$ transform under $\boldsymbol{x} \in \mathbb{R}^3$ and $A \in \mathrm{O}(3)$. For $\boldsymbol{x} \in \mathbb{R}^3$ the transform is simple, namely

$$\boldsymbol{x} : T_j \mapsto T_j + ix_j \,\mathrm{Id}_k \,.$$

This gives the interpretation of the centre of the corresponding monopole as $(i\,\mathrm{Tr}(T_j)) \in \mathbb{R}^3$ as we have previously seen in Example 3.2.56. For transformations $A \in \mathrm{SO}(3)$ the situation is more complicated. In order to preserve the form of $\Delta^\dagger$, one requires $T_i \mapsto A_{ij}T_j$, but this rotates the residues of the $T_i$, so there must be a corresponding conjugation by the unitary matrix $U = U(A)$ representing the image of $A$ in $\mathrm{SU}(k)$ under the $\mathrm{SU}(2)$-representation determined by the irreducible $k$-dimensional $\mathfrak{su}(2)$-representation given by the residues. Hence the overall transform is

$$A : T_i \mapsto A_{ij}T_j^{U(A)} = A_{ij}\left[U(A)T_jU(A)^{-1}\right].$$

The conjugation part of the action does not represent a physical transform of the monopole, but rather a gauge transform. I will not discuss the action of elements of $\mathrm{O}(3) \setminus \mathrm{SO}(3)$ just yet.

The action of the Euclidean group also induces an action on minitwistor space, and so we can work out the corresponding action on the coordinates of $T\mathbb{P}^1$. To do so it is helpful to be explicit about the map from $\mathbb{MT}$ to $T\mathbb{P}^1$: writing

$$\mathbb{MT} = \left\{ (\boldsymbol{u}, \boldsymbol{v}) \in S^2 \times \mathbb{R}^3 \,|\, \boldsymbol{u} \cdot \boldsymbol{v} = 0 \right\}$$

we take $\zeta = \frac{u_1 + iu_2}{1-u_3}$, $\eta = [(iv_1 - v_2) + 2v_3\zeta + (iv_1 + v_2)\zeta^2]$ [Hit82, §3]. This slightly strange looking choice of convention ensures agreement with those already set for the Nahm matrices as we will see shortly. Note that indeed the orientation-reversing involution on $\mathbb{MT}$ given by $(\boldsymbol{u}, \boldsymbol{v}) \mapsto (-\boldsymbol{u}, \boldsymbol{v})$ corresponds to the anti-holomorphic involution $(\zeta, \eta) \mapsto (-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)$ as claimed in §2.1.3. For $\boldsymbol{x} \in \mathbb{R}^3$ the translation transform is likewise simple as it was in the Nahm data picture, given by

$$\boldsymbol{x} : \eta \mapsto \eta + \left[ (ix_1 - x_2) + 2x_3\zeta + (ix_1 + x_2)\zeta^2 \right]. \tag{3.24}$$

**Remark 3.3.1.** *To see that this is the correct result to agree with the transform of the Nahm matrices, use the charge-1 case. Taking the Nahm matrices $T_j = ic_j$ for $c_j \in \mathbb{R}$, the corresponding spectral curve is*

$$0 = \det(\eta + L) = \eta + \left[ (ic_1 - c_2) + 2c_3\zeta + (ic_1 + c_2)\zeta^2 \right].$$

*Now a translation by $\boldsymbol{x}$ sends $T_j$ to $T_j + ix_j$ and so the spectral curve to*

$$0 = \eta + \left[ (ic_1 - c_2) + 2c_3\zeta + (ic_1 + c_2)\zeta^2 \right] + \left[ (ix_1 - x_2) + 2x_3\zeta + (ix_1 + x_2)\zeta^2 \right].$$

*This recovers the transform of Equation 3.24.*

For $A \in \mathrm{SO}(3)$ corresponding to a rotation about $\boldsymbol{n} \in S^2$ by angle $\theta$, let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & q \\ -\bar{q} & \bar{p} \end{smallmatrix}\right)$ be the corresponding matrix in $\mathrm{PSU}(2) \cong \mathrm{SO}(3)$ given by

$$p = \cos(\theta/2) + in_3 \sin(\theta/2), \quad q = n_2 \sin(\theta/2) - in_1 \sin(\theta/2).$$

This acts as the Möbius transformation [MS04, (8.219)]

$$A : (\zeta, \eta) \mapsto \left( \frac{a\zeta + b}{c\zeta + d}, \frac{\eta}{(c\zeta + d)^2} \right).$$

**Example 3.3.2.** *If $A$ is a rotation about $(0, 0, 1)$ by $2\pi/k$ radians then $p = e^{i\pi/k}, q = 0$, and $(\zeta, \eta) \mapsto (e^{2\pi i/k}\zeta, e^{2\pi i/k}\eta)$. At certain points throughout I shall call this map $s = s_k$. Furthermore, the rotation* $\mathrm{diag}(1, -1, -1)$ *corresponds to the map $r : (\zeta, \eta) \mapsto (1/\zeta, -\eta/\zeta^2)$.*

**Remark 3.3.3.** *The full Möbius group* $\mathrm{PSL}_2(\mathbb{C})$ *acts on $\zeta$ and hence $\eta$, and if this transform sends $\mathcal{C}$ to $\mathcal{C}'$ the two curves will have the same period matrix, but only the subgroup* $\mathrm{PSU}(2)$ *preserves the reality property of invariance under $\tau$ [HMR00, Bra11]. The action of a translation on $\zeta, \eta$ also commutes with $\tau$. The full action of the Euclidean group on Nahm matrices preserves the property $T_i = -T_i^\dagger$.*

The action of $\mathrm{O}(3)$ is antiholomorphic and can be worked out from the definition with, for example,

$$-\mathrm{Id} : (\zeta, \eta) \mapsto (-1/\bar{\zeta}, \bar{\eta}/\bar{\zeta}^2).$$

**Example 3.3.4.** *Composing the transformation for* $-\mathrm{Id}$ *with the rotation* $s_2$ *we see that the reflection* $R_3 = \mathrm{diag}(1, 1, -1)$ *corresponds to the map* $(\zeta, \eta) \mapsto (1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)$*, and further composing with the map* $r$ *we see that* $\mathrm{diag}(1, -1, 1)$ *corresponds to* $t : (\zeta, \eta) \mapsto (\bar{\zeta}, \bar{\eta})$.

**Remark 3.3.5.** *The map* $-\mathrm{Id}$ *is called* ***inversion*** *in [HS96a, Bie20], whereas the map* $R_3$ *is called inversion in [HMM95].*

These two descriptions of the action on Nahm matrices and on $T\mathbb{P}^1$ are compatible (up to conventions that shall not be material). Through the action of a translation we can always centre a monopole, and this corresponds to being able to make the $T_i$ traceless, equivalently setting the $\eta^{k-1}$ coefficient of the spectral curve to zero. As one would expect, the property of being centred is preserved under the action of $\mathrm{O}(3)$.

Having determined how the $T\mathbb{P}^1$ coordinates and the Nahm matrices transform under the Euclidean group we can ask what is the resulting effect on the spectral curve given by $P(\zeta, \eta) = \det [\eta + L(\zeta)]$. Clearly conjugating the Nahm matrices has no effect on the spectral curve, and so the holomorphic action of $\mathbb{R}^3 \rtimes \mathrm{O}(3)$ on the $T\mathbb{P}^1$ coordinates must correspond to the action on the $\mathbb{R}^3$ factor of the $\mathbb{R}^3 \otimes \mathfrak{su}(k)$ representation space supporting the $T_i$. As such, invariance of the spectral curve under $A \in \mathrm{SO}(3)$ which maps $(\zeta, \eta) \mapsto (\tilde{\zeta}, \tilde{\eta})$ can equally be rephrased as the condition $\det [\eta + L(\zeta)] = 0 = \det \left[\tilde{\eta} + L^{U(A)}(\tilde{\zeta})\right]$, where the Lax matrix $L^U$ is constructed from the transformed Nahm matrices $T_i^U$.

We can apply the same logic to the action of $A \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$ on the Nahm data. We know from [Hit83] that the antiholomorphic involution acts on the Nahm matrices as $T_i \to T_i^\dagger$, but we also know it acts on the $T\mathbb{P}^1$ coordinates as $(\zeta, \eta) \mapsto (-1/\bar{\zeta}, -\bar{\eta}/\bar{\zeta}^2)$. This $T\mathbb{P}^1$ transformation maps

$$\begin{aligned} &\eta + \left[(T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2\right] \\ \mapsto &-\frac{1}{\bar{\zeta}^2}\overline{\left\{\eta + \left[(\bar{T}_1 + i\bar{T}_2) - 2i\bar{T}_3 + (\bar{T}_1 - i\bar{T}_2)\zeta^2\right]\right\}}, \end{aligned}$$

and thus maps $T_i$ to $\bar{T}_i$. This means that the action of $\tau$ on the $\mathfrak{su}(k)$ component of the Nahm matrices' representation space is the transpose. Note that $A \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$ is orientation-reversing on $\mathbb{P}^1$, and so must correspond to an antiholomorphic automorphism of $T\mathbb{P}^1$, hence composing with $\tau$ corresponds to a holomorphic automorphism. One can check that under the map $T_i \mapsto A_{ij}T_j$ a solution to Nahm's equations is mapped to a solution of the anti-Nahm equations [CDL+22, Remark 2.7]

$$\frac{dT_i}{ds} = -\frac{1}{2}\sum_{j,l=1}^{3} \epsilon_{ijl}\left[T_j, T_l\right],$$

so composing with the transpose one regains a solution to Nahm's equations.

### The Moduli Space of Symmetric Monopoles

As mentioned, we will go on to see clear examples of how the action of symmetries on monopole data was used to partially classify 3-monopoles in §3.4, but before that I will present one generality of this approach, namely the ability to gain information about the dimension of the moduli space of monopoles with a given symmetry.

Let us suppose that we are considering the moduli space of charge-$k$ monopoles invariant under a fixed action of finite group $G$, that is $(M_k)^G$. For $G \leq \mathrm{O}(3)$, it is known that $(M_k)^G$ is a totally geodesic submanifold of $M_k$, and likewise for $(M_k^0)^G$ as a submanifold of $M_k^0$ [HMM95, Bie20]. This moduli space we shall distinguish from $(M_k^0)^{[G]}$ which is the moduli space of monopoles invariant under an element of the conjugacy class of $G$ inside $\mathrm{O}(3)$; clearly $(M_k^0)^G \subset (M_k^0)^{[G]}$ and when $G$ is discrete we have $\dim_{\mathbb{R}}(M_k^0)^G - \dim_{\mathbb{R}} N_{\mathrm{O}(3)}(G) = \dim_{\mathbb{R}}(M_k^0)^{[G]} - \dim_{\mathbb{R}} \mathrm{O}(3)$, where $N_{\mathrm{O}(3)}(G)$ is the normaliser.

Some results on such submanifolds are known for specific simple choices of $G$, for example the following two propositions. Recall the two different definitions of inversion as $R_3$ or $-\mathrm{Id}$ from Remark 3.3.5.

**Proposition 3.3.6** ([HMM95], Proposition 2)**.** $\dim_{\mathbb{R}}(M_k)^{\langle R_3\rangle} = 2k$, *and moreover* $\dim_{\mathbb{R}}(M_k^0)^{\langle R_3\rangle} = 2k-2$.

**Corollary 3.3.7.** *$k = 2$ is the only charge for which every centred monopole is symmetric under a reflection in a plane*

*Proof.* The $\mathrm{SO}(3)$ (and hence $\mathrm{O}(3)$) orbit of $R_3$ in $\mathrm{O}(3)$ is the set of all reflections in a plane through the origin, and this is a 2-dimensional submanifold of $\mathrm{O}(3)$. Equivalently this is saying $\dim N_{\mathrm{O}(3)}(\langle R_3\rangle) = 1$, corresponding to the rotations about the $x_3$ axis. As such if every centred monopole is invariant under a reflection in a plane one must have

$$\dim_{\mathbb{R}} M_k^0 = \dim_{\mathbb{R}}(M_k^0)^{\langle R_3\rangle} + 2 \Leftrightarrow 4k-4 = 2k-2+2 \Leftrightarrow k=2.$$

□

**Proposition 3.3.8** ([Bie20], Proposition 1.1)**.** $\dim_{\mathbb{R}}(M_k^0)^{\langle -\mathrm{Id}\rangle} = 4\lfloor k/2\rfloor$.

For a generic choice of $G$ we can say more. If $\mathcal{C}$ is the spectral curve corresponding to a monopole in $(M_k^0)^G$ we must have that $G \leq \mathrm{Aut}(\mathcal{C})$, and so we get ramification data $(g_0; c_1, \ldots, c_r)$ associated with the quotient $\mathcal{C} \to \mathcal{C}/G$ as in §2.1.4. This ramification data gives us a bound on the dimension of the symmetric moduli space as follows.

**Proposition 3.3.9** ([BDH23])**.** *For $k \geq 3$ and $G \leq \mathrm{O}(3)$ discrete (i.e. finite),* $\dim_{\mathbb{R}}(M_k^0)^G - \dim_{\mathbb{R}} N_{\mathrm{O}(3)}(G) \leq 3g_0 - 3 + r$.

*Proof.* [MSSV02, Lemma 3.1] gives that the complex dimension of each component of the locus of equivalence classes of genus $g \geq 2$ curves admitting an action of a group isomorphic to $G$ with signature $c$ is (provided it is nonempty) $\delta(g, G, c) := 3(g_0 - 1) + r = \dim_{\mathbb{C}} \mathcal{M}_{g_0,r}$ the moduli space of genus $g_0$ curves with $r$ marked points. The action of $\mathrm{SO}(3)$ on $(M_k^0)^{[G]}$ is trivial on the moduli space of curves because it induces a birational isomorphism. The result then follows as the $\mathrm{SO}(3)$ orbits of the moduli space of monopoles will form a component of this locus, hence[17] $\dim_{\mathbb{R}}(M_k^0)^G - \dim_{\mathbb{R}} N_{\mathrm{SO}(3)}(G) = \dim_{\mathbb{R}}(M_k^0)^{[G]} - \dim_{\mathbb{R}} \mathrm{SO}(3) \leq \dim_{\mathbb{R}}\left(\mathcal{M}_{g_0,r}^{\tau}\right)$, and using the fact from Teichmüller theory that $\dim_{\mathbb{R}}\left(\mathcal{M}_{g_0,r}^{\tau}\right) = \dim_{\mathbb{C}} \mathcal{M}_{g_0,r}$ [Ear71, §3.1]. □

**Remark 3.3.10.** *Note in the statement and proof of Proposition 3.3.9 we could have used $H \leq G$ and its corresponding signature, but this would have given a weaker bound as $\delta(g, G, c_G) \leq \delta(g, H, c_H)$ [MSSV02].*

The remarkable fact about Proposition 3.3.9 is that the bound depends on the signature only.

The curves of [HMM95], with group of rotational automorphisms $H$ of order $2k(k-1)$ for $k = 3, 4, 6$, have signature $c_H = (1; k-1)$ [HMM95, Proposition 4]. For these curves, and all known monopole spectral curves of charge 3 with a quotient by $H \leq \mathrm{Aut}(\mathcal{C})$ to an elliptic curve (see §3.4) where $H$ is generated by rotations, we have $\dim_{\mathbb{R}}(M_k^0)^H - \dim_{\mathbb{R}} N_{\mathrm{SO}(3)}(H) = \delta(g, H, c_H)' - 1$, where $\delta' \leq \delta$ is the dimension of the moduli space of curves in the linear system $|\pi^*\mathcal{O}(2k)|$ with the corresponding ramification type. As such one might make the following conjecture.

**Conjecture 3.3.11.** *When $H \leq \mathrm{SO}(3)$ acts on charge-$k$ monopole spectral curves $\mathcal{C}$ such that $g(\mathcal{C}/H) = 1$, the dimension of the moduli space of centred $H$-invariant $k$-monopoles is given by*

$$\dim_{\mathbb{R}}(M_k^0)^H - \dim_{\mathbb{R}} N_{\mathrm{SO}(3)}(H) = \delta(g, H, c_H)' - 1.$$

Requiring $H \leq \mathrm{SO}(3)$ ensures that the ES vector is a pullback from the Jacobian of the quotient elliptic curve as described in Remark 3.2.31. The requirement that the ES vector is a pullback is strict for this conjecture, as we will see later in §3.4.3 in the context of $\langle R_3 \rangle$-invariant 3-monopoles.

It is certainly true that the larger moduli space of $H$-invariant nonsingular $\tau$-invariant centred curves in the linear system $|\pi^*\mathcal{O}(2k)|$ satisfying the ES constraint will satisfy $\dim_{\mathbb{R}}(M_{k,ES}^0)^H - \dim_{\mathbb{R}} N_{\mathrm{SO}(3)}(H) = \delta' - 1$ because when the ES vector is a pullback the ES conditions become constraints on the $\mathrm{Jac}(\mathcal{C}/H)$ factor of the Jacobian only. As such the conjecture reduces to showing that the codimension of $(M_k^0)^H \subset (M_{k,ES}^0)$ is 0. We have seen previously that this is certainly the case when $H$ is trivial. One might wonder whether it is further possible that one would have $\dim(M_{k,ES}^0)^H - \dim_{\mathbb{R}} N_{\mathrm{SO}(3)}(H) = \delta' - g_0$, appealing to the connection with the ES constraints. We know that rotationally cyclically symmetric charge-$k$ monopoles satisfy the pullback condition with $H = C_k$, $g_0 = k-1$ and the quotient is unramified [Bra11, §2]. For such curves one can count that $\delta' = k \leq 3k - 6 = \delta$ when $k \geq 3$. The extended conjecture would then suggest that $\dim_{\mathbb{R}}(M_{k,ES}^0)^H - 1 = \delta' - g_0 = 1$, and this is the result found in [HMM95, §12]. As such there is some evidence for the extended conjecture.

[17] As described in Earle, points in the moduli space of genus $g \geq 2$ Riemann surfaces correspond to equivalence classes of complex structures on a fixed smooth surface of the corresponding genus. There is a natural conjugate involution on this moduli space of complex structures, such that any complex structure fixed under an orientation-reversing involution of the surface corresponds to a fixed point of the conjugate involution of the moduli space. Hence we may consider the moduli space $\mathcal{M}_{g_0,r}^{\tau} \subset \mathcal{M}_{g_0,r}$ of curves invariant under $\tau$ by this procedure.

### 3.3.2 Constructing Symmetric Monopole Data

Historically, the first attempts to construct symmetric monopoles involved imposing an ansatz of continuous rotational symmetry on the gauge fields themselves, as in [tH74, PS75, HM77, Ju78, WB79, MIM82], but this quickly proved to be difficult. [Koi82] took an interesting approach which is closer to that we shall develop in this subsection, namely showing that imposing a spherical symmetry ansatz on the BPS equation allows it to be formulated as an ODE with a Lax pair where the $r$ coordinate represents 'time', and then solving this Lax equation with the inverse scattering method. [Ath83] also showed how one could impose continuous symmetries on the data of Corrigan and Goddard. Here I shall present a method which uses the representation theory of the Nahm matrices in order to construct monopoles with discrete groups of rotational symmetries. This method can also be extended to construct Nahm data with continuous symmetries [BCG$^+$83, CDL$^+$22].

#### The HMM Representation Space

We will now want to adapt the work of [HMM95] in thinking of $\mathbb{R}^3 \otimes \mathfrak{u}(k)$ as the SU(2)-representation space of the Nahm triple $\mathcal{T}$. In fact, we shall only work with centred Nahm matrices, so that the representation space is actually the real Lie algebra $\mathbb{R}^3 \otimes \mathfrak{su}(k)$, which I will call the **HMM representation space**. Considering the action of SO(3) on $\mathcal{T}$ described in §3.3.1, one finds that in terms of the irreducible SU(2) representation spaces of §3, $\mathbb{R}^3 \cong \mathbb{S}^2$ and

$$\mathfrak{u}(k) \cong \mathbb{S}^{k-1} \otimes \mathbb{S}^{k-1} = \mathbb{S}^{2k-2} \oplus \mathbb{S}^{2k-4} \oplus \cdots \oplus \mathbb{S}^0 \Rightarrow \mathfrak{su}(k) \cong \mathbb{S}^{2k-2} \oplus \cdots \oplus \mathbb{S}^2.$$

Clearly for this to be valid we require $k \geq 2$, and so I will assume that to always be through in this section. These isomorphisms are actually as complex Lie algebras, so initially I will work with these, and then implicitly restrict to real Lie algebras when working with $\mathbb{R}^3 \otimes \mathfrak{su}(k)$ and imposing the condition $T_i = -T_i^\dagger$. Combining the two isomorphisms we see that the HMM representation space is isomorphic to

$$\mathbb{HMM} := (\mathbb{S}^{2k}_{-1} \oplus \mathbb{S}^{2k-2}_0 \oplus \mathbb{S}^{2k-4}_1) \oplus \cdots \oplus (\mathbb{S}^4_{-1} \oplus \mathbb{S}^2_0 \oplus \mathbb{S}^0_1), \tag{3.25}$$

where I have used $\mathbb{S}^{2r}_i$ to denote the copy of $\mathbb{S}^{2r}$ in the representation space coming from the tensor[18] $\mathbb{S}^2 \otimes \mathbb{S}^{2(r+i)}$. We will want to interpret these $\mathbb{S}^{2r}$ factors as corresponding to the homogeneous degree-$2r$ polynomials appearing in the spectral curve, which we will do later concretely in Proposition 3.3.24.

The reason for introducing the HMM representation space is that it will be a clarifying framework for the construction of monopole solutions with symmetry, namely for $G \leq \mathrm{PSU}(2)$ we want to construct the $G$-invariant subspace of the HMM representation space $(\mathbb{R}^3 \otimes \mathfrak{su}(k))^G$. Trivially we know $(\mathbb{R}^3 \otimes \mathfrak{su}(k))^{\{e\}} = \mathbb{R}^3 \otimes \mathfrak{su}(k)$, and moreover $(\mathbb{R}^3 \otimes \mathfrak{su}(k))^{\mathrm{PSU}(2)} = \langle (\rho_i) \rangle$ where $(\rho_i)$ is the triple of matrices spanning the $\mathbb{S}^0_1$ subspace. In practice, these $\rho_i$ generate the $k$-dimensional irreducible representation of $\mathfrak{su}(2)$ containing the residues of the $T_i$.

[18] [HS96c] will use the letters $u$, $m$, and $l$ to distinguish between these.

**Example 3.3.12.** *In the case $k = 2$ one can take $\rho_j = \frac{\sigma_j}{i}$, whereas for $k = 3$ one can take*

$$\rho_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We shall want to find additional triples $(S_i^{(j)}) \in \mathbb{R}^3 \otimes \mathfrak{su}(k)$ such that

$$\left\langle (\rho_i), (S_i^{(j)}) \right\rangle = \left(\mathbb{R}^3 \otimes \mathfrak{su}(k)\right)^G.$$

This is a hard task at the level of matrices, but if we have an equivariant map that takes a homogeneous polynomial in $\mathbb{HMM}$ to a triple in $\mathbb{R}^3 \otimes \mathfrak{su}(k)$ the problem is greatly simplified as the action of $\mathrm{SU}(2)$ on $\mathbb{S}^{2r}$ is understood simply. As such I will describe this algorithm now.

The idea behind this approach will be to identify highest-weight subspaces of the two representation spaces, thinking of them now as $\mathfrak{su}(2)$ representations. In particular I will construct an isomorphism

$$\mathbb{S}_i^{2r} \to \mathbb{S}_{-1}^{2r} \overset{\text{Polar}}{\to} \mathbb{S}^2 \otimes \mathbb{S}^{2r-2}\big|_{\mathbb{S}_{-1}^{2r}} \overset{\text{h.w.v}}{\to} \mathbb{S}^2 \otimes \mathbb{S}^{2(r+i)}\big|_{\mathbb{S}_i^{2r}} \overset{\text{h.w.v}}{\to} \mathbb{R}^3 \otimes \mathfrak{su}(k)\big|_{\mathbb{S}_i^{2r}}, \tag{3.26}$$

where h.w.v. indicates that the corresponding isomorphisms are constructed by identifying highest-weight vectors and Polar is a map that will be defined shortly in Equation 3.27. To get three matrices from an element of $\mathbb{R}^3 \otimes \mathfrak{su}(k)$ I will then provide a method that is consistent with the above definition of the $\rho_i$ as the triple of matrices spanning $\mathbb{S}_1^0$.

I shall start with some notation. Fixing a scale[19] on the $\rho_i$ giving the representation $\mathfrak{su}(2) \to \mathfrak{su}(k)$ such that

$$[\rho_1, \rho_2] = 2\rho_3, \quad [\rho_2, \rho_3] = 2\rho_1, \quad [\rho_3, \rho_1] = 2\rho_2,$$

we may realise the principal 3-dimensional simple subalgebra of $\mathfrak{su}(k)$ to be $\mathfrak{a}_0 = \langle X_0, Y_0, H_0 \rangle$ given by

$$X_0 = \frac{1}{2}(\rho_1 - i\rho_2), \quad Y_0 = -\frac{1}{2}(\rho_1 + i\rho_2), \quad H_0 = -i\rho_3,$$

which is a copy of $\mathfrak{su}(2)$.

**Remark 3.3.13.** *Notice we have*

$$X_0^\dagger = \frac{1}{2}(\rho_1^\dagger + i\rho_2^\dagger) = \frac{1}{2}(\rho_1 + i\rho_2) = Y_0, \quad H_0^\dagger = i\rho_3^\dagger = H_0,$$

*if $\rho_i^\dagger = -\rho_i$, as in the cases given in Example 3.3.12. Indeed using the irreducible representations described in [Hal15, §4.6] we can always ensure that $X_0, Y_0, H_0$ are real and satisfy $X_0^T = Y_0$, $H_0 = H_0^T$.*

[19]The scale taken here is nonstandard for representations of $\mathfrak{su}(2)$, but has been taken to align with the choice in [HMM95].

By a theorem of Kostant [Kos59], $\mathfrak{su}(k)$ decomposes into a direct sum of irreducible representations of $\mathfrak{a}_0$ acting via the adjoint action, and so we can use this to identify copies of $\mathbb{S}^{2r}$ inside the representation space if we know its highest-weight vectors.

**Lemma 3.3.14.** *$X_0^r \in \mathfrak{su}(k)$ is a highest-weight vector of weight $2r$. Moreover, we have highest-weight vectors $v_i \in \mathbb{S}_i^{2r} \subset \mathbb{R}^3 \otimes \mathfrak{su}(k)$ of weight $2r$ given by*

$$
\begin{aligned}
v_{-1} &= X \otimes X_0^{r-1}, \\
v_0 &= \operatorname{ad}_Y X \otimes X_0^r - \frac{1}{r} X \otimes \operatorname{ad}_{Y_0} X_0^r, \\
v_1 &= \operatorname{ad}_Y^2 X \otimes X_0^{r+1} - \frac{1}{r+1} \operatorname{ad}_Y X \otimes \operatorname{ad}_{Y_0} X_0^{r+1} \\
&\quad + \frac{1}{(r+1)(2r+1)} X \otimes \operatorname{ad}_{Y_0}^2 X_0^{r+1}.
\end{aligned}
$$

*Proof.* This follows simply from computation. □

**Remark 3.3.15.** *Here I am taking the notation that $\operatorname{ad}_A(B) = [A, B]$. This is the opposite sign notation to [HMM95, HS96c], which I will denote as $\widetilde{\operatorname{ad}}_A(B) = [B, A]$ to avoid ambiguity.*

**Remark 3.3.16.** *In proving Lemma 3.3.14, one only needs properties of $\mathfrak{su}(2)$ and highest-weight vectors, not anything specific to $\mathfrak{su}(k)$. As a result this gives a generic formulation for the highest-weight vectors of weight $2r$ in $\mathbb{S}^2 \otimes V$ for some $\mathfrak{su}(2)$ representation space $V$ for which a highest-weight vector of weight $2(r+i)$ is known.*

With these conventions set, we can describe how to construct a triple of matrices in $\mathbb{R}^3 \otimes \mathfrak{su}(k)$ from a polynomial $Q \in \mathbb{S}_i^{2r}$, by describing the maps from Equation 3.26. We have the morphism

$$
\begin{aligned}
&\operatorname{Polar} : \mathbb{S}^{2r} \hookrightarrow \mathbb{S}^2 \otimes \mathbb{S}^{2r-2}, \\
&Q(\zeta_0, \zeta_1) \mapsto \zeta_0^2 \otimes \frac{\partial^2 Q}{\partial \zeta_0^2} + 2\zeta_0\zeta_1 \otimes \frac{\partial^2 Q}{\partial \zeta_0 \partial \zeta_1} + \zeta_1^2 \otimes \frac{\partial^2 Q}{\partial \zeta_1^2}, \\
&\qquad = (\zeta_0 \partial_{\zeta_1})^2 \zeta_1^2 \otimes \frac{1}{2}\frac{\partial^2 Q}{\partial \zeta_0^2} + (\zeta_0 \partial_{\zeta_1}) \zeta_1^2 \otimes \frac{\partial^2 Q}{\partial \zeta_0 \partial \zeta_1} + \zeta_1^2 \otimes \frac{\partial^2 Q}{\partial \zeta_1^2}.
\end{aligned}
\tag{3.27}
$$

Call $\operatorname{Polar}(Q)$ the **polarisation** of $Q$. Note this maps the highest-weight vector $\zeta_1^{2r} \in \mathbb{S}^{2r}$ of weight $2r$ to the highest-weight vector $2r(2r-1)v_{-1} = 2r(2r-1)\zeta_1^2 \otimes \zeta_1^{2r-2} \in \mathbb{S}^2 \otimes \mathbb{S}^{2r-2}$ of the same weight, and so the inclusion Polar is an isomorphism onto its image $\mathbb{S}^2 \otimes \mathbb{S}^{2r-2}|_{\mathbb{S}_{-1}^{2r}}$.

Moreover by identifying highest-weight vectors and the action of the lowering operator $Y \in \mathfrak{su}(2)$ one get the isomorphism of the subspaces

$$
\begin{aligned}
\phi_{i,j} : \mathbb{S}^2 \otimes \mathbb{S}^{2(r+i)}\big|_{\mathbb{S}_i^{2r}} &\to \mathbb{S}^2 \otimes \mathbb{S}^{2(r+j)}\big|_{\mathbb{S}_j^{2r}}, \\
(Y\cdot)^k v_i &\mapsto (Y\cdot)^k v_j,
\end{aligned}
$$

so the highest-weight isomorphism $\mathbb{S}^2 \otimes \mathbb{S}^{2r-2}|_{\mathbb{S}^{2r}_{-1}} \overset{\text{h.w.v}}{\to} \mathbb{S}^2 \otimes \mathbb{S}^{2(r+i)}\big|_{\mathbb{S}^{2r}_i}$ is $\phi_{-1,i}$. The highest-weight isomorphism $\mathbb{S}^2 \otimes \mathbb{S}^{2(r+i)}\big|_{\mathbb{S}^{2r}_i} \overset{\text{h.w.v.}}{\to} \mathbb{R}^3 \otimes \mathfrak{su}(k)|_{\mathbb{S}^{2r}_i}$ is given likewise, but here only on the second component of the tensor product.

Having constructed this isomorphism we need a way to read off the corresponding triple of matrices from an element of $\mathbb{R}^3 \otimes \mathfrak{su}(k)$. To do this, note that the element

$$\begin{aligned}
v &= X \otimes \mathrm{ad}^2_{Y_0} X_0 - \mathrm{ad}_Y X \otimes \mathrm{ad}_{Y_0} X_0 + \mathrm{ad}^2_Y X \otimes X_0, \\
&= -2X \otimes Y_0 - H \otimes H_0 - 2Y \otimes X_0, \\
&= X \otimes (\rho_1 + i\rho_2) + \mathrm{ad}_Y X \otimes (-i\rho_3) + \mathrm{ad}^2_Y X \otimes \frac{1}{2}(\rho_1 - i\rho_2),
\end{aligned}$$

spans the $\mathbb{S}^0$ subspace of $\mathbb{R}^3 \otimes \mathfrak{su}(k)$. Reading off the second components of the final expression as $\rho_1', \rho_2', \rho_3'$ respectively, we would find the relations

$$\rho_1 = \frac{1}{2}\rho_1' + \rho_3', \quad \rho_2 = \frac{-i}{2}\rho_1' + i\rho_3', \quad \rho_3 = i\rho_2'.$$

To generalise this, if we take a polynomial $Q \in \mathbb{S}^{2r}$ and from it construct the vector $X \otimes S_1' + \mathrm{ad}_Y X \otimes S_2' + \mathrm{ad}^2_Y X \otimes S_3'$ then the corresponding triple of matrices is $(S_i)$ given by

$$S_1 = \frac{1}{2}S_1' + S_3', \quad S_2 = \frac{-i}{2}S_1' + iS_3', \quad S_3 = -iS_2'. \tag{3.28}$$

**Remark 3.3.17.** *If we work instead with $\widetilde{\mathrm{ad}}$ when creating the $S_i'$, and modify the basis change to have $S_3 = -iS_2'$, we recreate exactly the same results in all the cases presented in [HMM95].*

**Remark 3.3.18.** *The freedom of choice in the $\rho_i$ represents the freedom of choice in the orientation of the corresponding Nahm matrices.*

**Example 3.3.19.** *We can be very explicit about how this works in the case of finding the matrices corresponding to $Q \in \mathbb{S}^{2r}_{-1}$ where the algorithm is more simple because $\phi_{-1,-1}$ is the identity. The process is as follows.*

1. *Polarise $Q(\zeta_0, \zeta_1)$ (i.e. compute* $\mathrm{Polar}(Q)$*) as*

$$\zeta_0^2 \otimes Q_{00}(\zeta_0, \zeta_1) + 2\zeta_0\zeta_1 \otimes Q_{01}(\zeta_0, \zeta_1) + \zeta_1^2 \otimes Q_{11}(\zeta_0, \zeta_1).$$

2. *Find univariate polynomials $\tilde{Q}_{00}, \tilde{Q}_{01}, \tilde{Q}_{11}$ such that*

$$\begin{aligned}
\frac{1}{2}Q_{00}(\zeta_0, \zeta_1) &= \tilde{Q}_{00}\left(\zeta_0 \frac{\partial}{\partial \zeta_1}\right)\zeta_1^{2r-2}, \\
Q_{01}(\zeta_0, \zeta_1) &= \tilde{Q}_{01}\left(\zeta_0 \frac{\partial}{\partial \zeta_1}\right)\zeta_1^{2r-2}, \\
Q_{11}(\zeta_0, \zeta_1) &= \tilde{Q}_{11}\left(\zeta_0 \frac{\partial}{\partial \zeta_1}\right)\zeta_1^{2r-2}.
\end{aligned}$$

3. *Calculate matrices $S'_i$ as*

$$S'_1 = \tilde{Q}_{11}(\mathrm{ad}_{Y_0})X_0^{r-1}$$

*and likewise for $S'_2$ with $\tilde{Q}_{01}$, $S'_3$ with $\tilde{Q}_{00}$.*

4. *Calculate the $S_i$ from the $S'_i$ with Equation 3.28.*

*We shall call this the* ***upper-invariant algorithm***. *In the case of $Q \in \mathbb{S}_0^{2r}$, the situation would be more complicated as we must implement $\phi_{-1,0}$, namely we get the following steps.*

1. *Calculate the polarisation of $Q$ in $\mathbb{S}^2 \otimes \mathbb{S}^{2r-2}$ as in step 1 of the upper-invariant algorithm.*

2. *Find the univariate polynomial $\tilde{Q}$ such that*

$$\mathit{Polar}(Q) = \tilde{Q}(\zeta_0\partial_1 \otimes 1 + 1 \otimes \zeta_0\partial_1)(\zeta_1^2 \otimes \zeta_1^{2r-2}).$$

3. *Find univariate polynomials $\tilde{Q}_{00}, \tilde{Q}_{01}, \tilde{Q}_{11}$ such that*

$$\begin{aligned}\tilde{Q}(\zeta_0\partial_1 \otimes 1 + 1 \otimes \zeta_0\partial_1)(2\zeta_0\zeta_1 \otimes \zeta_1^{2r} - 2\zeta_1^2 \otimes \zeta_0\zeta_1^{2r-1}) = 2\zeta_0^2 &\otimes \tilde{Q}_{00}(\zeta_0\partial_1)\zeta_1^{2r} \\ &+ 2\zeta_0\zeta_1 \otimes \tilde{Q}_{01}(\zeta_0\partial_1)\zeta_1^{2r} \\ &+ \zeta_1^2 \otimes \tilde{Q}_{11}(\zeta_0\partial_1)\zeta_1^{2r}.\end{aligned}$$

4. *Apply steps 3 and 4 of the upper-invariant algorithm.*

*This shall give us the* ***middle-invariant algorithm***.

*There is likewise a process for the lower invariant. In all of these we find a univariate polynomial $\tilde{Q}$ such that $\mathit{Polar}(Q) = \tilde{Q}(Y\cdot)v_{-1}$, then write $\tilde{Q}(Y\cdot)v_i = 2\zeta_0^2 \otimes \tilde{Q}_{00}(Y\cdot)\zeta_1^{2(r+i)} + \dots$.*

**Example 3.3.20.** *Let us see Example 3.3.19 in action.*

*First consider $Q = \zeta_0^2$. Polarising we get $\zeta_0^2 \otimes 2$, so $Q_{00} = 2$, $Q_{01} = 0 = Q_{11}$. As such $\tilde{Q}_{00} = 1$, $\tilde{Q}_{01} = 0 = \tilde{Q}_{11}$ and $(S'_i) = (0, 0, \mathrm{Id})$.*

*Next consider $Q = \zeta_0\zeta_1$. Polarising would give $2\zeta_0\zeta_1 \otimes 1$, hence $Q_{01} = 1$, $Q_{00} = 0 = Q_{11}$, and so $\tilde{Q}_{01} = 1$, $\tilde{Q}_{00} = 0 = \tilde{Q}_{11}$. As such we find $(S'_i) = (0, \mathrm{Id}, 0)$.*

*Finally take $Q = \zeta_1^2$. Polarising yields $\zeta_1^2 \otimes 2$, so $Q_{11} = 2$, $Q_{00} = 0 = Q_{01}$, and $\tilde{Q}_{11} = 2$, $\tilde{Q}_{00} = 0 = \tilde{Q}_{01}$. Now we find $(S'_i) = (2\,\mathrm{Id}, 0, 0)$.*

**Example 3.3.21.** *Suppose now we have the vector $(S_i) \in \mathbb{R}^3 \otimes \mathfrak{su}(k)$ constructed from a polynomial $Q$, what is the polynomial corresponding to $(\tilde{S}_i) := (S_i^\dagger)$? We can compute*

$$\tilde{S}'_1 = \tilde{S}_1 + i\tilde{S}_2 = S_1^\dagger + iS_2^\dagger = (S_1 - iS_2)^\dagger = 2(S'_3)^\dagger,$$

*and likewise $\tilde{S}'_2 = -(S'_2)^\dagger$, $\tilde{S}'_3 = \frac{1}{2}(S'_1)^\dagger$.*

*Now we know, for example $S'_1 = \tilde{Q}_{11}(\mathrm{ad}_{Y_0})X_0^{r+i}$, and so by Remark 3.3.13 $\tilde{S}'_3 = \frac{1}{2}\overline{\tilde{Q}_{11}}(-\mathrm{ad}_{X_0})Y_0^{r+i}$, likewise for $\tilde{S}'_2$ and $\tilde{S}'_1$. Identifying $X_0$ with $\zeta_1^2$, $Y_0$ with*

$\zeta_0^2$, $\mathrm{ad}_{X_0}$ *with* $\zeta_1 \frac{\partial}{\partial \zeta_0}$, *and* $\mathrm{ad}_{Y_0}$ *with* $\zeta_0 \frac{\partial}{\partial \zeta_1}$, *the action of conjugate transpose on the* $\mathfrak{su}(k)$ *factor of* $\mathbb{R}^3 \otimes \mathfrak{su}(k)$ *corresponds to* $[\zeta_0 : \zeta_1] \mapsto [-\bar{\zeta}_1 : \bar{\zeta}_0]$ *in* $\mathbb{HMM}$ *under the isomorphism described. This is exactly what we would expect from §3.3.1 as both actions correspond to the antiholomorphic involution* $\tau$.

Note that the construction described here can all be computed explicitly in Sage, as is done in the code `nahm_data.py`. An example computation implementing the construction is given in the Sage notebook `charge_3_V4_symmetric_potential_monopole.ipynb`.

Given now some set of $d$ polynomials $Q^{(j)}$ as inputs for the algorithms we get triples $(S_i^{(j)})$ for $j = 1, \ldots, d$, and so because we know

$$\begin{aligned} \mathbb{R}^3 \otimes \mathfrak{su}(k) &= \left(\mathbb{R}^3 \otimes \mathfrak{su}(k)\right)^{\{e\}}, \\ \langle (\rho_i) \rangle &= \left(\mathbb{R}^3 \otimes \mathfrak{su}(k)\right)^{\mathrm{PSU}(2)}, \end{aligned}$$

we know there exist subgroups $H \leq G \leq \mathrm{SO}(3)$ such that

$$\left(\mathbb{R}^3 \otimes \mathfrak{su}(k)\right)^G \subseteq \left\langle (\rho_i), (S_i^{(j)}) \right\rangle \subseteq \left(\mathbb{R}^3 \otimes \mathfrak{su}(k)\right)^H,$$

and moreover $H$ (respectively $G$) is maximal (respectively minimal) with respect to inclusion. Identifying $H$ with a subgroup of $\mathrm{PSU}(2)$, we know that each $Q^{(j)}$ is in $(\mathbb{S}^{\deg Q^{(j)}})^H$. We shall often seek to find $Q^{(j)}$ such that $H = G$, which we may achieve by taking a basis of invariant polynomials as the inputs to the algorithm, which by equivariance must give a basis of invariant triples. Note this need not be true if we instead considered a group $G \leq \mathrm{O}(3)$ not generated entirely by rotations, and we will see an example of this in §3.4.3.

### Nahm's Equations on a Basis

We suppose now that we are given triples of matrices $(\rho_i)$, $(S_i^{(j)})$ for $j = 1, \ldots, d$, which in practice come from the algorithm described in the previous section. I shall describe the process of constructing solutions to Nahm's equations where $(T_i), (T_i') \in \left\langle (\rho_i), (S_i^{(j)}) \right\rangle$, modifying the method of [HMM95] to allow for the possibility that such a solution does not exist.

To start we shall recall the concept of **vectorisation**. The details of this will not be important, but it is a linear map $\mathrm{Vec} := V$ sending $m \times n$ matrices to vectors of length $mn$ such that

$$V(ABC) = (C^T \otimes A)V(B).$$

We can check that this makes sense: if $A, B, C$ are $k \times l$, $l \times m$, and $m \times n$ matrices respectively, then $V(ABC)$ is vector of length $kn$, $C^T \otimes A$ is a $kn \times lm$ matrix, and $V(B)$ is a vector of length $lm$. Letting $\pi(A) = 1 \otimes A - A^T \otimes 1$, this means we have

$$V([A, B]) = \pi(A)V(B).$$

We can thus write out Nahm's equations as a matrix equation, and introducing the notation for vertically concatenated triples of vectors that with $a$ being a lower index, $b = a+1 \mod 3$ and $c = a+2 \mod 3$, they become

$$\left(V(T'_a)\right) = \left(\pi(T_b)V(T_c)\right).$$

The vectors are of length $3k^2$ and the matrix has dimensions $3k^2 \times 3k^2$. As mentioned, I will want to consider the case where the vectors are in the span of $V(\rho_a)$, $V(S_a^{(j)})$. Suppose we have a solution to Nahm's equations given by

$$T_i(s) = x(s)\rho_i + y_j(s)S_i^{(j)}, \quad \text{(summing over } j\text{)},$$

then

$$\begin{aligned}
\left(V(T'_a)\right) &= \left(V(\rho_a), \quad V(S_a^{(j)})\right)\begin{pmatrix} x' \\ y'_j \end{pmatrix} = M\begin{pmatrix} x' \\ y'_j \end{pmatrix}, \\
\left(\pi(T_b)V(T_c)\right) &= \left(\left[x\pi(\rho_b) + y_l\pi(S_b^{(l)})\right]\left[xV(\rho_c) + y_mV(S_c^{(m)})\right]\right), \\
&= x^2\left(\pi(\rho_b)V(\rho_c)\right) + xy_l\left(\pi(\rho_b)V(S_c^{(l)}) + \pi(S_b^{(l)})V(\rho_c)\right) \\
&\quad + y_ly_m\left(\pi(S_b^{(l)})V(S_c^{(m)})\right).
\end{aligned}$$

where I have let $M$ be the $3k^2\times(d+1)$ matrix $\left(V(\rho_a), \quad V(S_a^{(j)})\right)$ and we presently sum over all $l, m = 1, \dots, d$. Note we can rewrite

$$\begin{aligned}
\sum_{l,m} y_ly_m\left(\pi(S_b^{(l)})V(S_c^{(m)})\right) &= \sum_l y_l^2\left(\pi(S_b^{(l)})V(S_c^{(l)})\right) \\
&\quad + \sum_{l<m} y_ly_m\left(\pi(S_b^{(l)})V(S_c^{(m)}) + \pi(S_b^{(m)})V(S_c^{(l)})\right), \\
&= \sum_{l\le m}\frac{1}{1+1_{l=m}}y_ly_m\left(\pi(S_b^{(l)})V(S_c^{(m)}) + \pi(S_b^{(m)})V(S_c^{(l)})\right),
\end{aligned}$$

Note here $1_{l=m}$ is the indicator function[20]. As such, introducing the vectors

$$\begin{aligned}
\boldsymbol{X}_{\alpha,\beta}^{(l)} &= \left(\pi(\rho_b)V(S_c^{(l)}) + \pi(S_b^{(l)})V(\rho_c)\right), \\
\hat{\boldsymbol{X}}_{\gamma,\delta}^{(l,m)} &= \frac{1}{1+1_{l=m}}\left(\pi(S_b^{(l)})V(S_c^{(m)}) + \pi(S_b^{(m)})V(S_c^{(l)})\right),
\end{aligned}$$

and using the known commutation relations of the $\rho_i$ we have

$$M\begin{pmatrix} x' \\ y'_j \end{pmatrix} = 2x^2\left[M\begin{pmatrix} 1 \\ \boldsymbol{0} \end{pmatrix}\right] + xy_l\boldsymbol{X}_{\alpha,\beta}^{(l)} + y_ly_m\hat{\boldsymbol{X}}_{\gamma,\delta}^{(l,m)}, \tag{3.29}$$

restricting the double sum to be over $l \le m$. The existence of a solution to Nahm's equations in $\left\langle(\rho_i), (S_i^{(j)})\right\rangle$ is then a question of consistency of some linear

[20]Here the notation used is such that the indicator function $1_A$ takes the value 1 if the statement $A$ is true, and 0 otherwise.

inhomogeneous equations (and analytic questions about existence of solutions to inhomogeneous ODEs which I shall not discuss). Certainly if we can find constants $\alpha^{(j)}, \beta^{(j,k)}$, $j,k = 1,\ldots,d$ which solve the $d$ equations

$$M \begin{pmatrix} \alpha^{(j)} \\ \beta^{(j,1)} \\ \vdots \\ \beta^{(j,d)} \end{pmatrix} = \boldsymbol{X}^{(j)}_{\alpha,\beta} \tag{3.30}$$

and constants $\gamma^{(j,k)}, \delta^{(j,k,l)}$, $j,k,l = 1,\ldots,d$, $j \le k$ which solve the $\frac{1}{2}d(d+1)$ equations

$$M \begin{pmatrix} \gamma^{(j,k)} \\ \delta^{(j,k,1)} \\ \vdots \\ \delta^{(j,k,d)} \end{pmatrix} = \frac{1}{1+1_{j=k}} \boldsymbol{X}^{(j,k)}_{\gamma,\delta} = \hat{\boldsymbol{X}}^{(j,k)}_{\gamma,\delta} \tag{3.31}$$

then the full commutation relations have a solution. Equations 3.30 and 3.31 correspond to the equations defining $\alpha,\beta,\gamma,\delta$ in [HMM95, p. 679]. The total number of constants is $d(1+d)+\frac{1}{2}d(d+1)(1+d) = \frac{1}{2}d(d+1)(d+3)$, and Nahm's equations then become

$$\begin{aligned} x' &= 2x^2 + \alpha^{(k)} x y_k + \gamma^{(k,l)} y_k y_l, \\ y_j' &= \beta^{(k,j)} x y_k + \delta^{(k,l,j)} y_k y_l. \end{aligned}$$

Conversely, provided that $\operatorname{rank}(M) = d+1$ we could perform Gaussian elimination to put $M$ in echelon form, and then we would get consistent ODEs on a submanifold specified by the linear constraints in $xy_l$ and $y_l y_m$. The rank condition also ensures that these ODEs would be unique. If the rank was less than $d+1$, $M$ would have a right kernel, and the number of degrees of freedom if a solution exists would be $\frac{1}{2}d(d+3)\dim \operatorname{Ker}_{\text{right}} M$.

**Example 3.3.22.** *Suppose that one generated the set of $S_i^{(j)}$ by applying the algorithm described in §3.3.2 to all of the HMM representation space $\mathbb{HMM}$, then*

$$d = \underbrace{(2n+1)}_{\mathbb{S}^{2n}_{-1}} + \underbrace{2(2n-1)}_{\mathbb{S}^{2n-2}_{-1,0}} + \sum_{k=2}^{n-1} \underbrace{3(2n-2k+1)}_{\mathbb{S}^{2n-2k}_{-1,0,1}} = 3n^2 - 1,$$

*and so $M$ is square and there are no possible constraints if $M$ is full rank.*

**Remark 3.3.23.** *This linearisation approach is different to that considered in [HMM95], where the procedure is presented as consistent commutation relations, and no process for finding the constants $\alpha,\beta,\gamma,\delta$ is presented. By describing the process in terms of vectorisation, it makes it clear how to implement the process of computing the constants.*

The main usage for this method of creating Nahm matrices from a basis is the construction of monopoles invariant under $G \le \mathrm{SO}(3)$, as in [HMM95, HS96d, HS96c, HS96a, HS96b, Sut97b]. In these cases, a basis of polynomials in $(\mathbb{S}^{2r})^G$, $r = 1, \dots, k$ are chosen as the input to the algorithm of §3.3.2, which provides a basis of $(\mathbb{R}^3 \otimes \mathfrak{su}(k))^G$. From this basis Nahm matrices are constructed, and due to the symmetry enforced by $G$ solving the resulting equations in the $x, y_j$ are typically easier. We will see some examples of this later in §3.4.

It is helpful conceptually to notice that, if we construct Nahm matrices from polynomials $\left\{ Q_r^{(j)} \,|\, r = 1, \dots, k,\, j = 1, \dots, n_r \right\}$ such that $\left\{ Q_r^{(j)} \,|\, j = 1, \dots, n_r \right\}$ forms a basis of $(\mathbb{S}^{2r})^G$ for some fixed $G \leq \mathrm{SO}(3)$, then the corresponding spectral curve will be of the form $\eta^k + \sum_{r=1}^k \eta^{k-r} \sum_{j=1}^{n_r} c_{r,j} Q_r^{(j)}$ for some $c_{r,j} \in \mathbb{C}$. This follows simply from symmetry considerations; if the Nahm matrices are invariant under $G$, the spectral curve will be invariant under $G$, and so will be built from the corresponding basis of invariant polynomials. In general, if the Nahm matrices are given by $T_i(s) = x(s)\rho_i + y_j(s)S_i^{(j)}$ then the coefficient of the polynomial $Q^{(j)}$ in the spectral curve will not just be $y_j$ because the determinant is a nonlinear construct, and we see examples of this in §3.4. In the case of the leading order $\eta^{k-1}$, the behaviour is actually very regular, as described by the next result.

**Proposition 3.3.24.** *Take the Nahm matrices $T_i(s) = x(s)\rho_i + y_j(s)S_i^{(j)}$, where the triples $(S_i^{(j)})$ are constructed from some polynomials $Q^{(j)}$. Without loss of generality assume that $Q^{(1)} = \zeta_0^2$, $Q^{(2)} = \zeta_0\zeta_1$, and $Q^{(3)} = \zeta_1^2$. The coefficient of $\eta^{k-1}$ in the spectral curve is then $2k(y_3 + y_2\zeta + y_1\zeta^2)$.*

*Proof.* Taking the spectral curve to be given by $P(\zeta, \eta) = \det[\eta + L(\zeta)]$ where

$$L = (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2,$$

we get that the coefficient of $\eta^{k-1}$ in $P$ is $\mathrm{Tr}(L)$, which we shall now rewrite. As in Remark 3.3.13, we will take the $\rho_i$ to be real satisfying $\rho_i^\dagger = -\rho_i$. This means they are traceless, and we have

$$\begin{aligned} \mathrm{Tr}(L) &= y_j \left[ (S_1^{(j)} + iS_2^{(j)}) - 2iS_3^{(j)}\zeta + (S_1^{(j)} - iS_2^{(j)})\zeta^2 \right], \\ &= y_j \left[ S_1^{\prime,(j)} + 2S_2^{\prime,(j)}\zeta + 2S_3^{\prime,(j)}\zeta^2 \right]. \end{aligned}$$

Now the $S_i^{\prime,(j)}$ will be given by $\tilde{Q}(\mathrm{ad}_{Y_0})(X_0^{r+i})$ for some univariate polynomial $Q$ when taking the triple of matrices corresponding to $Q^{(j)} \in \mathbb{S}_i^{2r}$, and this can only have nonzero trace when $r + i = 0$, that is $r = 1$ and $i = -1$. Hence the only contribution comes from applying the upper-invariant algorithm to $Q^{(j)}$, $j = 1, 2, 3$. We calculated the result of applying the algorithm to these input polynomials in Example 3.3.20, and then taking the trace gives the desired result. □

### 3.3.3 $C_k$-invariant Nahm Matrices

I will now briefly discuss one particular class of symmetric monopoles, first studied from the Nahm perspective in [Sut96b] for their connection to Seiberg-Witten theory, and later in [Sut97b, Bra11]. This will serve as a springboard for a later construction in §3.4.1.

The specific symmetry considered is that of a rotation about an axis by $2\pi/k$ radians. By an overall rotation we may choose the axis of rotation to be $(0,0,1)$, so the SO(3) element generating the symmetry is $\begin{pmatrix} \cos(2\pi/k) & -\sin(2\pi/k) & 0 \\ \sin(2\pi/k) & \cos(2\pi/k) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with corresponding transform on $\zeta, \eta$ given by $s = s_k : (\zeta, \eta) \mapsto (e^{2\pi i/k}\zeta, e^{2\pi i/k}\eta)$ as per Example 3.3.2. The generic form of a spectral curve invariant under $C_k = \langle s \rangle$ (with reality imposed) is given by

$$\mathcal{C} : P(\zeta,\eta) := \eta^k + \alpha_2\eta^{k-2}\zeta^2 + \alpha_3\eta^{k-3}\zeta^3 + \ldots + \alpha_{k-1}\eta\,\zeta^{k-1} + \alpha_k\zeta^k + \beta\left[\zeta^{2k} + (-1)^k\right] = 0, \tag{3.32}$$

where $\alpha_i, \beta \in \mathbb{R}$. The work of [Bra11] shows $\mathcal{C}$ given in Equation 3.32 is the unbranched cover of the genus-$(k-1)$ hyperelliptic curve

$$\mathcal{C}' : y^2 = (x^k + \alpha_2 x^{k-2} + \alpha_3 x^{k-3} + \ldots + \alpha_k)^2 - (-1)^k 4\beta^2, \tag{3.33}$$

where $x = \eta/\zeta$, $y = \beta[\zeta^k - (-1)^k\zeta^{-k}]$, and the covering map is given by the quotient $\pi : \mathcal{C} \to \mathcal{C}' := \mathcal{C}/C_k$. The curve determined by Equation 3.32 also has the symmetry $t : (\zeta,\eta) \mapsto (-1/\zeta, -\eta/\zeta^2)$ and $G = \langle s, t\rangle \cong D_k$ is the full automorphism group. The transformation $t$ corresponds up to an action of $\tau$ to a reflection $\mathrm{diag}(1,-1,1) \in \mathrm{O}(3)$ (see Example 3.3.4) and it becomes the hyperelliptic involution $t : (x,y) \to (x,-y)$ on the quotient curve. The relation of the linear flow in $\mathrm{Jac}(\mathcal{C})$ to a flow in the Jacobian of the quotient curve was made precise in the following theorem.

**Theorem 3.3.25** ([Bra11], Theorem 4.2)**.** *Recall the definition of the winding vector $\boldsymbol{U} \in \mathrm{Jac}(\mathcal{C})$ from Definition 3.2.21 and Proposition 3.2.24, which gives the direction of the flow in the Jacobian. The winding vector is invariant under the action of $C_k$, that is*

$$\boldsymbol{U} = \pi^*\boldsymbol{U}'$$

*for some $\boldsymbol{U}' \in \mathrm{Jac}(\mathcal{C}')$. Equivalently*

$$\frac{\zeta^{k-2-s}\eta^s d\zeta}{\partial_\eta P} = \pi^*\left(\frac{-1}{k}\frac{x^s dx}{y}\right),$$

*that is the $C_k$-invariant differentials on $\mathcal{C}$ reduce to hyperelliptic differentials.*

**Remark 3.3.26.** *The equivalence of the conditions in Theorem 3.3.25 may not be immediately clear but follows from the proof of Proposition 3.2.24.*

**Remark 3.3.27.** *[Bra11, Lemma 4.1, Theorem 4,2] actually show together that for any $G \leq \mathrm{Aut}(\mathcal{C})$ generated by rotations with quotient $\pi : \mathcal{C} \to \mathcal{C}' := \mathcal{C}/G$ we will have $\boldsymbol{U} = \pi^*\boldsymbol{U}'$ for some $\boldsymbol{U}' \in \mathrm{Jac}(\mathcal{C}')$. We discussed this earlier in Remark 3.2.31. Note that as part of the proof it is shown that there is a particular differential on a monopole spectral curve that is invariant under any rotation, and so the quotient genus is necessarily always positive.*

In [Bra11], following an ansatz of [Sut96b], Braden shows that Nahm's equations for charge-$k$ monopoles with $C_k$ rotational symmetry are equivalent to the

$A_{k-1}^{(1)}$ affine Toda equations given in real Flaschka variables by

$$a_i' = \frac{1}{2}a_i(b_i - b_{i+1}), \quad b_i' = a_i^2 - a_{i-1}^2, \tag{3.34}$$

where $i$ is taken mod $k$, and we use $'$ to denote $\frac{d}{ds}$ as with the Nahm matrices. This is done by putting the associated Nahm matrices in an explicit form from where the ODEs can be read off. We are able to do this explicitly with the Nahm matrices we get from the algorithm of §3.3.2 via the following steps.

1. Take the input polynomials to be $Q^{(i)} = \zeta_0^i \zeta_1^i, i = 1, \dots, k$ and $Q^{(k+1)} = \zeta_0^{2k} - \zeta_1^{2k}$, acted on by $C_k$ as $(\zeta_0, \zeta_1) \mapsto (e^{\pi i/k}\zeta_0, e^{-\pi i/k}\zeta_1)$.

2. Construct the invariant vectors $(\rho_i), (S_i^{(j)})$, and scale them so they are all anti-Hermitian, which can clearly be done by Example 3.3.21. We can calculate that the number of $S$ invariant vectors will be

$$d = 2 \times 1 + 1 \times 2 + (k-2) \times 3 = 3k - 2.$$

3. Calculate the (now anti-Hermitian) $T_i$. Note the variables $x, y_j$ are now always real-valued.

4. Diagonalise the matrix $T_3$ with a unitary matrix $U$ whose columns are the normalised eigenvectors of $T_3$, that is construct $U^{-1}T_3U$. As $T_3$ is anti-Hermitian and linear in the invariant vectors $\rho_3, S_3^{(j)}$, the diagonal entries which are the eigenvalues will be pure-imaginary and linear in $\{x, y_j\}$.

5. From [Bra11], conjugate by the same unitary matrix to give

$$U^{-1}(T_1 + iT_2)U = \sum_{j=1}^k a_j E_{j,j+1}, \quad U^{-1}(T_1 - iT_2)U = \sum_{j=1}^k -\bar{a}_j E_{j+1,j},$$

for some $a_1, \dots, a_k \in \mathbb{C}$, where $(E_{ij})_{ab} = \delta_{ai}\delta_{bj}$.

6. Writing $a_j = r_j e^{i\phi_j}$ (as generically $a_j \neq 0$) and solving

$$\begin{aligned} \phi_j + \theta_{j+1} - \theta_j = 0, \quad j = 1, \dots k-1, \\ \sum_j \theta_j = 0, \end{aligned} \tag{3.35}$$

for $\theta_1, \dots, \theta_k \in \mathbb{R}$, defines a unitary matrix $D = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$. Conjugating by $D$ preserves the form of $T_3$, but acts to make each $a_j$ real for $j = 1, \dots, k-1$ (as it multiplies $a_j$ by $e^{i(\theta_{j+1}-\theta_j)}$). Note that generically we could not have made all the $a_j$ real as Equations 3.35 would be overdetermined; the effect on $a_k$ is to multiply it by $e^{i(\theta_1 - \theta_k)} = e^{i(\phi_1 + \dots + \phi_{k-1})} = \prod_{j=1}^{k-1}(a_j/r_j)$. After quotienting by this action the number of independent variables we have is $3k - 2 + 1 - (k-1) = 2k$ (as we are using $k-1$ variables, but there is a redundancy in the solutions of the $\theta$ to pick arbitrary $\theta_1$ and get a solution). This is the correct count for Sutcliffe's ansatz.

7. Thinking now about the permutations contributing to the determinant
$$\det(\eta + L(\zeta)) = \det\left(\eta + \left[(T_1 + iT_2) - 2i\zeta T_3 + (T_1 - iT_2)\zeta^2\right]\right),$$
the only such permutations with no fixed points (that is, giving no $\eta$ contributions) are $\sigma = (1, 2, \ldots, k)$ and $\sigma^{-1}$. These contribute terms
$$(-1)^{k-1}\left[\prod_{j=1}^{k-1} r_j \times a_k \prod_{j=1}^{k-1}(a_j/r_j) + \prod_{j=1}^{k-1} -\zeta^2 r_j \times -\zeta^2 \overline{a_k \prod_{j=1}^{k-1}(a_j/r_j)}\right],$$
which can be written as
$$(-1)^{k-1}\left[\beta + (-1)^k \bar{\beta}\zeta^{2k}\right],$$
where $\beta = \prod_{j=1}^{k} a_j$. Now looking at the spectral curve, we have set $\beta$ to be real in the beginning, through a rotation viewed as acting on $\zeta, \eta$.

These variables are the Flaschka coordinates for the periodic Toda system.

**Example 3.3.28.** *It will be helpful to demonstrate how this process works in the case of $k = 2$, where the representation $\mathfrak{su}(2) \to \mathfrak{su}(k)$ is simply the identity. Here we want the three input polynomials $\zeta_0\zeta_1$, $\zeta_0^2\zeta_1^2$ and $\zeta_0^4 - \zeta_1^4$. Polarising these three we get*
$$2\zeta_0\zeta_1 \otimes 2, \quad \zeta_0^2 \otimes 2\zeta_1^2 + 2\zeta_0\zeta_1 \otimes 4\zeta_0\zeta_1 + \zeta_1^2 \otimes 2\zeta_0^2, \quad \zeta_0^2 \otimes 12\zeta_0^2 + \zeta_1^2 \otimes -12\zeta_1^2,$$
*which at level -1 correspond to triples of polynomials $(\tilde{Q}_{11}(x), \tilde{Q}_{01}(x), \tilde{Q}_{00}(x))$*
$$(0, 1, 0), \quad (x^2, 2x, 1), \quad (-12, 0, 3x^2).$$
*At this level there is no isomorphism to effect, so we simply get the triples $(S_i')$*
$$(0, \mathrm{Id}, 0), \quad (\mathrm{ad}_Y^2 X, 2\,\mathrm{ad}_Y X, X), \quad (-12X, 0, 3\,\mathrm{ad}_Y^2 X).$$
*Using $\mathrm{ad}_Y X = -H$, $\mathrm{ad}_Y^2 X = -2Y$, and converting to the triple $(S_i)$ we get*
$$(0, 0, i\,\mathrm{Id}), \quad (-Y + X, iY + iX, -2iH), \quad (-6X - 6Y, 6iX - 6iY, 0).$$
*At level 0 we also get a contribution from the first polynomial. We can see*
$$(\zeta_0\partial_1 \otimes 1 + 1 \otimes \zeta_0\partial_1)(\zeta_1^2 \otimes 1) = 2\zeta_0\zeta_1 \otimes 1,$$
$$(\zeta_0\partial_1 \otimes 1 + 1 \otimes \zeta_0\partial_1)(2\zeta_0\zeta_1 \otimes \zeta_1^2 - 2\zeta_1^2 \otimes \zeta_0\zeta_1) = 2\zeta_0^2 \otimes \zeta_1^2 - 2\zeta_1^2 \otimes \zeta_0^2,$$
*so at level 0 from the first polynomial we get the triple of univariate polynomials $(-x^2, 0, 1)$, corresponding to the triple of matrices $(S_i) = (Y + X, -iY + iX, 0)$. Using*
$$X = \frac{1}{2}(\rho_1 - i\rho_2), \quad Y = -\frac{1}{2}(\rho_1 + i\rho_2), \quad H = -i\rho_3,$$
*we recover that these triples of matrices are, up to a scale to set the vectors to*

*have anti-Hermitian matrices as factors,*

$$(0, 0, i\,\mathrm{Id}), \quad (\rho_1, \rho_2, -2\rho_3), \quad (\rho_2, \rho_1, 0), \quad (-\rho_2, \rho_1, 0)$$

*At charge-2, the $\rho_j$ correspond to $\sigma_j/i$ the scaled Pauli matrices (with some reindexing to account for a sign), but this gives Nahm matrices*

$$(T_i) = ((x+y_1)\rho_1 + (y_2 - y_3)\rho_2, (x+y_1)\rho_2 + (y_2+y_3)\rho_1, (x-2y_1)\rho_3 + iy_0\,\mathrm{Id}).$$

*Setting $y_0 = 0$ corresponds to centring so we can do this. It moreover turns out that one can set $y_3 = 0$ consistently. If one acts with a rotation matrix $A = \left(\begin{smallmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$, which corresponds to the $\mathrm{SU}(2)$ matrix $U = \mathrm{diag}(e^{i\theta/2}, e^{-i\theta/2})$, $T_3$ is fixed. One can check that*

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \Rightarrow \rho_1^U = \cos(\theta)\rho_1 + \sin(\theta)\rho_2, \quad \rho_2^U = \cos(\theta)\rho_2 - \sin(\theta)\rho_1,$$

*and so under the action of $A$ we have*

$$\begin{aligned} T_1 \mapsto \cos(\theta)\left[(x+y_1)\left(\cos(\theta)\rho_1 + \sin(\theta)\rho_2\right) + y_2\left(\cos(\theta)\rho_2 - \sin(\theta)\rho_1\right)\right] \\ - \sin(\theta)\left[(x+y_1)\left(\cos(\theta)\rho_2 - \sin(\theta)\rho_1\right) + y_2\left(\cos(\theta)\rho_1 + \sin(\theta)\rho_2\right)\right], \end{aligned}$$

*and likewise for $T_2$. Hence choosing $\theta = \pi/4$ gives the standard ansatz for the Nahm data of a charge-2 monopole of $T_j(s) = f_j(s)\rho_j$ for some real functions $f_j$ [BE21, §3.6].*

## 3.4 Classifying Charge-3 Monopoles

> The upshot: monopole spectral curves are interesting, encoding real as well as complex algebraic geometry and number theory but they are difficult to lay hands on. ... Victor's love of classical curves had him always looking for new candidates, and he would be delighted if progress could be made here.
>
> – Harry Braden
> Victor Enolski Memorial Tribute

Combining the ideas presented in §3.3.1, I will now present a partial classification of charge-3 monopoles based upon their possible symmetry, work completed in collaboration with Harry Braden which has been published [BDH23]. In particular, I will prove the following theorem.

**Theorem 3.4.1.** *Let $\mathcal{C} \subset T\mathbb{P}^1$ be a smooth charge-3 monopole spectral curve with $H \leq \mathrm{Aut}(\mathcal{C})$ such that the quotient genus $g(\mathcal{C}/H) = 1$. Then, up to an automorphism of $T\mathbb{P}^1$, the curve is given by the vanishing of one of the following 5 forms:*

1. $\eta^3+\eta[(a+ib)\zeta^4+c\zeta^2+(a-ib)]+[(d+ie)\zeta^6+(f+ig)\zeta^4-(f-ig)\zeta^2-(d-ie)]$,
2. $\eta^3+\eta[a(\zeta^4+1)+b\zeta^2]+ic\zeta(\zeta^4-1)$,
3. $\eta^3+a\eta\zeta^2+ib\zeta(\zeta^4-1)$,
4. $\eta^3+a\eta\zeta^2+b(\zeta^6-1)$,
5. $\eta^3+ia\zeta(\zeta^4-1)$,

*where* $a,b,c,d,e,f,g\in\mathbb{R}$.

The result does not itself guarantee the existence of monopole spectral curves in these families, and the classes intersect (for example, 5 is a special case of 3). In previous works monopole spectral curves of the form 3 and 5 have been understood in [HS96b] and [HMM95] as corresponding to charge-3 twisted line scattering and the tetrahedrally-symmetric monopole respectively, while one special case of the form 2 was understood in [HS96a] as the class of inversion-symmetric monopoles, with another in [Hit83] as the axially-symmetric 3-monopole (where the SO(2) symmetry of rotations is given by $(\zeta,\eta)\mapsto(e^{i\theta}\zeta,e^{i\theta}\eta)$). Curves of the form 2 had been observed in [Hou97, (3.71)], but the Hitchin constraints were only imposed for a restricted subset. I will deal with curves of the form 4 in §3.4.1 and curves of the form 2 in §3.4.2, providing explicit Nahm data for parametrisations of both cases. Along with the class of charge-3 monopoles described via an implicit condition in [BDE11], these form all the charge-3 monopole spectral curves currently known, which fit together as shown in Figure 3.2 for some parameter values. Figure 3.3 shows the relations between the symmetry groups of the curves, not specifying any constraints on the parameters.

$$\begin{tikzcd}
 & \eta^3+\alpha_2\eta\zeta^2+\alpha_3\zeta^3+\beta(\zeta^6-1),\text{[BDE11]} \arrow[dl, "\alpha_3=0"'] \arrow[d, "\alpha_2=0\text{ and rotation}"] \\
\eta^3+\alpha_2\eta\zeta^2+\beta(\zeta^6-1),\text{here} \arrow[d, "\beta=0"'] & \eta^3+c\zeta(\zeta^4-1),\text{[HMM95]} \\
\eta[\eta^2+\pi^2\zeta^2],\text{[Hit83]} & \eta^3+b\eta\zeta^2+c\zeta(\zeta^4-1),\text{[HS96b]} \arrow[l, "c=0"'] \arrow[u, "b=0"'] \\
\eta\left[\eta^2+a(\zeta^4+1)+b\zeta^2\right],\text{[HS96a]} \arrow[u, "a=0"] & \eta^3+\eta[a(\zeta^4+1)+b\zeta^2]+c\zeta(\zeta^4-1),\text{here.} \arrow[l, "c=0"'] \arrow[u, "a=0"]
\end{tikzcd}$$

**Fig. 3.2** Known charge-3 spectral curves and their relations

$$\begin{tikzcd}
 & S_3 \arrow[dl, "\alpha_3=0"'] \arrow[d, "\alpha_2=0 \text{ and rotation}"] \\
S_3, C_6 \leq D_6 \arrow[d, "\beta=0"'] & A_4 \leq C_3 \times S_4 \\
C_2 \times \mathrm{SO}(2) & C_4 \leq D_4 \arrow[l, "c=0"] \arrow[u, "b=0"'] \\
V_4 \arrow[u, "a=0"] & V_4 \leq V_4. \arrow[l, "c=0"] \arrow[u, "a=0"]
\end{tikzcd}$$

**Fig. 3.3** Automorphism groups of known charge-3 spectral curves and their relations, presented as $G$ or $H \leq G$ where $G$ is the full automorphism group and $H$ is the subgroup quotienting to an elliptic curve when it exists

The starting point for this classification is the identification made earlier in Remark 3.2.10 and Proposition 2.1.63 that all smooth charge-3 monopole spectral curves are non-hyperelliptic genus-4 curves whose canonical embedding in $\mathbb{P}^3$ lies in the quadric cone. One can also reach this conclusion going via del Pezzo surfaces of degree 1 [CKRSN19]. Moreover, we saw in §2.2.2 that in 1895 Wiman [Wim95b] classified all non-hyperelliptic genus-4 curves by their automorphism group and gave explicit defining equations for these. Wiman's classification had two families: curves arose either as the intersections of a cubic surface and nonsingular quadric in $\mathbb{P}^3$, or as the intersection of a cubic surface and quadric cone in $\mathbb{P}^3$. Thus charge-3 monopole spectral curves with automorphism group must lie in Wiman's second family. (The two rulings of the nonsingular quadric of Wiman's first family lead to projections from the curve to $\mathbb{P}^1 \times \mathbb{P}^1$, which is relevant for spectral curves of hyperbolic monopoles; this will be developed elsewhere.) In Table 3.1 we give those curves in Wiman's classification which lie on a cone presenting[21] these in terms of a curve given by the vanishing of a polynomial $P(x, z)$. We also write down their full automorphism group $G := \mathrm{Aut}(\mathcal{C})$ and the corresponding signature $c := c_G = (g_0; c_1, \ldots, c_r)$ giving the quotient genus $g_0 = g(\mathcal{C}/G)$ and the ramification indices $c_i$ of the quotient map $\mathcal{C} \to \mathcal{C}/G$ (as in §2.1.4). These have been calculated with the help of the information available from [LMF23]. We make some remarks about Table 3.1.

- These curves have been noted previously in Table 2.3 during the discussion of theta characteristics.
- The polynomial $P(x, z)$ given is a single representative of the orbit of the curve under the action of the whole automorphism group of $T\mathbb{P}^1$, not just the subgroup which preserves invariance under $\tau$.
- As in §2.2.2 the label $D_n$ refers to the dihedral group of order $2n$.

[21] We set $y = 1$ in Wiman's notation so as to make clear the connection to monopole spectral curves.

- Wiman's parameters are to be understood as generic: there may exist specific values of the parameters for which the automorphism group is larger than that indicated.

- Wiman provides a form where the $z^2$ term is always zero, equivalent to centring the monopole.

- All the curves given are irreducible, so we can only find reducible spectral curves as limiting members of the families provided.

- We recognise the curve with $C_3 \times S_4$ symmetry as corresponding to the tetrahedrally-symmetric monopole.

Table 3.1: Potential charge-3 monopole spectral curves with nontrivial automorphism group and those (with subgroups) quotienting to genus 1

| $P$ | $G$ | $c_G$ | $H$ | $c_H$ |
|---|---|---|---|---|
| $z^3 + z(ax^4 + bx^2 + c) + (dx^6 + ex^4 + fx^2 + g)$ | $C_2$ | $(1; 2^6)$ | $C_2$ | $(1; 2^6)$ |
| $z^3 + z(ax^4 + bx^2 + c) + dx(x^4 + ex^2 + f)$ | $C_2$ | $(2; 2^2)$ | | |
| $z^3 + z[a(x^4 + 1) + bx^2] + x[c(x^4 + 1) + dx^2]$ | $V_4$ | $(0; 2^7)$ | | |
| $z^3 + z[a(x^4 + 1) + bx^2] + x(x^4 - 1)$ | $V_4$ | $(1; 2^3)$ | $V_4$ | $(1; 2^3)$ |
| $z^3 + azx^2 + x(x^4 + 1)$ | $D_4$ | $(0; 2^4, 4)$ | $C_4$ | $(1; 4^2)$ |
| $z^3 + z(x^4 + a) + (bx^4 + c)$ | $C_4$ | $(0; 2, 4^4)$ | | |
| $z^3 + azx^2 + x^6 + bx^3 + 1$ | $S_3$ | $(0; 2^6)$ | | |
| $z^3 + azx^2 + x^6 + 1$ | $D_6$ | $(0; 2^5)$ | $S_3$, $C_6$ | $(1; 2^2)$ |
| $z^3 + z(ax^3 + b) + (x^6 + cx^3 + d)$ | $C_3$ | $(1; 3^3)$ | $C_3$ | $(1; 3^3)$ |
| $z^3 + az(x^3 + 1) + (x^6 + 20x^3 - 8)$ | $A_4$ | $(0; 2, 3^3)$ | | |
| $z^3 + az + x^6 + b$ | $C_6$ | $(0; 2, 6^3)$ | | |
| $z^3 + z + x^6$ | $C_{12}$ | $(0; 4, 6, 12)$ | | |
| $z^3 + az + x^5 + b$ | $C_5$ | $(0; 5^4)$ | | |
| $z^3 + z + x^5$ | $C_{10}$ | $(0; 5, 10^2)$ | | |
| $z^3 - (x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f)$ | $C_3$ | $(0; 3^6)$ | | |
| $z^3 - (x^6 + ax^4 + bx^2 + 1)$ | $C_6$ | $(0; 2^2, 3^3)$ | | |
| $z^3 - x(x^4 + ax^2 + 1)$ | $C_6 \times C_2$ | $(0; 2^2, 3, 6)$ | | |
| $z^3 - (x^6 + ax^3 + 1)$ | $C_3 \times S_3$ | $(0; 2^2, 3^2)$ | | |
| $z^3 - (x^5 + 1)$ | $C_{15}$ | $(0; 3, 5, 15)$ | | |
| $z^3 - (x^6 + 1)$ | $C_6 \times S_3$ | $(0; 2, 6^2)$ | | |
| $z^3 - x(x^4 + 1)$ | $C_3 \times S_4$ | $(0; 2, 3, 12)$ | $A_4$ | $(1; 2)$ |

Not all curves on the list will yield monopoles spectral curves, for example by the following result.

**Proposition 3.4.2** ([BE10a]). *There are only two curves in the family $\eta^3 + \chi(\zeta^6 + b\zeta^3 + 1) = 0$, $\chi, b \in \mathbb{R}$, that correspond to BPS monopoles; these are tetrahedrally-symmetric monopole spectral curves.*

### Genus-1 Reductions

Table 3.1 gives us a list of putative spectral curves with symmetry before we have imposed the further constraints of Hitchin. We saw in §3.2.2 that Nahm's equations correspond to a linear flow in the Jacobian of the corresponding spectral curve $\mathcal{C}$; the direction of this linear flow given by the winding vector $\boldsymbol{U}$. Braden [Bra11] has shown that when we have a symmetry group $G$ we may be able to reduce to the quotient curve $\mathcal{C} \xrightarrow{\pi} \mathcal{C}' := \mathcal{C}/G$ and reduced winding vector $\boldsymbol{U}'$ when $\boldsymbol{U} = \pi^*\boldsymbol{U}'$. For example charge-$k$ monopoles with $C_k$ symmetry reduce to questions about a genus-$(k-1)$ hyperelliptic curve, as seen in §3.3.3. The $k = 3$ case was studied in [BDE11]. The list of Table 3.1 is too general for current methods to make progress and hence we require a further criterion to reduce this. Here we adopt the following: does the genus-4 spectral curve (assumed with real structure) quotient (either by $\mathrm{Aut}(\mathcal{C})$ or a subgroup) to an elliptic curve? The rationale for this is that the remaining of Hitchin's conditions are most straightforwardly answered for elliptic curves; equivalently the Ercolani-Sinha constraint becomes one on the real period of an elliptic curve. There are also a number of curves known with this property [HMM95, HS96d, HS96a, HS96b], and the usefulness of this property has been identified previously [Hou97, Sut97a].

Thus we seek curves $\mathcal{C}$ with real structure from Wiman's list for which there exists $H \leq \mathrm{Aut}(\mathcal{C})$ such that $g(\mathcal{C}/H) = 1$. Here we may use the database of [LMF23] which has enumerated all the possible $H$ and the corresponding signatures for genus-4 curves. We may then use our knowledge of the explicit forms of the curves to match up these cases, which leaves us with the reduced list in the final two columns of Table 3.1. As previously noted, the $H = A_4$ case corresponds to the tetrahedrally-symmetric monopole [HMM95], and the $H = C_4$ case has already been solved in [HS96b]. We also see that the cases $H = S_3$ and $H = C_6$ arise from the same curve, indicating that the curve has two distinct quotients to an elliptic curve.

In the following sections we will investigate in more detail the two new cases $H = C_6$ (or equivalently $H = S_3$) with full automorphism group $G = D_6$, and $H = V_4$ (with full automorphism group $G = V_4$). In these cases we will see that the ES cycle on the spectral curve is invariant under the action of $H$ and so corresponds to a cycle on the quotient curve, hence we may impose the Hitchin constraints on the quotient elliptic curve. This will not remain true for the curve with $C_2$ symmetry, and so though I shall make a few comments I will not construct fully its Nahm data. I will begin with the $D_6$ case which is both illustrative and simpler, though ultimately the new solutions and their scattering family are less interesting.

Before turning to these however we may complete the proof of Theorem 3.4.1. With the exception of the $H = C_3$ curve, imposing reality on the curves with groups $H$ listed in Table 3.1 yields the curves of Theorem 3.4.1 (and in the same order). Note not all $M \in \mathrm{GL}_2(\mathbb{C})/\langle -1 \rangle \leq \mathrm{Aut}(\mathbb{P}(1:1:2))$ will commute with the action of $\tau$, but $S \in \mathrm{SU}(2)$ will. One may use Schur decomposition to write $M = STS^{-1}$ for some $S \in \mathrm{SU}(2)$, $T$ an upper-triangular matrix, and so when imposing reality on Wiman's normal forms one should consider the orbits under upper-triangular matrices. The only real forms present in the orbit of the

$G = H = C_3$ family have $a = b = 0$ in the corresponding defining equation $P$; the resulting curve then lies in the family described by Proposition 3.4.2. Only the tetrahedrally-symmetric monopole within this family quotients to an elliptic curve and by a rotation this may be written as $\eta^3 + ia\zeta(\zeta^4 - 1) = 0$, the final entry of the theorem. We have thus established Theorem 3.4.1.

### 3.4.1 $D_6$

In this section I will now construct explicitly Nahm data for all $D_6$-symmetric 3-monopoles. This will involve 3 steps:

1. construct (anti-Hermitian) matrices with the right symmetry to reduce Nahm's matrix ODEs to a system of ODEs in (real) functions,
2. solve the ODEs in terms of elliptic functions, and
3. impose the reality and boundary conditions of §3.2.2 to make the solutions of Nahm's equations into Nahm data

We shall complete these in order.

#### Constructing Symmetric Nahm Matrices

In order to construct Nahm data for the $D_6$ monopoles, it is helpful to recall the $C_k$-invariant monopole spectral curves described in §3.3.3. Taking $k = 3$ yields the curve in Table 3.1 with full automorphism group $G = S_3$ ($= D_3$ in [LMF23] notation). We saw in §3.3.2 a general discussion of how to use the algorithms to construct Nahm matrices, and we put these into practice now yielding Nahm matrices

$$T_1 = \begin{pmatrix} 0 & iy_0 & iy_1 + y_5 + iy_6 \\ iy_0 & 0 & -2x - y_2 + iy_3 \\ iy_1 - y_5 + iy_6 & 2x + y_2 + iy_3 & 0 \end{pmatrix},$$
$$T_2 = \begin{pmatrix} iy_0 & 0 & 2x + y_2 - iy_3 \\ 0 & -iy_0 & iy_1 + y_5 + iy_6 \\ -2x - y_2 - iy_3 & iy_1 - y_5 + iy_6 & 0 \end{pmatrix},$$
$$T_3 = \begin{pmatrix} iy_1 + iy_4 - \frac{2}{3}iy_6 & -2x + 2y_2 & 0 \\ 2x - 2y_2 & iy_1 + iy_4 - \frac{2}{3}iy_6 & 0 \\ 0 & 0 & -2iy_1 + iy_4 + \frac{4}{3}iy_6 \end{pmatrix}.$$

with accompanying ODEs in 8 real variables

$$
\begin{aligned}
x' &= 2x^2 - \frac{1}{3}y_0^2 - \frac{5}{6}y_1^2 - \frac{1}{2}y_2^2 + \frac{1}{6}y_3^2 + \frac{1}{6}y_5^2 + \frac{5}{6}y_6^2,\\
y_0' &= -4xy_0 + 4y_0y_2,\\
y_1' &= -4xy_1 - \frac{16}{5}y_1y_2 - \frac{6}{5}y_3y_5 - \frac{6}{5}y_2y_6,\\
y_2' &= \frac{2}{3}y_0^2 - \frac{4}{3}y_1^2 - 2xy_2 - y_2^2 - \frac{1}{3}y_3^2 - \frac{1}{3}y_5^2 - y_1y_6 + \frac{1}{3}y_6^2,\\
y_3' &= 2xy_3 - 2y_2y_3 - 3y_1y_5 + 2y_5y_6,\\
y_4' &= 0,\\
y_5' &= -3y_1y_3 + 2xy_5 - 2y_2y_5 + 2y_3y_6,\\
y_6' &= -\frac{9}{5}y_1y_2 + \frac{6}{5}y_3y_5 + 6xy_6 + \frac{6}{5}y_2y_6.
\end{aligned}
$$

The associated spectral curve is

$$
\eta^3 + \alpha_1\eta^2\zeta + \alpha_2\eta\zeta^2 + \alpha_3\zeta^3 + \beta\zeta^6 - \bar{\beta} = 0,
$$

where

$$
\begin{aligned}
\alpha_1 &= -6y_4,\\
\alpha_2 &= 4y_0^2 - 8y_1^2 + 48xy_2 - 12y_2^2 + 4y_3^2 + 12y_4^2 + 4y_5^2 + 24y_1y_6 - \frac{4}{3}y_6^2,\\
\alpha_3 &= -160x^2y_1 + 16y_0^2y_1 + 8y_1^3 + 128xy_1y_2 - 40y_1y_2^2 - 8y_1y_3^2 - 8y_0^2y_4\\
&\quad + 16y_1^2y_4 - 96xy_2y_4 + 24y_2^2y_4 - 8y_3^2y_4 - 8y_4^3 - 32xy_3y_5 + 32y_2y_3y_5\\
&\quad - 8y_1y_5^2 - 8y_4y_5^2 - \frac{32}{3}y_0^2y_6 - \frac{128}{3}y_1^2y_6 - 32xy_2y_6 + 80y_2^2y_6 + \frac{16}{3}y_3^2y_6\\
&\quad - 48y_1y_4y_6 + \frac{16}{3}y_5^2y_6 + 24y_1y_6^2 + \frac{8}{3}y_4y_6^2 + \frac{16}{27}y_6^3,\\
\beta &= -16x^2y_0 + 4y_0y_1^2 - 16xy_0y_2 - 4y_0y_2^2 + 8jy_0y_1y_3 - 4y_0y_3^2 - 16ixy_0y_5\\
&\quad - 8iy_0y_2y_5 + 4y_0y_5^2 + 8y_0y_1y_6 + 8iy_0y_3y_6 + 4y_0y_6^2.
\end{aligned}
$$

In order to make the variables real we have imposed the anti-Hermiticity condition required of the Nahm matrices at the beginning, by making the invariant vectors corresponding to each variable anti-Hermitian.

We may consistently set $y_3 = 0 = y_5$, which we may view as using the conjugation action of diagonal matrices $\operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$, $\theta_1 + \theta_2 + \theta_3 = 0$. This leaves us with the $2 \times 3 = 6$ real variables we would expect to have from the corresponding Toda. Note that because $\alpha_1' = 0$, the centre of mass of the Toda system is already fixed. Moreover, we may centre to consistently set $y_4 = 0$, and so we now have the equations in the remaining five variables as

$$
\begin{aligned}
x' &= 2x^2 - \frac{1}{3}y_0^2 - \frac{5}{6}y_1^2 - \frac{1}{2}y_2^2 + \frac{5}{6}y_6^2,\\
y_0' &= -4xy_0 + 4y_0y_2,\\
y_1' &= -4xy_1 - \frac{16}{5}y_1y_2 - \frac{6}{5}y_2y_6,\\
y_2' &= \frac{2}{3}y_0^2 - \frac{4}{3}y_1^2 - 2xy_2 - y_2^2 - y_1y_6 + \frac{1}{3}y_6^2,\\
y_6' &= -\frac{9}{5}y_1y_2 + 6xy_6 + \frac{6}{5}y_2y_6,
\end{aligned}
$$

with conserved quantities

$$
\begin{aligned}
\alpha_2 &= 4y_0^2 - 8y_1^2 + 48xy_2 - 12y_2^2 + 24y_1y_6 - \frac{4}{3}y_6^2,\\
\alpha_3 &= -160x^2y_1 + 16y_0^2y_1 + 8y_1^3 + 128xy_1y_2 - 40y_1y_2^2\\
&\quad - \frac{32}{3}y_0^2y_6 - \frac{128}{3}y_1^2y_6 - 32xy_2y_6 + 80y_2^2y_6\\
&\quad + 24y_1y_6^2 + \frac{16}{27}y_6^3,\\
\beta &= -16x^2y_0 + 4y_0y_1^2 - 16xy_0y_2 - 4y_0y_2^2\\
&\quad + 8y_0y_1y_6 + 4y_0y_6^2.
\end{aligned}
$$

At this stage the resulting ODEs are somewhat opaque and we may use the connection to Toda to clarify, putting the Nahm Lax pair in Toda form

$$
\begin{aligned}
T_1 + iT_2 &= \begin{pmatrix} 0 & -2\sqrt{2}x - \sqrt{2}y_1 - \sqrt{2}y_2 - \sqrt{2}y_6 & 0\\ 0 & 0 & 2\sqrt{2}x - \sqrt{2}y_1 + \sqrt{2}y_2 - \sqrt{2}y_6\\ 2y_0 & 0 & 0 \end{pmatrix},\\
T_1 - iT_2 &= \begin{pmatrix} 0 & 0 & -2y_0\\ 2\sqrt{2}x + \sqrt{2}y_1 + \sqrt{2}y_2 + \sqrt{2}y_6 & 0 & 0\\ 0 & -2\sqrt{2}x + \sqrt{2}y_1 - \sqrt{2}y_2 + \sqrt{2}y_6 & 0 \end{pmatrix},\\
-2iT_3 &= \begin{pmatrix} -4x + 2y_1 + 4y_2 - \frac{4}{3}y_6 & 0 & 0\\ 0 & -4y_1 + \frac{8}{3}y_6 & 0\\ 0 & 0 & 4x + 2y_1 - 4y_2 - \frac{4}{3}y_6 \end{pmatrix}.
\end{aligned}
\tag{3.36}
$$

This gives us variables

$$
a_0 = 2y_0, \quad a_1 = -2\sqrt{2}x - \sqrt{2}y_1 - \sqrt{2}y_2 - \sqrt{2}y_6, \quad a_2 = 2\sqrt{2}x - \sqrt{2}y_1 + \sqrt{2}y_2 - \sqrt{2}y_6,
$$
$$
b_1 = 4x - 2y_1 - 4y_2 + \frac{4}{3}y_6, \quad b_2 = 4y_1 - \frac{8}{3}y_6, \quad b_3 = -4x - 2y_1 + 4y_2 + \frac{4}{3}y_6.
$$

These may be inverted, with any 6-tuple satisfying $\sum_i b_i = 0$ giving valid $x, y_j$. In these new variables we get the Toda equations 3.34 together with the constants

$$\begin{aligned}\alpha_2 &= b_1b_2 + b_1b_3 + b_2b_3 + a_0^2 + a_1^2 + a_2^2,\\ \alpha_3 &= b_1b_2b_3 + b_1a_2^2 + b_2a_0^2 + b_3a_1^2,\\ \beta &= a_0a_1a_2.\end{aligned}$$

At this stage we have six variables and three constraints. The spectral curve is

$$\eta^3 + \alpha_2\eta\zeta^2 + \alpha_3\zeta^3 + \beta(\zeta^6 - 1) = 0, \quad \alpha_2, \alpha_3, \beta \in \mathbb{R},$$

which covers by the $C_3$ quotient the hyperelliptic curve

$$y^2 = (x^3 + \alpha_2 x + \alpha_3)^2 + 4\beta^2. \tag{3.37}$$

One could in principle solve these explicitly using the fact that the flow linearises on the Jacobian of the associated hyperelliptic curve as in [vM76, Theorem 5.1] (equivalently by Theorem 3.3.25); for $k = 3$ this was the approach taken in [BDE11] where a family of monopoles including the tetrahedrally-symmetric monopole was investigated.

Following our starting strategy we will restrict to the curve with the $D_6$ symmetry which quotients to an elliptic curve. I shall now give four different perspectives on how this gives a simplification which eventually allows us to construct Nahm data explicitly.

1. We may use Gröbner bases in Sage to utilise the constants $\alpha_2, \alpha_3, 0 = \sum b_i$ to eliminate the $b_i$, and we get the equations,

$$\begin{aligned}0 &= \sum_{i=0}^{2} a_i^2 - \alpha_2 - \frac{1}{3}(d_1^2 + d_1d_2 + d_2^2),\\ 0 &= a_1^2 d_2 - a_2^2 d_1 + \alpha_3 + \frac{1}{3}\alpha_2(d_1 - d_2) + \frac{1}{27}(d_1 - d_2)^3,\end{aligned}$$

where we have introduced $d_i = \frac{2a_i'}{a_i}$. Using $\beta = a_0a_1a_2$ to eliminate $a_0$ we then have two nonlinear ODEs in two variables, the maximal reduction one can achieve with generic $\alpha_i$ and $\beta$. One simplification which can be achieved is by attempting to make the second equation a polynomial in $d_1 - d_2$. To do this we would need $a_1^2 = a_2^2$. We can calculate that

$$\begin{aligned}\frac{d}{ds}(a_1^2 - a_2^2) &= 2\left[a_1\left(\frac{1}{2}a_1(b_1 - b_2)\right) - a_2\left(\frac{1}{2}a_2(b_2 - b_3)\right)\right],\\ &= a_1^2(b_1 - b_2) - a_2^2(b_2 - b_3),\\ &= a_1^2(b_1 - 2b_2 + b_3) + (a_1^2 - a_2^2)(b_2 - b_3),\\ &= -3b_2a_1^2 + (a_1^2 - a_2^2)(b_2 - b_3).\end{aligned}$$

Hence we can consistently set $a_1^2 - a_2^2 = 0$ provided $b_2a_1^2 = 0$. As $b_2' = a_2^2 - a_1^2$, this means we can consistently set $a_1^2 = a_2^2$ and $b_2 = 0$. Making these restrictions we can now eliminate the one remaining equation to find

$$0 = a_0^2 + 2a_1^2 - \alpha_2 - d_1^2 \Rightarrow a_1^2 \left(2\frac{da_1}{ds}\right)^2 = \beta^2 + 2a_1^6 - \alpha_2 a_1^4.$$

Upon setting $u = a_1^2$ this becomes

$$\left(\frac{du}{ds}\right)^2 = \beta^2 + 2u^3 - \alpha_2 u^2, \tag{3.38}$$

to which we shall return. We record that the $j$-invariant of the associated elliptic curve $y^2 = \beta^2 + 2u^3 - \alpha_2 u^2$ is $16\alpha_2^6/(\beta^2[\alpha_2^3 - 27\beta^2])$. Note the restrictions $a_1^2 = a_2^2$, $b_2 = 0$, make $\alpha_3 = 0$, and so the symmetry of the spectral curve is enhanced to the $D_6$ desired.

2. When Sutcliffe introduced the Toda ansatz for cyclic monopoles in [Sut96b], he showed that for $k = 2$ Nahm data could be constructed, but for $k = 3$ although he could solve the equations he could not find solutions with the correct pole behaviour. The solution was obtained from the infinite chain solution as follows. We have from

$$\left(\log a_j^2\right)'' = -a_{j-1}^2 + 2a_j^2 - a_{j+1}^2$$

and the standard elliptic function identity for the Weierstrass $\wp$ function

$$\frac{d^2}{du^2}\log[\wp(u) - \wp(v)] = -\wp(u+v) + 2\wp(u) - \wp(u-v)$$

that taking $u = ju_0 + t + t_0$ and $v = u_0$ gives

$$\begin{aligned}\frac{d^2}{dt^2}\log[\wp(ju_0 + t + t_0) - \wp(u_0)] = &-\wp([j+1]u_0 + t + t_0) + 2\wp(ju_0 + t + t_0)\\ &- \wp([j-1]u_0 + t + t_0),\end{aligned}$$

and we may identify $a_j = \wp(ju_0 + t + t_0) - \wp(u_0)$. This yields the solution for the infinite chain and we must still impose periodicity to obtain a solution. Imposing periodicity yields (for $k = 3$) that $a_j = \wp(2jK/3 + t) - \wp(2K/3)$ which is equivalent to the solution of [Sut96b] which is given in Jacobi elliptic functions.[22] The ansatz employed here forces only one of the $a_j$ to be singular at any point, and this means the pole condition on irreducibility

[22] To make connection with [Sut96b] we use Lawden's notation [Law89, §6.3.1, §6.9]. Thus for $k = 3$ we take $u_0 = 2K/3$. Now

$$dc^2(u) = \frac{\wp(u) - e_2}{\wp(u) - e_1} = 1 + \frac{e_1 - e_2}{\wp(u) - e_1} = 1 + \frac{1}{e_1 - e_3}\left[\wp(u + \omega_1) - e_1\right] = 1 + \left[\wp(u + \omega_1) - e_1\right],$$

$cs^2(2K/3) = \wp(2K/3) - e_1$.

Note Sutcliffe's '$q_j^2$' is our $a_j$. Then [Sut96b, 3.41] is $dc^2(u) - 1 - cs^2(2K/3) = \wp(u + \omega_1) - \wp(2K/3) = \wp(u + K) - \wp(2K/3)$ and so with $u = 2jK/3 + t + K$ (his choice of $\delta$) we get $a_j = \wp(2jK/3 + t) - \wp(2K/3)$ and the corresponding asymptotics given in [Sut96b, 3.42-45].

of the residues of the Nahm matrices cannot be satisfied. If we are to find an alternative solution that does indeed yield a monopole then this would suggest that one appropriate route would be to pick a simplification which forces multiple variables to have poles simultaneously. Such is the case when $a_1^2 = a_2^2$, the condition we found previously.

One can more rigorously approach this idea, of searching for algebraic relations between the variables in order to have the correct pole structure, by using Gröbner bases. Write the Toda ODEs 3.34 schematically in terms of variables $u_i$, $i = 1, \dots, r$ (here $r = 6$) as $u_i' = q_i(u_1, \dots, u_r)$ for some quadrics $q_i$, and denote the constants of motion $c_i$, $i = 1, \dots, s$ (here $s = 4$ corresponding to $\sum b_i$, $\alpha_{2,3}$, and $\beta$), as $c_i = c_i(u_1, \dots, u_r)$. We can then expand each $u_i$ as a Laurent series about $s = 0$ as $u_i(s) = \sum_{j \geq -1} u_{i,j} s^j$ and require that the lowest power terms in $s$ cancel out, which gives the equations

$$u_{i,-1} + q_i(u_{1,-1}, \dots, u_{r,-1}) = 0 = c_i(u_{1,-1}, \dots, u_{r,-1}).$$

Note these equations are given entirely in terms of $u_{i,-1}$. Using Gröbner bases we may get a better understanding of the corresponding variety, which informs us about what the possible residues look like. Applying this procedure to the Toda equations and constants one finds that (up to permuting indices) taking $a_{1,-1}^2 = a_{2,-1}^2$ is the only way to get an irreducible representation for the residues at $s = 0$.

3. We next show that $a_1^2 = a_2^2$ follows from imposing symmetry under the action of $A := \mathrm{diag}(1, -1, -1) \in \mathrm{SO}(3)$, which we saw in Example 3.3.2 corresponded to $r : (\zeta, \eta) \mapsto (1/\zeta, -\eta/\zeta^2)$. The Nahm matrices defined in Equation 3.36 are, in Toda form,

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & a_1 & -a_0 \\ -a_1 & 0 & a_2 \\ a_0 & -a_2 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2i}\begin{pmatrix} 0 & a_1 & a_0 \\ a_1 & 0 & a_2 \\ a_0 & a_2 & 0 \end{pmatrix}, \quad T_3 = \frac{-i}{2}\begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}, \tag{3.39}$$

and so

$$T_1^A = \frac{1}{2}\begin{pmatrix} 0 & -a_1 & a_0 \\ a_1 & 0 & a_2 \\ -a_0 & -a_2 & 0 \end{pmatrix}, \quad T_2^A = \frac{1}{2i}\begin{pmatrix} 0 & -a_1 & -a_0 \\ -a_1 & 0 & a_2 \\ -a_0 & a_2 & 0 \end{pmatrix}, \quad T_3^A = \frac{-i}{2}\begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}.$$

Recalling §3.3.1, equivalence of the spectral curves means that there exists a constant invertible matrix $C$ such that

$$C(-T_1^A + iT_2^A)C^{-1} = T_1 + iT_2, \quad C(-T_3^A)C^{-1} = T_3, \quad C(-T_1^A - iT_2^A)C^{-1} = T_1 - iT_2. \tag{3.40}$$

Because $T_3^A = T_3$ is diagonal and traceless, the only way to achieve this is if at least one of the $b_i$ is 0 and $C$ permutes the other two. By conjugating with a permutation matrix we can without loss of generality pick $b_2 = 0$ so

$b_1 = -b_3$, which gives that the generic $C$ is $C = \left(\begin{smallmatrix} 0 & 0 & a \\ 0 & b & 0 \\ c & 0 & 0 \end{smallmatrix}\right)$. Picking a generic $a, b, c$, Equation 3.40 becomes

$$a_0(a-c) = a_1 a + a_2 b = a_1 c + a_2 b = a_1 b + a_2 c = a_2 a + a_1 b = 0.$$

To avoid having an $a_i = 0$ we required $a = c$, and so these reduce to

$$a_1 a + a_2 b = 0 = a_1 b + a_2 a,$$

and consequently $(a/b)^2 = 1 \Rightarrow a_1 = \pm a_2$, yielding the desired $a_1^2 = a_2^2$.

4. In order to get the curve with $D_6$ symmetry of Table 3.1 we must set $\alpha_3 = 0$. We have seen that for $k = 3$ this is a consequence of the symmetry $r : (\zeta, \eta) \to (1/\zeta, -\eta/\zeta^2)$. For general $k$ this means we keep only the even terms of Equation 3.32,

$$\eta^k + \alpha_2 \eta^{k-2}\zeta^2 + \alpha_4 \eta^{k-4}\zeta^4 + \ldots + \beta[\zeta^{2k} + (-1)^k] = 0. \tag{3.41}$$

The full automorphism group of this curve is $D_k \times C_2$; for $k = 3$ this is the curve with full automorphism group $D_6 \cong D_3 \times C_2$ that we are interested in. Now on the hyperelliptic curve $\mathcal{C}/C_k$ given by Equation 3.33 $r$ acts as $r : (x, y) \to (-x, (-1)^{k-1}y)$; thus $y$ is invariant under $r$ only for $k$ odd, in which case it will be a function on the quotient curve $\mathcal{C}/\langle s, r\rangle$; for $k$-even $v = xy$ is invariant. Thus we can write the quotient hyperelliptic curve as

$$\begin{aligned} v^2 &= x^2(x^k + \alpha_2 x^{k-2} + \alpha_4 x^{k-4} + \ldots + \alpha_k)^2 - 4\beta^2 x^2, && k \text{ even}, \\ y^2 &= (x^k + \alpha_2 x^{k-2} + \alpha_4 x^{k-4} + \ldots + \alpha_{k-1}x)^2 + 4\beta^2, && k \text{ odd}. \end{aligned}$$

Setting $k = 2l$ or $k = 2l-1$ for the even and odd cases of the curves then with $u = xa_1^2$ we have these curves covering $2:1$ the curves

$$v^2 = u(u^l + \alpha_2 u^{l-1} + \alpha_4 u^{l-2} + \ldots + \alpha_k)^2 - 4\beta^2 u, \qquad k \text{ even}, \tag{3.42}$$
$$y^2 = u(u^{l-1} + \alpha_2 u^{l-2} + \alpha_4 u^{l-3} + \ldots + \alpha_{k-1})^2 + 4\beta^2, \qquad k \text{ odd}. \tag{3.43}$$

The first has genus $l$ and the second has genus $l-1$. From Theorem 3.3.25

$$\frac{\eta^{k-2} d\zeta}{\partial_\eta P} = \pi^* \left( -\frac{1}{k} \frac{x^{k-2} dx}{y} \right)$$

for the curve given by Equation 3.33, and we observe that this differential is invariant under $r$ for $k$ both even and odd. Furthermore

$$\frac{x^{k-2}dx}{y} = \begin{cases} \dfrac{x^{2l-2}dx}{y} = \dfrac{x^{2l-2}du}{2xy} = \dfrac{u^{l-1}du}{2v}, \\ \dfrac{x^{2l-3}dx}{y} = \dfrac{x^{2l-4}du}{2y} = \dfrac{u^{l-2}du}{2y}. \end{cases}$$

In each case we obtain the differential on the corresponding hyperelliptic curve of maximum degree in $u$ and the work of [Bra11] tells us the winding vector, if it is 2-torsion, will reduce to one on the quotient curve.

In particular the $k=3$ curve $y^2=(x^3+\alpha_2 x)^2+4\beta^2$ covers the elliptic curve $\mathcal{E}=\mathcal{C}/H$,

$$y^2=u(u+\alpha_2)^2+4\beta^2,$$

with $H=\langle s,r\rangle\cong S_3$. The $j$-invariant of this curve is $j_{\mathcal{E}}=16\alpha_2^6/(\beta^2[\alpha_2^3-27\beta^2])$, the value observed earlier. We note that the genus-2 curve also covers the elliptic curve $\mathcal{E}'=\mathcal{C}/H'$,

$$w^2=u^2(u+\alpha_2)^2+4\beta^2 u,$$

where now $H'=\langle s,rt\rangle\cong C_6$ with $w=xy$ the invariant coordinate. Because $\pi^*(du/(2w))=dx/y$ does not pull back to the differential appearing in the Ercolani-Sinha constraint we cannot solve the Hitchin constraints in terms of $\mathcal{E}'$. We record that the curve is in general distinct $j_{\mathcal{E}'}=[\alpha_2^3(\alpha_2^3-24\beta^2)^3]\,/\,[\beta^6(\alpha_2^3-27\beta^2)]$. We have that $\mathcal{E}$ and $\mathcal{E}'$ are the two quotients identified in Table 3.1.

We remark that the reduction of the spectral curve we have just described may be understood directly in terms of the Toda equations and 'folding'. For the $k=3$ case at hand set $e^{\rho_i}:=a_i^2=\beta^{2/3}e^{q_i-q_{i+1}}$ (so that $a_0a_1a_2=\beta$) and again take $b_j=q_j'$. Then the Toda equations take the form

$$\rho_i''=2e^{\rho_i}-e^{\rho_{i-1}}-e^{\rho_{i+1}}=\overline{K}_{ij}e^{\rho_j}$$

where $\overline{K}_{ij}$ is the extended Cartan matrix of $A_2$. Folding [OT83] corresponds to the action $\rho_i\to\rho_{\sigma(i)}$ by a diagram automorphism $\sigma$ of the extended Dynkin diagram: this retains integrability and here corresponds to identifying $\rho_1=\rho_2:=\rho_{12}$, equivalently $a_1^2=a_2^2$. Using $e^{\rho_0}=\beta^2e^{-2\rho_{12}}$ the equations of motion $\rho_{12}''=e^{\rho_{12}}-e^{\rho_0}$ and $\rho_0''=2(e^{\rho_0}-e^{\rho_{12}})$ reduce to the one equation,

$$\rho_{12}''=e^{\rho_{12}}-\beta^2e^{-2\rho_{12}},$$

the ODE reduction of the Bullough-Dodd equation, a known integrable equation which may be directly integrated. With $u=e^{\rho_{12}}$ we obtain precisely Equation 3.38. More generally we are seeing the reduction by folding $A_{2l-1}^{(1)}\to C_l^{(1)}$ for $k=2l$ even, and $A_{2[l-1]}^{(1)}\to A_{2[l-1]}^{(2)}$ for $k=2l-1$ odd, both coming from an order-2 symmetry of the Dynkin diagram.

### Solving Nahm's Equations

A number of different arguments have then lead us to an elliptic reduction of the Toda equations for $k=3$ with corresponding ODE given in Equation 3.38. The aim shall now be to show that Nahm data can be constructed from this. In doing so we will use properties of hypergeometric functions, using some details laid out in §2.1.5.

We have seen that the reduction leads to $a_1^2=a_2^2$ and $b_2=0$. In continuing to solve for the Nahm data one finds that the choice of sign of $a_2$ relative to $a_1$ does not affect the ability to impose the Hitchin constraints. Indeed, changing the choice of sign merely corresponds to changing the sign of $\beta$, and again as we will see this does not restrict the spectral curve. As such we take $a_2 = -a_1$ in what follows. Now setting $\tilde{u} = u - \frac{\alpha_2}{6}$ and $\tilde{s} = s/\sqrt{2}$ we may transform Equation 3.38 into standard Weierstrass form with solution

$$\tilde{u} = \wp((s - s_0)/\sqrt{2}; g_2, g_3),$$

where $g_2 = \frac{\alpha_2^2}{3}$ and $g_3 = \frac{\alpha_2^3}{27} - 2\beta^2$. Here we assume $\Delta := g_2^3 - 27g_3^2 = 4\beta^2(\alpha_2^3 - 27\beta^2) \neq 0$ to avoid nonsingularity, commenting on the singular limits at the appropriate junctures. The $j$-invariant of the elliptic curve is as we have already seen

$$j = 1728\frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{16\alpha_2^6}{\beta^2(\alpha_2^3 - 27\beta^2)}.$$

To be Nahm data we require that the Nahm matrices have a pole at $s = 0$ which can be achieved by setting $s_0 = 0$. We can then express all the Flaschka variables as

$$\begin{aligned} a_1 &= \pm\sqrt{\wp(s/\sqrt{2}; g_2, g_3) + \frac{\alpha_2}{6}}, & a_2 &= -a_1, & a_0 &= \frac{\beta}{a_1 a_2}, \\ b_1 &= \pm\sqrt{2a_1^2 + a_0^2 - \alpha_2}, & b_2 &= 0, & b_3 &= -b_1. \end{aligned} \tag{3.44}$$

We have some signs of the square roots to set in Equation 3.44.

(i) Using that, around $s = 0$, $\wp(s/\sqrt{2}; g_2, g_3) \sim 2s^{-2} \Rightarrow a_1^2 \sim \frac{2}{s^2}$, we have $a_0 \sim \frac{\beta s^2}{2}$. The ODE for $a_0'$, with $b_3 = -b_1$, gives

$$b_1 = -\frac{a_0'}{a_0} \sim -\frac{(\beta s)}{(\beta s^2/2)} = -\frac{2}{s}.$$

This requires us to take the negative square root for $b_1$ around $s = 0$. We will want residues at $s = 2$, and it will turn out by applying similar analysis that we need the positive root around $s = 2$. These swap over when $b_1 = 0$, which corresponds to $a_1' = 0$. As we see later this must happen at $s = 1$. Alternatively one can see this from the observation that $a_1$ is even about $s = 1$ by a judicious choice of period, and so $b_1 = \frac{2a_1'}{a_1}$ is odd about the same point.

(ii) The sign of $a_1$ is a free choice, and does not affect the geometry of the monopole, hence in what follows below we always take the positive sign.

### Imposing Boundary Conditions

The corresponding Nahm matrices (Equation 3.39) have residues at $s = 0$ given by

$$R_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad R_3 = i\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which yield a 3-dimensional irreducible representation of $\mathfrak{su}(2)$.

Next we require a simple pole at $s = 2$ again forming a 3-dimensional irreducible representation. There are two ways to achieve a residue at $s = 2$:

(i) have that $2/\sqrt{2} = \sqrt{2}$ is in the lattice corresponding to the values $g_2, g_3$, or

(ii) have that around $s = 2$, $\wp(s/\sqrt{2}; g_2, g_3) \sim -\frac{\alpha_2}{6} + \mathcal{O}(s-2)$.

These correspond to having $a_1$ and $a_0$ be singular at $s = 2$ respectively. (Because of the constant $\beta$ they cannot both be singular.) One can check that the second condition would give a reducible representation at $s = 2$ (as again only one of the $a_i$ have a pole here) and so we discount it.

Focusing then on the first condition, one way to fix the real period of the associated lattice is to invert the $j$-invariant of the elliptic curve corresponding to $g_2, g_3$ to give the period $\tau$. Here this is most readily achieved by solving the quadratic (for example, see [BBG95])

$$4\alpha(1-\alpha) = \frac{1728}{j} = 108(\beta^2/\alpha_2^3)\left[1 - 27(\beta^2/\alpha_2^3)\right],$$

for which we see the two solutions are $\alpha = \frac{27\beta^2}{\alpha_2^3}$, and $1-\alpha$. The corresponding normalised period is

$$\tau = \tau(\alpha) := i\frac{{}_2F_1(1/6, 5/6, 1; 1-\alpha)}{{}_2F_1(1/6, 5/6, 1; \alpha)}.$$

Some analytic properties of this function we need were given in §2.1.5.

**Remark 3.4.3.** *If we had taken the other root $\alpha$ in the numerator of the hypergeometric function then this would give the period $-1/\tau$.*

As we want the lattice corresponding to $g_2, g_3$ to be $\sqrt{2}\mathbb{Z} + \sqrt{2}\tau\mathbb{Z}$, we get the transcendental equations

$$\frac{1}{3}\alpha_2^2 = \frac{1}{4}g_2(1,\tau), \quad \frac{1}{27}\alpha_2^3 - 2\beta^2 = \frac{1}{8}g_3(1,\tau).$$

For any given value of $\alpha \in (0,1)$, let $\alpha_2^2 = \frac{3}{4}g_2(1,\tau)$. We then have two equations defining $\beta$:

$$\beta^2 = \frac{\alpha\alpha_2^3}{27}, \quad \beta^2 = \frac{1}{2}\left[\frac{1}{27}\alpha_2^3 - \frac{1}{8}g_3(1,\tau)\right].$$

To have a valid solution we must have that the two equations are consistent with each other. We see that $\alpha_2$ must be the same sign as $\alpha$ to get $\beta \in \mathbb{R}$. Moreover, as $g_2(1,\tau) > 0$ because $\alpha_2$ is real, we can check that

$$\begin{aligned}
\frac{1}{2}\left[\frac{1}{27}\alpha_2^3 - \frac{1}{8}g_3(1,\tau)\right] &= \frac{1}{2}\left[\frac{\operatorname{sgn}(\alpha_2)}{27}\left(3g_2(1,\tau)/4\right)^{3/2} - \frac{1}{8}g_3(1,\tau)\right], \\
&= \frac{\operatorname{sgn}(\alpha_2)g_2(1,\tau)^{3/2}}{16\sqrt{27}}\left[1 - \frac{\operatorname{sgn}(g_3(1,\tau))}{\operatorname{sgn}(\alpha_2)}\sqrt{\frac{27g_3^2}{g_2^3}}\right], \\
&= \frac{\operatorname{sgn}(\alpha_2)\left(4\alpha_2^2/3\right)^{3/2}}{16\sqrt{27}}\left[1 - \frac{\operatorname{sgn}(g_3(1,\tau))}{\operatorname{sgn}(\alpha_2)}\left(1-\frac{1728}{j}\right)^{1/2}\right], \\
&= \frac{\alpha_2^3}{2\times 27}\left[1 - \frac{\operatorname{sgn}(g_3(1,\tau))}{\operatorname{sgn}(\alpha_2)}\left(1-4\alpha(1-\alpha)\right)^{1/2}\right], \\
&= \frac{\alpha_2^3}{2\times 27}\left[1 - \frac{\operatorname{sgn}(g_3(1,\tau))}{\operatorname{sgn}(\alpha_2)\operatorname{sgn}(1-2\alpha)}(1-2\alpha)\right], \\
&= \frac{\alpha\alpha_2^3}{27} \quad \text{if} \quad \operatorname{sgn}(g_3(1,\tau)) = \operatorname{sgn}(\alpha_2)\operatorname{sgn}(1-2\alpha).
\end{aligned}$$

Hence the two equations are consistent, provided the stated sign condition holds, or if $\alpha = 0$. A consideration of the information given about $\tau$ and $g_3$ in §2.1.5 tells us that we only get solutions in the region $\alpha \in [0,1]$, where $\alpha = 0, 1$ really correspond to the limits $\lim_{\epsilon\to 0^+}\epsilon, 1-\epsilon$ respectively.

In order to exclude the possibility of other poles of the Nahm matrices in the region $s \in (0,2)$, it is necessary that for all $s \in (0,2)$

$$\wp(s/\sqrt{2}; g_2, g_3) + \frac{\alpha_2}{6} > 0.$$

We know that (i) $\wp$ takes its minimum at $s = 1$; (ii) that the minimum value is the most-positive root of the corresponding cubic $4\wp^3 - g_2\wp - g_3 = 0$; (iii) that this root is positive [DLMF, §23.5]. Therefore there are no other poles in $(0,2)$. Further as $\alpha_2 \neq 0$ has the same sign as $\alpha$, then $\alpha_2 > 0$ so $a_1^2 > 0$. Therefore we know that $a_1$ is always real, and hence so are all the Flaschka variables, thus giving all the Nahm variables being real as desired.

The remaining condition required for valid Nahm data is that $T_i(s) = T_i(2-s)^T$. The nature of the Weierstrass $\wp$ is such that $\wp((2-s)/\sqrt{2}; g_2, g_2) = \wp(s/\sqrt{2}; g_2, g_3)$, so we automatically have that $a_1(2-s) = a_1(s)$, $a_0(s) = a_0(2-s)$. Moreover, because of the change in the sign of the square root giving $b_1$ at $s = 1$, we have that $b_1(2-s) = -b_1(s)$. Taken together these ensure the desired symmetry of the Nahm matrices and we have a 1-parameter family of new solutions. As such, we have now proven the following theorem.

**Theorem 3.4.4** ([BDH23])**.** *Given $\alpha \in [0,1]$, define*

$$\tau = \tau(\alpha) = i\frac{{}_2F_1(1/6, 5/6, 1; 1-\alpha)}{{}_2F_1(1/6, 5/6, 1; \alpha)}.$$

*Solving*

$$\frac{1}{3}\alpha_2^2 = \frac{1}{4}g_2(1,\tau), \quad \frac{1}{27}\alpha_2^3 - 2\beta^2 = \frac{1}{8}g_3\left(1,\tau\right),$$

*with* $\mathrm{sgn}(\alpha_2) = \mathrm{sgn}(\alpha)$ *yields a monopole spectral curve with* $D_6$ *symmetry*

$$\eta^3 + \alpha_2\eta\zeta^2 + \beta(\zeta^6 - 1) = 0.$$

*Moreover, the Nahm data is given explicitly in terms of* $\wp$ *functions by Equation 3.39 and Equation 3.44.*

### Distinguished Curves

Having solved for general $\alpha \in [0,1]$ we now investigate the special values of $\alpha = 0, 1/2, 1$.

- ($\alpha = 0^+$) The limit $\alpha \to 0$ corresponds to $\tau \to +i\infty$, and we have using the asymptotic expansion of the Eisenstein series that $g_2(1,\tau) \to \frac{4\pi^4}{3}$, $g_3(1,\tau) \to \frac{8\pi^6}{27}$, so $\alpha = 0$ is indeed a solution with $\beta = 0$, $\alpha_2 = \pi^2$. This recreates the well known axially-symmetric monopole with spectral curve $\eta(\eta^2+\pi^2\zeta^2) = 0$ [Hit82, Hit83].

  If we had $\beta = 0$ from the beginning (and so $\Delta = 0$, and for $\alpha_2 \neq 0$ then $\alpha = 0$), we would have found a singular elliptic curve

  $$4\tilde{u}^3 - \frac{1}{3}\alpha_2^2\tilde{u} - \frac{1}{27}\alpha_2^3 = 4\left(\tilde{u} + \frac{\alpha_2}{6}\right)^2\left(\tilde{u} - \frac{\alpha_2}{3}\right),$$

  with solution to the corresponding ODE (using known integrals) given by

  $$\tilde{u} = \frac{\alpha_2}{3} + \frac{\alpha_2}{2}\tan^2\left[\frac{\sqrt{\alpha_2}}{2}(s-s_0)\right], \quad a_1 = \sqrt{\frac{\alpha_2}{2}}\sec\left[\frac{\sqrt{\alpha_2}}{2}(s-s_0)\right].$$

  We could then manufacture the right residue at $s = 0$ by having $s_0 = \frac{\pi}{2}\cdot\frac{2}{\sqrt{\alpha_2}}$. To get the correct periodicity, we would require that $\frac{\pi}{2} = \frac{\sqrt{\alpha_2}}{2}(2 - s_0)$ and consequently that $\alpha_2 = \pi^2$ again giving the axially-symmetric monopole.

- ($\alpha = 1^-$) To get this limit, we use $\tau(1^-) = -1/\tau(0^+)$, so

  $$g_2(1,\tau(1^-)) = g_2(1,-1/\tau(0^+)) = \tau(0^+)^4 g_2(1,\tau(0^+)) = \frac{1}{\tau(1^-)^4}\frac{4\pi^4}{3},$$

  and likewise for $g_3$. Solving gives

  $$\alpha_2 \sim -\left(\frac{\pi}{\tau}\right)^2, \quad \beta \sim \pm\frac{i}{3\sqrt{3}}\left(\frac{\pi}{\tau}\right)^3,$$

  or equivalently writing $\tau = i\epsilon$ for $0 < \epsilon \ll 1$,

  $$\alpha_2 \sim 3\left(\frac{\pi}{\sqrt{3}\epsilon}\right)^2, \quad \beta \sim \pm\left(\frac{\pi}{\sqrt{3}\epsilon}\right)^3,$$

The corresponding spectral curve thus factorises as

$$0 = \eta^3 + 3\left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right)^2 \eta\zeta^3 - \left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right)^3 (\zeta^6 - 1),$$
$$= \left[\eta - \left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right)(\zeta^2 - 1)\right]\left[\eta - \left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right)(\omega\zeta^2 - \omega^2)\right]\left[\eta - \left(\frac{\mp\pi}{\sqrt{3}\epsilon}\right)(\omega^2\zeta^2 - \omega)\right].$$

This corresponds to three well-separated 1-monopoles on the vertices of an equilateral triangle in the $x, y$-plane with side length $\frac{\pi}{\epsilon}$ [HMM95]. As $\epsilon$ tends to zero these three vertices tend to the point $\infty$, the singular degeneration to the cuspidal elliptic curve with $\Delta = 0$ and $\alpha = 1$.

- ($\alpha = 1/2$) In this case $\tau = i$, and the lattice is the square lattice. The values of $g_2, g_3$ for this lattice are known explicitly [DLMF, 23.5.8], giving the equations

$$\frac{1}{3}\alpha_2^2 = \frac{1}{4}\frac{\Gamma(1/4)^8}{16\pi^2}, \quad \frac{1}{27}\alpha_2^3 - 2\beta^2 = 0 \Rightarrow \alpha_2 = \frac{\sqrt{3}\Gamma(1/4)^4}{8\pi}, \quad \beta = \pm\frac{\Gamma(1/4)^6}{32(\sqrt{3}\pi)^{3/2}}.$$

  The coefficients seen here are the same, up to a sign, as those of a distinguished monopole found in [HS96b]. This is no accident, but arises because the square lattice is behind the distinguished "twisted figure-of-eight" monopole, as we show later in §3.4.2.

**Scattering**

To complete our understanding of these monopoles we discuss the corresponding scattering. This has already been described using the rational map approach in [Sut97b]. The $D_6$-symmetric monopoles described here corresponds to the prismatic subgroup $D_{3h}$ of O(3): this confines the monopoles (as located by zero of the Higgs field) to lie in a plane, and thus any scattering observed must be planar. Note for each value of $\alpha \neq 0$ there are two choices of $\beta$ from the defining equations, and these two branches coalesce where $\beta = 0 \Leftrightarrow \alpha = 0$. This gives us a view of scattering from $\alpha = 1$ with three initially well-separated 1-monopoles with a choice of sign. They move inwards along the axes of symmetry of the corresponding equilateral triangle through $\alpha = 0$ where the 3-monopoles instantaneously takes the configuration of the axially-symmetric monopole. Here we change branch (i.e. sign of $\beta$), and move back out to $\alpha = 1$ where now because of the change of sign these three well-separated 1-monopoles are deflected by $\pi/3$ radians. Note that as with the planar scattering of 2-monopoles [Ati87], because of symmetry one cannot associate a given in-going monopole with an out-going one but rather interpret the scattering process as the three in-going monopoles splitting into thirds which then recombine to form the out-going monopoles.

### 3.4.2 $V_4$

I shall now construct Nahm data for all $V_4$-symmetric 3-monopoles analogously to §3.4.1.

**Constructing Symmetric Nahm Matrices**

For our curve with $V_4$ symmetry the generators of the automorphism group are $(\zeta, \eta) \mapsto (-\zeta, -\eta)$ and $(\zeta, \eta) \mapsto (-1/\zeta, \eta/\zeta^2)$; equivalently these correspond to the rotations $\mathrm{diag}(-1,-1,1)$ and $\mathrm{diag}(-1,1,-1)$ whose product is the $r$ defined earlier in Example 3.3.2. Imposing further the involution $(\zeta, \eta) \mapsto (\zeta, -\eta)$ as a symmetry (the composition of the action of $-\mathrm{Id}$ with the antiholomorphic involution) restricts to the case of the inversion-symmetric 3-monopoles known in [HS96a]. There they solve for Nahm matrices given in terms of three real-valued functions $f_i$ satisfying $f_1' = f_2 f_3$ (and cyclic), the Euler top equations, with the corresponding spectral curve being

$$\eta^3 + \eta \left[ \left( f_1^2 - f_2^2 \right) (\zeta^4 + 1) + (2f_1^2 + 2f_2^2 - 4f_3^3)\zeta^2 \right] = 0.$$

We shall want to keep this in mind when constructing Nahm matrices for the more general $V_4$ monopoles.

Taking the polynomials $\zeta_0\zeta_1(\zeta_0^4 - \zeta_1^4)$, $\zeta_0^2\zeta_1^2$, and $\zeta_0^4 + \zeta_1^4$ as the inputs to the procedure of outlined in §3.3.2 gives the ODES in the six real-valued variables

$$\begin{aligned}
x' &= 2x^2 - \frac{1}{6}y_0^2 + \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 + \frac{1}{6}y_3^2 - \frac{1}{2}y_4^2, & y_2' &= \frac{1}{3}y_0^2 + y_1^2 - 2xy_2 - y_2^2 - \frac{1}{3}y_3^2 - y_1y_4, \\
y_0' &= -2xy_0 + 2y_0y_2 + 2y_1y_3 + y_3y_4, & y_3' &= 2y_0y_1 + 2xy_3 - 2y_2y_3 + y_0y_4, \\
y_1' &= 2xy_1 + 2y_1y_2 + \frac{2}{3}y_0y_3 - y_2y_4, & y_4' &= -2y_1y_2 + \frac{2}{3}y_0y_3 - 4xy_4,
\end{aligned}$$

with the corresponding spectral curve

$$\mathcal{C}: \quad \eta^3 + \eta \left[ a(\zeta^4 + 1) + b\zeta^2 \right] + c\zeta(\zeta^4 - 1) = 0, \tag{3.45}$$

where

$$\begin{aligned}
a &= 8xy_0 + 4y_0y_2 - 4y_1y_3 + 4y_3y_4, \\
b &= 4y_0^2 - 12y_1^2 + 48xy_2 - 12y_2^2 + 4y_3^2 - 24y_1y_4, \\
c &= -8iy_0^2y_1 - 8iy_1^3 + 48ixy_1y_2 + 24iy_1y_2^2 - 16ixy_0y_3 + 16iy_0y_2y_3 \\
&\quad + 8iy_1y_3^2 + 48ix^2y_4 - 4iy_0^2y_4 + 12iy_1^2y_4 - 12iy_2^2y_4 + 4iy_3^2y_4 - 4iy_4^3.
\end{aligned}$$

The full Nahm matrices are

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\bar{f}_1 \\ 0 & f_1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & f_2 \\ 0 & 0 & 0 \\ -\bar{f}_2 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -\bar{f}_3 & 0 \\ f_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.46}$$

where the $f_i$ are given by

$$\begin{aligned}
f_1 &= 2x + y_0 - iy_1 + y_2 + iy_3 + iy_4, \\
f_2 &= 2x - y_0 - iy_1 + y_2 - iy_3 + iy_4, \\
f_3 &= 2x + 2iy_1 - 2y_2 + iy_4,
\end{aligned}$$

One can check that setting $y_1 = y_3 = y_4 = 0$ is consistent, and corresponds to the inversion-symmetric case. In terms of the $f_i$ we can write the ODEs giving Nahm's equations as a complex generalisation of the Euler top equations[23]

$$\bar{f_1}' = f_2 f_3 \quad \text{(and cyclic)}, \tag{3.47}$$

with the constants in the spectral curve given as

$$a = |f_1|^2 - |f_2|^2, \quad b = 2|f_1|^2 + 2|f_2|^2 - 4|f_3|^3, \quad c = 2(f_1 f_2 f_3 - \bar{f}_1 \bar{f}_2 \bar{f}_3).$$

**Remark 3.4.5.** *We observe that Equations 3.47 come from the Poisson structure (as defined in [LGMV11])* $\{f_i, \bar{f}_j\} = \delta_{ij}$, *with Hamiltonian* $c/2 = f_1 f_2 f_3 - \bar{f}_1 \bar{f}_2 \bar{f}_3$. *This complex extension of the Euler equations is integrable.*

**Remark 3.4.6.** *We have not fully used up the gauge symmetry available to us. Namely, if we conjugate the* $T_i$ *by* $U = \operatorname{diag}(u_1, u_2, u_3)$ *where* $u_j = e^{i\phi_j}$ *and* $\sum \phi_j = 0$, *we get*

$$f_1 \mapsto u_3 u_2^{-1} f_1, \quad f_2 \mapsto u_1 u_3^{-1} f_2, \quad f_3 \mapsto u_2 u_1^{-1} f_3,$$

*which preserves the form of the equations.*

A consequence of Remark 3.4.6 and the form of the Nahm matrices in Equation 3.46 is that for the $T_i$ to have residues which form an irreducible representation of $\mathfrak{su}(2)$ it is sufficient for the $f_i$ to have simple poles at $s = 0, 2$.

### Solving Nahm's Equations

In order to find a solution we note that $a_{ij} = |f_i|^2 - |f_j|^2$ and $c = 2(f_1 f_2 f_3 - \bar{f}_1 \bar{f}_2 \bar{f}_3)$ are now constants. As $c$ is imaginary it will be useful to introduce $\tilde{c} := -ic$. Setting $F = |f_1|$ we have

$$\begin{aligned}
(F')^2 &= \left\{ \left[ \left(f_1 \bar{f}_1\right)^{1/2} \right]' \right\}^2 = \left\{ \frac{1}{2} (f_1 \bar{f}_1)' (f_1 \bar{f}_1)^{-1/2} \right\}^2 = \frac{1}{4} (f_1 f_2 f_3 + \bar{f}_1 \bar{f}_2 \bar{f}_3)^2 F^{-2}, \\
&= \frac{1}{4} F^{-2} \left[ (c/2)^2 + 4|f_1|^2 |f_2|^2 |f_3|^2 \right], \\
&= \frac{1}{4} F^{-2} \left[ (c/2)^2 + 4F^2 (F^2 - a_{12})(F^2 + a_{31}) \right],
\end{aligned}$$

and so with $G = F^2$ we get

$$(G')^2 = \frac{1}{4} c^2 + 4G(G - a_{12})(G + a_{31}),$$

which then has solutions in terms of elliptic functions. In terms of the coefficients of the spectral curve we already have $a_{12} = a$, and we can moreover find $a_{31} =$

[23] These equations are also found in [Hou97, (3.57)] where they are attributed to [ABDP62]; they also appear as the $x$-independent solutions in the description of 3-wave scattering [NMPZ84, (17), p. 177]. We thank Pol Vanhaecke and Sasha Mikhailov for this latter reference.

$\frac{-1}{4}(b+2a)$, so we can rewrite the equation as

$$(\tilde{G}')^2 = 4\tilde{G}^3 - g_2\tilde{G} - g_3, \tag{3.48}$$

where $\tilde{G} = G - \frac{b+6a}{12}$, $g_2 = a^2 + \frac{b^2}{12}$, and $g_3 = \frac{b(b^2-36a^2)}{216} + \frac{1}{4}\tilde{c}^2$. Then $\tilde{G} = \wp$, the Weierstrass $\wp$ function. The $j$-invariant for this elliptic curve is

$$j = 1728\frac{g_2^3}{g_2^3 - 27g_3^3} = \frac{(12a^2+b^2)^3}{\left(a^6 - \frac{1}{2}a^4b^2 + \frac{1}{16}a^2b^4 + \frac{9}{4}a^2b\tilde{c}^2 - \frac{1}{16}b^3\tilde{c}^2 - \frac{27}{16}\tilde{c}^4\right)}, \tag{3.49}$$

which is precisely that of the quotient of the full $V_4$-symmetric curve (Equation 3.45) by the $V_4$ symmetry. We also note that the pull-back of the invariant differential of this quotient is exactly that needed when discussing the Ercolani-Sinha constraint.

#### Imposing Boundary Condition

Before going on to solve Equation 3.47 completely, let us recall what remains to be shown to get a monopole spectral curve (i.e. to have our Nahm matrices satisfy all the conditions to give Nahm data). We need to have that the $\wp$ function associated with the elliptic curve determined by Equation 3.48 has real period 2, but we will be able to impose this by tuning the coefficients. As the right-hand side of

$$\wp = |f_1|^2 - \frac{b+6a}{12} \tag{3.50}$$

is always real this requires $\wp$ to be real and so to be taken on a rectangular or rhombic lattice. Also for reality we need that

$$G(s) = \wp(s) + \frac{b+6a}{12}, \quad G(s) - a_{12} = \wp(s) + \frac{b-6a}{12}, \quad G(s) + a_{31} = \wp(s) - \frac{b}{6}$$

are always positive. Once we have achieved these we will have regularity in the region $(0,2)$, and so get the right pole structure. The final condition is symmetry about $s=1$, which is enforced on the $|f_j|$ (because $|f_j| \sim \sqrt{\wp}$), and so the remaining Nahm constraint $T_j(s) = T_j(2-s)^T$ becomes simply $f_j(s) = -\bar{f}_j(2-s)$: that is we require $\arg f_j(s) = \pm\pi - \arg f_j(2-s)$.

Indeed writing $f_j = |f_j|\, e^{i\theta_j}$ we can work out the equations for the angles, using

$$f_j' = \left(|f_j|' + i\theta_j'\,|f_j|\right)e^{i\theta_j} = \left(\frac{|f_j|'}{|f_j|} + i\theta_j'\right)f_j \Rightarrow \theta_j' = \frac{1}{i}\left[\frac{f_j'}{f_j} - \frac{|f_j|'}{|f_j|}\right] = \frac{-\tilde{c}}{4\,|f_j|^2}. \tag{3.51}$$

The $\theta_j$ are thus strictly monotonic (unless $\tilde{c}=0$, in which case they are constant), and symmetry about $s=1$ of $|f_j|$ then necessitates that $\theta_j(s) - \theta_j(1)$ is antisymmetric about $s=1$.

We also have that

$$\tilde{c} = 4\,|f_1|\,|f_2|\,|f_3|\sin(\theta_1+\theta_2+\theta_3) = \sqrt{\tilde{c}^2+4(G')^2}\sin(\theta_1+\theta_2+\theta_3).$$

At $s=1$ where $G'(s)=0$ we need $\sin(\theta_1+\theta_2+\theta_3)=1$, and by our gauge freedom we can choose $\theta_1(1)=\pi/2=\theta_2(1)$ and so $\theta_3(1)=-\pi/2$, thus enforcing our condition of symmetry about $s=1$. We then see that the antisymmetry of $\theta_j(s)-\theta_j(1)$ about $s=1$ enforces the remaining reality condition. We also note that as $|f_j(s)|^2=\wp(s)-c_j:=\wp(u)-\wp(v_j)$ for appropriate $s$ and $v_j=\int_\infty^{c_j}\left[4u^3-g_2u-g_3\right]^{-1/2}du$ we have [Law89, (6.14.6)]

$$\int\frac{du}{\wp(u)-\wp(v)}=\frac{1}{\wp'(v)}\left[2u\zeta(v)+\log\frac{\sigma(u-v)}{\sigma(u+v)}\right], \tag{3.52}$$

where $\zeta,\sigma$ are the corresponding Weierstrass functions, allowing us to find the $\theta_j(s)$ explicitly. We note that $|f_j(s)|^2=\wp(s)-c_j:=\wp(s)-\wp(v_j)$ does not fix the sign of $v_j$ for $\wp(\pm v_j)=c_j$. We fix the sign as follows. First observe that

$$(\tilde{G}'(s))^2=(G'(s))^2=\frac{1}{4}c^2+4[\wp(s)-\wp(v_1)][\wp(s)-\wp(v_2)][\wp(s)-\wp(v_3)],$$

and so $\wp'^2(v_i)=c^2/4$; we fix the sign so that $\wp'(v_i)=c/2=i\tilde{c}/2$. Further consider the elliptic function $\wp'(s)-c/2$ with three zeros (at $s\in\{v_1,v_2,v_3\}$) and three poles (at $s=0$). Then with the base of the Abel-Jacobi map at $s=0$ (as is standard) we have that $\sum_i v_i$ is a lattice point. Also observe that

$$\zeta(v_i)+\zeta(v_j)=\zeta(v_i+v_j).$$

We find from Equations 3.51 and 3.52 that

$$\theta_i(s):=\theta_i(1)+i\left[s\zeta(v_j)+\frac{1}{2}\log\frac{\sigma(s-v_j)\sigma(1+v_j)}{\sigma(s+v_j)\sigma(1-v_j)}\right]. \tag{3.53}$$

Then $\theta_i(-s)-\theta_i(1)=-\left[\theta_i(s)-\theta_i(1)\right]$ is antisymmetric as required. Using the Legendre relation we find that $\sin(\theta_1+\theta_2+\theta_3)$ is periodic in $s$ as required for consistency.

It remains to fix the real period of the corresponding elliptic curve. We describe two methods. The first makes use of the Jacobi elliptic functions to express the lattice invariants in terms of complete elliptic functions [AS72, §18.9]. Given the discriminant $\Delta$ for the cubic defining $\wp$ [AS72, §18.9] gives equations for the lattice invariants in terms of complete elliptic functions. These are related to the complete elliptic function $K$ by $g_2=12\left(K(m)^2/3\right)^2q_1(m)$ and $g_3=4\left(K(m)^2/3\right)^3(2m-1)q_2(m)$. With sgn $=$ sgn$(\Delta)$ we have for $\Delta>0$

$$g_2=12\left(\frac{K^2}{3\omega_1}\right)^2\left(1-m+m^2\right),\quad g_3=4\left(\frac{K^2}{3\omega_1}\right)^3(m-2)(2m-1)(m+1),$$

whereas for $\Delta < 0$

$$g_2 = 12\left(\frac{K^2}{3\omega_2}\right)^2\left(1-16m+16m^2\right), \quad g_3 = 8\left(\frac{K^2}{3\omega_2}\right)^3(2m-1)(32m^2-32m-1).$$

Here $m = k^2 \in (0,1)$ is the argument of $K$, the underlying lattice has periods $2\omega$, $2\omega'$, $\omega_1 = \omega$ and $\omega_2 = \omega+\omega'$. Fixing 2 as a period of the lattice, and that the lattice is real, sets $\omega_1 = 1$ for $\Delta > 0$ and $\omega_2 = 1$ for $\Delta < 0$. Observe that for $\operatorname{sgn}(\Delta) = \pm 1$ that $g_2(m)$ takes its minimum value at $m = 1/2$ while for $m \in (0, 1/2)$ we have $g_3(m) > 0$.

Our elliptic curve gave the equations $a^2 + \frac{b^2}{12} = g_2$, $\frac{b(b^2-36a^2)}{216} + \frac{1}{4}\tilde{c}^2 = g_3$. These equations are underdetermined, but we may substitute for $a^2$ and take $\alpha = -27\tilde{c}^2/b^3$ to find

$$(4-2\alpha)\tilde{b}^3 - g_2\tilde{b} - g_3 = 0, \tag{3.54}$$

where $\tilde{b} = \frac{b}{6}$. The discriminant of this cubic is

$$\Delta_\alpha(m) = 4(4-2\alpha)g_2^3 - 27(4-2\alpha)^2 g_3^2 = 4(4-2\alpha)\left[g_2^3 - 27(1-\alpha/2)g_3^2\right].$$

Note $\Delta_0 = \Delta$. For a given generic value of $\alpha$ in some region we may solve Equation 3.54, determining $b$, $\tilde{c}$ and $a$ in turn.

In order to get Nahm data, we require that $b$, $\tilde{c}$, and $a$ are real. We know that this cubic has real coefficients, and so there will always be a real root of the cubic. To get reality of $\tilde{c}$, we need that this real root $\tilde{b}$ satisfies $\operatorname{sgn}(\tilde{b}) = -\operatorname{sgn}(\alpha)$, and for reality of $a$ we need $\left|\tilde{b}\right| \le g_2/3$. This establishes the following theorem.

**Theorem 3.4.7.** *Given $\alpha \in \mathbb{R}$, $m \in [0,1]$, and $\operatorname{sgn} = \pm 1$, define $g_2, g_3$ by $g_2 = 12\left(K(m)^2/3\right)^2 q_1(m)$, $g_3 = 4\left(K(m)^2/3\right)^3(2m-1)q_2(m)$, where*

$$q_1(m) = \begin{cases} 1-m+m^2, & \operatorname{sgn} = 1, \\ 1-16m+16m^2, & \operatorname{sgn} = -1, \end{cases} \qquad q_2(m) = \begin{cases} (m-2)(m+1), & \operatorname{sgn} = 1, \\ 2(32m^2-32m-1), & \operatorname{sgn} = -1. \end{cases}$$

*If $m$ is such that $g_2 > 0$ and the polynomial $(4-2\alpha)x^3 - g_2 x - g_3$ has a real root $x_*$ with $|x_*| < \sqrt{g_2/3}$ and $\operatorname{sgn}(x_*) = -\operatorname{sgn}(\alpha)$, then we may solve*

$$a^2 + \frac{b^2}{12} = g_2, \quad \frac{b(b^2-36a^2)}{216} + \frac{\tilde{c}^2}{4} = g_3,$$

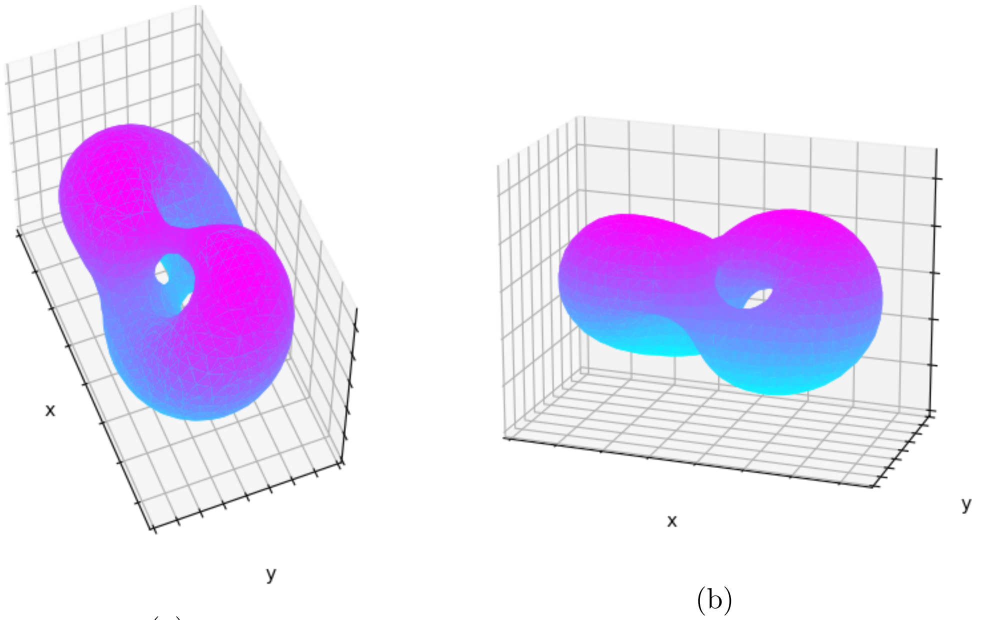
*for $a, b, \tilde{c} \in \mathbb{R}$. Then*

$$\eta^3 + \eta\left[a(\zeta^4+1) + b\zeta^2\right] + i\tilde{c}\zeta(\zeta^4-1) = 0$$

*is a monopole spectral curve with $V_4$ symmetry. Moreover the Nahm data is given explicitly in terms of elliptic functions by Equations 3.46, 3.50 and 3.53.*

In contrast to the $D_6$ monopoles found in §3.4.1, the $V_4$ monopoles parametrised in Theorem 3.4.7 can take surprising and intricate configurations not previously seen studying monopoles, for example that seen in Figure 3.4.

**Remark 3.4.8.** *Figures 3.4 and 3.6 were plotted in Python using a modification of code provided to me by Paul Sutcliffe, first described in [HS96d]. Example*




(a) (b)

**Fig. 3.4** Surface of constant energy density $\mathcal{E} = 0.18$ for the $V_4$ monopole given by the parameters $m = 0.6$, $\alpha = -2.0$, sgn $= 1$

*code showing how this may be done is provided, namely* **`plotting_script.sh`**, **`monopole_plotting.py`**, **`V4_nahmdata.py`**, *and* **`minimal_plotting_from_file.ipynb`**. *Note the code work with the convention that the norm on* $\mathfrak{su}(2)$ *is* $|X|^2 = -\frac{1}{2}\operatorname{Tr}(X^2)$ *to be consistent with the Nahm data being defined for* $s \in [0, 2]$.

A second approach to fixing the correct real period to give Nahm data is to invert the $j$-invariant of Equation 3.49 as done in the earlier $D_6$ case. Though we are unable to invert in terms of a single rational $\alpha$ as with the $D_6$-symmetric monopole, we may use [DI08, (4)] which gives

$$\tau = i\left[\frac{2\sqrt{\pi}}{\Gamma(7/12)\Gamma(11/12)}\frac{{}_2F_1(1/12, 5/12, 1/2; x)}{{}_2F_1(1/12, 5/12, 1; 1-x)} - 1\right],$$

where $x = 1 - \frac{1728}{j} = \frac{(1-2\alpha-3\gamma)^2}{(1+\gamma)^3}$, with $\alpha = -\frac{27\tilde{c}^2}{b^3}$, $\gamma = \frac{12a^2}{b^2}$. One may then fix the real period of the lattice, which will give solutions consistent with the definition of $x$ for some range of the parameters $\alpha, \gamma$. We investigate one particular restriction of this kind subsequently in §3.4.2. We remark that [Hou97] solved the associated Nahm data only for the (1-parameter) case $\Delta = 0$ in which the elliptic curve degenerates and has trigonometric solutions.

### Restrictions on Elliptic Function Parameters

Theorem 3.4.7 implicitly restricts the range of $\alpha$ and $m$ in order to ensure the corresponding coefficient of the spectral curve are real. Necessary conditions to find such solutions are as follows.

First consider $\Delta > 0$. Then $g_2 > 0$ and $g_3$ is monotonically decreasing for $m \in (0, 1)$ with $\operatorname{sgn}(g_3) = -\operatorname{sgn}(m - 1/2)$. We have the following properties.

(i) If $\alpha > 2$ the discriminant $\Delta_\alpha(m) < 0$. Then Equation 3.54 has one real root whose sign is opposite that of $g_3$. Now $\operatorname{sgn}(\tilde{b}) = -\operatorname{sgn}(\alpha) < 0$ is opposite that of $g_3$; hence we require $g_3 > 0$ and so $m \in (0, 1/2)$.

(ii) For $\alpha \in (0, 2)$ the discriminant $\Delta_\alpha(m) > 0$ upon comparison with $\Delta = g_2^3 - 27g_3^2 > 0$. Then, because the sum of the roots is zero, they cannot all be the same sign.

(iii) When $\alpha < 0$, from the derivative of the cubic we know it will have a local maxima and minima at $\tilde{b} = \pm\sqrt{\frac{g_2}{3(4-2\alpha)}}$; it is the minima when the sign is positive. Recalling that we require a root with sign $\operatorname{sgn} \tilde{b} = -\operatorname{sgn} \alpha = 1$, the local minima must be nonpositive, and the value at this $\tilde{b}$ is $-\frac{2}{3}\tilde{b}g_2 - g_3$. As the value at this minima is monotonically increasing for $m > 1/2$, and negative at $m = 1/2$, then the value at the minima is negative for all $m < m_*$, the value for which is it zero. Solving, one gets the condition $\Delta_\alpha(m_*) = 0$, taking the root greater than $1/2$.

Therefore necessary conditions for a real root of the right sign to exist for $\Delta > 0$ are that

- if $\alpha > 2$, $m < 1/2$,
- if $\alpha \in (0, 2)$, any $m$ is valid,
- if $\alpha < 0$, $m < m_2(\alpha)$, where $m_2$ is the root of $\Delta_\alpha(m) = 0$ in $(1/2, 1)$.

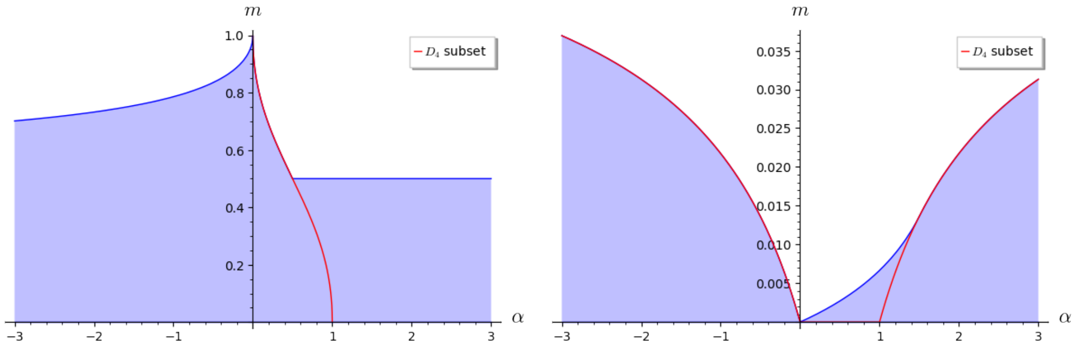
To get Nahm data we require that this real root is bounded in magnitude by $\sqrt{g_2/3}$, with the case that it is equal corresponding to $a = 0$, i.e. to the $D_4$ monopoles of [HS96b]. Figures showing these parameter regions are given in Figure 3.5.

In the case $\Delta < 0$, in order to get real roots of the right sign one analogously gets restrictions on $m$ relative to $\alpha$ such that

- if $\alpha < 0$, $m < 1/2$,
- if $\alpha \in (0, 2)$, $m > 1/2$ or $m < m_1(\alpha)$, defined to be the root $< 1/2$ of the polynomial $\Delta_\alpha(m) = 0$,
- if $\alpha > 2$, $m < m_2(\alpha)$, now defined to be the root $> 1/2$ of the polynomial $\Delta_\alpha(m) = 0$.

Fixing the size of the root in this case requires more work, complicated by the fact that $g_2$ is real only if $|m - 1/2| > \sqrt{3}/4$. Using explicitly formulas for the roots $\tilde{b}$ from Cardano's formula one can achieve explicit bounds, but here we omit these. In practice, when using this approach to plot monopoles, numerical methods can be used to find the appropriate $m$ region for a given $\alpha$, as done to generate Figure 3.5.

Note that, for certain admissible $\alpha, m$ there may be two possible monopoles because two roots of the cubic defining $b$ satisfy the required conditions. Numerical investigations indicates that this phenomenon only occurs for $\Delta > 0$.




(a) $\Delta > 0$ (b) $\Delta < 0$

**Fig. 3.5** Valid parameter regions for $V_4$ monopoles, with the subset corresponding to $D_4$ monopoles highlighted

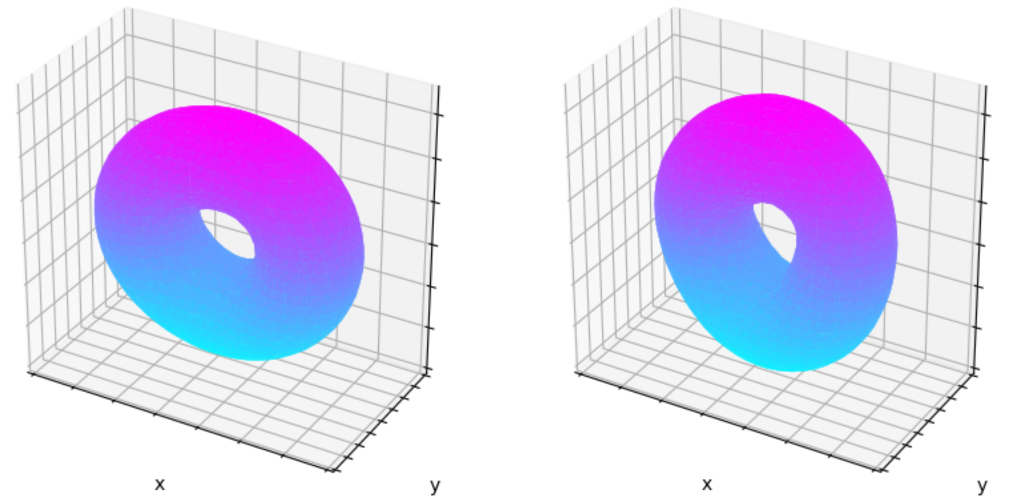
We plotted two examples of this, seen in Figure 3.6 to investigate the difference in the associated monopoles. The visual difference between the configurations suggests that the parameters $\alpha, m$ do not provide the most physically significant parametrisation of the moduli space. It is worth remarking that these regions where there are multiple solutions where $\Delta > 0$ stitch together along with the regions where $\Delta \leq 0$ so as to make the moduli space connected.

### $D_4$ Monopoles

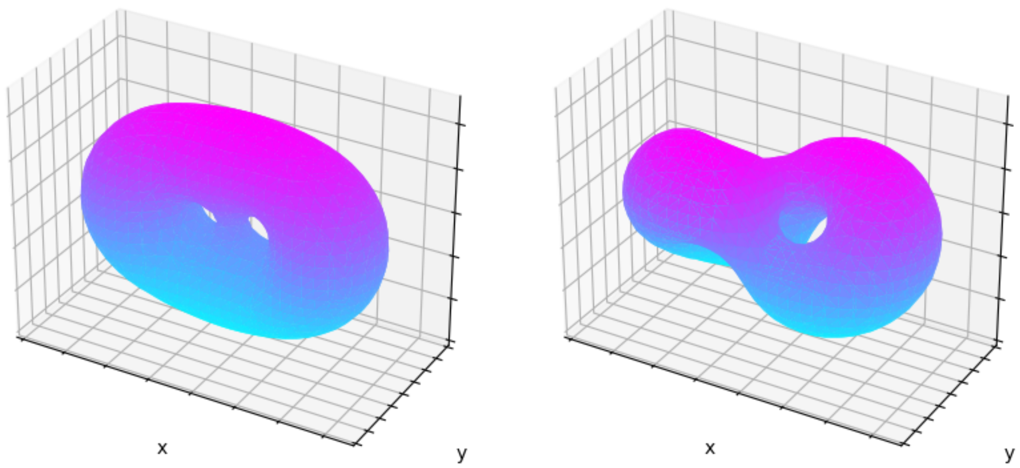
In [HS96b] a subfamily of the Nahm matrices in Equation 3.46 with $D_4$ symmetry was studied. To the existing $V_4$ symmetries is appended the order-4 element $(\zeta, \eta) \mapsto (i\zeta, -i\eta)$ (corresponding to the composition of inversion with a rotation of $\pi/2$ in the $xy$-plane). This symmetry then requires $a = 0$. By a dimension argument we expect the $j$-invariant inversion to yield a 1-parameter family for the enlarged symmetry group corresponding to a geodesic in the moduli space, and this was the case considered in [HS96b] where the $C_4$ quotient yields an elliptic curve. Placing this curve in our $V_4$ family allows us a different approach to this family of curves. The restriction $a = 0$ means that $\frac{1728}{j} = 4\alpha(1-\alpha)$ with $\alpha = -\frac{27\tilde{c}^2}{b^3}$ and we can then fix the real period via the same approach as for the $D_6$ monopole. The equations we get are

$$\frac{b^2}{3} = \frac{1}{4} g_2(1, \tau), \quad \frac{b^3}{27} + 2\tilde{c}^2 = \frac{1}{8} g_3(1, \tau),$$

with these being consistent with the definition of $\alpha$ provided $\operatorname{sgn}(g_3(1, \tau)) = \operatorname{sgn}(b) \operatorname{sgn}(1 - 2\alpha)$. To also have that $\tilde{c}$ is real, we must have $\operatorname{sgn}(b) = -\operatorname{sgn}(\alpha)$ and hence our consistency condition is $\operatorname{sgn}(g_3(1, \tau)) = -\operatorname{sgn}(\alpha) \operatorname{sgn}(1 - 2\alpha)$. We thus have solutions in the region $\alpha \in (0, 1/2)$ if $\operatorname{sgn}(g_3(1, \tau)) < 0$, which requires $\tau = -1/\tau(\alpha)$. We can extend this to $\alpha \in (1/2, 1)$ still taking $\tau = -1/\tau(\alpha)$. Moreover, for $\alpha < 0$, we require $g_3(1, \tau) > 0$, which can be achieved taking $\tau = \tau(\alpha)$. Finally, for $\alpha > 0$, we require $\operatorname{sgn}(g_1(1, \tau)) > 0$, achievable with $\tau = -1/\tau(\alpha)$. As such the parameter region in this case is the whole of $\mathbb{R}$. A




(a) $k = 0.45$, $\alpha = 0.2$, $\Delta > 0$, $b = -3.21$ (b) $k = 0.45$, $\alpha = 0.2$, $\Delta > 0$, $b = -7.19$




(c) $k = 0.77$, $\alpha = -2.0$, $\Delta > 0$, $b = 1.42$ (d) $k = 0.77$, $\alpha = -2.0$, $\Delta > 0$, $b = 7.24$

**Fig. 3.6** Comparison of two pairs of monopoles with equal values of $\alpha, m$, taking $\mathcal{E} = 0.17$

case-by-case consideration shows that $G$, $G - a_{12}$, $G + a_{31}$ are always positive on the interval $[0, 2]$, so we do indeed get Nahm data as desired.

As with the $D_6$-symmetric monopoles we may identify special values of $\alpha$ and the curves they give. A similar analysis gives those found in [HS96b], namely

- $\alpha = \pm\infty$ gives the tetrahedrally-symmetric monopole,
- $\alpha = 0^+, 0^-$ gives three well-separated 1-monopoles and the axially-symmetric monopole respectively, and
- $\alpha = 1/2$ gives the "twisted figure-of-eight" monopole. Note $\alpha = 1/2$ corresponds to the square lattice we saw as distinguished for the $D_6$ monopole.

We additionally see the curve with $\alpha = 1$ as distinguished in our parametrisation, which gives the curve

$$\eta^3 - \pi^2 \eta \zeta^2 \pm \frac{i}{\sqrt{27}} \pi^3 \zeta (\zeta^4 - 1) = 0.$$

In terms of the parameters $a, \epsilon$ of [HS96b], this curve is given by $a = 2\sqrt{2}$, $\epsilon = -1$.

**Scattering**

As such we can now understand our scattering as starting at $\alpha = 0^+$ with three well-separated 1-monopoles. As $\alpha$ increases to $\infty$ we have to pick a choice of $\tilde{c}$ continuously (though there is no specific choice at $\alpha = 0^+$ as the map $\zeta \mapsto -\zeta$ which swaps the choice of $\tilde{c}$ is a symmetry of our well separated configuration), and we pass through two distinguished curves, arriving at the tetrahedrally-symmetric monopoles in one orientation. We match that to $\alpha = -\infty$ taking the tetrahedrally-symmetric monopole with the same orientation there, allowing $\alpha$ to then increase up to $0^-$ where it takes the configuration of the axially-symmetric monopole. Here the two branches of $\tilde{c}$ coalesce, we change branch and do the process in reverse.

### 3.4.3 $C_2$

Here I shall now consider spectral curves of the form

$$P(\zeta, \eta) = \eta^3 + \eta(a\zeta^4 + b\zeta^2 + c) + (d\zeta^6 + e\zeta^4 + f\zeta^2 + g) = 0$$

which have the $C_2$ symmetry $\zeta \mapsto -\zeta$ which corresponds to the reflection $R_3 = \operatorname{diag}(1, 1, -1) \in \mathrm{O}(3)$. This signature of the corresponding quotient is $(1; 2^6)$. We have already seen in §3.3.1 that the dimension of the moduli space of $R_3$-invariant 3-monopoles is 4, so we cannot have $\dim_{\mathbb{R}}(M_3^0)^{\langle R_3 \rangle} - \dim_{\mathbb{R}} N_{\mathrm{O}(3)}(\langle R_3 \rangle) = \delta' - 1$, but this is consistent with the conjecture of §3.3.1 as under the action of $R_3$ the differential $\Omega^{(1)} = \frac{\eta d\zeta}{\partial_\eta P}$ has eigenvalue $-1$, and so by the argument of [Bra11, p. 9] the Ercolani-Sinha cycles $\mathfrak{es}$ is also anti-invariant. Now using the method of [BSZ19] one can check (see Sage notebook `isogeny_decomposition_C2_3-monopole.ipynb`) that for a generic $\mathcal{C}$ invariant under $C_2$ the isogeny decomposition of the Jacobian is $\operatorname{Jac}(\mathcal{C}) \sim \operatorname{Jac}(\mathcal{C}/C_2) \times A_3$, where $A_3$ is some irreducible abelian 3-fold. By the previous computation the winding vector $\boldsymbol{U}$ lives in the irreducible $A_3$ factor, as this abelian variety is defined precisely to be the $(-1)$-eigenspace of the $C_2$ symmetry, and as such one cannot solve for Nahm data in terms of elliptic functions. If one could solve in terms of elliptic functions, this would say that there was an elliptic curve as an algebraic subvariety of the 3-fold $A_3$ upon which Nahm's equations have linearised, contradicting the irreducibility of $A_3$.

This final point deserves some amplification. Suppose that Nahm data could be written down in terms of algebraic combinations of elliptic functions on the same elliptic curve, with arguments linear in $s$, interpreted now as a coordinate on the complex torus corresponding to the elliptic curve. To see the impact of this we need to unpack exactly what the 'linearisation' process I have so far described is. I shall follow the treatment of [Gri85, AvMV04]; to have agreement on conventions transform the spectral curve by $\eta \mapsto -\eta$ so it is given by the characteristics polynomial $\det[\eta \operatorname{Id}_k - L] = 0$ (recall §3.2.2). To a Lax pair $L(\zeta, s), M(\zeta, s)$ associate the vector $\boldsymbol{\xi}(\zeta, \eta, s)$, the eigenvector of $L$ with eigenvalue $\eta$ normalised such that $\xi_1 = 1$; this will be given by $\xi_l = \frac{\Delta_{1l}}{\Delta_{11}}$ for $l = 1, \dots, k$, where $\Delta_{ij}(\zeta, \eta, s)$ is the cofactor corresponding to the $(i,j)$ entry of $\eta \operatorname{Id}_k - L(\zeta, s)$. By isospectrality, the eigenvector will have $s$ dependence given by

$$\boldsymbol{\xi}' + M\boldsymbol{\xi} = \lambda \boldsymbol{\xi} \tag{3.55}$$

for some scalar function $\lambda(\zeta, \eta, s)$, given by

$$\lambda = \sum_{j=1}^{k} M_{1j} \frac{\Delta_{1j}}{\Delta_{11}}.$$

At any value $s$, we associated to $L$ the divisor $D_L$ on the spectral curve defined to be the minimal effective divisor such that

$$\forall\, l = 1, \dots, k, \quad (\xi_l) + D_L \geq 0.$$

The 'linearisation' map is then map into the Jacobian $\iota(s) = \mathcal{A}_*(D_{L(s)} - D_{L(0)})$. Moreover, away from the support of $D_{L(s)}$, $\lambda$ has poles only where $M$ has poles, which for the $M$ of Equation 3.18 will be only where $\zeta = \infty$. Assuming that generically the point $\infty_j$ are not in the support of $D_{L(s)}$, $\lambda$ then determines a Laurent tail divisor supported at the $\infty_j$ that Griffiths denotes with $\rho(M)$. I will denote this set of Laurent tails with $\left\{ r_{\infty_j} = \lambda_j \zeta \right\}$. The fundamental theorem describing the linearisation procedure is then the following.

**Theorem 3.4.9** ([AvMV04], Theorem 6.39, [Gri85], Theorem 7.7)**.** *Fixing a basis of differentials $\{\nu_j\}$, the derivative of the linearisation map is given by*

$$\frac{d}{ds}\mathcal{A}_*(D_{L(s)} - D_{L(0)})_i = \sum_j \operatorname{Res}_{\infty_j}(\lambda_j \zeta \nu_i) = \sum_j \operatorname{Res}_{\infty_j}(\lambda \nu_i).$$

As such when $L, M$, and hence $\lambda$, are given in terms of meromorphic functions on a complex torus $\mathcal{E}$ with variable $s$ we get a map from this complex torus into

$\mathrm{Jac}(\mathcal{C})$ given by

$$\begin{aligned}&\mathcal{E} \to \mathrm{Jac}(\mathcal{C}),\\ &s \to \int_0^s \sum_j \mathrm{Res}_{\infty_j}(\lambda(s')\boldsymbol{\nu})\, ds',\end{aligned}$$

where I have written $\boldsymbol{\nu}$ for the vector $(\nu_1, \ldots, \nu_g)$. When the flow on the Jacobian linearises, as occurs with the Lax pair for Nahm's equations, this map becomes

$$s \mapsto s \sum_j \mathrm{Res}_{\infty_j}(\lambda(0)\boldsymbol{\nu}) = s\boldsymbol{U}.$$

Now because the linearisation map is a function of $s$ only through the Lax matrix $L$, and by assumption $L$ depends on $s$ only through elliptic functions linear in $s$, we have that the linearisation map is well-defined on the complex torus $\mathcal{E}$. That is, if we vary $s$ by a period of the elliptic curve, then the image in $\mathrm{Jac}(\mathcal{C})$ differs by a period in this Jacobian. This would not be true in general for arbitrarily chosen $\boldsymbol{U}$, and means that that the image of the linearisation map $\mathbb{C}\boldsymbol{U}$ is a complex subtorus of the Jacobian, hence an abelian subvariety [BL04, Proposition 4.1.1].

We can thus not expect to find an expression for the Nahm data of the $C_2$ monopole that is meromorphic and given in terms of elliptic functions. Nevertheless, for completeness I shall now present Nahm's equations for curves of this form in a somewhat simplified form. In order to get the $C_2$-invariant matrices, it turns out that starting with only polynomials semi-invariant under $\zeta_0 \to -\zeta_0$ in $\mathbb{S}^{2r}$, $r = 1, 2, 3$ as the input to the method of §3.3.2 gives that the matrix equation 3.29 is inconsistent. This is not in contradiction to how we constructed the algorithm, as the map $\zeta_0 \to -\zeta_0$ comes from a reflection $R_3 \in \mathrm{O}(3) \setminus \mathrm{SO}(3)$. Instead one should start with the full basis of $\mathbb{HMM}$ (recall Equation 3.25), and then impose the $C_2$ symmetry and centring after the fact, similar to how we imposed the additional involution on the $C_3$-invariant Nahm matrices in §3.4.1. This gives Nahm matrices

$$\begin{aligned}
T_1 &= \begin{pmatrix} ig_1 & ig_2 & g_3 \\ ig_2 & ig_4 & g_5 \\ -g_3 & -g_5 & -i(g_1+g_4) \end{pmatrix},\\
T_2 &= \begin{pmatrix} ih_1 & ih_2 & h_3 \\ ih_2 & ih_4 & h_5 \\ -h_3 & -h_5 & -i(h_1+h_4) \end{pmatrix},\\
T_3 &= \begin{pmatrix} 0 & f_1 & if_2 \\ -f_1 & 0 & if_3 \\ if_2 & if_3 & 0 \end{pmatrix},
\end{aligned}$$

where the $f_i, g_i, h_i$ are real functions that can themselves be given in terms of real functions $x, y_i$. The ODEs the $f_i, g_i, h_i$, satisfy are

$$\begin{aligned}
f_1' &= g_2h_1 - g_1h_2 + g_4h_2 + g_5h_3 - g_2h_4 - g_3h_5,\\
f_2' &= -2g_3h_1 - g_5h_2 + 2g_1h_3 + g_4h_3 - g_3h_4 + g_2h_5,\\
f_3' &= -g_5h_1 - g_3h_2 + g_2h_3 - 2g_5h_4 + g_1h_5 + 2g_4h_5,\\
g_1' &= -2f_1h_2 + 2f_2h_3,\\
g_2' &= f_1h_1 + f_3h_3 - f_1h_4 + f_2h_5,\\
g_3' &= -2f_2h_1 - f_3h_2 - f_2h_4 - f_1h_5,\\
g_4' &= 2f_1h_2 + 2f_3h_5,\\
g_5' &= -f_3h_1 - f_2h_2 + f_1h_3 - 2f_3h_4,
\end{aligned}$$

and the $h_i$ ODEs being the same as the $g_i$ ODEs switching $h_i \leftrightarrow g_i$ and picking up an extra minus sign, that is if $g_i' = p(f,h)$, $h_i' = -p(f,g)$. Note that because of the symmetry of the matrices, the ODEs for $f_3, g_4, g_5$ are the same as those for $f_2, g_1, g_3$ respectively changing $f_1 \leftrightarrow -f_1$, $f_2 \leftrightarrow f_3$, $h_1 \leftrightarrow h_4$, and $h_3 \leftrightarrow h_5$. The ODEs for $g_2$ is necessarily invariant under this involution (for $f_1$ anti-invariant).

### 3.4.4 Further Investigations

In this section we have proven Theorem 3.4.1, and for those 3-monopole with reductions to elliptic curves for which the ES cycle pushes down appropriately, we have constructed the Nahm data explicitly. This work then has two natural extensions which should be properly investigated.

1. The higher-charge monopoles with reductions to elliptic curves should be classified, and for those where the ES cycle pushes down the Nahm data should be explicitly computed. Partial work in this direction was done in [Hou97], for example it was shown that no charge-$k$ monopole with $D_{k-1}$ generalising the $V_4$ symmetry described in §3.4.2 can quotient by this $D_{k-1}$ to an elliptic curve. At present the possible signatures for group actions with elliptic quotients in genus 9 and higher have not been computed in the LMFDB, and so a first step would be the tabulation of those results.

2. Spectral curves corresponding to hyperbolic monopoles live in the minitwistor space of hyperbolic space, which is isomorphic[24] to $\mathbb{P}^1 \times \mathbb{P}^1$, and specifically charge-k hyperbolic monopoles are bidegree-$(k,k)$ curves in this surface [Ati87]. As $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the nonsingular quadric in $\mathbb{P}^3$, and bidegree-$(3,3)$ curves in this correspond to the other class of non-hyperelliptic curves classified by Wiman, the work in this section has the potential to classifying certain hyperbolic monopole spectral curves. Work in this direction has already been completed in [NR07, Proposition 5.2], where the authors classify all hyperbolic monopoles with $G = A_4$, $S_4$, or $A_5$ rotational symmetry where the quotient by $G$ is an elliptic curve.

[24] One can see this (at least partially) by observing that any geodesic of hyperbolic space is determined by its endpoints, and that $\partial H^3 \cong \mathbb{P}^1$.

# Chapter 4

# Conclusion

> Finally: It was stated at the outset, that this system would not be here, and at once, perfected. You cannot but plainly see that I have kept my word. But now I leave my cetological System standing thus unfinished, even as the great Cathedral of Cologne was left, with the crane still standing upon the top of the uncompleted tower. For small erections may be finished by their first architects; grand ones, true ones, ever leave the copestone to posterity. God keep me from ever completing anything. This whole book is but a draught - nay, but the draught of a draught. Oh, Time, Strength, Cash, and Patience!
>
> – Herman Melville
> Moby Dick

As I stated at the beginning of this thesis, the unifying thread running throughout is the concept of symmetry. I shall conclude this thesis then with a brief recollection of how this symmetry manifested itself in each key section, and my estimations of which directions may prove ripe for further study.

In §2.2 the role of symmetry was in permuting the theta characteristics on the curve; this orbit decomposition reveals intrinsic geometry through its relation to the arrangement of tangent hyperplanes of the canonical embedding and the theta divisor. I highlighted multiple ways to compute this decomposition both theoretically and practically, including algorithmic considerations, and enumerated many examples for curves of small genus. The theory determining when there are invariant characteristics is in its nascent stages, I have developed this further and laid down sketches of the future form of results but leave much room for the proving of new theorems pinning down these structures and verifying conjectures.

In §2.3 we looked at a particular curve with many symmetries. This symmetry let us write down explicitly the period matrix, get geometric intuition for distinguished orbits, understand a rich structure of interleaving quotients, and analytically understand part of the orbit decomposition of theta characteristics. There remained unexplained 'coincidences' in this structure, namely isomorphisms between quotient elliptic curves and a 'threeness' of the orbit decomposition that we were unable to explain which further investigation may clarify. It is possible that these are related to the modular aspects of Bring's curve which were left unexplored in this thesis.

In §3.3 we were able to utilise the connection between magnetic monopoles and their various algebraic manifestations to constrain the possible monopole configurations with certain symmetries in O(3). This included constraining the dimension of the moduli space using ramification data of the action on spectral curves, and providing algorithms implemented in Sage for constructing Nahm data invariant under certain symmetries. There remain questions to be answered on both of these techniques in order to strengthen the existing results I have given. Moreover, I have not investigated what role the distinguished theta characteristics on monopole spectral curves play in providing insight into their geometry and as such the geometry of the monopole. It is likely that in such a study the real geometry of the curve and its Jacobian will be important.

Finally, §3.4 unified the mathematics of the rest of this thesis in classifying charge-3 monopoles with elliptic quotients. Using the known symmetry data I computed Nahm data and numerical visualisations for previously unknown monopoles, a task not achieved in 25 years. Pushing these results to both higher charge and to hyperbolic monopoles represents a natural continuation and one which will further intertwine mathematical physics and algebraic geometry.

# Appendix A

# Differences in Conventions

## A.1 The Norm on $\mathfrak{su}(2)$

As mentioned, the monopole boundary conditions (Definition 3.1.5) desired depend on the choice of norm used on $\mathfrak{su}(2)$. Here we collate a record of a variety of sources and the convention they use, in the form $|X|^2 = -\alpha\,\mathrm{Tr}(X^2)$, $|\phi| \sim 1-m/r$, denoting with $k$ the corresponding monopole charge.

| Source | $\alpha$ | $m$ |
|---|---|---|
| [AH88] | $1/2$ | $k/2$ |
| [BDH23] | $1/2$ | $k/2$ |
| [BE18] | $1/2$ | $k/2$ |
| [BE21] | $1/2$ | $k/2$ |
| [CG81] | $2$ | $k$ |
| [Cor82] | $2$ | $k$ |
| [Hit82] | $1/2$ | $k$ |
| [Hit83] | $1/2$ | $k$ |
| [Hit87] | $1/2$ | $k/2$ |
| [Hou97] | $1/2$ | $k/2$ |
| [HS96d] | $1/2$ | $k$ |
| [HS96a] | $1/2$ | $k/2$ |
| [HS96b] | $1/2$ | $k$ |
| [Hur83] | $1/2$ | $k/2$ |
| [Hur85a] | $1/2$ | $k/2$ |
| [JT80] | $2$ | $k$ |
| [MS04] | $1/2$ | $k/2$ |
| [PR81] | $2$ | $k$ |
| [Pra81] | $2$ | $k$ |
| [Stu94] | $1/2$ | $k/2$ |
| [War81a] | $1/2$ | $k/2$ |
| [War81c] | $1/2$ | $k/2$ |
| [WW91] | $1/2$ | $k$ |

Note here the papers [Hit82, Hit83, HS96d, HS96b, WW91] stand out for using the combination $(1/2, k)$.

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