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NON-LINEAR DYNAMICS OF A TEST PARTICLE NEAR THE LAGRANGE POINTS L_4 AND L_5 (EARTH-MOON AND SUN-EARTH CASE)

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Abstract

The two-bodies problem can be fully solved, and was solved by Kepler (1609) and Newton (1687). The general three-body problem is often given as an example of a mathematical problem that 'can't be solved'. So, there is no general analytical solution. This problem can be significant and a special case of this problem is the Circular Restricted Three-Body Problem (CRTBP), which can be applied to the Earth-Moon system with a spacecraft, the Sun-Earth system with an asteroid, etc. In this paper, let's focus on the motion of a test particle near the triangular Lagrange points L_4 and L_5 in the Earth-Moon and the Sun-Earth systems. Studying the movement of an object around these points is especially important for space mission design. To generate a trajectory around these points, the non-linear equations of motion for the circular restricted three-body problem were numerically integrated into MATLAB®2023 software and the results are presented in the plane (x, y) and the phase plane (x, v_x) and (y, v_y). By numerical orbit integration, it is possible to investigate what happens when the displacement is relatively large or short from the Lagrange points. Then the small astronomical body may vibrate around these points. The results in this paper are shown in the rotating and inertia axes. Various initial positions near the Lagrange points and velocities are used to produce various paths the test particle can take. The same examples of numerical studies of trajectories associated with Lagrange points are shown in the inertial and the rotating coordinates system and are discussed. From the results of the numerical tests performed in MATLAB®2023, it is possible to saw that there are different types of periodic, quasi-periodic, and chaotic orbits.

Keywords: Lagrange points, periodic orbit, phase plane, chaotic motion, trojan asteroids.

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1. Introduction

The general three-body problem has been studied by many mathematicians and physicists such as Euler, Lagrange, Laplace, Jacobi, Le Verrier, Hamilton, and so on, but the problem does not have a close general analytical solution [1, 2]. In 1890 it was proved that, unlike the two-body problem it is integrable, and thus its solutions are completely understood, the three-body problem is not integrable, and the trajectories of three-body systems depend sensitively on the initial conditions of the problem. In this problem, there are different types of motion such as periodic motion, quasi-periodic motion, and chaotic motion. Chaotic trajectories were discovered by Henri P., in 1890 and confirmed again by Lorenz in 1963 and Stone & Leigh in 2019 [2]. In 2017, Xiaoming Li, Yipeng Jing, and Shijun Liao presented 1349 families of Newtonian periodic planar three-body orbits with unequal mass and zero angular momentum and the initial conditions in this case of isosceles collinear configurations. Among these 1223 families are entirely new [146]. All the new orbits found are periodic [2, 3].

The general three-body problem is extremely complex. If the mass of one body is neglected concerning the masses of the other two bodies, the general three-body problem is called the restricted three-body problem. If two bodies move in coplanar circular orbits around their center of mass the problem is called the Circular Restricted Three Body Problem (CRTBP) [4]. In this paper let's consider CRTBP to study the motion of a test particle under the action of the gravitational field of two massive bodies such as the Earth-Moon system and the Earth-Sun system.

Euler and Lagrange have discovered five equilibrium points which are called Lagrange points, L_1, L_2, and L_3 (collinear equilibrium points), and L_4 and L_5 (triangular equilibrium points) [5]. From linear stability analysis, the Lagrange points L_4, and L_5 are stable for the Earth-Moon and Sun-Earth systems. The study of the movement around the Lagrange points in these systems is very important, especially for space mission projects and other applications in astronomical observation [6].

At the triangular Lagrange points L_4 and L_5 in the Earth-Moon system there are large concentrations of dust (Kordylewski clouds) [7]. These clouds were discovered to exist in 2018 [5, 8, 9]. After this discovery, several astronomers, physicists, and other scientists observed these dusts. Another interesting case for study is the case of Earth Trojans in the Earth-Sun system. In January 2010, a 300 m diameter asteroid (2010 TK7) was discovered using the WISE and (614689) 2020 XL5, another Earth trojan asteroid discovered on December 12, 2020. Trojan asteroids are small bodies orbiting around the L_4 or L_5 Lagrange points of a Sun-planet system [10].

Studies about this problem are of great importance. On the one hand, these studies are important in astronomy and spaceflight, and on the other hand, they extend the theory of dynamical systems. These findings are of interest to physicists and astronomers in general, but can also be used in the field of advanced engineering of astronomical observations. For example, the equilateral point, L_5, provides an ideal position to monitor the space weather and L_4 or L_5 equilateral Lagrange points in the Earth-Sun system or Earth-Moon system may harbor Earth Trojans and space dust that are of significant interest to the scientific community [11].

This paper aims to generate different trajectories around triangular Lagrange points for different initial conditions of the coordinate and velocity of a test particle. Let's perform this using a numerical method, performing numerical integration in the MATLAB®2023 software.

2. Materials and methods

The general three-body problem can be significant and a special case of this problem is the *Circular Restricted Three-Body Problem* (CRTBP) [12]. Euler (1767) was the first to formulate this problem in a rotating coordinate system [13]. In this paper, let's use the model of the CRTBP, which can be applied to the Earth-Moon system with a test «particle». This method can also be used in the Sun-Earth system with a «test particle» (an asteroid, a comet, or a spacecraft, etc.). In this problem, two massive bodies (or primaries) orbit around their center of mass in circular and coplanar orbits, with the same angular velocity [14]. Another small particle (for example a small astronomical body) moves into the plane defined by the two revolving primaries. The motions of the primary bodies are assumed to be unaffected by the particle [5, 15, 16].

To simplify the problem, let's use the non-dimensional system. The distance of primaries is a unit. The angular velocity is unity [12]. The non-dimensional period of the primaries T is 2π. The primaries are located at positions of coordinates, $(-\mu, 0, 0)$ and $(1-\mu, 0, 0)$, respectively. The quantity $\mu = m_2/(m_1+m_2)$ – the mass ratio of the primaries (mass parameter) [17, 18].

The relations between the accelerations, velocities, and coordinates are called the equations of motion. Equations of motions for the CRTBP in the non-dimensional form are:

$$a_x = 2v_y+\partial U/\partial x;\ a_y = -2v_x+\partial U/\partial y;\ a_z = \partial U/\partial z, \tag{1}$$

where a_x, a_y, and a_z are the acceleration of the test particle along the three directions x, y, and z, respectively and v_x, v_y, and v_z are the components of its velocity according to these directions [19].

The pseudo-potential function $U = U(x, y, z)$ is given by:

$$U = 1/2(x^2+y^2)+((1-\mu)/r_1+\mu/r_2)+1/2\mu(1-\mu), \tag{2}$$

where r_1 and r_2 are the distances from the primaries to the test particle, respectively [20, 21]. Jacobi's integral is given by [22]:

$$C = 2U-(v_x^2+v_y^2+v_z^2). \tag{3}$$

This quantity is conserved and is very important for the CRTBP [23, 24].

From analytical solutions of the CRTBP, Euler, and Lagrange have identified the existence of five equilibrium points, the so-called Lagrange points denoted L_i, ($i = 1, 2, 3, 4, 5$) [25]. The Lagrange points L_1, L_2, L_3, L_4, and L_5 of the CRTBP are stationary only in the rotating frame and are critical points of the function $U = U(x, y, z)$, where a third body will remain in equilibrium under the gravitational attraction of the other two massive bodies [7]. The study of the movement of objects around these points is of great importance for science and humanity, especially in recent times [14, 26]. The values of the location for Lagrange points L_1, L_2, and L_3 are determined numerically. For this, let's use Iterative Methods (Newton's method). Locations of the triangular Lagrange points L_4, and L_5 are calculated based on an exact solution. Let's perform the numerical computation of the trajectories with the program in MATLAB®2023 using integration options ode set ('RelTol',1e-11, 'AbsTol',1e-08) for the time step. MATLAB's ode45, Runge-Kutta-Fehlberg (RKF 45) solver is used. This is a method of order four with an error estimator of order fifth. This method is an algorithm that can be used to solve numerically the system of equations (1). The error in the solutions can be estimated and controlled by using a higher-order embedded method that allows for an adaptive step size to be determined automatically [13].

3. Results and discussions

In this paper, the motion of the test particle is modeled in an inertial system and a rotating system for Earth-Moon and Sun-Earth systems. For convenience, the problem is formulated in canonical or non-dimensional units. The motion of a test particle started from different initial conditions of coordinates and velocities.

Let's use MATLAB®2023 software to numerically integrate the differential equations for CRTBP and the results are shown in the phase plane.

The locations of the five Lagrange points for the Earth-Moon system are shown in **Fig. 1**. The L_1 point lies between the Earth and the Moon, the L_2 point lies outside the Moon, and the L_3 point lies on the negative *x-axis*. The triangular Lagrange points L_4 and L_5 are located at the corners of two equilateral triangles, with the other corners defined by the locations of the two primaries. It turns out that Lagrange points L_1, L_2, and L_3 on the apse line are unstable, whereas L_4, and L_5 – 60° ahead of and behind the Moon in its orbit – are stable.

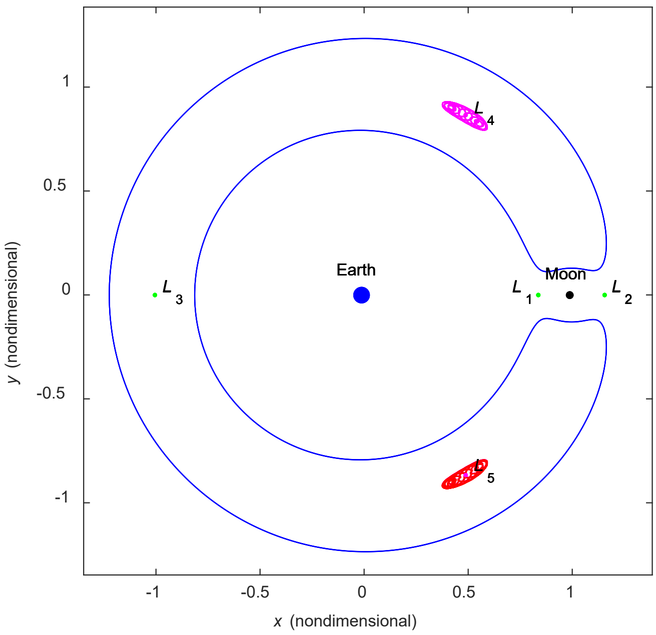

Fig. 1. The trajectories (tadpole orbits) of the test particle (purple around L_4 and red around L_5) in the rotating frame

The trajectories of the test particle (purple around L_4 and red around L_5) are shown in **Fig. 1** in the Earth-Moon system in the plane (x, y), with initial conditions: $x(0) = 0.4878$, $y(0) = \pm 0.863$, $z(0) = 0$, $v_x(0) = 0$, $v_y(0) = 0.004582530195192$, $v_z(0) = 0$ and $C = 3$. These trajectories are called tadpole orbits. The horseshoe orbit in blue is the zero-velocity curve for $C = 3.14$. Zero velocity curves are the geometric locations of points where the velocity and kinetic energy is zero. The test particle can move only within this «trajectory», where kinetic energy is positive while on the other side, it is negative and the movement of the test particle is not possible in this area.

From the numerical experiment, it is found that even in this case, the trajectories around the points L_4 and L_5 remain connected to these points all the time, so they are stable trajectories. It is also possible to present this result in the phase plane (x, v_x) or (y, v_y) as in **Fig. 2, 3** respectively. Moderately stringent tolerances are necessary to reproduce the qualitative behavior of the trajectories. Suitable values are 1e-11 for RelTol and 1e-11 for AbsTol. **Fig. 4–6** show the results of numerical tests for different initial conditions in the rotating system (trajectory in magenta) and the inertial system (trajectory in red). Initial conditions used in the **Fig. 4** are: $x(0) = 0.4878$, $y(0) = 0.8655555$, $v_x(0) = 0$, $v_y(0) = 7.472402584399912\times10^{-4}$, $v_z(0) = 0$ and period is 555 units of time.

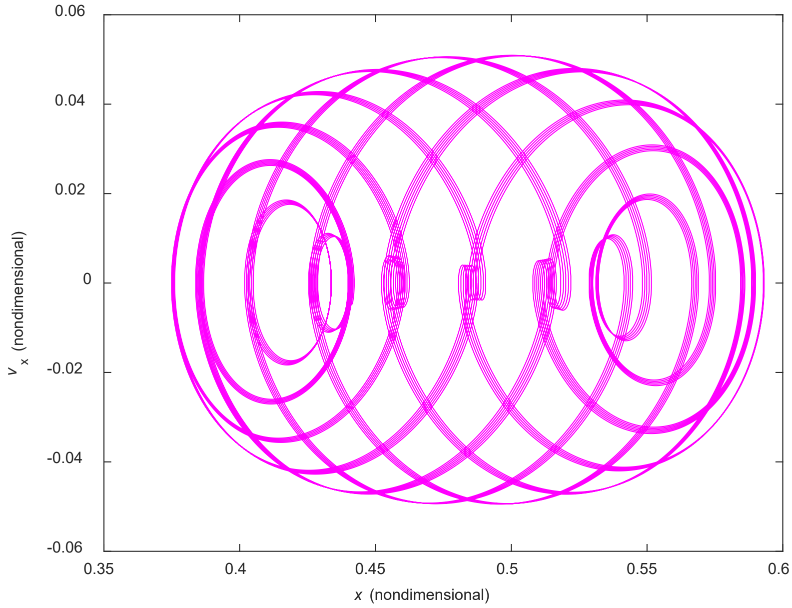

Fig. 2. The trajectories of the test particle in the Earth-Moon system in the phase plane (x, v_x) with initial conditions: $x(0) = 0.4878$, $y(0) = 0.863$, $z(0) = 0$, $v_x(0) = 0$, $v_y(0) = 0.004582530195192$, $v_z(0) = 0$

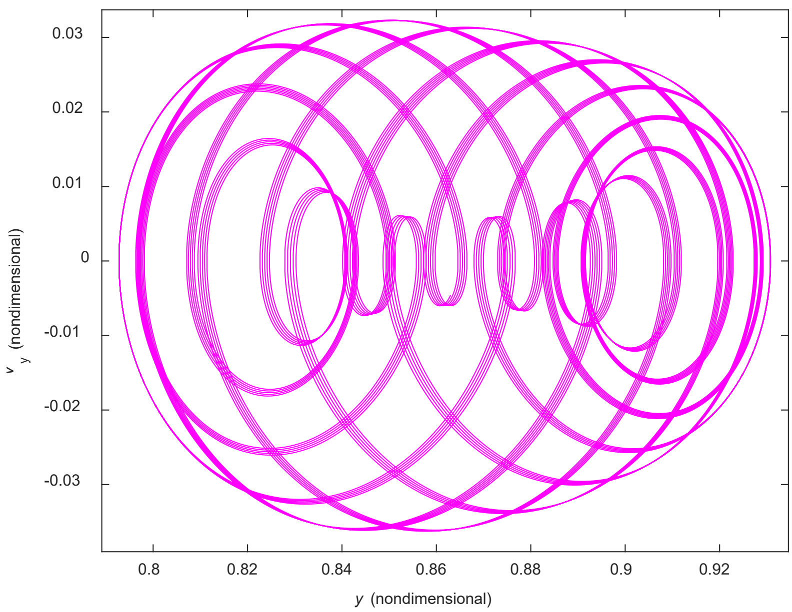

Fig. 3. The trajectories of the test particle in the Earth-Moon system in the phase plane (y, v_y) with initial conditions: $x(0) = 0.4878$, $y(0) = 0.863$, $z(0) = 0$, $v_x(0) = 0$, $v_y(0) = 0.004582530195192$, $v_z(0) = 0$

The result in **Fig. 4** shows that the orbit is stable and periodic and in the results in **Fig. 5, 6** there is a chaotic orbit in the rotating system and the inertial system. Let's notice that for a small change in the initial conditions for the coordinate, there is a significant change in the shape of the trajectory of the test particle.

Fig. 7 shows a quasi-periodic orbit of a test particle around the L_4 point in the Sun-Earth system rotating axis (left) and the inertia axis (right). Its period is approximately 365.0473 days.

Fig. 8 shows this trajectory (with the same initial conditions) for 31950.3548 days in the rotating axis.

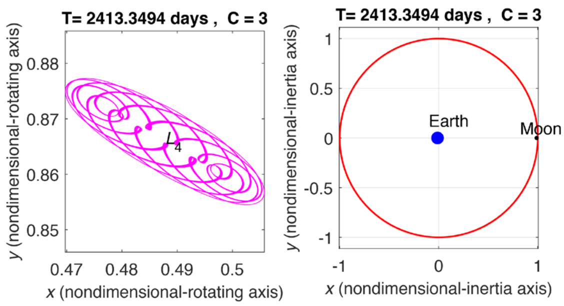

Fig. 4. Left is the trajectory of the test particle in the rotating frame and the right is the trajectory of the test particle in the reference frame

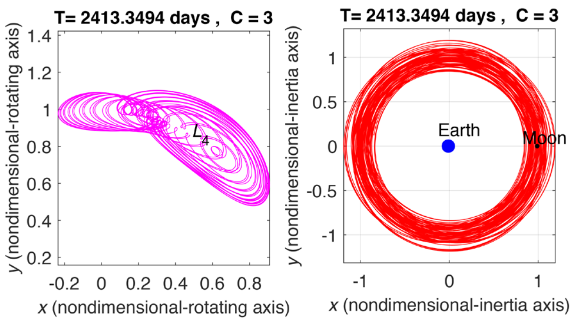

Fig. 5. Left is the trajectory of the test particle for $x(0) = 0.477$, $y(0) = 0.862$, $z(0) = 0$, $v_x(0) = 0$, $v_y(0) = 0.015343469818984$, $v_z(0) = 0$ in the rotating frame and the right is the trajectory of the test particle in the reference frame

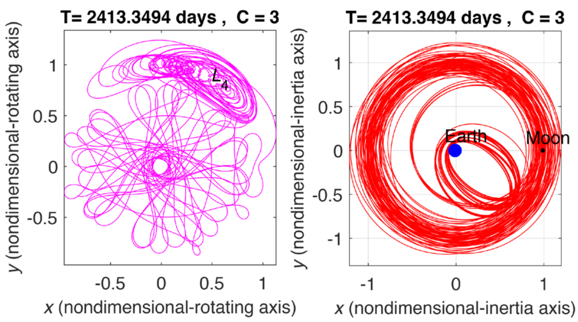

Fig. 6. Left is the chaotic trajectory of the test particle with initial conditions: $x(0) = 0.4878$, $y(0) = 0.89$, $z(0) = 0$, $v_x(0) = 0$, $v_y(0) = 0.035796673734005$, $v_z(0) = 0$ in the rotating frame and the right is the trajectory of the test particle in the reference frame

Fig. 9 shows a periodic orbit of a test particle around the L_4 point in the Sun-Earth system rotating axis (left) and inertia axis (right). Jacobi integral is 2.96. **Fig. 10** shows this trajectory (with the same initial conditions) with a larger time of integration than the previous one, approximately 11618.3108 days in the rotating axis.

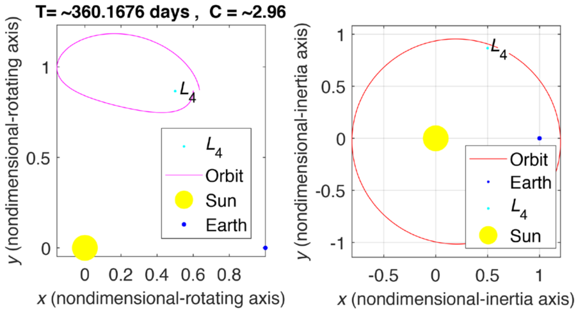

Fig. 7. Left is a trajectory of a test particle for $x(0) = 0.6$, $y(0) = 0.869$, $z(0) = 0$, $v_x(0) = 0$, $v_y(0) = 0.2215$, $v_z(0) = 0$ in the rotating frame in the Sun-Earth system, and on the right is this trajectory in the inertial frame

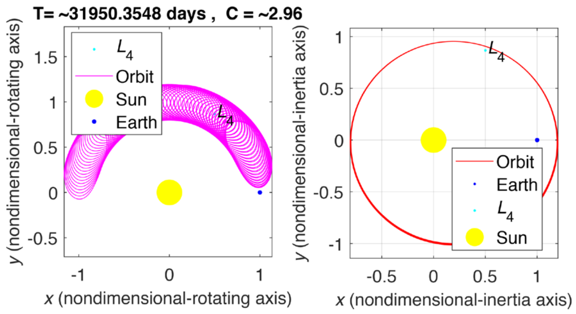

Fig. 8. A trajectory of a test particle with the same initial conditions in **Fig. 7** but the time of integration is much bigger, approximately 31950.3548 days

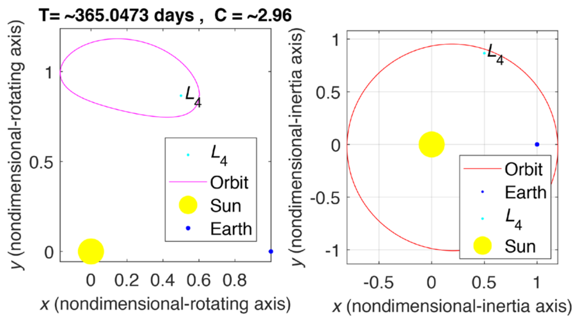

Fig. 9. Left is a trajectory of a test particle with initial conditions: $x(0) = 0.6$, $y(0) = 0.875$, $z(0) = 0$, $v_x(0) = 0$, $v_y(0) = 0.2207$, $v_z(0) = 0$, and the right is this trajectory in the inertial reference frame

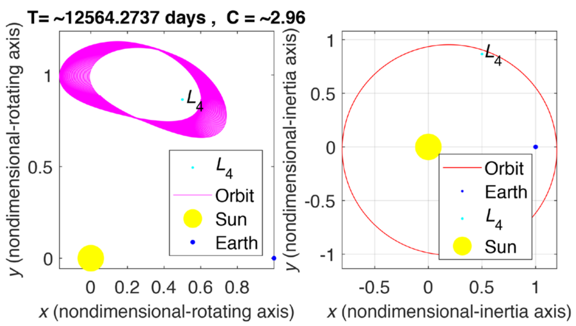

Fig. 10. Left is a trajectory of a test particle with the same initial conditions as the result in **Fig. 9** and the time of integration is approximately 11618.3108 days

Since in this work, a gravitational method is considered based on Newton's gravity, then these results are limited to the high speed of the test point (third body) movement. These results are reproducible and is very important to know the model used in this paper and to know any numerical methods to solve the equations of motion, which do not have a general analytical solution for this problem that it is used in this study. These results can be applied in Astronomy and advanced engineering of astronomical observations. In the future, this study can be applied in the direction of the study of different systems (for example Sun-Jupiter system or Sun-Mars system, etc.) in solar systems and different exoplanetary systems.

4. Conclusions

From many numerical tests, it is possible to conclude that the trajectories around the Lagrange triangular or equilateral points, L_4 and L_5 remain connected to this point all the time except for the result in which the trajectory of a test particle is chaotic. For different initial conditions, there are different types of motion. Such as periodic motion, quasi-periodic motion, and chaotic motion. It is discovered by many numerical experiments in MATLAB®2023. From one result in this paper, it is possible to conclude that: there is non-periodical behavior for a long time. From this result, it is possible to say that the examined system is a deterministic system that shows sensitivity to the initial conditions.

Conflict of interest

The authors declare that they have no conflict of interest about this research, whether financial, personal, authorship, or otherwise, that could affect the research and its results presented in this paper.

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The study was performed without financial support.

Data availability

Data will be made available on reasonable request.

Use of artificial intelligence

The authors confirm that they did not use artificial intelligence technologies when creating the current work.

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