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Manal H. Alabdulhadi
Department of Mathematics, College of Science and Arts, Qassim University, Bukairiayh 51452, Saudi Arabia, m_elgarhy85@sva.edu.eg

Ahmed R. El-Saeed
Department of Basic Sciences, Obour High Institute for Management & Informatics, Egypt, m_elgarhy85@sva.edu.eg

Mohammed Elgarhy
Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni-Suef 62521, Egypt\\ Department of Basic Sciences, Higher Institute of Administrative Sciences, Belbeis, AlSharkia, Egypt, m_elgarhy85@sva.edu.eg

Abd El-Hamid Eisa
Department of Statistics, Faculty of Commerce, Al-Azhar University, Cairo, Egypt, m_elgarhy85@sva.edu.eg

Doaa A. Abdo
Department of Applied Statistics and Insurance, Faculty of Commerce, Mansoura University, Mansoura, Egypt, m_elgarhy85@sva.edu.eg

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Statistical Inference of Marshall-Olkin Extended Exponential Distribution Based on Progressively Type-I Censored Data

Manal H. Alabdulhadi[1], *Ahmed R. El-Saeed*[2], *Mohammed Elgarhy*[3,4,*], *Abd El-Hamid Eisa*[5] *and Doaa A. Abdo*[6]

[1]Department of Mathematics, College of Science and Arts, Qassim University, Bukairiayh 51452, Saudi Arabia
[2]Department of Basic Sciences, Obour High Institute for Management & Informatics, Egypt
[3]Mathematics and Computer Science Department, Faculty of Science, Beni-Suef University, Beni-Suef 62521, Egypt
[4]Department of Basic Sciences, Higher Institute of Administrative Sciences, Belbeis, AlSharkia, Egypt
[5]Department of Statistics, Faculty of Commerce, Al-Azhar University, Cairo, Egypt
[6]Department of Applied Statistics and Insurance, Faculty of Commerce, Mansoura University, Mansoura, Egypt

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Abstract: In this article, we intend to study the progressive Type-I censoring (PT-TC) that has been examined, employing the Marshall-Olkin extended exponential (MOEE) distribution as the fundamental lifetime distribution. The censoring technique is believed to be independent and non-informative. Because maximum likelihood (ML) estimators cannot be derived in closed form, ML estimates (MLEs) are calculated via Newton-Raphson method approaches. In this approach, MLEs and asymptotic confidence intervals for unknown parameters are produced. Under squared error and linear exponential (LINEX) loss functions, the Bayes estimations of unknown parameters with gamma priors are evaluated. Once both parameters are unknown, the Bayes estimators cannot be computed explicitly. Then, the Markov Chain Monte Carlo (MCMC) technique is employed to construct Bayes estimates using the Metropolis-Hasting (MH) algorithm. The highest posterior density (HPD) credible intervals of the unknown parameter are calculated. Simulation studies are carried out to explore the finite sample effectiveness of the recommended estimators, as well as data set analyses at various schemes of PT-TC samples.

Keywords: Bayesian estimation, Marshall-Olkin extended exponential distribution, progressive Type-I censoring scheme, maximum likelihood estimation, Markov Chain Monte Carlo.

1 Introduction

The use of a progressive censoring scheme in life testing and reliability research has received a significant amount of emphasis in the literature. In several life testing tests, the experimenter is necessitated to remove several live units at a set period in the experiment, or maybe some units may be unexpectedly lost from the experiment. Examples of this kind are common in medical investigations, as well as in numerous industrial and agricultural operations. Typically, units are removed from the experiment as part of such investigations in order to reduce the time and costs connected with testing. Ref. [1] originally described progressive censoring as a valuable strategy for supplying inferential findings to data derived from such studies. As a censoring strategy, we considered PT-IC samples in this research. In summary, this strategy is seen when a pre-determined number of life test units are constantly withdrawn from the experiment at the conclusion of each of the pre-specified time periods. It gives the practical characteristic of knowing the termination time as well as greater flexibility to the experimenter during the design phase by allowing the test units to be eliminated at non-terminal time points [2].

Assume n units are subjected to a life-testing experiment. Let $X_1, X_2, \ldots, X_n$ represent the lifespan of these n units selected from a population with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Suppose $x_{(1)} < x_{(2)} < \ldots < x_{(n)}$ represents the ordered lifetimes recorded from the life test. When R_i items are deleted the surviving items at the

* Corresponding author e-mail: m_elgarhy85@sva.edu.eg

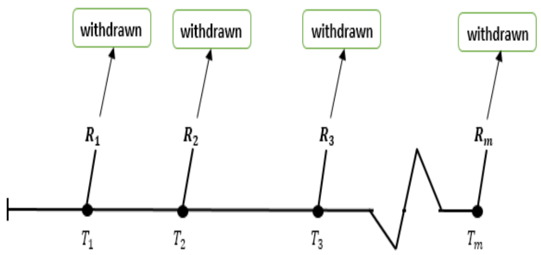

Fig. 1: PT-IC Scheme

preset time of censoring T_{q_i} corresponding to the q_ith quantiles, $i = 1,2,\ldots,m$, where m is the number of levels in the test, $T_{q_i} > T_{q_{i-1}}$ and $n = r + \sum_{i=1}^{m} R_i$, PT-IC is noticed. The likelihood function under PT-IC is provided via

$$L(\theta) \propto \prod_{i=1}^{r} f(x_{(i)};\theta) \prod_{j=1}^{m} \left(1 - F(T_{(q_j)};\theta)\right)^{R_j} \qquad (1)$$

where $x_{(i)}$ is the observed lifespan of the ith order statistic (OS) [1]. Figure 1 depicts this censoring technique [3]. Thus, Type-I censoring and complete samples can be considered special cases of PT-IC. For more details and implementation of this kind of censoring, one may mention [2,3,4].

Recently, various works for the PT-IC scheme have been investigated such as: Ref. [5] derived the MLEs for the parameters of the generalized inverted exponential (GIE) model. Ref. [6] studied the statistical inference of the inverse Weibull distribution. For the Nadaraj-Haghighi distribution, Ref. [7] studied the MLEs and Bayesian estimates (BEs) in the presence of competing risks model under PT-IC for the GIE distribution. For the Weibull distribution, [4] proposed a competing risks model under PT-IC.

Ref. [8] proposed a way of inserting a positive parameter to broaden a family of distributions for greater flexibility or to build co-variate models. They demonstrated that the approach has a stability quality and that the distribution family is geometric and exceptionally stable. They investigated extended exponential and Weibull distributions. According to Marshall-Olkin, if $\bar{F}(x) = 1 - F(x)$ denotes the reliability function (RF) of a continuous random variable X, then the resulting distribution is described in the form of RE function is provided by

$$\bar{F}(x) = \frac{\alpha \bar{G}(x)}{1 - \bar{\alpha}\bar{G}(x)}; \qquad -\infty < x < \infty, \quad \alpha > 0, \qquad (2)$$

where $\bar{\alpha} = 1 - \alpha$. If we take the RF of the exponential (E) distribution which is given as $\bar{G}(x) = \exp(-\lambda x)$ in equation (2), we get the MOEE distribution with RF given as:

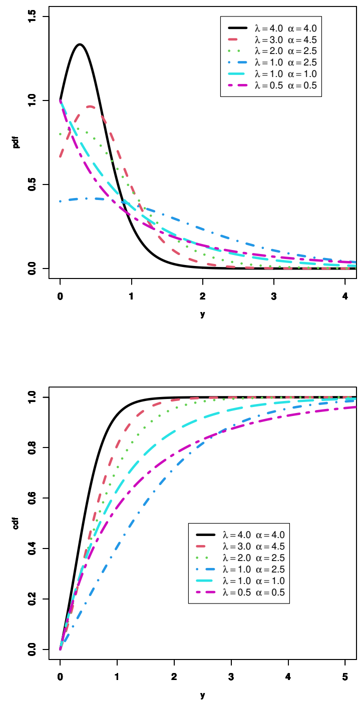

Fig. 2: Pdf and cdf of MOEE distribution for some values of α and λ

$$\bar{F}(x;\alpha,\lambda) = \frac{\alpha}{\exp(\lambda x) - \bar{\alpha}}; \quad x > 0, \quad \alpha,\lambda > 0, \qquad (3)$$

thus the pdf and cdf of the MOEE distribution are provided via

$$f(x;\alpha,\lambda) = \frac{\alpha\lambda \exp(-\lambda x)}{[1 - \bar{\alpha}\exp(-\lambda x)]^2}; \qquad x > 0, \quad \alpha,\lambda > 0 \qquad (4)$$

and

$$F(x;\alpha,\lambda) = \frac{1 - \exp(-\lambda x)}{1 - \bar{\alpha}\exp(-\lambda x)} \qquad (5)$$

where α represents the shape parameter and λ is the scale parameter. If $\alpha = 1$ reduces to the E distribution [9]. Figure 2 illustrated the behavior of the pdf and cdf for the MOEE distribution at some various values of α and λ.

The hazard rate function (hrf) of the MOEE distribution is provided via

$$h(x;\alpha,\lambda) = \frac{\lambda}{1 - \bar{\alpha}\exp(\lambda x)}; \quad x \geq 0.$$

The behavior of the hrf and RF for the MOEE distribution are illustrated in Figure 3 at some various values of α and λ.

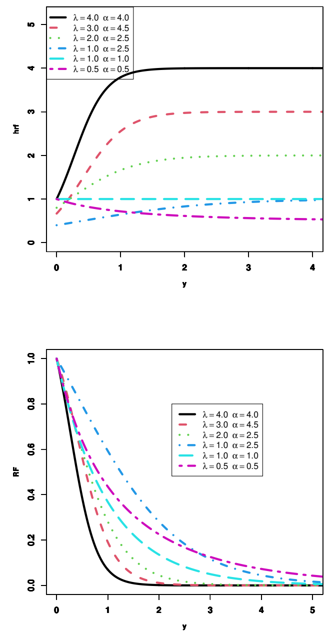

Fig. 3: Hazard rate function and the RF of MOEE distribution for some values of α and λ

Several authors studied the characteristics and properties of the MOEE distribution. For example, Ref. [10] studied the BE under squared error loss function and both informative and non-informative priors. Ref. [11] used the different estimations utilized as, ML, maximum product spacing approach, least square, and weighted least square approaches to assess the estimation of the parameters, reliability function and hazard function. Reliability test plans have been studied for MOEE distribution by [9]. Ref. [12] studied progressive Type-II censoring scheme. They studied the estimation and prediction problems for the MLEs and BEs.

The main goal of this article, is to estimate the unknown parameters for the MOEE distribution under PT-IC employing both a classical and a Bayesian point and interval estimation. We have organized the rest of the paper as follows. Supposing that the lifetime of the test units are independently MOEE distributed and using PT-IC, the MLEs of unknown parameters are discussed in Section 2. Construction of the asymptotic confidence intervals are also demonstrated in this section. In Section 3, BEs and associated highest posterior density interval estimates are obtained with respect to two different loss functions; namely squared error and LINEX loss functions. We have applied the MCMC method and utilized Metropolis-Hasting algorithm to evaluate these BEs. In section 4, a simulation study has been performed for comparison purposes using Monte Carlo simulations, and real-life data is analyzed to illustrate the proposed estimation methods. Finally, a conclusion is given in section 5.

2 Maximum Likelihood Approach

In this part, we use PT-IC data to calculate MLEs for the unknown parameters of the MOEE distribution. As a result, we may get the PT-IC samples $\mathbf{x} = (x_{(1)}, x_{(2)}, \ldots, x_{(r)})$ that reflect the observed lives of the n units under this censoring strategy. Given the observed data $\mathbf{x}$, the related probability function of α and λ may be expressed as

$$L(\alpha,\lambda)\& \propto \prod_{i=1}^{r}\left(\frac{\alpha\lambda\exp\left(-\lambda x_{(i)}\right)}{\left[1-\bar{\alpha}\exp\left(-\lambda x_{(i)}\right)\right]^{2}}\right) \prod_{j=1}^{m}\left(\frac{\alpha\exp\left(-\lambda T_{q_j}\right)}{1-\bar{\alpha}\exp\left(-\lambda T_{q_j}\right)}\right)^{R_j}. \tag{6}$$

By applying the logarithm of $L(\alpha,\lambda)$ to obtain log-likelihood $\ln L$ which is represented as ℓ

$$\ell(\alpha,\lambda) \propto r\ln(\alpha)+r\ln(\lambda) + \sum_{i=1}^{r}\left(-\lambda x_{(i)}-2\ln\left[1-\bar{\alpha}\exp(-\lambda x_{(i)})\right]\right) + \sum_{j=1}^{m}R_j\left(\ln(\alpha)-\lambda T_{q_j}-\ln\left[1-\bar{\alpha}\exp(-\lambda T_{q_j})\right]\right). \tag{7}$$

The first partial derivatives of ℓ with regard to α and λ are:

$$\frac{\partial\ell}{\partial\alpha} = \frac{r}{\alpha}+\sum_{i=1}^{r}\left(\frac{2\alpha\exp\left(-\lambda x_{(i)}\right)}{1-\bar{\alpha}\exp\left(-\lambda x_{(i)}\right)}\right) + \sum_{j=1}^{m}R_j\left(\frac{1}{\alpha}-\frac{\exp\left(-\lambda T_{q_j}\right)}{1-\bar{\alpha}\exp\left(-\lambda T_{q_j}\right)}\right) = \frac{r}{\alpha}+\sum_{i=1}^{r}\left(\frac{2\alpha}{\exp\left(\lambda x_{(i)}\right)-\bar{\alpha}}\right)+ \sum_{j=1}^{m}R_j\left(\frac{1}{\alpha}-\frac{1}{\exp\left(\lambda T_{q_j}\right)-\bar{\alpha}}\right)$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{r}{\lambda}+\sum_{i=1}^{r}\left(-x_{(i)}+\frac{2\bar{\alpha}x_{(i)}\exp\left(-\lambda x_{(i)}\right)}{1-\bar{\alpha}\exp\left(-\lambda x_{(i)}\right)}\right) + \sum_{j=1}^{m}R_j\left(-T_{q_j}-\frac{T_{q_j}\bar{\alpha}\exp\left(-\lambda T_{q_j}\right)}{1-\bar{\alpha}\exp\left(-\lambda T_{q_j}\right)}\right) = \frac{r}{\lambda}+\sum_{i=1}^{r}\left(-x_{(i)}+\frac{2\bar{\alpha}x_{(i)}}{\exp\left(\lambda x_{(i)}\right)-\bar{\alpha}}\right) + \sum_{j=1}^{m}R_j\left(-T_{q_j}-\frac{T_{q_j}\bar{\alpha}}{\exp\left(\lambda T_{q_j}\right)-\bar{\alpha}}\right)$$

Equating $\frac{\partial \ell}{\partial \alpha}|_{\alpha=\hat{\alpha}}$ and $\frac{\partial \ell}{\partial \lambda}|_{\lambda=\hat{\lambda}}$ to zero as follows:

$$\frac{r}{\hat{\alpha}}+\sum_{i=1}^{r}\left(\frac{2\hat{\alpha}}{\exp(\hat{\lambda}x_{(i)})-\hat{\alpha}}\right)+\sum_{j=1}^{m}R_j\left(\frac{1}{\hat{\alpha}}-\frac{1}{\exp(\hat{\lambda}T_{q_j})-\hat{\alpha}}\right)=0 \quad (8)$$

$$\frac{r}{\hat{\lambda}}+\sum_{i=1}^{r}\left(-x_{(i)}+\frac{2\hat{\hat{\alpha}}x_{(i)}}{\exp(\hat{\lambda}x_{(i)})-\hat{\alpha}}\right)+\sum_{j=1}^{m}R_j\left(-T_{q_j}-\frac{T_{q_j}\hat{\hat{\alpha}}}{\exp(\hat{\lambda}T_{q_j})-\hat{\alpha}}\right)=0 \quad (9)$$

The above two equations (8) and (9) cannot be obtained numerically. To acquire the appropriate MLEs for the above equations, we must use a suitable numerical approach, such as Newton-Raphson. The MLEs of α and λ are the numerical solutions of the preceding system of equations for $\hat{\lambda}$ and $\hat{\alpha}$, respectively.

The asymptotic properties of MLEs imply that the pair $(\hat{\alpha},\hat{\lambda})$ is approximately distributed as a bivariate normal random variable with a mean of (α,λ) and a variance-covariance matrix of $I_X^{-1}(\hat{\alpha},\hat{\lambda})$. Here, $I_X(\cdot)$ represents the Fisher information matrix. The individual elements of the Fisher information matrix are calculated as follows:

$$I_X(\alpha,\lambda)=\begin{bmatrix}-E\left(\frac{\partial^2 \ln\ell}{\partial\alpha^2}\right) & -E\left(\frac{\partial^2 \ln\ell}{\partial\alpha\partial\lambda}\right)\\ -E\left(\frac{\partial^2 \ln\ell}{\partial\lambda\partial\alpha}\right) & -E\left(\frac{\partial^2 \ln\ell}{\partial\lambda^2}\right)\end{bmatrix}$$

where

$$\frac{\partial^2\ell}{\partial\alpha^2}=\frac{-r}{\alpha^2}+\sum_{i=1}^{r}\left(\frac{2(\exp(\lambda x_{(i)})-\bar{\alpha}-\alpha)}{[\exp(\lambda x_{(i)})-\bar{\alpha}]^2}\right)-\sum_{j=1}^{m}R_j\left(\frac{1}{\alpha^2}+\frac{1}{[\exp(\lambda T_{q_j})-\bar{\alpha}]^2}\right),$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=\frac{-r}{\lambda^2}+\sum_{i=1}^{r}\left(\frac{2\bar{\alpha}x_{(i)}^2\exp(\lambda x_{(i)})}{[\exp(\lambda x_{(i)})-\bar{\alpha}]^2}\right)+\sum_{j=1}^{m}R_j\left(\frac{T_{q_j}^2\bar{\alpha}\exp(\lambda T_{q_j})}{[\exp(\lambda T_{q_j})-\bar{\alpha}]^2}\right),$$

and

$$\frac{\partial^2\ln\ell}{\partial\alpha\partial\lambda}=\sum_{i=1}^{r}\left(\frac{-2x_{(i)}(\exp(\lambda x_{(i)})-\bar{\alpha})-2\bar{\alpha}x_{(i)}}{[\exp(\lambda x_{(i)})-\bar{\alpha}]^2}\right)-\sum_{j=1}^{m}R_j\left(\frac{(\exp(\lambda T_{q_j})-\bar{\alpha})T_{q_j}-T_{q_j}\bar{\alpha}}{[\exp(\lambda T_{q_j})-\bar{\alpha}]^2}\right).$$

Consequently, the pivotal quantities $\frac{\hat{\alpha}-\alpha}{\sqrt{\sigma_{11}}}$ and $\frac{\hat{\lambda}-\lambda}{\sqrt{\sigma_{22}}}$ are approximately distributed as standard normal. Therefore, $100(1-\tau)\%$ approximate confidence intervals of α and λ are then obtained as $\hat{\alpha}\pm Z_{\tau/2}\sqrt{\sigma_{11}}$ and $\hat{\lambda}\pm Z_{\tau/2}\sqrt{\sigma_{22}}$ where $Z_{\tau/2}$ is the $(\tau/2)^{th}$ upper percentile of the standard normal distribution. Finally, the corresponding coverage probabilities (CP),

$$CP_\alpha=P\left[\left|\frac{\hat{\alpha}-\alpha}{\sqrt{\sigma_{11}}}\right|\le Z_{\tau/2}\right],\quad CP_\lambda=P\left[\left|\frac{\hat{\lambda}-\lambda}{\sqrt{\sigma_{22}}}\right|\le Z_{\tau/2}\right]$$

can be computed using the Monte Carlo simulations.

3 Bayesian Approach

In this part, we will look at the Bayesian estimate of the MOEE distribution's unknown parameters using the PT-IC technique. To get BEs of the parameters α and λ for Bayesian parameter estimation, the square error (SE) and LINEX loss functions are used. Consequently, the BEs cannot always be articulated explicitly. As a result, estimated BEs are generated employing numerical techniques under informative prior.

Assume that each unknown parameter is stochastically independent. Assume that the prior density for the parameter α, are a gamma $(\mu_1,1)$ and the parameter λ, the prior distribution is taken to be a gamma (μ_2,ν_2). As a result, the joint prior distribution for α and λ is provided via

$$\pi(\alpha,\lambda)=\pi_1(\alpha)\pi_2(\lambda)$$

$$\pi(\alpha,\lambda)=\alpha^{\mu_1-1}\lambda^{\mu_2-1}\exp\left(-\left\{\frac{\alpha}{\nu_1}+\frac{\lambda}{\nu_2}\right\}\right)$$

where the hyper-parameters μ_1,ν_1,μ_2,ν_2 are chosen to represent previous information of the unknown parameters. To choose the values of the hyper-parameters μ_1,ν_1,μ_2,ν_2 we use the method of hyper-parameter elicitation proposed by [16].
Given the observed data $\mathbf{x}=(x_{(1)},x_{(2)},\ldots,x_{(r)})$, the associated posterior density (PD) may be represented as

$$\pi(\alpha,\lambda|\mathbf{x})=\frac{\pi(\alpha,\lambda)L(\alpha,\lambda)}{\int_0^\infty\int_0^\infty\pi(\alpha,\lambda)L(\alpha,\lambda)d\lambda d\alpha}.$$

The PD function may be described as follows:

$$\pi(\alpha,\lambda|\mathbf{x})=K^{-1}\left[\alpha^{\mu_1-1}\lambda^{\mu_2-1}\exp\left(-\left\{\frac{\alpha}{\nu_1}+\frac{\lambda}{\nu_2}\right\}\right)\times\prod_{i=1}^{r}\left(\frac{\alpha\lambda\exp(-\lambda x_{(i)})}{[1-\bar{\alpha}\exp(-\lambda x_{(i)})]^2}\right)\times\prod_{j=1}^{m}\left(\frac{\alpha\exp(-\lambda T_{q_j})}{1-\bar{\alpha}\exp(-\lambda T_{q_j})}\right)^{R_j}\right].$$

where k is a normalize constant. As a result, the PD may be expressed as

$$\pi(\alpha,\lambda|\mathbf{x}) \propto \alpha^{r+\mu_1-1}\lambda^{r+\mu_2-1}\exp\left(-\left\{\frac{\alpha}{\nu_1}+\frac{\lambda}{\nu_2}\right\}\right) \times \prod_{i=1}^{r}\left(\frac{\exp\left(-\lambda x_{(i)}\right)}{\left[1-\bar{\alpha}\exp\left(-\lambda x_{(i)}\right)\right]^2}\right) \times \prod_{j=1}^{m}\left(\frac{\alpha}{\exp\left(\lambda T_{q_j}\right)-\bar{\alpha}}\right)^{R_j}. \quad (10)$$

Under the SE loss function, the BEs of any function, namely $g(\alpha,\lambda)$ is provided via

$$\tilde{g}_{BS} = E(g(\alpha,\lambda|\mathbf{x})) = \int_0^\infty\int_0^\infty g(\alpha,\lambda)\pi(\alpha,\lambda|\mathbf{x})d\lambda d\alpha. \quad (11)$$

Under the LINEX loss function, the BEs of any function, namely $g(\alpha,\lambda)$ is provided via

$$\tilde{g}_{BL} = E\left(\exp\left(-cg(\alpha,\lambda)\right) \mid \mathbf{x}\right) = -\frac{1}{c}\ln\left[\int_0^\infty\int_0^\infty \exp\left(-cg(\alpha,\lambda)\right)\pi(\alpha,\lambda\mid\mathbf{x})d\boldsymbol{\alpha}d\boldsymbol{\lambda}\right] \quad (12)$$

It can be observed that the estimates provided by (11) and (12) cannot be reduced into closed-form expressions. As a result, we use the most commonly used approximation MCMC to generate the necessary estimations.

The stages of the MH method to draw a sample from the PD provided equation (11) are as described in the following:

The first stage: Establish the initial value of θ as $\theta^{(0)} = (\hat{\alpha},\hat{\lambda})$.

The second stage: For $i = 1,2,\ldots,M$ repeat the next stages:

1. Let $\theta = \theta^{(i-1)}$.
2. Generate a new candidate parameter value δ from $N_2(\ln\theta, S_\theta)$.
3. Set $\theta' = \exp(\delta)$.
4. Compute $\beta = \frac{\pi(\theta'|x)}{\pi(\theta|x)}$, where $\pi(\cdot)$ is the posterior distribution in equation (**??**).
5. Generate a sample u from the uniform $U(0,1)$ distribution.
6. Accept or reject the new candidate θ':

$$\begin{cases} \text{If } u \le \beta & \text{set } \theta^{(i)} = \theta' \\ \text{Otherwise} & \text{set } \theta^{(i)} = \theta. \end{cases}$$

Furthermore, part of the initial samples chosen from the posterior density can be removed (burn-in), and the remaining samples can be used to generate BEs using the loss functions SE and LINEX. The equation (11) can be approximated more precisely as

$$\tilde{g}_{SE}(\alpha,\lambda) = \frac{1}{M-l_B}\sum_{i=l_B}^{M} g(\alpha_i,\lambda_i), \quad (13)$$

$$\tilde{g}_{L}(\alpha,\lambda) = \frac{-1}{c}\ln\left(\frac{1}{M-l_B}\sum_{i=l_B}^{M}\exp\left(-cg(\alpha_i,\lambda_i)\right)\right) \quad (14)$$

where l_B represents the number of burn-in samples.

4 Numerical Outcomes

The purpose of this section is to compare the performance of the various estimating methods outlined in previous sections. For illustrative purposes, we investigate a real data set; moreover, a simulation study is used to evaluate the behavior of the suggested approaches as well as to assess the statistical performances of the estimators under the PT-IC scheme. For calculations, we utilized R, a statistical programming language. In addition, the *bbmle* and *HDInterval* packages may be used to compute MLEs and HPD intervals in R-language.

4.1 Real data analysis

In this part, we examine a real-world data set provided by [20]. The original data set consists of 16 observations and shows the failure times of software releases in hours, with an average lifetime of 1000 hours from the start of program execution. The data is as described in the following:

0.519 0.968 1.430 1.893 2.490 3.058 3.625 4.442
5.218 5.823 6.539 7.083 7.485 7.846 8.205 8.564

Ref. [10], verified that the MOEE distribution provides a good fit for the given data set. The calculated Kolmogorov-Smirnov (K-S) distance between the empirical and the fitted extended for the MOEE distribution was 0.1753 and its p-value is 0.647 where $\hat{\alpha} = 2.689$ and $\hat{\lambda} = 0.3614$.

From the original data, six PT-IC schemes are generated with different m stages and removed items R_j at CT T_j, where $j = 1,2$, lifetime. These various schemes are mentioned in Table 1. Note that: $R_m = n - (\sum_{j=1}^{m-1} R_j + r)$ and r is the amount of failure items. Also, when $T_m = \max(x)$ and $R_1 = R_2 = \ldots = R_m = 0$. Type-I censoring scheme, Scheme 6, may be seen as a special case of PT-IC, and complete sampling can be regarded as a particular case of PT-IC.

We compute the MLEs of the parameters α and λ and their associated 95 % asymptotic confidence interval estimates. We also compute BEs employing the MH

Table 1: Different schemes for progressively Type-I censored samples

Scheme	m	CT (T_j)	Removed items (R_i)
1	4	(1, 2, 4, 7)	(3, 0, 0 ,R_m)
2	4	(1, 2, 4, 7)	(1, 1, 1, R_m)
3	4	(1, 2, 4, 7)	(0, 0, 0, 3, R_m)
4	5	(1, 1.5, 3, 5, 7)	(3, 0, 0, 0, R_m)
5	5	(1, 1.5, 3, 5, 7)	(0, 0, 0, 3, R_m)
6	5	(1, 1.5, 3, 5, 7)	(0, 0, 0, 0, $n-r$)
7		(1, 1.5, 3, 5, 8.564)	(0,0,0,0,0)

algorithm under the non-informative prior where $\mu_1 = \nu_1 = \mu_2 = \nu_2 = 0$. It is said that while using the MH method to generate samples from the posterior distribution, actual values of (λ, α) are considered as $(\lambda^{(0)}, \alpha^{(0)}) = (\hat{\lambda}, \hat{\alpha})$, where $\hat{\lambda}$ and $\hat{\alpha}$ are the MLEs of the parameters α and λ respectively. Thus, we considered the variance-covariance matrix S_θ of $(\ln(\hat{\lambda}), \ln(\hat{\alpha}))$, that can be easily investigated employing the delta method. Furthermore, we removed 2000 burn-in samples from the total 10000 samples generated by the posterior density and calculated BEs and HPD interval estimations using [14].

All the estimated values of MLEs and associated interval estimates (Asymptotic CI) and standard errors (St.Er) are presented in Table 2. Also, Bayesian estimation using MCMC by applying MH algorithm and associated HPD intervals and St.Er are computed.

4.2 Simulation Study

In this part, we use Monte Carlo simulation research to assess the performance of estimation approaches, specifically MLE and Bayesian estimation, under the PT-IC scheme for MOEE distribution. We produce 1000 data points from the MOEE distribution for the MLEs under the following assumptions:

1. $\alpha = 1.5$ and $\lambda = 2.5$, i.e. $MOEE(1.5, 2.5)$.
2. Sample sizes are $n = 25$, $n = 50$ and $n = 100$.
3. Number of stages of PT-IC are $m = 4, 5$.
4. CTs T_j are proposed as follows:
   - $CT - I = (0.05, 0.15, 0.25, 0.50)$
   - $CT - II = (0.10, 0.30, 0.50, 1.00)$
   - $CT - III = (0.05, 0.15, 0.25, 0.40, 0.60)$
   - $CT - IV = (0.10, 0.30, 0.50, 0.75, 1.00)$

   where $j = 1, \ldots, m$. The patterns of CT can be classified according to m. In our study, $CT - I$ and $CT - II$ are used when $m = 3$ and $CT - III$ and $CT - IV$ are used when $m = 5$.
5. Removed items R_j are assumed at different sample size n as shown in Table 3 where $R_m = n - (\sum_{j=1}^{m-1} R_j + r)$ and r is the number of failure items.

It is indicate that scheme R_1 and R_{11} are represent Type-I censoring scheme as a special case with number of failure items $R_m = n - r$ and CT is T_m. We compute MLEs and the accompanying 95% asymptotic CI based on the produced data. When calculating MLEs, the initial estimate values are assumed to be the same as the genuine parameter values.

We calculate BEs for the Bayesian estimating technique utilizing the MOEE algorithm with informative priors. As in previous examples, we construct 1000 complete samples of size 60 from the $MOEE(1.5, 2.5)$ distribution, and the hyper parameter values are $\mu_1 = 5.17, \nu_1 = 2.90, \mu_2 = 20.47, \nu_2 = 7.69$.

The aforementioned informative prior values are used to compute the required estimations. We use the MH method with the MLEs as starting guess values and the related variance-covariance matrix S_θ of $(\ln(\hat{\alpha}), ln(\hat{\lambda}))$. Finally, we removed 2000 burn-in samples from the total 10000 samples generated by the posterior density and calculated BEs and HPD interval estimations using [14].

All of the mean estimates for both approaches are presented in Tables 4.a and 4.b for sample sizes $n = 50$, and $n = 100$, respectively. Furthermore, the first row shows average estimations (Avg.), whereas the second row reflects corresponding means square errors (MSEs). We have asymptotic confidence intervals for MLEs and HPD for BEs based on MCMC, which are provided in Tables 5.a and 5.b for sample sizes $n = 50$, and $n = 100$, respectively. In addition, the first row indicates average interval lengths (AILs), whereas the second row reflects corresponding coverage probabilities (CPs).

According to the tabulated figures, greater values of n lead to better estimates dependent on MSEs. It has also been shown that MLEs compete effectively with informative BEs. The AILs for BEs are better than theses in MLs.The increasing in time censoring points, the more efficient of estimates for all proposed methods of estimation. Furthermore, MSEs and AILs of linked interval estimations are often lower when units are eliminated early in the process.

Two images depict the convergence of MCMC estimates for α and λ. First; Figure 4 for $m = 4$ and pattern of censoring R_3 and $CT - I$ for choosing sample size $n = 50$. Second; Figure 5 for $m = 5$ and pattern of censoring R_{10} and $CT - IV$ for choosing sample size $n = 50$.

5 Concluding Remarks

In this study, we looked at the challenge of estimating the parameters for the MOEE distribution under PT-IC from both a classical and a Bayesian standpoint. We estimated MLEs and related asymptotic confidence intervals for the MOEE distribution's unknown parameters. Then, utilizing informative priors, we generated BEs using MCMC and the associated HPD interval estimates for two loss functions: SE and LINEX loss. Furthermore, when an informative prior is considered, a discussion of how to

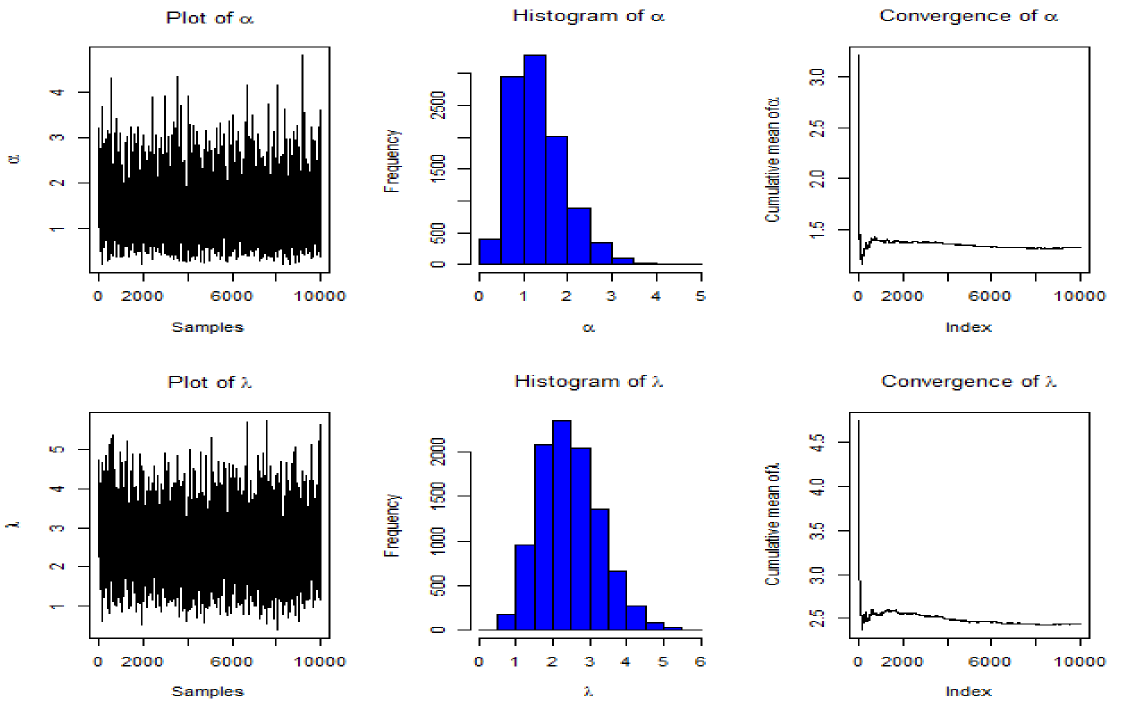

Fig. 4: Distribution and convergence of MCMC estimates for α and λ using MH algorithm under R_3 and $CT-I$ where $m=4$ and $n=50$

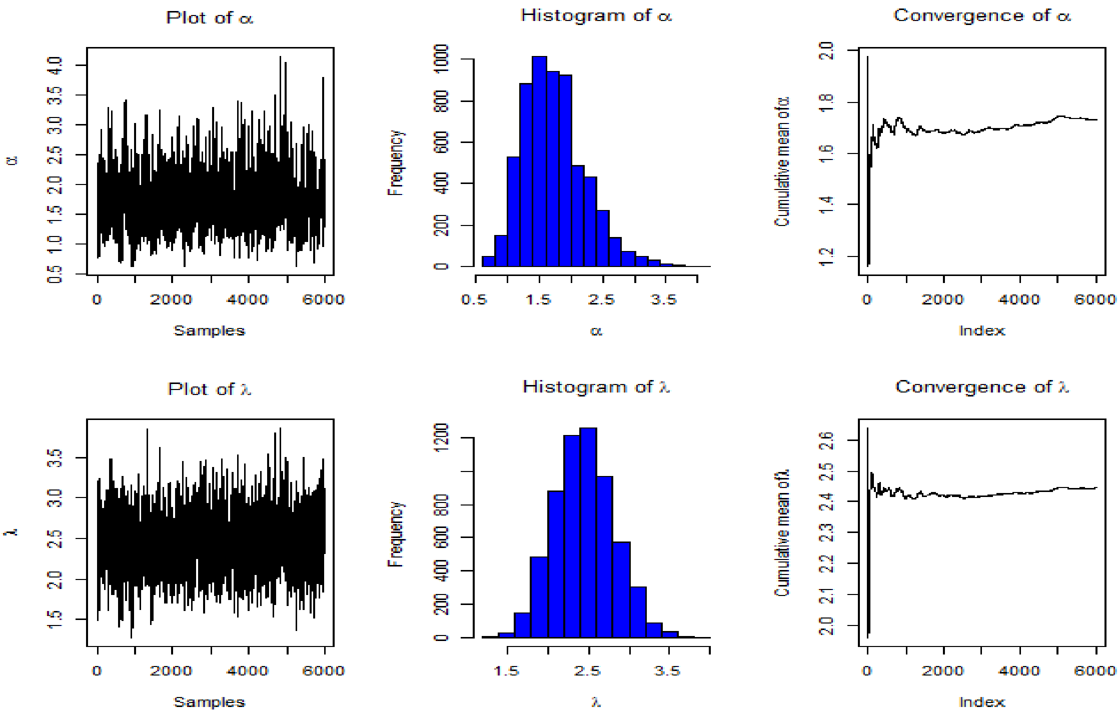

Fig. 5: Distribution and convergence of MCMC estimates for α and λ using MH algorithm under R_{13} and $CT-IV$ where $m=5$ and $n=50$

Table 2: ML and BEs with associated St.Er (in practices) and CIs based on different PT-IC schemes for given real data set at different number of stages

Sch.	Parm.	MLE		Bayesian: SE		Bayesian: LINEX	
		Estimate (St.Er)	Asy CI	Estimate (St.Er)	HPD	Estimate (St.Er)	HPD
1	α	1.339 (2.307)	(0.774, 1.904)	1.345 (0.0004)	(1.301, 1.385)	1.533 (0.0002)	(1.504, 1.562)
	λ	0.158 (0.186)	(0.112, 0.203)	0.109 (0.0009)	(0.070, 0.175)	0.167 (0.0006)	(0.123, 0.210)
2	α	2.237 (3.039)	(1.575, 2.899)	2.214 (0.0019)	(2.153, 2.302)	1.916 (0.0043)	(1.809, 2.042)
	λ	0.244 (0.194)	(0.202, 0.287)	0.260 (0.0002)	(0.235, 0.290)	0.260 (0.0004)	(0.225, 0.304)
3	α	1.506 (2.450)	(0.972, 2.039)	1.499 (0.0012)	(1.436, 1.543)	1.500 (0.0007)	(1.447, 1.547)
	λ	0.187 (0.205)	(0.142, 0.232)	0.175 (0.0011)	(0.118, 0.234)	0.171 (0.0008)	(0.134, 0.229)
4	α	1.181 (2.165)	(0.650, 1.712)	1.136 (0.0008)	(1.085, 1.182)	1.654 (0.0009)	(1.588, 1.691)
	λ	0.147 (0.192)	(0.101, 0.195)	0.179 (0.0006)	(0.132, 0.229)	0.217 (0.0005)	(0.177, 0.263)
5	α	1.154 (2.158)	(0.684, 1.624)	1.186 (0.0008)	(1.132, 1.237)	1.162 (0.0040)	(1.030, 1.239)
	λ	0.149 (0.201)	(0.105, 0.193)	0.148 (0.0005)	(0.117, 0.201)	0.191 (0.0028)	(0.111, 0.269)
6	α	2.701 (3.184)	(2.133, 3.268)	2.739 (0.0020)	(2.663, 2.833)	2.734 (0.0009)	(2.677, 2.782)
	λ	0.277 (0.171)	(0.247, 0.307)	0.346 (0.0005)	(0.301, 0.387)	0.253 (0.0007)	(0.210, 0.302)
7	α	2.647 (1.184)	(2.145, 3.061)	2.678 (0.0018)	(2.654, 2.762)	2.677 (0.0016)	(2.652, 2.771)
	λ	0.270 (0.165)	(0.241, 0.296)	0.235 (0.0003)	(0.211, 0.287)	0.236 (0.0004)	(0.220, 0.256)

Note: Sch.-Scheme, Parm.-Parameter, St.E-Standard error.

Table 3: Different patterns for removing items from life test at different number of stages

m	Scheme	Patterns		
		$n = 25$	$n = 50$	$n = 100$
4	R_1	$(0^{(3)}, R_m)$	$(0^{(3)}, R_m)$	$(0^{(3)}, R_m)$
	R_2	$(1^{(3)}, R_m)$	$(3^{(3)}, R_m)$	$(5^{(3)}, R_m)$
	R_3	$(2^{(3)}, R_m)$	$(5^{(3)}, R_m)$	$(10^{(3)}, R_m)$
	R_4	$(3^{(3)}, R_m)$	$(8^{(3)}, R_m)$	$(15^{(3)}, R_m)$
	R_5	$(4, 0^{(2)}, R_m)$	$(9, 0^{(2)}, R_m)$	$(15, 0^{(2)}, R_m)$
	R_6	$(8, 0^{(2)}, R_m)$	$(15, 0^{(2)}, R_m)$	$(30, 0^{(2)}, R_m)$
	R_7	$(12, 0^{(2)} R_m)$	$(24, 0^{(2)}, R_m)$	$(45, 0^{(2)}, R_m)$
	R_8	$(0^{(2)}, 4, R_m)$	$(0^{(2)}, 9, R_m)$	$(0^{(2)}, 15, R_m)$
	R_9	$(0^{(2)}, 8, R_m)$	$(0^{(2)}, 15, R_m)$	$(0^{(2)}, 30, R_m)$
	R_{10}	$(0^{(2)}, 12, R_m)$	$(0^{(2)}, 24, R_m)$	$(0^{(2)}, 45, R_m)$
5	R_{11}	$(0^{(4)}, R_m)$	$(0^{(4)}, R_m)$	$(0^{(4)}, R_m)$
	R_{12}	$(1^{(4)}, R_m)$	$(2^{(4)}, R_m)$	$(4^{(4)}, R_m)$
	R_{13}	$(2^{(4)}, R_m)$	$(4^{(4)}, R_m)$	$(8^{(4)}, R_m)$
	R_{14}	$(3^{(4)}, R_m)$	$(6^{(4)}, R_m)$	$(12^{(3)}, R_m)$
	R_{15}	$(4, 0^{(3)}, R_m)$	$(8, 0^{(3)}, R_m)$	$(16, 0^{(3)}, R_m)$
	R_{16}	$(8, 0^{(3)}, R_m)$	$(16, 0^{(3)}, R_m)$	$(32, 0^{(3)}, R_m)$
	R_{17}	$(12, 0^{(3)}, R_m)$	$(24, 0^{(3)}, R_m)$	$(48, 0^{(3)}, R_m)$
	R_{18}	$(0^{(3)}, 4, R_m)$	$(0^{(3)}, 8, R_m)$	$(0^{(3)}, 16, R_m)$
	R_{19}	$(0^{(3)}, 8, R_m)$	$(0^{(3)}, 12, R_m)$	$(0^{(3)}, 32, R_m)$
	R_{20}	$(0^{(3)}, 12, R_m)$	$(0^{(3)}, 20, R_m)$	$(0^{(3)}, 48, R_m)$

Here, $(1^{(3)}, 0)$, for example, means that the censoring scheme employed is $(1, 1, 1, 0)$.

choose the values of hyper-parameters in Bayesian estimation based on historical samples is reviewed. The simulation results show that MLEs informative BEs utilizing MCMC outperform MLEs. In future study, we will apply Bayesian estimation using MCMC; however, alternative approaches such as Lindely's approximation or significance sampling can be applied with PT-IC. Furthermore, maximum product spacing might be utilized as an alternative to conventional estimates (MLEs). Furthermore, the current methodology might be expanded to the development of optimal progressive censoring as well as other censoring approaches. For future works, many authors can use the MOEE distribution for

Table 4.a: Average estimated values and MSEs of the ML and BEs for MOEE distribution with $\alpha = 1.5$and $\lambda = 2.5$under different censoring schemes and sample size $n = 50$

m	Sch.		$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}_{SE}$	$\tilde{\lambda}_{SE}$	$\tilde{\alpha}_L$	$\tilde{\lambda}_L$	$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}_{SE}$	$\tilde{\lambda}_{SE}$	$\tilde{\alpha}_L$	$\tilde{\lambda}_L$
4			CT-I = (0.05, 0.15, 0.25, 0.50)						CT-II = (0.10, 0.30, 0.50, 1.00)					
	R_1	Avg.	2.194	2.792	1.542	2.502	1.461	2.421	1.920	2.636	1.503	2.436	1.417	2.363
		MSE	4.455	2.072	0.063	0.038	0.089	0.067	1.858	0.847	0.077	0.055	0.069	0.065
	R_2	Avg.	2.309	2.839	1.547	2.496	1.460	2.405	1.979	2.709	1.501	2.440	1.415	2.360
		MSE	5.016	2.240	0.077	0.031	0.066	0.037	2.125	1.117	0.070	0.055	0.064	0.065
	R_3	Avg.	2.440	2.992	1.552	2.506	1.466	2.413	2.038	2.772	1.492	2.447	1.408	2.365
		MSE	5.357	2.668	0.077	0.031	0.087	0.041	2.473	1.362	0.063	0.045	0.062	0.058
	R_4	Avg.	2.793	3.304	1.550	2.522	1.467	2.430	2.433	3.032	1.504	2.444	1.417	2.351
		MSE	8.480	3.914	0.071	0.027	0.104	0.084	4.673	2.400	0.068	0.035	0.064	0.052
	R_5	Avg.	2.286	2.863	1.538	2.503	1.447	2.411	1.988	2.708	1.503	2.447	1.416	2.368
		MSE	4.847	2.068	0.069	0.030	0.058	0.035	2.187	1.059	0.072	0.055	0.066	0.064
	R_6	Avg.	2.402	2.891	1.541	2.501	1.452	2.412	1.963	2.690	1.485	2.440	1.400	2.357
		MSE	6.582	2.633	0.073	0.031	0.070	0.041	2.536	1.346	0.079	0.055	0.072	0.066
	R_7	Avg.	2.840	3.200	1.556	2.516	1.462	2.419	2.174	2.866	1.498	2.455	1.409	2.368
		MSE	10.400	3.553	0.077	0.032	0.070	0.035	3.625	1.931	0.066	0.043	0.062	0.067
	R_8	Avg.	2.317	2.864	1.545	2.493	1.461	2.407	1.962	2.691	1.495	2.439	1.410	2.359
		MSE	5.350	2.239	0.069	0.036	0.076	0.047	2.000	1.080	0.065	0.045	0.065	0.058
	R_9	Avg.	2.450	3.005	1.550	2.507	1.457	2.409	2.073	2.725	1.476	2.432	1.391	2.347
		MSE	5.564	2.659	0.067	0.027	0.062	0.036	3.473	1.686	0.062	0.045	0.062	0.060
	R_{10}	Avg.	2.838	3.237	1.569	2.493	1.492	2.398	2.148	2.763	1.478	2.437	1.396	2.352
		MSE	10.174	4.261	0.073	0.023	0.138	0.035	4.059	1.848	0.060	0.037	0.073	0.064
5			CT-III = (0.05, 0.15, 0.25, 0.40, 0.60)						CT-IV = (0.10, 0.30, 0.50, 0.75, 1.00)					
	R_{11}	Avg.	1.868	2.596	1.522	2.479	1.440	2.403	1.801	2.612	1.555	2.533	1.462	2.455
		MSE	1.979	1.181	0.060	0.038	0.054	0.044	1.453	0.871	0.087	0.059	0.069	0.055
	R_{12}	Avg.	1.976	2.665	1.521	2.489	1.441	2.412	1.928	2.678	1.568	2.541	1.473	2.458
		MSE	2.843	1.427	0.063	0.035	0.067	0.046	2.112	1.056	0.093	0.050	0.072	0.046
	R_{13}	Avg.	2.225	2.793	1.554	2.488	1.469	2.406	2.107	2.790	1.583	2.548	1.490	2.468
		MSE	4.472	1.858	0.063	0.029	0.054	0.034	3.410	1.595	0.087	0.048	0.089	0.077
	R_{14}	Avg.	2.493	3.042	1.549	2.502	1.468	2.417	2.207	2.840	1.579	2.544	1.479	2.446
		MSE	5.891	2.840	0.067	0.029	0.088	0.042	4.469	2.225	0.085	0.039	0.066	0.034
	R_{15}	Avg.	1.998	2.661	1.529	2.487	1.446	2.410	1.860	2.649	1.572	2.533	1.475	2.451
		MSE	2.840	1.477	0.063	0.037	0.061	0.049	1.463	0.959	0.086	0.054	0.065	0.050
	R_{16}	Avg.	2.197	2.797	1.543	2.499	1.458	2.418	1.951	2.699	1.573	2.544	1.476	2.456
		MSE	4.195	1.752	0.069	0.036	0.061	0.040	2.296	1.255	0.092	0.052	0.070	0.046
	R_{17}	Avg.	2.265	2.832	1.530	2.498	1.446	2.412	2.104	2.798	1.584	2.554	1.486	2.458
		MSE	5.745	2.107	0.073	0.032	0.112	0.043	3.294	1.952	0.090	0.047	0.078	0.047
	R_{18}	Avg.	1.956	2.661	1.520	2.488	1.439	2.410	1.953	2.669	1.576	2.532	1.478	2.448
		MSE	2.624	1.476	0.061	0.036	0.054	0.041	2.269	1.131	0.093	0.052	0.072	0.050
	R_{19}	Avg.	2.269	2.841	1.539	2.496	1.459	2.415	2.097	2.767	1.587	2.543	1.490	2.455
		MSE	4.853	2.127	0.065	0.031	0.066	0.036	3.025	1.506	0.080	0.047	0.059	0.042
	R_{20}	Avg.	2.268	2.820	1.529	2.486	1.450	2.407	1.960	2.629	1.576	2.526	1.477	2.435
		MSE	4.913	2.222	0.063	0.031	0.064	0.038	2.829	1.495	0.074	0.041	0.057	0.043

estimating its parameters under different types of ranked set sampling.

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Table 4.b: Average estimated values and MSEs of the ML and BEs for MOEE distribution with $\alpha = 1.5$ and $\lambda = 2.5$ under different censoring schemes and sample size $n = 100$

m	Sch.		$\hat{\alpha}$	$\hat{\lambda}$	$\widetilde{\alpha}_{SE}$	$\widetilde{\lambda}_{SE}$	$\widetilde{\alpha}_{L}$	$\widetilde{\lambda}_{L}$	$\hat{\alpha}$	$\hat{\lambda}$	$\widetilde{\alpha}_{SE}$	$\widetilde{\lambda}_{SE}$	$\widetilde{\alpha}_{L}$	$\widetilde{\lambda}_{L}$
4			CT-I = (0.05, 0.15, 0.25, 0.50)						CT-II = (0.10, 0.30, 0.50, 1.00)					
	R_1	Avg.	1.844	2.663	1.508	2.469	1.441	2.398	1.687	2.590	1.533	2.496	1.463	2.437
		MSE	1.478	1.028	0.049	0.037	0.045	0.043	0.590	0.450	0.078	0.063	0.067	0.064
	R_2	Avg.	1.815	2.605	1.508	2.457	1.439	2.382	1.671	2.566	1.520	2.485	1.450	2.420
		MSE	1.506	1.205	0.049	0.037	0.048	0.050	0.662	0.531	0.076	0.058	0.067	0.060
	R_3	Avg.	1.819	2.625	1.498	2.456	1.433	2.381	1.733	2.605	1.532	2.488	1.458	2.417
		MSE	1.549	1.361	0.048	0.032	0.049	0.042	0.855	0.705	0.073	0.054	0.062	0.056
	R_4	Avg.	1.958	2.739	1.515	2.466	1.445	2.383	1.803	2.667	1.522	2.490	1.447	2.409
		MSE	2.208	1.745	0.049	0.027	0.044	0.042	1.230	1.095	0.061	0.044	0.053	0.048
	R_5	Avg.	1.786	2.590	1.500	2.456	1.434	2.385	1.644	2.530	1.525	2.469	1.454	2.406
		MSE	1.450	1.108	0.051	0.034	0.049	0.044	0.585	0.477	0.071	0.058	0.061	0.063
	R_6	Avg.	1.881	2.657	1.519	2.459	1.452	2.383	1.678	2.548	1.522	2.479	1.449	2.410
		MSE	1.727	1.283	0.051	0.032	0.062	0.042	0.807	0.667	0.081	0.061	0.070	0.064
	R_7	Avg.	1.895	2.623	1.505	2.458	1.435	2.379	1.688	2.573	1.517	2.480	1.442	2.403
		MSE	2.322	1.625	0.055	0.028	0.052	0.040	0.874	0.837	0.072	0.055	0.062	0.059
	R_8	Avg.	1.846	2.595	1.514	2.449	1.448	2.378	1.675	2.566	1.516	2.484	1.445	2.418
		MSE	1.820	1.225	0.054	0.033	0.051	0.045	0.726	0.535	0.075	0.054	0.066	0.057
	R_9	Avg.	1.887	2.642	1.517	2.453	1.448	2.377	1.715	2.561	1.519	2.472	1.445	2.399
		MSE	1.900	1.445	0.050	0.030	0.047	0.040	0.995	0.778	0.071	0.049	0.062	0.055
	R_{10}	Avg.	2.014	2.789	1.499	2.470	1.444	2.394	1.843	2.626	1.539	2.481	1.462	2.400
		MSE	2.659	2.233	0.047	0.034	0.160	0.129	1.556	1.008	0.066	0.039	0.054	0.047
5			CT-III = (0.05, 0.15, 0.25, 0.40, 0.60)						CT-IV = (0.10, 0.30, 0.50, 0.75, 1.00)					
	R_{11}	Avg.	1.687	2.565	1.481	2.447	1.408	2.375	1.646	2.538	1.500	2.455	1.435	2.399
		MSE	0.823	0.636	0.068	0.055	0.065	0.066	0.564	0.397	0.068	0.053	0.062	0.059
	R_{12}	Avg.	1.723	2.567	1.488	2.441	1.414	2.367	1.637	2.530	1.486	2.452	1.421	2.390
		MSE	0.996	0.791	0.070	0.055	0.065	0.065	0.672	0.510	0.068	0.053	0.064	0.059
	R_{13}	Avg.	1.697	2.538	1.471	2.428	1.397	2.346	1.726	2.565	1.513	2.448	1.443	2.378
		MSE	1.102	0.965	0.068	0.053	0.066	0.070	0.942	0.700	0.062	0.048	0.056	0.057
	R_{14}	Avg.	1.928	2.720	1.505	2.450	1.428	2.363	1.836	2.663	1.508	2.453	1.436	2.378
		MSE	1.879	1.481	0.063	0.042	0.059	0.054	1.356	1.095	0.058	0.045	0.054	0.047
	R_{15}	Avg.	1.689	2.512	1.489	2.425	1.413	2.352	1.702	2.573	1.514	2.460	1.447	2.399
		MSE	1.005	0.759	0.069	0.058	0.064	0.070	0.669	0.489	0.070	0.054	0.062	0.060
	R_{16}	Avg.	1.773	2.599	1.493	2.447	1.414	2.368	1.712	2.596	1.507	2.469	1.440	2.402
		MSE	1.235	0.913	0.073	0.052	0.067	0.063	0.793	0.593	0.065	0.049	0.062	0.055
	R_{17}	Avg.	1.892	2.673	1.511	2.452	1.428	2.367	1.800	2.651	1.513	2.470	1.442	2.395
		MSE	1.723	1.149	0.072	0.047	0.065	0.061	1.176	0.915	0.073	0.049	0.064	0.055
	R_{18}	Avg.	1.756	2.588	1.495	2.441	1.420	2.366	1.710	2.602	1.504	2.470	1.437	2.407
		MSE	1.100	0.786	0.069	0.052	0.064	0.062	0.729	0.562	0.064	0.056	0.058	0.060
	R_{19}	Avg.	1.816	2.597	1.492	2.436	1.416	2.356	1.765	2.617	1.500	2.462	1.431	2.393
		MSE	1.759	1.130	0.066	0.052	0.062	0.064	1.049	0.760	0.060	0.048	0.056	0.055
	R_{20}	Avg.	1.850	2.624	1.491	2.434	1.419	2.354	1.704	2.559	1.481	2.453	1.414	2.383
		MSE	1.902	1.213	0.059	0.038	0.082	0.074	1.071	0.835	0.058	0.048	0.056	0.056

Table 5.a: Average interval lengths and CPs(in%) of the ML and BEs for MOEE distribution with $\alpha = 1.5$ and $\lambda = 2.5$ under different censoring schemes and sample size $n = 50$

m	Sch.		$\hat{\alpha}$	$\hat{\lambda}$	$\widetilde{\alpha}_{SE}$	$\widetilde{\lambda}_{SE}$	$\widetilde{\alpha}_{L}$	$\widetilde{\lambda}_{L}$	$\hat{\alpha}$	$\hat{\lambda}$	$\widetilde{\alpha}_{SE}$	$\widetilde{\lambda}_{SE}$	$\widetilde{\alpha}_{L}$	$\widetilde{\lambda}_{L}$
4			CT-I = (0.05, 0.15, 0.25, 0.50)						CT-II = (0.10, 0.30, 0.50, 1.00)					
	R_1	AIL	5.739	5.718	0.975	0.759	0.901	0.733	4.220	3.599	1.051	0.832	0.948	0.799
		CP	92.3	95.2	95.1	95.9	94.5	95.0	92.9	95.1	95.6	95.0	94.7	95.1
	R_2	AIL	6.261	6.073	1.048	0.672	0.957	0.635	4.531	4.140	1.008	0.861	0.904	0.820
		CP	93.3	96.3	96.1	95.6	96.1	95.8	93.2	94.9	95.9	95.0	95.6	95.2
	R_3	AIL	6.766	6.521	1.046	0.674	0.968	0.636	4.834	4.704	0.933	0.794	0.841	0.751
		CP	93.9	96.1	96.6	95.0	96.9	95.3	93.2	95.7	94.7	93.9	94.4	96.3
	R_4	AIL	8.015	7.526	1.013	0.629	0.947	0.599	6.266	6.146	1.032	0.694	0.935	0.651
		CP	93.3	96.8	94.8	96.4	95.8	95.8	93.8	95.8	95.3	95.6	95.8	95.4
	R_5	AIL	6.213	6.051	0.996	0.663	0.889	0.623	4.506	4.008	1.048	0.874	0.949	0.830
		CP	93.9	96.6	95.0	96.0	95.8	95.7	92.6	94.3	94.7	94.8	94.8	95.4
	R_6	AIL	6.835	6.310	1.019	0.717	0.924	0.682	4.640	4.397	1.073	0.876	0.959	0.823
		CP	93.5	95.7	95.4	95.9	95.0	95.6	93.8	94.8	94.2	94.4	94.0	94.4
	R_7	AIL	8.205	7.117	1.045	0.699	0.939	0.656	5.323	5.346	0.998	0.774	0.903	0.729
		CP	93.0	96.2	96.0	95.6	96.0	95.6	92.1	94.1	94.7	95.3	94.6	95.1
	R_8	AIL	6.290	6.139	0.989	0.672	0.933	0.645	4.538	4.243	0.972	0.783	0.880	0.746
		CP	93.4	96.4	94.3	95.1	94.4	95.7	93.4	95.5	94.9	95.9	94.8	96.1
	R_9	AIL	6.806	6.620	0.954	0.626	0.904	0.597	5.056	4.941	0.963	0.741	0.870	0.704
		CP	92.9	96.4	95.3	95.6	96.0	95.5	92.4	94.2	96.0	95.5	95.2	95.2
	R_{10}	AIL	8.630	7.722	0.990	0.597	0.919	0.571	5.616	5.663	0.935	0.704	0.855	0.667
		CP	93.6	95.2	95.4	94.9	95.5	95.1	93.5	96.1	94.6	94.2	94.4	94.2
5			CT-III = (0.05, 0.15, 0.25, 0.40, 0.60)						CT-IV = (0.10, 0.30, 0.50, 0.75, 1.00)					
	R_{11}	AIL	4.492	4.517	0.949	0.749	0.864	0.719	3.959	3.614	1.125	0.918	1.021	0.879
		CP	93.4	96.0	94.6	94.7	94.0	95.2	93.0	95.4	94.8	94.9	95.2	94.7
	R_{12}	AIL	4.885	4.995	0.962	0.717	0.889	0.690	4.384	4.084	1.140	0.875	1.024	0.830
		CP	92.6	95.4	95.1	94.7	95.4	94.2	92.3	95.0	95.9	96.9	95.9	95.3
	R_{13}	AIL	5.693	5.629	0.977	0.660	0.890	0.629	5.024	4.820	1.062	0.826	0.970	0.778
		CP	93.4	96.0	94.3	95.2	96.1	96.1	92.9	94.4	95.5	95.0	95.5	94.9
	R_{14}	AIL	6.728	6.487	0.995	0.656	0.920	0.616	5.753	5.915	1.065	0.740	0.965	0.680
		CP	92.9	96.0	95.7	95.0	95.8	94.7	92.6	94.9	95.8	94.5	95.1	94.6
	R_{15}	AIL	4.944	4.898	0.964	0.732	0.881	0.712	4.215	3.950	1.108	0.890	0.992	0.847
		CP	92.5	95.5	96.1	95.1	95.9	95.0	93.2	95.8	95.5	95.7	95.4	95.4
	R_{16}	AIL	5.632	5.425	1.000	0.704	0.916	0.678	4.580	4.477	1.151	0.837	1.037	0.788
		CP	92.5	95.7	95.1	95.4	95.2	95.3	93.1	95.4	94.8	95.7	95.3	95.5
	R_{17}	AIL	6.137	5.941	1.042	0.683	0.957	0.655	5.193	5.356	1.107	0.816	1.014	0.769
		CP	93.3	95.9	95.4	95.8	95.6	95.0	92.8	95.0	95.7	95.0	94.4	94.6
	R_{18}	AIL	4.856	5.030	0.953	0.734	0.874	0.692	4.478	4.121	1.147	0.841	1.035	0.801
		CP	93.1	95.3	94.7	96.1	95.5	95.9	91.9	94.7	95.3	95.0	95.6	94.3
	R_{19}	AIL	5.891	5.741	0.971	0.683	0.903	0.646	5.082	4.839	1.023	0.814	0.918	0.758
		CP	92.4	95.2	95.3	94.3	95.1	94.3	93.5	94.9	94.7	94.7	94.7	94.5
	R_{20}	AIL	6.240	6.144	0.945	0.610	0.882	0.580	4.939	5.122	1.008	0.757	0.924	0.726
		CP	93.0	96.5	96.4	95.0	96.1	95.2	93.0	95.1	94.9	94.5	95.6	94.7

Table 5.b: Average interval lengths and CPs(in%) of the ML and BEs for MOEE distribution with $\alpha = 1.5$ and $\lambda = 2.5$ under different censoring schemes and sample size $n = 100$

m	Sch.		$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}_{SE}$	$\tilde{\lambda}_{SE}$	$\tilde{\alpha}_{L}$	$\tilde{\lambda}_{L}$	$\hat{\alpha}$	$\hat{\lambda}$	$\tilde{\alpha}_{SE}$	$\tilde{\lambda}_{SE}$	$\tilde{\alpha}_{L}$	$\tilde{\lambda}_{L}$
4			CT-I = (0.05, 0.15, 0.25, 0.50)						CT-II = (0.10, 0.30, 0.50, 1.00)					
	R_1	AIL	4.030	4.115	0.864	0.714	0.793	0.683	2.875	2.530	1.059	0.944	0.974	0.919
		CP	94.0	95.5	95.4	95.0	95.5	93.6	93.9	95.3	96.3	94.8	96.1	94.8
	R_2	AIL	4.105	4.460	0.849	0.707	0.778	0.671	3.050	2.819	1.051	0.906	0.968	0.871
		CP	92.9	96.2	95.2	95.2	95.6	95.8	93.2	95.5	94.8	95.3	95.2	95.3
	R_3	AIL	4.268	4.939	0.847	0.637	0.788	0.609	3.453	3.268	1.026	0.883	0.938	0.849
		CP	94.0	96.3	95.5	94.4	95.7	94.9	93.0	95.3	95.7	96.4	95.8	95.4
	R_4	AIL	4.792	5.582	0.817	0.633	0.749	0.600	3.877	4.118	0.932	0.800	0.846	0.759
		CP	94.1	96.3	94.5	95.2	95.1	95.6	93.3	95.1	95.7	95.6	95.8	95.4
	R_5	AIL	4.053	4.413	0.875	0.648	0.806	0.622	2.986	2.739	1.007	0.920	0.925	0.892
		CP	93.9	96.3	94.9	94.6	95.9	95.0	93.5	96.2	96.0	94.9	95.6	95.1
	R_6	AIL	4.399	4.809	0.848	0.679	0.784	0.657	3.231	3.055	1.106	0.928	1.011	0.892
		CP	93.5	96.6	95.3	94.1	95.3	95.3	93.2	95.6	95.0	94.7	94.9	94.6
	R_7	AIL	4.630	5.310	0.919	0.632	0.857	0.614	3.470	3.574	1.034	0.921	0.941	0.880
		CP	92.7	95.9	95.4	95.1	95.5	95.0	93.6	93.9	95.1	95.9	95.1	95.7
	R_8	AIL	4.187	4.504	0.895	0.683	0.826	0.663	3.114	2.885	1.060	0.877	0.981	0.849
		CP	92.7	96.1	95.5	95.1	95.0	95.0	91.6	95.3	94.7	95.3	94.4	95.4
	R_9	AIL	4.459	5.081	0.852	0.643	0.786	0.613	3.525	3.459	1.041	0.850	0.958	0.816
		CP	92.5	95.9	95.6	95.5	95.9	94.5	93.1	94.2	94.3	94.5	94.9	94.6
	R_{10}	AIL	5.018	5.816	0.834	0.624	0.813	0.621	4.029	4.085	0.933	0.745	0.850	0.724
		CP	93.3	95.9	96.0	95.5	95.1	96.7	93.7	95.7	95.1	96.2	95.4	96.1
5			CT-III = (0.05, 0.15, 0.25, 0.40, 0.60)						CT-IV = (0.10, 0.30, 0.50, 0.75, 1.00)					
	R_{11}	AIL	3.378	3.170	0.979	0.897	0.888	0.869	2.829	2.524	0.997	0.824	0.924	0.803
		CP	93.6	95.9	94.5	95.2	94.7	95.6	93.2	95.8	93.9	95.3	93.7	94.9
	R_{12}	AIL	3.559	3.490	1.023	0.854	0.926	0.821	3.022	2.839	1.013	0.862	0.931	0.833
		CP	92.9	95.3	95.7	94.9	95.6	95.0	92.9	95.1	93.8	94.6	93.9	95.2
	R_{13}	AIL	3.679	3.985	1.009	0.818	0.914	0.784	3.489	3.335	0.986	0.830	0.904	0.807
		CP	93.5	95.8	94.7	95.3	95.0	95.4	93.1	94.9	94.5	95.2	94.6	94.6
	R_{14}	AIL	4.394	4.811	0.963	0.748	0.875	0.705	4.028	4.314	0.959	0.703	0.885	0.661
		CP	93.3	95.1	94.6	96.1	94.8	96.0	94.1	96.1	95.8	94.7	96.0	94.6
	R_{15}	AIL	3.510	3.433	1.024	0.864	0.931	0.835	3.081	2.762	1.001	0.908	0.924	0.883
		CP	93.2	95.0	93.9	95.6	93.9	95.6	93.1	95.6	94.8	95.1	93.7	95.2
	R_{16}	AIL	3.794	3.795	1.053	0.864	0.951	0.820	3.312	3.115	0.996	0.847	0.927	0.820
		CP	92.9	95.2	94.3	93.3	94.4	94.5	93.8	95.4	94.2	93.7	94.3	94.0
	R_{17}	AIL	4.228	4.325	1.040	0.786	0.942	0.755	3.712	3.709	1.038	0.844	0.949	0.803
		CP	93.1	95.6	95.3	95.0	95.1	95.4	92.9	95.4	95.2	94.6	95.2	94.4
	R_{18}	AIL	3.635	3.527	1.004	0.835	0.915	0.801	3.172	2.882	0.982	0.924	0.902	0.888
		CP	93.6	95.4	94.3	94.7	94.6	94.9	94.1	95.1	93.9	94.4	94.0	93.9
	R_{19}	AIL	3.952	4.073	1.005	0.841	0.910	0.796	3.607	3.420	0.938	0.868	0.866	0.835
		CP	92.0	94.8	94.8	94.6	95.1	94.8	92.2	94.5	94.8	94.6	94.6	94.6
	R_{20}	AIL	4.278	4.714	0.943	0.694	0.860	0.666	3.570	3.596	0.945	0.845	0.870	0.811
		CP	92.7	96.3	96.2	94.5	95.4	95.4	93.1	95.5	95.4	95.2	95.8	94.8

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Manal H. Alabdulhadi Assistant professor, department of mathematics, Qassim University, SA. Interested in statistics-imprecise probability- inferences.

Ahmed R. El-Saeed is a lecturer of statistics, Department of Basic Sciences, Obour High Institute for Management & Informatics, Obour city, Egypt. He received the M.Sc. and PhD degrees in statistics from Mathematical Statistics Department, Faculty of Graduated Studies for Statistical Research, Cairo University, Egypt, in 2015 and 2020, respectively. The main research interests are in the life testing and reliability models and their applications in data analysis.

Mohammed Elgarhy is Assistant Professor of Statistics. He received MSc and Ph.D. in Statistics in 2014 and 2017 from Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt. His current research interests include generalized classes of distributions, distribution theory, statistical inference, entropy, ranked set sampling and lifetime test. He has published more than 150 research articles in reputed international journals of mathematical and engineering sciences. He is referee and editor of mathematical journals.

Abd El-Hamid Eisa Dr. Abd Elhamid Eisa completed his PhD in Statistics and its subject is Parameters Estimation Of A Class Of The Odd Generalized Exponential Family Of Distributions With Applications from the Faculty of Commerce for Boys - Al-Azhar University. He also works as a teacher in the same faculty and has published many researches in the field of applied and mathematical statistics and has many contributions in the statistical analysis of theses for researchers.

Doaa A. Abdo is a lecturer of applied statistics and insurance department at the faculty of commerce Mansoura university in Egypt. She completed her PhD in applied statistics and its title is (suggested methods to solve publication bias problems in meta analysis) (Applied Study). She has published many articles in the field of applied and mathematical statistics.