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Research article

Dynamics of a stochastic hybrid delay food chain model with jumps in an impulsive polluted environment

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Abstract: In this paper, a stochastic hybrid delay food chain model with jumps in an impulsive polluted environment is investigated. We obtain the sufficient and necessary conditions for persistence in mean and extinction of each species. The results show that the stochastic dynamics of the system are closely correlated with both time delays and environmental noises. Some numerical examples are introduced to illustrate the main results.

Keywords: stochastic hybrid delay system; food chain model; impulsive polluted environment; Lévy jumps

1. Introduction

The predator-prey model is one of the hotspots in biomathematics. For example, Yavuz and Sene [1] considered a fractional predator-prey model with harvesting rate, Chatterjee and Pal [2] studied a predator-prey model for the optimal control of fish harvesting through the imposition of a tax and Ghosh et al. [3] presented a three-component model consisting of one prey and two predator species using imprecise biological parameters as interval numbers and applied a functional parametric form in the proposed prey-predator system. Because of its important role in the ecosystem, the food chain model has been extensively studied [4–8]. Specifically, the classical four-species food chain model can be expressed as follows:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - a_{11}x_1(t) - a_{12}x_2(t)\right]\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + a_{21}x_1(t) - a_{22}x_2(t) - a_{23}x_3(t)\right]\mathrm{d}t, \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + a_{32}x_2(t) - a_{33}x_3(t) - a_{34}x_4(t)\right]\mathrm{d}t, \\ \mathrm{d}x_4(t) = x_4(t)\left[-r_4 + a_{43}x_3(t) - a_{44}x_4(t)\right]\mathrm{d}t, \end{cases} \tag{1.1}$$

where $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$ represent the densities of prey, primary predator, intermediate predator and top predator at time t, respectively. r_1 is the growth rate of prey, r_2, r_3 and r_4 are the death rates of primary predator, intermediate predator and top predator, respectively. a_{ij} and a_{ji} $(i < j)$ are the capture rates and food conversion rates, respectively. a_{ii} are the intraspecific competition rates of species i. All parameters in system (1.1) are positive constants.

In ecology, biology, physics, engineering and other areas of applied sciences, continuous-time models, fractional-order models as well as discrete-time models have been widely adopted [9, 10]. However, "time delays occur so often that to ignore them is to ignore reality" [11, 12], and in the models of population dynamics, the delay differential equations are much more realistic [13–15]. We know that systems with discrete time delays and those with continuously distributed time delays do not contain each other but systems with S-type distributed time delays contain both. Introducing S-type distributed time delays into system (1.1) yields

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t)\right]\mathrm{d}t, \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t)\right]\mathrm{d}t, \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t) - \mathcal{D}_{34}(x_4)(t)\right]\mathrm{d}t, \\ \mathrm{d}x_4(t) = x_4(t)\left[-r_4 + \mathcal{D}_{43}(x_3)(t) - \mathcal{D}_{44}(x_4)(t)\right]\mathrm{d}t, \end{cases} \tag{1.2}$$

where $\mathcal{D}_{ji}(x_i)(t) = a_{ji}x_i(t) + \int_{-\tau_{ji}}^{0} x_i(t+\theta)\mathrm{d}\mu_{ji}(\theta)$, $\int_{-\tau_{ji}}^{0} x_i(t+\theta)\mathrm{d}\mu_{ji}(\theta)$ are Lebesgue-Stieltjes integrals, $\tau_{ji} > 0$ are time delays, $\mu_{ji}(\theta)$ are nondecreasing bounded variation functions defined on $[-\tau, 0]$, $\tau = \max\left\{\tau_{ji}\right\}$.

On the other hand, the deterministic system has its limitation in mathematical modeling of ecosystems since the parameters involved in the system are unable to capture the influence of environmental noises [16, 17]. Introducing Gaussian white noises into the corresponding deterministic model is one common way to characterize environmental noises [18–25]. As we all know, Gaussian white noise $\xi(t)$ is a stationary and ergodic stochastic process with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(s) \rangle = \sigma^2\delta(t-s)$, where σ^2 is the noise intensity [26]. The readers can refer to [27–34] for more related works. In this paper, we assume that r_i are affected by Gaussian white noises, i.e., $r_1 \hookrightarrow r_1 + \sigma_1\dot{W}_1(t)$, $-r_2 \hookrightarrow -r_2 + \sigma_2\dot{W}_2(t)$, $-r_3 \hookrightarrow -r_3 + \sigma_3\dot{W}_3(t)$ and $-r_4 \hookrightarrow -r_4 + \sigma_4\dot{W}_4(t)$. Then, system (1.2) becomes

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1 - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t)\right]\mathrm{d}t + \sigma_1 x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2 + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t)\right]\mathrm{d}t + \sigma_2 x_2(t)\mathrm{d}W_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3 + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t) - \mathcal{D}_{34}(x_4)(t)\right]\mathrm{d}t + \sigma_3 x_3(t)\mathrm{d}W_3(t), \\ \mathrm{d}x_4(t) = x_4(t)\left[-r_4 + \mathcal{D}_{43}(x_3)(t) - \mathcal{D}_{44}(x_4)(t)\right]\mathrm{d}t + \sigma_4 x_4(t)\mathrm{d}W_4(t), \end{cases} \tag{1.3}$$

where $W_i(t)$ are mutually independent standard Wiener processes defined on a complete probability space $(\Omega, \mathcal{F}, P)$ satisfying the usual statistical properties, namely $\langle \mathrm{d}W_i(t) \rangle = 0$ and $\langle \mathrm{d}W_i(t)\mathrm{d}W_j(s) \rangle = \delta_{ij}\delta(t-s)\mathrm{d}t$ [35].

Besides, population system may be affected by telephone noises which can cause the system to switch from one environmental regime to another [36–38]. So, telephone noises should be taken into consideration in system (1.3), resulting the following model:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)\left[r_1(\rho(t)) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t)\right]\mathrm{d}t + \sigma_1(\rho(t))x_1(t)\mathrm{d}W_1(t), \\ \mathrm{d}x_2(t) = x_2(t)\left[-r_2(\rho(t)) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t)\right]\mathrm{d}t + \sigma_2(\rho(t))x_2(t)\mathrm{d}W_2(t), \\ \mathrm{d}x_3(t) = x_3(t)\left[-r_3(\rho(t)) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t) - \mathcal{D}_{34}(x_4)(t)\right]\mathrm{d}t + \sigma_3(\rho(t))x_3(t)\mathrm{d}W_3(t), \\ \mathrm{d}x_4(t) = x_4(t)\left[-r_4(\rho(t)) + \mathcal{D}_{43}(x_3)(t) - \mathcal{D}_{44}(x_4)(t)\right]\mathrm{d}t + \sigma_4(\rho(t))x_4(t)\mathrm{d}W_4(t), \end{cases} \tag{1.4}$$

where $\rho(t)$ is a continuous time Markov chain with finite state space $\mathbb{S} = \{1, 2, ..., S\}$, which describes the telephone noises.

Moreover, the behaviour of real biological species, in different ecosystems, is affected by Lévy noises [39]. Lévy processes are characterized by stationary independent increments [40]. Assume that $L(t)$ $(t \geq 0)$ is a Lévy process, using the decomposition [41]

$$L(t) = L\left(\frac{t}{n}\right) + \left[L\left(\frac{2t}{n}\right) - L\left(\frac{t}{n}\right)\right] + \cdots + \left[L\left(\frac{nt}{n}\right) - L\left(\frac{(n-1)t}{n}\right)\right],$$

one can observe that the probability distribution of $L(t)$ is infinitely divisible. The most general expression for the characteristic function of $L(t)$ is

$$\varphi(k) = \exp\left\{ik\mu - |\sigma k|^{\alpha}\left[1 - i\beta sgn(k)\Phi\right]\right\},$$

where $sgn(k)$ is the sign function with

$$\Phi = \begin{cases} \tan(\pi\alpha/2), & for\ all\ \alpha \neq 1, \\ -(2/\pi)\log|k|, & for\ all\ \alpha = 1, \end{cases}$$

where $\alpha \in (0, 2]$ is the stability parameter, σ is the scale parameter, σ^{α} is the noise intensity, $\mu \in \mathbb{R}$ is the location parameter and $\beta \in [-1, 1]$ is the skewness parameter [39]. In addition, Lévy noises are statistically independent with zero mean. Now, let us further improve system (1.4) by considering Lévy noises. Some scholars pointed out that Lévy noises can be used to describe some sudden environmental perturbations, for instance, earthquakes and hurricanes [42–47]. In the context of an epidemic situation, random jumps could refer to sudden and significant increases in the number of cases or spread of the disease that occur unpredictably [48]. System (1.4) with Lévy noises can be expressed as follows:

$$\begin{cases} \mathrm{d}x_1(t) = x_1(t)[(r_1(\rho(t)) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t, \rho(t))], \\ \mathrm{d}x_2(t) = x_2(t)[(-r_2(\rho(t)) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t, \rho(t))], \\ \mathrm{d}x_3(t) = x_3(t)[(-r_3(\rho(t)) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t) - \mathcal{D}_{34}(x_4)(t))\,\mathrm{d}t + \mathcal{S}_3(t, \rho(t))], \\ \mathrm{d}x_4(t) = x_4(t)[(-r_4(\rho(t)) + \mathcal{D}_{43}(x_3)(t) - \mathcal{D}_{44}(x_4)(t))\,\mathrm{d}t + \mathcal{S}_4(t, \rho(t))], \end{cases} \tag{1.5}$$

where $\mathcal{S}_i(t, \rho(t)) = \sigma_i(\rho(t))\mathrm{d}W_i(t) + \int_{\mathbb{Z}} \gamma_i(\mu, \rho(t))\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu)$, N is a Poisson counting measure with characteristic measure λ on a measurable subset $\mathbb{Z}$ of $[0, +\infty)$, where $\lambda(\mathbb{Z}) < +\infty$ and $\widetilde{N}(\mathrm{d}t, \mathrm{d}\mu) = N(\mathrm{d}t, \mathrm{d}\mu) - \lambda(\mathrm{d}\mu)\mathrm{d}t$, $\gamma_j(\mu, \rho(t)) > -1$ $(\mu \in \mathbb{Z})$ are bounded functions $(j = 1, 2, 3, 4)$.

Finally, environmental pollution caused by agriculture, industries and other human activities has become a big challenge that is commonly concerned by international society. For example, with the rapid development of industrial and agricultural production, some chemical plants and other industries often periodically discharge sewage or other pollutants into rivers, soil and air [49]. These pollutants can cause direct damage to ecosystems, such as species extinction, desertification and the greenhouse effect. Hence, we extend system (1.5) into the following form:

$$\begin{cases} \left.\begin{aligned} &\mathrm{d}x_1(t) = x_1(t)\left[(r_1(\rho(t)) - r_{11}C_{10}(t) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right],\\ &\mathrm{d}x_2(t) = x_2(t)\left[(-r_2(\rho(t)) - r_{22}C_{20}(t) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right],\\ &\mathrm{d}x_3(t) = x_3(t)\left[(-r_3(\rho(t)) - r_{33}C_{30}(t) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t) - \mathcal{D}_{34}(x_4)(t))\,\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\right],\\ &\mathrm{d}x_4(t) = x_4(t)\left[(-r_4(\rho(t)) - r_{44}C_{40}(t) + \mathcal{D}_{43}(x_3)(t) - \mathcal{D}_{44}(x_4)(t))\,\mathrm{d}t + \mathcal{S}_4(t,\rho(t))\right],\\ &\mathrm{d}C_{i0}(t) = \left[k_iC_e(t) - (g_i + m_i)C_{i0}(t)\right]\mathrm{d}t,\\ &\mathrm{d}C_e(t) = -hC_e(t)\mathrm{d}t, \end{aligned}\right\} t \neq n\gamma,\\ \Delta x_i(t) = 0,\ \Delta C_{i0}(t) = 0,\ \Delta C_e(t) = b,\ t = n\gamma,\ n \in \mathbb{Z}^+\ (i = 1,2,3,4), \end{cases} \tag{1.6}$$

where $\Delta x_i(t) = x_i(t^+) - x_i(t)$, $\Delta C_{i0}(t) = C_{i0}(t^+) - C_{i0}(t)$ and $\Delta C_e(t) = C_e(t^+) - C_e(t)$. For other parameters in system (1.6), see Table 1.

To the best of our knowledge to date, results about a stochastic hybrid delay four-species food chain model with jumps have not been reported. So, in this paper we investigate the dynamics of a stochastic hybrid delay four-species food chain model with jumps in an impulsive polluted environment. The organization of this paper is as follows: In Section 2, some basic preliminaries are presented. In Section 3, the sufficient and necessary conditions for stochastic persistence in mean and extinction of each species are obtained. In Section 4, some numerical examples are provided to illustrate our main results. Finally, we conclude the paper with a brief conclusion and discussion in Section 5.

Table 1. Definition of some parameters in system (1.6).

Parameter	**Definition**
$C_{i0}(t)$	the toxicant concentration in the organism of species i at time t
$C_e(t)$	the toxicant concentration in the environment at time t
r_{ii}	the dose-response rate of species i to the organismal toxicant
k_i	the toxin uptake rate per unit biomass
g_i	the organismal net ingestion rate of toxin
m_i	the organismal deportation rate of toxin
h	the rate of toxin loss in the environment
γ	the period of the impulsive toxicant input
b	the toxicant input amount at every time

2. Preliminaries

We have four fundamental assumptions for system (1.6).

Assumption 1. *$W_1(t)$, $W_2(t)$, $W_3(t)$, $W_4(t)$, $\rho(t)$ and* N *are mutually independent. $\rho(t)$, taking values in $\mathbb{S} = \{1, 2, ..., S\}$, is irreducible with one unique stationary distribution $\pi = (\pi_1, \pi_2, ..., \pi_S)^{\mathrm{T}}$.*

Assumption 2. *$r_j(i) > 0$, $a_{jk} > 0$ and there exist $\gamma_j^*(i) \geq \gamma_{j*}(i) > -1$ such that $\gamma_{j*}(i) \leq \gamma_j(\mu, i) \leq \gamma_j^*(i)$ ($\mu \in \mathbb{Z}$), $\forall i \in \mathbb{S}$, $j, k = 1, 2, 3, 4$.*

Remark 1. *Assumption 2 implies that the intensities of Lévy jumps are not too big to ensure that the solution will not explode in finite time.*

Assumption 3. $0 < k_i \le g_i + m_i$ $(i = 1, 2, 3, 4)$, $0 < b \le 1 - e^{-h\gamma}$.

Remark 2. *Assumption 3 means* $0 \le C_{i0}(t) < 1$ *and* $0 \le C_e(t) < 1$, *which must be satisfied to be realistic because* $C_{i0}(t)$ *and* $C_e(t)$ *are concentrations of the toxicant* $(i = 1, 2, 3, 4)$.

Assumption 4. $A_{22}A_{33}A_{44}\,|\mathbf{A}|\,|\boldsymbol{\Xi}| > A_{12}A_{21}A_{23}A_{32}A_{44}\,|\boldsymbol{\Xi}| + A_{23}A_{32}A_{34}A_{43}\,|\mathbf{A}|^2$.

Lemma 1. *[50, 51]* $C_{i0}(t)$ *involved in system (1.6) satisfies*

$$\lim_{t\to+\infty} t^{-1}\int_0^t C_{i0}(s)\mathrm{d}s = \frac{k_i b}{h\,(g_i + m_i)\,\gamma} = K_i \quad (i = 1, 2, 3, 4).$$

3. Persistence in mean and extinction

Denote

$$\begin{cases} A_{ij} = a_{ij} + \displaystyle\int_{-\tau_{ij}}^{0} \mathrm{d}\mu_{ij}(\theta), \;\; K_i = \frac{k_i b}{h\,(g_i + m_i)\,\gamma}, \\ b_1(\cdot) = r_1(\cdot) - \dfrac{\sigma_1^2(\cdot)}{2} - \displaystyle\int_{\mathbb{Z}} \left[\gamma_1(\mu, \cdot) - \ln(1 + \gamma_1(\mu, \cdot))\right] \lambda(\mathrm{d}\mu), \\ b_j(\cdot) = r_j(\cdot) + \dfrac{\sigma_j^2(\cdot)}{2} + \displaystyle\int_{\mathbb{Z}} \left[\gamma_j(\mu, \cdot) - \ln(1 + \gamma_j(\mu, \cdot))\right] \lambda(\mathrm{d}\mu) \;\; (j = 2, 3, 4), \\ \Sigma_1 = \displaystyle\sum_{i=1}^{S} \pi_i b_1(i) - r_{11}K_1, \;\; \Sigma_j = -\sum_{i=1}^{S} \pi_i b_j(i) - r_{jj}K_j \;\; (j = 2, 3, 4), \\ B_1 = \Sigma_1, \;\; B_2 = \Sigma_2 + \dfrac{A_{21}}{A_{11}} B_1, \;\; B_3 = \Sigma_3 + \dfrac{A_{32}}{A_{22}} B_2, \;\; B_4 = \Sigma_4 + \dfrac{A_{43}}{A_{33}} B_3, \\ \mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ -A_{21} & A_{22} \end{pmatrix}, \;\; \boldsymbol{\Xi} = \begin{pmatrix} A_{11} & A_{12} & 0 \\ -A_{21} & A_{22} & A_{23} \\ 0 & -A_{32} & A_{33} \end{pmatrix}, \;\; \boldsymbol{\Delta} = \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ -A_{21} & A_{22} & A_{23} & 0 \\ 0 & -A_{32} & A_{33} & A_{34} \\ 0 & 0 & -A_{43} & A_{44} \end{pmatrix}. \end{cases}$$

Denote $\boldsymbol{\Sigma}^{(2)} = (\Sigma_1, \Sigma_2)^{\mathrm{T}}$, $\boldsymbol{\Sigma}^{(3)} = (\Sigma_1, \Sigma_2, \Sigma_3)^{\mathrm{T}}$, $\boldsymbol{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)^{\mathrm{T}}$. Denote $\mathbf{A_j}$ is $\mathbf{A}$ with column j replaced by $\boldsymbol{\Sigma}^{(2)}$ $(j = 1, 2)$; $\boldsymbol{\Xi}_\mathbf{j}$ is $\boldsymbol{\Xi}$ with column j replaced by $\boldsymbol{\Sigma}^{(3)}$ $(j = 1, 2, 3)$; $\boldsymbol{\Delta}_\mathbf{j}$ is $\boldsymbol{\Delta}$ with column j replaced by $\boldsymbol{\Sigma}$ $(j = 1, 2, 3, 4)$.

Theorem 1. *For any initial condition* $\phi \in C\left([-\tau, 0], \mathbb{R}_+^4\right)$, *system (1.6) has a unique global solution* $(x_1(t), x_2(t), x_3(t), x_4(t))^{\mathrm{T}} \in \mathbb{R}_+^4$ *on* $t \in \mathbb{R}_+$ *a.s. Moreover, for any constant* $p > 0$, *there exists* $K_i(p) > 0$ *such that* $\sup_{t\in\mathbb{R}_+} \mathbb{E}\left[x_i^p(t)\right] \le K_i(p)$ $(i = 1, 2, 3, 4)$.

Proof. The proof is rather standard and hence is omitted (see e.g., [52]). □

Lemma 2. *[53] Suppose* $Z(t) \in C(\Omega \times [0, +\infty), \mathbb{R}_+)$ *and* $\lim_{t\to+\infty} \frac{o(t)}{t} = 0$.

(i) *If there exists constant $\delta_0 > 0$ such that for $t \gg 1$,*

$$\ln Z(t) \le \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + o(t),$$

then

$$\begin{cases} \limsup\limits_{t\to+\infty} t^{-1} \displaystyle\int_0^t Z(s)\mathrm{d}s \le \frac{\delta}{\delta_0} \ a.s. & (\delta \ge 0)\,; \\ \lim\limits_{t\to+\infty} Z(t) = 0 \ a.s. & (\delta < 0)\,. \end{cases}$$

(ii) *If there exist constants $\delta > 0$ and $\delta_0 > 0$ such that for $t \gg 1$,*

$$\ln Z(t) \ge \delta t - \delta_0 \int_0^t Z(s)\mathrm{d}s + o(t),$$

then

$$\liminf_{t\to+\infty} t^{-1} \int_0^t Z(s)\mathrm{d}s \ge \frac{\delta}{\delta_0} \ a.s.$$

Lemma 3. *If $|\mathbf{\Delta_4}| > 0$, then $\left|\mathbf{\Delta_j}\right| > 0$ ($j = 1, 2, 3$).*

Proof. Compute

$$A_{44}\,|\mathbf{\Delta_4}| - A_{43}\,|\mathbf{\Delta_3}| = [(A_{33}A_{44} + A_{34}A_{43})\,|\mathbf{A}| + A_{11}A_{23}A_{32}A_{44}]\,\Sigma_4.$$

$$A_{32}\,|\mathbf{\Delta_2}| = A_{33}\,|\mathbf{\Delta_3}| + A_{34}\,|\mathbf{\Delta_4}| - [(A_{33}A_{44} + A_{34}A_{43})\,|\mathbf{A}| + A_{11}A_{23}A_{32}A_{44}]\,\Sigma_3.$$

$$A_{21}\,|\mathbf{\Delta_1}| = A_{22}\,|\mathbf{\Delta_2}| + A_{23}\,|\mathbf{\Delta_3}| - [A_{11}A_{44}\,(A_{22}A_{33} + A_{23}A_{32}) + A_{34}A_{43}\,|\mathbf{A}| + A_{12}A_{21}A_{33}A_{44}]\,\Sigma_2.$$

Noting that $\Sigma_j < 0$ ($j = 2, 3, 4$), we obtain the desired assertion. □

First, let us consider the following auxiliary system:

$$\begin{cases} \left.\begin{array}{l} \mathrm{d}X_1(t) = X_1(t)\left[(r_1(\rho(t)) - r_{11}C_{10}(t) - \mathcal{D}_{11}(X_1)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right], \\ \mathrm{d}X_2(t) = X_2(t)\left[(-r_2(\rho(t)) - r_{22}C_{20}(t) + \mathcal{D}_{21}(X_1)(t) - \mathcal{D}_{22}(X_2)(t))\,\mathrm{d}t + \mathcal{S}_2(t,\rho(t))\right], \\ \mathrm{d}X_3(t) = X_3(t)\left[(-r_3(\rho(t)) - r_{33}C_{30}(t) + \mathcal{D}_{32}(X_2)(t) - \mathcal{D}_{33}(X_3)(t))\,\mathrm{d}t + \mathcal{S}_3(t,\rho(t))\right], \\ \mathrm{d}X_4(t) = X_4(t)\left[(-r_4(\rho(t)) - r_{44}C_{40}(t) + \mathcal{D}_{43}(X_3)(t) - \mathcal{D}_{44}(X_4)(t))\,\mathrm{d}t + \mathcal{S}_4(t,\rho(t))\right], \\ \mathrm{d}C_{i0}(t) = \left[k_iC_e(t) - (g_i + m_i)C_{i0}(t)\right]\mathrm{d}t, \\ \mathrm{d}C_e(t) = -hC_e(t)\mathrm{d}t, \end{array}\right\} t \ne n\gamma, \\ \Delta X_i(t) = 0,\ \Delta C_{i0}(t) = 0,\ \Delta C_e(t) = b,\ t = n\gamma,\ n \in \mathbb{Z}^+\ (i = 1, 2, 3, 4). \end{cases} \tag{3.1}$$

Lemma 4. *System (3.1) satisfies Table 2, where*

$$\overline{\mathbf{X}^{\mathrm{T}}(\infty)} = \lim_{t\to+\infty} t^{-1}\left(\int_0^t X_1(s)\mathrm{d}s, \int_0^t X_2(s)\mathrm{d}s, \int_0^t X_3(s)\mathrm{d}s, \int_0^t X_4(s)\mathrm{d}s\right).$$

Table 2. Stochastic persistence in mean and extinction of system (3.1).

B_4	B_3	B_2	B_1	$\overline{\mathbf{X}^{\mathrm{T}}(\infty)}$
≥ 0	≥ 0	≥ 0	≥ 0	$\left(\frac{B_1}{A_{11}}, \frac{B_2}{A_{22}}, \frac{B_3}{A_{33}}, \frac{B_4}{A_{44}}\right)$
< 0	≥ 0	≥ 0	≥ 0	$\left(\frac{B_1}{A_{11}}, \frac{B_2}{A_{22}}, \frac{B_3}{A_{33}}, 0\right)$
	< 0	≥ 0	≥ 0	$\left(\frac{B_1}{A_{11}}, \frac{B_2}{A_{22}}, 0, 0\right)$
		< 0	≥ 0	$\left(\frac{B_1}{A_{11}}, 0, 0, 0\right)$
			< 0	$(0, 0, 0, 0)$

Proof. Consider the following stochastic hybrid delay logistic model with Lévy jump in an impulsive polluted environment:

$$\begin{cases} \left.\begin{array}{l} \mathrm{d}X_1(t) = X_1(t)\left[(r_1(\rho(t)) - h_1 - r_{11}C_{10}(t) - \mathcal{D}_{11}(X_1)(t))\,\mathrm{d}t + \mathcal{S}_1(t,\rho(t))\right], \\ \mathrm{d}C_{10}(t) = \left[k_1 C_e(t) - (g_1 + m_1)\,C_{10}(t)\right]\mathrm{d}t, \\ \mathrm{d}C_e(t) = -hC_e(t)\mathrm{d}t, \end{array}\right\} t \neq n\gamma, \\ \Delta X_1(t) = 0,\ \Delta C_{10}(t) = 0,\ \Delta C_e(t) = b,\ t = n\gamma,\ n \in \mathbb{N}_+. \end{cases} \tag{3.2}$$

Thanks to Lemma 1 and Lemma 2.3 in [54], system (3.2) satisfies

$$\begin{cases} \lim\limits_{t\to+\infty} X_1(t) = 0 \quad a.s. & (B_1 < 0)\,; \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t X_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} \quad a.s. & (B_1 \geq 0)\,. \end{cases} \tag{3.3}$$

By Itô's formula, we compute

$$\ln \mathbf{X}(t) = \mathbf{\Sigma}\, t - \mathbf{A_0}\int_0^t \mathbf{X}(s)\mathrm{d}s + \begin{pmatrix} -\mathcal{T}_{11}(X_1)(t) \\ \mathcal{T}_{21}(X_1)(t) - \mathcal{T}_{22}(X_2)(t) \\ \mathcal{T}_{32}(X_2)(t) - \mathcal{T}_{33}(X_3)(t) \\ \mathcal{T}_{43}(X_3)(t) - \mathcal{T}_{44}(X_4)(t) \end{pmatrix} + \mathbf{o(t)}, \tag{3.4}$$

where

$$\ln \mathbf{X}(t) = \begin{pmatrix} \ln X_1(t) \\ \ln X_2(t) \\ \ln X_3(t) \\ \ln X_4(t) \end{pmatrix}, \quad \int \mathbf{X}(s)\mathrm{d}s = \begin{pmatrix} \int X_1(s)\mathrm{d}s \\ \int X_2(s)\mathrm{d}s \\ \int X_3(s)\mathrm{d}s \\ \int X_4(s)\mathrm{d}s \end{pmatrix},$$

$$\mathbf{A_0} = \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ -A_{21} & A_{22} & 0 & 0 \\ 0 & -A_{32} & A_{33} & 0 \\ 0 & 0 & -A_{43} & A_{44} \end{pmatrix}, \quad \mathbf{o(t)} = o(t)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\mathcal{T}_{ji}(X_i)(t) = \int_{-\tau_{ji}}^0 \int_\theta^0 X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta) - \int_{-\tau_{ji}}^0 \int_{t+\theta}^t X_i(s)\mathrm{d}s\mathrm{d}\mu_{ji}(\theta).$$

$\mathfrak{Case}$ (1) : $B_1 < 0$. Based on Eq (3.3), $\lim_{t\to+\infty} X_1(t) = 0$ a.s. Therefore, for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln X_2(t) \leq (\Sigma_2 + \epsilon) - a_{22}\int_0^t X_2(s)\mathrm{d}s.$$

Since $\Sigma_2 < 0$, then $\lim_{t\to+\infty} X_2(t) = 0$ a.s. Similarly, $\lim_{t\to+\infty} X_j(t) = 0$ a.s. $(j = 3, 4)$.

$\mathfrak{Case}$ (2) : $B_1 \geq 0$. Then,

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} \ a.s. \tag{3.5}$$

Consider the following auxiliary function:

$$\mathrm{d}\widetilde{X_2(t)} = \widetilde{X_2(t)} \left[\left(-r_2(\rho(t)) - r_{22}C_{20}(t) + \mathcal{D}_{21}(X_1)(t) - a_{22}\widetilde{X_2(t)} \right) \mathrm{d}t + \mathcal{S}_2(t, \rho(t)) \right].$$

Then $X_2(t) \leq \widetilde{X_2(t)}$ a.s. By Itô's formula, we get

$$\ln \widetilde{X_2(t)} = B_2 t - a_{22} \int_0^t \widetilde{X_2(s)}\mathrm{d}s + o(t).$$

In view of Lemma 2, we obtain:
If $B_1 \geq 0$, $B_2 < 0$, then $\lim_{t\to+\infty} \widetilde{X_2(t)} = 0$ a.s.
If $B_1 \geq 0$, $B_2 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1} \int_0^t \widetilde{X_2(s)}\mathrm{d}s = \frac{B_2}{a_{22}} \ a.s.$$

Therefore, for arbitrary $\zeta > 0$, we have

$$\lim_{t\to+\infty} t^{-1} \int_{t-\zeta}^t X_i(s)\mathrm{d}s = 0 \ \ a.s. \ \ (i = 1, 2). \tag{3.6}$$

According to system (3.4) and Eq (3.6), we obtain

$$\ln X_2(t) = B_2 t - A_{22} \int_0^t X_2(s)\mathrm{d}s + o(t).$$

Thanks to Lemma 2, we deduce:
If $B_1 \geq 0$, $B_2 < 0$, then $\lim_{t\to+\infty} X_j(t) = 0$ a.s. $(j = 2, 3, 4)$.
If $B_1 \geq 0$, $B_2 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1} \int_0^t X_2(s)\mathrm{d}s = \frac{B_2}{A_{22}} \ a.s.$$

$\mathfrak{Case}$ (3) : $B_1 \geq 0$, $B_2 \geq 0$. Consider the following SDE:

$$\mathrm{d}\widetilde{X_3(t)} = \widetilde{X_3(t)} \left[\left(-r_3(\rho(t)) - r_{33}C_{30}(t) + \mathcal{D}_{32}(X_2)(t) - a_{33}\widetilde{X_3(t)} \right) \mathrm{d}t + \mathcal{S}_3(t, \rho(t)) \right].$$

Then $X_3(t) \leq \widetilde{X_3(t)}$ a.s. By Itô's formula, we get

$$\ln \widetilde{X_3(t)} = B_3 t - a_{33} \int_0^t \widetilde{X_3(s)}\mathrm{d}s + o(t).$$

In the light of Lemma 2, we obtain:
If $B_1 \geq 0$, $B_2 \geq 0$, $B_3 < 0$, then $\lim_{t\to+\infty} \widetilde{X_3(t)} = 0$ a.s.
If $B_1 \geq 0$, $B_2 \geq 0$, $B_3 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1} \int_0^t \widetilde{X_3(s)}\mathrm{d}s = \frac{B_3}{a_{33}} \ a.s.$$

Hence, for arbitrary $\zeta > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\zeta}^{t} X_i(s)\mathrm{d}s = 0 \quad a.s. \quad (i = 1, 2, 3). \tag{3.7}$$

Thanks to system (3.4) and Eq (3.7), we obtan

$$\ln X_3(t) = B_3 t - A_{33}\int_0^t X_3(s)\mathrm{d}s + o(t).$$

Based on Lemma 2, we obtain:
If $B_1 \geq 0$, $B_2 \geq 0$, $B_3 < 0$, then $\lim_{t\to+\infty} X_j(t) = 0$ a.s. $(j = 3, 4)$.
If $B_1 \geq 0$, $B_2 \geq 0$, $B_3 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1}\int_0^t X_3(s)\mathrm{d}s = \frac{B_3}{A_{33}} \ a.s.$$

$\mathfrak{Case}$ (4) : $B_1 \geq 0$, $B_2 \geq 0$, $B_3 \geq 0$. Consider the following SDE:

$$\mathrm{d}\widetilde{X_4(t)} = \widetilde{X_4(t)}\left[\left(-r_4(\rho(t)) - r_{44}C_{40}(t) + \mathcal{D}_{43}(X_3)(t) - a_{44}\widetilde{X_4(t)}\right)\mathrm{d}t + \mathcal{S}_4(t,\rho(t))\right].$$

Then $X_4(t) \leq \widetilde{X_4(t)}$ a.s. By Itô's formula, we get

$$\ln\widetilde{X_4(t)} = B_4 t - a_{44}\int_0^t \widetilde{X_4(s)}\mathrm{d}s + o(t).$$

In view of Lemma 2, we obtain:
If $B_1 \geq 0$, $B_2 \geq 0$, $B_3 \geq 0$, $B_4 < 0$, then $\lim_{t\to+\infty}\widetilde{X_4(t)} = 0$ a.s.
If $B_1 \geq 0$, $B_2 \geq 0$, $B_3 \geq 0$, $B_4 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1}\int_0^t \widetilde{X_4(s)}\mathrm{d}s = \frac{B_4}{a_{44}} \ a.s.$$

Hence, for arbitrary $\zeta > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\zeta}^{t} X_i(s)\mathrm{d}s = 0 \quad a.s. \quad (i = 1, 2, 3, 4). \tag{3.8}$$

Thanks to systems (3.4) and (3.8), we deduce

$$\ln X_4(t) = B_4 t - A_{44}\int_0^t X_4(s)\mathrm{d}s + o(t).$$

Based on Lemma 2 and the arbitrariness of ϵ, we obtain:
If $B_1 \geq 0$, $B_2 \geq 0$, $B_3 < 0$, $B_4 < 0$, then $\lim_{t\to+\infty} X_4(t) = 0$ *a.s.*
If $B_1 \geq 0$, $B_2 \geq 0$, $B_3 \geq 0$, $B_4 \geq 0$, then

$$\lim_{t\to+\infty} t^{-1}\int_0^t X_4(s)\mathrm{d}s = \frac{B_4}{A_{44}} \ a.s.$$

The proof is complete. □

Lemma 5. *For system (1.6):*
(i) $\limsup_{t\to+\infty} t^{-1}\ln x_i(t) \le 0$ *a.s.* ($i = 1,2,3,4$).
(ii) $\lim_{t\to+\infty} x_i(t) = 0 \Rightarrow \lim_{t\to+\infty} x_j(t) = 0$ *a.s.* ($1 \le i < j \le 4$).

Proof. From Lemma 4, system (3.1) satisfies $\lim_{t\to+\infty} t^{-1}\ln X_i(t) = 0$ a.s. ($i = 1,2,3,4$). By the stochastic comparison theorem, we obtain the desired assertion (i). The proof of (ii) is similar to that of Lemma 4 and here is omitted. □

Theorem 2. *Under Assumption 4 system (1.6) satisfies Table 3, where*

$$\overline{\mathbf{x}^{\mathrm{T}}(\infty)} = \lim_{t\to+\infty} t^{-1}\left(\int_0^t x_1(s)\mathrm{d}s, \int_0^t x_2(s)\mathrm{d}s, \int_0^t x_3(s)\mathrm{d}s, \int_0^t x_4(s)\mathrm{d}s\right).$$

Table 3. Stochastic persistence in mean and extinction of system (1.6).

$\lvert\mathbf{\Delta_4}\rvert$	$\lvert\mathbf{\Xi_3}\rvert$	$\lvert\mathbf{A_2}\rvert$	B_1	$\overline{\mathbf{x}^{\mathrm{T}}(\infty)}$
+				$\left(\frac{\lvert\mathbf{\Delta_1}\rvert}{\lvert\mathbf{\Delta}\rvert}, \frac{\lvert\mathbf{\Delta_2}\rvert}{\lvert\mathbf{\Delta}\rvert}, \frac{\lvert\mathbf{\Delta_3}\rvert}{\lvert\mathbf{\Delta}\rvert}, \frac{\lvert\mathbf{\Delta_4}\rvert}{\lvert\mathbf{\Delta}\rvert}\right)$
−	+			$\left(\frac{\lvert\mathbf{\Xi_1}\rvert}{\lvert\mathbf{\Xi}\rvert}, \frac{\lvert\mathbf{\Xi_2}\rvert}{\lvert\mathbf{\Xi}\rvert}, \frac{\lvert\mathbf{\Xi_3}\rvert}{\lvert\mathbf{\Xi}\rvert}, 0\right)$
	−	+		$\left(\frac{\lvert\mathbf{A_1}\rvert}{\lvert\mathbf{A}\rvert}, \frac{\lvert\mathbf{A_2}\rvert}{\lvert\mathbf{A}\rvert}, 0, 0\right)$
		−	+	$\left(\frac{B_1}{A_{11}}, 0, 0, 0\right)$
			−	$(0,0,0,0)$

Proof. Compute $\lvert\mathbf{\Delta_4}\rvert < A_{43}\lvert\mathbf{\Xi_3}\rvert < A_{32}A_{43}\lvert\mathbf{A_2}\rvert < A_{21}A_{32}A_{43}B_1$. By Eq (3.8), for any $\zeta > 0$,

$$\lim_{t\to+\infty} t^{-1}\int_{t-\zeta}^t x_i(s)\mathrm{d}s = 0 \ \textit{a.s.}\ (i = 1,2,3,4).$$

By Itô's formula, we compute

$$\ln\mathbf{x}(t) = \mathbf{\Sigma}\, t - \mathbf{\Delta}\int_0^t \mathbf{x}(s)\mathrm{d}s + \mathbf{o(t)}. \tag{3.9}$$

$\mathfrak{Case}$ (i) : $\lvert\mathbf{\Delta_4}\rvert > 0$. According to system (3.9), we compute

$$\lim_{t\to+\infty} t^{-1}\left(A_{21}A_{32}A_{43}\ln x_1(t) + A_{11}A_{32}A_{43}\ln x_2(t) + A_{43}\lvert\mathbf{A}\rvert\ln x_3(t) + \lvert\mathbf{\Xi}\rvert\ln x_4(t) + \lvert\mathbf{\Delta}\rvert\int_0^t x_4(s)\mathrm{d}s\right) = \lvert\mathbf{\Delta_4}\rvert. \tag{3.10}$$

In view of Lemma 5 (i) and Lemma 2, we get

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s \ge \frac{\lvert\mathbf{\Delta_4}\rvert}{\lvert\mathbf{\Delta}\rvert}\ \textit{a.s.} \tag{3.11}$$

Based on system (3.9), we compute

$$\lim_{t\to+\infty} t^{-1}\left(A_{22}A_{43}\ln x_1(t) - A_{12}A_{43}\ln x_2(t) - A_{12}A_{23}\ln x_4(t) + A_{43}\lvert\mathbf{A}\rvert\int_0^t x_1(s)\mathrm{d}s - A_{12}A_{23}A_{44}\int_0^t x_4(s)\mathrm{d}s\right)$$
$$= A_{43}\lvert\mathbf{A_1}\rvert - A_{12}A_{23}\Sigma_4. \tag{3.12}$$

By Lemma 5 (i) and Eq (3.12), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$A_{22}A_{43}\ln x_1(t) \le \left(A_{43}\,|\mathbf{A_1}| - A_{12}A_{23}\Sigma_4 + A_{12}A_{23}A_{44}\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s + \epsilon\right)t - A_{43}\,|\mathbf{A}|\int_0^t x_1(s)\mathrm{d}s.$$

In view of Eq (3.11), we deduce

$$\begin{aligned}
&A_{43}\,|\mathbf{A_1}| - A_{12}A_{23}\Sigma_4 + A_{12}A_{23}A_{44}\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s\\
\ge&A_{43}\,|\mathbf{A_1}| - A_{12}A_{23}\Sigma_4 + A_{12}A_{23}A_{44}\liminf_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s\\
\ge&A_{43}\,|\mathbf{A_1}| - A_{12}A_{23}\Sigma_4 + A_{12}A_{23}A_{44}\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|} = A_{43}\,|\mathbf{A}|\,\frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|},
\end{aligned} \tag{3.13}$$

where $|\mathbf{A}| > 0$ and $|\mathbf{\Delta}| > 0$. From Lemma 3, we have $|\mathbf{\Delta_1}| > 0$. Based on Lemma 2 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le A_{43}^{-1}\,|\mathbf{A}|^{-1}\left(A_{43}\,|\mathbf{A_1}| - A_{12}A_{23}\Sigma_4 + A_{12}A_{23}A_{44}\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s\right) \triangleq \Gamma_{x_1}^{sup} \quad a.s. \tag{3.14}$$

According to system (3.9), we compute

$$\lim_{t\to+\infty} t^{-1}\left(A_{21}A_{32}\ln x_1(t) + A_{11}A_{32}\ln x_2(t) + |\mathbf{A}|\ln x_3(t) + A_{34}\,|\mathbf{A}|\int_0^t x_4(s)\mathrm{d}s + |\mathbf{\Xi}|\int_0^t x_3(s)\mathrm{d}s\right) = |\mathbf{\Xi_3}|\,. \tag{3.15}$$

Thanks to Lemma 5 (i) and Eq (3.15), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$|\mathbf{A}|\ln x_3(t) \ge \left(|\mathbf{\Xi_3}| - A_{34}\,|\mathbf{A}|\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s - \epsilon\right)t - |\mathbf{\Xi}|\int_0^t x_3(s)\mathrm{d}s.$$

If

$$|\mathbf{\Xi_3}| - A_{34}\,|\mathbf{A}|\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s > 0,$$

then by Lemma 2 and the arbitrariness of ϵ, we obtain

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge |\mathbf{\Xi}|^{-1}\left(|\mathbf{\Xi_3}| - A_{34}\,|\mathbf{A}|\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s\right) \triangleq \Gamma_{x_3}^{inf} \quad a.s. \tag{3.16}$$

If

$$|\mathbf{\Xi_3}| - A_{34}\,|\mathbf{A}|\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s \le 0,$$

since $\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge 0$, we also obtain Eq (3.16).

According to system (3.9), Eq (3.14) and Eq (3.16), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_2(t) \le \left(\Sigma_2 + A_{21}\Gamma_{x_1}^{sup} - A_{23}\Gamma_{x_3}^{inf} + \epsilon\right)t - A_{22}\int_0^t x_2(s)\mathrm{d}s.$$

On the basis of Eq (3.13), Eq (3.14) and Eq (3.16), we have

$$\begin{aligned}&\Sigma_2 + A_{21}\Gamma_{x_1}^{sup} - A_{23}\Gamma_{x_3}^{inf}\\ \geq&\Sigma_2 + A_{21}\frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} - A_{23}\,|\mathbf{\Xi}|^{-1}\left(|\mathbf{\Xi_3}| - A_{34}\,|\mathbf{A}|\liminf_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s\right)\\ \geq&\Sigma_2 + A_{21}\frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} - A_{23}\,|\mathbf{\Xi}|^{-1}\left(|\mathbf{\Xi_3}| - A_{34}\,|\mathbf{A}|\,\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|}\right) = A_{22}\frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|}.\end{aligned} \tag{3.17}$$

From Lemma 3, we have $|\mathbf{\Delta_2}| > 0$. By Lemma 2 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \leq A_{22}^{-1}\left(\Sigma_2 + A_{21}\Gamma_{x_1}^{sup} - A_{23}\Gamma_{x_3}^{inf}\right) \triangleq \Gamma_{x_2}^{sup} \quad a.s. \tag{3.18}$$

By system (3.9), Eq (3.11) and Eq (3.18), for $\forall\epsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_3(t) \leq \left(\Sigma_3 + A_{32}\Gamma_{x_2}^{sup} - A_{34}\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|} + \epsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s.$$

In view of Eq (3.17) and Eq (3.18), we obtain

$$\Sigma_3 + A_{32}\Gamma_{x_2}^{sup} - A_{34}\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|} \geq \Sigma_3 + A_{32}\frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} - A_{34}\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|} = A_{33}\frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|}. \tag{3.19}$$

In the light of Lemma 3, we have $|\mathbf{\Delta_3}| > 0$. Thanks to Lemma 2 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \leq A_{33}^{-1}\left(\Sigma_3 + A_{32}\Gamma_{x_2}^{sup} - A_{34}\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|}\right) \triangleq \Gamma_{x_3}^{sup} \quad a.s. \tag{3.20}$$

By system (3.9) and Eq (3.20), for $\forall\epsilon \in (0, 1)$ and $t \gg 1$,

$$\ln x_4(t) \leq \left(\Sigma_4 + A_{43}\Gamma_{x_3}^{sup} + \epsilon\right)t - A_{44}\int_0^t x_4(s)\mathrm{d}s.$$

Thanks to Eq (3.19), we obtain

$$\Sigma_4 + A_{43}\Gamma_{x_3}^{sup} \geq \Sigma_4 + A_{43}\frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = A_{44}\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|}.$$

In the light of Lemma 2 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s \leq A_{44}^{-1}\left(\Sigma_4 + A_{43}\Gamma_{x_3}^{sup}\right) \quad a.s.$$

In other words, we have

$$\begin{aligned}&\frac{A_{22}A_{33}A_{44}\,|\mathbf{A}|\,|\mathbf{\Xi}| - A_{12}A_{21}A_{23}A_{32}A_{44}\,|\mathbf{\Xi}| - A_{23}A_{32}A_{34}A_{43}\,|\mathbf{A}|^2}{A_{22}A_{33}\,|\mathbf{A}|\,|\mathbf{\Xi}|}\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s\\ \leq&\Sigma_4 + A_{43}A_{33}^{-1}\left[\Sigma_3 + A_{32}A_{22}^{-1}\left(\Sigma_2 + A_{21}\frac{|\mathbf{A_1}|}{|\mathbf{A}|} + \frac{A_{12}A_{21}A_{23}\Sigma_4}{A_{43}\,|\mathbf{A}|} - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right) - A_{34}\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|}\right].\end{aligned} \tag{3.21}$$

According to Eq (3.21) and Assumption 4, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s \le \frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|}\ a.s. \tag{3.22}$$

Combining Eq (3.11) and Eq (3.22) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s = \frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|}\ a.s. \tag{3.23}$$

Substituting Eq (3.22) into Eq (3.14) yields

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le A_{43}^{-1}\,|\mathbf{A}|^{-1}\left(A_{43}\,|\mathbf{A_1}| - A_{12}A_{23}\Sigma_4 + A_{12}A_{23}A_{44}\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|}\right) = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|}. \tag{3.24}$$

Substituting Eq (3.22) into Eq (3.16) yields

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge |\mathbf{\Xi}|^{-1}\left(|\mathbf{\Xi_3}| - A_{34}\,|\mathbf{A}|\,\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|}\right) = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|}. \tag{3.25}$$

Substituting Eq (3.24) and Eq (3.25) into Eq (3.18) yields

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \le A_{22}^{-1}\left(\Sigma_2 + A_{21}\frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} - A_{23}\frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|}\right) = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|}. \tag{3.26}$$

Substituting Eq (3.26) into Eq (3.20) yields

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \le A_{33}^{-1}\left(\Sigma_3 + A_{32}\frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} - A_{34}\frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|}\right) = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|}. \tag{3.27}$$

Combining Eq (3.25) and Eq (3.27) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|}\ a.s. \tag{3.28}$$

In view of system (3.9), we have

$$\lim_{t\to+\infty} t^{-1}\left(\ln x_1(t) + A_{11}\int_0^t x_1(s)\mathrm{d}s + A_{12}\int_0^t x_2(s)\mathrm{d}s\right) = B_1. \tag{3.29}$$

By Eq (3.26) and Eq (3.29), for $\forall\epsilon\in(0,1)$ and $t\gg 1$,

$$\ln x_1(t) \ge \left(B_1 - A_{12}\frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} - \epsilon\right)t - A_{11}\int_0^t x_1(s)\mathrm{d}s,$$

where $B_1 - A_{12}\frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} = A_{11}\frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|}$. From Lemma 3, we have $|\mathbf{\Delta_1}| > 0$. According to Lemma 2 and the arbitrariness of ϵ, we have

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \ge \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|}\ a.s. \tag{3.30}$$

Combining Eq (3.24) with Eq (3.30) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|}\ a.s. \tag{3.31}$$

Substituting Eq (3.31) into system (3.9) yields

$$\lim_{t\to+\infty} t^{-1}\left(\ln x_2(t) + A_{22}\int_0^t x_2(s)\mathrm{d}s\right) = A_{22}\frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|}\ a.s.$$

From Lemma 3, we have $|\mathbf{\Delta_2}| > 0$. By Lemma 2, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|}\ a.s. \tag{3.32}$$

$\mathfrak{Case}$ (ii) : $|\mathbf{\Xi_3}| > 0 > |\mathbf{\Delta_4}|$. Thanks to Eq (3.10), we deduce

$$\limsup_{t\to+\infty} t^{-1}\ln\left(x_1^{A_{21}A_{32}A_{43}}(t)x_2^{A_{11}A_{32}A_{43}}(t)x_3^{A_{43}|\mathbf{A}|}(t)x_4^{|\mathbf{\Xi}|}(t)\right) \le |\mathbf{\Delta_4}| < 0\ a.s.$$

which implies

$$\lim_{t\to+\infty} x_1^{A_{21}A_{32}A_{43}}(t)x_2^{A_{11}A_{32}A_{43}}(t)x_3^{A_{43}|\mathbf{A}|}(t)x_4^{|\mathbf{\Xi}|}(t) = 0\ a.s. \tag{3.33}$$

From Lemma 5 (*ii*) and Eq (3.33), we obtain

$$\lim_{t\to+\infty} x_4(t) = 0\ a.s. \tag{3.34}$$

In other words,

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s = 0\ a.s.$$

According to system (3.9), we compute

$$\lim_{t\to+\infty} t^{-1}\left(A_{21}A_{32}\ln x_1(t) + A_{11}A_{32}\ln x_2(t) + |\mathbf{A}|\ln x_3(t) + |\mathbf{\Xi}|\int_0^t x_3(s)\mathrm{d}s\right) = |\mathbf{\Xi_3}|\,. \tag{3.35}$$

Combining Lemma 5 (*i*) with Lemma 2 yields

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \ge \frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\ a.s. \tag{3.36}$$

Based on system (3.9), we compute

$$\lim_{t\to+\infty} t^{-1}\left(A_{22}\ln x_1(t) - A_{12}\ln x_2(t) + |\mathbf{A}|\int_0^t x_1(s)\mathrm{d}s - A_{12}A_{23}\int_0^t x_3(s)\mathrm{d}s\right) = |\mathbf{A_1}|\,. \tag{3.37}$$

By Lemma 5 (*i*) and Eq (3.37), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$A_{22}\ln x_1(t) \le \left(|\mathbf{A_1}| + A_{12}A_{23}\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s + \epsilon\right)t - |\mathbf{A}|\int_0^t x_1(s)\mathrm{d}s.$$

On the basis of Eq (3.36), we deduce

$$|\mathbf{A_1}| + A_{12}A_{23}\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \geq |\mathbf{A_1}| + A_{12}A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|} = |\mathbf{A}|\,\frac{|\mathbf{\Xi_1}|}{|\mathbf{\Xi}|} > 0.$$

In view of Lemma 2 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \leq |\mathbf{A}|^{-1}\left(|\mathbf{A_1}| + A_{12}A_{23}\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s\right) \triangleq \Upsilon_{x_1}^{sup} \quad a.s. \tag{3.38}$$

According to system (3.9), Eq (3.36) and Eq (3.38), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_2(t) \leq \left(\Sigma_2 + A_{21}\Upsilon_{x_1}^{sup} - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|} + \epsilon\right)t - A_{22}\int_0^t x_2(s)\mathrm{d}s. \tag{3.39}$$

Combining Eq (3.38) with system (3.39) yields

$$\Sigma_2 + A_{21}\Upsilon_{x_1}^{sup} - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|} \geq \Sigma_2 + A_{21}\,|\mathbf{A}|^{-1}\left(|\mathbf{A_1}| + A_{12}A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right) - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|} = A_{22}\frac{|\mathbf{\Xi_2}|}{|\mathbf{\Xi}|} > 0. \tag{3.40}$$

In the light of Lemma 2 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \leq A_{22}^{-1}\left(\Sigma_2 + A_{21}\Upsilon_{x_1}^{sup} - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right) \quad a.s. \tag{3.41}$$

From system (3.9) and Eq (3.41), for $\forall\epsilon \in (0,1)$ and $t \gg 1$,

$$\ln x_3(t) \leq \left(\Sigma_3 + \frac{A_{32}}{A_{22}}\left(\Sigma_2 + A_{21}\Upsilon_{x_1}^{sup} - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right) + \epsilon\right)t - A_{33}\int_0^t x_3(s)\mathrm{d}s.$$

Thanks to Eq (3.40), we obtain

$$\Sigma_3 + \frac{A_{32}}{A_{22}}\left(\Sigma_2 + A_{21}\Upsilon_{x_1}^{sup} - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right) \geq \Sigma_3 + A_{32}\frac{|\mathbf{\Xi_2}|}{|\mathbf{\Xi}|} = A_{33}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|} > 0.$$

In the light of Lemma 2 and the arbitrariness of ϵ, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \leq A_{33}^{-1}\left(B_3 - \frac{A_{21}A_{32}}{A_{11}A_{22}}B_1 + \frac{A_{32}}{A_{22}}\left(A_{21}\Upsilon_{x_1}^{sup} - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right)\right) \quad a.s.$$

In other words, we have

$$\frac{A_{22}A_{33}\,|\mathbf{A}| - A_{12}A_{21}A_{23}A_{32}}{A_{22}\,|\mathbf{A}|}\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \leq B_3 - \frac{A_{21}A_{32}}{A_{11}A_{22}}B_1 + \frac{A_{32}}{A_{22}}\left(A_{21}\frac{|\mathbf{A_1}|}{|\mathbf{A}|} - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right). \tag{3.42}$$

In view of Eq (3.42) and Assumption 4, we obtain

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s \leq \frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|} \quad a.s. \tag{3.43}$$

Combining Eq (3.36) and Eq (3.43) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\ a.s. \tag{3.44}$$

Substituting Eq (3.44) into Eq (3.38) yields

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s \le |\mathbf{A}|^{-1}\left(|\mathbf{A_1}| + A_{12}A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right) = \frac{|\mathbf{\Xi_1}|}{|\mathbf{\Xi}|}. \tag{3.45}$$

Substituting Eq (3.45) into Eq (3.41) yields

$$\limsup_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \le A_{22}^{-1}\left(\Sigma_2 + A_{21}\frac{|\mathbf{\Xi_1}|}{|\mathbf{\Xi}|} - A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right) = \frac{|\mathbf{\Xi_2}|}{|\mathbf{\Xi}|}. \tag{3.46}$$

In view of system (3.9), we compute

$$\lim_{t\to+\infty} t^{-1}\left(A_{21}\ln x_1(t) + A_{11}\ln x_2(t) + |\mathbf{A}|\int_0^t x_2(s)\mathrm{d}s + A_{11}A_{23}\int_0^t x_3(s)\mathrm{d}s\right) = |\mathbf{A_2}|\,. \tag{3.47}$$

By Lemma 5 (i) and Eq (3.47), for $\forall\epsilon\in(0,1)$ and $t \gg 1$,

$$A_{11}\ln x_2(t) \ge \left(|\mathbf{A_2}| - A_{11}A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|} - \epsilon\right)t - |\mathbf{A}|\int_0^t x_2(s)\mathrm{d}s. \tag{3.48}$$

From Eq (3.40), we have $|\mathbf{\Xi_2}| > 0$ and $|\mathbf{\Xi}| > 0$. Based on system (3.48) and Lemma 2,

$$\liminf_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s \ge |\mathbf{A}|^{-1}\left(|\mathbf{A_2}| - A_{11}A_{23}\frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|}\right) = \frac{|\mathbf{\Xi_2}|}{|\mathbf{\Xi}|}\ a.s. \tag{3.49}$$

Combining Eq (3.46) with Eq (3.49) yields

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Xi_2}|}{|\mathbf{\Xi}|}\ a.s. \tag{3.50}$$

Substituting Eq (3.50) into system (3.9) yields

$$\lim_{t\to+\infty} t^{-1}\left(\ln x_1(t) + A_{11}\int_0^t x_1(s)\mathrm{d}s\right) = A_{11}\frac{|\mathbf{\Xi_1}|}{|\mathbf{\Xi}|}\ a.s.$$

From Eq (3.38), we have $|\mathbf{\Xi_1}| > 0$ and $|\mathbf{\Xi}| > 0$. Therefore, by Lemma 2, we obtain

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Xi_1}|}{|\mathbf{\Xi}|}\ a.s. \tag{3.51}$$

𝔆𝔞𝔰𝔢 (iii) : $|\mathbf{A_2}| > 0 > |\mathbf{\Xi_3}|$. Then $\lim_{t\to+\infty} x_4(t) = 0$ a.s. Thanks to Eq (3.35), we deduce

$$\limsup_{t\to+\infty} t^{-1}\ln\left(x_1^{A_{21}A_{32}}(t)x_2^{A_{11}A_{32}}(t)x_3^{|\mathbf{A}|}(t)\right) \le |\mathbf{\Xi_3}| < 0\ a.s.$$

which implies

$$\limsup_{t\to+\infty} x_1^{A_{21}A_{32}}(t)x_2^{A_{11}A_{32}}(t)x_3^{|\mathbf{A}|}(t) = 0\ a.s. \tag{3.52}$$

According to Lemma 5 (*ii*) and Eq (3.52), we obtain

$$\lim_{t\to+\infty} x_3(t) = 0 \ a.s. \tag{3.53}$$

In other words, we derive

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s = 0 \ a.s. \tag{3.54}$$

In view of Eq (3.47) and Eq (3.54), we get

$$\lim_{t\to+\infty} t^{-1} \left(A_{21} \ln x_1(t) + A_{11} \ln x_2(t) + |\mathbf{A}| \int_0^t x_2(s)\mathrm{d}s \right) = |\mathbf{A_2}|\,. \tag{3.55}$$

Based on Lemma 5 (*i*) and Lemma 2, we have

$$\liminf_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s \geq \frac{|\mathbf{A_2}|}{|\mathbf{A}|} \ \ a.s. \tag{3.56}$$

In the light of Eq (3.37) and Eq (3.54), we obtain

$$\lim_{t\to+\infty} t^{-1} \left(A_{22} \ln x_1(t) - A_{12} \ln x_2(t) + |\mathbf{A}| \int_0^t x_1(s)\mathrm{d}s \right) = |\mathbf{A_1}|\,.$$

By Lemma 5 (*i*) and Lemma 2, we obtain

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s \leq \frac{|\mathbf{A_1}|}{|\mathbf{A}|} \ a.s. \tag{3.57}$$

Substituting Eq (3.54) and Eq (3.57) into system (3.9) yields

$$\ln x_2(t) \leq \left(\Sigma_2 + A_{21} \frac{|\mathbf{A_1}|}{|\mathbf{A}|} + \epsilon \right) t - A_{22} \int_0^t x_2(s)\mathrm{d}s.$$

On the basis of Lemma 2 and the arbitrariness of ϵ, we have

$$\limsup_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s \leq A_{22}^{-1} \left(\Sigma_2 + A_{21} \frac{|\mathbf{A_1}|}{|\mathbf{A}|} \right) = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} \ a.s. \tag{3.58}$$

Combining Eq (3.56) with Eq (3.58) yields

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} \ a.s. \tag{3.59}$$

By system (3.9) and Eq (3.59), we compute

$$\lim_{t\to+\infty} t^{-1} \left(\ln x_1(t) + A_{11} \int_0^t x_1(s)\mathrm{d}s \right) = B_1 - A_{12} \frac{|\mathbf{A_2}|}{|\mathbf{A}|} = A_{11} \frac{|\mathbf{A_1}|}{|\mathbf{A}|} \ a.s.$$

In the light of Lemma 2, we obtain

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} \ a.s. \tag{3.60}$$

$\mathfrak{Case}$ (iv) : $B_1 > 0 > |\mathbf{A_2}|$. Then

$$\lim_{t\to+\infty} x_i(t) = 0 \ a.s. \ (i = 3, 4). \tag{3.61}$$

According to Eq (3.55), we gain

$$\limsup_{t\to+\infty} t^{-1} \ln\left(x_1^{A_{21}}(t) x_2^{A_{11}}(t)\right) \le |A_2| < 0 \ a.s. \tag{3.62}$$

Hence, $\limsup_{t\to+\infty} x_1^{A_{21}}(t) x_2^{A_{11}}(t) = 0$. By Lemma 5 (*ii*) and Eq (3.62),

$$\lim_{t\to+\infty} x_2(t) = 0 \ a.s. \tag{3.63}$$

In other words, we derive

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = 0 \ a.s. \tag{3.64}$$

Substituting Eq (3.64) into system (3.9) yields

$$\ln x_1(t) = B_1 t - A_{11} \int_0^t x_1(s)\mathrm{d}s + o(t).$$

On the basis of Lemma 2 and the arbitrariness of ϵ, we obtain

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} \ a.s. \tag{3.65}$$

$\mathfrak{Case}$ (v) : $B_1 < 0$. Compute

$$\lim_{t\to+\infty} t^{-1} \left(\ln x_1(t) + A_{11} \int_0^t x_1(s)\mathrm{d}s + A_{12} \int_0^t x_2(s)\mathrm{d}s\right) = B_1. \tag{3.66}$$

By Eq (3.66), we have

$$\limsup_{t\to+\infty} t^{-1} \left(\ln x_1(t) + A_{11} \int_0^t x_1(s)\mathrm{d}s\right) \le B_1.$$

In view of Lemma 2, we obtain $\lim_{t\to+\infty} x_1(t) = 0$ a.s. According to Lemma 5 (*ii*), we get

$$\lim_{t\to+\infty} x_j(t) = 0 \ a.s. \ (j = 2, 3, 4). \tag{3.67}$$

The proof is complete. □

4. Numerical examples

In this section we introduce some numerical examples to illustrate our main results. For simplicity, we suppose that $\mathbb{S} = \{1, 2\}$. Then system (1.6) is a hybrid system of the following two subsystems:

$$\begin{cases} \left.\begin{aligned} &\mathrm{d}x_1(t) = x_1(t)\left[(r_1(1) - r_{11}C_{10}(t) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1(t,1)\right],\\ &\mathrm{d}x_2(t) = x_2(t)\left[(-r_2(1) - r_{22}C_{20}(t) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2(t,1)\right],\\ &\mathrm{d}x_3(t) = x_3(t)\left[(-r_3(1) - r_{33}C_{30}(t) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t) - \mathcal{D}_{34}(x_4)(t))\,\mathrm{d}t + \mathcal{S}_3(t,1)\right],\\ &\mathrm{d}x_4(t) = x_4(t)\left[(-r_4(1) - r_{44}C_{40}(t) + \mathcal{D}_{43}(x_3)(t) - \mathcal{D}_{44}(x_4)(t))\,\mathrm{d}t + \mathcal{S}_4(t,1)\right],\\ &\mathrm{d}C_{i0}(t) = \left[0.1C_e(t) - (0.1 + 0.1)\,C_{i0}(t)\right]\mathrm{d}t,\\ &\mathrm{d}C_e(t) = -0.5C_e(t)\mathrm{d}t, \end{aligned}\right\} t \ne 12n,\\ \Delta x_i(t) = 0,\ \Delta C_{i0}(t) = 0,\ \Delta C_e(t) = 0.6,\ t = 12n,\ n \in \mathbb{N}_+ \ (i = 1, 2, 3, 4), \end{cases} \tag{4.1}$$

and

$$\left\{\begin{array}{l}\mathrm{d}x_1(t) = x_1(t)\left[(r_1(2) - r_{11}C_{10}(t) - \mathcal{D}_{11}(x_1)(t) - \mathcal{D}_{12}(x_2)(t))\,\mathrm{d}t + \mathcal{S}_1\,(t,2)\right],\\ \mathrm{d}x_2(t) = x_2(t)\left[(-r_2(2) - r_{22}C_{20}(t) + \mathcal{D}_{21}(x_1)(t) - \mathcal{D}_{22}(x_2)(t) - \mathcal{D}_{23}(x_3)(t))\,\mathrm{d}t + \mathcal{S}_2\,(t,2)\right],\\ \mathrm{d}x_3(t) = x_3(t)\left[(-r_3(2) - r_{33}C_{30}(t) + \mathcal{D}_{32}(x_2)(t) - \mathcal{D}_{33}(x_3)(t) - \mathcal{D}_{34}(x_4)(t))\,\mathrm{d}t + \mathcal{S}_3\,(t,2)\right],\\ \mathrm{d}x_4(t) = x_4(t)\left[(-r_4(2) - r_{44}C_{40}(t) + \mathcal{D}_{43}(x_3)(t) - \mathcal{D}_{44}(x_4)(t))\,\mathrm{d}t + \mathcal{S}_4\,(t,2)\right],\\ \mathrm{d}C_{i0}(t) = \left[0.1C_e(t) - (0.1+0.1)\,C_{i0}(t)\right]\mathrm{d}t,\\ \mathrm{d}C_e(t) = -0.5C_e(t)\mathrm{d}t,\end{array}\right\} t \neq 12n,$$
$$\Delta x_i(t) = 0,\ \Delta C_{i0}(t) = 0,\ \Delta C_e(t) = 0.6,\ t = 12n,\ n \in \mathbb{N}_+\ (i = 1,2,3,4), \tag{4.2}$$

with initial conditions $x_1(\theta) = 2\mathrm{e}^{\theta}$, $x_2(\theta) = 1.5\mathrm{e}^{\theta}$, $x_3(\theta) = 0.8\mathrm{e}^{\theta}$ $x_4(\theta) = 0.5\mathrm{e}^{\theta}$ and $\theta \in [-\ln 2, 0]$.

Let $r_{ii} = 0.3$, $\tau_{ji} = \ln 2$, $\mu_{ji}(\theta) = \mu_{ji}\mathrm{e}^{\theta}$, $\gamma_j(\mu, i) = \gamma_j(i)$ and $\lambda(\mathbb{Z}) = 1$, see Table 4. Denote

$$\text{Param(i)} = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \mu_{11} & \mu_{12} & 0 & 0 & \sigma_1(i) & \gamma_1(i)\\ a_{21} & a_{22} & a_{23} & 0 & \mu_{21} & \mu_{22} & \mu_{23} & 0 & \sigma_2(i) & \gamma_2(i)\\ 0 & a_{32} & a_{33} & a_{34} & 0 & \mu_{32} & \mu_{33} & \mu_{34} & \sigma_3(i) & \gamma_3(i)\\ 0 & 0 & a_{43} & a_{44} & 0 & 0 & \mu_{43} & \mu_{44} & \sigma_4(i) & \gamma_4(i)\end{pmatrix}.$$

Then system (1.6) may be regarded as the result of regime switching between subsystems (4.1) and (4.2) with the following estimated parameters, respectively,

$$\text{Param(1)} = \begin{pmatrix} 0.2 & 0.1 & 0 & 0 & 0.2 & 0.1 & 0 & 0 & 0.1 & 0.1\\ 0.5 & 0.3 & 0.1 & 0 & 0.2 & 0.1 & 0.1 & 0 & 0.1 & 0.1\\ 0 & 0.4 & 0.3 & 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.1 & 0.1\\ 0 & 0 & 0.4 & 0.3 & 0 & 0 & 0.1 & 0.2 & 0.1 & 0.1\end{pmatrix},$$

$$\text{Param(2)} = \begin{pmatrix} 0.2 & 0.1 & 0 & 0 & 0.2 & 0.1 & 0 & 0 & 1.2 & 0.2\\ 0.5 & 0.3 & 0.1 & 0 & 0.2 & 0.1 & 0.1 & 0 & 0.2 & 0.2\\ 0 & 0.4 & 0.3 & 0.2 & 0 & 0.2 & 0.2 & 0.2 & 0.2 & 0.2\\ 0 & 0 & 0.4 & 0.3 & 0 & 0 & 0.1 & 0.2 & 0.2 & 0.2\end{pmatrix}.$$

Table 4. Source of some parameter values in system (1.6).

Parameter	Value	Source
k_i	0.1	[55]
g_i	0.1	[55]
m_i	0.1	[55]
h	0.5	[55]
γ	12	[55]
b	0.6	[55]
τ_{ji}	$\ln 2$	[56]
$\lambda(\mathbb{Z})$	1	[56]

Compute $|\mathbf{\Delta}| = 0.066525$, $|\mathbf{\Xi}| = 0.1005$ and $|\mathbf{A}| = 0.195$. Denote

$$\overrightarrow{\gamma}(1) = (\gamma_1(1), \gamma_2(1), \gamma_3(1), \gamma_4(1)),\ \overrightarrow{\mathbf{r}}(j) = (r_1(j), r_2(j), r_3(j), r_4(j))\ \ (j = 1, 2).$$

4.1. The effects of Markovian switching on the persistence in mean and extinction

Let $\overrightarrow{\mathbf{r}}(1) = (0.9, 0.5, 0.3, 0.2)$. Compute

$$|\boldsymbol{\Delta_1}| = 0.1384,\ |\boldsymbol{\Delta_2}| = 0.1113,\ |\boldsymbol{\Delta_3}| = 0.0614,\ |\boldsymbol{\Delta_4}| = 0.0317 > 0.$$

Based on Theorem 2, all species in subsystem (4.1) are persistent in mean and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Delta_1}|}{|\boldsymbol{\Delta}|} = 2.0811 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Delta_2}|}{|\boldsymbol{\Delta}|} = 1.6731 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Delta_3}|}{|\boldsymbol{\Delta}|} = 0.9226 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Delta_4}|}{|\boldsymbol{\Delta}|} = 0.4762 \quad a.s. \end{cases} \tag{4.3}$$

Let $\overrightarrow{\mathbf{r}}(2) = (0.6, 0.3, 0.2, 0.1)$. Then $B_1 = -0.1527 < 0$. From Theorem 2, all species in subsystem (4.2) are extinctive.

Case 1 : $(\pi_1, \pi_2) = (0.8, 0.2)$. Compute

$$|\boldsymbol{\Delta_1}| = 0.1096,\ |\boldsymbol{\Delta_2}| = 0.0779,\ |\boldsymbol{\Delta_3}| = 0.0390,\ |\boldsymbol{\Delta_4}| = 0.0090 > 0.$$

By Theorem 2, all species in system (1.6) are persistent in mean and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Delta_1}|}{|\boldsymbol{\Delta}|} = 1.6469 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Delta_2}|}{|\boldsymbol{\Delta}|} = 1.1709 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Delta_3}|}{|\boldsymbol{\Delta}|} = 0.5870 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_4(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Delta_4}|}{|\boldsymbol{\Delta}|} = 0.1346 \quad a.s. \end{cases} \tag{4.4}$$

Case 2 : $(\pi_1, \pi_2) = (0.6, 0.4)$. Compute

$$|\boldsymbol{\Delta_4}| = -0.0138 < 0,\ |\boldsymbol{\Xi_1}| = 0.1205,\ |\boldsymbol{\Xi_2}| = 0.0700,\ |\boldsymbol{\Xi_3}| = 0.0132 > 0.$$

From Theorem 2, $x_1(t)$, $x_2(t)$ and $x_3(t)$ are persistent in mean, while $x_4(t)$ is extinctive and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Xi_1}|}{|\boldsymbol{\Xi}|} = 1.1988 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_2(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Xi_2}|}{|\boldsymbol{\Xi}|} = 0.6965 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\int_0^t x_3(s)\mathrm{d}s = \dfrac{|\boldsymbol{\Xi_3}|}{|\boldsymbol{\Xi}|} = 0.1309 \quad a.s. \end{cases} \tag{4.5}$$

Case 3 : $(\pi_1, \pi_2) = (0.5, 0.5)$. Compute

$$|\mathbf{\Xi_3}| = -0.0137 < 0,\ |\mathbf{A_1}| = 0.1923,\ |\mathbf{A_2}| = 0.0852 > 0.$$

Thanks to Theorem 2, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ and $x_4(t)$ are extinctive and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} = 0.9860 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} = 0.4368 \quad a.s. \end{cases} \tag{4.6}$$

Case 4 : $(\pi_1, \pi_2) = (0.3, 0.7)$. Compute

$$|\mathbf{A_2}| = -0.0279 < 0,\ B_1 = 0.1557 > 0.$$

Based on Theorem 2, $x_1(t)$ is persistent in mean, while $x_2(t)$, $x_3(t)$ and $x_4(t)$ are extinctive and

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} = 0.5191 \quad a.s. \tag{4.7}$$

Case 5 : $(\pi_1, \pi_2) = (0.1, 0.9)$. Compute $B_1 = -0.0499 < 0$. On the basis of Theorem 2, all species in system (1.6) are extinctive.

4.2. The effects of Lévy jumps on the persistence in mean and extinction

Let $\overrightarrow{\mathbf{r}}(1) = (0.7, 0.5, 0.3, 0.2)$. We study the effects of Lévy jumps on the persistence in mean and extinction of system (4.1) by changing the values of $\gamma_j(1)$ and setting the remaining parameters of the examples to be the same as those in system (4.1). Denote $I_4 = \{-0.3, 0.4\}$, $\alpha_4 \in I_4$; $I_3 = \{-0.6, 1.1\}$, $\alpha_3 \in I_3$; $I_2 = \{-0.9, 1.9\}$, $\alpha_2 \in I_2$; $I_1 = \{-0.8, 1.7\}$, $\alpha_1 \in I_1$.

4.2.1. The effects of $\gamma_j(1)$ on the persistence in mean and extinction of system (4.1)

Case 1 : Let $\overrightarrow{\gamma}(1) = (0.1, 0.1, 0.1, \boldsymbol{\alpha_4})$. Then $|\mathbf{\Delta_4}| < 0$, $|\mathbf{\Xi_3}| = 0.0606 > 0$. According to Theorem 2, $x_1(t)$, $x_2(t)$ and $x_3(t)$ are persistent in mean, while $x_4(t)$ is extinctive.

Let $\overrightarrow{\gamma}(1) = (0.1, 0.1, 0.1, \mathbf{0.1})$. Then $|\mathbf{\Delta_4}| = 0.0047 > 0$. By Theorem 2, all species in system (4.1) are persistent in mean. See Table 5.

Table 5. Changes of $\gamma_4(1)$ when $\gamma_1(1) = \gamma_2(1) = \gamma_3(1) = 0.1$.

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\gamma_4(1)$	$\overline{\mathbf{x}^{\mathrm{T}}(\infty)}$
0.1	0.1	0.1	$\boldsymbol{\alpha_4}$	$(1.6852, 1.1316, 0.6027, \mathbf{0})$
0.1	0.1	0.1	**0.1**	$(1.6805, 1.1410, 0.5618, \mathbf{0.0703})$

Case 2 : Let $\overrightarrow{\gamma}(1) = (0.1, 0.1, \boldsymbol{\alpha_3}, \alpha_4)$. Then $|\mathbf{\Xi_3}| < 0$, $|\mathbf{A_2}| = 0.2478 > 0$. Based on Theorem 2, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ and $x_4(t)$ are extinctive. See Table 6.

Case 3 : Let $\overrightarrow{\gamma}(1) = (0.1, \boldsymbol{\alpha_2}, \alpha_3, \alpha_4)$. Then $|\mathbf{A_2}| < 0$, $B_1 = 0.6753 > 0$. From Theorem 2, $x_1(t)$ is persistent in mean, while $x_2(t)$, $x_3(t)$ and $x_4(t)$ are extinctive. See Table 7.

Case 4 : Let $\overrightarrow{\gamma}(1) = (\boldsymbol{\alpha_1}, \alpha_2, \alpha_3, \alpha_4)$. Then $B_1 < 0$. Thanks to Theorem 2, all species are extinctive. See Table 8.

Table 6. Changes of $\gamma_3(1)$ when $\gamma_1(1) = \gamma_2(1) = 0.1$ and $\gamma_4(1) \in I_4$.

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\gamma_4(1)$	$\overline{\mathbf{x}^{\mathrm{T}}(\infty)}$
0.1	0.1	$\boldsymbol{\alpha_3}$	$\in I_4$	$(1.6157, 1.2707, \mathbf{0}, 0)$
0.1	0.1	**0.1**	$\in I_4$	$(1.6852, 1.1316, \mathbf{0.6027}, 0)$

Table 7. Changes of $\gamma_2(1)$ when $\gamma_1(1) = 0.1$, $\gamma_3(1) \in I_3$ and $\gamma_4(1) \in I_4$.

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\gamma_4(1)$	$\overline{\mathbf{x}^{\mathrm{T}}(\infty)}$
0.1	$\boldsymbol{\alpha_2}$	$\in I_3$	$\in I_4$	$(2.2510, \mathbf{0}, 0, 0)$
0.1	**0.1**	$\in I_3$	$\in I_4$	$(1.6157, \mathbf{1.2707}, 0, 0)$

4.2.2. The effects of $\gamma_1(1)$ on the persistence in mean and extinction of system (4.1)

Case 1 : Let $\gamma_1(1) = -0.8$. Then $B_1 = -0.1294 < 0$. According to Theorem 2, all species in system (4.1) are extinctive.

Let $\gamma_1(1) = -0.7$. Then $|\mathbf{A_2}| = -0.0518 < 0$, $B_1 = 0.1760 > 0$. By Theorem 2, $x_1(t)$ is persistent in mean, while $x_2(t)$, $x_3(t)$ and $x_4(t)$ are extinctive and

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} = 0.5868 \quad a.s. \tag{4.8}$$

Let $\gamma_1(1) = -0.6$. Then $|\mathbf{\Xi_3}| = -0.0329 < 0$, $|\mathbf{A_2}| = 0.0608 > 0$. Based on Theorem 2, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ and $x_4(t)$ are extinctive and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} = 1.0564 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} = 0.3119 \quad a.s. \end{cases} \tag{4.9}$$

Let $\gamma_1(1) = -0.3$. Then $|\mathbf{\Delta_4}| = -0.0023 < 0$, $|\mathbf{\Xi_3}| = 0.0450 > 0$. In view of Theorem 2, $x_1(t)$, $x_2(t)$ and $x_3(t)$ are persistent in mean, while $x_4(t)$ is extinctive and

$$\begin{cases} \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Xi_1}|}{|\mathbf{\Xi}|} = 1.5740 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Xi_2}|}{|\mathbf{\Xi}|} = 1.0074 \quad a.s. \\ \lim\limits_{t\to+\infty} t^{-1}\displaystyle\int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|} = 0.4476 \quad a.s. \end{cases} \tag{4.10}$$

Table 8. Changes of $\gamma_1(1)$ when $\gamma_2(1) \in I_2$, $\gamma_3(1) \in I_3$ and $\gamma_4(1) \in I_4$.

$\gamma_1(1)$	$\gamma_2(1)$	$\gamma_3(1)$	$\gamma_4(1)$	$\overline{\mathbf{x}^{\mathrm{T}}(\infty)}$
$\boldsymbol{\alpha_1}$	$\in I_2$	$\in I_3$	$\in I_4$	$(\mathbf{0}, 0, 0, 0)$
0.1	$\in I_2$	$\in I_3$	$\in I_4$	$(\mathbf{2.2510}, 0, 0, 0)$

Let $\gamma_1(1) = -0.1$. Then $|\mathbf{\Delta_4}| = 0.0046 > 0$. From Theorem 2, all species are persistent in mean and

$$\begin{cases} \lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} = 1.6792 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} = 1.1392 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = 0.5606 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_4(s)\mathrm{d}s = \frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|} = 0.0690 \quad a.s. \end{cases} \tag{4.11}$$

Case 2 : Let $\gamma_1(1) = 0.2$. Then $|\mathbf{\Delta_4}| = 0.0029 > 0$. Thanks to Theorem 2, all species are persistent in mean and

$$\begin{cases} \lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Delta_1}|}{|\mathbf{\Delta}|} = 1.6545 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Delta_2}|}{|\mathbf{\Delta}|} = 1.1065 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Delta_3}|}{|\mathbf{\Delta}|} = 0.5384 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_4(s)\mathrm{d}s = \frac{|\mathbf{\Delta_4}|}{|\mathbf{\Delta}|} = 0.0440 \quad a.s. \end{cases} \tag{4.12}$$

Let $\gamma_1(1) = 0.6$. Then $|\mathbf{\Delta_4}| = -0.0122 < 0$, $|\mathbf{\Xi_3}| = 0.0230 > 0$. On the basis of Theorem 2, $x_1(t)$, $x_2(t)$ and $x_3(t)$ are persistent in mean, while $x_4(t)$ is extinctive and

$$\begin{cases} \lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{\Xi_1}|}{|\mathbf{\Xi}|} = 1.4172 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{\Xi_2}|}{|\mathbf{\Xi}|} = 0.8323 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_3(s)\mathrm{d}s = \frac{|\mathbf{\Xi_3}|}{|\mathbf{\Xi}|} = 0.2287 \quad a.s. \end{cases} \tag{4.13}$$

Let $\gamma_1(1) = 0.9$. Then $|\mathbf{\Xi_3}| = -0.0155 < 0$, $|\mathbf{A_2}| = 0.0957 > 0$. By Theorem 2, $x_1(t)$ and $x_2(t)$ are persistent in mean, while $x_3(t)$ and $x_4(t)$ are extinctive and

$$\begin{cases} \lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{|\mathbf{A_1}|}{|\mathbf{A}|} = 1.1608 \quad a.s. \\ \lim_{t\to+\infty} t^{-1} \int_0^t x_2(s)\mathrm{d}s = \frac{|\mathbf{A_2}|}{|\mathbf{A}|} = 0.4908 \quad a.s. \end{cases} \tag{4.14}$$

Let $\gamma_1(1) = 1.3$. Then $|\mathbf{A_2}| = -0.0297 < 0$, $B_1 = 0.2129 > 0$. From Theorem 2, $x_1(t)$ is persistent in mean, while $x_2(t)$, $x_3(t)$ and $x_4(t)$ are extinctive and

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)\mathrm{d}s = \frac{B_1}{A_{11}} = 0.7097 \quad a.s. \tag{4.15}$$

Let $\gamma_1(1) = 1.7$. Then $B_1 = -0.0267 < 0$. In view of Theorem 2, all species are extinctive.

5. Discussion and conclusions

This paper concerns the dynamics of a stochastic hybrid delay food chain model with jumps in an impulsive polluted environment. Theorem 2 establishes sufficient and necessary conditions for persistence in mean and extinction of each species. Our results reveal that the stochastic dynamics of the system is closely correlated with both time delays and environmental noises.

Some interesting topics deserve further investigation, for instance, it is meaningful to consider the optimal harvesting problem of the stochastic hybrid delay food chain model with Lévy noises in an impulsive polluted environment. One may also propose some more realistic systems, such as considering the generalized functional response and the influences of impulsive perturbations. We will leave investigation of these problems to the future.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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