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Research article

L_2/L_1 induced norm and Hankel norm analysis in sampled-data systems

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Abstract: This paper is concerned with the L_2/L_1 induced and Hankel norms of sampled-data systems. In defining the Hankel norm, the h-periodicity of the input-output relation of sampled-data systems is taken into account, where h denotes the sampling period; past and future are separated by the instant $\Theta \in [0, h)$, and the norm of the operator describing the mapping from the past input in L_1 to the future output in L_2 is called the quasi L_2/L_1 Hankel norm at Θ. The L_2/L_1 Hankel norm is defined as the supremum over $\Theta \in [0, h)$ of this norm, and if it is actually attained as the maximum, then a maximum-attaining Θ is called a critical instant. This paper gives characterization for the L_2/L_1 induced norm, the quasi L_2/L_1 Hankel norm at Θ and the L_2/L_1 Hankel norm, and it shows that the first and the third ones coincide with each other and a critical instant always exists. The matrix-valued function $H(\varphi)$ on $[0, h)$ plays a key role in the sense that the induced/Hankel norm can be obtained and a critical instant can be detected only through $H(\varphi)$, even though φ is a variable that is totally irrelevant to Θ. The relevance of the induced/Hankel norm to the H_2 norm of sampled-data systems is also discussed.

Keywords: sampled-data systems; intersample behavior; induced norm; quasi Hankel norm; Hankel norm; H_2 norm; impulse response
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1. Introduction

Control theory usually deals with dynamical systems whose current output depends not only on the current value of the input, but also on its past values. Such dynamical systems are often affected by an unexpected input called a disturbance, and failing to suppress its effect adequately on the output despite the use of feedback control means poor control performance. Thus, it is important in such systems to evaluate how the input affects the output in the worst case. For stable continuous-time linear time-invariant (LTI) systems, the L_q/L_p induced norm is a quantitative measure of the worst effect of the input $w \in L_p[0, \infty)$ on the output z, viewed as an element in $L_q[0, \infty)$, where $L_p(\mathcal{I})$

denotes the Lebesgue space of vector-valued functions on the interval $\mathcal{I}$ endowed with the associated L_p norm. Similarly, the L_q/L_p Hankel norm is a quantitative measure of the worst effect of the past input $w \in L_p(-\infty, 0)$ on the future output z, regarded as an element in $L_q[0, \infty)$. These two norms play an important role in evaluating the properties and performance of dynamical systems and control systems; thus, characterization of these norms and relevant issues have been studied intensively [1–8].

Because most control systems nowadays employ digital controllers in their implementation, sampled-data systems constitute a class of important dynamical systems; such a control system with a discrete-time controller for a continuous-time plant is called a sampled-data system, especially when the primary attention is paid to the intersample behavior of the continuous-time signals in the plant, rather than their discrete-time behavior that is viewed on the sampling instants associated with the sampling period h that is inherent to the digital controller (even though the case of non-periodic sampling has also been studied extensively [9]). The viewpoint on this intersample behavior in the periodic sampling case leads us to viewing sampled-data systems as h-periodic systems from the input/output relation viewed in continuous time, even when the continuous-time plant is LTI and the discrete-time controller is also LTI (see Section 2, above (2.3), for the precise meaning of this h-periodicity). This aspect has inspired many important and interesting studies on sampled-data systems, such as dealing with the topics on the H_2 norms that are suitably defined for stable sampled-data systems [10–14], as well as the induced norm from L_2 to L_2 [15–19], the induced norm from L_2 to L_∞ [14, 20] and the induced norm from L_∞ to L_∞ [21–24].

These studies on the induced norms of stable sampled-data systems with periodic sampling have recently been extended to studies on the Hankel norm from L_2 to L_∞ [25, 26], the Hankel norm from L_2 to L_2 [27] and the Hankel norm from L_∞ to L_∞ [28], through a novel viewpoint focused on amending the somewhat insufficient arguments in the pioneering study [29]. More precisely speaking, the h-periodic nature of sampled-data systems was shown to give rise, in the Hankel norm analysis, to introducing the novel notions of what are called the associated quasi L_∞/L_2, L_2/L_2 and L_∞/L_∞ Hankel operators/ norms at each Θ in the sampling interval $[0, h)$. Then, the supremum of the quasi L_2/L_2 Hankel norms over the sampling interval $[0, h)$ is defined as the L_2/L_2 Hankel norm, and this is similar for the L_∞/L_2 Hankel norm and the L_∞/L_∞ Hankel norm.

Among these operator-theoretic studies, those on the L_∞/L_2 induced norm and the L_∞/L_2 Hankel norm for sampled-data systems have shown that they actually coincide with each other [25, Corollary 3.6], [26]. On the other hand, it is known that these two norms coincide with each other also for continuous-time LTI systems [2, Corollary], [3, Theorem 2], in which case they also equal their H_2 norm [8], as far as single-output systems are concerned [4, Corollary 3.1 and Remark 3.3]. In this sense, the L_∞/L_2 induced and Hankel norms for sampled-data systems can be regarded as an important quantity that has a close connection with the H_2 norm of sampled-data systems introduced in [11, 12], but with a slightly different viewpoint taken for dealing with the intersample behavior. Indeed, such an aspect has been discussed extensively in [14].

With the above situation in mind, the present paper is concerned with alternatively studying the induced norm from L_1 to L_2 and the Hankel norm from L_1 to L_2 for stable sampled-data systems for the following reason. For continuous-time LTI systems, these two norms coincide with each other [2, Corollary], [3, Theorem 2] and equal their H_2 norm as far as single-input systems are concerned [4, Corollary 3.1 and Remark 3.3]. In this sense, the L_2/L_1 induced and Hankel norms could also be regarded as an alternative quantity that has a close connection with the H_2 norm [11, 12] for sampled-data systems. Hence, the L_2/L_1 induced and Hankel norms for sampled-data systems could be regarded as dealing with the intersample behavior of sampled-data systems through yet another viewpoint, and thus would deserve an independent study in addition to the existing studies for the L_∞/L_2 case.

This paper is organized as follows. The definitions of the L_2/L_1 induced norm and the L_2/L_1 Hankel norm are given for stable sampled-data systems in Section 2. For the operator-theoretic treatment of sampled-data systems for these norms, we employ the lifting treatment [17, 18, 30], which allows us to deal with the continuous-time input-output relation of sampled-data systems in a discrete-time fashion. This treatment, together with the associated operator-based discrete-time representation of sampled-data systems, is also briefly reviewed in this section. Section 3 gives the characterization of the L_2/L_1 induced norm of stable sampled-data systems through the use of a matrix-valued function denoted by $H(\varphi), \varphi \in [0, h)$, which further admits the numerical computation of this norm as well. Section 4 first gives the characterization of the norm of the quasi L_2/L_1 Hankel operator, i.e., the quasi L_2/L_1 Hankel norm at $\Theta \in [0, h)$, and this result is further extended to the characterization of the L_2/L_1 Hankel norm for stable sampled-data systems. These arguments lead directly to the numerical computation methods for the quasi L_2/L_1 Hankel norm for each $\Theta \in [0, h)$, as well as the L_2/L_1 Hankel norm. In particular, $H(\varphi)$ also plays a key role for the latter, through the use of which we can establish the fact that the L_2/L_1 induced norm (for the analysis of which the matrix-valued function $H(\varphi)$ was first introduced) and the L_2/L_1 Hankel norms also coincide with each other for sampled-data systems. Furthermore, this section tackles the problem of whether a critical instant exists in the L_2/L_1 analysis of sampled-data systems, where a critical instant is defined as the maximum-attaining point $\Theta \in [0, h)$—if one exists—at which the associated quasi L_2/L_1 Hankel norm equals the L_2/L_2 Hankel norm, i.e., the supremum of the quasi L_2/L_2 Hankel norms over the sampling interval $[0, h)$. It is also discussed how $H(\varphi)$ could be used to detect a critical instant, and some relevant issues are further studied. Section 5 then discusses the relationship between the L_2/L_1 induced/Hankel norm and the H_2 norms for sampled-data systems, paying attention to the fact that there are different definitions for the H_2 norm of sampled-data systems in the literature (for example, in addition to the aforementioned one given in [11, 12], other definitions have been given in [10, 14]). In particular, it is shown that the L_2/L_1 viewpoint could lead to introducing further different definitions for the H_2 norm, and that the L_2/L_1 induced norm has a closer relationship to all these H_2 norms than the L_∞/L_2 induced norm in sampled-data systems. Section 6 gives numerical examples illustrating the arguments developed in this paper and confirm that a critical instant can not only be zero, but also nonzero, depending on the given sampled-data systems. Finally, concluding remarks and future topics are stated in Section 7.

The notation in this paper is as follows. We use $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ to mean the set of n-dimensional real vectors and that of $n \times m$ real matrices, respectively. The 2-norm and 1-norm of a finite-dimensional vector are denoted by $|\cdot|_2$ and $|\cdot|_1$, respectively, i.e., $|v|_2 := (v^T v)^{1/2}$ and $|v|_1 := \sum_{i=1}^{m} |v_i|$ for $v \in \mathbb{R}^m$. We use $\|\cdot\|_{L_{1,p}(\mathcal{I})}$ and $\|\cdot\|_{L_2(\mathcal{I})}$ to mean the $L_{1,p}$ and L_2 norms

$$\|w(\cdot)\|_{L_{1,p}(\mathcal{I})} := \int_{\mathcal{I}} |w(t)|_p dt \ \ (p = 1, 2), \quad \|z(\cdot)\|_{L_2(\mathcal{I})} := \left(\int_{\mathcal{I}} |z(t)|_2^2 dt \right)^{1/2} \tag{1.1}$$

respectively, for real-vector-valued functions w and z on the interval $\mathcal{I}$ such that the associated right-hand side is well-defined. To simplify the notation and facilitate the arguments, $L_1(\mathcal{I})$ is used to mean either or both of $L_{1,1}(\mathcal{I})$ or $L_{1,2}(\mathcal{I})$. For example, we would mean the associate statement with $L_1(\mathcal{I})$

replaced by $L_{1,p}(\mathcal{I})$ with $p = 1$, as well as the statement with $p = 2$ at the same time. Otherwise, the statement would refer to some viewpoint that simultaneously applies to both $L_{1,1}(\mathcal{I})$ and $L_{1,2}(\mathcal{I})$. The usage and distinction would be clear from the context. In addition to those in (1.1), this paper uses many other relevant norm symbols, among which most important ones are summarized in Table 1 to facilitate the understanding of the arguments in this paper. Finally, $\mathrm{sq}(X)$ with a matrix X is a shorthand notation for $X^T X$, and $\mu_1(\cdot)$ and $\mu_2(\cdot)$ denote the maximum diagonal entry and maximum eigenvalue of a positive semidefinite matrix, respectively.

Table 1. Main norm symbols used in this paper.

Symbol	Meaning	Eq No.
$\|\cdot\|_{L_{1,p}(\mathcal{I})}$ or $\|\cdot\|_{L_1(\mathcal{I})}$	L_1 norm on the interval $\mathcal{I}$	(1.1)
$\|\cdot\|_1$	shorthand notation of the above when $\mathcal{I} = [0, h)$	
$\|\cdot\|_{L_2(\mathcal{I})}$	L_2 norm on the interval $\mathcal{I}$	(1.1)
$\|\cdot\|_2$	shorthand notation of the above when $\mathcal{I} = [0, h)$	
$\|\Sigma_{\mathrm{SD}}\|_{2/(1,p)}$ or $\|\Sigma_{\mathrm{SD}}\|_{2/1}$	L_2/L_1 induced norm of Σ_{SD}	(2.2)
$\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}$	L_2/L_1 Hankel norm of Σ_{SD}	(2.4)
$\|\widehat{w}\|_{1,0+}$	$L_1[0, \infty)$ norm of w represented in terms of its lifting $\widehat{w}$	(3.2)
$\|\widehat{z}\|_{2,0+}$	$L_2[0, \infty)$ norm of z represented in terms of its lifting $\widehat{z}$	(3.2)
	or $L_2[\Theta, \infty)$ norm of z represented in terms of its lifting $\widehat{z}$ and $\mathbf{\Phi}_\Theta$	(4.3)
$\|\widehat{w}\|_{1,0-}$	$L_1[-\infty, \Theta)$ norm of w represented in terms of its lifting $\widehat{w}$	(4.3)

2. The L_2/L_1 induced norm, the L_2/L_1 Hankel norm and the lifting treatment of sampled-data systems

Consider the stable sampled-data system Σ_{SD} shown in Figure 1, where P denotes the continuous-time generalized plant, and Ψ, $\mathcal{H}$ and $\mathcal{S}$ denote the discrete-time controller, the zero-order hold and the ideal sampler, respectively, operating with the sampling period h. We suppose that P and Ψ are LTI and are respectively described by

$$P : \begin{cases} \frac{dx}{dt} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{12} u \\ y = C_2 x \end{cases} \qquad \Psi : \begin{cases} \psi_{k+1} = A_\Psi \psi_k + B_\Psi y_k \\ u_k = C_\Psi \psi_k + D_\Psi y_k \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$, $u(t) \in \mathbb{R}^{n_u}$, $z(t) \in \mathbb{R}^{n_z}$, $y(t) \in \mathbb{R}^{n_y}$, $\psi_k \in \mathbb{R}^{n_\Psi}$, $y_k = y(kh)$, $u(t) = u_k$ $(kh \leq t < (k+1)h)$; note that the time instants at which the sampler takes its actions are assumed to be the integer multiples of the sampling period h, and the zero-order hold is assumed to operate in synchrony with the sampler. Throughout the paper, we are interested in how the effect of $w \in L_1$ is suppressed in the response $z \in L_2$ in the sampled-data system Σ_{SD}, and, for the quantitative analysis developed in this paper, this section defines the L_2/L_1 induced norm and the L_2/L_1 Hankel norm for Σ_{SD}.

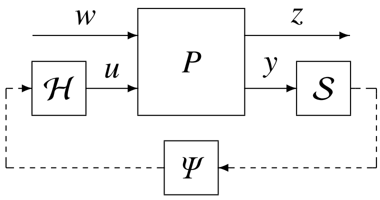

Figure 1. Sampled-data system $\Sigma_{\rm SD}$.

We first define the L_2/L_1 induced norm of $\Sigma_{\rm SD}$ as a quantitative measure for the worst effect of the input $w \in L_{1,p}[0,\infty)$ affecting the output z that is viewed as an element in $L_2[0,\infty)$. With the operator $\mathbf{T}$ denoting the associated input/output mapping from w to z in $\Sigma_{\rm SD}$, its $L_2/L_{1,p}$ (or, simply, L_2/L_1) induced norm is defined by

$$\|\Sigma_{\rm SD}\|_{2/(1,p)} := \sup_{\|w\|_{L_{1,p}[0,\infty)}\le 1} \|\mathbf{T}w\|_{L_2[0,\infty)} \quad (p=1,2) \tag{2.2}$$

which is also denoted by $\|\Sigma_{\rm SD}\|_{2/1}$ for simplicity. This definition follows exactly the same line as that in the case of continuous-time LTI systems, for which it is known that the L_2/L_1 induced norm coincides with the H_2 norm [8] in the single-input case [4, Remark 3.3]. Since the H_2 norm is a very important measure in feedback control, so is the L_2/L_1 induced norm for continuous-time LTI systems for the same reason. This paper is focused on developing generalized arguments to cover sampled-data systems in the context of the L_2/L_1 induced norm (as well as the L_2/L_1 Hankel norm).

Next, we define the L_2/L_1 Hankel norm in sampled-data systems along the same line as the preceding studies [25–28] on the L_∞/L_2, L_2/L_2 and L_∞/L_∞ Hankel norms of sampled-data systems. In sampled-data systems, the input/output behavior between w and z is h-periodic because of the h-periodic actions of the sampler $\mathcal{S}$, the zero-order hold $\mathcal{H}$ and discrete-time controller Ψ. What is meant precisely is that the mapping $\mathbf{T}$ from w to z (under the treatment of the initial condition given shortly) satisfies that $\mathbf{T}\mathbf{S}_\tau = \mathbf{S}_\tau\mathbf{T}$ for the shift operator $\mathbf{S}_\tau$ by the delay $\tau > 0$ (i.e., $(\mathbf{S}_\tau f)(t) = f(t-\tau)$) when τ is an integer multiple of the sampling period h. Thus, it matters quite significantly, in the studies of the Hankel norms of sampled-data systems, when to take the time instant separating past and future (even though a similar issue with respect to when to take the initial time that has arisen in the studies of induced norms of sampled-data systems does not make any difference after all). For this reason, we introduce $\Theta \in [0,h)$ and consider separating past and future at Θ, taking into account the success in such treatment in [27] to amend the treatment of considering only $\Theta = 0$ in the pioneering study [29] in the L_2/L_2 setting, as well as the success, e.g., in [25, 26] in the L_∞/L_2 setting. We then consider the mapping from the past input $w \in L_{1,p}(-\infty,\Theta)$ to the future output z that is viewed as an element in $L_2[\Theta,\infty)$, assuming that the state at $t=-\infty$ is at the origin. This mapping is denoted by $\mathbf{H}_{2/(1,p)}^{[\Theta]}$, and we define it as the quasi $L_2/L_{1,p}$ (or, simply, L_2/L_1) Hankel operator at Θ. In addition, we refer to the norm of $\mathbf{H}_{2/(1,p)}^{[\Theta]}$, defined as

$$\|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| := \sup_{\|w\|_{L_{1,p}(-\infty,\Theta)}\le 1} \|z\|_{L_2[\Theta,\infty)} \quad (p=1,2) \tag{2.3}$$

as the quasi $L_2/L_{1,p}$ (or L_2/L_1) Hankel norm at Θ. Finally, we define *the* $L_2/L_{1,p}$ (or L_2/L_1) Hankel norm of the sampled-data system $\Sigma_{\rm SD}$ by

$$\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} := \sup_{0\le\Theta<h} \|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| \quad (p=1,2) \tag{2.4}$$

by taking the standpoint that the Hankel norm should be defined as a quantitative measure for the worst effect of the past input w on the future output. If the right-hand side of (2.4) is attained as the maximum over $\Theta \in [0, h)$, each maximum-attaining point Θ is basically called a critical instant; however, due to some subtle issues and the motivation suggested by some of the main results up to the early part of Section 4, a more precise definition of this term is deferred to the last subsection of Section 4 (i.e., after some of the main results are presented explicitly).

The present paper is devoted to characterizing the L_2/L_1 induced norm and the L_2/L_1 Hankel norm for the sampled-data system $\Sigma_{\rm SD}$, as well as further clarifying their mutual relationship and their relationship to the several types of H_2 norms of the sampled-data system $\Sigma_{\rm SD}$ that are discussed in the existing literature [10–14]. To facilitate these arguments, we employ the lifted representation of sampled-data systems [17, 18, 30]; the remaining part of this section is devoted to its brief description.

Given the function $f(t)$, its lifting under the sampling period h is defined by the sequence of

$$\widehat{f_k}(\theta) = f(kh+\theta)\ (0 \le \theta < h) \tag{2.5}$$

in k. By applying this lifting treatment to the input w and the output z, we obtain the lifted representation of the input-output relation of the sampled-data system $\Sigma_{\rm SD}$ given by

$$\begin{cases} \xi_{k+1} = \mathcal{A}\xi_k + \mathcal{B}\widehat{w_k} \\ \ \widehat{z_k} \ = \mathcal{C}\xi_k + \mathcal{D}\widehat{w_k} \end{cases} \tag{2.6}$$

with $\xi_k := [x_k^T\ \psi_k^T]^T$ $(x_k = x(kh))$. Basically, this representation follows immediately as a result of the expression for the solution $x(t)$ for (2.1) under the given w and u. More precisely, the matrix $\mathcal{A}$ and the operators $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ are respectively given by

$$\mathcal{A} = \begin{bmatrix} A_d + B_{2d}D_\Psi C_{2d} & B_{2d}C_\Psi \\ B_\Psi C_{2d} & A_\Psi \end{bmatrix} : \mathbb{R}^{n+n_\Psi} \to \mathbb{R}^{n+n_\Psi} \tag{2.7}$$

$$\mathcal{B} = J_\Sigma \mathbf{B}_1 : L_1[0,h) \to \mathbb{R}^{n+n_\Psi} \tag{2.8}$$

$$\mathcal{C} = \mathbf{M}_1 C_\Sigma : \mathbb{R}^{n+n_\Psi} \to L_2[0,h) \tag{2.9}$$

$$\mathcal{D} = \mathbf{D}_{11} : L_1[0,h) \to L_2[0,h) \tag{2.10}$$

with

$$A_d := \exp(Ah),\ B_{2d} := \int_0^h \exp(A\tau)B_2 d\tau,\ C_{2d} = C_2$$

$$\mathbf{B}_1 w = \int_0^h \exp\{A(h-\tau)\}B_1 w(\tau)d\tau$$

$$(\mathbf{D}_{11}w)(\theta) = \int_0^\theta C_1 \exp\{A(\theta-\tau)\}B_1 w(\tau)d\tau$$

$$J_\Sigma := \begin{bmatrix} I \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+n_\Psi)\times n},\ C_\Sigma := \begin{bmatrix} I & 0 \\ D_\Psi C_2 & C_\Psi \end{bmatrix}$$

$$M_1 := \begin{bmatrix} C_1 & D_{12} \end{bmatrix},\ A_2 := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}$$

$$\left(\mathbf{M}_1 \begin{bmatrix} x \\ u \end{bmatrix}\right)(\theta) = M_1 \exp(A_2\theta) \begin{bmatrix} x \\ u \end{bmatrix}.$$

The matrix $\mathcal{A}$ is Schur-stable. This is simply because the standing assumption on the stability of $\Sigma_{\rm SD}$ actually refers to the Schur stability assumption of this matrix, precisely speaking. Hence, $\mathcal{A}$ has all of its eigenvalues in the open unit disc.

To facilitate the notation in the following arguments, $\|\widehat{w}_k\|_{L_{1,p}[0,h)}$ is briefly denoted by $\|\widehat{w}_k\|_{(1,p)}$, or, more simply, by $\|\widehat{w}_k\|_1$, and $\|\widehat{z}_k\|_{L_2[0,h)}$ is simply denoted by $\|\widehat{z}_k\|_2$.

3. Characterizing the L_2/L_1 induced norm of sampled-data systems

This section tackles the L_2/L_1 induced norm of sampled-data systems by clarifying the "worst input" associated with the L_2/L_1 induced norm of $\Sigma_{\rm SD}$. The arguments eventually lead to an explicit computational method of the L_2/L_1 induced norm.

The L_2/L_1 induced norm is associated with the assumption that $x(0) = 0$ and $\psi_0 = 0$ (i.e., $\xi_0 = 0$), and the associated input/output relation of $\Sigma_{\rm SD}$ is described by its lifted representation in (2.6) as follows:

$$\begin{bmatrix} \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \widehat{z}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathcal{D} & 0 & \cdots & & \\ \mathcal{C}\mathcal{B} & \mathcal{D} & 0 & \cdots & \\ \mathcal{C}\mathcal{A}\mathcal{B} & \mathcal{C}\mathcal{B} & \mathcal{D} & 0 & \cdots \\ \mathcal{C}\mathcal{A}^2\mathcal{B} & \mathcal{C}\mathcal{A}\mathcal{B} & \mathcal{C}\mathcal{B} & \ddots & \ddots \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_1 \\ \widehat{w}_2 \\ \widehat{w}_3 \\ \vdots \end{bmatrix}. \tag{3.1}$$

By defining $\widehat{w} := [\widehat{w}_0^T\ \widehat{w}_1^T\ \widehat{w}_2^T\ \ \cdots]^T, \widehat{z} := [\widehat{z}_0^T\ \ \widehat{z}_1^T\ \ \widehat{z}_2^T\ \ \cdots]^T$ and

$$\|\widehat{w}\|_{1,0+} := \sum_{k=0}^{\infty} \|\widehat{w}_k\|_1,\ \|\widehat{z}\|_{2,0+} := \left(\sum_{k=0}^{\infty} \|\widehat{z}_k\|_2^2\right)^{1/2} \tag{3.2}$$

we are immediately led to the norm-preserving properties of lifting, i.e., $\|\widehat{z}\|_{2,0+} = \|z\|_{L_2[0,\infty)}$, as well as $\|\widehat{w}\|_{1,0+} = \|w\|_{L_1[0,\infty)}$, where the latter holds for the underlying $p = 1$ and $p = 2$. Applying the triangle inequality and the properties of the relevant norms to (3.1) leads to

$$\begin{aligned} \|\widehat{z}\|_{2,0+} &= \left\| \begin{bmatrix} \mathcal{D} \\ \mathcal{C}\mathcal{B} \\ \mathcal{C}\mathcal{A}\mathcal{B} \\ \vdots \end{bmatrix} \widehat{w}_0 + \begin{bmatrix} 0 \\ \mathcal{D} \\ \mathcal{C}\mathcal{B} \\ \vdots \end{bmatrix} \widehat{w}_1 + \cdots \right\|_{2,0+} \\ &\le \left\| \begin{bmatrix} \mathcal{D} \\ \mathcal{C}\mathcal{B} \\ \mathcal{C}\mathcal{A}\mathcal{B} \\ \vdots \end{bmatrix} \widehat{w}_0 \right\|_{2,0+} + \left\| \begin{bmatrix} 0 \\ \mathcal{D} \\ \mathcal{C}\mathcal{B} \\ \vdots \end{bmatrix} \widehat{w}_1 \right\|_{2,0+} + \cdots \\ &= \|\mathcal{F}\widehat{w}_0\|_{2,0+} + \|\mathcal{F}\widehat{w}_1\|_{2,0+} + \cdots \\ &\le \|\mathcal{F}\| \cdot \|\widehat{w}_0\|_1 + \|\mathcal{F}\| \cdot \|\widehat{w}_1\|_1 + \cdots \\ &= \|\mathcal{F}\| \cdot \|\widehat{w}\|_{1,0+} \end{aligned} \tag{3.3}$$

where $\mathcal{F}$ is the first column of the operator matrix in (3.1) and we use the shorthand notation $\|\mathcal{F}\|$ to mean $\sup_{\|\cdot\|_1\le 1} \|\mathcal{F}(\cdot)\|_{2,0+}$ (see Appendix A for more rigorous arguments). This implies that the L_2/L_1 induced norm $\|\Sigma_{\rm SD}\|_{2/1}$ does not exceed $\|\mathcal{F}\|$.

On the other hand, it is not hard to see that

$$\sup_{\|\widehat{w}\|_{1,0+}\le 1} \|\widehat{z}\|_{2,0+} = \sup_{\|\widehat{w}\|_{1,0+}\le 1} \left\| \mathcal{F}\widehat{w}_0 + \begin{bmatrix} 0 \\ \mathcal{F} \end{bmatrix} \widehat{w}_1 \cdots \right\|_{2,0+} \ge \sup_{\|\widehat{w}_0\|_1\le 1} \|\mathcal{F}\widehat{w}_0\|_{2,0+} \tag{3.4}$$

by considering the case in which $\widehat{w}_k = 0$ $(k \ge 1)$. Since the rightmost side of (3.4) is nothing but $\|\mathcal{F}\|$, this, together with (3.3), immediately leads to

$$\|\Sigma_{\rm SD}\|_{2/1} = \|\mathcal{F}\| = \sup_{\|\widehat{w}_0\|_1\le 1} \|\mathcal{F}\widehat{w}_0\|_{2,0+}. \tag{3.5}$$

To realize further and more explicit characterization of $\|\Sigma_{\rm SD}\|$ and thus the above norm $\|\mathcal{F}\|$, we next employ the fast-lifting treatment [31] by taking an $N \in \mathbb{N}$. The idea is to introduce $h' := h/N$ to divide the interval $[0, h)$ into N subintervals, and, given the function $f(t)$ on this interval, its fast-lifting representation is defined by the collection of

$$f_m(\theta') := f((m-1)h' + \theta') \qquad (0 \le \theta' < h',\ m = 1, \ldots, N). \tag{3.6}$$

We apply fast-lifting to $\widehat{w}_0$ and consider the portion of $\mathcal{F}$ that acts on the resulting $\widehat{w}_{0,m}$, which we denote by $\mathcal{F}_m$ (i.e., for f' defined on $[0, h')$, the definition of $\mathcal{F}_m$ precisely implies that $\mathcal{F}_m f' = \mathcal{F} f$, with f on $[0, h)$ the fast-lifting representation of which satisfies that $f_{m'} = \delta_{mm'} f'$ $(m' = 1, \ldots, N)$ with the Kronecker delta $\delta_{mm'}$, yielding $\mathcal{F}\widehat{w}_0 = \sum_{m=1}^{N} \mathcal{F}_m \widehat{w}_{0,m}$). It then follows from $\|\mathcal{F}\widehat{w}_0\|_{2,0+} \le \sum_{m=1}^{N} \|\mathcal{F}_m\| \cdot \|\widehat{w}_{0,m}\|_{L_1[0,h')}$ that $\|\mathcal{F}\|$ satisfies

$$\|\mathcal{F}\| \le \max_{m=1,\ldots,N} \|\mathcal{F}_m\| \tag{3.7}$$

as a result of the triangle inequality and the properties of the relevant norms (in a similar fashion to the derivation of (3.3)), where the shorthand notation $\|\mathcal{F}_m\|$ means that $\sup_{\|f'\|_{L_1[0,h')}\le 1} \|\mathcal{F}_m f'\|_{2,0+}$ $(m = 1, \ldots, N)$. Since the right-hand side of the above inequality implies consideration of the confined situation in which the input $\widehat{w}_0$ of $\mathcal{F}$ satisfies that $\widehat{w}_{0,m} = 0$ $(m \ne m^\star)$, where $m^\star$ denotes $\arg\max_m \|\mathcal{F}_m\|$, it readily follows that $\|\mathcal{F}\| \ge \max_{m=1,\ldots,N} \|\mathcal{F}_m\|$. Since the preceding arguments are true regardless of N, this inequality, together with (3.7), leads to

$$\|\mathcal{F}\| = \max_{m=1,\ldots,N} \|\mathcal{F}_m\| \quad (\forall N \in \mathbb{N}). \tag{3.8}$$

Hence, increasing N leads to the fact that we can associate the value of the induced norm with the situation in which the input is concentrated on an infinitesimally small interval around the worst timing in the interval $[0, h)$.

The remaining part of the arguments for characterizing $\|\Sigma_{\rm SD}\|_{2/1}$ is carried out separately for $p = 1$ and $p = 2$.

The case of $p = 1$. Define $\mathcal{F}_m^{(j)}$ as the portion of $\mathcal{F}_m$ (i.e., its jth column) that acts on the jth entry of the input so that $\mathcal{F}_m\widehat{w}_{0,m} = \sum_{j=1}^{n_w} \mathcal{F}_m^{(j)}\widehat{w}_{0,m}^{(j)}$, where we further define $\widehat{w}_{0,m}^{(j)}$ as the jth entry of the associated input $\widehat{w}_{0,m}$. Applying once again the triangle inequality $\|\mathcal{F}_m\widehat{w}_{0,m}\| \leq \sum_{j=1}^{n_w} \|\mathcal{F}_m^{(j)}\| \cdot \|\widehat{w}_{0,m}^{(j)}\|_{L_1[0,h')}$ and the properties of the relevant norms to the treatment of $\|\mathcal{F}_m\|$ leads to

$$\|\mathcal{F}_m\| \leq \max_{j=1,\ldots,n_w} \|\mathcal{F}_m^{(j)}\| \tag{3.9}$$

in quite a similar fashion to (3.3), but entirely due to a key property of $L_{1,1}[0,h')$, i.e., $\sum_{j=1}^{n_w} \|(\cdot)^{(j)}\|_{L_{1,1}[0,h')} = \|\cdot\|_{L_{1,1}[0,h')}$ (because $\sum_{j=1}^{n_w} \|(\cdot)^{(j)}\|_{L_{1,2}[0,h')} \neq \|\cdot\|_{L_{1,2}[0,h')}$, we cannot have parallel arguments for $p = 2$ in this context, which is why the arguments for $p = 1$ and $p = 2$ are developed separately). Since the right-hand side of the above inequality (3.9) implies consideration of the confined situation in which the input $\widehat{w}_{0,m}$ of $\mathcal{F}_m$ satisfies that $\widehat{w}_{0,m}^{(j)} = 0$ ($j \neq j^\star$), where $j^\star$ denotes $\arg\max_j \|\mathcal{F}_m^{(j)}\|$, it is immediate that we also obtain that $\|\mathcal{F}_m\| \geq \max_{j=1,\ldots,n_w} \|\mathcal{F}_m^{(j)}\|$. This, together with (3.9), leads to

$$\|\mathcal{F}_m\| = \max_{j=1,\ldots,n_w} \|\mathcal{F}_m^{(j)}\| \tag{3.10}$$

by which we can associate the value of the induced norm with the situation in which the input is concentrated on the worst single (the $j^\star$th) input (as well as at the worst timing in $[0,h)$ in the sense that corresponds to the interpretation below (3.8)).

To develop more explicit characterization for the $L_2/L_{1,1}$ induced norm of Σ_{SD} on the basis of the above arguments, we consider the L_2 norm of the output z for the situation in which $\widehat{w}_0$ is zero, except for its jth entry, and is applied only on the interval $[\varphi, \varphi+\epsilon)$ ($\varphi \in [0,h)$), with $\epsilon > 0$ that is small enough, instead of directly dealing with an infinitesimally small interval mentioned below (3.8). Even though we skip the details for the moment, this situation with the $L_{1,1}$ norm of the input being 1 essentially corresponds to that with a unit impulse that is applied to the jth entry at $t = \varphi$ (see Appendix B for more detailed arguments). It is easy to see, e.g., from the arguments in [14]*, that the corresponding output z is described by $\widehat{z}_0(\theta) = D_\theta(\varphi)e_j$ and $\widehat{z}_k(\theta) = C_\theta\mathcal{A}^k B_h(\varphi)e_j$ for $k \geq 1$, where e_j denotes the jth column of the identity matrix in $\mathbb{R}^{n_w \times n_w}$, and

$$B_h(\tau) := J_\Sigma \exp\{A(h-\tau)\}B_1 \quad (\tau \in [0,h)) \tag{3.11}$$

$$C_\theta := M_1 \exp(A_2\theta)C_\Sigma \quad (\theta \in [0,h)) \tag{3.12}$$

$$D_\theta(\tau) := C_1 \exp\{A(\theta-\tau)\}B_1 \mathbf{1}(\theta-\tau) \qquad (\theta, \tau \in [0,h)) \tag{3.13}$$

with $\mathbf{1}(\cdot)$ denoting the unit step function. Hence, the square of the L_2 norm of the corresponding output for such a situation is described by

$$\|z\|_{L_2[0,\infty)}^2 = \|\widehat{z}\|_{2,0+}^2 = \int_0^h \mathrm{sq}(D_\theta(\varphi)e_j)d\theta + \sum_{k=0}^{\infty} \int_0^h \mathrm{sq}(C_\theta\mathcal{A}^k B_h(\varphi)e_j)d\theta = H^{(j)}(\varphi) \tag{3.14}$$

*In fact, this is once again simply an immediate consequence of the expression for the the solution $x(t)$ of (2.1) for given w and u.

where $\mathrm{sq}(X) := X^T X$, and $H^{(j)}(\varphi)$ denotes the jth diagonal entry of the positive semidefinite matrix $H(\varphi)$ that is defined as

$$H(\varphi) = \int_0^h \mathrm{sq}(D_\theta(\varphi))d\theta + \sum_{k=0}^{\infty} \int_0^h \mathrm{sq}(C_\theta \mathcal{A}^k B_h(\varphi))d\theta. \tag{3.15}$$

Recalling that we have to take the worst timing φ, as well as the worst entry j of the input w, we are readily led to one of the main results on the induced norm given as Theorem 1 below, where the matrix-valued function $H(\varphi)$ can be regarded as reflecting the periodically time-varying nature of Σ_{SD}. Note that the second term of $H(\varphi)$ in (3.15) can readily be obtained by computing $\sum_{k=0}^{\infty} \int_0^h \mathrm{sq}(C_\theta \mathcal{A}^k)d\theta$, which (does converge by the Schur stability of $\mathcal{A}$), in turn, can be obtained as the solution X_h to the discrete-time Lyapunov equation $\mathcal{A}^T X_h \mathcal{A} - X_h + \int_0^h \mathrm{sq}(C_\theta)d\theta = 0$, where the integrals on the right-hand side and the second term of $H(\varphi)$ in (3.15) can also be computed through the well-known technique [32, Equation (2.2)]. These arguments are quite standard; thus the details are omitted.

Theorem 1. *The $L_2/L_{1,1}$ induced norm of the sampled-data system Σ_{SD} is given by*

$$\|\Sigma_{\mathrm{SD}}\|_{2/(1,1)} = \sup_{0 \le \varphi < h} \mu_1^{1/2}(H(\varphi)) \tag{3.16}$$

where $\mu_1(\cdot)$ denotes the maximum diagonal entry.

The case of $p = 2$. As noted in the arguments for $p = 1$, we cannot have (3.9) for the case of $p = 2$. Hence, we have to develop different arguments for computing $\|\mathcal{F}\|$ when $p = 2$, but the basic idea is the same as that in the case of $p = 1$ in the sense that we only have to consider the situation in which the input is concentrated on an infinitesimally small interval around the worst timing φ in $[0, h)$. Since (3.9) does not hold for $p = 2$, the only difference is that we are led (eventually in an equivalent fashion) to considering the output z for the unit impulse applied at $t = \varphi$ for all entries of w. More precisely, we take $v \in \mathbb{R}^{n_w}$ such that $|v|_2 = 1$ and $w(t) = v\delta(t - \varphi)$. Then, the corresponding output in the lifted form is given by $\widehat{z_0}(\theta) = D_\theta(\varphi)v$ and $\widehat{z_k}(\theta) = C_\theta \mathcal{A}^k B_h(\varphi)v$ for $k \ge 1$. By the construction of $H(\varphi)$, it is easy to see that the square of the L_2 norm of the corresponding output z for this situation is described by $v^T H(\varphi)v$, and its maximum over $|v|_2 = 1$ is given by the maximum eigenvalue of $H(\varphi)$ by the well-known Rayleigh quotient property. Hence, we are led to another main result on the induced norm, as follows.

Theorem 2. *The $L_2/L_{1,2}$ induced norm of the sampled-data system Σ_{SD} is given by*

$$\|\Sigma_{\mathrm{SD}}\|_{2/(1,2)} = \sup_{0 \le \varphi < h} \mu_2^{1/2}(H(\varphi)) \tag{3.17}$$

where $\mu_2(\cdot)$ denotes the maximum eigenvalue.

Remark 1. *Since $|v|_1 \ge |v|_2$ for each $v \in \mathbb{R}^{n_w}$ and thus $\|w\|_{L_{1,1}[0,\infty)} \le 1$ is more restrictive than $\|w\|_{L_{1,2}[0,\infty)} \le 1$, it is obvious that*

$$\|\Sigma_{\mathrm{SD}}\|_{2/(1,1)} \le \|\Sigma_{\mathrm{SD}}\|_{2/(1,2)} \tag{3.18}$$

(with the equality being true, obviously, when $n_w = 1$). This inequality can also be confirmed by the properties of $\mu_p(\cdot)$, $p = 1, 2$, i.e., $\mu_2(X) \geq \mu_1(X)$ for each positive semidefinite matrix X. Theorems 1 and 2, by the way, clearly show that Proposition 2 of [33] that is stated without proof is completely wrong.

Remark 2. *The arguments developed in Appendix B for $p = 1$, justifying the treatment through the unit impulse response, are based on the situation in which the unit impulse is applied to a single entry of w. Nevertheless, the arguments therein are still valid for the case of $p = 2$, in which an impulse is applied to all entries of w through the unit vector v (i.e., $|v|_2 = 1$). To see this, consider $\Sigma_{\rm SD}$ with B_1 replaced by $B_1 V$, where V is an orthogonal matrix whose first column equals v. The $L_2/L_{1,2}$ induced norm remains unchanged by the introduction of such V, while applying the unit impulse at $t = \varphi$ to all entries of w of the original $\Sigma_{\rm SD}$ through v is exactly the same as applying the unit impulse at $t = \varphi$ to the first (and thus single) entry of w to this modified $\Sigma_{\rm SD}$.*

4. Characterizing the L_2/L_1 Hankel norm of sampled-data systems

This section tackles the L_2/L_1 Hankel norm of the sampled-data system $\Sigma_{\rm SD}$ and gives its characterization in such a way that the norm can readily be computed explicitly. In particular, it is shown that it actually coincides with the L_2/L_1 induced norm, as in the case of the continuous-time LTI case [2, Corollary], [3, Theorem 2]. In addition, after a rigorous definition is given for a critical instant, we further investigate the problem of whether or not a critical instant always exists in the L_2/L_1 Hankel norm analysis. Toward these directions, we start with the arguments on the quasi L_2/L_1 Hankel norm at $\Theta \in [0, h)$.

4.1. Characterizing the quasi L_2/L_1 Hankel norm at Θ

The quasi L_2/L_1 Hankel operator at Θ is associated with the assumption that $x(-\infty) = 0$, $\psi_{-\infty} = 0$ (i.e., $\xi_{-\infty} = 0$) and $w(t) \equiv 0$ $(t \geq \Theta)$. To facilitate the consideration of the relation between the past input in $L_1(-\infty, \Theta)$ and the future output in $L_2[\Theta, \infty)$ under this assumption, we introduce the operator $\mathbf{\Phi}_\Theta : L_2[0, h) \to L_2[0, h)$, given by

$$(\mathbf{\Phi}_\Theta a)(\theta) := \begin{cases} 0 & (0 \leq \theta < \Theta) \\ a(\theta) & (\Theta \leq \theta < h) \end{cases} \quad a \in L_2[0, h). \tag{4.1}$$

Then, the relation between the past input and future output of $\Sigma_{\rm SD}$ is described by (2.6) as

$$\begin{bmatrix} \mathbf{\Phi}_\Theta \widehat{z}_0 \\ \widehat{z}_1 \\ \widehat{z}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_\Theta \mathcal{D} & \mathbf{\Phi}_\Theta \mathcal{C}\mathcal{B} & \mathbf{\Phi}_\Theta \mathcal{C}\mathcal{A}\mathcal{B} & \cdots \\ \mathcal{C}\mathcal{B} & \mathcal{C}\mathcal{A}\mathcal{B} & \mathcal{C}\mathcal{A}^2\mathcal{B} & \cdots \\ \mathcal{C}\mathcal{A}\mathcal{B} & \mathcal{C}\mathcal{A}^2\mathcal{B} & \ddots & \\ \vdots & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} \widehat{w}_0 \\ \widehat{w}_{-1} \\ \widehat{w}_{-2} \\ \vdots \end{bmatrix} \tag{4.2}$$

(where the operator matrix on the right-hand side has the Toeplitz structure if $\mathbf{\Phi}_\Theta$ is deleted). By defining $\widehat{w}_{\Theta-} := [\widehat{w}_0^T \ \ \widehat{w}_{-1}^T \ \ \cdots]^T$, $\widehat{z}_{\Theta+} := [(\mathbf{\Phi}_\Theta \widehat{z}_0)^T \ \ \widehat{z}_1^T \ \ \cdots]^T$ and

$$\|\widehat{w}_{\Theta-}\|_{1,0-} := \sum_{k=0}^{\infty} \|\widehat{w}_{-k}\|_1, \ \|\widehat{z}_{\Theta+}\|_{2,0+} := \left(\|\mathbf{\Phi}_\Theta \widehat{z}_0\|_2^2 + \sum_{k=1}^{\infty} \|\widehat{z}_k\|_2^2 \right)^{1/2} \tag{4.3}$$

and noting the assumption that $w(t) \equiv 0$ $(t \geq \Theta)$, we once again have the norm-preserving properties of lifting, i.e., $||\widehat{z}_{\Theta+}||_{2,0+} = ||z||_{L_2[\Theta,\infty)}$, as well as $||\widehat{w}_{\Theta-}||_{1,0+} = ||w||_{L_1[-\infty,\Theta)}$, where the latter holds for $p = 1$ and $p = 2$. In addition, with the columns of the operator matrix in (4.2), we define $\mathcal{F}_k$ $(k = 0, 1, 2, \ldots)$ as

$$\mathcal{F}_0 := \begin{bmatrix} \mathbf{\Phi}_\Theta \mathcal{D} \\ \mathcal{C}\mathcal{B} \\ \mathcal{C}\mathcal{A}\mathcal{B} \\ \vdots \end{bmatrix}, \quad \ldots, \quad \mathcal{F}_k := \begin{bmatrix} \mathbf{\Phi}_\Theta \mathcal{C}\mathcal{A}^{k-1}\mathcal{B} \\ \mathcal{C}\mathcal{A}^{k}\mathcal{B} \\ \mathcal{C}\mathcal{A}^{k+1}\mathcal{B} \\ \vdots \end{bmatrix}, \quad \cdots \tag{4.4}$$

Then, applying the triangle inequality and the properties of the relevant norms to (4.2) (in a similar fashion to the derivation of (3.3)) leads to

$$\begin{aligned} ||z||_{L_2[\Theta,\infty)} &= ||\widehat{z}_{\Theta+}||_{2,0+} \\ &\leq \sum_{k=0}^{\infty} ||\mathcal{F}_k^-|| \cdot ||\widehat{w}_{-k}^-||_{L_1[0,\Theta)} + \sum_{k=1}^{\infty} ||\mathcal{F}_k^+|| \cdot ||\widehat{w}_{-k}^+||_{L_1[\Theta,h)} \end{aligned} \tag{4.5}$$

where $\mathcal{F}_k^-$ denotes the portion of $\mathcal{F}_k$ that acts on the input on the interval $[0, \Theta)$, while $\mathcal{F}_k^+$ denotes the portion of $\mathcal{F}_k$ that acts on the input on the interval $[\Theta, h)$ (the precise definitions follow essentially the same technique used in defining $\mathcal{F}_m$ from $\mathcal{F}$ in Section 3); also, for a vector-valued function on $[0, h)$, its restrictions onto the subintervals $[0, \Theta)$ and $[\Theta, h)$ are denoted by $(\cdot)^-$ and $(\cdot)^+$, respectively. Furthermore, $||\mathcal{F}_k^-||$ and $||\mathcal{F}_k^+||$ are defined accordingly in the obvious fashion. Here, we readily see that $\mathcal{F}_k$ corresponds to $\mathcal{F}_0$ with the first k entries removed and with (the norm-contracting operator) $\mathbf{\Phi}_\Theta$ applied on the resulting first entry. This immediately leads to $||\mathcal{F}_0^-|| \geq ||\mathcal{F}_k^-||$ $(k \geq 1)$ and $||\mathcal{F}_1^+|| \geq ||\mathcal{F}_k^+||$ $(k \geq 2)$; thus, it is easy to see from (4.5) that

$$||z||_{L_2[\Theta,\infty)} \leq \max\left(||\mathcal{F}_0^-||, ||\mathcal{F}_1^+||\right) \tag{4.6}$$

because

$$\sum_{k=0}^{\infty} ||\widehat{w}_{-k}^-||_{L_1[0,\Theta)} + \sum_{k=1}^{\infty} ||\widehat{w}_{-k}^+||_{L_1[\Theta,h)} = ||w||_{L_1(-\infty,\Theta)} \leq 1 \tag{4.7}$$

where the inequality follows by hypothesis.

On the other hand, we readily have that $||z||_{L_2[\Theta,\infty)} \geq \max\left(||\mathcal{F}_0^-||, ||\mathcal{F}_1^+||\right)$ when $||w||_{L_1(-\infty,\Theta)} \leq 1$ by essentially the same arguments as those leading to (3.4). This, together with (4.6), implies that

$$\sup_{||w||_{L_1(-\infty,\Theta)} \leq 1} ||z||_{L_2[\Theta,\infty)} = \max\left(||\mathcal{F}_0^-||, ||\mathcal{F}_1^+||\right). \tag{4.8}$$

To realize further and more explicit characterization of the above quantity (i.e., the quasi L_2/L_1 Hankel norm at Θ), we apply the fast-lifting treatment as in the case of the L_2/L_1 induced norm studied in the preceding section, but with a slightly modified fashion. That is, for the treatment of $||\mathcal{F}_0^-||$, the interval $[0, \Theta)$ is divided into N subintervals, and, for the treatment of $||\mathcal{F}_1^+||$, the interval $[\Theta, h)$ is divided into N subintervals. Following essentially the same arguments as those in the preceding section leads to the observation that the quasi L_2/L_1 Hankel norm at Θ can be associated with the situation in which the past input is concentrated on an infinitesimally small interval around the worst timing in the interval $[0, \Theta)$ or $[\Theta - h, 0)$. This is essentially the same as the situation in which, given $\Theta \in [0, h)$, a unit impulse is applied at the worst timing $\Theta - \phi$ ($\in [\Theta - h, \Theta)$) for some $\phi \in (0, h]$; also, the quasi L_2/L_1 Hankel norm at Θ corresponds to the L_2 norm of the corresponding worst output z that is evaluated over the interval after Θ, i.e., $[\Theta, \infty)$. Now, let

$$\begin{aligned} &\varphi := \Theta - \phi \in [0, \Theta) \ \text{ if } \ \Theta - \phi \geq 0 \\ &\varphi := \Theta - \phi + h \in [\Theta, h) \ \text{ if } \ \Theta - \phi < 0 \end{aligned} \tag{4.9}$$

for the underlying $\Theta \in [0, h)$. That is[†], instead of representing the worst timing relative to Θ through ϕ, we introduce φ to represent the worst timing relative to the sampling instant 0, taking into account the use of the lifting treatment. Then, $\varphi \in [0, h)$ is such that it is the worst timing for the unit impulse in the aforementioned sense when $\varphi \in [0, \Theta)$, and $\varphi - h$ is the worst timing when $\varphi \in [\Theta, h)$. Hence, similar to the arguments about (3.14) in the preceding section (where the response z was considered in the lifted form by using (2.6) for the impulse input w applied at $\varphi \in [0, h)$), we see that, if $\varphi \in [0, \Theta)$, then the corresponding output $\widehat{z}_0(\theta)$ can be represented by $D_\theta(\varphi)$ and $\widehat{z}_k(\theta)$ can be represented by $C_\theta \mathcal{A}^{k-1} B_h(\varphi)$ for $k \geq 1$. Similarly, if $\varphi \in [\Theta, h)$ (for which the worst timing $\varphi - h$ is negative but no smaller than $-h$, so that $D_\theta(\varphi)$ becomes irrelevant and only $C_\theta \mathcal{A}^k B_h(\varphi)$ plays a role in representing the output z after $t = \Theta \geq 0$), then the corresponding output $\widehat{z}_k$ can be represented by $C_\theta \mathcal{A}^k B_h(\varphi)$ for $k \geq 0$ as an immediate result of applying (2.6). Through the above observations, we are led to introducing the positive semidefinite matrices

$$F_{\Theta,0}(\varphi) = \int_\Theta^h \mathrm{sq}\left(D_\theta(\varphi)\right) d\theta + \sum_{k=0}^{\infty} \int_0^h \mathrm{sq}\left(C_\theta \mathcal{A}^k B_h(\varphi)\right) d\theta \tag{4.10}$$

$$F_{\Theta,1}(\varphi) = \int_\Theta^h \mathrm{sq}\left(C_\theta B_h(\varphi)\right) d\theta + \sum_{k=1}^{\infty} \int_0^h \mathrm{sq}\left(C_\theta \mathcal{A}^k B_h(\varphi)\right) d\theta \tag{4.11}$$

for $\varphi \in [0, \Theta)$ and $[\Theta, h)$, respectively, which can also be computed easily by using the solution of an appropriate discrete-time Lyapunov equation (see essentially the same remark before Theorem 1). The unit impulse mentioned above, by the way, actually corresponds to $v\delta(t - \varphi)$ (or $v\delta(t - (\varphi - h))$ if $\varphi \in [\Theta, h)$) with the delta function $\delta(t)$, where $v = e_j$ for some $j = 1, \ldots, n_w$ for $p = 1$, while $|v|_2 = 1$ for $p = 2$. In addition, it is obvious from the construction of the above $F_{\Theta,0}(\varphi)$ and $F_{\Theta,1}(\varphi)$ (taking account of the aforementioned $\widehat{z}_k$) that $\|z\|^2_{L_2[\Theta,\infty)} = \|\widehat{z}_{\Theta+}\|^2_{2,0+}$ corresponds to $v^T F_{\Theta,0}(\varphi) v$ or $v^T F_{\Theta,1}(\varphi) v$ in such a situation with v; we further have to consider the worst v among those mentioned above. Since the quasi L_2/L_1 Hankel norm at Θ is also associated with the worst timing φ with respect to Θ, we are led to the following result on the quasi L_2/L_1 Hankel norm, where $\mu_p(\cdot)$ ($p = 1, 2$) are as stated in Theorems 1 and 2, respectively.

Theorem 3. *The quasi L_2/L_1 Hankel norm at Θ of the sampled-data system Σ_{SD} is given by*

$$\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\| = \max\left(\sup_{0 \leq \varphi < \Theta} \mu_p^{1/2}(F_{\Theta,0}(\varphi)),\ \sup_{\Theta \leq \varphi < h} \mu_p^{1/2}(F_{\Theta,1}(\varphi)) \right). \tag{4.12}$$

[†]We remark that, in the treatment throughout this paper, the timing $\varphi = 0$ (or $\Theta - \phi = 0$) corresponds to the instant immediately after the sampler takes its action, while $\varphi = h - 0$ (or $\Theta - \phi = -0$) corresponds to the instant immediately before the sampler takes its action.

4.2. *Characterizing the L_2/L_1 Hankel norm*

Since the quasi L_2/L_1 Hankel norm at Θ has been characterized by Theorem 3, we have, in principle, also successfully characterized the L_2/L_1 Hankel norm through the use of (2.4). Such characterization, however, obviously leads to a double supremum in $\Theta \in [0, h)$, as well as $\varphi \in [0, h)$. This is not only inconvenient, it is actually redundant in a sense. Indeed, Theorem 4, given shortly, shows through the following lemma that the L_2/L_1 Hankel norm of $\Sigma_{\rm SD}$ can actually be characterized by simply applying a single supremum in $\varphi \in [0, h)$.

Lemma 1. *For the positive semidefinite matrices X and Y of the same size, we have*

$$\mu_p(X) \le \mu_p(X + Y) \quad (p = 1, 2). \tag{4.13}$$

We skip the proof of this lemma because (the proof is quite simple and) it is well known.

Theorem 4. *The L_2/L_1 Hankel norm of the sampled-data system $\Sigma_{\rm SD}$ is given by*

$$\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = \sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi)) \quad (p = 1, 2). \tag{4.14}$$

Before proceeding to the proof, we note that $F_{\Theta,0}(\varphi)$ as defined in (4.10) for $\varphi \in [0, \Theta)$, has a continuous extension for $\varphi = \Theta$ by the continuity of $B_h(\cdot)$ on $[0, h)$ and that of $D_\theta(\cdot)$ on $[0, \theta]$; also, $F_{\Theta,1}(\varphi)$ as defined in (4.11) for $\varphi \in [\Theta, h)$, has a continuous extension for $\varphi = h$. With this observation taken into account, the subsequent part of this paper takes the standpoint that we may refer to $F_{\Theta,0}(\Theta)$ and $F_{\Theta,1}(h)$. Similarly, $H(\varphi)$ as defined in (3.15) for $\varphi \in [0, h)$, has a continuous extension for $\varphi = h$; thus, we further take the standpoint that we may refer to $H(h)$, which is nothing but $\lim_{\varphi\to h-0} H(\varphi)$. Under this standpoint, it readily follows from (3.15) and (4.10) that

$$F_{\varphi,0}(\varphi) = H(\varphi), \quad \varphi \in [0, h] \tag{4.15}$$

since $D_\theta(\varphi) = 0$ for $0 \le \theta < \varphi$. We further note from the direct comparisons with (3.15), (4.10) and (4.11) that[‡]

$$F_{\Theta,0}(\varphi) \le H(\varphi), \quad \varphi \in [0, \Theta) \tag{4.16}$$
$$F_{\Theta,1}(\varphi) \le H(\varphi), \quad \varphi \in [\Theta, h) \tag{4.17}$$
$$F_{h,0}(\varphi) = F_{0,1}(\varphi), \quad \varphi \in [0, h] \tag{4.18}$$
$$F_{0,0}(0) = H(0), \quad F_{h,0}(h) = F_{0,1}(h) = H(h) \tag{4.19}$$
$$F_{h,1}(h) \le F_{h,0}(h) \tag{4.20}$$

where the range for Θ is also extended in a similar fashion to $[0, h]$ with respect to $F_{\Theta,0}(\varphi)$ and $F_{\Theta,1}(\varphi)$ in (4.18)–(4.20).

Proof of Theorem 4. It readily follows from (4.12) that

$$\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\| \ge \sup_{0\le\varphi<\Theta} \mu_p^{1/2}(F_{\Theta,0}(\varphi)) \quad (\Theta > 0). \tag{4.21}$$

[‡]To see the equalities regarding $H(h)$ in (4.19), note that $D_\theta(h) = 0$ for $\theta \in [0, h)$ by (3.13); thus, the first term of $H(h)$ in (3.15) vanishes. Other relations follow quite immediately from (3.15), (4.10) and (4.11).

It further follows from the aforementioned arguments on continuous extension, together with (4.15), that

$$\|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| \geq \sup_{0\leq\varphi\leq\Theta} \mu_p^{1/2}(F_{\Theta,0}(\varphi)) \geq \mu_p^{1/2}(F_{\Theta,0}(\Theta)) = \mu_p^{1/2}(H(\Theta)) \quad (\Theta > 0) \tag{4.22}$$

and this, together with (2.4), implies that

$$\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} \geq \sup_{0<\Theta<h} \|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| \geq \sup_{0<\Theta<h} \mu_p^{1/2}(H(\Theta)) = \sup_{0\leq\varphi<h} \mu_p^{1/2}(H(\varphi)) \tag{4.23}$$

where the last equality follows since $H(\varphi)$ is continuous at $\varphi = 0$.

To show the converse inequality, we note (4.16). Then, applying Lemma 1 leads to

$$\sup_{0\leq\varphi<\Theta} \mu_p^{1/2}(F_{\Theta,0}(\varphi)) \leq \sup_{0\leq\varphi<\Theta} \mu_p^{1/2}(H(\varphi)) \leq \sup_{0\leq\varphi<h} \mu_p^{1/2}(H(\varphi)). \tag{4.24}$$

Similarly, by noting (4.17), we readily have

$$\sup_{\Theta\leq\varphi<h} \mu_p^{1/2}(F_{\Theta,1}(\varphi)) \leq \sup_{\Theta\leq\varphi<h} \mu_p^{1/2}(H(\varphi)) \leq \sup_{0\leq\varphi<h} \mu_p^{1/2}(H(\varphi)). \tag{4.25}$$

Hence, the converse inequality of (4.23) has been established by (4.12) together with (2.4), and this completes the proof. □

Furthermore, Theorem 4, together with Theorems 1 and 2, immediately leads to the following result.

Corollary 1. *Let $p = 1$ or $p = 2$. The $L_2/L_{1,p}$ Hankel norm coincides with the $L_2/L_{1,p}$ induced norm in the sampled-data system $\Sigma_{\rm SD}$.*

Just for reference, we note that parallel results for sampled-data systems have also been obtained between the L_∞/L_2 induced norm and Hankel norm [25, Corollary 3.6], [26], as well as between the L_∞/L_∞ induced norm and Hankel norm [28, Corollary 2] for the case in which the direct feedthrough matrix D_{11} from w to z in the continuous-time plant P is zero[§].

4.3. Critical instant and its existence

The remaining part of this section is devoted to the arguments on critical instants in the L_2/L_1 Hankel norm analysis, such as whether or not a critical instant always exists, together with the relevance of such arguments to the properties of the matrix-valued function $H(\varphi)$. These aspects studied in this subsection are mutually related in quite a deep fashion; thus, some preliminary comments on the mutual relevance will be crucial before proceeding with the arguments in this subsection.

First, a rough (but not entirely rigorous) definition of a critical instant is, as stated after (2.4), an instant $\Theta = \Theta^\star \in [0, h)$—if one exists—such that *the* L_2/L_1 Hankel norm given by the left-hand side of (2.4) equals the quasi L_2/L_1 Hankel norm at $\Theta^\star$, i.e., $\|\mathbf{H}_{2/(1,p)}^{[\Theta^\star]}\|$. Despite the definition of the L_2/L_1 Hankel norm through the use of the supremum over $\Theta \in [0, h)$ in (2.4), on the other hand, we have established that it can actually be obtained through the treatment of the matrix-valued function $H(\varphi)$ that does not involve Θ at all. This motivates us to tackle an interesting problem of whether (not merely the L_2/L_1 Hankel norm but as an addition) a "critical instant" can also be somehow detected only through the analysis of some properties of the matrix-valued function $H(\varphi)$. To be helpful in such a direction of arguments, it turns out that it makes sense to employ the following rigorous definition of a critical instant by modifying the aforementioned rough definition.

[§]This matrix must be zero for the L_2/L_1 and L_∞/L_2 settings so that the associated induced norm is well-defined.

Definition 1. *In the $L_2/L_{1,p}$ Hankel norm analysis, the instant $\Theta^\star \in (0, h)$ is called a critical instant if*

$$\sup_{0\le\Theta<h} \|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| = \|\mathbf{H}_{2/(1,p)}^{[\Theta^\star]}\| \tag{4.26}$$

while the instant $\Theta^\star = 0$ is called a critical instant if

$$\sup_{0\le\Theta<h} \|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| = \lim_{\Theta\to+0} \|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| =: \|\mathbf{H}_{2/(1,p)}^{[+0]}\|. \tag{4.27}$$

Note that, underlying the above definition, in some sense, is a later-shown fact that $\|\mathbf{H}_{2/(1,p)}^{[0]}\| \neq \|\mathbf{H}_{2/(1,p)}^{[+0]}\|$, in general (see Remark 3 given later); however, there are also two main reasons why the case of the critical instant $\Theta^\star$ at 0 is treated separately (as opposed to the L_∞/L_2 Hankel norm analysis [25, 26] and the L_2/L_2 Hankel norm analysis [27]), as follows. First, when (4.26) holds, it roughly implies that the "worst input" in the L_2/L_1 Hankel norm analysis is close to an impulse applied at $t = \Theta^\star$, but, if such $\Theta^\star$ is 0, then this instant is a sampling instant; thus, an intuitive interpretation of applying an impulse at this specific instant becomes somewhat hard due to the possible two interpretations of (i) applying it just before the sampler takes its action, and (ii) applying it just after that action. In this respect, the treatment in (4.27) corresponds to dealing with $\Theta^\star = 0$ only as the limit $\Theta^\star \to +0$; thus, it helps us to circumvent the aforementioned subtleties in intuitive understanding. The second reason is more mathematical, and it is crucial in leading us to more sophisticated overall arguments that take into account whether a critical instant can be detected only through the analysis of the matrix-valued function $H(\varphi)$. By the end of this subsection, the rationale for the above definition would become much clearer.

With the above precise definition of a critical instant, we investigate the issue of whether a critical instant $\Theta = \Theta^\star$ always exists for the sampled-data system $\Sigma_{\rm SD}$, taking into account the relevant studies on the quasi L_∞/L_2 Hankel norm analysis [25, 26] and the quasi L_2/L_2 Hankel norm analysis [27]. We first state the following result, where we say that $\mu_p^{1/2}(H(\varphi))$ is maximum-attaining on $[0, h)$ (for the underlying $p = 1$ or $p = 2$) if there exists a maximum-attaining point $\varphi^\star \in [0, h)$ such that $\mu_p^{1/2}(H(\varphi^\star)) = \sup_{\varphi\in[0,h)} \mu_p^{1/2}(H(\varphi))$.

Theorem 5. *Let $p = 1$ or $p = 2$, and suppose that $\mu_p^{1/2}(H(\varphi))$ is maximum-attaining on $[0, h)$. Then, for each maximum-attaining point $\varphi^\star \in [0, h)$ of $\mu_p^{1/2}(H(\varphi))$, the instant $\Theta = \varphi^\star$ is a critical instant of $\Sigma_{\rm SD}$ in the $L_2/L_{1,p}$ Hankel norm analysis.*

Proof. By the hypothesis of this theorem, together with Corollary 1 and Theorems 1 and 2, we have

$$\sup_{0\le\Theta<h} \|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = \|\Sigma_{\rm SD}\|_{2/(1,p)} = \sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi)) = \mu_p^{1/2}(H(\varphi^\star)). \tag{4.28}$$

On the other hand, it readily follows from (4.22) that

$$\|\mathbf{H}_{2/(1,p)}^{[+0]}\| \ge \mu_p^{1/2}(H(0)) \tag{4.29}$$

by the continuity of $H(\varphi)$ at $\varphi = 0$. Now, if $\varphi^\star \neq 0$, then let $\Theta = \varphi^\star$ in (4.22) and apply (4.28). If $\varphi^\star = 0$, on the other hand, then apply (4.28) to (4.29). Then, we have

$$\|\mathbf{H}_{2/(1,p)}^{[\varphi^\star]}\| \ge \sup_{0\le\Theta<h} \|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| \quad (\text{if } \varphi^\star \in (0, h)), \qquad \|\mathbf{H}_{2/(1,p)}^{[+0]}\| \ge \sup_{0\le\Theta<h} \|\mathbf{H}_{2/(1,p)}^{[\Theta]}\| \quad (\text{if } \varphi^\star = 0) \tag{4.30}$$

which obviously implies that both sides actually coincide with each other (i.e., the inequality can, in fact, be replaced by an equality) in both inequalities; thus, $\Theta = \varphi^\star$ is a critical instant. This completes the proof. □

Note that Theorem 5 leads us to be interested in whether or not $H(h) = \lim_{\varphi \to h-0} H(\varphi)$ coincides with $H(0)$. This is because, if they coincide with each other, then it is obvious that $\mu_p^{1/2}(H(\varphi))$ is maximum-attaining on $[0, h)$; thus, the existence of a critical instant is ensured by this theorem. However, the definition of $H(\varphi)$ in (3.15) does not lead to such a general relation. To get around the difficulty, we can derive the following result.

Proposition 1. *The following inequalities hold for $p = 1, 2$, where $\|\mathbf{H}_{2/(1,p)}^{[h]}\|$ is a shorthand notation for $\lim_{\Theta \to h-0} \|\mathbf{H}_{2/(1,p)}^{[\Theta]}\|$.*

$$\mu_p^{1/2}(H(h)) \le \|\mathbf{H}_{2/(1,p)}^{[+0]}\| \le \sup_{0 \le \varphi < h} \mu_p^{1/2}(H(\varphi)) = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)} \tag{4.31}$$

$$\mu_p^{1/2}(H(h)) \le \|\mathbf{H}_{2/(1,p)}^{[h]}\| \le \sup_{0 \le \varphi < h} \mu_p^{1/2}(H(\varphi)) = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}. \tag{4.32}$$

Once this result is established, we immediately have the following result (Proposition 1 immediately leads to the first assertion; in particular, we are led, under the hypothesis of the following proposition, to $\|\mathbf{H}_{2/(1,p)}^{[+0]}\| = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}$, which is nothing but the second assertion).

Proposition 2. *Suppose that $p = 1$ or $p = 2$. If $\sup_{0 \le \varphi < h} \mu_p^{1/2}(H(\varphi)) = \mu_p^{1/2}(H(h))$, then $\|\mathbf{H}_{2/(1,p)}^{[h]}\| = \|\mathbf{H}_{2/(1,p)}^{[+0]}\|$ and $\Theta = 0$ is a critical instant.*

The above result leads to the following result, which covers the situation that is not handled by Theorem 5.

Theorem 6. *Let $p = 1$ or $p = 2$, and suppose that $\mu_p^{1/2}(H(\varphi))$ is not maximum-attaining on $[0, h)$. Then, $\Theta = 0$ is a critical instant of Σ_{SD} in the $L_2/L_{1,p}$ Hankel norm analysis.*

Proof. It is obvious that, if $\mu_p^{1/2}(H(\varphi))$ is not maximum-attaining on $[0, h)$, then $\sup_{0 \le \varphi < h} \mu_p^{1/2}(H(\varphi)) = \mu_p^{1/2}(H(h))$. Hence, the assertion follows from Proposition 2. □

Finally, the following result is an immediate consequence of Theorems 5 and 6.

Corollary 2. *The sampled-data system Σ_{SD} always has a critical instant in the L_2/L_1 Hankel norm analysis for $p = 1, 2$. In particular, at least one critical instant can be obtained through the analysis of $H(\varphi)$ on $[0, h)$.*

The first assertion of the above result is similar to that for the quasi L_2/L_2 Hankel norm analysis [27, Theorem 3], but it is in sharp contrast to that for the quasi L_∞/L_2 Hankel norm analysis [25, 26] for the sampled-data system Σ_{SD}, where a counterexample is provided. For the second assertion, it should be quite important to note that a critical instant $\Theta = \Theta^\star$ can be detected from $H(\varphi)$, whose definition does not involve Θ at all.

Proof of Proposition 1. The equalities in (4.31) and (4.32) are nothing but (4.14); thus, the second inequalities in (4.31) and (4.32) are obvious. With the arguments about continuous extension that are stated below Theorem 4, it follows from (4.12), together with (4.15), that

$$\|\mathbf{H}^{[+0]}_{2/(1,p)}\| = \max\left(\mu_p^{1/2}(F_{0,0}(0)), \lim_{\Theta\to+0} \sup_{\Theta\le\varphi<h} \mu_p^{1/2}(F_{\Theta,1}(\varphi))\right) = \max\left(\mu_p^{1/2}(H(0)), \sup_{0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi))\right) \tag{4.33}$$

$$\ge \sup_{0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi)) = \sup_{0\le\varphi\le h} \mu_p^{1/2}(F_{0,1}(\varphi)) \ge \mu_p^{1/2}(F_{0,1}(h)) = \mu_p^{1/2}(H(h)) \tag{4.34}$$

where the first equality follows from (4.12), and we used (4.19) in the second equality. On the other hand, the fact that $F_{\Theta,1}(\varphi) \to F_{0,1}(\varphi)$ as $\Theta \to +0$ uniformly with respect to $\varphi \in [0, h)$ was used in the second equality, and (4.19) was used again in the last equality. Similarly, it follows from (4.12) that

$$\|\mathbf{H}^{[h]}_{2/(1,p)}\| = \max\left(\lim_{\Theta\to h-0} \sup_{0\le\varphi<\Theta} \mu_p^{1/2}(F_{\Theta,0}(\varphi)), \mu_p^{1/2}(F_{h,1}(h))\right) = \max\left(\sup_{0\le\varphi<h} \mu_p^{1/2}(F_{h,0}(\varphi)), \mu_p^{1/2}(F_{h,1}(h))\right)$$

$$= \max\left(\sup_{0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi)), \mu_p^{1/2}(F_{h,1}(h))\right) = \max\left(\sup_{0\le\varphi\le h} \mu_p^{1/2}(F_{0,1}(\varphi)), \mu_p^{1/2}(F_{h,1}(h))\right)$$

$$= \sup_{0\le\varphi\le h} \mu_p^{1/2}(F_{0,1}(\varphi)) \tag{4.35}$$

$$\ge \mu_p^{1/2}(F_{0,1}(h)) = \mu_p^{1/2}(H(h)) \tag{4.36}$$

where the second equality holds since $F_{\Theta,0}(\varphi) \to F_{h,0}(\varphi)$ as $\Theta \to h - 0$ uniformly with respect to $\varphi \in [0, h)$, the third equality follows from (4.18) and the fourth equality follows from the continuous extension arguments. To see the fifth equality, we note that $F_{0,1}(h) = F_{h,0}(h) \ge F_{h,1}(h)$ by (4.19) and (4.20), and the last equality follows again from (4.19). This completes the proof. □

Remark 3. *If we directly consider* $\|\mathbf{H}^{[0]}_{2/(1,p)}\|$ *rather than* $\|\mathbf{H}^{[+0]}_{2/(1,p)}\|$*, we readily see from (4.12) that* $\mu_p^{1/2}(F_{0,0}(0))$ $(= \mu_p^{1/2}(H(0)))$ *should be treated as an empty object in (4.33). This implies that we have established*

$$\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \sup_{0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi)), \quad \|\mathbf{H}^{[+0]}_{2/(1,p)}\| = \max\left(\mu_p^{1/2}(H(0)), \|\mathbf{H}^{[0]}_{2/(1,p)}\|\right) \tag{4.37}$$

and, thus,

$$\|\mathbf{H}^{[0]}_{2/(1,p)}\| \le \|\mathbf{H}^{[+0]}_{2/(1,p)}\|. \tag{4.38}$$

In particular, the first inequality of (4.34) would change into an equality if the left-hand side were $\|\mathbf{H}^{[0]}_{2/(1,p)}\|$*, and this implies, by (4.35), that we have established*

$$\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \|\mathbf{H}^{[h]}_{2/(1,p)}\|. \tag{4.39}$$

Supposing the standpoint that we are interested in $\|\mathbf{H}^{[h]}_{2/(1,p)}\|$*, the above equality implies that it is redundant to also be interested in* $\|\mathbf{H}^{[0]}_{2/(1,p)}\|$*, and it would be more informative to be interested in* $\|\mathbf{H}^{[+0]}_{2/(1,p)}\|$ *instead.*

We have the following result that is relevant to Proposition 1 and Remark 3; this corollary can also lead immediately to the first assertion of Corollary 2, but it does not seem helpful in the derivation of some of the preceding results, e.g., Theorem 6 and thus the second assertion of Corollary 2.

Corollary 3. *The following inequality holds for $p = 1$ and $p = 2$.*

$$\|\mathbf{H}^{[+0]}_{2/(1,p)}\| \geq \|\mathbf{H}^{[h]}_{2/(1,p)}\|. \tag{4.40}$$

Proof. The assertion follows immediately from (4.33) and (4.35). □

Furthermore, we have the following result.

Theorem 7. *Let $p = 1$ or $p = 2$, and suppose that $\mu_p^{1/2}(H(\varphi))$ is either (i) not maximum-attaining on $[0, h)$, or (ii) maximum-attaining on $[0, h)$ and $(\mu_p^{1/2}(H(h)) =)\ \lim_{\varphi \to h-0} \mu_p^{1/2}(H(\varphi))$ equals the maximum. Then, the $L_2/L_{1,p}$ Hankel norm $\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}$ is given by $\|\mathbf{H}^{[0]}_{2/(1,p)}\|$ $(= \|\mathbf{H}^{[h]}_{2/(1,p)}\|)$. In other words, the right-hand side of (2.4) is attained as the maximum at $\Theta = 0$.*

Proof. It follows from (4.37) that

$$\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \sup_{0 \leq \varphi \leq h} \mu_p^{1/2}(F_{0,1}(\varphi)) \geq \mu_p^{1/2}(F_{0,1}(h)) = \mu_p^{1/2}(H(h)) \tag{4.41}$$

where the continuous extension arguments and (4.19) are used. On the other hand, the hypothesis (i) implies that $\mu_p^{1/2}(H(h)) = \sup_{0 \leq \varphi < h} \mu_p^{1/2}(H(\varphi))$; thus,

$$\mu_p^{1/2}(H(h)) = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)} \tag{4.42}$$

by (4.14), while the hypothesis (ii) also implies (4.42) immediately, because the maximum must equal the quasi $L_2/L_{1,p}$ Hankel norm by (4.14). It then follows from (4.41) and (4.42), together with (2.4), that the inequality in (4.41) must, in fact, be an equality. This, together with (4.39) and (4.42), completes the proof. □

Note that the above theorem is closely related to Theorem 6, but that the theorem was relevant to the limit $\Theta \to +0$ by virtue of Definition 1 for a critical instant (at 0), while the above theorem is associated with the direct treatment of $\Theta = 0$; the latter implies that the statement of Theorem 6 would remain valid even if the definition of a critical instant were modified in such a way that only (4.26) was used also for the case of $\Theta^\star = 0$. Similarly, under such an "alternative" definition for a critical instant, it readily follows from Theorem 7 (under the hypothesis (ii)) that the statement of Theorem 5 would remain valid if the slight additional requirement of $\mu_p^{1/2}(H(h)) = \mu_p^{1/2}(H(0))$ were added in the hypothesis for the case of $\varphi^\star = 0$. More specifically, we have the following result also covering the case in which the condition about this additional requirement does not hold.

Theorem 8. *Let $p = 1$ or $p = 2$, and suppose that $\mu_p^{1/2}(H(\varphi))$ attains the maximum over $\varphi \in [0, h)$ at $\varphi = 0$. If $\mu_p^{1/2}(H(h)) = \mu_p^{1/2}(H(0))$, then $\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\|$ attains the maximum over $\Theta \in [0, h)$ at $\Theta = 0$. Otherwise, i.e., if $\mu_p^{1/2}(H(h)) < \mu_p^{1/2}(H(0))$, then the following properties (I) and (II) hold:*
(I) (Not only for $\Theta_0 = 0$, but) for every $\Theta_0 \in [0, h)$, the quasi $L_2/L_{1,p}$ Hankel norm $\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\|$ attains the maximum over $\Theta \in [0, h)$ at $\Theta = \Theta_0$, provided that the following condition is satisfied for the subsystem, denoted by P_{11}, of the generalized plant P restricted to the input w and the output z:
(i-1) When $p = 1$: There exists $j^\star$ such that $P_{11}^{(j^\star)} = 0$ and $\mu_p(H(0))$ equals the $j^\star$th diagonal entry of $H(0)$, where $P_{11}^{(j)}$ denotes the jth column of P_{11} associated with the jth entry of $w(t)$;
(i-2) When $p = 2$: There exists $v^\star$ such that $P_{11}v^\star = 0$, $|v^\star|_2 = 1$ and $\mu_p(H(0))$ equals $(v^\star)^T H(0) v^\star$.

(II) The quasi $L_2/L_{1,p}$ Hankel norm at 0, i.e., $\|\mathbf{H}^{[0]}_{2/(1,p)}\|$, is less than the $L_2/L_{1,p}$ Hankel norm $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ of $\Sigma_{\rm SD}$, provided that the following conditions are satisfied for P_{11}:
(ii-1) When $p = 1$, there exists no j such that $P^{(j)}_{11} = 0$;
(ii-2) When $p = 2$, there exists no $v \neq 0$ such that $P_{11}v = 0$.

Proof. We first prove the first assertion. If $\mu_p^{1/2}(H(h)) = \mu_p^{1/2}(H(0))$, then the hypothesis of this theorem implies that $\mu_p^{1/2}(H(h))$ equals the maximum of $\mu_p^{1/2}(H(\varphi))$ over $\varphi \in [0, h)$. Hence, the assertion follows from Theorem 7 (under its hypothesis (ii)).

We next prove part (I) of the second assertion. Note that, if $\mu_p^{1/2}(H(h)) \neq \mu_p^{1/2}(H(0))$, then $\mu_p^{1/2}(H(h)) > \mu_p^{1/2}(H(0))$ can never be the case given the hypothesis that $\mu_p^{1/2}(H(\varphi))$ attains the maximum over $\varphi \in [0, h)$ at $\varphi = 0$; thus, the opposite inequality holds (as asserted): $\mu_p^{1/2}(H(h)) < \mu_p^{1/2}(H(0))$. Comparing the definition of $H(\varphi)$ in (3.15) and that of $F_{0,1}(\varphi)$ in (4.11), we see that

$$H(\varphi) = \int_0^h {\rm sq}(D_\theta(\varphi))d\theta + F_{0,1}(\varphi) \quad (\varphi \in [0, h)). \tag{4.43}$$

We first consider the case of $p = 1$. By the hypothesis (i-1) on P_{11} for $p = 1$, taking the $j^\star$th diagonal entry of both sides of (4.43) with $\varphi = 0$ leads to

(a1) $\mu_p^{1/2}(H(0)) = F^{(j^\star)}_{0,1}(0)^{1/2}$

by the definition of $j^\star$, where $F^{(j)}_{\Theta,i}(\varphi)$ denotes the jth diagonal entry of $F_{\Theta,i}(\varphi)$ for $i = 0, 1$. Similarly, since the $j^\star$th diagonal entry of the first term of $F_{\Theta,0}(\varphi)$ in (4.10) vanishes under the hypothesis on P_{11}, it readily follows that

(b1) $F^{(j^\star)}_{\Theta,0}(\varphi)$ is independent of Θ and equals $F^{(j^\star)}_{0,1}(\varphi)$ for every $\varphi \in [0, h)$

by (4.11). Hence, for $\Theta > 0$, we have

$$\begin{aligned}\max\left(\sup_{0\le\varphi<\Theta} F^{(j^\star)}_{\Theta,0}(\varphi)^{1/2}, \sup_{\Theta\le\varphi<h} F^{(j^\star)}_{\Theta,1}(\varphi)^{1/2}\right) &\ge \sup_{0\le\varphi<\Theta} F^{(j^\star)}_{\Theta,0}(\varphi)^{1/2} \ge F^{(j^\star)}_{\Theta,0}(0)^{1/2} = F^{(j^\star)}_{0,1}(0)^{1/2} \\ &= \mu_p^{1/2}(H(0)) = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}\end{aligned} \tag{4.44}$$

where the first equality follows from (b1), the second equality follows from (a1) and the last equality follows from the hypothesis of the theorem and (4.14). For $\Theta = 0$, on the other hand, we also have

$$\begin{aligned}\max\left(\sup_{0\le\varphi<\Theta} F^{(j^\star)}_{\Theta,0}(\varphi)^{1/2}, \sup_{\Theta\le\varphi<h} F^{(j^\star)}_{\Theta,1}(\varphi)^{1/2}\right) &= \sup_{0\le\varphi<h} F^{(j^\star)}_{0,1}(\varphi)^{1/2} \ge F^{(j^\star)}_{0,1}(0)^{1/2} \\ &= \mu_p^{1/2}(H(0)) = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}\end{aligned} \tag{4.45}$$

in a similar fashion. The above observations, together with (4.12), imply that $\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\| \ge \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ for each $\Theta \in [0, h)$, which, in turn, implies that $\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\| = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ for each $\Theta \in [0, h)$ by (2.4). This completes the proof of the first assertion for $p = 1$.

We next prove part (I) of the second assertion for $p = 2$, following similar steps to those for the case of $p = 1$. It follows from (4.43) that

(a2) $\mu_p^{1/2}(H(0)) = (v^\star)^T F_{0,1}(0)v^\star$

by the hypothesis (i-2) on P_{11} for $p = 2$ and the associated definition of $v^\star$; the comparison of (4.10) and (4.11) leads to the following:

(b2) $(v^\star)^T F_{\Theta,0}(\varphi)v^\star$ is independent of Θ and equals $(v^\star)^T F_{0,1}(\varphi)v^\star$ for every $\varphi \in [0, h)$

Taking $(v^\star)^T F_{\Theta,0}(\varphi)v^\star$ and $(v^\star)^T F_{\Theta,1}(\varphi)v^\star$ instead of $F_{\Theta,0}^{(j^\star)}(\varphi)^{1/2}$ and $F_{\Theta,1}^{(j^\star)}(\varphi)^{1/2}$, respectively, in the above arguments for $p = 1$, we can readily complete the proof.

We next prove part (II) of the second assertion. Noting that $\|\mathbf{H}_{2/(1,p)}^{[0]}\|$ cannot exceed $\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}$ according to (2.4), suppose that $\|\mathbf{H}_{2/(1,p)}^{[0]}\| = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}$. By showing that we are then led to contradiction, the proof will be completed.

Assuming the above equality means that we should have $\|\mathbf{H}_{2/(1,p)}^{[0]}\| = \|\mathbf{H}_{2/(1,p)}^{[+0]}\|$, because of (4.38), together with (2.4). Hence, it follows from (4.37) that

$$\mu_p^{1/2}(H(0)) \le \sup_{0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi)). \tag{4.46}$$

This, together with (4.17) with Θ set to 0, implies that

$$\mu_p^{1/2}(H(0)) \le \sup_{0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi)) \le \sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi)) = \mu_p^{1/2}(H(0)) \tag{4.47}$$

where the last equality follows from the hypothesis of this theorem. Thus, the inequalities must, in fact, be equalities, i.e.,

$$\sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi)) = \sup_{0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi)). \tag{4.48}$$

We first consider the case of $p = 1$. By the condition (ii-1), it follows from (4.43) that the jth diagonal entry of $H(\varphi)$ satisfies that $H^{(j)}(\varphi) > F_{0,1}^{(j)}(\varphi)$ for each $j = 1, \ldots, n_w$ and every $\varphi \in [0, h)$; thus,

$$\mu_p^{1/2}(H(\varphi)) > \mu_p^{1/2}(F_{0,1}(\varphi)) \quad (\varphi \in [0, h)). \tag{4.49}$$

Even though $[0, h)$ is not a compact set, we can show that taking the supremum over $[0, h)$ on both sides of the above inequality leads to

$$\sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi)) > \sup_{0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi)) \tag{4.50}$$

by taking account of the present hypothesis that $\mu_p^{1/2}(H(\varphi))$ attains the maximum over $\varphi \in [0, h)$ at φ and $\mu_p^{1/2}(H(h)) < \mu_p^{1/2}(H(0))$. To see this, we take $h_0 \in [0, h)$ such that $\mu_p^{1/2}(H(\varphi)) \le \mu_p^{1/2}(H(0)) - d/2$ whenever $\varphi \in [h_0, h)$, where $d := \mu_p^{1/2}(H(0)) - \mu_p^{1/2}(H(h)) > 0$. This, together with (4.49), leads to $\mu_p^{1/2}(H(0)) > \mu_p^{1/2}(H(0)) - d/2 \ge \sup_{h_0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi))$, while $\sup_{0\le\varphi\le h_0} \mu_p^{1/2}(H(\varphi)) > \sup_{0\le\varphi\le h_0} \mu_p^{1/2}(F_{0,1}(\varphi))$ immediately follows from (4.49). Hence, we have that $\sup_{0\le\varphi<h} \mu_p^{1/2}(F_{0,1}(\varphi)) < \max(\mu_p^{1/2}(H(0)), \sup_{0\le\varphi\le h_0} \mu_p^{1/2}(H(\varphi))) \le \max(\mu_p^{1/2}(H(0)), \sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi))) = \sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi))$, where the last equality follows from the hypothesis of this theorem. We have thus established (4.50) as claimed, but it obviously contradicts (4.48). This completes the proof for the case of $p = 1$.

We finally prove part (II) of the second assertion for $p = 2$. By the condition (ii-2), it follows from (4.43) that $v^T H(\varphi)v > v^T F_{0,1}(\varphi)v$ for every v such that $|v|_2 = 1$; thus, (4.49) also holds for $p = 2$. Then, (4.50) follows also for $p = 2$ by exactly the same arguments as those for $p = 1$. Hence, the proof is completed by the contradiction between (4.48) and (4.50). □

Regarding the above theorem, neither of the conditions (i-1) or (ii-1) holds if $P_{11}^{(j)} = 0$ for some $j = 1, \ldots, n_w$, but, for none of such j, the jth diagonal entry of $H(0)$ is the maximum among all of the diagonal entries; a similar situation exists where neither of the conditions (i-2) or (ii-2) holds. We have the following result in connection with such a situation. The proof is given in Appendix C.

Corollary 4. *The assertion (II) of Theorem 8 remains valid even if the conditions (ii-1) and (ii-2) are replaced by the following conditions, respectively:*
(ii-1) when $p = 1$, the condition (i-1) fails and $\varphi = 0$ is the only maximum-attaining point of $\mu_p^{1/2}(H(\varphi))$ on $[0, \varphi)$;
(ii-2) when $p = 2$, the condition (i-2) fails and $\varphi = 0$ is the only maximum-attaining point of $\mu_p^{1/2}(H(\varphi))$ on $[0, \varphi)$.

Since we can apply such further alternative arguments through the use of the present definition of a critical instant, i.e., Definition 1 with the separate treatment (4.27), it is regarded as reasonable and useful.

5. Relationship among the L_2/L_1 induced/Hankel norm and H_2 norms of Σ_{SD}

We have characterized the L_2/L_1 induced norm, as well as the L_2/L_1 Hankel norm for the stable sampled-data system Σ_{SD} in the preceding sections. More precisely, we have clarified their numerical computation methods for $p = 1$ and $p = 2$, and it has been shown that the L_2/L_1 induced/Hankel norms are related to (or can alternatively be interpreted as) evaluating the L_2 norm of the output z for some unit impulse applied to the input w at the worst timing $\varphi \in [0, h)$.

In the stable continuous-time LTI systems, on the other hand, the H_2 norm plays an important role as a qualitative measure for evaluating the worst effect of the input to the output. Furthermore, it is well known that the time-domain interpretation of the H_2 norm is also relevant to the evaluation of the L_2 norm of the output for the unit impulse input (which may be assumed to be applied at $t = 0$ due to the time-invariance). In particular, the H_2 norm is known to coincide with the L_2/L_1 induced norm for the single-input case [4, Remark 3.3]. These observations naturally lead one to be interested in the relationship between the L_2/L_1 induced norm and the H_2 norm also for the sampled-data system Σ_{SD}, and this section is devoted to studying such a relationship, taking account the fact that has been established in the preceding sections that the L_2/L_1 Hankel norm coincides with the L_2/L_1 induced norm, both for $p = 1$ and $p = 2$, for the sampled-data system Σ_{SD} (i.e., we will not directly mention the L_2/L_1 Hankel norm of $\|\Sigma_{\mathrm{SD}}\|_{2/1}$ in the following paragraphs, even though we do mention the quasi L_2/L_1 Hankel norms for reasons that will become clear later).

Before proceeding to such arguments, however, it should be first noted that there are several different time-domain definitions of the H_2 norms for Σ_{SD} because of the h-periodicity of the associated input-output mapping, depending on how to take this periodicity into account when defining the H_2 norm of Σ_{SD}. We begin by summarizing these definitions (just for reference, some comments on the frequency-domain treatment of sampled-data systems and their H_2 norm are given in the remark at the end of this section).

The first definition was introduced in [10], where the unit impulse was assumed to be applied only at $t = 0$ (or, more precisely, $t = -0$, i.e., just before the sampler takes its action at $t = 0$) to the input, in spite of the h-periodicity of the input-output mapping. When Σ_{SD} is a multi-input system, then such a unit impulse is applied independently to each entry of the input, and the square of the L_2 norm of the associated response is summed over the independently handled unit impulse inputs. The square root of the sum was then defined as the H_2 norm of $\Sigma_{\rm SD}$. In what follows, this definition of the H_2 norm is denoted by $||\Sigma_{\rm SD}||_{H_2}^{[-0]}$ (which, as well as all of the H_2 norms introduced below, has no distinction between $p = 1$ and $p = 2$), and it turns out that it can be represented as[¶]

$$||\Sigma_{\rm SD}||_{H_2}^{[-0]} = \lim_{\varphi \to h-0} {\rm tr}^{1/2}(H(\varphi)) \tag{5.1}$$

with $H(\varphi)$ given by (3.15), where ${\rm tr}(\cdot)$ denotes the trace of a matrix.

The second definition was introduced in [11, 12] by modifying the somewhat unnatural treatment in the above definition that considers that the unit impulse is applied only at (or just before) the sampling instant $t = 0$. The modification was made by considering the situations in which the unit impulse is applied at $t = \varphi \in [0, h)$ rather than only at $t = 0$. The sum of the squares of the L_2 norms is then considered for each $\varphi \in [0, h)$, and the square root of the average of such sums was defined as the H_2 norm of $\Sigma_{\rm SD}$, which we denote by $||\Sigma_{\rm SD}||_{H_2}^{[0,h)}$. In other words,

$$||\Sigma_{\rm SD}||_{H_2}^{[0,h)} = \left(\frac{1}{h} \int_0^h {\rm tr}\,(H(\varphi))\, d\varphi \right)^{1/2}. \tag{5.2}$$

The third definition was introduced in [14] in a similar fashion to the above second definition, but with the supremum taken instead of the average over $\varphi \in [0, h)$. The associated H_2 norm is denoted by[‖] $||\Sigma_{\rm SD}||_{H_2}^{[\varphi^\star]}$, where $[\varphi^\star]$ should be regarded as a "non-numeric" symbol, referring to the standpoint of considering the "worst φ" in the sampling interval $[0, h)$. Namely,

$$||\Sigma_{\rm SD}||_{H_2}^{[\varphi^\star]} = \sup_{0 \le \varphi < h} {\rm tr}^{1/2}\,(H(\varphi)). \tag{5.3}$$

We could also introduce other relevant quantities for $\Sigma_{\rm SD}$ by replacing the square of the L_2 norm for φ in the second and third H_2 norm definitions with the square of the quasi L_2/L_1 Hankel norm at φ. This is because (i) $\Sigma_{\rm SD}$ reduces to a continuous-time LTI system when $\Psi = 0$ (in which case the quasi L_2/L_1 Hankel norm at φ is obviously independent of φ and is nothing but the L_2/L_1 Hankel norm itself), (ii) the L_2/L_1 Hankel norm coincides with the L_2/L_1 induced norm for continuous-time LTI systems [2, Corollary], [3, Theorem 2] and (iii) the L_2/L_1 induced norm coincides with the H_2 norm for continuous-time single-input LTI systems [4, Remark 3.3]. Hence, we see that each of these two quantities does reduce to the standard H_2 norm of continuous-time systems for the special case with $\Psi = 0$ and a single input; thus, they could be meaningful also for a single-input sampled-data system. The above observation would suggest two other definitions of the H_2 norm of $\Sigma_{\rm SD}$ that are not confined to the single-input case once we introduce the following quantity instead of (4.12) (with Θ replaced by φ in accordance with the situation in the present arguments, which thus involves replacing the original φ in (4.12) with θ):

$$f_{\rm tr}^{1/2}(\varphi) = \max\{ \sup_{0 \le \theta < \varphi} {\rm tr}^{1/2}(F_{\varphi,0}(\theta)),\ \sup_{\varphi \le \theta < h} {\rm tr}^{1/2}(F_{\varphi,1}(\theta))\}. \tag{5.4}$$

[¶]Even though H is not an h-periodic function, the input-output behavior of the sampled-data system $\Sigma_{\rm SD}$ is h-periodic. Hence, note that an impulse input applied at $t = h - 0$ is essentially the same as that at $t = -0$.

[‖]We remark, just in case, that the exact notation used in [14] was $||\Sigma_{\rm SD}||^{[\tau^\star]}$, but the present paper employs $\varphi^\star$ instead of $\tau^\star$ so that the relevance of this third definition to the arguments of the present paper with the variable φ becomes much clearer.

More precisely, further new definitions of the H_2 norm for $\Sigma_{\rm SD}$ could be introduced (through the viewpoint of the quasi L_2/L_1 Hankel operator at φ) by replacing with the above $f_{\rm tr}(\varphi)$ the L_2 norm for φ in the second and third definitions. The resulting definitions are denoted by $||\Sigma_{\rm SD}||_{H_2}^{{\rm quasi},[0,h)}$ and $||\Sigma_{\rm SD}||_{H_2}^{{\rm quasi},[\varphi^\star]}$, respectively, and are given as follows:

$$||\Sigma_{\rm SD}||_{H_2}^{{\rm quasi},[0,h)} = \left(\frac{1}{h}\int_0^h f_{\rm tr}(\varphi)d\varphi\right)^{1/2} \tag{5.5}$$

$$||\Sigma_{\rm SD}||_{H_2}^{{\rm quasi},[\varphi^\star]} = \sup_{0\le\varphi<h} f_{\rm tr}^{1/2}(\varphi). \tag{5.6}$$

Regarding the relationship among these possible definitions of the H_2 norm and the L_2/L_1 induced norm for the sampled-data system $\Sigma_{\rm SD}$, we have the following result.

Theorem 9. *The L_2/L_1 induced norm and H_2 norms of the sampled-data system $\Sigma_{\rm SD}$ satisfy the following relations:*

$$(1/\sqrt{n_w})||\Sigma_{\rm SD}||_{H_2}^{[\varphi^\star]} \le ||\Sigma_{\rm SD}||_{2/(1,p)} \le ||\Sigma_{\rm SD}||_{H_2}^{[\varphi^\star]} \quad (p=1,2) \tag{5.7}$$

$$||\Sigma_{\rm SD}||_{H_2}^{[0,h)} \le ||\Sigma_{\rm SD}||_{H_2}^{{\rm quasi},[0,h)} \le ||\Sigma_{\rm SD}||_{H_2}^{[\varphi^\star]} = ||\Sigma_{\rm SD}||_{H_2}^{{\rm quasi},[\varphi^\star]} \tag{5.8}$$

$$||\Sigma_{\rm SD}||_{H_2}^{[-0]} \le ||\Sigma_{\rm SD}||_{H_2}^{[\varphi^\star]}. \tag{5.9}$$

In particular, for $\Sigma_{\rm SD}$ with a single input (i.e., when $n_w = 1$), the following relation holds:

$$||\Sigma_{\rm SD}||_{2/(1,p)} = ||\Sigma_{\rm SD}||_{H_2}^{[\varphi^\star]} \quad (p=1,2). \tag{5.10}$$

Proof. The proof is mostly straightforward if we note the relationship between the average and the supremum, as well as that between $\mu_p(\cdot)$ $(p = 1,2)$ and ${\rm tr}(\cdot)$. Indeed, we are thus led to (5.7), (5.9) and (5.10), as well as $||\Sigma_{\rm SD}||_{H_2}^{{\rm quasi},[0,h)} \le ||\Sigma_{\rm SD}||_{H_2}^{{\rm quasi},[\varphi^\star]}$. Furthermore, the equality assertion in (5.8) follows from essentially the same arguments as the proof of Theorem 4 by the use of the same inequalities of $F_{\varphi,0}(\theta) \le H(\theta)$ and $F_{\varphi,1}(\theta) \le H(\theta)$ used therein. Similarly, it follows from (5.4) that

$$f_{\rm tr}^{1/2}(\varphi) \ge \sup_{0\le\theta<\varphi} {\rm tr}^{1/2}(F_{\varphi,0}(\theta)) = \sup_{0\le\theta\le\varphi} {\rm tr}^{1/2}(F_{\varphi,0}(\theta)) \ge {\rm tr}^{1/2}(F_{\varphi,0}(\varphi)) = {\rm tr}^{1/2}(H(\varphi)) \tag{5.11}$$

where the two equalities follow from essentially the same arguments as those in the earlier half of the proof of Theorem 4. Hence, we are led to $||\Sigma_{\rm SD}||_{H_2}^{{\rm quasi},[0,h)} \ge ||\Sigma_{\rm SD}||_{H_2}^{[0,h)}$. Combining these results leads to (5.8), and this completes the proof. □

Remark 4. *Some of the above inequalities have been derived already in [14], where the relationship between the second H_2 norm and the L_∞/L_2 induced norm was discussed for the sampled-data system $\Sigma_{\rm SD}$ by introducing*

$$F(\theta) = \int_0^h {\rm sq}(D_\theta(\varphi)^T)d\varphi + \sum_{k=0}^{\infty}\int_0^h {\rm sq}((C_\theta \mathcal{A}^k B_h(\varphi))^T)d\varphi \tag{5.12}$$

$$\widetilde{G}(\varphi) = \int_0^h {\rm sq}(D_\theta(\varphi)^T)d\theta + \sum_{k=0}^{\infty}\int_0^h {\rm sq}((C_\theta \mathcal{A}^k B_h(\varphi))^T)d\theta. \tag{5.13}$$

We then have

$$\int_0^h \mathrm{tr}(H(\varphi))d\varphi = \int_0^h \mathrm{tr}(\widetilde{G}(\varphi))d\varphi = \int_0^h \mathrm{tr}(F(\theta))d\theta \tag{5.14}$$

and thus the second H_2 norm can be represented by the use of $F(\theta)$, while the L_∞/L_2 induced norm (which equals the L_∞/L_2 Hankel norm) is also represented by the use of $F(\theta)$ [14, 25, 26]. Even though such an observation plays a key role in studying the relationship between the second H_2 norm and the L_∞/L_2 induced norm in [14], it is not easy to relate (the non-integrated) $\mathrm{tr}(F(\theta))$ and $\mathrm{tr}(H(\varphi))$ directly. Hence, no definite relations exist between the L_∞/L_2 induced (Hankel) norm and the first H_2 norm, as well as between the former and the third H_2 norm (see Examples 1–3 in [14]). Since we can combine (5.8) or (5.9) with (5.7), we could say that the L_2/L_1 induced (Hankel) norm is more deeply related than the L_∞/L_2 induced (Hankel) norm to different definitions of the H_2 norms for the sampled-data system Σ_{SD}.

Remark 5. *There are frequency-domain studies of sampled-data systems, such as [34, 35], which are entirely free from the time-domain lifting technique; the H_2 norm is also defined in such a context of studies as in [13]. Alternatively, the transfer operator can be introduced to sampled-data systems as in [30] by applying the lifting technique used in the present paper, through which the H_2 norm can also be introduced. In fact, the second definition [12] corresponds to an equivalent time-domain interpretation of such a definition.*

6. Numerical examples

This section studies some numerical examples to demonstrate the theoretical results on the quasi L_2/L_1 Hankel norms and the L_2/L_1 Hankel norm, as well as to confirm the relations among the L_2/L_1 induced norm and H_2 norms of sampled-data systems.

In the following numerical analysis, the supremum of a function over $[0, h)$ is computed as the maximum over $\Phi_M := \{0, h', \cdots, Mh'\}$ with $h' := h/M$ and $M = 200$, which includes h. This is because each relevant function has a continuous extension to the closed interval $[0, h]$ and the above maximum tends to the relevant supremum as $M \to \infty$. For the same reason, the H_2 norm defined by (5.1) is computed as $\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[-0]} = \mathrm{tr}^{1/2}(H(h))$. Similarly, the averages in (5.2) and (5.5) are computed approximately by using those for the $M + 1$ values of φ in Φ_M.

Example 1. *Consider the stable two-input ($n_w = 2$) sampled-data system with $h = 0.02$ and*

$$A = \begin{bmatrix} -3 & 5 \\ -4 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$A_\Psi = \begin{bmatrix} -1.0386 & 0.0308 \\ 0.0001 & -0.0092 \end{bmatrix}, \quad B_\Psi = \begin{bmatrix} 0.0382 \\ -0.0000 \end{bmatrix}, \quad C_\Psi = \begin{bmatrix} -481.8809 & 6.9907 \end{bmatrix}, \quad D_\Psi = 8.4184. \tag{6.1}$$

Example 2. *Consider the stable three-input ($n_w = 3$) sampled-data system with $h = 0.02$ and*

$$A = \begin{bmatrix} -6 & 1 & 2 & 3 \\ 8 & 4 & 5 & -6 \\ 0 & -1 & 4 & -7 \\ -2 & 3 & 2 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -3 & 3 & 9 \\ -3 & 0 & -1 \\ 2 & 5 & 1 \\ 2 & -1 & 7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 5 \\ 0 & 1 \\ 3 & 1 \\ -1 & 2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 8 & 3 & 1 & 2 \\ -6 & -4 & 0 & -1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -9 & -3 & 1 & 4 \\ -4 & 8 & -2 & 2 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 5 & 4 \\ -6 & -2 \end{bmatrix},$$

$$A_\Psi = \begin{bmatrix} 0.2779 & -0.0234 & 0.0220 & -0.0066 \\ -0.5834 & 0.8358 & 0.0152 & -0.0056 \\ -0.0003 & 0.0013 & -0.0550 & -0.0574 \\ -0.0018 & 0.0004 & -0.0131 & -0.1293 \end{bmatrix}, \quad B_\Psi = \begin{bmatrix} 0.0782 & -0.1315 \\ 0.0497 & -0.1161 \\ 0.0001 & -0.0001 \\ 0.0002 & -0.0003 \end{bmatrix}$$

$$C_\Psi = \begin{bmatrix} -4.6254 & 0.0880 & 0.1081 & -0.0237 \\ 7.6625 & 2.9260 & -0.2137 & 0.0732 \end{bmatrix}, \quad D_\Psi = \begin{bmatrix} 0.4585 & -0.8368 \\ -0.5921 & 1.5766 \end{bmatrix}. \tag{6.2}$$

For these examples, different types of norms considered in this paper are computed as shown in Tables 2 and 3. We can confirm from these tables the relations in (5.7)–(5.9), as well as (3.18). In addition, the L_2/L_1 Hankel norm can be computed (not only by (4.14) as in Tables 2 and 3, but also) by applying (2.4) through the use of the quasi L_2/L_1 Hankel norms for $\Theta \in [0, h)$ that are shown in Figures 2 and 3 for Example 1 (to which Figure 4 is also relevant, as mentioned later) and in Figures 5 and 6 for Example 2**, and can be confirmed to coincide with the L_2/L_1 induced norm in both examples (see Corollary 1), as seen from Tables 2 and 3. Furthermore, we can confirm the inequality $\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\| \geq \mu_p^{1/2}(H(\Theta))$ in (4.22) in all of these figures.

Furthermore, we can confirm from Figures 2, 3, 5 and 6 the inequality (4.22), i.e., $\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\| \geq \mu_p^{1/2}(H(\varphi))$ for $\varphi = \Theta$. Nevertheless, taking the supremum for $\varphi = \Theta \in [0, h)$ on both sides yields the identical value, as seen from these figures, and this is consistent with (2.4) and (4.14) on the L_2/L_1 Hankel norm. In connection with this consistency, we note that (2.4) with $\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\|$ is more directly related to determining the existence (and the value, if one exists) of a critical instant, and we see from the relevant curves in Figures 2 and 3, as well as Figures 5 and 6, that $\Sigma_{\rm SD}$ has a critical instant at $\Theta = 0$ in Example 1 and Example 2, both for $p = 1$ and $p = 2$ (we can confirm in Figures 5 and 6 that the quasi L_2/L_1 Hankel norm at Θ tends to that at $\Theta = 0$ as $\Theta \to h - 0$). In particular, the observation associated with Example 1 confirms the assertion in Theorem 5, which tackles the existence of a critical instant through the less relevant quantity $H(\varphi)$. The observation associated with Example 2, on the other hand, implies that a converse type of the assertion in Theorem 5 does not hold, in general. That is, a critical instant can exist even if $\mu_p^{1/2}(H(\varphi))$ is not maximum-attaining on $[0, h)$; this is asserted in a more precise manner in Theorem 6, and Figures 5 and 6 confirm that $\Theta = 0$ is indeed a critical instant for such a case, both for $p = 1$ and $p = 2$. To summarize, Examples 1 and 2 are consistent with Corollary 2. Furthermore, Figures 2, 3, 5 and 6 are consistent with Corollary 3.

As an additional remark, we note that, for each interval of Θ $(= \varphi)$ in Figures 2, 3, 5 and 6 such that the strict version of the inequality (i.e., $\|\mathbf{H}^{[\Theta]}_{2/(1,p)}\| > \mu_p^{1/2}(H(\varphi))$) actually holds in (4.22), it has the meaning that, for each Θ in the interval, the worst timing at which a unit impulse is (virtually) applied in the sense of the quasi L_2/L_1 Hankel norm at Θ is not immediately before Θ; more precisely, for Θ in such an interval, the corresponding $\phi \in (0, h]$ introduced above (4.9) is given by $\phi > 0$ (rather than "$\phi = +0$"). This can be seen by noting that $F_{\varphi,0}(\varphi) = H(\varphi)$ on the right-hand side of (4.12).

On the other hand, we can confirm from Figures 4 and 7 (respectively for Examples 1 and 2, and relevant to the comparison between the L_2/L_1 induced norm and the H_2 norm) the inequality $f_{\rm tr}^{1/2}(\varphi) \geq$

**More precisely, these figures (as well as the figure for Example 3) for the quasi L_2/L_1 Hankel norms are actually drawn in such a way that the value at $\Theta = 0$ actually corresponds to $\|\mathbf{H}^{[+0]}_{2/(1,p)}\|$ rather than $\|\mathbf{H}^{[0]}_{2/(1,p)}\|$; by directly taking the value of $\|\mathbf{H}^{[0]}_{2/(1,p)}\|$ instead, we have confirmed in all examples that (4.38) and (4.39) hold, but the curve of the quasi L_2/L_1 Hankel norms then becomes messy, and this is why the figures are drawn in the aforementioned fashion.

$\mathrm{tr}^{1/2}(H(\varphi))$ given in (5.11). In spite of this, taking the supremum in φ over $[0, h)$ on both sides leads to the identical value, as seen from these figures, which corresponds to the equality in (5.8) about two different types of H_2 norm of Σ_{SD}. For each interval of ϕ in Figures 4 and 7 such that the strict version of the inequality (i.e., $f_{\mathrm{tr}}^{1/2}(\varphi) > \mathrm{tr}^{1/2}(H(\varphi))$) actually holds in (5.11), it has a similar meaning to what was mentioned above. That is, for each Θ in the interval, the worst timing before Θ at which a unit impulse is applied (in the sense of evaluating the square root of $\sum_{i=1}^{n_w} \|z_i\|^2_{L_2[\Theta,\infty)}$, with z_i being the response for the unit impulse applied to the ith input) is not immediately before Θ. This can be seen by noting that $F_{\varphi,0}(\varphi) = H(\varphi)$ on the right-hand side of (5.4).

We finally give an example in which a nonzero critical instant exists.

Table 2. The L_2/L_1 induced (Hankel) norm and H_2 norms in Example 1.

norm type	value
$\|\Sigma_{\mathrm{SD}}\|_{2/(1,1)} = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,1)}$	1.5543
$\|\Sigma_{\mathrm{SD}}\|_{2/(1,2)} = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,2)}$	1.6280
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[-0]}$	1.5982
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0,h)}$	1.6193
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[\varphi^\star]}$	1.6391
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{\mathrm{quasi},[0,h)}$	1.6214
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{\mathrm{quasi},[\varphi^\star]}$	1.6391

Table 3. The L_2/L_1 induced (Hankel) norm and H_2 norms in Example 2.

norm type	value
$\|\Sigma_{\mathrm{SD}}\|_{2/(1,1)} = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,1)}$	31.3038
$\|\Sigma_{\mathrm{SD}}\|_{2/(1,2)} = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,2)}$	44.0854
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[-0]}$	50.3686
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[0,h)}$	50.0664
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{[\varphi^\star]}$	50.3686
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{\mathrm{quasi},[0,h)}$	50.0816
$\|\Sigma_{\mathrm{SD}}\|_{H_2}^{\mathrm{quasi},[\varphi^\star]}$	50.3686

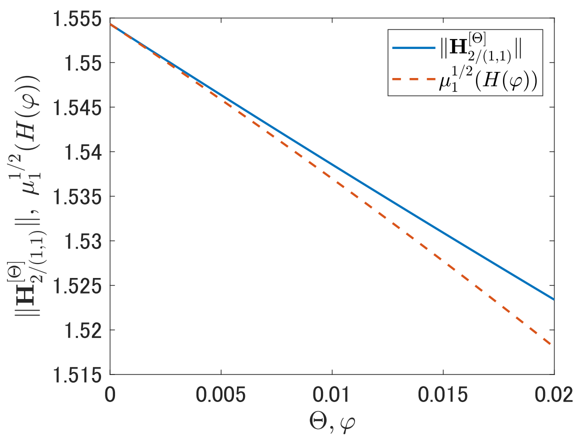

Figure 2. Variations of $\mu_1^{1/2}(H(\varphi))$ and $\|\mathbf{H}_{2/(1,1)}^{[\Theta]}\|$ for $\varphi, \Theta \in [0, h]$ in Example 1.

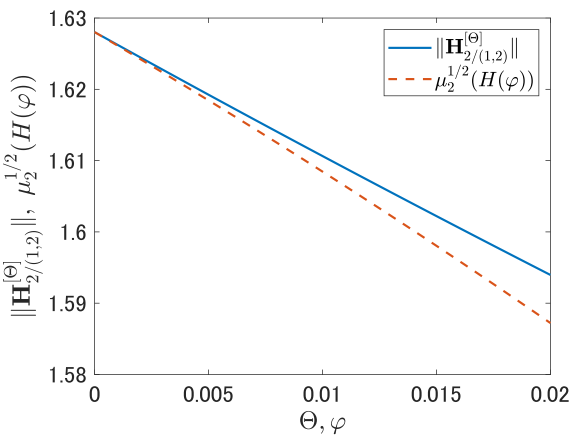

Figure 3. Variations of $\mu_2^{1/2}(H(\varphi))$ and $\|\mathbf{H}_{2/(1,2)}^{[\Theta]}\|$ for $(\varphi, \Theta \in [0, h]$ in Example 1.

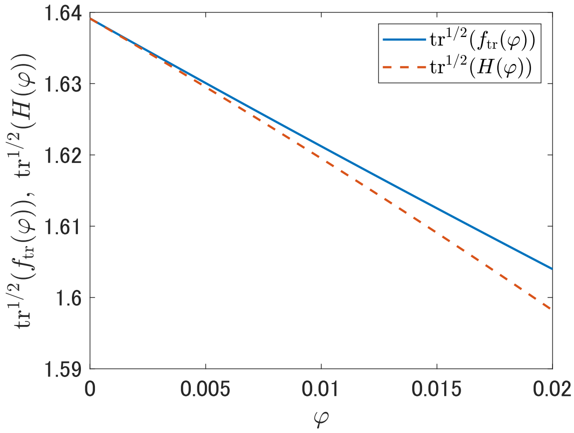

Figure 4. Variations of $\mathrm{tr}^{1/2}(H(\varphi))$ and $\mathrm{tr}^{1/2}(f_{\mathrm{tr}}(\varphi))$ for $\varphi \in [0, h]$ in Example 1.

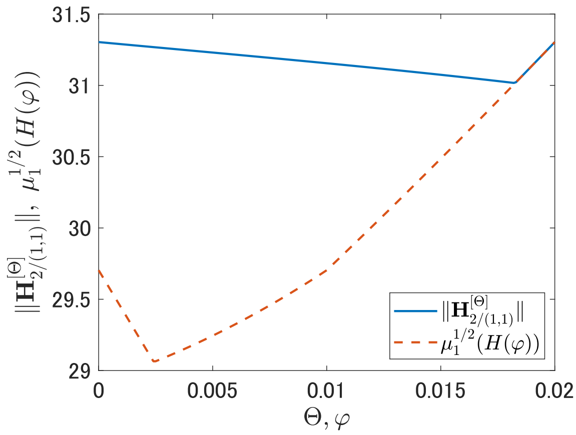

Figure 5. Variations of $\mu_1^{1/2}(H(\varphi))$ and $\|\mathbf{H}^{[\Theta]}_{2/(1,1)}\|$ for $\varphi, \Theta \in [0, h]$ in Example 2.

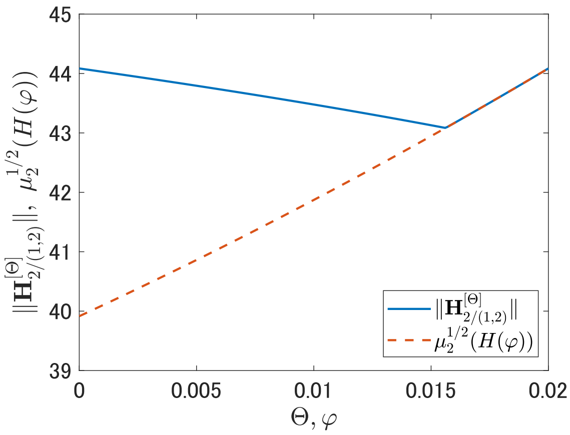

Figure 6. Variations of $\mu_2^{1/2}(H(\varphi))$ and $\|\mathbf{H}^{[\Theta]}_{2/(1,2)}\|$ for $\varphi, \Theta \in [0, h]$ in Example 2.

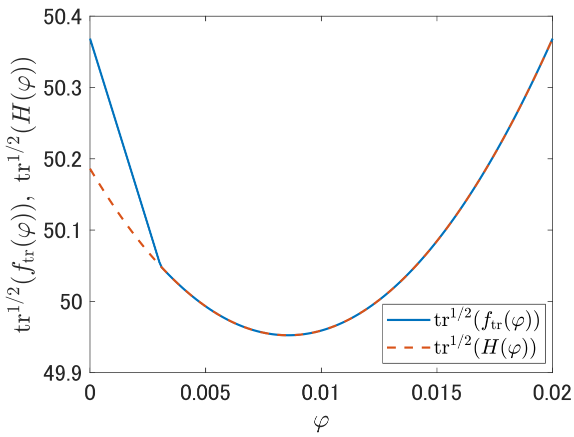

Figure 7. Variations of $\mathrm{tr}^{1/2}(H(\varphi))$ and $\mathrm{tr}^{1/2}(f_{\mathrm{tr}}(\varphi))$ for $\varphi \in [0, h]$ in Example 2.

Example 3. *Consider the stable single-input ($n_w = 1$) LTI sampled-data system with $h = 0.2$ and*

$$A = \begin{bmatrix} -3 & 5 \\ -4 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0.17 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$A_\Psi = \begin{bmatrix} -1.0980 & 0.1395 \\ 0.0000 & -0.5665 \end{bmatrix}, \quad B_\Psi = \begin{bmatrix} 0.1048 \\ -0.0000 \end{bmatrix}, \quad C_\Psi = \begin{bmatrix} -11.1422 & 0.0628 \end{bmatrix}, \quad D_\Psi = 0.1633. \tag{6.3}$$

We can readily ascertain from Figure 8 that this sampled-data system has a nonzero critical instant, unlike the two preceding examples for which only $\Theta = 0$ is a critical instant. In addition, we see that both inequalities in (4.22) hold as equalities for every $\Theta \in [0, h)$ in this example.

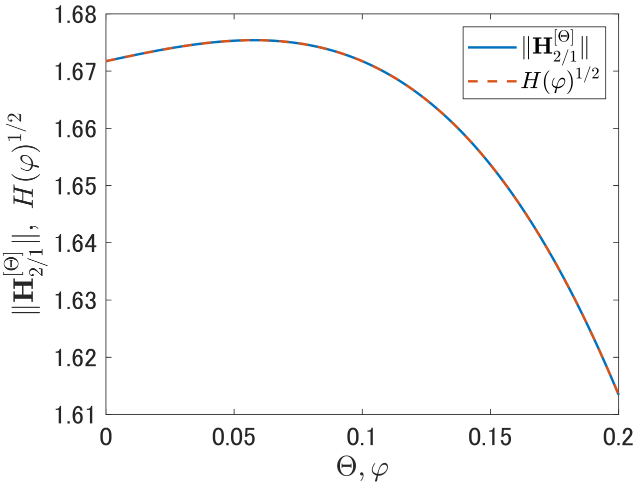

Figure 8. Variations of $H(\varphi)^{1/2}$ and $\|\mathbf{H}_{2/1}^{[\Theta]}\|$ for $\Theta \in [0, h]$ in Example 3.

7. Discussions and conclusions

This paper studied the L_2/L_1 induced norm and the L_2/L_1 Hankel norm for stable sampled-data systems through the application of an operator-theoretic treatment of their input-output relation via the lifting technique [17, 18, 30]. These norms have been characterized through the use of the positive-semidefinite matrix-valued function $H(\varphi)$ in (3.15) that is defined over the sampling interval $[0, h)$ (in such a way that the numerical computation of these norms can readily be carried out), as well as shown to coincide with each other, as in the case of stable continuous-time LTI systems. Very importantly, it is the intrinsic h-periodic nature of the input-output relation of sampled-data systems that leads us to the treatment of such a matrix-valued function, and, in this respect, it would be quite helpful to note the following standpoint of this paper in contrast to the L_∞/L_2 viewpoint, which led to another matrix-valued function $F(\theta)$ in (5.12).

Namely, this study was motivated by the fundamental facts for stable continuous-time LTI systems, particularly that (i) the L_∞/L_2 induced norm and the L_∞/L_2 Hankel norm coincide with each other and, for the single-output case, further equal another important quantity of the H_2 norm, (ii) the L_2/L_1 induced norm and the L_2/L_1 Hankel norm coincide with each other and, for the single-input case, also equal the H_2 norm, together with the further fact for sampled-data systems that (iii) the L_∞/L_2

induced norm and the L_∞/L_2 Hankel norm again coincide with each other, and their relationship to one definition of the H_2 norm for sampled-data systems (given in [11, 12]) has been discussed in an earlier study [14]. More specifically, this paper was naturally motivated by these series of facts, and we were led to studying the L_2/L_1 induced and Hankel norms as well for stable sampled-data systems. In other words, as a result of applying different treatments through another alternative viewpoint of tackling the h-periodic nature of the input-output relation of sampled-data systems from the L_2/L_1 viewpoint, we were instead led to $H(\varphi)$, rather than $F(\theta)$.

Regarding the motivation and standpoint mentioned above, we have indeed arrived at clarifying new aspects on the H_2 norms for sampled-data systems and their relevant norms, as discussed in Section 5 (in particular, Remark 4), and whose brief summary is as follows. In the study of the L_∞/L_2 induced/Hankel norms for sampled-data systems [14, 25, 26], their relationship has been clarified only with respect to the specific type of H_2 norm of sampled-data systems given in [11, 12]. However, it is important to note that there are other possible alternatives for the definition of the H_2 norm for sampled-data systems. In this regard, the arguments of this paper have indeed suggested the introduction of further alternative possible definitions of the H_2 norm through the L_2/L_1 viewpoint, as well as showed that some explicit relations can be established among the L_2/L_1 induced/Hankel norm and different definitions of the H_2 norms, that is, if the L_2/L_1 viewpoint is taken. In this sense, it was shown that the L_2/L_1 induced norm has a closer relationship to all these H_2 norms than the L_∞/L_2 induced norm in sampled-data systems.

Regarding the main issue of the L_2/L_1 Hankel norm analysis studied in this paper, the quasi L_2/L_1 Hankel norm at each $\Theta \in [0, h)$ was also characterized, and a relevant problem on the existence of a critical instant was also tackled. In particular, it was shown that a critical instant Θ always exists in the L_2/L_1 Hankel norm analysis, as in the L_2/L_2 Hankel norm analysis [27], but unlike a seemingly more related problem of the L_∞/L_2 Hankel norm analysis [25, 26]. It was also shown that $H(\varphi)$ can give some information on the locations of critical instants Θ. However, it remains open whether all of the critical instants can be detected only through the analysis of $H(\varphi)$, whose definition *does not* involve Θ at all. In the L_∞/L_2 Hankel norm analysis, positive answers to a parallel question in terms of $F(\theta)$ have been derived in [26] under mild assumptions, and investigating whether a similar result can be derived in the L_2/L_1 setting will be a future topic.

Finally, we remark that, by following the same line as previous studies [36, 37] on the analysis and synthesis of sampled-data systems under the specific H_2 norm given in [14], we can tackle the problem of controller synthesis stabilizing the closed-loop system and minimizing the L_2/L_1 induced/Hankel norm for sampled-data systems. The details will be reported independently as a future topic.

Appendix

A. Rigorous arguments on (3.3)

This appendix is devoted to a more rigorous derivation of (3.3).

Since $\sum_{k=0}^{K} \|\widehat{z}_k\|_2 \le \sum_{k=0}^{K'} \|\widehat{z}_k\|_2$ whenever $K \le K'$, it is obvious that

$$
\begin{aligned}
\left(\sum_{k=0}^{K}\|\widehat{z}_k\|_2\right)^{1/2} &\le \left\| \begin{bmatrix} \mathcal{D} \\ \mathcal{C}\mathcal{B} \\ \mathcal{C}\mathcal{A}\mathcal{B} \\ \vdots \\ \mathcal{C}\mathcal{A}^{K-1}\mathcal{B} \\ \vdots \end{bmatrix} \widehat{w}_0 + \begin{bmatrix} 0 \\ \mathcal{D} \\ \mathcal{C}\mathcal{B} \\ \vdots \\ \mathcal{C}\mathcal{A}^{K-1}\mathcal{B} \\ \vdots \end{bmatrix} \widehat{w}_1 + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathcal{D} \\ \vdots \end{bmatrix} \widehat{w}_K \right\|_{2,0+} \\
&\le \left\| \begin{bmatrix} \mathcal{D} \\ \mathcal{C}\mathcal{B} \\ \mathcal{C}\mathcal{A}\mathcal{B} \\ \vdots \\ \mathcal{C}\mathcal{A}^{K-1}\mathcal{B} \\ \vdots \end{bmatrix} \widehat{w}_0 \right\|_{2,0+} + \left\| \begin{bmatrix} 0 \\ \mathcal{D} \\ \mathcal{C}\mathcal{B} \\ \vdots \\ \mathcal{C}\mathcal{A}^{K-1}\mathcal{B} \\ \vdots \end{bmatrix} \widehat{w}_1 \right\|_{2,0+} + \cdots + \left\| \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \mathcal{D} \\ \vdots \end{bmatrix} \widehat{w}_K \right\|_{2,0+} \\
&\le \|\mathcal{F}\| \cdot \|\widehat{w}_0\|_1 + \|\mathcal{F}\| \cdot \|\widehat{w}_1\|_1 + \cdots + \|\mathcal{F}\| \cdot \|\widehat{w}_K\|_1 \\
&\le \|\mathcal{F}\| \cdot \|\widehat{w}\|_{1,0+}\,.
\end{aligned} \tag{7.1}
$$

Since this is true for each nonnegative integer K, letting $K \to \infty$ leads to (3.3).

Remark 6. *The above arguments implicitly assume that $\mathcal{F}$ is a bounded operator, but the independent arguments after (3.5) indeed establish that this is indeed the case.*

B. Supplementary arguments on the impulse response treatment

This appendix is devoted to showing how dealing with L_1 rigorously in our problem setting eventually leads, as a sort of limit, to an interpretation that is equivalent to the impulse response treatment suggested in Section 3.

Recall that the arguments therein were confined to the situation in which the input is concentrated on a single entry of w, as well as on an infinitesimally small interval around the worst timing φ in the interval $[0, h)$. Thus, we assume that only the jth entry of w is nonzero, and this input, denoted by $w^{(j)}$, is possibly nonzero only on the interval $[\varphi, \varphi + \epsilon)$ for a given $\varphi \in [0, h)$ and a sufficiently small $\epsilon > 0$. The corresponding response of z can be described by $\mathcal{B}\widehat{w}_0$ (or by $\mathcal{C}\mathcal{A}^k\mathcal{B}\widehat{w}_0$, more precisely speaking) and $\mathcal{D}\widehat{w}_0$ by (2.6), and thus more explicitly by $\int_{\varphi}^{\varphi+\epsilon} B_h^{(j)}(\tau)\widehat{w}_0^{(j)}(\tau)d\tau$ and $\int_{\varphi}^{\varphi+\epsilon} D_\theta^{(j)}(\tau)\widehat{w}_0^{(j)}(\tau)d\tau$, because of (2.8), (2.10), (3.11) and (3.13), where $B_h^{(j)}(\cdot)$ and $D_\theta^{(j)}(\cdot)$ denote the jth columns of $B_h(\cdot)$ and $D_\theta(\cdot)$, respectively.

The subsequent arguments follow three steps. In the first step, we give the motivation and rationale for the approximations in which $B_h^{(j)}(\tau)$ and $D_\theta^{(j)}(\tau)$ are equivalently regarded as taking the constant values $B_h^{(j)}(\varphi)$ and $D_\theta^{(j)}(\varphi)$, respectively, or, more precisely,

$$
\int_{\varphi}^{\varphi+\epsilon} B_h^{(j)}(\tau)\widehat{w}_0^{(j)}(\tau)d\tau \simeq B_h^{(j)}(\varphi)\int_{\varphi}^{\varphi+\epsilon} \widehat{w}_0^{(j)}(\tau)d\tau \tag{7.2}
$$

and

$$
\int_{\varphi}^{\varphi+\epsilon} D_\theta^{(j)}(\tau)\widehat{w}_0^{(j)}(\tau)d\tau \simeq D_\theta^{(j)}(\varphi)\int_{\varphi}^{\varphi+\epsilon} \widehat{w}_0^{(j)}(\tau)d\tau \tag{7.3}
$$

(and state why such an approach could roughly be interpreted as an impulse response treatment). These two approximations are handled separately in Steps 1a and 1b. In the second step, we discuss how the treatment of $B_h^{(j)}(\tau)$ and $D_\theta^{(j)}(\tau)$ through the use of $B_h^{(j)}(\varphi)$ and $D_\theta^{(j)}(\varphi)$, respectively, leads to approximation errors in the framework of rigorous treatment of L_1. The two cases for $B_h^{(j)}(\tau)$ and $D_\theta^{(j)}(\tau)$ are handled separately in Steps 2a and 2b. Finally, the third step completes the arguments by combining the preceding arguments.

Step 1a. Regarding $\int_\varphi^{\varphi+\epsilon} B_h^{(j)}(\tau)\widehat{w}_0^{(j)}(\tau)d\tau$, we readily see that

$$\begin{aligned}\left|\int_\varphi^{\varphi+\epsilon} B_h^{(j)}(\tau)\widehat{w}_0^{(j)}(\tau)d\tau - B_h^{(j)}(\varphi)\int_\varphi^{\varphi+\epsilon}\widehat{w}_0^{(j)}(\tau)d\tau\right|_2 &\le \int_\varphi^{\varphi+\epsilon}\left|B_h^{(j)}(\tau)-B_h^{(j)}(\varphi)\right|_2\cdot|\widehat{w}_0^{(j)}(\tau)|d\tau\\ &\le \left(\sup_{\varphi\le\tau<\varphi+\epsilon}\left|B_h^{(j)}(\tau)-B_h^{(j)}(\varphi)\right|_2\right)\int_\varphi^{\varphi+\epsilon}|\widehat{w}_0^{(j)}(\tau)|d\tau. \quad (7.4)\end{aligned}$$

By the continuity of $B_h(\cdot)$ on $[0,h)$, we have

$$\lim_{\epsilon\to+0}\sup_{\varphi\le\tau<\varphi+\epsilon}\left|B_h^{(j)}(\tau)-B_h^{(j)}(\varphi)\right|_2\to 0. \quad (7.5)$$

Roughly speaking, this implies that we can have the aforementioned approximation (7.2) for a sufficiently small $\epsilon > 0$, wherein $B_h^{(j)}(\tau)$, $\tau\in[\varphi,\varphi+\epsilon)$ are replaced by a single quantity $B_h^{(j)}(\varphi)$. Furthermore, in the sense that we only deal with $B_h^{(j)}(\varphi)$, this approximation can be interpreted as considering the situation in which $\widehat{w}_0^{(j)}$ is the unit impulse applied at $t=\varphi$.

Step 1b. On the other hand, we also have

$$\begin{aligned}\left|\int_\varphi^{\varphi+\epsilon} D_\theta^{(j)}(\tau)\widehat{w}_0^{(j)}(\tau)d\tau - D_\theta^{(j)}(\varphi)\int_\varphi^{\varphi+\epsilon}\widehat{w}_0^{(j)}(\tau)d\tau\right|_2 &\le \int_\varphi^{\varphi+\epsilon}\left|D_\theta^{(j)}(\tau)-D_\theta^{(j)}(\varphi)\right|_2\cdot|\widehat{w}_0^{(j)}(\tau)|d\tau\\ &\le \left(\sup_{\varphi\le\tau<\varphi+\epsilon}\left|D_\theta^{(j)}(\tau)-D_\theta^{(j)}(\varphi)\right|_2\right)\int_\varphi^{\varphi+\epsilon}|\widehat{w}_0^{(j)}(\tau)|d\tau.\end{aligned} \quad (7.6)$$

For $\theta<\varphi$ $(\le\tau)$, both sides of the above inequality are zero by the definition (3.13) of $D_\theta(\tau)$, while, for $\theta>\varphi$, it follows from the continuity of $D_\theta(\tau)$ on the sufficiently small interval $[\varphi,\varphi')$ that

$$\lim_{\epsilon\to+0}\sup_{\varphi\le\tau<\varphi+\epsilon}\left|D_\theta^{(j)}(\tau)-D_\theta^{(j)}(\varphi)\right|_2\to 0 \quad (7.7)$$

for each $\theta\in[0,h)$ and every $\varphi\in[0,h)$. This implies, roughly speaking, that we can have the aforementioned approximation (7.3) for a sufficiently small $\epsilon>0$ (except at $\theta=\varphi$, or as long as we consider integrals with respect to θ as in the computation of the L_2 norm of z), and, in the sense that we only deal with $D_\theta^{(j)}(\varphi)$, this approximation can be interpreted as considering the same situation as above, i.e., $\widehat{w}_0^{(j)}$ is the unit impulse applied at $t=\varphi$.

More precisely, our arguments in L_1 proceed as follows on the basis of the above approximation ideas.

Step 2a. Regarding (7.4), it follows readily from (3.11) that there exists a positive function $\eta_{B,\varphi}(\epsilon)$ such that

$$\sup_{\varphi\le\tau<\varphi+\epsilon}\left|B_h^{(j)}(\tau)-B_h^{(j)}(\varphi)\right|_2\le\eta_{B,\varphi}(\epsilon) \quad (7.8)$$

and $\eta_{B,\varphi}(\epsilon) \to 0$ uniformly with respect to $\varphi \in [0,h)$ as $\epsilon \to +0$. Hence, we have

$$\left| \int_{\varphi}^{\varphi+\epsilon} \left(B_h^{(j)}(\tau) - B_h^{(j)}(\varphi) \right) \widehat{w}_0^{(j)}(\tau) d\tau \right|_2 \le \eta_{B,\varphi}(\epsilon) \int_{\varphi}^{\varphi+\epsilon} |\widehat{w}_0^{(j)}(\tau)| d\tau \tag{7.9}$$

so that, if ($\widehat{w}_0^{(j)}(\tau) = 0$ for $\tau \notin [\varphi, \varphi+\epsilon)$ and) $\int_{\varphi}^{\varphi+\epsilon} |\widehat{w}_0^{(j)}(\tau)| d\tau \le 1$, then the triangle inequality leads to

$$\begin{aligned}
\left| \mathcal{B}\widehat{w}_0 \right|_2 &= \left| \int_{\varphi}^{\varphi+\epsilon} B_h^{(j)}(\tau) \widehat{w}_0^{(j)}(\tau) d\tau \right|_2 = \left| \int_{\varphi}^{\varphi+\epsilon} \left[B_h^{(j)}(\varphi) + \left(B_h^{(j)}(\tau) - B_h^{(j)}(\varphi) \right) \right] \widehat{w}_0^{(j)}(\tau) d\tau \right|_2 \\
&= \left| B_h^{(j)}(\varphi) \int_{\varphi}^{\varphi+\epsilon} \widehat{w}_0^{(j)}(\tau) d\tau + \int_{\varphi}^{\varphi+\epsilon} \left(B_h^{(j)}(\tau) - B_h^{(j)}(\varphi) \right) \widehat{w}_0^{(j)}(\tau) d\tau \right|_2 \\
&= \left| B_h^{(j)}(\varphi) \right|_2 \left| \int_{\varphi}^{\varphi+\epsilon} \widehat{w}_0^{(j)}(\tau) d\tau \right| + \alpha_{B,\epsilon,\varphi}(\widehat{w}_0^{(j)}) \eta_{B,\varphi}(\epsilon)
\end{aligned} \tag{7.10}$$

for some $\alpha_{B,\epsilon,\varphi}(\cdot)$ such that $|\alpha_{B,\epsilon,\varphi}(\widehat{w}_0^{(j)})| \le 1$. Essentially the same arguments lead to

$$\left| (C\mathcal{A}^k \mathcal{B}\widehat{w}_0)(\theta) \right|_2 = \left| C_\theta \mathcal{A}^k \mathcal{B}\widehat{w}_0 \right|_2 = \left| C_\theta \mathcal{A}^k B_h^{(j)}(\varphi) \right|_2 \left| \int_{\varphi}^{\varphi+\epsilon} \widehat{w}_0^{(j)}(\tau) d\tau \right| + \alpha_{B,\epsilon,\varphi,\theta,k}(\widehat{w}_0^{(j)}) \eta_{B,\varphi,\theta,k}(\epsilon) \tag{7.11}$$

for some $\alpha_{B,\epsilon,\varphi,\theta,k}(\cdot)$ such that $|\alpha_{B,\epsilon,\varphi,\theta,k}(\widehat{w}_0^{(j)})| \le 1$ if $\int_{\varphi}^{\varphi+\epsilon} |\widehat{w}_0^{(j)}(\tau)| d\tau \le 1$, and $\eta_{B,\varphi,\theta,k}(\epsilon) \to 0$ uniformly with respect to $\varphi \in [0,h)$ and $\theta \in [0,h)$ as $\epsilon \to +0$. Hence, both $\int_0^h \eta_{B,\varphi,\theta,k}(\epsilon) d\theta$ and $\int_0^h \eta_{B,\varphi,\theta,k}(\epsilon)^2 d\theta$ tend to 0 uniformly with respect to $\varphi \in [0,h)$ as $\epsilon \to +0$.

Step 2b. Regarding (7.6), on the other hand, we first note that, if $\tau \le \theta$, then $\theta \ge \varphi$; thus, it readily follows from (3.13) that there exists a positive function $\eta_{D,\varphi,\theta}(\epsilon)$ such that

$$\sup_{\tau} \left| D_\theta^{(j)}(\tau) - D_\theta^{(j)}(\varphi) \right|_2 \le \eta_{D,\varphi,\theta}(\epsilon) = \eta_{D,\varphi,\theta}(\epsilon) \mathbf{1}(\theta - \varphi) \tag{7.12}$$

where the left-hand side is taken for τ within the intersection of $[\varphi, \varphi+\epsilon)$ and $\tau \le \theta$ (i.e., the interval $[\varphi, \theta]$ if $\theta \in [\varphi, \varphi+\epsilon)$, and the interval $[\varphi, \varphi+\epsilon)$ if $\theta \ge \varphi+\epsilon$); also note that $\eta_{D,\varphi,\theta}(\epsilon) \to 0$ uniformly with respect to $\varphi \in [0,h)$ and $\theta \in [0,h)$ as $\epsilon \to +0$. If $\tau > \theta$, on the other hand, we have that $D_\theta^{(j)}(\tau) = 0$; thus, it readily follows again from (3.13) that there exists a positive constant $\eta_{D',\varphi,\theta}$ such that

$$\sup_{\tau} \left| D_\theta^{(j)}(\tau) - D_\theta^{(j)}(\varphi) \right|_2 \le \eta_{D',\varphi,\theta} \mathbf{1}(\theta - \varphi) \tag{7.13}$$

where the left-hand side is taken for τ within the intersection of $[\varphi, \varphi+\epsilon)$ and $\tau > \theta$ (i.e., the interval $(\theta, \varphi+\epsilon)$, assuming that $\theta \in [\varphi, \varphi+\epsilon)$) and $\eta_{D',\varphi,\theta}$ is bounded with respect to $\varphi \in [0,h)$ and $\theta \in [0,h)$.

Combining the above arguments for $\tau \in [\varphi, \theta]$ and $\tau \in (\theta, \varphi+\epsilon)$ for $\theta \in [\varphi, \varphi+\epsilon)$, and noting that (7.12) holds also for $\theta \ge \varphi+\epsilon$, it follows that

$$\eta_{D'',\varphi}(\epsilon) := \int_{\varphi}^{\varphi+\epsilon} \sup_{\tau \in [\varphi,\theta] \cup (\theta,\varphi+\epsilon)} \left| D_\theta^{(j)}(\tau) - D_\theta^{(j)}(\varphi) \right|_2 d\theta + \int_{\varphi+\epsilon}^{h} \sup_{\tau \in [\varphi,\varphi+\epsilon)} \left| D_\theta^{(j)}(\tau) - D_\theta^{(j)}(\varphi) \right|_2 d\theta \tag{7.14}$$

satisfies that $\eta_{D'',\varphi}(\epsilon) \le \max_{m=1,2} \max \left(\epsilon(\eta_{D,\varphi,\theta}(\epsilon) + \eta_{D',\varphi,\theta})^m, h\eta_{D,\varphi,\theta}(\epsilon)^m \right)$; thus, $\eta_{D'',\varphi}(\epsilon) \to 0$ uniformly with respect to φ as $\epsilon \to +0$. Furthermore, it follows from the triangle inequality (together with (7.12) and (7.13) and essentially the same technique for the derivation of (7.10)) that, if $\int_{\varphi}^{\varphi+\epsilon} |\widehat{w}_0^{(j)}(\tau)| d\tau \le 1$, then

$$\left|(\mathcal{D}\widehat{w}_0)(\theta)\right|_2 = \left|\int_{\varphi}^{\varphi+\epsilon} D_\theta^{(j)}(\tau)\cdot \widehat{w}_0^{(j)}(\tau)d\tau\right|_2 = \left|D_\theta^{(j)}(\varphi)\right|_2 \left|\int_{\varphi}^{\varphi+\epsilon} \widehat{w}_0^{(j)}(\tau)d\tau\right| + \alpha_{D,\epsilon,\varphi,\theta}(\widehat{w}_0^{(j)})\bar{\eta}_{D,\varphi,\theta}(\epsilon)\mathbf{1}(\theta-\varphi) \tag{7.15}$$

for some $\alpha_{D,\epsilon,\varphi,\theta}(\cdot)$ such that $|\alpha_{D,\epsilon,\varphi,\theta}(\widehat{w}_0^{(j)})| \le 1$ whenever $\int_{\varphi}^{\varphi+\epsilon} |\widehat{w}_0^{(j)}(\tau)| d\tau \le 1$. Here, the aforementioned uniform convergence property of $\eta_{D'',\varphi}(\epsilon)$ about (7.14) implies that $\bar{\eta}_{D,\varphi,\theta}(\epsilon)$ is such a function that both $\int_{\varphi}^{h} \bar{\eta}_{D,\varphi,\theta}(\epsilon)d\theta$ and $\int_{\varphi}^{h} \bar{\eta}_{D,\varphi,\theta}(\epsilon)^2 d\theta$ tend to 0 uniformly with respect to $\varphi \in [0,h)$ as $\epsilon \to +0$. For the case of (7.15), we have thus arrived at a situation that is parallel to what has been established in Step 2a for the case of (7.11).

Step 3. We are in the final position to complete the justification of the treatment developed in Section 3. The square of the L_2 norm of the output z corresponding to $\widehat{w}_0^{(j)}$ can be computed by using (7.11) and (7.15), where the maximum of $\left|\int_{\varphi}^{\varphi+\epsilon} \widehat{w}_0^{(j)}(\tau)d\tau\right|$ under the assumption that $\int_{\varphi}^{\varphi+\epsilon} |\widehat{w}_0^{(j)}(\tau)|d\tau \le 1$ is (attained and) obviously 1. Furthermore, recall that the L_2/L_1 induced norm $\|\Sigma_{\mathrm{SD}}\|_{2/1}$ was associated with the situation with an infinitesimally small ε. Thus, by considering $\widehat{w}_0^{(j)}$ such that $\int_{\varphi}^{\varphi+\epsilon} |\widehat{w}_0^{(j)}(\tau)|d\tau = 1$ and $\left|\int_{\varphi}^{\varphi+\epsilon} \widehat{w}_0^{(j)}(\tau)d\tau\right| = 1$ for each ϵ, and by taking the limit $\epsilon \to +0$ with the uniform convergence stated at the end of Steps 2a and 2b taken into account, it follows that the square of the aforementioned L_2 norm tends to

$$H^{(j)}(\varphi) = \int_0^h \left|D_\theta^{(j)}(\varphi)\right|_2^2 d\theta + \sum_{k=0}^{\infty} \int_0^h \left|C_\theta \mathcal{A}^k B_h^{(j)}(\varphi)\right|_2^2 d\theta \tag{7.16}$$

since $D_\theta^{(j)}(\varphi) = 0$ for $\theta < \varphi$ according to (3.13), where the notation on the left-hand side is used because the right-hand side equals the jth diagonal entry of the matrix $H(\varphi)$ defined in (3.15). Furthermore, it is easy to see that this value cannot be exceeded through the use of $\widehat{w}_0^{(j)}$ such that $|\int_{\varphi}^{\varphi+\epsilon} \widehat{w}_0^{(j)}(\tau)d\tau| < 1$. Since the L_2/L_1 induced norm was further associated with the worst timing $\varphi \in [0,h)$, together with the worst j among $j = 1,\ldots,n_w$, we have justified Theorem 1 through no use of an actual treatment of directly applying an impulse input.

C. Proof of Corollary 4

This appendix is devoted to the proof of Corollary 4. We first consider the case of $p = 1$. The proof proceeds in three steps.

Step 1_1**.** Suppose that the condition (i-1) fails, i.e., suppose that either of the following conditions holds:

(a-1) there exists no $j = 1,\ldots,n_w$ such that $P_{11}^{(j)} = 0$;
(b-1) (a-1) is not true, and the jth diagonal entry of $H(0)$ satisfies that $H^{(j)}(0) \ne \mu_p(H(0))$ whenever $P_{11}^{(j)} = 0$.

For the case (a-1), the assertion (II) of Theorem 8 implies that

$$\|\mathbf{H}_{2/(1,p)}^{[0]}\| < \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}. \tag{7.17}$$

Hence, the proof for $p = 1$ will be completed by showing that the same inequality follows also for the case (b-1), which can be rephrased equivalently as

(b-1)' (a-1) is not true, and $(H^{(j)}(0))^{1/2} < \mu_p^{1/2}(H(0)) = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ whenever $P_{11}^{(j)} = 0$

because $H^{(j)}(0) \le \mu_p(H(0))$ according to the definition of $\mu_p(\cdot)$ for $p = 1$, and $\mu_p^{1/2}(H(0)) = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ by the hypothesis of Theorem 8 (which obviously is included implicitly in the hypothesis of this corollary) and (4.14). Now, let j be as required in (b-1)', i.e., $P_{11}^{(j)} = 0$ and $(H^{(j)}(0))^{1/2} < \mu_p^{1/2}(H(0)) = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$, and we show that

$$\sup_{0\le\varphi<h} (H^{(j)}(\varphi))^{1/2} < \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} \tag{7.18}$$

under the hypothesis of this corollary (which includes the hypothesis about the assertion (II) of Theorem 8). To this end, suppose the contrary to (7.18), which, obviously, yields $\sup_{0\le\varphi<h}(H^{(j)}(\varphi))^{1/2} = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ by (4.14). Since $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = \mu_p^{1/2}(H(0)) > \mu_p^{1/2}(H(h))$ by the hypothesis, we have that $\mu_p^{1/2}(H(0)) > (H^{(j)}(h))^{1/2}$; thus, assuming that $\sup_{0\le\varphi<h}(H^{(j)}(\varphi))^{1/2} = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ leads to $\sup_{0\le\varphi<h}(H^{(j)}(\varphi))^{1/2} = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} > (H^{(j)}(h))^{1/2}$. This obviously implies the existence of $\varphi_0 \in [0, h)$ such that $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = (H^{(j)}(\varphi_0))^{1/2}$, but such φ_0 cannot be zero according to (b-1)', i.e., $\varphi_0 \in (0, h)$. On the other hand, since $(H^{(j)}(\varphi_0))^{1/2} \le \mu_p^{1/2}(H(\varphi_0))$, we have that $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} \le \mu_p^{1/2}(H(\varphi_0))$ given the fact that $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = (H^{(j)}(\varphi_0))^{1/2}$ mentioned just above; however, the strict inequality cannot hold because of (4.14), which implies that $\varphi_0 \in (0, h)$ is a maximum-attaining point of $\mu_p^{1/2}(H(\varphi))$ (and further implies that $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = \mu_p^{1/2}(H(\varphi_0))$). However, this contradicts the hypothesis of this corollary; thus, we have established (7.18).

Step 2_1**.** The inequality (7.18) implies that the existence of the jth input channel of $\Sigma_{\rm SD}$ has no direct contribution to the value of $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$, and this fact, in turn, implies that the existence of the jth input channel has no influence on whether or not $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ is attained as $\|\mathbf{H}_{2/(1,p)}^{[0]}\|$. From this observation, let us consider the sampled-data system $\Sigma_{\rm SD}$ with its jth input channel removed for each j such that $P_{11}^{(j)} = 0$ and $(H^{(j)}(0))^{1/2} < \mu_p^{1/2}(H(0))$. The set of all such j is denoted by J, and the resulting sampled-data system is denoted by $\Sigma_{{\rm SD}\langle J\rangle}$, where it should be noted that J can never coincide with the entire set $\{1, \ldots, n_w\}$; this is because $(H^{(j)}(0))^{1/2} < \mu_p^{1/2}(H(0))$ for all $j = 1, \ldots, n_w$ obviously contradicts the definition of $\mu_p(\cdot)$ for $p = 1$. Hence, $\Sigma_{{\rm SD}\langle J\rangle}$ does make sense. Let the $H(\varphi)$ corresponding to $\Sigma_{{\rm SD}\langle J\rangle}$ be denoted by $H_{\langle J\rangle}(\varphi)$. It is then obvious from the above arguments, together with (4.14), that

$$\sup_{0\le\varphi<h} \mu_p^{1/2}(H_{\langle J\rangle}(\varphi)) = \|\Sigma_{{\rm SD}\langle J\rangle}\|_{{\rm H},2/(1,p)} = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = \sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi)) = \mu_p^{1/2}(H(0)) \tag{7.19}$$

where the second equality is a direct consequence of the above arguments, together with (4.14), and the last equality follows from the hypothesis of Theorem 8. Furthermore, the inequality in (b-1)' implies that "removing the jth input channel for $j \in J$ does not affect $\mu_p^{1/2}(H(0))$," which, precisely speaking, means that $\mu_p^{1/2}(H(0)) = \mu_p^{1/2}(H_{\langle J\rangle}(0))$, because $H_{\langle J\rangle}(\varphi)$ is nothing but the submatrix of $H(\varphi)$ that is obtained by removing its jth row and column for all $j \in J$. Hence, it follows from (7.19) that

$$\sup_{0\le\varphi<h} \mu_p^{1/2}(H_{\langle J\rangle}(\varphi)) = \mu_p^{1/2}(H_{\langle J\rangle}(0)) \tag{7.20}$$

(i.e., $\mu_p^{1/2}(H_{\langle J\rangle}(\varphi))$ attains the maximum over $\varphi \in [0, h)$ at $\varphi = 0$) and, also,

$$\mu_p^{1/2}(H_{\langle J\rangle}(0)) = \mu_p^{1/2}(H(0)) > \mu_p^{1/2}(H(h)) \ge \mu_p^{1/2}(H_{\langle J\rangle}(h)) \tag{7.21}$$

where the strict inequality follows from the hypothesis about the assertion (II) of Theorem 8.

Step 3_1**.** Now, (7.20) and (7.21) imply that $\Sigma_{\mathrm{SD}\langle J\rangle}$ satisfies the hypothesis corresponding to the assertion (II) of Theorem 8; thus, we can apply it to $\Sigma_{\mathrm{SD}\langle J\rangle}$. Noting that the present proof was initiated under the hypothesis of this corollary that (i-1) fails for the original sampled-data system Σ_{SD} (which means either (a-1) or (b-1)), and further noting that the construction of $\Sigma_{\mathrm{SD}\langle J\rangle}$ rules out the condition (b-1), we see that the condition (ii-1) (corresponding to the condition (a-1)) must be satisfied for $\Sigma_{\mathrm{SD}\langle J\rangle}$; thus, we are led to $\|\mathbf{H}^{[0]}_{2/(1,p)\langle J\rangle}\| < \|\Sigma_{\mathrm{SD}\langle J\rangle}\|_{\mathrm{H},2/(1,p)}$, where the right-hand side denotes the quasi $L_2/L_{1,p}$ Hankel norm at 0 for $\Sigma_{\mathrm{SD}\langle J\rangle}$. This immediately implies that

$$\|\mathbf{H}^{[0]}_{2/(1,p)\langle J\rangle}\| < \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)} \tag{7.22}$$

by (7.19). We finally derive (7.17) from the above inequality by supposing the contrary to (7.17), which is obviously $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}$ by (2.4), and showing that we are then led to contradiction. To this end, we first note from (4.12) that $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \sup_{0\le\varphi<h}\mu_p^{1/2}(F_{0,1}(\varphi))$ and $\|\mathbf{H}^{[0]}_{2/(1,p)\langle J\rangle}\| = \sup_{0\le\varphi<h}\mu_p^{1/2}(F_{0,1\langle J\rangle}(\varphi))$, where $F_{0,1\langle J\rangle}(\varphi)$ is the submatrix of $F_{0,1}(\varphi)$ that is obtained by removing its jth row and column for all $j \in J$. Hence, if $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}$ in spite of (7.22), then there must exist some $j \in J$ such that $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \sup_{0\le\varphi<h}(F^{(j)}_{0,1}(\varphi))^{1/2}$. By (4.17), this implies that $\|\mathbf{H}^{[0]}_{2/(1,p)}\| \le \sup_{0\le\varphi<h}(H^{(j)}(\varphi))^{1/2}$; thus, $\|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)} \le \sup_{0\le\varphi<h}(H^{(j)}(\varphi))^{1/2}$ by the assumption $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \|\Sigma_{\mathrm{SD}}\|_{\mathrm{H},2/(1,p)}$, but this contradicts (7.18). This completes the proof for $p = 1$.

We next give the proof for the case of $p = 2$. Even though it follows some similar arguments, it does not necessarily follow parallel steps to the arguments for $p = 1$ to avoid some intrinsic issues relevant to the case of $p = 2$, e.g., dealing with infinitely many v in connection with the treatment of $\mu_p(H(0))$ through the use of $v^T H(0)v$.

Step 1_2**.** Let us denote the maximum eigenvalue of $H(0)$ by $\overline{\lambda}$. We first note that v with $|v|_2 = 1$ satisfies that $v^T H(0)v = \mu_p(H(0))$ $(= \overline{\lambda})$ if and only if v belongs to the eigenspace $W_{\overline{\lambda}}$ of $H(0)$ that corresponds to the eigenvalue $\overline{\lambda}$. In other words, v satisfies that $v^T H(0)v = \overline{\lambda}|v|_2^2$ if and only if $v \in W_{\overline{\lambda}}$, which implies that the set of v such that $v^T H(0)v = \overline{\lambda}|v|_2^2$ forms a linear space, which is nothing but $W_{\overline{\lambda}}$. On the other hand, the set of v such that $P_{11}v = 0$ is also a linear space, which we denote by W_0. The hypothesis of this corollary that (i-2) fails for the sampled-data system Σ_{SD} means that $W_{\overline{\lambda}} \cap W_0 = \{0\}$. Let v_i $(i = 1, \ldots, r_{\overline{\lambda}})$ be orthonormal vectors spanning $W_{\overline{\lambda}}$, and let us take an orthogonal matrix $V = [\overline{V}\ \underline{V}]$ such that the columns of $\overline{V}$ are given by these vectors. We further consider the sampled-data system Σ_{SD} with B_1 replaced by $B_1 V$, which means that P_{11} is replaced by $P_{11}V$ (and P_{21} is replaced by $P_{21}V$). The resulting sampled-data system is denoted by $\Sigma_{\mathrm{SD}}V$. Noting that the orthogonal matrix V does not affect the quasi $L_2/L_{1,p}$ Hankel norm and the $L_2/L_{1,p}$ Hankel norm for the case of $p = 2$, it follows that it suffices to justify the assertion of this corollary when the given sampled-data system is actually the modified sampled-data system $\Sigma_{\mathrm{SD}}V$. Note that v' satisfies that $(P_{11}V)v' = 0$ for the modified sampled-data system $\Sigma_{\mathrm{SD}}V$ if and only if $Vv' \in W_0$, while $H(\varphi)$ corresponding to $\Sigma_{\mathrm{SD}}V$, denoted by $H_V(\varphi)$, satisfies that $v'H_V(0)v' = \mu_p(H_V(0))|v'|_2^2$ $(= \overline{\lambda}|v'|_2^2)$ if and only if $Vv' \in W_{\overline{\lambda}}$, because $H_V(\varphi)$ is obviously given by $V^T H(\varphi)V$, i.e., because $v'H_V(0)v' = (Vv')^T H(0)(Vv')$. These observations imply the natural consequence that the condition (i-2) of Theorem 8 also fails in the modified sampled-data system $\Sigma_{\mathrm{SD}}V$, because $W_{\overline{\lambda}} \cap W_0 = \{0\}$, as mentioned above.

Here, we may assume, without loss of generality, that $r_{\overline{\lambda}} < n_w$. This is because, otherwise, $W_{\overline{\lambda}}$ equals $\mathbb{R}^{n_w}$, i.e., every v such that $|v|_2 = 1$ satisfies that $v^T H(0)v = \mu_p(H(0))$ according to the definition of $W_{\overline{\lambda}}$, which in turn implies that the failing of the condition (i-2) in the original sampled-data system $\Sigma_{\rm SD}$ immediately implies that it satisfies the condition (ii-2), leading to the conclusion that $\|\mathbf{H}^{[0]}_{2/(1,p)}\| < \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ by the assertion (II) of Theorem 8; hence, the proof of this corollary is completed in such a case.

With the above arguments in mind, let $J := \{r_{\overline{\lambda}} + 1, \dots, n_w\}$ and consider the modified sampled-data system $\Sigma_{\rm SD}V$ with the jth input channel removed for $j \in J$ (which is nothing but $\Sigma_{\rm SD}\overline{V}$). Furthermore, this is also nothing but $\Sigma_{{\rm SD}\langle J\rangle}$ in the notation introduced in the arguments for $p = 1$, provided that the modified sampled-data system $\Sigma_{\rm SD}V$ is identified with the original sampled-data system $\Sigma_{\rm SD}$ (for the aforementioned reason that the orthogonal matrix V does not affect the quasi $L_2/L_{1,p}$ Hankel norm and the $L_2/L_{1,p}$ Hankel norm for the case of $p = 2$), and we indeed do so in the following arguments. It is then obvious that $H(\varphi)$ corresponding to $\Sigma_{{\rm SD}\langle J\rangle}$, denoted by $H_{\langle J\rangle}(\varphi)$, is given by $\overline{V}^T H(\varphi)\overline{V}$, with $H(\varphi)$ being the one that is defined for the original sampled-data system $\Sigma_{\rm SD}$.

Before dealing with $\Sigma_{{\rm SD}\langle J\rangle}$, however, we first establish

$$\sup_{0\le\varphi<h} (v^T H(\varphi)v)^{1/2} < \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}, \quad \forall v \notin W_{\overline{\lambda}},\ |v|_2 = 1 \tag{7.23}$$

by following the arguments similar to those used in the case of $p = 1$ for the derivation of (7.18). To this end, suppose the contrary to the above inequality for some $v \notin W_{\overline{\lambda}}$ with $|v|_2 = 1$. Then, by using this v and replacing $H^{(j)}(\varphi)$ with $v^T H(\varphi)v$ (including the case in which φ is actually h or φ_0) in the arguments for $p = 1$, and further noting that $\mu_p^{1/2}(H(h)) \ge (v^T H(h)v)^{1/2}$ because $|v|_2 = 1$, we can readily see that $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = (v^T H(\varphi_0)v)^{1/2}$ for some $\varphi_0 \in [0, h)$. However, since $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = \mu_p^{1/2}(H(0))$ by the hypothesis of Theorem 8, such φ_0 cannot be zero because $v \notin W_{\overline{\lambda}}$. Hence, we have that $\varphi_0 > 0$, but this obviously contradicts the hypothesis of this corollary. Thus, we have established (7.23).

Step 2_2**.** The inequality (7.23) implies that, in an equivalent impulse response interpretation for the quasi L_2/L_1 Hankel norm and the L_2/L_1 Hankel norm, "applying an impulse input from the direction that does not belong to $W_{\overline{\lambda}}$ does not contribute to the values of these norms", which, more precisely, implies (the second equality of) (7.19) for the case of $p = 2$. More formally, we see that the maximum of $\sup_{0\le\varphi<h}(v^T H(\varphi)v)^{1/2}$ over v with $|v|_2 = 1$ is attained at $v \in W_{\overline{\lambda}}$ because $\sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi)) = \mu_p^{1/2}(H(0))$ by the hypothesis of Theorem 8; this, in turn, implies that $\sup_{0\le\varphi<h} \mu_p^{1/2}(H_{\langle J\rangle}(\varphi)) = \sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi))$ because $H_{\langle J\rangle}(\varphi) = \overline{V}^T H(\varphi)\overline{V}$ and ${\rm Im}\,\overline{V} = W_{\overline{\lambda}}$, where ${\rm Im}\,\overline{V}$ denotes the image of $\overline{V}$. Thus, we are led to

$$\|\Sigma_{{\rm SD}\langle J\rangle}\|_{{\rm H},2/(1,p)} = \sup_{0\le\varphi<h} \mu_p^{1/2}(H_{\langle J\rangle}(\varphi)) = \sup_{0\le\varphi<h} \mu_p^{1/2}(H(\varphi)) = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} = \mu_p^{1/2}(H(0)) \tag{7.24}$$

where the last equality follows again from the hypothesis. Furthermore, (7.21) follows for the same reason (i.e., $H_{\langle J\rangle}(0) = \overline{V}^T H(0)\overline{V}$). In particular, the equality in (7.21), together with (7.24), leads to (7.20).

Step 3_2**.** As in the case of $p = 1$, it is important to note that (7.20) and (7.21) established in the above step imply that the hypothesis about the assertion (II) of Theorem 8 is satisfied for $\Sigma_{{\rm SD}\langle J\rangle}$. It is also important to note that, since the condition (i-2) fails for the original $\Sigma_{\rm SD}$ by the hypothesis of this corollary, it suffices to consider only $\Sigma_{{\rm SD}\langle J\rangle}$ for the original sampled-data system $\Sigma_{\rm SD}$ satisfying the following condition:

(b-2) for each $v \in W_{\overline{\lambda}}$ (or, equivalently, $(v^T H(0)v)^{1/2} = \mu_p^{1/2}(H(0)) = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$) with $|v|_2 = 1$, the condition $P_{11}v \neq 0$ is satisfied.

Since $H(\varphi)$ corresponding to $\Sigma_{{\rm SD}\langle J\rangle}$ is given by $\overline{V}^T H(\varphi)\overline{V}$ and ${\rm Im}\,\overline{V} = W_{\overline{\lambda}}$, and since P_{11} corresponding to $\Sigma_{{\rm SD}\langle J\rangle}$ is given by $P_{11}\overline{V}$, with P_{11} being the one that is defined for the original sampled-data system $\Sigma_{\rm SD}$, we readily see, under the above condition (b-2), that the condition (i-2) fails, but the condition (ii-2) does hold when Theorem 8 is applied to $\Sigma_{{\rm SD}\langle J\rangle}$. Hence, we are led to $\|\mathbf{H}^{[0]}_{2/(1,p)\langle J\rangle}\| < \|\Sigma_{{\rm SD}\langle J\rangle}\|_{{\rm H},2/(1,p)}$, and this, in turn, together with (7.24), implies (7.22) for the case of $p = 2$, i.e., $\|\mathbf{H}^{[0]}_{2/(1,p)\langle J\rangle}\| < \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$, as in the case of $p = 1$.

We finally establish (7.17), i.e., $\|\mathbf{H}^{[0]}_{2/(1,p)}\| < \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$, also for $p = 1$, by supposing its contrary, which, obviously, is $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$, and by showing that we are then led to contradiction. To this end, we first note from (4.12) that $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \sup_{0\le\varphi<h}\mu_p^{1/2}(F_{0,1}(\varphi))$; thus, $\|\mathbf{H}^{[0]}_{2/(1,p)\langle J\rangle}\| = \sup_{0\le\varphi<h}\mu_p^{1/2}(F_{0,1\langle J\rangle}(\varphi))$, where $F_{0,1\langle J\rangle}(\varphi) = \overline{V}^T F_{0,1}(\varphi)\overline{V}$. Hence, if $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$ in spite of (7.22), i.e., $\|\mathbf{H}^{[0]}_{2/(1,p)\langle J\rangle}\| < \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$, then it implies that

$$\sup_{0\le\varphi<h}\mu_p^{1/2}(\overline{V}^T F_{0,1}(\varphi)\overline{V}) < \sup_{0\le\varphi<h}\mu_p^{1/2}(F_{0,1}(\varphi)) = \|\mathbf{H}^{[0]}_{2/(1,p)}\|. \tag{7.25}$$

Since ${\rm Im}\,\overline{V} = W_{\overline{\lambda}}$, the above inequality implies that the maximum of $\sup_{0\le\varphi<h}(v^T F_{0,1}(\varphi)v)^{1/2}$ over v with $|v|_2 = 1$ is attained at $v \notin W_{\overline{\lambda}}$, and such v further satisfies that $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \sup_{0\le\varphi<h}(v^T F_{0,1}(\varphi)v)^{1/2}$ by the above equality. By (4.17), this implies that $\|\mathbf{H}^{[0]}_{2/(1,p)}\| \le \sup_{0\le\varphi<h}(v^T H(\varphi)v)^{1/2}$; thus, $\|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)} \le \sup_{0\le\varphi<h}(v^T H(\varphi)v)^{1/2}$ by the assumption that $\|\mathbf{H}^{[0]}_{2/(1,p)}\| = \|\Sigma_{\rm SD}\|_{{\rm H},2/(1,p)}$. However, since $v \notin W_{\overline{\lambda}}$ and $|v|_2 = 1$, this contradicts (7.23); hence, the proof for $p = 2$ is completed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest that may influence the publication of this paper.

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